Loops of Lagrangian submanifolds and pseudoholomorphic discs

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Abstract

We examine three invariants of exact loops of Lagrangian submanifolds that are modelled on invariants introduced by Polterovich for loops of Hamiltonian symplectomorphisms. One of these is the minimal Hofer length in a given Hamiltonian isotopy class. We determine the exact values of these invariants for loops of projective Lagrangian planes. The proof uses the Gromov invariants of an associated symplectic fibration over the 2-disc with a Lagrangian subbundle over the boundary.

1 Introduction

In this paper we study the Hofer geometry for exact loops of Lagrangian submanifolds of a symplectic manifold (M,ω). Think of such a loop as a connected submanifold $\Lambda \subset S^1 \times M$ such that the projection $\Lambda \to S^1$ is a submersion and

$$\Lambda_t := \{z \in M \mid (e^{2\pi i t}, z) \in \Lambda\}$$

is a Lagrangian submanifold of M for every t. The loop is called exact if there exists a Hamiltonian isotopy ψ_t of M such that $\psi_t(\Lambda_0) = \Lambda_t$ for every t. The Hofer length of an exact Lagrangian loop Λ is defined by

$$\ell(\Lambda) := \int_0^1 \left(\max_{\Lambda_t} H_t - \min_{\Lambda_t} H_t \right) dt,$$

where the Hamiltonian functions $H_t : M \to \mathbb{R}$ are chosen such that the corresponding Hamiltonian isotopy $\psi_t : M \to M$ satisfies $\psi_t(\Lambda_0) = \Lambda_t$. It is interesting to minimize the Hofer length over the Hamiltonian isotopy class of Λ. This infimum will be denoted by

$$\nu(\Lambda) = \nu(\Lambda; M, \omega) := \inf_{\Lambda \sim \Lambda'} \ell(\Lambda').$$

As an explicit example consider the space $\mathcal{L} = \mathcal{L}(\mathbb{C}P^n, \mathbb{R}P^n)$ of Lagrangian submanifolds of $\mathbb{C}P^n$ that are diffeomorphic to $\mathbb{R}P^n$. It contains the finite dimensional manifold $\mathrm{PL}(n+1)$ of projective Lagrangian planes. The space $\mathrm{PL}(n+1)$ is the orbit of $\mathbb{R}P^n$ under the action of $\mathrm{PU}(n+1)$ and its fundamental group is isomorphic to $\mathbb{Z}_{n+1}$. Consider the loop $\Lambda^k \subset S^1 \times \mathbb{C}P^n$ defined by

$$\Lambda^k := \bigcup_{t \in \mathbb{R}} \{e^{2\pi i t}\} \times \phi_{kt}(\mathbb{R}P^n), \tag{1}$$

where $\phi_t([z_0 : \cdots : z_n]) := [e^{\pi i t} z_0 : z_1 : \cdots : z_n]$ and $k \in \mathbb{Z}$. The loops Λ^j and Λ^k are homotopic in $\mathrm{PL}(n+1)$ (as based loops) if and only if they are Hamiltonian isotopic (as free loops) if and only if $k - j$ is divisible by $n+1$. If $k - j$ is not divisible by $n+1$ then Λ^j and Λ^k can be distinguished by the Maslov index. More precisely, every Lagrangian loop $\Lambda \subset S^1 \times \mathbb{C}P^n$, with fibres Λ_t Lagrangian isotopic to $\mathbb{R}P^n$, has a well defined Maslov index $\mu(\Lambda) \in \mathbb{Z}_{n+1}$. It is defined as the Maslov index of a smooth map $u : D = \{z \in \mathbb{C} \,|\, |z| \le 1\} \to M$ such that $u(e^{2\pi i t}) \in \Lambda_t$. Such maps u always exist and the Maslov indices of any two such maps differ by an integer multiple of $n+1$. It turns out that

$$\mu(\Lambda^k) \equiv k \bmod n+1. \tag{2}$$

In the case $n = 1$ the loop Λ^1 is obtained by rotating a great circle on the 2-sphere through 180 degrees around an axis that passes through the circle. The result is an embedding of the Klein bottle into $S^1 \times S^2$. The image of this embedding is a Lagrangian submanifold of $D \times S^2$ with respect to a suitable symplectic form. In contrast Λ^0 is a Lagrangian torus in $D \times S^2$. In general, the cases where n is even and where n is odd are topologically different. If n is even, then Λ^k is diffeomorphic to $S^1 \times \mathbb{R}P^n$ for every k. If n is odd then Λ^j is diffeomorphic to Λ^k if and only if $k - j$ is even, and Λ^k is orientable if and only if k is even. In particular, Λ^k is diffeomorphic to $\Lambda^0 = S^1 \times \mathbb{R}P^n$ whenever k is even.

Fix $k \in \{1,\dots,n\}$ and consider the exact Lagrangian loop

$$\Lambda := \bigcup_{t\in\mathbb{R}} \{e^{2\pi it}\} \times \psi_t(\mathbb{R}P^n),$$

where

$$\psi_t([z_0:\cdots:z_n]) := ([z_0 : e^{\pi it}z_1 : \cdots : e^{\pi it}z_k : z_{k+1} : \cdots : z_n]).$$

This loop is Hamiltonian isotopic to Λ^k and it has Hofer length $1/2$, whereas Λ^k has Hofer length $k/2$. The next theorem asserts that Λ minimizes the Hofer length in its Hamiltonian isotopy class and hence is a geodesic for the Hofer metric.

Theorem A *Let $\omega \in \Omega^2(\mathbb{C}P^n)$ denote the Fubini-Study form that satisfies the normalization condition $\int_{\mathbb{C}P^n} \omega^n = 1$. Then*

$$\nu(\Lambda^k; \mathbb{C}P^n, \omega) = \frac{1}{2}$$

for $k = 1,\dots,n$ and $\nu(\Lambda^0) = 0$.

This is a Lagrangian analogue of a theorem by Polterovich [20] about loops of Hamiltonian symplectomorphisms of complex projective space. Following [20] we consider two other invariants of exact Lagrangian loops $\Lambda \subset S^1 \times M$ that can be expressed in terms of Hamiltonian connection 2-forms τ on the trivial bundle $D \times M$ that vanish over Λ. Let $\mathcal{T}(\Lambda) \subset \Omega^2(D\times M)$ denote the space of such connection 2-forms. The **relative K-area** $\chi(\Lambda)$ is obtained by minimizing the Hofer norm of the curvature Ω_τ over $\mathcal{T}(\Lambda)$. The third invariant is related to the relative cohomology classes $[\tau] \in H^2(D\times M, \Lambda; \mathbb{Z})$ of $\tau \in \mathcal{T}(\Lambda)$. These form a 1-dimensional affine space parallel to the subspace generated by the integral cohomology class $\sigma := [dx\wedge dy/\pi]$. For $\tau_0, \tau_1 \in \mathcal{T}(\Lambda)$ define $s(\tau_1,\tau_0) \in \mathbb{R}$ by $s(\tau_1,\tau_0)\sigma = [\tau_1] - [\tau_0]$. The invariant $\varepsilon(\Lambda)$ is defined by

$$\varepsilon(\Lambda) := \varepsilon^+(\tau_0,\Lambda) - \varepsilon^-(\tau_0,\Lambda),$$

for $\tau_0 \in \mathcal{T}(\Lambda)$, where

$$\varepsilon^+(\tau_0,\Lambda) := \inf\{s(\tau,\tau_0) \mid \tau \in \mathcal{T}(\Lambda),\ \tau^{n+1} > 0\},$$

$$\varepsilon^-(\tau_0,\Lambda) := \sup\{s(\tau,\tau_0) \mid \tau \in \mathcal{T}(\Lambda),\ \tau^{n+1} < 0\}.$$

Theorem B *For every exact Lagrangian loop* $\Lambda \subset S^1 \times M$

$$\varepsilon(\Lambda) \leq \chi(\Lambda) = \nu(\Lambda).$$

A lower bound for $\varepsilon(\Lambda)$ can sometimes be obtained by studying pseudoholomorphic sections of $D \times M$ with boundary values in Λ. We assume that the pair (M, Λ_0) is monotone and fix a class $A \in H_2(D \times M, \Lambda; \mathbb{Z})$ that satisfies

$$n \pm \mu_\Lambda(A) \leq N - 2,$$

where $n = \dim \Lambda_0 = \dim M/2$, N denotes the minimal Maslov number of the pair (M, Λ_0), and μ_Λ denotes the Maslov class. Under these assumptions we define Gromov invariants

$$\mathrm{Gr}_A^\pm(\Lambda) \in H_{n \pm \mu_\Lambda(A)}(\Lambda_0; \mathbb{Z}_2).$$

A connection 2-form $\tau \in \mathcal{T}(\Lambda)$ and an ω-compatible almost complex structure J on M determine an almost complex structure $\tilde{J} = \tilde{J}(\tau, J)$ on $D \times M$. Under our assumptions the moduli space of $\tilde{J}(\tau, \pm J)$-holomorphic sections of $D \times M$ is, for a generic τ, a compact smooth manifold of dimension $n \pm \mu_\Lambda(A)$. The Gromov invariant is defined as the image of the mod-2 fundamental class under the evaluation map $u \mapsto u(1)$. Now let $\Lambda^k \subset S^1 \times \mathbb{C}P^n$ be given by (1) with $1 \leq k \leq n$. Let $A^\pm \in H_2(D \times \mathbb{C}P^n, \Lambda^k; \mathbb{Z})$ be the homology classes of the constant sections $u^+(x, y) \equiv [1 : 0 : \cdots : 0]$ and $u^-(x, y) \equiv [0 : \cdots : 0 : 1]$.

Theorem C $\mathrm{Gr}_{A^\pm}^\pm(\Lambda^k) \neq 0$.

Theorem C can be interpreted as an existence result for pseudoholomorphic sections and we shall use this to prove that $\varepsilon(\Lambda^k) \geq 1/2$. On the other hand the Hamiltonian isotopy class of Λ^k contains a loop of length equal to $1/2$. Hence Theorem A follows from Theorem B.

We expect that the same techniques can be used to obtain similar results for general symplectic quotients of $\mathbb{C}^n$ by subgroups of $\mathrm{U}(n)$. These quotients will not, in general, satisfy our assumption of monotonicity for the definition of the Gromov invariants. However, it should be possible to derive the same conclusions by using the invariants introduced in Cieliebak–Gaio–Salamon [4] instead. This programme will be carried out elsewhere.

In [20, 21, 22, 23] Polterovich studied the Hofer length of loops $\psi_t = \psi_{t+1} : M \to M$ of Hamiltonian symplectomorphisms. Let $P \to S^2$ denote the Hamiltonian fibration associated to the Hamiltonian loop. Poltervich introduced invariants $\nu^\pm(P)$, $\chi^\pm(P)$, and $\varepsilon^\pm(P)$ on which our invariants are modelled. Here $\nu^+(P)$ is obtained by minimizing the positive part of the Hofer length in a given Hamiltonian isotopy class, the K-area $\chi^+(P)$ is a symplectic analogue of an invariant introduced by Gromov [10], and the invariant $\varepsilon^+(P)$ is based on the coupling construction of Guillemin–Lerman–Sternberg [11]. In [20, 21] Polterovich proves that these invariants are equal:

$$\varepsilon^\pm(P) = \chi^\pm(P) = \nu^\pm(P).$$

We adopt the convention $\pm\nu^\pm(P) \geq 0$. Let us denote by $\nu(P)$, $\chi(P)$, and $\varepsilon(P)$ the Hamiltonian analogues of our invariants of Lagrangian loops. These were also considered by Polterovich and he noted that

$$\varepsilon(P) = \varepsilon^+(P) - \varepsilon^-(P) = \nu^+(P) - \nu^-(P) \leq \nu(P).$$

This is the Hamiltonian analogue of Theorem B. Now consider the Lagrangian loop $\Lambda \subset S^1 \times \bar{M} \times M$ given by

$$\Lambda_t = \mathrm{graph}(\psi_t).$$

The invariants introduced by Polterovich are related to our invariants by

$$\nu(\Lambda) \leq \nu(P), \qquad \varepsilon(\Lambda) \leq \varepsilon(P).$$

The Gromov invariants of the fibration P associated to a Hamiltonian loop were independently studied by Seidel [28, 29, 30] and his results were used by Lalonde–McDuff–Polterovich [15] to prove that Hamiltonian loops act trivially on homology. Our results on the Gromov invariants can be viewed as Lagrangian analogues of results in [20, 28] on the Gromov invariants of symplectic fibrations.

The present paper is organized as follows. In Section 2 we discuss background material about the Hofer metric. The space of Lagrangian submanifolds is naturally foliated by Hamiltonian isotopy classes and the Hofer metric is defined on each leaf of this foliation. In Section 3 we introduce the invariants $\nu(\Lambda)$, $\chi(\Lambda)$, and $\varepsilon(\Lambda)$ of exact Lagrangian loops and give a proof of Theorem B. In the 2-dimensional case the invariant $\nu(\Lambda)$ can sometimes be computed explicitly. This is done in Section 4 for the 2-torus. In Section 5 we introduce the Gromov invariants and in Section 6 we prove Theorems A and C. In Appendix A we prove a result about Hamiltonian isotopy on Riemann surfaces which is used in Section 4.

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2 The Hofer metric for Lagrangian submanifolds

Let (M, ω) be a $2n$-dimensional symplectic manifold and L be a compact connected n-manifold without boundary. Denote by

$$\mathcal{X} = \{\iota \in \mathrm{Emb}(L, M) \,|\, \iota^*\omega = 0\}$$

the space of Lagrangian embeddings of L into M. The group $\mathcal{G} = \mathrm{Diff}(L)$ acts on this space by $\iota \mapsto \iota \circ \phi$ for $\phi \in \mathcal{G}$. Two Lagrangian embeddings $\iota_0, \iota_1 \in \mathcal{X}$ lie in the same $\mathcal{G}$-orbit if and only if they have the same image $\Lambda = \iota_0(L) = \iota_1(L)$. Hence the quotient space

$$\mathcal{L} := \mathcal{X}/\mathcal{G}$$

can be naturally identified with the set of Lagrangian submanifolds of M that are diffeomorphic to L. A function $\mathbb{R} \to \mathcal{L} : t \mapsto \Lambda_t$ is called **smooth** if there exists a smooth function $\mathbb{R} \times L \to M : (t, q) \mapsto \iota_t(q)$ such that $\iota_t(L) = \Lambda_t$ for all t. One can think of $\mathcal{L}$ as an infinite dimensional manifold.

Lemma 2.1 *The tangent space of $\mathcal{L}$ at a point $\Lambda \in \mathcal{L}$ can be naturally identified with the space of closed 1-forms on Λ:*

$$T_\Lambda \mathcal{L} = \left\{\beta \in \Omega^1(\Lambda) \,|\, d\beta = 0\right\}$$

Proof: Let $\mathbb{R} \times L \to M : (t, q) \mapsto \iota_t(q)$ be a smooth function such that $\iota_t \in \mathcal{X}$ for all t and define

$$\alpha_t := \omega(v_t, d\iota_t \cdot) \in \Omega^1(L), \qquad v_t := \partial_t \iota_t \in \mathcal{C}^\infty(L, {\iota_t}^*TM). \tag{3}$$

Then

$$0 = \partial_t {\iota_t}^* \omega = d\alpha_t$$

and hence the tangent space of $\mathcal{X}$ at ι is given by

$$T_\iota \mathcal{X} = \left\{v \in \mathcal{C}^\infty(L, \iota^*TM) \,|\, \omega(v, d\iota \cdot) \in \Omega^1(L) \text{ is closed}\right\}.$$

The tangent space to the $\mathcal{G}$-orbit consists of all vector fields of the form $v = d\iota \circ \xi$, where $\xi \in \mathrm{Vect}(L)$. The map $v \mapsto \omega(v, d\iota \cdot)$ identifies the quotient space $T_\iota \mathcal{X} / T_\iota(\iota \cdot \mathcal{G})$ with the space of closed 1-forms on L.

If $\iota_t, \iota_t' \in \mathcal{X}$ are two smooth paths in $\mathcal{X}$ that satisfy $\iota_t' = \iota_t \circ \phi_t$ for some path $\phi_t \in \mathcal{G}$ then the vector fields $v_t := \partial_t \iota_t$ and $v_t' := \partial_t \iota_t'$ are related by

$$v_t' = v_t \circ \phi_t + d\iota_t \circ \xi_t \circ \phi_t$$

where $\xi_t \in \mathrm{Vect}(L)$ generates the diffeomorphism ϕ_t via $\partial_t \phi_t = \xi_t \circ \phi_t$. Hence the 1-forms $\alpha_t := \omega(v_t, d\iota_t \cdot)$ and $\alpha_t' := \omega(v_t', d\iota_t' \cdot)$ are related by

$$\alpha_t' = {\phi_t}^* \alpha_t.$$

Hence two closed 1-forms $\alpha, \alpha' \in \Omega^1(L)$ corresponding to two Lagrangian embeddings ι and $\iota' = \iota \circ \phi$ represent the same tangent vector of $\mathcal{L}$ if and only if $\alpha' = \phi^* \alpha$ or, equivalently, $\iota_* \alpha = {\iota'}_* \alpha'$. This proves the lemma. □

Let $\mathbb{R} \to \mathcal{L} : t \mapsto \Lambda_t$ be a smooth path of Lagrangian submanifolds. We define the derivative of this path at time t by

$$\partial_t \Lambda_t := \iota_{t*} \alpha_t,$$

where the path $\mathbb{R} \to \mathcal{X} : t \mapsto \iota_t$ is chosen such that $\iota_t(L) = \Lambda_t$ for every t and α_t is defined by (3). The proof of Lemma 2.1 shows that the 1-form $\beta_t = \iota_{t*} \alpha_t \in \Omega^1(\Lambda_t)$ is closed and is independent of the choice of the lift $t \mapsto \iota_t$ used to define it.

