More notes on the Octonions

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1 Cross products

Assume throughout that V is a finite-dimensional real vector space. A skew-symmetric bilinear map

$$V \times V \to V : (u, v) \mapsto u \times v \tag{1.1}$$

is called a **cross product** if it satisfies the following two axioms.

(A) $u \times (u \times v) \in \mathrm{span}\{u, v\}$ for all $u, v \in V$.

(B) If $u, v \in V$ are linearly independent, then so are $u, v, u \times v$.

The next observation is discussed in Donaldson's lecture [1].

Theorem 1.1. *Assume* $\dim(V) > 1$ *and let* (1.1) *be a skew-symmetric bilinear map. Then the map* (1.1) *satisfies (A) and (B) if and only if there exists an inner product on* V *that satisfies the equations*

$$\langle u \times v, u\rangle = \langle u \times v, v\rangle = 0, \tag{1.2}$$

$$|u \times v|^2 = |u|^2|v|^2 - \langle u, v\rangle^2 \tag{1.3}$$

for all $u, v \in V$. *Morover, if such an inner product exists, it is uniquely determined by the cross product and is given by the formula*

$$\langle u, v\rangle = \frac{\mathrm{trace}\,(A_u A_v)}{1 - \dim(V)}, \qquad A_u v := u \times v, \tag{1.4}$$

for $u, v \in V$.

Proof. See page 2. □

Remark 1.2. **(i)** It follows from Theorem 1.1 and [7, Theorem 2.5] that V admits a cross product if and only if its dimension is either 0, 1, 3, or 7.

(ii) The formula $x \times y := (x_1y_2 - x_2y_1)e_2$ defines a skew-symmetric bilinear map on $\mathbb{R}^2$ that satisfies (A) and (1.3), but not (B) and (1.2).

(iii) The formula $x \times y := z_1e_1 + z_2e_2 + (z_3 + z_2)e_3$ with $z_1 := x_2y_3 - x_3y_2$, $z_2 := x_3y_1 - x_1y_3$, $z_3 := x_1y_2 - x_2y_1$, defines a skew-symmetric bilinear map on $\mathbb{R}^3$ that satisfies (B) but not (A).

(iv) Let $(u, v) \mapsto u \times v$ be the skew-symmetric bilinear map on $\mathbb{R}^4$ defined by $e_0 \times e_i = e_i$ for $i \neq 0$ and $e_i \times e_j = e_k$ for each cyclic permutation i, j, k of $1, 2, 3$. This map satisfies (1.3) but not (A), (B), and (1.2).

Proof of Theorem 1.1. The proof has five steps.

Step 1. *Let* $\langle\cdot,\cdot\rangle$ *be an inner product that satisfies* (1.2) *and* (1.3)*. Then*

$$u \times (u \times v) = \langle u, v\rangle u - |u|^2 v \qquad \text{for all } u, v \in V, \tag{1.5}$$

the map (1.1) *is a cross product, and the inner product is given by* (1.4)*.*

Equation (1.5) was established in [7, Lemma 2.9]. It implies that (1.1) satisfies (A) and (B) and that $\mathrm{trace}(A_u^2) = (1 - \dim(V))|u|^2$ for all $u \in V$. This proves Step 1. Throughout the remainder of the proof we assume that our skew-symmetric bilinear map (1.1) is a cross product.

Step 2. *There exists a map* $q : V \to \mathbb{R}$ *and a map* $V \to V^* : u \mapsto \Lambda_u$ *such that* $q(0) = 0$*,* $\Lambda_0 = 0$*,* $q(u) > 0$ *for* $0 \neq u \in V$*, and for all* $u, v \in V$

$$u \times (u \times v) = \Lambda_u(v)u - q(u)v. \tag{1.6}$$

Fix a nonzero vector $u \in V$. Since $A_uu = 0$ by skew-symmetry, the linear map $A_u : V \to V$ descends to an endomorphism $\overline{A}_u : \overline{V}_u \to \overline{V}_u$ of the quotient space $\overline{V}_u := V/\mathbb{R}u$. Let $\pi_u : V \to \overline{V}_u$ denote the canonical projection and fix a vector $v \in V$ such that u and v are linearly independent. Then by (A) there exists a real number $q(u, v)$ such that $A_uA_uv \in -q(u, v)v + \mathbb{R}u$. Hence the vector $0 \neq \overline{v} := \pi_u(v) \in \overline{V}_u$ satisfies $\overline{A}_u\overline{A}_u\overline{v} = -q(u, v)\overline{v}$ and so each nonzero vector in $\overline{V}_u$ is an eigenvector of $\overline{A}_u\overline{A}_u$. Thus $q(u) := q(u, v)$ is independent of v and $A_uA_uv + q(u)v \in \mathbb{R}u$ for every $v \in V$. Hence there exists a linear functional $\Lambda_u : V \to \mathbb{R}$ such that $A_uA_uv + q(u)v = \Lambda_u(v)u$ for all $v \in V$. Since the bases u, v, A_uv and u, A_uv, A_uA_uv induce the same orientation on the 3-dimensional subspace $\Lambda := \mathrm{span}\{u, v, A_uv\}$ whenever u and v are linearly independent, it follows that $q(u) > 0$. This proves Step 2.