We wish to study Hamiltonian isotopies of Lagrangian submanifolds. This corresponds to paths in $\mathcal{L}$ that are tangent to the subbundle

$$\mathcal{H} = \left\{ (\Lambda, \beta) \in T\mathcal{L} \,|\, \Lambda \in \mathcal{L},\ \beta \in \Omega^1(\Lambda) \text{ is exact} \right\}.$$

Abstractly, one can think of $\mathcal{H}$ as a distribution on $\mathcal{L}$. It follows from Weinstein's Lagrangian neighbourhood theorem that this distribution is integrable. We shall see that the leaf through $\Lambda_0 \in \mathcal{L}$ consists of all Lagrangian submanifolds of M that are Hamiltonian isotopic to Λ_0 . To be more precise, let $\mathbb{R} \times M \to \mathbb{R} : (t, z) \mapsto H_t(z)$ be a smooth Hamiltonian function and denote by $\mathbb{R} \times M \to M : (t, z) \mapsto \psi_t(z)$ the Hamiltonian isotopy generated by H via

$$\frac{d}{dt} \psi_t = X_t \circ \psi_t, \qquad \iota(X_t)\omega = dH_t, \qquad \psi_0 = \mathrm{id}. \tag{4}$$

Lemma 2.2 *Let $\mathbb{R} \to \mathcal{L} : t \mapsto \Lambda_t$ be a smooth path of Lagrangian submanifolds and ψ_t be a Hamiltonian isotopy on M generated by the Hamiltonian functions $H_t : M \to \mathbb{R}$ via (4). Then $\Lambda_t = \psi_t(\Lambda_0)$ for every t if and only if*

$$\partial_t \Lambda_t = dH_t|_{\Lambda_t}$$

for every t.

Proof: Choose a smooth path $\mathbb{R} \to \mathcal{X} : t \mapsto \iota_t$ such that $\iota_t(L) = \Lambda_t$ for every t and let $\alpha_t \in \Omega^1(L)$ be defined by (3). Then $\partial_t \Lambda_t = dH_t|_{\Lambda_t}$ if and only if $d(H_t \circ \iota_t) = \alpha_t$. It follows from the definitions that this is equivalent to

$$X_t(\iota_t(q)) - \partial_t \iota_t(q) \in \operatorname{im} d\iota_t(q)$$

for all t and all q. This means that there exists a smooth family of vector fields $\xi_t \in \mathrm{Vect}(L)$ such that

$$X_t \circ \iota_t = \partial_t \iota_t + d\iota_t \circ \xi_t.$$

Equivalently, $\psi_t \circ \iota_0 = \iota_t \circ \phi_t$, where the isotopy $\phi_t \in \mathrm{Diff}(L)$ is generated by ξ_t via $\partial_t \phi_t = \xi_t \circ \phi_t$ and $\phi_0 = \mathrm{id}$. This proves the lemma. □

The previous lemma shows that every path in $\mathcal{L}$ that is generated by a Hamiltonian isotopy is tangent to $\mathcal{H}$. The converse is proved next.

Lemma 2.3 *A smooth path $[0, 1] \to \mathcal{L} : t \mapsto \Lambda_t$ is tangent to $\mathcal{H}$ if and only if there exists a Hamiltonian isotopy $t \mapsto \psi_t$ such that $\psi_t(\Lambda_0) = \Lambda_t$ for every t.*

Proof: The "if" part was proved in Lemma 2.2. Suppose that the path $t \mapsto \Lambda_t$ is tangent to $\mathcal{H}$. Choose a smooth function $[0, 1] \to \mathcal{X} : t \mapsto \iota_t$ such that $\iota_t(L) = \Lambda_t$ for every t and let $\alpha_t \in \Omega^1(L)$ be defined by (3). By assumption, α_t is exact for every t. Fix a smooth path $q_t \in L$ and, for every t, choose $h_t : L \to \mathbb{R}$ such that

$$dh_t = \alpha_t, \qquad h_t(q_t) = 0.$$

Then the function $\mathbb{R} \times L \to \mathbb{R} : (t, q) \mapsto h_t(q)$ is smooth. We construct a smooth function $[0, 1] \times M \to \mathbb{R} : (t, z) \mapsto H_t(z)$ such that

$$H_t \circ \iota_t = h_t. \tag{5}$$

Choose an almost complex structure J on M that is compatible with ω. Let $\varepsilon > 0$ be so small that, for every $t \in [0, 1]$, the map

$$T\Lambda_t \to M : (z, v) \mapsto \exp_z(Jv)$$

restricts to a diffeomorphism from the ε-neighbourhood of the zero section in $T\Lambda_t$ onto the open neighbourhood

$$U_t := \{\exp_z(Jv) \mid z \in \Lambda_t,\ v \in T_z\Lambda_t,\ |v| < \varepsilon\}$$

of Λ_t in M. Choose a cutoff function $\rho : [0, \varepsilon] \to [0, 1]$ such that $\rho(r) = 1$ for $r < \varepsilon/3$ and $\rho(r) = 0$ for $r > 2\varepsilon/3$. Define $H_t : M \to \mathbb{R}$ by

$$H_t(\exp_z(Jv)) := \rho(|v|) h_t \circ {\iota_t}^{-1}(z)$$

for $z \in \Lambda_t$ and $v \in T_z\Lambda_t$ with $|v| < \varepsilon$, and by $H_t(z) := 0$ for $z \in M \setminus U_t$. Then H_t satisfies (5) and hence

$$dH_t|_{\Lambda_t} = \iota_{t*} dh_t = \iota_{t*}\alpha_t = \partial_t \Lambda_t.$$

By Lemma 2.2, the Hamiltonian isotopy ψ_t generated by H_t satisfies $\psi_t(\Lambda_0) = \Lambda_t$ for every t. This proves the lemma. □

Remark 2.4 The Hamiltonian functions constructed in Lemma 2.3 satisfy

$$\max H_t = \max h_t, \qquad \min H_t = \min h_t \tag{6}$$

for every t. With a slightly more sophisticated argument one can show that the Hamiltonian functions can be chosen such that the Hamiltonian vector fields X_t satisfy $\partial_t \iota_t = X_t \circ \iota_t$ and hence the resulting Hamiltonian isotopy satisfies

$$\psi_t \circ \iota_0 = \iota_t. \tag{7}$$

However, in general there does not exist a Hamiltonian isotopy that satisfies both (6) and (7).

Lemma 2.5 *Let $\mathbb{R} \to \mathcal{L} : t \mapsto \Lambda_t$ be a smooth path of Lagrangian submanifolds. Let $\mathbb{R} \to \mathrm{Diff}(M, \omega) : t \mapsto \psi_t$ be a symplectic isotopy and define $\beta_t \in \Omega^1(M)$ by $\beta_t := \iota(Y_t)\omega$, where $\partial_t \psi_t = Y_t \circ \psi_t$. Then β_t is closed and the path $\Lambda'_t := {\psi_t}^{-1}(\Lambda_t)$ satisfies*

$$\partial_t \Lambda'_t = {\psi_t}^* \left(\partial_t \Lambda_t - \beta_t|_{\Lambda_t} \right).$$

Proof: Choose a lift $\mathbb{R} \to \mathcal{X} : t \mapsto \iota_t$ of $t \mapsto \Lambda_t$ and denote

$$\iota'_t := {\psi_t}^{-1} \circ \iota_t, \qquad \alpha_t := \omega(\partial_t \iota_t, d\iota_t \cdot), \qquad \alpha'_t := \omega(\partial_t \iota'_t, d\iota'_t \cdot).$$

Then $\alpha'_t = \alpha_t - {\iota_t}^* \beta_t$ and hence

$$\partial_t \Lambda'_t = \iota'_{t*} \alpha'_t = {\psi_t}^* \iota_{t*} \alpha'_t = {\psi_t}^* \left(\iota_{t*} \alpha_t - \beta_t \right) = {\psi_t}^* \left(\partial_t \Lambda_t - \beta_t \right)$$

as claimed. □

The subbundle $\mathcal{H} \subset T\mathcal{L}$ carries a natural norm. Following Hofer [12] we define the norm of an exact 1-form $\alpha = dh \in \Omega^1(\Lambda)$ by

$$\|dh\| := \max h - \min h.$$

This norm gives rise to a distance function on each leaf of the foliation determined by $\mathcal{H}$. Let $\mathcal{L}_0$ be such a leaf. By Lemma 2.3, $\mathcal{L}_0$ is the Hamiltonian isotopy class of any Lagrangian submanifold $\Lambda \in \mathcal{L}_0$. Let $[0,1] \to \mathcal{L}_0 : t \mapsto \Lambda_t$ be a smooth path in $\mathcal{L}_0$. The length of this path is defined by

$$\ell(\{\Lambda_t\}) := \int_0^1 \|\partial_t \Lambda_t\| \, dt.$$

Lemma 2.3 and Remark 2.4 show that

$$\ell(\{\Lambda_t\}) = \inf_{\psi_t(\Lambda_0)=\Lambda_t} \ell(\{\psi_t\}), \tag{8}$$

where the infimum runs over all Hamiltonian isotopies $t \mapsto \psi_t$ that satisfy $\psi_t(\Lambda_0) = \Lambda_t$ for all t and $\ell(\{\psi_t\})$ denotes the Hofer length (cf. [12]).

Now let $\Lambda, \Lambda' \in \mathcal{L}_0$ and denote by $\mathcal{P}(\Lambda, \Lambda')$ the space of all smooth paths $[0,1] \to \mathcal{L}_0 : t \mapsto \Lambda_t$ that connect $\Lambda_0 = \Lambda$ to $\Lambda_1 = \Lambda'$. The distance between Λ and Λ' is defined by

$$d(\Lambda, \Lambda') := \inf_{\{\Lambda_t\} \in \mathcal{P}(\Lambda,\Lambda')} \ell(\{\Lambda_t\}). \tag{9}$$

It follows immediately from (8) that

$$d(\Lambda, \Lambda') = \inf_{\psi(\Lambda)=\Lambda'} d(\mathrm{id}, \psi) \tag{10}$$

where the infimum runs over all Hamiltonian symplectomorphisms ψ of M that satisfy $\psi(\Lambda) = \Lambda'$ and $d(\mathrm{id}, \psi)$ denotes the Hofer distance (cf. [12]). The function (9) is obviously nonnegative, symmetric, and satisfies the triangle inequality. That it defines a metric is a deep theorem due to Chekanov [3].

Theorem 2.6 (Chekanov) *If $\Lambda \neq \Lambda'$ then $d(\Lambda, \Lambda') > 0$.*

Remark 2.7 In [18] Milinković studied geodesics in the space of Lagrangian submanifolds. Generalizing a result by Bialy and Polterovich [2], he proved that the distance of two exact Lagrangian submanifolds $\Lambda = \mathrm{graph}(dS)$ and $\Lambda' = \mathrm{graph}(dS')$ of the cotangent bundle T^*L is given by

$$d(\Lambda, \Lambda') = \|d(S - S')\|.$$

3 Invariants of Lagrangian loops

In this section we shall consider exact loops of Lagrangian submanifolds. In the terminology of the previous section this corresponds to loops inside a leaf of the foliation of $\mathcal{L}$ determined by $\mathcal{H}$. We shall construct three invariants of Hamiltonian isotopy classes of such loops and study the relations between them.

3.1 The minimal length

Continue the notation of Section 2. A **Lagrangian loop** in M is a smooth function $\mathbb{R} \to \mathcal{L} : t \mapsto \Lambda_t$ such that

$$\Lambda_{t+1} = \Lambda_t$$

for all $t \in \mathbb{R}$. Such a loop determines a subset $\Lambda \subset S^1 \times M$ defined by

$$\Lambda := \left\{ (e^{2\pi i t}, z) \,|\, t \in \mathbb{R}, z \in \Lambda_t \right\}. \tag{11}$$

Note that a loop $\mathbb{R} \to \mathcal{L} : t \mapsto \Lambda_t$ is smooth if and only if this set Λ is a smooth submanifold of $S^1 \times M$. We shall frequently identify the loop $\mathbb{R} \to \mathcal{L} : t \mapsto \Lambda_t$ with the corresponding submanifold $\Lambda \subset S^1 \times M$.

A Lagrangian loop $t \mapsto \Lambda_t$ is called **exact** if it is tangent to $\mathcal{H}$, i.e. $\partial_t \Lambda_t \in \Omega^1(\Lambda_t)$ is exact for every t. Two exact Lagrangian loops $t \mapsto \Lambda_t$ and $t \mapsto \Lambda'_t$ are called **Hamiltonian isotopic** if there exists a smooth function $[0,1] \times \mathbb{R} \to \mathcal{L} : (s,t) \mapsto \Lambda_{s,t}$ such that

$$\Lambda_{0,t} = \Lambda_t, \qquad \Lambda_{1,t} = \Lambda'_t,$$

the map $t \mapsto \Lambda_{s,t}$ is an exact Lagrangian loop for every s, and $\partial_s \Lambda_{s,t} \in \Omega^1(\Lambda_{s,t})$ is exact for all s and t. Here the function $[0,1] \times \mathbb{R} \to \mathcal{L} : (s,t) \mapsto \Lambda_{s,t}$ is called smooth if there exists a smooth function $[0,1] \times \mathbb{R} \times L \to M : (s,t,q) \mapsto \iota_{s,t}(q)$ such that $\iota_{s,t}(L) = \Lambda_{s,t}$ for all s and t. Let $\Lambda, \Lambda' \subset S^1 \times M$ be two exact Lagrangian loops. We write $\Lambda \sim \Lambda'$ iff Λ is Hamiltonian isotopic to Λ'. A Hamiltonian isotopy class corresponds to a component in the free loop space of a leaf $\mathcal{L}_0 \subset \mathcal{L}$ of the foliation determined by $\mathcal{H}$. To every such Hamiltonian isotopy class we assign the real number

$$\nu(\Lambda) := \inf_{\Lambda' \sim \Lambda} \ell(\Lambda').$$

So $\nu(\Lambda)$ is obtained by minimizing the Hofer length over all exact Lagrangian loops that are Hamiltonian isotopic to Λ.

3.2 The relative K-area

Following Polterovich [21] we introduce the notion of relative K-area. This invariant is defined in terms of Hamiltonian connections on the symplectic fibre bundle $D \times M \to D$ that preserve the subbundle $\Lambda \subset D \times M$ defined by (11). Here $D \subset \mathbb{C}$ denotes the closed unit disc. We begin by recalling the basic notions of symplectic connections and curvature (cf. [11, 16]). Think of a connection on $D \times M$ as a horizontal distribution. Any such connection is determined by a **connection 2-form** on $D \times M$ of the form

$$\tau = \omega + \alpha \wedge dx + \beta \wedge dy + f dx \wedge dy$$

where $\alpha = \alpha_{x,y} \in \Omega^1(M)$, $\beta = \beta_{x,y} \in \Omega^1(M)$, and $f = f_{x,y} \in \Omega^0(M)$ depend smoothly on $x+iy \in D$. The horizontal subspace is the τ-orthogonal complement of the vertical subspace. Explicitly, the horizontal lifts of $\partial/\partial x$ and $\partial/\partial y$ at $(x+iy, z) \in D \times M$ are the vectors $(1, X_{x,y}(z))$ and $(i, Y_{x,y}(z))$, respectively, where the vector fields $X = X_{x,y}, Y = Y_{x,y} \in \mathrm{Vect}(M)$ are defined by

$$\iota(X)\omega = \alpha, \qquad \iota(Y)\omega = \beta.$$

Thus the connection associated to τ is independent of f. It is called **symplectic** if $\alpha_{x,y}$ and $\beta_{x,y}$ are closed for all $x+iy \in D$, and **Hamiltonian** if $\alpha_{x,y}$ and $\beta_{x,y}$ are exact for all $x+iy \in D$ and τ is closed.[1] Thus a Hamiltonian connection 2-form has the form

$$\tau = \omega + dF \wedge dx + dG \wedge dy + (\partial_x G - \partial_y F + c) dx \wedge dy, \qquad (12)$$

where $F, G : D \times M \to \mathbb{R}$ and $c : D \to \mathbb{R}$ are smooth maps such that the functions $F_{x,y} = F(x+iy, \cdot)$ and $G_{x,y} = G(x+iy, \cdot)$ have mean value zero:

$$\int_M F_{x,y}\omega^n = \int_M G_{x,y}\omega^n = 0.$$

In (12) the d in dF denotes the differential on M, i.e. dF denotes the smooth family $x+iy \mapsto dF_{x,y}$ of 1-forms on M, and similarly for dG. We shall only consider Hamiltonian connections with the property that parallel transport along the boundary preserves Λ.

[1] In [16] a connection is called Hamiltonian if parallel transport along every **loop** in the base is a Hamiltonian symplectomorphism. In the case of a simply connected base this is equivalent to the existence of a closed 2-form τ that represents this connection. In contrast, we call a connection Hamiltonian if parallel transport along every **path** is a Hamiltonian symplectomorphism. This notion only makes sense when the structure group of the bundle in question is the group of Hamiltonian symplectomorphisms.

Lemma 3.1 *Let τ be a Hamiltonian connection 2-form on $D \times M$ of the form (12) and denote*

$$H_t := -2\pi \sin(2\pi t) F_{\cos(2\pi t),\sin(2\pi t)} + 2\pi \cos(2\pi t) G_{\cos(2\pi t),\sin(2\pi t)}. \qquad (13)$$

Let $\mathbb{R} \to \mathcal{L} : t \mapsto \Lambda_t$ be an exact Lagrangian loop, let $\Lambda \subset D \times M$ be defined by (11), and choose a smooth function $\iota : \mathbb{R} \times L \to D \times M$ such that $\iota(t,q) = (e^{2\pi i t}, \iota_t(q))$ and $\iota_t(L) = \Lambda_t$. Then the following are equivalent.

(i) *Parallel transport of τ along the boundary preserves Λ.*

(ii) $\iota^*\tau = 0$.

(iii) $dH_t|_{\Lambda_t} = \partial_t \Lambda_t$ *for every $t \in \mathbb{R}$.*

Proof: The parallel transport of τ along a curve $t \mapsto x(t) + iy(t)$ is determined by the Hamiltonian functions

$$H_t = \dot{x}(t) F_{x(t),y(t)} + \dot{y}(t) G_{x(t),y(t)}$$

via (4). The functions H_t in (13) correspond to the path $t \mapsto e^{2\pi i t}$. By Lemma 2.2, the Hamiltonian isotopy determined by H_t preserves Λ if and only if $dH_t|_{\Lambda_t} = \partial_t \Lambda_t$ for every t. This shows that (i) is equivalent to (iii).

To prove the equivalence of (ii) and (iii) note that

$$\begin{aligned} \iota^*\tau &= dt \wedge \omega(\partial_t \iota_t - X_t \circ \iota_t, d\iota_t \cdot) \\ &= dt \wedge (\alpha_t - \iota_t^* dH_t) \end{aligned}$$

where $X_t \in \mathrm{Vect}(M)$ denotes the Hamiltonian vector field of H_t as in (4). The right hand side vanishes if and only if $dH_t|_{\Lambda_t} = \partial_t \Lambda_t$. This proves the lemma. □

For every exact Lagrangian loop $\mathbb{R} \to \mathcal{L} : t \mapsto \Lambda_t$ let us denote the set of Hamiltonian connections that preserve Λ by

$$\mathcal{T}(\Lambda) = \left\{ \tau \in \Omega^2(D \times M) \,|\, \tau \text{ has the form (12)}, \, \tau|_{T\Lambda} = 0 \right\}.$$

We shall prove in Lemma 3.2 below that this set is nonempty. Let $\mathbb{R} \to \mathcal{L} : t \mapsto \Lambda'_t$ be another exact Lagrangian loop. A diffeomorphism

$$\Psi : (D \times M, \Lambda) \to (D \times M, \Lambda')$$

is called a **fibrewise (Hamiltonian) symplectomorphism** if it has the form $\Psi(x+iy,z) = (x+iy, \psi_{x,y}(z))$, where $\psi_{x,y} : M \to M$ is a (Hamiltonian) symplectomorphism for all x, y. In the case $\Lambda = \Lambda'$ we denote by $\mathcal{G}(\Lambda)$ the group of fibrewise Hamiltonian symplectomorphisms of $(D \times M, \Lambda)$. This group acts on $\mathcal{T}(\Lambda)$ by $\tau \mapsto \Psi^*\tau$. The **curvature** of a connection 2-form τ of the form (12) is the function $\Omega_\tau : D \times M \to \mathbb{R}$ defined by

$$\Omega_\tau(x,y,z) := \{F_{x,y}, G_{x,y}\}(z) + \partial_y F_{x,y}(z) - \partial_x G_{x,y}(z) \tag{14}$$

for $x + iy \in D$ and $z \in M$. It is sometimes useful to think of the curvature as a 2-form $\Omega_\tau dx \wedge dy$ on $D \times M$ rather than a function.

Lemma 3.2 **(i)** *For every exact Lagrangian loop $\mathbb{R} \to \mathcal{L} : t \mapsto \Lambda_t$ the set $\mathcal{T}(\Lambda)$ is nonempty.*

(ii) *Two exact Lagrangian loops Λ and Λ' are Hamiltonian isotopic if and only if the corresponding pairs $(D\times M, \Lambda)$ and $(D\times M, \Lambda')$ are fibrewise Hamiltonian symplectomorphic.*

(iii) *If τ is a Hamiltonian connection 2-form on $D\times M$ and Ψ is a fibrewise Hamiltonian symplectomorphism of $D \times M$ then*

$$\Omega_{\Psi^*\tau} = \Omega_\tau \circ \Psi.$$

Proof: Let $\phi_t \in \mathrm{Diff}(L)$ be defined by

$$\iota_{t+1} \circ \phi_t = \iota_t.$$

Since L is connected there exists a smooth path $\mathbb{R} \to L : t \mapsto q_t$ such that, for every $t \in \mathbb{R}$,

$$q_{t+1} = \phi_t(q_t). \tag{15}$$

For example choose q_t in the interval $0 \le t \le 1$ such that $q_t = q_1$ for $1 - \varepsilon \le t \le 1$ and $q_t = {\phi_t}^{-1}(q_1)$ for $0 \le t \le \varepsilon$. Then define q_t for $t \in \mathbb{R}$ such that (15) is satisfied. Let $h_t : L \to \mathbb{R}$ be defined by

$$dh_t = \alpha_t := \omega(\partial_t\iota_t, d\iota_t\cdot), \qquad h_t(q_t) = 0.$$

By (15), the function $t \mapsto \iota_t(q_t)$ is 1-periodic in t and the proof of Lemma 2.1 shows that the 1-forms $\iota_{t*}\alpha_t$ are 1-periodic in t. Hence the functions $h_t \circ {\iota_t}^{-1}$

are 1-periodic in t and hence, so are the functions H_t defined in the proof of Lemma 2.3. Now define

$$\widetilde{H}_t(z) := H_t(z) - \frac{\int_M H_t \omega^n}{\int_M \omega^n}.$$

Let $\rho : [0,1] \to [0,1]$ be a smooth cutoff function such that $\rho(r) = 0$ for $r < \varepsilon$ and $\rho(r) = 1$ for $r > 1 - \varepsilon$ and define τ by

$$\Phi^* \tau = \omega + \rho(r) d\widetilde{H}_t dt + \dot{\rho}(r) \widetilde{H}_t dr \wedge dt, \tag{16}$$

where $\Phi : [0,1] \times [0,1] \times M \to D \times M$ is given by $\Phi(r,t,z) = (re^{2\pi it}, z)$. Explicitly, τ has the form (12) where $F, G : D \times M \to \mathbb{R}$ are given by

$$F_{x,y} = \frac{-\sin(2\pi t)\rho(r)}{2\pi r} \widetilde{H}_t, \qquad G_{x,y} = \frac{\cos(2\pi t)\rho(r)}{2\pi r} \widetilde{H}_t, \tag{17}$$

for $x + iy = re^{2\pi it}$. These functions have mean value zero and satisfy (13) with H_t replaced by $\widetilde{H}_t$. Since $H_t \circ \iota_t = h_t$ it follows as in the proof of Lemma 2.3 that

$$d\widetilde{H}_t|_{\Lambda_t} = dH_t|_{\Lambda_t} = \partial_t \Lambda_t.$$

By Lemma 3.1, the parallel transport of τ along the boundary preserves Λ. Hence τ is an element of $\mathcal{T}(\Lambda)$. This proves (i).

We prove (ii). Assume first that there exists a fibrewise Hamiltonian symplectomorphism of the form $\Psi(x+iy, z) = (x+iy, \psi_{x+iy}(z))$ such that

$$\psi_{e^{2\pi it}}(\Lambda_t) = \Lambda'_t$$

for every t. Define

$$\psi_{s,t} := \psi_{se^{2\pi it}}, \qquad \Lambda_{s,t} := \psi_{s,t}(\Lambda_t)$$

for $0 \le s \le 1$ and $t \in \mathbb{R}$. Then $t \mapsto \Lambda_{s,t}$ is an exact Lagrangian loop for every s and $\partial_s \Lambda_{s,t} \in \Omega^1(\Lambda_{s,t})$ is exact for all s and t. Hence the Lagrangian loop $\Lambda_{1,t} = \Lambda'_t$ is Hamiltonian isotopic to $\Lambda_{0,t} = \psi_0(\Lambda_t)$. Since ψ_0 is a Hamiltonian symplectomorphism, the loop $t \mapsto \psi_0(\Lambda_t)$ is Hamiltonian isotopic to $t \mapsto \Lambda_t$. Conversely, suppose that $t \mapsto \Lambda_t$ and $t \mapsto \Lambda'_t$ are two exact Lagrangian loops that are Hamiltonian isotopic. Choose an exact isotopy $(s,t) \mapsto \Lambda_{s,t}$ such that $\Lambda_{0,t} = \Lambda_t$, $\Lambda_{1,t} = \Lambda'_t$, and $\partial_s \Lambda_{s,t} = 0$ for $s \le 1/2$. As in the proof of (i), one can construct a smooth family of Hamiltonian functions $H_{s,t}: M \to \mathbb{R}$ such that

$$H_{s,t+1} = H_{s,t}, \qquad dH_{s,t}|_{\Lambda_{s,t}} = \partial_s \Lambda_{s,t}.$$

Define the Hamiltonian symplectomorphisms $\psi_{s,t}: M \to M$ by

$$\partial_s \psi_{s,t} = X_{s,t} \circ \psi_{s,t}, \qquad \iota(X_{s,t})\omega = dH_{s,t}, \qquad \psi_{0,t} = \mathrm{id}.$$

Then $\psi_{s,t} = \mathrm{id}$ for $s \leq 1/2$ and the required fibrewise Hamiltonian symplectomorphism is given by $\Psi(se^{2\pi it}, z) := (se^{2\pi it}, \psi_{s,t}(z))$.

We prove (iii). Let τ be given by (12) and suppose that

$$\Psi(x+iy, z) = (x+iy, \psi_{x,y}(z))$$

is a fibrewise Hamiltonian symplectomorphism. Choose smooth functions $A, B : D \times M \to \mathbb{R}$ such that the functions $A_{x,y} := A(x+iy, \cdot)$ and $B_{x,y} := B(x+iy, \cdot)$ have mean value zero and the Hamiltonian vector fields $X_A = X_{A_{x,y}}$ and $X_B = X_{B_{x,y}}$ satisfy

$$\partial_x \psi = X_A \circ \psi, \qquad \partial_y \psi = X_B \circ \psi. \tag{18}$$

Then

$$\Psi^*\tau = \omega + d\widetilde{F} \wedge dx + d\widetilde{G} \wedge dy + (\partial_x \widetilde{G} - \partial_y \widetilde{F} + c) dx \wedge dy,$$

where

$$\widetilde{F} = (F - A) \circ \Psi, \qquad \widetilde{G} = (G - B) \circ \Psi.$$

Hence

$$\begin{aligned}
\Omega_{\Psi^*\tau} &= \partial_x \widetilde{G} - \partial_y \widetilde{F} - \{\widetilde{F}, \widetilde{G}\} \\
&= \partial_x (G - B) \circ \Psi + d(G - B) \circ X_A \circ \Psi \\
&\quad - \partial_y (F - A) \circ \Psi - d(F - A) \circ X_B \circ \Psi \\
&\quad - \{(F - A), (G - B)\} \circ \Psi \\
&= (\partial_x G - \partial_y F - \{F, G\}) \circ \Psi \\
&\quad - (\partial_x B - \partial_y A - \{A, B\}) \circ \Psi \\
&= \Omega_\tau \circ \Psi.
\end{aligned}$$

The last equality follows from the definition of A and B in (18). This proves the lemma. $\square$

The **relative K-area** of an exact Lagrangian loop Λ is defined by

$$\chi(\Lambda) := \inf_{\tau \in \mathcal{T}(\Lambda)} \|\Omega_\tau\|,$$

where

$$\|\Omega_\tau\| := \int_D \left(\max_{z \in M} \Omega_\tau(x, y, z) - \min_{z \in M} \Omega_\tau(x, y, z) \right) dxdy.$$

Theorem 3.3 *For every exact Lagrangian loop $\Lambda \subset S^1 \times M$*

$$\chi(\Lambda) = \nu(\Lambda).$$

Proof: Let $\mathbb{R} \to \mathcal{L} : t \mapsto \Lambda_t$ be an exact Lagrangian loop. Let $\tau \in \Omega^2(D \times M)$ be the connection 2-form defined by (16) in the proof of Lemma 3.2, where the cutoff function $\rho : [0, 1] \to [0, 1]$ is chosen to be nondecreasing. Then

$$\Phi^*(Fdx + Gdy) = \rho \widetilde{H}_t dt,$$

where $F, G : D \times M \to \mathbb{R}$ are given by (17) and $\Phi(r, t, z) = (re^{2\pi it}, z)$. Taking the differential of this 1-form on $[0, 1]^2 \times M$ we find

$$\Phi^*((\partial_x G - \partial_y F) dx \wedge dy) = \dot{\rho} \widetilde{H}_t dr \wedge dt.$$

Since $\{F, G\} = 0$ and $\Phi^*(dx \wedge dy) = 2\pi r dr \wedge dt$ we obtain

$$\Omega_\tau(re^{2\pi it}, z) = -\frac{\dot{\rho}(r)}{2\pi r} \widetilde{H}_t(z).$$

Moreover,

$$\left\| \widetilde{H}_t \right\| = \max_M \widetilde{H}_t - \min_M \widetilde{H}_t = \max_{\Lambda_t} \widetilde{H}_t - \min_{\Lambda_t} \widetilde{H}_t,$$

and hence

$$\|\Omega_\tau\| = \int_0^1 \int_0^1 \dot{\rho}(r) \left\| \widetilde{H}_t \right\| drdt = \int_0^1 \left\| \widetilde{H}_t \right\| dt = \ell(\Lambda).$$

This implies $\chi(\Lambda) \le \ell(\Lambda)$. If Λ and Λ' are Hamiltonian isotopic then, by Lemma 3.2 (ii), there exists a fibrewise Hamiltonian symplectomorphism Ψ of $D \times M$ such that $\Psi(\Lambda) = \Lambda'$. Hence $\tau \in \mathcal{T}(\Lambda')$ if and only if $\Psi^*\tau \in \mathcal{T}(\Lambda)$. By Lemma 3.2 (iii), $\chi(\Lambda) = \chi(\Lambda') \le \ell(\Lambda')$. Hence $\chi(\Lambda) \le \nu(\Lambda)$.