Step 3. *Let $q : V \to \mathbb{R}$ and $V \to V^* : u \mapsto \Lambda_u$ be as in Step 2. Then the formula* (1.4) *defines an inner product on V and, for all $u, v \in V$,*

$$|u|^2 = \langle u, u\rangle = q(u), \qquad u \times (u \times v) = \Lambda_u(v)u - |u|^2 v, \tag{1.7}$$

$$|u \times v|^2 = |u|^2|v|^2 - \Lambda_u(v)^2, \qquad \Lambda_u(v) = \Lambda_v(u), \qquad \Lambda_{u\times v}(u) = 0. \tag{1.8}$$

Fix a nonzero vector $u \in V$. Since $A_u u = 0$, it follows directly from (1.6) that $\mathrm{trace}(A_u A_u) = (1 - \dim(V))q(u)$. Hence the bilinear map

$$V \times V \to \mathbb{R} : (u, v) \mapsto \langle u, v\rangle := \frac{\mathrm{trace}(A_u A_v)}{1 - \dim(V)}$$

satisfies $\langle u, u\rangle = q(u) > 0$ for every nonzero vector $u \in V$ and therefore is an inner product satisfying (1.7). Use (1.7) repeatedly to obtain

$$\begin{aligned}
\Lambda_{u\times v}(u)u \times v &= |u \times v|^2 u + (u \times v) \times \big((u \times v) \times u\big) \\
&= |u \times v|^2 u + (u \times v) \times \big(|u|^2 v - \Lambda_u(v)u\big) \\
&= |u \times v|^2 u + |u|^2\big(v \times (v \times u)\big) + \Lambda_u(v)\big(u \times (u \times v)\big) \\
&= |u \times v|^2 u + |u|^2\big(\Lambda_v(u)v - |v|^2 u\big) + \Lambda_u(v)\big(\Lambda_u(v)u - |u|^2 v\big) \\
&= \big(|u \times v|^2 + \Lambda_u(v)^2 - |u|^2|v|^2\big)u + |u|^2\big(\Lambda_v(u) - \Lambda_u(v)\big)v.
\end{aligned}$$

If u, v are linearly independent, this implies (1.8) by (B). Next observe that $\Lambda_{tu} = t\Lambda_u$ and $\Lambda_u(u) = |u|^2$ for $u \in V$ and $t \in \mathbb{R}$ by (1.7). Thus (1.8) continues to hold when u, v are linearly dependent, and this proves Step 3.

Step 4. *Let $V \to V^* : u \mapsto \Lambda_u$ be as in Step 2 and let $\langle\cdot,\cdot\rangle$ be the inner product in Step 3. Then $\Lambda_u(v) = \langle u, v\rangle$ for all $u, v \in V$.*

When u, v are linearly dependent, this follows directly from (1.6) and (1.7). Thus assume that u, v are linearly independent. Then $\Lambda := \mathrm{span}\{u, v, u \times v\}$ is a three-dimensional subspace of V by (B) and is invariant under the cross product by (A). Define the linear maps $A, B : \Lambda \to \Lambda$ by

$$Aw := u \times w, \qquad Bw := v \times w$$

for $w \in \Lambda$ and abbreviate $\lambda := \Lambda_u(v) = \Lambda_v(u)$ (see (1.8) in Step 3). Then

$$\begin{aligned}
&AB(u \times v) = BA(u \times v) = -\lambda(u \times v), \\
&ABw + BAw + 2\langle u, v\rangle w \in \mathrm{span}\{u, v\}
\end{aligned} \tag{1.9}$$

for all $w \in \Lambda$ by (1.7). Take $w = u \times v$ and use (B) to obtain $\lambda = \langle u, v\rangle$. This proves Step 4.