We prove that $\nu(\Lambda) \le \chi(\Lambda)$. Let $\tau \in \mathcal{T}(\Lambda)$. We shall construct an exact Lagrangian loop Λ' that is Hamiltonian isotopic to Λ and satisfies

$$\ell(\Lambda') \le \|\Omega_\tau\|. \tag{19}$$

Suppose that τ has the form (12). Since the function c in (12) has no effect on the curvature we may assume, without loss of generality, that $c \equiv 0$. Define $H = H_{r,t} : M \to \mathbb{R}$ and $K = K_{r,t} : M \to \mathbb{R}$ by the formula

$$\Phi^*\tau = \omega + dK \wedge dr + dH \wedge dt + (\partial_r H - \partial_t K) dr \wedge dt.$$

Explicitly,

$$K_{r,t} = \cos(2\pi t) F_{re^{2\pi it}} + \sin(2\pi t) G_{re^{2\pi it}},$$

$$H_{r,t} = 2\pi r \cos(2\pi t) G_{re^{2\pi it}} - 2\pi r \sin(2\pi t) F_{re^{2\pi it}}.$$

Define the Hamiltonian symplectomorphisms $\psi_{r,t} : M \to M$ by

$$\partial_r \psi_{r,t} = X_{K_{r,t}} \circ \psi_{r,t}, \qquad \psi_{0,t} = \mathrm{id}.$$

Then the loop

$$\Lambda'_t = {\psi_{1,t}}^{-1}(\Lambda_t)$$

is evidently Hamiltonian isotopic to Λ. We shall prove that it satisfies (19). To see this, denote by Ψ the fibrewise Hamiltonian symplectomorphism of $[0,1]^2 \times M$ given by

$$\Psi(r,t,z) = (r,t,\psi_{r,t}(z)).$$

Then, as in the proof of Lemma 3.2, we obtain

$$\Psi^*\Phi^*\tau = \omega + dH' \wedge dt + \partial_r H' dr \wedge dt,$$

where $H'_{r,t} = (H_{r,t} - B_{r,t}) \circ \psi_{r,t}$ and $B_{r,t} : M \to \mathbb{R}$ is defined by $\partial_t \psi_{r,t} = X_{B_{r,t}} \circ \psi_{r,t}$. These functions satisfy

$$\|\Omega_\tau\| = \int_0^1 \int_0^1 \|\partial_r H'_{r,t}\| \, dr dt, \qquad H'_{0,t} = 0.$$

Moreover, by Lemma 2.5, we have

$$\begin{aligned}
\partial_t \Lambda'_t &= {\psi_{1,t}}^* \left(\partial_t \Lambda_t - dB_{1,t}|_{\Lambda_t} \right) \\
&= {\psi_{1,t}}^* (dH_{1,t}|_{\Lambda_t}) - d(B_{1,t} \circ \psi_{1,t})|_{\Lambda'_t} \\
&= dH'_{1,t}|_{\Lambda'_t}.
\end{aligned}$$

Hence the length of Λ' is given by

$$\begin{aligned}
\ell(\Lambda') &= \int_0^1 \left(\max_{\Lambda'_t} H'_{1,t} - \min_{\Lambda'_t} H'_{1,t} \right) dt \\
&\leq \int_0^1 \left(\max_M H'_{1,t} - \min_M H'_{1,t} \right) dt \\
&= \int_0^1 \left(\max_M \left(\int_0^1 \partial_r H'_{r,t}\, dr \right) - \min_M \left(\int_0^1 \partial_r H'_{r,t}\, dr \right) \right) dt \\
&\leq \int_0^1 \int_0^1 \left(\max_M \partial_r H'_{r,t} - \min_M \partial_r H'_{r,t} \right) dr dt \\
&= \int_0^1 \int_0^1 \left\| \partial_r H'_{r,t} \right\| dr dt \\
&= \|\Omega_\tau\|.
\end{aligned}$$

Thus we have proved that for every $\tau \in \mathcal{T}(\Lambda)$ there exists an exact Lagrangian loop Λ' that is Hamiltonian isotopic to Λ and satisfies $\ell(\Lambda') \leq \|\Omega_\tau\|$. Hence $\chi(\Lambda) \leq \nu(\Lambda)$ and this proves the theorem. □

3.3 The non-symplectic interval

Let $\Lambda \subset D \times M$ be an exact Lagrangian loop and $\tau \in \mathcal{T}(\Lambda)$ be a Hamiltonian connection 2-form. Since τ is closed and vanishes on Λ (see Lemma 3.1) it determines a relative cohomology class

$$[\tau] \in H^2(D \times M, \Lambda; \mathbb{R}).$$

Let Σ be a compact oriented Riemann surface with (possibly empty) boundary $\partial\Sigma$. A smooth map $v : (\Sigma, \partial\Sigma) \to (D \times M, \Lambda)$ determines a 2-dimensional relative homology class

$$[v] := v_*[\Sigma] \in H_2(D \times M, \Lambda; \mathbb{Z}).$$

The pairing of this class with $[\tau]$ is given by

$$\langle [\tau], [v] \rangle = \int_\Sigma v^*\tau.$$

Since every 2-dimensional integral homology class of the pair $(D \times M, \Lambda)$ can be represented by a smooth map v as above, the cohomology class $[\tau]$ is uniquely determined by these pairings. Define $\sigma \in H^2(D \times M, \Lambda; \mathbb{R})$ by

$$\langle \sigma, [v] \rangle = \deg(\pi \circ v) \tag{20}$$

for every $v : (\Sigma, \partial\Sigma) \to (D \times M, \Lambda)$, where

$$\pi : (D \times M, \Lambda) \to (D, \partial D)$$

denotes the obvious projection. In (20) the degree of a smooth map $v_0 : (\Sigma, \partial\Sigma) \to (D, \partial D)$ is understood as the degree of its restriction to the boundary. It agrees with the number of preimages of an interior regular value, counted with appropriate signs (cf. Milnor [19]). Note that

$$\sigma = \frac{1}{\pi}[dx \wedge dy]$$

and hence σ agrees with the pullback of the positive integral generator of $H^2(D, \partial D; \mathbb{R})$ under the projection π.

Lemma 3.4 *Let $\tau_0, \tau_1 \in \mathcal{T}(\Lambda)$. Then there exists a constant $s = s(\tau_1, \tau_0) \in \mathbb{R}$ such that*

$$[\tau_1] - [\tau_0] = s\sigma.$$

Proof: Let τ_i be given by (12) with F, G, c replaced by F_i, G_i, c_i for $i = 0, 1$. Denote

$$F := F_1 - F_0, \qquad G := G_1 - G_0, \qquad c := c_1 - c_0,$$

and let $H_t : M \to \mathbb{R}$ be defined by (13). Since $\tau_0, \tau_1 \in \mathcal{T}(\Lambda)$ it follows from Lemma 3.1 that there exists a function $h : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ such that

$$H_t|_{\Lambda_t} \equiv h(t)$$

for every $t \in \mathbb{R}$. We shall prove that the required identity holds with

$$s := \int_0^1 h(t)\, dt + \int_D c\, dxdy.$$

To see this note that, by (13),

$$(Fdx + Gdy)|_\Lambda = \pi^*\alpha_h. \tag{21}$$

where $\alpha_h \in \Omega^1(S^1)$ denotes the pushforward of the 1-form $hdt \in \Omega^1(\mathbb{R}/\mathbb{Z})$ under the diffeomorphisms $\mathbb{R}/\mathbb{Z} \to S^1 : [t] \mapsto e^{2\pi i t}$. Let Σ be a compact oriented Riemann surface and $v : \Sigma \to D \times M$ be a smooth function such that $v(\partial\Sigma) \subset \Lambda$. Denote $v_0 := \pi \circ v : (\Sigma, \partial\Sigma) \to (D, \partial D)$. Then

$$\begin{aligned}\int_\Sigma v^*(\tau_1 - \tau_0) &= \int_\Sigma v^*\left(dF \wedge dx + dG \wedge dy + (\partial_x G - \partial_y F + c) dx \wedge dy\right) \\ &= \int_{\partial\Sigma} v^*(F dx + G dy) + \int_\Sigma {v_0}^*(c dx \wedge dy) \\ &= \int_{\partial\Sigma} {v_0}^*\alpha_h + \int_\Sigma {v_0}^*(c dx \wedge dy) \\ &= s \deg(v_0).\end{aligned}$$

The penultimate equality follows from (21) and the last from the identities

$$\int_{\partial\Sigma} {v_0}^*\alpha_h = \deg(v_0) \int_{S^1} \alpha_h \tag{22}$$

and

$$\int_\Sigma {v_0}^*(c dx \wedge dy) = \deg(v_0) \int_D c dx \wedge dy. \tag{23}$$

Here (22) is the degree theorem for maps between compact 1-manifolds and (23) is the degree theorem for maps between 2-manifolds with boundary. More precisely, if the function $c : D \to \mathbb{R}$ has mean value zero then there exists a 1-form $\alpha \in \Omega^1(D)$ such that $d\alpha = c dx \wedge dy$ and $\alpha|_{TS^1} = 0$. This implies that the left hand side of (23) vanishes. Hence it suffices to establish (23) for constant functions c and this reduces to (22). This proves the lemma. □

Let $\tau_0 \in \mathcal{T}(\Lambda)$. We shall now address the question which cohomology classes $[\tau_0] + s\sigma$ can be represented by nondegenerate Hamiltonian connection 2-forms. Such a 2-form is a symplectic form on $D \times M$ with respect to which Λ is a Lagrangian submanifold. Denote

$$\mathcal{T}^\pm(\Lambda) := \left\{\tau \in \mathcal{T}(\Lambda) \mid \pm\tau^{n+1} > 0\right\}.$$

Here the inequality $\tau^{n+1} > 0$ means that $\tau^{n+1} = f\, dx \wedge dy \wedge \omega^n$, where $f : D \times M \to \mathbb{R}$ is a positive function. For $\tau_0 \in \mathcal{T}(\Lambda)$ we define

$$\varepsilon^+(\tau_0, \Lambda) := \inf\left\{s(\tau, \tau_0) \mid \tau \in \mathcal{T}^+(\Lambda)\right\},$$

$$\varepsilon^-(\tau_0, \Lambda) := \sup\left\{s(\tau, \tau_0) \mid \tau \in \mathcal{T}^-(\Lambda)\right\}.$$

The proof of Theorem 3.5 below shows that the class $[\tau_0] + s\sigma$ can be represented by a symplectic form $\tau \in \mathcal{T}^{\pm}(\Lambda)$ for $\pm s$ sufficiently large and hence $\pm\varepsilon^{\pm}(\tau_0, \Lambda) < \infty$. Evidently, $\varepsilon^{\pm}(\tau_1, \Lambda) - \varepsilon^{\pm}(\tau_0, \Lambda) = s(\tau_1, \tau_0)$. Hence the number

$$\varepsilon(\Lambda) := \varepsilon^+(\tau_0, \Lambda) - \varepsilon^-(\tau_0, \Lambda)$$

is independent of the connection 2-form $\tau_0 \in \mathcal{T}(\Lambda)$ used to define it. This number is called the **width of the nonsymplectic interval**.

Theorem 3.5 *For every exact Lagrangian loop $\Lambda \subset D \times M$*

$$\varepsilon(\Lambda) \leq \chi(\Lambda).$$

Proof: Let $\mathbb{R} \to \mathcal{L} : t \mapsto \Lambda_t$ be an exact Lagrangian loop and $F, G : D \times M \to \mathbb{R}$ be smooth functions such that the functions $H_t : M \to \mathbb{R}$ defined by (13) satisfy $dH_t|_{\Lambda_t} = \partial_t \Lambda_t$ for every t. For every smooth function $c : D \to \mathbb{R}$ let $\tau_c \in \mathcal{T}(\Lambda)$ be given by (12). In particular, τ_0 is given by (12) with $c = 0$. We shall prove that

$$\varepsilon^+(\tau_0, \Lambda) \leq \int_D \max_{z \in M} \Omega_{\tau_0}(x, y, z)\, dxdy, \tag{24}$$

$$\varepsilon^-(\tau_0, \Lambda) \geq \int_D \min_{z \in M} \Omega_{\tau_0}(x, y, z)\, dxdy. \tag{25}$$

To see this, note that

$$ndF \wedge dG \wedge \omega^{n-1} = \{F, G\}\omega^n$$

and hence

$$\begin{array}{rcl} {\tau_c}^{n+1} & = & (n+1)(\partial_x G - \partial_y F + c)dx \wedge dy \wedge \omega^n \\ & & +\, n(n+1)dF \wedge dx \wedge dG \wedge dy \wedge \omega^{n-1} \\ & = & (n+1)(\partial_x G - \partial_y F - \{F, G\} + c)dx \wedge dy \wedge \omega^n \\ & = & (n+1)(c - \Omega_{\tau_0})dx \wedge dy \wedge \omega^n. \end{array} \tag{26}$$

This shows that τ_c is nondegenerate if and only if $c(x, y) \neq \Omega_{\tau_0}(x, y, z)$ for all $(x + iy, z) \in D \times M$. Fix a number

$$s > \int_D \max_{z \in M} \Omega_{\tau_0}(x, y, z)\, dxdy.$$

Choose a smooth function $c : D \to \mathbb{R}$ such that

$$c(x,y) > \max_{z\in M} \Omega_{\tau_0}(x,y,z)$$

for all $x+iy \in D$ and

$$\int_D c\, dxdy = s.$$

Then τ_c is nondegenerate and represents the class $[\tau_c] = [\tau_0] + s\sigma$. This proves (24) and (25) follows from a similar argument. It follows from (24) and (25) that

$$\begin{aligned}
\varepsilon(\Lambda) &= \varepsilon^+(\tau_0,\Lambda) - \varepsilon^-(\tau_0,\Lambda) \\
&\le \int_D \left(\max_{z\in M} \Omega_{\tau_0}(x,y,z) - \min_{z\in M} \Omega_{\tau_0}(x,y,z) \right) dxdy \\
&= \|\Omega_{\tau_0}\|.
\end{aligned}$$

Since the curvature of τ_0 is equal to the curvature of τ_c for every c it follows that $\varepsilon(\Lambda) \le \|\Omega_\tau\|$ for every $\tau \in \mathcal{T}(\Lambda)$ and hence $\varepsilon(\Lambda) \le \chi(\Lambda)$. This proves the theorem. □

Remark 3.6 Let us denote

$$T(\Lambda) := \left\{ [\tau] \in H^2(D\times M, \Lambda; \mathbb{R}) \,|\, \tau \in \mathcal{T}(\Lambda) \right\}. \tag{27}$$

By Lemma 3.4, this set is a 1-dimensional affine subspace of $H^2(D\times M, \Lambda; \mathbb{R})$. Denote

$$T^\pm(\Lambda) := \left\{ [\tau] \,|\, \tau \in \mathcal{T}^\pm(\Lambda) \right\}.$$

These sets are open and connected. To prove connectedness, let $\tau_i \in \mathcal{T}^+(\Lambda)$ be given by (12) with F, G, c replaced by F_i, G_i, c_i for $i = 0, 1$. By (26), $c_i > \Omega_{\tau_i}$. Assume without loss of generality that $s(\tau_1, \tau_0) \ge 0$. Then the path $[0,1] \to T^+(\Lambda) : t \mapsto [\tau_0] + ts(\tau_1,\tau_0)\sigma$ connects $[\tau_0]$ with $[\tau_1]$. This shows that the sets $T^\pm(\Lambda)$ are connected. The complement $T(\Lambda) \setminus (T^-(\Lambda) \cup T^+(\Lambda))$ is compact and connected. It can be expressed in the form

$$T(\Lambda) \setminus (T^-(\Lambda) \cup T^+(\Lambda)) = \left\{ [\tau_0] + s\sigma \,|\, \varepsilon^-(\tau_0,\Lambda) \le s \le \varepsilon^+(\tau_0,\Lambda) \right\}$$

for every $\tau_0 \in \mathcal{T}(\Lambda)$. We do not know if this complement is always nonempty or, equivalently, if $\varepsilon(\Lambda)$ is always nonnegative.

4 Loops on the 2-torus

Consider the torus $M = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with the standard symplectic form

$$\omega = dx \wedge dy$$

and let $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ denote the projection. Let

$$B_r = \{(s,t) \in \mathbb{R}^2 \,|\, s^2 + t^2 \leq r^2\}$$

and suppose that $S \subset \mathbb{T}^2$ is the image of an embedding $B_1 \to \mathbb{T}^2$. Define

$$\Lambda_t := \Lambda_t(S) := \{[x, y+t] \,|\, [x,y] \in \partial S\} \tag{28}$$

(see Figure 1).

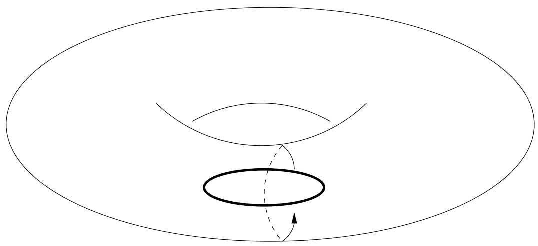

Figure 1: A Lagrangian loop on the 2-torus

Theorem 4.1 *Let $S \subset \mathbb{T}^2$ be a closed embedded disc and $t \mapsto \Lambda_t$ be the exact Lagrangian loop defined by (28). Then*

$$\nu(\Lambda) = \mathrm{area}(S).$$

Proof: We prove that $\nu(\Lambda) \leq \mathrm{area}(S)$. To see this, choose smooth functions $x, y : \mathbb{R} \to \mathbb{R}$ such that

$$x(\theta+1) = x(\theta), \qquad y(\theta+1) = y(\theta),$$

and the map $\iota_t : \mathbb{R}/\mathbb{Z} \to \mathbb{T}^2$ defined by

$$\iota_t(\theta) := [x(\theta), y(\theta) + t]$$

is an embedding with $\iota_t(\mathbb{R}/\mathbb{Z}) = \Lambda_t$. Then

$$\alpha_t := \omega(\partial_t \iota_t, d\iota_t \cdot) = -\dot{x} d\theta \in \Omega^1(\mathbb{R}/\mathbb{Z}).$$

Hence $\alpha_t = dh_t$ where $h_t = -x : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$. Hence

$$\|\partial_t \Lambda_t\| = \|dh_t\| = \max x - \min x$$

and this implies

$$\ell(\Lambda) = \max x - \min x.$$

By Proposition A.1 in the appendix, two loops $t \mapsto \Lambda_t(S)$ and $t \mapsto \Lambda_t(S')$, associated to two embedded discs $S, S' \subset \mathbb{T}^2$ via (28), are Hamiltonian isotopic if and only if S and S' have the same area. Now for every $\delta > 0$ there exists an embedded disc S' (as illustrated in Figure 2) such that

$$\operatorname{area}(S) = \operatorname{area}(S'), \qquad \max x' - \min x' < \operatorname{area}(S) + \delta,$$

where $x', y' : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ are chosen such that the map $\iota'(\theta) = [x'(\theta), y'(\theta)]$ defines an embedding $\mathbb{R}/\mathbb{Z} \to \mathbb{T}^2$ whose image is $\partial S'$. Hence the length of the loop $t \mapsto \Lambda_t(S')$ is bounded above by $\operatorname{area}(S) + \delta$. Thus we have proved that

$$\nu(\Lambda) \le \operatorname{area}(S).$$

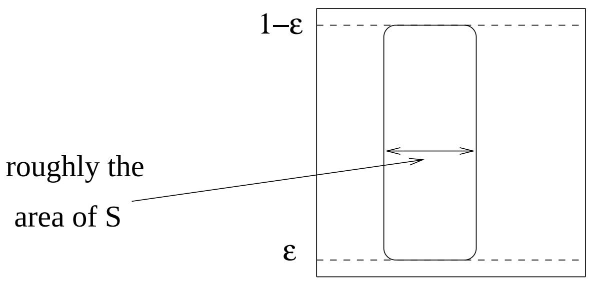

Figure 2: Minimizing the length

To show the reverse inequality let $t \mapsto \Lambda'_t$ be an exact Lagrangian loop that is Hamiltonian isotopic to Λ. Then

$$\Lambda'_0 = \partial S',$$

where $S' \subset \mathbb{T}^2$ is a smoothly embedded closed disc of the same area as S. Let $\psi_t : \mathbb{T}^2 \to \mathbb{T}^2$ be a Hamiltonian isotopy such that

$$\psi_t(\Lambda'_0) = \Lambda'_t.$$

We shall prove that

$$\operatorname{area}(S) \le \ell\left(\{\psi_t\}_{0\le t\le 1}\right). \tag{29}$$

To see this, choose an embedded closed discs $\tilde{S} \subset \mathbb{R}^2$ such that $\pi(\tilde{S}) = S'$ and let $\tilde{\psi}_t : \mathbb{R}^2 \to \mathbb{R}^2$ be a lift of ψ_t. Since Λ' is Hamiltonian isotopic to Λ we have $\tilde{\psi}_{t+1}(\tilde{S}) = \tilde{\psi}_t(\tilde{S}) + (0,1)$ and hence

$$\tilde{\psi}_1(\tilde{S}) \cap \tilde{S} = \emptyset.$$

Let $\tilde{H}_t : \mathbb{R}^2 \to \mathbb{R}$ be the Hamiltonian functions that generate $\tilde{\psi}_t$ and have mean value zero over the fundamental domain $[0,1]^2$. Choose $R > 1$ such that $\tilde{\psi}_t(\tilde{S}) \subset B_R$ for every $t \in [0,1]$ and let $\beta : \mathbb{R}^2 \to [0,1]$ be a compactly supported cutoff function such that $\beta|_{B_R} \equiv 1$. Then the functions

$$\hat{H}_t := \beta \tilde{H}_t$$

generate a compactly supported Hamiltonian isotopy $\hat{\psi}_t$ of $\mathbb{R}^2$ that satisfies

$$\hat{\psi}_1(\tilde{S}) \cap \tilde{S} = \emptyset.$$

Now it follows from the *energy-capacity inequality* in Hofer [12] that the *displacement energy* of $\tilde{S}$ is bounded below by the area. Hence

$$\operatorname{area}(\tilde{S}) \le d(\mathrm{id}, \hat{\psi}_1) \le \ell(\{\hat{\psi}_t\}_{0\le t\le 1}) = \ell(\{\psi_t\}_{0\le t\le 1}).$$

Since

$$\operatorname{area}(\tilde{S}) = \operatorname{area}(S') = \operatorname{area}(S),$$

this proves (29). It follows from (29) and (8) that

$$\operatorname{area}(S) \le \ell(\Lambda')$$

for every exact Lagrangian loop Λ' that is Hamiltonian isotopic to Λ. Hence $\operatorname{area}(S) \le \nu(\Lambda)$. □

Theorem 4.1 shows that the invariant $\nu(\Lambda)$ is not necessarily invariant under Lagrangian isotopy, but only under exact Lagrangian isotopy. The techniques of proof are specific to the 2-dimensional case. To establish lower bounds for our invariants in higher dimensions we shall use existence theorems for pseudoholomorphic discs.

5 Relative Gromov invariants

Throughout we assume that our symplectic manifold (M, ω) is compact. The relative Gromov invariants of an exact Lagrangian loop $\Lambda \subset D \times M$ are defined in terms of holomorphic sections of the bundle $D \times M \to D$ with boundary values in Λ. Let us denote by $\mathrm{Map}_\Lambda(D, M)$ the space of smooth functions $u : D \to M$ that satisfy $u(e^{2\pi it}) \in \Lambda_t$ for every $t \in \mathbb{R}$. The **Maslov class** is a function

$$\mu_\Lambda : \mathrm{Map}_\Lambda(D, M) \to \mathbb{Z}$$

defined as follows. Given $u \in \mathrm{Map}_\Lambda(D, M)$ choose a trivialization of the tangent bundle u^*TM. Then the tangent spaces $T_{u(e^{2\pi it})}\Lambda_t$ define a loop of Lagrangian subspaces in $(\mathbb{R}^{2n}, \omega_0)$ and $\mu_\Lambda(u)$ is defined as the Maslov index of this loop (cf. [24]). This integer is independent of the choice of the trivialization used to define it, and it depends only on the homology class of u in $H_2(D \times M, \Lambda; \mathbb{Z})$. We shall assume throughout that the pair (M, Λ_0) is **monotone**, i.e. there exists a $\lambda > 0$ such that, for every smooth map $v \in \mathrm{Map}_{\Lambda_0}(D, M)$,

$$\int_D v^*\omega = \lambda \mu_{\Lambda_0}(v).$$

Here μ_{Λ_0} denotes the Maslov class corresponding to the constant loop $t \mapsto \Lambda_0$. The **minimal Maslov number** of the pair (M, Λ_0) is defined by

$$N := \inf \{\mu_{\Lambda_0}(v) \mid v : (D, \partial D) \to (M, \Lambda_0),\ \mu_{\Lambda_0}(v) > 0\}.$$

We shall define relative Gromov invariants for every tuple $\mathbf{t} = (t_1, \ldots, t_k) \in \mathbb{R}^k$ with $0 \leq t_1 < \cdots < t_k < 1$ and every class $A \in H_2(D \times M, \Lambda; \mathbb{Z})$ that satisfies $n \pm \mu_\Lambda(A) \leq N - 2$. The invariants are homology classes

$$\mathrm{Gr}^\pm_{A,\mathbf{t}}(\Lambda) \in H_{n \pm \mu_\Lambda(A)}(\Lambda_\mathbf{t}; \mathbb{Z}_2),$$

where $\Lambda_\mathbf{t} := \Lambda_{t_1} \times \cdots \times \Lambda_{t_k}$. These homology classes arise from certain moduli spaces $\mathcal{M}_A(\tau, \pm J)$ of (anti-)holomorphic sections of the bundle $D \times M$ with boundary values in Λ that represent the class A. The points $(e^{2\pi it_1}, \ldots, e^{2\pi it_k})$ determine an evaluation map

$$\mathrm{ev}_\mathbf{t} : \mathcal{M}_A(\tau, \pm J) \to \Lambda_\mathbf{t}$$

and $\mathrm{Gr}^\pm_{A,\mathbf{t}}(\Lambda)$ is defined as the image of the fundamental cycle of $\mathcal{M}_A(\tau, \pm J)$ under the induced homomorphism on homology. We shall work with almost complex structures on $D \times M$ that are compatible with the fibration. Every such structure is determined by a family of almost complex structures on M and a connection 2-form $\tau \in \mathcal{T}(\Lambda)$.

5.1 J-holomorphic discs

Let $\Lambda \subset S^1 \times M$ be an exact Lagrangian loop and $\tau \in \mathcal{T}(\Lambda)$ be a Hamiltonian connection 2-form that preserves Λ. Throughout we shall denote by $\mathcal{J}(M, \omega)$ the space of almost complex structures on TM that are compatible with ω. Let $D \to \mathcal{J}(M, \omega) : (x, y) \mapsto J_{x,y}$ be a smooth family of such almost complex structures. Associated to the triple (τ, J, Λ) there is a natural boundary value problem for smooth functions $u : D \to M$:

$$\partial_x u - X_F(u) + J(\partial_y u - X_G(u)) = 0, \tag{30}$$

$$u(e^{2\pi i t}) \in \Lambda_t, \qquad t \in \mathbb{R}. \tag{31}$$

Here we abbreviate $J = J_{x,y}$, τ is given by (12), $X_F = X_F(x, y, \cdot) \in \mathrm{Vect}(M)$ denotes the Hamiltonian vector field of the function $F = F(x, y, \cdot) : M \to \mathbb{R}$, and similarly for X_G. Following Gromov [9] we observe that the solutions of (30) can be thought of as pseudo-holomorphic curves in $D \times M$.

Remark 5.1 Consider the almost complex structure $\tilde{J}$ on $D \times M$ given by

$$\tilde{J} = \tilde{J}(\tau, J) := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ -JX_F + X_G & -X_F - JX_G & J \end{pmatrix}.$$

Then $u : D \to M$ is a solution of (30) if and only if the function

$$\tilde{u}(x, y) = (x, y, u(x, y)) \tag{32}$$

is a $\tilde{J}$-holomorphic curve in $D \times M$, i.e.

$$\partial_x \tilde{u} + \tilde{J} \partial_y \tilde{u} = 0.$$

If τ is given by (12) then, for every $\tilde{\zeta} = (\xi, \eta, \zeta) \in T_{x,y,z}(D \times M)$,

$$\tau(\tilde{\zeta}, \tilde{J}\tilde{\zeta}) = |\zeta - \xi X_F - \eta X_G|^2 + (c - \Omega_\tau)(\xi^2 + \eta^2).$$

Hence $\tilde{J}$ is tamed by τ whenever $\tau \in \mathcal{T}^+(\Lambda)$ (see (26)). If $\tau \in \mathcal{T}^-(\Lambda)$ then $\tilde{J}(\tau, -J)$ is tamed by $-\tau$.

The **energy** of a solution u of (30) is defined by

$$E(u) := \int_D |\partial_x u - X_F(u)|^2 \, dxdy.$$

The next lemma shows that the solutions of (30) and (31) that represent a given homology class $A \in H_2(D \times M, \Lambda; \mathbb{Z})$ satisfy a uniform energy bound.

Lemma 5.2 *Let $u : D \to M$ be a smooth solution of (30) and (31) and denote by $A \in H_2(D \times M, \Lambda; \mathbb{Z})$ the homology class represented by the map $\tilde{u} : D \to D \times M$ defined by (32). Let $c : D \to \mathbb{R}$ be the function in (12). Then*

$$E(u) = \langle [\tau], A \rangle + \int_D (\Omega_\tau(x, y, u) - c(x, y)) \, dxdy.$$

Proof: We compute

$$\begin{aligned} E(u) &= \int_D \omega(\partial_x u - X_F(u), \partial_y u - X_G(u)) \, dxdy \\ &= \int_D \Big(\omega(\partial_x u, \partial_y u) - dF(u)\partial_y u + dG(u)\partial_x u + \{F, G\}(u) \Big) \, dxdy \\ &= \int_D \Big(\omega(\partial_x u, \partial_y u) - dF(u)\partial_y u + dG(u)\partial_x u \Big) \, dxdy \\ &\quad + \int_D \Big(\Omega_\tau(x, y, u) - (\partial_y F)(u) + (\partial_x G)(u) \Big) \, dxdy \\ &= \int_D \Big(\tau(\partial_x \tilde{u}, \partial_y \tilde{u}) - c(x, y) \Big) \, dxdy + \int_D \Omega_\tau(x, y, u) \, dxdy. \end{aligned}$$

This proves the lemma. □

Let us denote the moduli space of solutions of (30) and (31) that represent a given homology class $A \in H_2(D \times M, \Lambda; \mathbb{Z})$ by

$$\mathcal{M}_A(\tau, J) := \{ u : D \to M \mid u \text{ satisfies (30) and (31)}, \, [\tilde{u}] = A \}.$$

We shall prove that, for a generic pair (τ, J), this space is a smooth manifold of dimension $n + \mu_\Lambda(A)$. Moreover, if the pair (M, Λ_0) is monotone with minimal Maslov number N and $n + \mu_\Lambda(A) < N$, we shall prove that $\mathcal{M}_A(\tau, J)$ is compact, again for a generic pair (τ, J). The key tool for establishing compactness is the energy bound of Lemma 5.2. Under these asumptions the moduli spaces can be used to define Gromov invariants of Λ. The significance of these invariants for exact Lagrangian loops lies in the following observation.

Lemma 5.3 *Let Λ be an exact Lagrangian loop and $A \in H_2(D \times M, \Lambda; \mathbb{Z})$. Suppose that for every $\tau \in \mathcal{T}^+(\Lambda)$ there exists a J such that $\mathcal{M}_A(\tau, J) \neq \emptyset$. Then*

$$\varepsilon^+(\tau_0, \Lambda) + \langle [\tau_0], A \rangle \geq 0$$

for every $\tau_0 \in \mathcal{T}(\Lambda)$.

Proof: Let $\tau \in \mathcal{T}^+(\Lambda)$ and $u \in \mathcal{M}_A(\tau, J)$. Let $\tilde{u} : D \to D \times M$ be given by (32). Then $\tilde{u}$ is a $\tilde{J}(\tau, J)$-holomorphic curve. By Remark 5.1, $\tilde{J}(\tau, J)$ is tamed by τ. Hence

$$0 < \int_D \tilde{u}^* \tau = \langle [\tau], A \rangle = \langle [\tau_0], A \rangle + s(\tau, \tau_0).$$

The infimum of the numbers on the right is $\langle [\tau_0], A \rangle + \varepsilon^+(\tau_0, \Lambda)$. This proves the lemma. $\square$

A similar estimate for $\varepsilon^-(\tau_0, \Lambda)$ can be obtained by studying anti-holomorphic curves. These are solutions of the equation

$$\partial_x u - X_F(u) - J(\partial_y u - X_G(u)) = 0, \tag{33}$$

that satisfy the same boundary condition (31). Let us denote the moduli space of solutions by $\mathcal{M}_A(\tau, -J)$.

Lemma 5.4 *Let Λ be an exact Lagrangian loop and $A \in H_2(D \times M, \Lambda; \mathbb{Z})$. Suppose that for every $\tau \in \mathcal{T}^-(\Lambda)$ there exists a J such that $\mathcal{M}_A(\tau, -J) \neq \emptyset$. Then*

$$\varepsilon^-(\tau_0, \Lambda) + \langle [\tau_0], A \rangle \leq 0$$

for every $\tau_0 \in \mathcal{T}(\Lambda)$.

Proof: Let $\tau \in \mathcal{T}^-(\Lambda)$ and $u \in \mathcal{M}_A(\tau, -J)$. Let $\tilde{u} : D \to D \times M$ be given by (32). Then $\tilde{u}$ is a $\tilde{J}(\tau, -J)$-holomorphic curve. By Remark 5.1, $\tilde{J}(\tau, -J)$ is tamed by $-\tau$. Hence

$$0 > \int_D \tilde{u}^* \tau = \langle [\tau], A \rangle = \langle [\tau_0], A \rangle + s(\tau, \tau_0).$$

The supremum of the numbers on the right is $\langle [\tau_0], A \rangle + \varepsilon^-(\tau_0, \Lambda)$. This proves the lemma. $\square$

5.2 Fredholm theory

In this subsection we examine the moduli spaces $\mathcal{M}_A^{\pm}(\tau, J)$ in more detail and show that, for a generic J, these spaces are smooth manifolds of the predicted dimensions $n \pm \mu_\Lambda(A)$. The arguments are standard (cf. [7, 17]) and we shall only outline the main points. Fix an exact Lagrangian loop $\Lambda \subset S^1 \times M$, a homology class $A \in H_2(D \times M, \Lambda; \mathbb{Z})$, and a constant $p > 2$. Consider the Banach manifold

$$\mathcal{B} = W^{1,p}_{\Lambda,A}(D, M)$$

of all functions $u : D \to M$ of class $W^{1,p}$ that satisfy the boundary condition (31) and represent the class A. There is a natural vector bundle $\mathcal{E} \to \mathcal{B}$ with fibres

$$\mathcal{E}_u = L^p(D, u^*TM)$$

and the left hand sides of (30) and (33) define Fredholm sections $\mathcal{F}^{\pm} : \mathcal{B} \to \mathcal{E}$ given by

$$\mathcal{F}^{\pm}(u) := \mathcal{F}(u; \tau, \pm J) := \partial_x u - X_F(u) \pm J(\partial_y u - X_G(u)).$$

The moduli spaces $\mathcal{M}_A(\tau, \pm J)$ are the zero sets of these sections. The tangent space

$$T_u\mathcal{B} = W^{1,p}_{\Lambda}(D, u^*TM)$$

consists of all vector fields $\xi \in W^{1,p}(D, u^*TM)$ along u which are of class $W^{1,p}$ and satisfy the boundary condition $\xi(e^{2\pi it}) \in T_{u(e^{2\pi it})}\Lambda_t$. The vertical differential of $\mathcal{F}^{\pm}$ at a zero $u \in \mathcal{M}_A^{\pm}(\tau, J)$ is the linear operator

$$D_u^{\pm} = D\mathcal{F}^{\pm}(u) : W^{1,p}_{\Lambda}(D, u^*TM) \to L^p(D, u^*TM)$$

given by

$$D_u^{\pm}\xi = \nabla_x\xi - \nabla_\xi X_F(u) \pm J(\nabla_y\xi - \nabla_\xi X_G(u)) \pm (\nabla_\xi J)(\partial_y u - X_G(u)). \quad (34)$$

Here ∇ denotes the Levi-Civita connection of the Riemannian metric

$$\langle \cdot, \cdot \rangle_{x,y} = \omega(\cdot, J_{x,y}\cdot)$$

and thus depends on $x + iy \in D$. The expression ∇X_F denotes the covariant derivative of $X_F = X_{F_{x,y}}$ with respect to the Levi-Civita connection at the point $x + iy$. The next theorem follows from the Riemann-Roch theorem for discs (see for example [25] for a recent exposition) and the infinite dimensional implicit function theorem (see for example [26, Appendix B]). The proof is standard (see for example [17]) and will be omitted.