Step 5. *The inner product in Step 3 satisfies* (1.2) *and* (1.3)*.*

By Step 4 and (1.7) the inner product in Step 3 satisfies (1.5), i.e.

$$u \times (u \times v) = \langle u, v\rangle u - |u|^2 v$$

for all $u, v \in V$. This implies

$$\langle u, u \times (u \times v)\rangle = 0 \tag{1.10}$$

for all $u, v \in V$. Now fix a pair of vectors $u, v \in V$ such that $u \neq 0$ and define

$$w := -\frac{u \times v}{|u|^2}.$$

Then

$$u \times w = -\frac{u \times (u \times v)}{|u|^2} = v - \frac{\langle u, v\rangle}{|u|^2} u$$

by (1.5), hence $u \times (u \times w) = u \times v$, and hence $\langle u, u \times v\rangle = 0$ by (1.10). This shows that the inner product in Step 3 satisfies (1.2). That it also satisfies (1.3) follows from Step 4 and the identity $|u \times v|^2 = |u|^2|v|^2 - \Lambda_u(v)^2$ in (1.8) in Step 3. This proves Step 5 and Theorem 1.1. □

2 Volume forms

Let V be a seven-dimensional real vector space. Recall from [7, Section 3] that a 3-form $\phi \in \Lambda^3 V^*$ is called **nondegenerate** if, for every pair of linearly independent vectors $u, v \in V$ there exists a third vector $w \in V$ such that $\phi(u, v, w) \neq 0$. Call an inner product $\langle \cdot, \cdot \rangle$ compatible with a 3-form ϕ if the skew-symmetric bilinear map $V \times V \to V : (u, v) \mapsto u \times v$, defined by

$$\langle u \times v, w\rangle = \phi(u, v, w) \tag{2.1}$$

for $u, v, w \in V$, is a cross product that satisfies (1.2) and (1.3). Then [7, Theorem 3.2] asserts that a 3-form ϕ is nondegenerate if and only if it admits a compatible inner product, that this inner product is uniquely determined by ϕ in the nondegenerate case, and that it is characterized by the equation

$$6\langle u, v\rangle \mathrm{dvol} = \iota(u)\phi \wedge \iota(v)\phi \wedge \phi \qquad \text{for } u, v \in V, \tag{2.2}$$

where the orientation is chosen such that $\langle u, u\rangle > 0$ for $u \neq 0$, and the scaling factor is chosen such that $\mathrm{dvol} \in \Lambda^7 V^*$ is the volume form associated to the inner product and orientation. Conversely, Theorem 1.1 asserts that every cross product (1.1) on V uniquely determines a nondegenerate 3-form ϕ via (1.4) and (2.1). It is called the **associative calibration** [3].

Now let $\phi \in \Lambda^3 V^*$ be a nondegenerate 3-form and denote by

$$*_\phi : \Lambda^k V^* \to \Lambda^{7-k} V^*$$

the Hodge $*$-operator associated to the inner product and orientation determined by ϕ. Then the volume form associated to the inner product and orientation determined by ϕ is given by

$$\rho(\phi) := \mathrm{dvol}_\phi = \frac{1}{7}(*_\phi \phi) \wedge \phi \tag{2.3}$$

Thus ρ defines a map, equivariant under the action of the general linear group, from the space $\mathcal{P} \subset \Lambda^3 V^*$ of nondegenerate 3-forms to the space $\mathcal{V} \subset \Lambda^7 V^*$ of volume forms.

Theorem 2.1. *The derivative of the map $\rho : \mathcal{P} \to \mathcal{V}$ in* (2.3) *at an element $\phi \in \mathcal{P}$ in the direction $\widehat{\phi} \in T_\phi \mathcal{P} = \Lambda^3 V^*$ is given by*

$$d\rho(\phi)\widehat{\phi} := \frac{1}{3}\big(*_\phi \phi\big) \wedge \widehat{\phi} \tag{2.4}$$

Proof. Fix an associative calibration $\phi \in \mathcal{P}$ and denote by $\psi := *_\phi \phi \in \Lambda^4 V^*$ the corresponding coassociative calibration. Then there is a natural splitting

$$\Lambda^3 V^* = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27},$$

where $\Lambda^3_1 \subset \Lambda^3 V^*$ is the 1-dimensional subspace spanned by ϕ and the 7-dimensional subspace Λ^3_7 and the 27-dimensional subspace Λ^3_{27} are given by

$$\Lambda^3_7 := \big\{\iota(u)\psi \,\big|\, u \in V\big\}, \qquad \Lambda^3_{27} := \big\{\omega \in \Lambda^3 V^* \,\big|\, \phi \wedge \omega = 0,\, \psi \wedge \omega = 0\big\}.$$

This splitting is orthogonal for the inner product determined by ϕ and

$$\omega \in \Lambda^3_7 \oplus \Lambda^3_{27} \qquad \Longleftrightarrow \qquad \omega \wedge \psi = 0$$