Theorem 5.5 *For every $u \in W^{1,p}_{\Lambda,A}(D,M)$ the operators $D_u^\pm$ defined by (34) are Fredholm and their indices are*

$$\mathrm{index} D_u^\pm = n \pm \mu_\Lambda(u).$$

If $D_u^\pm$ is surjective for every $u \in \mathcal{M}_A(\tau, \pm J)$ then $\mathcal{M}_A(\tau, \pm J)$ is a smooth manifold of dimension

$$\dim \mathcal{M}_A(\tau, \pm J) = n \pm \mu_\Lambda(A).$$

Fix an exact Lagrangian loop Λ and denote by $\mathcal{J}(D; M, \omega)$ the space of all smooth families of almost complex structures $J : D \to \mathcal{J}(M, \omega)$. Given $J \in \mathcal{J}(D; M, \omega)$, a connection 2-form $\tau \in \mathcal{T}(\Lambda)$ is called **regular** for (30) if D_u^+ is surjective for every $A \in H_2(D \times M, \Lambda; \mathbb{Z})$ and every $u \in \mathcal{M}_A(\tau, J)$. Similarly, $\tau \in \mathcal{T}(\Lambda)$ is called **regular** for (33) if D_u^- is surjective for every $A \in H_2(D \times M, \Lambda; \mathbb{Z})$ and every $u \in \mathcal{M}_A(\tau, -J)$. We shall denote set of all connection 2-forms that are regular for (30), respectively (33), by

$$\mathcal{T}_{\mathrm{reg}}(\Lambda; \pm J) \subset \mathcal{T}(\Lambda).$$

The proof of the next theorem is a standard application of the Sard-Smale theorem (cf. [17]) and will be omitted.

Theorem 5.6 *The sets $\mathcal{T}_{\mathrm{reg}}(\Lambda; \pm J)$ are of the second category in $\mathcal{T}(\Lambda)$ in the sense of Baire, i.e. they are countable intersections of open and dense subsets of $\mathcal{T}(\Lambda)$. In particular, they are dense.*

Let $J_0, J_1 \in \mathcal{J}(D; M, \omega)$ and choose regular connection 2-forms

$$\tau_0 \in \mathcal{T}_{\mathrm{reg}}(\Lambda; \pm J_0), \qquad \tau_1 \in \mathcal{T}_{\mathrm{reg}}(\Lambda; \pm J_1).$$

By Theorem 5.5, the spaces $\mathcal{M}_A(\tau_0, \pm J_0)$ and $\mathcal{M}_A(\tau_1, \pm J_1)$ are smooth manifolds of the same dimension. These manifolds are cobordant. To construct a cobordism choose a smooth path $[0,1] \to \mathcal{J}(D; M, \omega) : \lambda \mapsto J_\lambda$ that connects J_0 to J_1. Let us denote by

$$\mathcal{T} = \mathcal{T}(\Lambda, \tau_0, \tau_1)$$

the space of smooth homotopies $[0,1] \to \mathcal{T}(\Lambda) : \lambda \mapsto \tau_\lambda$ that connect τ_0 to τ_1. Given $\{\tau_\lambda\} \in \mathcal{T}$ denote

$$\mathcal{W}_A(\{\tau_\lambda\}, \{\pm J_\lambda\}) = \{(\lambda, u) \,|\, 0 \le \lambda \le 1,\, u \in \mathcal{M}_A(\tau_\lambda, \pm J_\lambda)\}.$$

A homotopy $\{\tau_\lambda\} \in \mathcal{T}$ is called **regular** if, for every $A \in H_2(D \times M, \Lambda; \mathbb{Z})$ and every pair $(\lambda, u) \in \mathcal{W}_A(\{\tau_\lambda\}, \{\pm J_\lambda\})$,

$$\operatorname{im} D^{\pm}_{\lambda,u} + \mathbb{R}\xi^{\pm}_{\lambda,u} = L^p(D, u^*TM).$$

Here $D^{\pm}_{\lambda,u}$ is defined by (34) with τ and J replaced by τ_λ and J_λ, respectively, and $\xi^{\pm}_{\lambda,u} \in L^p(D, u^*TM)$ is given by

$$\xi^{\pm}_{\lambda,u} := X_{\partial_\lambda F_\lambda}(u) \pm J_\lambda(u) X_{\partial_\lambda G_\lambda}(u) \mp \partial_\lambda J_\lambda(u)(\partial_y u - X_{G_\lambda}(u)).$$

The set of all regular homotopies will be denoted by

$$\mathcal{T}_{\mathrm{reg}}(\Lambda, \tau_0, \tau_1; \{\pm J_\lambda\}) \subset \mathcal{T}.$$

The proof of the next theorem is again standard and will be omitted.

Theorem 5.7 *Let $[0,1] \to \mathcal{J}(D; M, \omega) : \lambda \mapsto J_\lambda$ be a smooth family of almost complex structures and suppose that $\tau_0 \in \mathcal{T}_{\mathrm{reg}}(\Lambda; \pm J_0)$ and $\tau_1 \in \mathcal{T}_{\mathrm{reg}}(\Lambda, \pm J_1)$. Then the sets $\mathcal{T}_{\mathrm{reg}}(\Lambda, \tau_0, \tau_1; \{\pm J_\lambda\}) \subset \mathcal{T}$ are of the second category in the sense of Baire. Moreover, if $\{\tau_\lambda\} \in \mathcal{T}_{\mathrm{reg}}(\Lambda, \tau_0, \tau_1; \{\pm J_\lambda\})$ then $\mathcal{W}_A(\{\tau_\lambda\}, \{\pm J_\lambda\})$ is a smooth manifold of dimension*

$$\dim \mathcal{W}_A(\{\tau_\lambda\}, \{\pm J_\lambda\}) = n \pm \mu_\Lambda(A) + 1$$

with boundary

$$\partial \mathcal{W}_A(\{\tau_\lambda\}, \{\pm J_\lambda\}) = \mathcal{M}_A(\tau_0, \pm J_0) \cup \mathcal{M}_A(\tau_1, \pm J_1).$$

5.3 Compactness

Theorem 5.8 *Let $\Lambda \subset S^1 \times M$ be an exact Lagrangian loop and suppose that the pair (M, Λ_0) is monotone. Let $A \in H_2(D \times M, \Lambda; \mathbb{Z})$ and denote by $N \in \mathbb{N}$ the minimal Maslov number of the pair (M, Λ_0).*

(i) *If*

$$n \pm \mu_\Lambda(A) \leq N - 1$$

then $\mathcal{M}_A(\tau, \pm J)$ is compact for every $J \in \mathcal{J}(D; M, \omega)$ and $\tau \in \mathcal{T}_{\mathrm{reg}}(\Lambda; \pm J)$.

(ii) *If*

$$n \pm \mu_\Lambda(A) \leq N - 2$$

then $\mathcal{W}_A(\{\tau_\lambda\}, \{\pm J_\lambda\})$ is compact for every smooth path $[0,1] \to \mathcal{J}(D; M, \omega) : \lambda \mapsto J_\lambda$, every $\tau_0 \in \mathcal{T}_{\mathrm{reg}}(\Lambda; \pm J_0)$, every $\tau_1 \in \mathcal{T}_{\mathrm{reg}}(\Lambda; \pm J_1)$, and every regular homotopy $\{\tau_\lambda\} \in \mathcal{T}_{\mathrm{reg}}(\Lambda, \tau_0, \tau_1; \{\pm J_\lambda\})$.

The proof of Theorem 5.8 relies on the following theorem about Gromov compactness for J-holomorphic discs. This result is implicitly contained in Gromov's original paper [9] and has been folk knowledge since then. However, the full details of the proof have not so far appeared in the literature. They were recently carried out by Frauenfelder [8] in his Diploma thesis. In his thesis Frauenfelder also discusses the corresponding notion of *stable maps* for pseudoholomorphic discs.

Theorem 5.9 (Gromov) *Let $(\tau^\nu, J^\nu) \in \mathcal{T}(\Lambda) \times \mathcal{J}(D; M, \omega)$ be a sequence that converges in the $\mathcal{C}^\infty$-topology to $(\tau, J) \in \mathcal{T}(\Lambda) \times \mathcal{J}(D; M, \omega)$. Let $A \in H_2(D \times M, \Lambda; \mathbb{Z})$ and $u^\nu \in \mathcal{M}_A(\tau^\nu, \pm J^\nu)$. If u^ν has no $\mathcal{C}^\infty$-convergent subsequence then there exist*

(i) *finitely many points $(x_i, y_i) \in D$ and maps $v_i : S^2 \to M$, $i = 1, \ldots, k$,*

(ii) *finitely many points $t_j \in \mathbb{R}$ and maps $w_j : D \to M$, $j = 1 \ldots, \ell$,*

(iii) *a map $u_0 : D \to M$,*

such that v_i is a nonconstant J_{x_i,y_i}-(anti)holomorphic sphere for $i = 1, \ldots, k$, w_j is a nonconstant $J_{e^{2\pi i t_j}}$-(anti)holomorphic disc with $w_j(\partial D) \subset \Lambda_{t_j}$ for $j = 1, \ldots, \ell$, $u_0 \in \mathcal{M}_{A_0}(\tau, \pm J)$ for some $A_0 \in H_2(D \times M, \Lambda; \mathbb{Z})$, and

$$A = A_0 + \sum_{i=1}^{k} [v_i] + \sum_{j=1}^{\ell} [w_j]. \tag{35}$$

Here $[v_i]$ and $[w_j]$ denote the induced homology classes in $H_2(D \times M, \Lambda; \mathbb{Z})$ and one of the integers k and ℓ is nonzero.

Remark 5.10 (i) Let $\tilde{M}$ be a compact manifold and $\tilde{L} \subset \tilde{M}$ be a compact submanifold of half the dimension. Suppose that $\tilde{\omega}^\nu$ is a sequence of symplectic forms on M that converges to $\tilde{\omega}$ in the $\mathcal{C}^\infty$-topology such that $\tilde{L}$ is a Lagrangian submanifold of $(\tilde{M}, \tilde{\omega}^\nu)$ for every ν. Suppose that $\tilde{J}^\nu$ is a sequence of $\tilde{\omega}^\nu$-tame almost complex structures on $\tilde{M}$ that converges in the $\mathcal{C}^\infty$-topology to $\tilde{J}$. In [8] Frauenfelder proves, in particular, that a sequence of $\tilde{J}^\nu$-holomorphic discs $\tilde{u}^\nu : (D, \partial D) \to (\tilde{M}, \tilde{L})$ that represent a fixed homology class $A \in H_2(\tilde{M}, \tilde{L}; \mathbb{Z})$ has a subsequence that converges (in a precisely defined sense) to a tree consisting of $\tilde{J}$-holomorphic spheres in $\tilde{M}$ and $\tilde{J}$-holomorphic discs in $\tilde{M}$ with boundary in $\tilde{L}$ such that the sum of their homology classes in $H_2(\tilde{M}, \tilde{L}; \mathbb{Z})$ is equal to A. The techniques in [8] are an adaptation of those in Hofer–Salamon [13] for holomorphic spheres to the case of holomorphic discs.

(ii) The moduli space $\mathcal{M}_A(\tau, \pm J)$ does not depend on the function $c : D \to M$ in (12). Hence we may assume without loss of generality that the connection forms τ^ν in Theorem 5.9 lie in $\mathcal{T}^\pm(\Lambda)$. Under this assumption the manifold $\tilde{M} = D \times M$, the submanifold $\tilde{L} = \Lambda$, the symplectic forms $\tilde{\omega}^\nu = \pm\tau^\nu$, the almost complex structures $\tilde{J}^\nu = \tilde{J}(\tau^\nu, \pm J^\nu)$ defined in Remark 5.1, and the functions $\tilde{u}^\nu$ given by (32) satisfy the requirements of (i).

(iii) Theorem 5.9 follows from (i) and (ii) since each bubble in the limit curve is contained in a fibre of the (trivial) fibration $D \times M$. To see this, note that each curve v_i appears as the limit of a sequence

$$v_i^\nu(x, y) = u^\nu(x_i^\nu + \varepsilon^\nu x, y_i^\nu + \varepsilon^\nu y),$$

where $x_i^\nu \to x_i$, $y_i^\nu \to y_i$, $\varepsilon^\nu \to 0$, and

$$\lim_{\nu\to\infty} \frac{\varepsilon^\nu}{1 - \sqrt{(x_i^\nu)^2 + (y_i^\nu)^2}} = 0.$$

One can show that, after passing to a suitable subsequence, the sequence v_i^ν converges to v_i in the $\mathcal{C}^\infty$-topology on the complement of some finite set. The functions v_i^ν satisfy

$$\partial_x v_i^\nu - \varepsilon^\nu X_{F^\nu} + J^\nu(\partial_y v_i^\nu - \varepsilon^\nu X_{G^\nu}) = 0,$$

where X_{F^ν}, X_{G^ν}, and J^ν are evaluated at the point $(x_i^\nu + \varepsilon^\nu x, y_i^\nu + \varepsilon^\nu y, v_i^\nu)$. It follows that the limit curve v_i extends to a J_{x_i,y_i}-holomorphic sphere. The holomorphic discs w_j appear as similar limits with $x_j + iy_j = e^{2\pi i t_j}$ and

$$\lim_{\nu\to\infty} \frac{\varepsilon^\nu}{1 - \sqrt{(x_j^\nu)^2 + (y_j^\nu)^2}} > 0.$$

A similar argument as above, with coordinates on the upper halfplane, then shows that the limit curve w_j is a $J_{e^{2\pi i t_j}}$-holomorphic disc with boundary values in Λ_{t_j}.

(iv) The limit curve in (i) is a stable map consisting of $\tilde{J}$-holomorphic discs and spheres. For closed curves this concept is due to Kontsevich [14]. Some of the components of the stable map may be constant. However, these do not contribute to the homology class and can be neglected for our purposes. If the original sequence $\tilde{u}^\nu$ does not have a $\mathcal{C}^\infty$-convergent subsequence, then the limit curve has more than one nonconstant component. This shows that in Theorem 5.9 either k or ℓ is nonzero.

Proof of Theorem 5.8: We prove (ii). Suppose, by contradiction, that $\mathcal{W}_A(\{\tau_\lambda\},\{\pm J_\lambda\})$ is not compact. Then there exists a sequence

$$(\lambda^\nu, u^\nu) \in \mathcal{W}_A(\{\tau_\lambda\},\{\pm J_\lambda\})$$

that has no convergent subsequence. We may assume without loss of generality that λ^ν converges to λ_0. Then, by Theorem 5.9, there exist nonconstant $J_{\lambda_0;x_i,y_i}$-(anti)holomorphic spheres $v_i : S^2 \to M$ for $i = 1,\dots,k$, nonconstant $J_{\lambda_0;e^{2\pi i t_j}}$-(anti)holomorphic discs $w_j : (D,\partial D) \to (M, L_{t_j})$ for $j = 1,\dots,\ell$, and an element $u_0 \in \mathcal{M}_{A_0}(\tau_{\lambda_0}, \pm J_{\lambda_0})$ for some $A_0 \in H_2(D\times M,\Lambda;\mathbb{Z})$ such that (35) is satisfied. Since the pair (M,Λ_t) is monotone with minimal Maslov number N for every t we have

$$\pm\mu_\Lambda(v_i) \ge N, \qquad \pm\mu_\Lambda(w_j) \ge N$$

for $i = 1,\dots,k$ and $j = 1,\dots,\ell$. Since either k or ℓ is nonzero this implies

$$\begin{aligned} n \pm \mu_\Lambda(A) &= n \pm \mu_\Lambda(A_0) \pm \sum_{i=1}^{k} \mu_\Lambda(v_i) \pm \sum_{j=1}^{\ell} \mu_\Lambda(w_j) \\ &\ge n \pm \mu_\Lambda(A_0) + N. \end{aligned}$$

Since $\{\tau_\lambda\} \in \mathcal{T}_{\mathrm{reg}}(\Lambda,\tau_0,\tau_1;\{\pm J_\lambda\})$ the moduli space $\mathcal{W}_{A_0}(\{\tau_\lambda\},\{\pm J_\lambda\})$ is a smooth manifolds of dimension

$$\begin{aligned} \dim \mathcal{W}_{A_0}(\{\tau_\lambda\},\{\pm J_\lambda\}) &= n \pm \mu_\Lambda(A_0) + 1 \\ &\le n \pm \mu_\Lambda(A) + 1 - N \\ &< 0. \end{aligned}$$

Hence

$$\mathcal{W}_{A_0}(\{\tau_\lambda\},\{\pm J_\lambda\}) = \emptyset,$$

in contradiction to the fact that

$$(\lambda_0, u_0) \in \mathcal{W}_{A_0}(\{\tau_\lambda\},\{\pm J_\lambda\}).$$

Thus we have proved (ii). The proof of (i) is almost word by word the same and will be left to the reader. □

5.4 Gromov invariants

Fix an exact Lagrangian loop $\Lambda \subset S^1 \times M$ and a class $A \in H_2(D\times M, \Lambda; \mathbb{Z})$. Throughout we shall assume that the pair (M, Λ_0) is monotone and

$$n \pm \mu_\Lambda(A) \leq N - 2, \tag{36}$$

where $N \in \mathbb{N}$ denotes the minimal Maslov number of the pair (M, Λ_0). Fix a tuple $\mathbf{t} = (t_1, \dots, t_k) \in \mathbb{R}^k$ such that $0 \leq t_1 < \cdots < t_k < 1$ and denote

$$\Lambda_{\mathbf{t}} := \Lambda_{t_1} \times \cdots \times \Lambda_{t_k}.$$

For $\tau \in \mathcal{T}(\Lambda)$ and $J \in \mathcal{J}(D; M, \omega)$ we define $\mathrm{ev}_{\mathbf{t}} : \mathcal{M}_A(\tau, \pm J) \to \Lambda_{\mathbf{t}}$ by

$$\mathrm{ev}_{\mathbf{t}}(u) := (u(e^{2\pi i t_1}), \dots, u(e^{2\pi i t_k})).$$

If $\tau \in \mathcal{T}_{\mathrm{reg}}(\Lambda; \pm J)$ then, by Theorems 5.5 and 5.8, the moduli space $\mathcal{M}^{\pm}_A(\tau, J)$ is a compact smooth manifolds(without boundary) of dimension $n \pm \mu_\Lambda(A)$. It is not necessarily orientable. Let

$$[\mathcal{M}_A(\tau, \pm J)] \in H_{n\pm\mu_\Lambda(A)}(\mathcal{M}_A(\tau, \pm J); \mathbb{Z}_2)$$

denote the fundamental cycle. The Gromov invariants are defined by

$$\mathrm{Gr}^{\pm}_{A,\mathbf{t}}(\Lambda) := \mathrm{ev}_{\mathbf{t}*}[\mathcal{M}^{\pm}_A(\tau, J)] \in H_{n\pm\mu_\Lambda(A)}(\Lambda_{\mathbf{t}}; \mathbb{Z}_2). \tag{37}$$

Lemma 5.11 *The homology classes $\mathrm{Gr}^{\pm}_{A,\mathbf{t}}(\Lambda) \in H_{n\pm\mu_\Lambda(A)}(\Lambda_{\mathbf{t}}; \mathbb{Z}_2)$ are independent of the choices of the almost complex structure $J \in \mathcal{J}(D; M, \omega)$ and the connection 2-form $\tau \in \mathcal{T}_{\mathrm{reg}}(\Lambda; \pm J)$ used to define them.*

Proof: Theorems 5.7 and 5.8 (ii). □

Corollary 5.12 *Let $A^{\pm} \in H_2(D \times M, \Lambda; \mathbb{Z})$ satisfy (36) and suppose that*

$$\mathrm{Gr}^{\pm}_{A^{\pm},\mathbf{t}^{\pm}}(\Lambda) \neq 0$$

for some $\mathbf{t}^{\pm}$. Then

$$\varepsilon^+(\tau, \Lambda) \geq -\langle [\tau], A^+ \rangle, \qquad \varepsilon^-(\tau, \Lambda) \leq -\langle [\tau], A^- \rangle$$

for every $\tau \in \mathcal{T}(\Lambda)$.

Proof: Lemmata 5.3 and 5.4. □

6 Complex projective space

In this section we shall use the Gromov invariants to compute the K-area of certain exact Lagrangian loops in $\mathbb{C}P^n$. The archetypal example is the half turn of a great circle in the 2-sphere. An explicit computation shows that the Hofer length of this loop is 1/2. We shall use Corollary 5.12 and Theorems 3.3 and 3.5 to show that this loop minimizes the Hofer length in its Hamiltonian isotopy class.

6.1 Rotations of real projective space

Consider the complex projective space

$$M = \mathbb{C}P^n$$

equipped with symplectic form ω that is induced by the Fubini-Study metric and satisfies the normalization condition

$$\int_{\mathbb{C}P^n} \omega^n = 1.$$

Let $L = \mathbb{R}P^n$ and fix an integer $k \in \{1, \ldots, n\}$. As in the introduction, we consider the exact Lagrangian loop

$$\Lambda := \bigcup_{t \in \mathbb{R}} \{e^{2\pi i t}\} \times \psi_t(\mathbb{R}P^n), \tag{38}$$

where

$$\psi_t([z_0 : \cdots : z_n]) = ([z_0 : e^{\pi i t} z_1 : \cdots : e^{\pi i t} z_k : z_{k+1} : \cdots : z_n]).$$

The Hamiltonian isotopy ψ_t is generated, via (4), by the the time independent Hamiltonian function $H_t = H : \mathbb{C}P^n \to \mathbb{R}$ given by

$$H([z_0 : \ldots : z_n]) = \frac{k}{2n+2} - \frac{|z_1|^2 + \cdots + |z_k|^2}{2(|z_0|^2 + \cdots + |z_n|^2)}. \tag{39}$$

This function has mean value zero and Hofer norm

$$\|H\| = \max H - \min H = \frac{1}{2}.$$

Since H attains its maximum and its minimum on $\Lambda_t = \psi_t(\mathbb{R}P^n)$ it follows that $\ell(\Lambda) = 1/2$.

6.2 The Maslov index

We prove that the minimal Maslov number of the pair $(\mathbb{C}P^n, \mathbb{R}P^n)$ is

$$N = n + 1. \tag{40}$$

For $n = 1$ this is obvious. For $n > 1$ consider the homology exact sequence of the pair $(\mathbb{C}P^n, \mathbb{R}P^n)$. It has the form

$$0 \to H_2(\mathbb{C}P^n; \mathbb{Z}) \to H_2(\mathbb{C}P^n, \mathbb{R}P^n; \mathbb{Z}) \to H_1(\mathbb{R}P^n; \mathbb{Z}) \to 0.$$

Now $\mathbb{R}P^n$ decomposes the line $\mathbb{C}P^1 \subset \mathbb{C}P^n$ into two discs that represent the same homotopy class in $\pi_2(\mathbb{C}P^n, \mathbb{R}P^n)$. Hence there is an element $A \in H_2(\mathbb{C}P^n, \mathbb{R}P^n; \mathbb{Z})$ such that $2A$ is equal to the image of the generator under the homomorphism

$$\mathbb{Z} \cong H_2(\mathbb{C}P^n; \mathbb{Z}) \to H_2(\mathbb{C}P^n, \mathbb{R}P^n; \mathbb{Z}).$$

This implies that A is the generator of $H_2(\mathbb{C}P^n, \mathbb{R}P^n; \mathbb{Z}) \cong \mathbb{Z}$. Since the Maslov class of $2A \in \pi_2(\mathbb{C}P^n, \mathbb{R}P^n)$ is equal to $2\langle c_1(T\mathbb{C}P^n), [\mathbb{C}P^1]\rangle = 2n + 2$ we have proved (40).

Lemma 6.1 *Let (M, ω) be a symplectic manifold and $\Lambda \subset S^1 \times M$ be an exact Lagrangian loop such that (M, Λ_0) is a monotone pair with minimal Maslov number N. Then*

$$\mu_\Lambda(u_1) \equiv \mu_\Lambda(u_0) \mod N$$

for all $u_0, u_1 \in \mathrm{Map}_\Lambda(D, M)$.

Proof: If $u_0(e^{2\pi it}) = u_1(e^{2\pi it})$ for every $t \in \mathbb{R}$ then u_0 (with reversed orientation) and u_1 form a sphere and the difference $\mu_\Lambda(u_1) - \mu_\Lambda(u_0)$ is equal twice the first Chern number of this sphere. Hence the difference of the Maslov numbers is an even multiple of N. This continues to hold whenever $u_0|_{\partial D}$ is homotopic to $u_1|_{\partial D}$ as a section of the bundle $\Lambda \to S^1$. For any two maps $u_0, u_1 \in \mathrm{Map}_\Lambda(D, M)$ there exists a smooth function $v : (D, \partial D) \to (M, \Lambda_0)$ such that $v(-1) = u_0(1)$ and the connected sum $u_0 \# v$ is homotopic to u_1 along the boundary. Hence, by what we have just proved,

$$\mu_\Lambda(u_1) - \mu_\Lambda(u_0) - \mu_{\Lambda_0}(v) \in 2N\mathbb{Z}.$$

Since $\mu_{\Lambda_0}(v)$ is an integer multiple of N, the lemma is proved. □

Returning to the loop $\Lambda \subset S^1 \times \mathbb{C}P^n$ we observe that (39) is a Morse-Bott function with critical manifolds

$$C^+ := \left\{[0 : z_1 : \cdots : z_k : 0 : \cdots : 0] \,|\, (z_1, \ldots, z_k) \in \mathbb{C}^k \setminus \{0\}\right\},$$

$$C^- := \left\{[z_0 : 0 : \cdots : 0 : z_{k+1} : \cdots : z_n] \,|\, (z_0, z_{k+1}, \ldots, z_n) \in \mathbb{C}^{n-k+1} \setminus \{0\}\right\}.$$

Note that H attains its minimum $(k-n-1)/(2n+2)$ on C^+ and its maximum $k/(2n+2)$ on C^-. Moreover, $C^\pm \cap \mathbb{R}P^n \subset \Lambda_t$ for every t. Let us denote by

$$A^\pm \in H_2(D \times \mathbb{C}P^n, \Lambda; \mathbb{Z})$$

the homology classes represented by the constant functions $D \to \mathbb{C}P^n$ with values in $C^\pm \cap \mathbb{R}P^n$. The next lemma shows that Λ has Maslov index $k \in \mathbb{Z}_{n+1}$ as claimed in the introduction (see (2)). It also shows that the homology classes $A^\pm \in H_2(D \times \mathbb{C}P^n, \Lambda; \mathbb{Z})$ satisfy the condition (36) for the definition of the Gromov invariants.

Lemma 6.2

$$\mu_\Lambda(A^+) = k - 1 - n, \qquad \mu_\Lambda(A^-) = k. \tag{41}$$

Proof: In the case of A^-, consider the constant function $u(x,y) \equiv p := [1 : 0 : \cdots : 0]$. Then a trivialization of the pullback tangent bundle $u^*T\mathbb{C}P^n$ is determined by the coordinate chart $[z_0 : \cdots : z_n] \mapsto (z_1/z_0, \ldots, z_n/z_0)$. In these coordinates the Hamiltonian flow is $\zeta \mapsto (e^{\pi i t}\zeta_1, \ldots, e^{\pi i t}\zeta_k, \zeta_{k+1}, \ldots, \zeta_n)$. Since $T_p\Lambda_0 \cong \mathbb{R}^n \subset \mathbb{C}^n \cong T_p\mathbb{C}P^n$, we see that the Maslov index of the loop $t \mapsto T_p\Lambda_t$ is equal to k. This proves the second equation in (41) and the first follows from a similar argument. □

6.3 Computation of the Gromov invariants

Since $N = n+1$ it follows from Lemma 6.2 that the classes $A^\pm$ satisfy (36) and hence the requirements of Theorem 5.8. The next theorem shows that the Gromov invariants $\mathrm{Gr}^\pm_{A^\pm,0}(\Lambda)$ are nonzero. Here the subscript 0 corresponds to the choice $\mathbf{t} = t_1 = 0$ for the evaluation map.

Theorem 6.3

$$\mathrm{Gr}^+_{A^+,0}(\Lambda) = [\mathbb{R}P^{k-1}] \in H_{k-1}(\mathbb{R}P^n; \mathbb{Z}_2),$$

$$\mathrm{Gr}^-_{A^-,0}(\Lambda) = [\mathbb{R}P^{n-k}] \in H_{n-k}(\mathbb{R}P^n; \mathbb{Z}_2).$$

Proof: Let $\tau \in \mathcal{T}(\Lambda)$ be given by (12) with $c = 0$ and

$$F_{x,y} = \frac{-\sin(2\pi t)\rho(r)}{2\pi r} H, \qquad G_{x,y} = \frac{\cos(2\pi t)\rho(r)}{2\pi r} H,$$

where $re^{2\pi it} = x + iy$ and H is given by (39). As in (17), $\rho : [0,1] \to [0,1]$ is a smooth nondecreasing cutoff function such that $\rho(r) = 0$ for r near 0 and $\rho(r) = 1$ for r near 1. The formula

$$\Omega_\tau(re^{2\pi it}, z) = -\frac{\dot\rho(r)}{2\pi r} H(z) \tag{42}$$

for $z \in \mathbb{C}P^n$ shows that $\Omega_\tau(x,y,z) \geq 0$ for $z \in C^+$ and $\Omega_\tau(x,y,z) \leq 0$ for $z \in C^-$. By (42) and Lemma 5.2 with $c = 0$ and $E(u) = 0$,

$$\langle [\tau], A^+ \rangle = \frac{k-1-n}{2n+2}, \qquad \langle [\tau], A^- \rangle = \frac{k}{2n+2}. \tag{43}$$

The explicit formulae for F and G show that $C^\pm$ consist entirely of critical points of $F_{x,y}$ and $G_{x,y}$ for all $x + iy \in D$. This shows that the constant functions $u : D \to \mathbb{C}P^n$ with values in $C^+ \cup C^-$ are horizontal for the symplectic connection determined by τ. In explicit terms $\partial_x u = X_F(u)$ and $\partial_y u = X_G(u)$. Hence these constant functions satisfy both equations (30) and (33) for every $J \in \mathcal{J}(D; \mathbb{C}P^n, \omega)$. The constant functions with values in $(C^+ \cup C^-) \cap \mathbb{R}P^n$ satisfy in addition the boundary condition (31). The formula (41) shows that the constant solutions with values in $C^+ \cap \mathbb{R}P^n$ and those with values in $C^- \cap \mathbb{R}P^n$ represent different homology classes.

We prove that, for every $J \in \mathcal{J}(D; \mathbb{C}P^n, \omega)$,

$$\mathcal{M}_{A^+}(\tau, J) = \left\{ u : D \to C^+ \cap \mathbb{R}P^n \,|\, du = 0 \right\}. \tag{44}$$

To see this, let $u \in \mathcal{M}_{A^+}(\tau, J)$. Then, by Lemma 5.2 and (42),

$$\begin{aligned}
0 &\leq E(u) \\
&= \langle [\tau], A^+ \rangle + \int_D \Omega_\tau(x, y, u(x,y))\, dxdy \\
&= \frac{k-n-1}{2n+2} - \int_0^1 \int_0^1 \dot\rho(r) H(u(re^{2\pi it}))\, drdt \\
&\leq \frac{k-n-1}{2n+2} - \int_0^1 \int_0^1 \dot\rho(r) \min H\, drdt \\
&= 0.
\end{aligned}$$

Hence every $u \in \mathcal{M}_{A^+}(\tau, J)$ satisfies $E(u) = 0$ and

$$\dot{\rho}(r) \neq 0 \qquad \Longrightarrow \qquad H(u(re^{2\pi it})) = \min H.$$

The latter implies that $u(x_0, y_0) \in C^+$ for some point $x_0 + iy_0 \in D$ and the former implies that u is a horizontal section of $D \times M$ with respect to τ. Now let $x_1 + iy_1 \in D$, choose a path $[0,1] \to D : t \mapsto x(t) + iy(t)$ that connects $x_0 + iy_0$ to $x_1 + iy_1$, and define $z : [0,1] \to M$ by

$$z(t) := u(x(t), y(t)).$$

Then $z(0) \in C^+$ and

$$\dot{z}(t) = \dot{x}(t) X_{F_{x(t),y(t)}}(z(t)) + \dot{y}(t) X_{G_{x(t),y(t)}}(z(t)).$$

Since C^+ consists of critical points of $F_{x,y}$ and $G_{x,y}$ for all $x + iy \in D$ it follows that $z(t) = z(0)$ for all $t \in [0,1]$. Hence u is constant. The boundary condition shows that this constant lies in $C^+ \cap \mathbb{R}P^n$. This proves (44). Hence $\mathcal{M}_{A^+}(\tau, J)$ is diffeomorphic to $\mathbb{R}P^{k-1}$. The linearized operator is surjective because it decomposes into a direct sum of n Cauchy–Riemann operators on complex line bundles over the disc; k-1 of these have boundary Maslov index zero and $n + 1 - k$ have boundary Maslov index -1; so all n operators are surjective. The evaluation map $u \mapsto u(1)$ is obviously an embedding of $\mathcal{M}_{A^+}(\tau, J) \cong \mathbb{R}P^{k-1}$ into $\mathbb{R}P^n$. A similar assertion holds for $\mathcal{M}_{A^-}(\tau, -J)$ and this proves the theorem. □

6.4 Invariants of projective Lagrangian loops

Let $\mathrm{PL}(n+1)$ denote the manifold of projective Lagrangian planes in $\mathbb{C}P^n$. There is a fibration

$$S^1/\{\pm 1\} \hookrightarrow \mathrm{L}(n+1) \to \mathrm{PL}(n+1),$$

where $\mathrm{L}(n+1)$ denotes the manifold of Lagrangian subspaces of $\mathbb{C}^{n+1}$ and S^1 acts by multiplication. The generator $t \mapsto e^{\pi it}$ of $\pi_1(S^1/\{\pm 1\})$ gives rise to a loop of Lagrangian subspaces of Maslov index $n + 1$. Hence the homotopy exact sequence of the fibration shows that the fundamental group of $\mathrm{PL}(n+1)$ is isomorphic to $\mathbb{Z}_{n+1}$. For $k \in \mathbb{Z}$ we denote by $\Lambda^k \subset S^1 \times \mathbb{C}P^n$ the exact Lagrangian loop defined by (1) in the introduction, i.e $\Lambda^k_t := \phi_{kt}(\mathbb{R}P^n)$, where $\phi_t([z_0 : \cdots : z_n]) = [e^{\pi it} z_0 : z_1 : \cdots : z_n]$. If k is divisible by $n+1$ then this loop is contractible. If $k \in \{1, \ldots, n\}$ and $k \equiv k' \bmod n+1$ then $\Lambda^{k'}$ is Hamiltonian isotopic to Λ^k.