(see [7, Theorem 8.5]). Hence $\omega \wedge \psi = \pi_1(\omega) \wedge \psi$ for all $\phi \in \mathcal{P}$ and $\omega \in \Lambda^3 V^*$. For $k = 1, 7, 27$ denote by $\pi_k : \Lambda^3 V^* \to \Lambda^3_k$ the ϕ-orthogonal projection. Then the derivative of the map

$$\mathcal{P} \to \Lambda^4 V^* : \phi \mapsto \Theta(\phi) := *_\phi \phi$$

at $\phi \in \mathcal{P}$ in the direction $\widehat{\phi} \in T_\phi \mathcal{P} = \Lambda^3 V^*$ is given by

$$d\Theta(\phi)\widehat{\phi} = *_\phi \left(\tfrac{4}{3}\pi_1(\widehat{\phi}) + \pi_7(\widehat{\phi}) + \pi_{27}(\widehat{\phi})\right) \tag{2.5}$$

(see [2] and [7, Theorem 8.18]).

Since $7\rho(\phi) = \phi \wedge \Theta(\phi)$, it follows from (2.5) that

$$\begin{aligned}
7d\rho(\phi)\widehat{\phi} &= \widehat{\phi} \wedge \Theta(\phi) + \phi \wedge d\Theta(\phi)\widehat{\phi} \\
&= \widehat{\phi} \wedge *_\phi\phi + \phi \wedge *_\phi \left(\tfrac{4}{3}\pi_1(\widehat{\phi}) + \pi_7(\widehat{\phi}) + \pi_{27}(\widehat{\phi})\right) \\
&= \widehat{\phi} \wedge *_\phi\phi + \left(\tfrac{4}{3}\pi_1(\widehat{\phi}) + \pi_7(\widehat{\phi}) + \pi_{27}(\widehat{\phi})\right) \wedge *_\phi\phi \\
&= \widehat{\phi} \wedge \psi + \left(\tfrac{4}{3}\pi_1(\widehat{\phi}) + \pi_7(\widehat{\phi}) + \pi_{27}(\widehat{\phi})\right) \wedge \psi \\
&= \widehat{\phi} \wedge \psi + \tfrac{4}{3}\widehat{\phi} \wedge \psi \\
&= \frac{7}{3}\widehat{\phi} \wedge *_\phi\phi
\end{aligned}$$

for all $\phi \in \mathcal{P}$ and all $\widehat{\phi} \in T_\phi\mathcal{P} = \Lambda^3 V^*$. This proves Theorem 2.1. □

3 The Hitchin functional

Let M be a closed oriented 7-manifold, fix a cohomology class $a \in H^3(M;\mathbb{R})$, and denote by $\mathscr{P}_a \subset \Omega^3(M)$ the space of closed 3-forms $\phi \in \Omega^3(M)$ that represent the cohomology class a and are nondegenerate and compatible with the orientation. Then every $\phi \in \mathscr{P}_a$ determines a volume form

$$\mathrm{dvol}_\phi = \frac{1}{7}(*_\phi\phi) \wedge \phi \in \Omega^7(M)$$

as in (2.3) and the total volume of M with respect to this volume form defines a functional $\mathscr{V}_a : \mathscr{P}_a \to \mathbb{R}$ given by

$$\mathscr{V}_a(\phi) := \int_M \mathrm{dvol}_\phi \qquad (3.1)$$

for $\phi \in \mathscr{P}_a$.

Theorem 3.1. *An element $\phi \in \mathscr{P}_a$ is a critical point of the volume functional $\mathscr{V}_a$ if and only if $d{*_\phi}\phi = 0$.*

Proof. By Theorem 2.1 the differential of the functional $\mathscr{V}_a$ at $\phi \in \mathscr{P}_a$ in the direction of an exact 3-form $\widehat{\phi} \in T_\phi\mathscr{P}_a$ is given by

$$d\mathscr{V}_a(\phi)\widehat{\phi} = \frac{1}{3}\int_M (*_\phi\phi) \wedge \widehat{\phi}.$$

This expression vanishes for every exact 3-form $\widehat{\phi}$ if and only if the 4-form $*_\phi\phi$ is closed. □

A nondegenerate 3-form ϕ on M is called a G_2**-structure** if it is closed and coclosed with respect to the Riemannian metric and orientation determined by ϕ. Thus an element $\phi \in \mathscr{P}_a$ is a G_2-structure if and only if it is a critical point of the volume functional $\mathscr{V}_a$. A theorem of Fernández and Gray [2] asserts that a nondegenerate 3-form ϕ is a G_2-structure if and only if the associated cross product is invariant under parallel transport for the associated Riemannian metric.

References

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