Corollary 6.4 *If k is not divisible by $n+1$ then*

$$\nu(\Lambda^k) = \chi(\Lambda^k) = \varepsilon(\Lambda^k) = \frac{1}{2}.$$

If k is divisible by $n+1$ then $\nu(\Lambda^k) = \chi(\Lambda^k) = 0$.

Proof: Let $k \in \{1, \dots, n\}$. Then the loop Λ, given by (38), is Hamiltonian isotopic to Λ^k and hence

$$\varepsilon(\Lambda^k) = \varepsilon(\Lambda), \qquad \chi(\Lambda^k) = \chi(\Lambda), \qquad \nu(\Lambda^k) = \nu(\Lambda).$$

By Theorem 6.3, $\mathrm{Gr}^+_{A^+,0}(\Lambda) \neq 0$ and $\mathrm{Gr}^-_{A^-,0}(\Lambda) \neq 0$. Hence, by Corollary 5.12 and (43),

$$\varepsilon^+(\tau, \Lambda) \geq -\langle [\tau], A^+ \rangle = \frac{n+1-k}{2n+2}, \quad \varepsilon^-(\tau, \Lambda) \leq -\langle [\tau], A^- \rangle = -\frac{k}{2n+2}.$$

Here $\tau \in \mathcal{T}(\Lambda)$ denotes the connection 2-form introduced in the proof of Theorem 6.3. Hence

$$\varepsilon(\Lambda) = \varepsilon^+(\tau, \Lambda) - \varepsilon^-(\tau, \Lambda) \geq \frac{1}{2}.$$

Since $\nu(\Lambda) \leq \ell(\Lambda) = 1/2$ the result follows from Theorems 3.3 and 3.5. □

Remark 6.5 Our invariants do not distinguish between Λ^j and Λ^k unless one of the numbers is divisible by $n+1$ and the other is not. However, if

$$\gcd(j, n+1) \neq \gcd(k, n+1)$$

then the iterated loops Λ^{mj} and Λ^{mk} have different invariants for some m. To see this suppose, without loss of generality, that $\gcd(j, n+1) < \gcd(k, n+1)$ and denote

$$m := \frac{n+1}{\gcd(k, n+1)} < \frac{n+1}{\gcd(j, n+1)}.$$

Then mk is divisible by $n+1$ whereas mj is not. By Corollary 6.4,

$$\nu(\Lambda^{mj}) \neq \nu(\Lambda^{mk}).$$

In the case of Hamiltonian loops the analogue of the line $\mathcal{T}(\Lambda)$ has a natural basepoint and in that case there are separate invariants $\varepsilon^+(P)$ and $\varepsilon^-(P)$ that contain finer information than their difference.

Remark 6.6 We conjecture that the constant loop $\Lambda^0 = S^1 \times \mathbb{R}P^n$ satisfies $\varepsilon(\Lambda^0) = 0$. This does not follow from the techniques of this paper. The homology class $A^0 \in H^2(D \times \mathbb{C}P^n, S^1 \times \mathbb{R}P^n; \mathbb{Z})$, represented by the constant maps $D \to \mathbb{R}P^n$, satisfies $\mu_{\Lambda^0}(A^0) = 0$. Hence A^0 does not satisfy our condition (36) for the definition of the Gromov invariants, although the arguments of Theorem 6.3 carry over to the constant loop Λ^0 with $A^+ = A^- = A^0$. It should be possible to circumvent the problems arising from Gromov compactness by using the invariants introduced in Cieliebak–Gaio–Salamon [4]. We expect that these techniques apply to the constant loop Λ^0 in $\mathbb{C}P^n$.

Remark 6.7 We expect that the techniques of [4] also apply to symplectic quotients of $\mathbb{C}^n$ that do not satisfy our monotonicity hypothesis. This should give rise to results similar to the ones in this section for general toric varieties.

Remark 6.8 Let (M, ω) be a symplectic $2n$-manifold and L be a closed n-manifold with $H^1(L; \mathbb{R}) = 0$. In [31] Weinstein considers the space of all pairs (Λ, ρ) where $\Lambda \subset M$ is a Lagrangian submanifold diffeomorphic to L and ρ is a volume form on Λ (or a smooth measure in the nonorientable case). He interpretes this space as the cotangent bundle of $\mathcal{L} = \mathcal{L}(M, \omega, L)$ and examines the symplectic action functional on the loop space of $T^*\mathcal{L}$. In [6] Donaldson interpretes this cotangent bundle as a symplectic quotient of the space of all embeddings $\iota : L \to M$ with vanishing cohomology class $\iota^*[\omega]$ by the group of volume preserving diffeomorphisms of L (with respect to any given smooth measure). The group action is Hamiltonian and the zero set of the moment map is the space of Lagrangian embeddings of L into (M, ω). It would be interesting to examine analogues of the invariants studied in the present paper for loops in $T^*\mathcal{L}$ and relate these to the work of Weinstein and Donaldson.

A Symplectic isotopy on Riemann surfaces

The following results are known. However, we could not find proofs in the literature and present them here for the sake of completeness.

Proposition A.1 *Let Σ be a compact connected oriented Riemann surface with area form ω and $S, S' \subset \Sigma$ be two closed embedded discs with the same area. Then there exists a Hamiltonian symplectomorphism $\psi : \Sigma \to \Sigma$ such that $\psi(S) = S'$.*

The proof relies on the following three lemmata. The first asserts that, in dimension 2, a symplectomorphism is smoothly isotopic to the identity if and only if it is symplectically isotopic to the identity. For the 2-torus this follows from the characterization of Hamiltonian symplectomorphisms in Conley–Zehnder [5, Theorem 6]. In general the proof is a parametrized version of Moser isotopy. The work of Seidel [27] shows that the result has no analogue in higher dimensions.

Lemma A.2 *Let Σ be a compact oriented Riemann surface with area form ω and $\psi : \Sigma \to \Sigma$ be a symplectomorphism. Then ψ is smoothly isotopic to the identity if and only if it is symplectically isotopic to the identity.*

Proof: Let $[0,1] \to \mathrm{Diff}(\Sigma) : t \mapsto \psi_t$ be a smooth isotopy such that $\psi_0 = \mathrm{id}$ and $\psi_1 = \psi$. Define

$$\omega_t := \psi_{t*}\omega, \qquad \omega_{s,t} := s\omega + (1-s)\omega_t$$

for $0 \le s, t \le 1$. Then $\omega_{s,0} = \omega_{s,1} = \omega_{1,t} = \omega$ and $\omega_{0,t} = \omega_t$ for all s and t. Fix a Riemannian metric on Σ with volume form ω and let $\alpha_t \in \Omega^1(\Sigma)$ be defined by

$$d\alpha_t = \omega_t - \omega, \qquad \alpha_t \in \mathrm{im}\, d^*.$$

Choose $X_{s,t} \in \mathrm{Vect}(\Sigma)$ such that $\iota(X_{s,t})\omega_{s,t} = \alpha_t$ and define $\psi_{s,t} \in \mathrm{Diff}(\Sigma)$ by

$$\partial_s \psi_{s,t} = X_{s,t} \circ \psi_{s,t}, \qquad \psi_{0,t} = \psi_t.$$

Then $\partial_s({\psi_{s,t}}^*\omega_{s,t}) = 0$ and ${\psi_{0,t}}^*\omega_{0,t} = \omega$. Hence ${\psi_{s,t}}^*\omega_{s,t} = \omega$ for all s and t. Moreover, $\psi_{s,0} = \mathrm{id}$ and $\psi_{s,1} = \psi$ for all s. Hence $t \mapsto \psi_{1,t}$ is the required symplectic isotopy from id to ψ. □

Lemma A.3 *Let Σ be a compact oriented Riemann surface, $S \subset \Sigma$ be an embedded closed disc, and $\omega_0, \omega_1 \in \Omega^2(\Sigma)$ be two area forms such that*

$$\int_\Sigma (\omega_1 - \omega_0) = \int_S (\omega_1 - \omega_0) = 0.$$

Then there exists a smooth isotopy $\psi_t : \Sigma \to \Sigma$ such that

$$\psi_0 = \mathrm{id}, \qquad {\psi_1}^*\omega_1 = \omega_0, \qquad \psi_t(S) = S$$

for every $t \in [0,1]$.

Proof: The result follows again from Moser isotopy. We prove that there exists a 1-form $\alpha \in \Omega^1(\Sigma)$ such that

$$d\alpha + \omega_1 - \omega_0 = 0, \qquad \alpha|_{T\partial S} = 0. \tag{45}$$

To see this, choose any 1-form $\beta \in \Omega^1(\Sigma)$ such that $d\beta + \omega_1 - \omega_0 = 0$. Then the integral of β over ∂S vanishes and so $\beta|_{\partial S}$ is exact. Hence there exists a smooth function $f : \Sigma \to \mathbb{R}$ such that $(\beta - df)|_{T\partial S} = 0$ and the 1-form $\alpha := \beta - df$ satisfies (45). Now let $\omega_t := t\omega_1 + (1-t)\omega_0$ and define $X_t \in \mathrm{Vect}(\Sigma)$ and $\psi_t \in \mathrm{Diff}(\Sigma)$ by

$$\partial_t \psi_t = X_t \circ \psi_t, \qquad \iota(X_t)\omega_t = \alpha, \qquad \psi_0 = \mathrm{id}.$$

Then X_t is tangent to ∂S for every t. Hence ψ_t preserves ∂S and ${\psi_t}^*\omega_t = \omega_0$ for every t. This proves the lemma. □

Lemma A.4 *Let Σ be a compact connected Riemann surface and $S, S' \subset \Sigma$ be two embedded discs. Then there exists a diffeomorphism $f : \Sigma \to \Sigma$ such that f is isotopic to the identity and $f(S) = S'$.*

Proof: Choose orientation preserving embeddings $\phi, \phi' : B_1 \to \Sigma$ such that $\phi(B_1) = S$ and $\phi'(B_1) = S'$. We prove the result in four steps.

Step 1: *There exists a diffeomorphism $g : \Sigma \to \Sigma$ that is isotopic to the identity and satisfies $g \circ \phi(0) = \phi'(0)$.*

Choose a path $\gamma : [0,1] \to \Sigma$ such that $\gamma(0) = \phi(0)$ and $\gamma(1) = \phi'(0)$. Next choose a smooth family of vector fields $X_t \in \mathrm{Vect}(\Sigma)$ such that $X_t(\gamma(t)) = \dot\gamma(t)$ for every t. Then the diffeomorphisms $g_t : \Sigma \to \Sigma$, defined by $\partial_t g_t = X_t \circ g_t$ and $g_0 = \mathrm{id}$, satisfy $g_t(\gamma(0)) = \gamma(t)$ for every t. Hence g_1 satisfies the requirements of Step 1.

Step 2: *ϕ can be chosen such that $d(g \circ \phi)(0) = d\phi'(0)$.*

Define $\Psi \in \mathbb{R}^{2\times 2}$ by

$$d(g \circ \phi)(0)\Psi = d\phi'(0).$$

Then $\det \Psi > 0$ and hence there exists a path $[0,1] \to \mathrm{GL}(2) : t \mapsto \Psi(t)$ such that $\Psi(0) = \mathbb{1}$ and $\Psi(1) = \Psi$. Choose a family of vector fields $X_t : B_1 \to \mathbb{R}^2$ that vanish near the boundary and satisfy

$$X_t(0) = 0, \qquad dX_t(0) = \dot\Psi(t)\Psi(t)^{-1}.$$

Let $\psi_t : B_1 \to B_1$ be the isotopy generated by X_t. Then

$$\psi_t(0) = 0, \qquad d\psi_t(0) = \Psi(t)$$

for every t. Now replace ϕ by $\phi \circ \psi_1$.

Step 3: *ϕ can be chosen such that $g \circ \phi(z) = \phi'(z)$ for $|z|$ sufficiently small.*

By Step 2, we may assume that $d(g \circ \phi)(0) = d\phi'(0)$. Choose $\delta > 0$ such that $\phi'(B_\delta) \subset g(S)$ and consider the function

$$h := \phi^{-1} \circ g^{-1} \circ \phi' : B_\delta \to B_1.$$

This function is an embedding and satisfies $dh(0) = \mathbb{1}$. Choose a smooth cutoff function $\beta : [0,1] \to [0,1]$ such that $\beta(r) = 1$ for $r \le 1/3$ and $\beta(r) = 0$ for $r \ge 2/3$. For $0 < \varepsilon < \delta$ define $h_\varepsilon : B_1 \to B_1$ by

$$h_\varepsilon(z) := \beta(|z|/\varepsilon)h(z) + (1 - \beta(|z|/\varepsilon))z.$$

Then h_ε is a diffeomorphism for $\varepsilon > 0$ sufficiently small and $g \circ \phi \circ h_\varepsilon(z) = \phi'(z)$ for $|z| < \varepsilon/3$. Hence the embedding $\phi \circ h_\varepsilon$ satisfies the requirements of Step 3 for $\varepsilon > 0$ sufficiently small.

Step 4: *We prove the lemma.*

By Step 3, there exist embeddings $\phi, \phi' : B_1 \to \Sigma$, a constant $\varepsilon > 0$, and a diffeomorphism $g : \Sigma \to \Sigma$ such that g is isotopic to the identity and

$$|z| < \varepsilon \qquad \Longrightarrow \qquad g \circ \phi(z) = \phi'(z).$$

Choose $\delta > 0$ such that ϕ and ϕ' extend to embeddings of $B_{1+\delta}$ into Σ. Choose a smooth function $\rho : [0, 1+\delta] \to [0, 1+\delta]$ such that $\dot\rho(r) > 0$ for every r and

$$\rho(r) = \begin{cases} r, & \text{for } r \le \varepsilon/2, \\ 1, & \text{for } r = \varepsilon, \\ r, & \text{for } r \ge 1 + \delta/2. \end{cases}$$

Let $f : \Sigma \to \Sigma$ be given by

$$f(\phi(z)) := \phi(\rho(|z|)z/|z|)$$

for $z \in B_{1+\delta}$ and by $f = \mathrm{id}$ in $\Sigma \setminus \phi(B_{1+\delta})$. Then f is isotopic to the identity and $f \circ \phi(B_\varepsilon) = S$. Similarly, there exists a diffeomorphism $f' : \Sigma \to \Sigma$ that is isotopic to the identity and satisfies $f' \circ \phi'(B_\varepsilon) = S'$. The diffeomorphism $f' \circ g \circ f^{-1}$ is isotopic to the identity and maps S to S'. This proves the lemma. □

Proof of Proposition A.1: By Lemma A.4, there exists a diffeomorphism $f : \Sigma \to \Sigma$ that is isotopic to the identity and satisfies $f(S) = S'$. Since S and S' have the same area, we obtain

$$\int_\Sigma (f^*\omega - \omega) = \int_S (f^*\omega - \omega) = 0.$$

By Lemma A.3, there exists a diffeomorphism $\psi : \Sigma \to \Sigma$ that is isotopic to the identity and satisfies

$$\psi^* f^* \omega = \omega, \qquad \psi(S) = S.$$

Hence $\phi := f \circ \psi$ is isotopic to the identity and

$$\phi^*\omega = \omega, \qquad \phi(S) = S'.$$

By Lemma A.2, ϕ is symplectically isotopic to the identity. Let $t \mapsto \phi_t$ be a symplectic isotopy such that $\phi_0 = \mathrm{id}$ and $\phi_1 = \phi$. Then the embedded discs $S_t := \phi_t(S)$ all have the same area and $S_0 = S$, $S_1 = S'$. Hence $t \mapsto \partial S_t$ is an exact Lagrangian path. By Lemma 2.3, there exists a Hamiltonian isotopy $t \mapsto \psi_t$ of Σ such that $\psi_t(\partial S_0) = \partial S_t$ for all t. Hence $\psi_1(S) = S'$ and this proves the proposition. □

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