Parabolic $L^p - L^q$ estimates

Dietmar A. Salamon
ETH Zürich

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Abstract

The purpose of this expository paper is to give a self-contained proof of maximal L^p/L^q regularity for the heat equation on $\mathbb{R}^n$, and to explain the role of the Besov space $B_q^{2-2/q,p}$ for the initial conditions.

Contents

1 Introduction

This is an expository paper. Its purpose is to give self-contained proofs of the following three theorems. For $n \in \mathbb{N}$ let $(t,x) = (t, x_1, \dots, x_n)$ be the coordinates on $\mathbb{R}^{n+1}$ and denote the Laplace operator on $\mathbb{R}^n$ by $\Delta := \sum_{i=1}^n \partial^2/\partial x_i^2$. Abbreviate $\partial_t := \partial/\partial t$ and $\partial_i := \partial/\partial x_i$ for $i = 1, \dots, n$. The gradient of a smooth function $f : \mathbb{R}^n \to \mathbb{C}$ is the function $\nabla f := (\partial_1 f, \dots, \partial_n f) : \mathbb{R}^n \to \mathbb{C}^n$.

Theorem 1.1. *For every positive integer n and every pair of real numbers $p, q > 1$ there exists a constant $c = c(n,p,q) > 0$ such that every compactly supported smooth function $u : \mathbb{R}^{n+1} \to \mathbb{C}$ satisfies the estimate*

$$\left(\int_{-\infty}^{\infty} \|\partial_t u\|_{L^p(\mathbb{R}^n)}^q \, dt \right)^{1/q} \le c \left(\int_{-\infty}^{\infty} \|\partial_t u - \Delta u\|_{L^p(\mathbb{R}^n)}^q \, dt \right)^{1/q}. \tag{1.1}$$

Proof. See page 90. □

Theorem 1.1 leads to the question under which assumption on the initial condition u_0 the solution $u : [0,\infty) \times \mathbb{R}^n \to \mathbb{C}$ of the heat equation

$$\partial_t u = \Delta u, \qquad u(0,\cdot) = u_0, \tag{1.2}$$

belongs to the space $W^{1,q}([0,\infty), L^p(\mathbb{R}^n, \mathbb{C})) \cap L^q([0,\infty), W^{2,p}(\mathbb{R}^n, \mathbb{C}))$. The answer involves the Besov space $B_q^{s,p}(\mathbb{R}^n, \mathbb{C})$ for $0 < s < 2$ and $p, q > 1$. This space is the completion of $C_0^\infty(\mathbb{R}^n, \mathbb{C})$ with respect to the norm

$$\|f\|_{B_q^{s,p}} := \|f\|_{L^p} + \|f\|_{b_q^{s,p}}, \tag{1.3}$$

for $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$, where

$$\|f\|_{b_q^{s,p}}^q := \int_0^\infty \frac{\sup_{|h| \le r} \left(\int_{\mathbb{R}^n} |f(x+h) - 2f(x) + f(x-h)|^p \, dx \right)^{q/p}}{r^{sq}} \frac{dr}{r}. \tag{1.4}$$

Theorem 1.2. *For every positive integer n and every pair of real numbers $p, q > 1$ there exists a constant $c = c(n, p, q) > 0$ with the following significance. Let $u_0 \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$ and suppose that $u : [0, \infty) \times \mathbb{R}^n \to \mathbb{C}$ is the unique solution of the heat equation (1.2) such that $u_t := u(t, \cdot)$ is square integrable for all $t \geq 0$. Define the number $0 < s < 2$ by*

$$s := 2 - 2/q.$$

Then

$$\frac{1}{c} \|u_0\|_{b_q^{s,p}} \leq \left(\int_0^\infty \|\partial_t u\|_{L^p(\mathbb{R}^n)}^q \, dt \right)^{1/q} \leq c \|u_0\|_{B_q^{s,p}} . \tag{1.5}$$

If $0 < s < 1$ (or, equivalently, $1 < q < 2$) then the norm $\|u_0\|_{B_q^{s,p}}$ on the right can be replaced by the norm $\|u_0\|_{b_q^{s,p}}$ of the homogeneous Besov space.

Proof. See page 131. □

Theorem 1.3. *For every positive integer n and every real number $p \geq 2$ there exists a constant $c = c(n, p) > 0$ such that every compactly supported smooth function $u : \mathbb{R}^{n+1} \to \mathbb{C}$ satisfies the estimate*

$$\|\nabla u(T, \cdot)\|_{L^p(\mathbb{R}^n)} \leq c \left(\int_{-\infty}^T \|\partial_t u - \Delta u\|_{L^p(\mathbb{R}^n)}^2 \, dt \right)^{1/2} \tag{1.6}$$

for all $T \in \mathbb{R}$.

Proof. See page 11. □

Theorem 1.1 is called **maximal regularity** and was proved in the sixties by deSimon [10] for $p = 2$ and Ladyshenskaya–Solonnikov–Uralćeva [21] for $p = q$. In [2] Benedek–Calderón–Panzone proved that the assertion is independent of q for general analytic semigroups (Theorem 9.3). In our proof for $p = q$ we follow the approach of Lamberton [20]. A proof of Theorem 1.1 for all p and q (which applies to general analytic semigroups and extends the result of Lamberton) can be found in Hieber–Prüss [15].

The Besov spaces $B_q^{s,p}(\mathbb{R}^n, \mathbb{C})$ were introduced in 1959 by Besov [3]. Theorem 1.2 is due to Peetre [31] and Triebel [38, 39] for $1 < q < 2$, and to Grigor'yan–Liu [13, Thm 1.5 & Rmk 1.8] for $q \geq 2$ (see also [17, Thm 6.7] and [23, Thm 5.8]). The present exposition follows the argument in [13].

We will use without proof the theory of strongly continuous semigroups and the basic properties of the Fourier transform.

Before entering into the proofs we formulate some consequences of these results. The inhomogeneous heat equation on $\mathbb{R}^n$ with a compactly supported smooth function $u_0 : \mathbb{R}^n \to \mathbb{C}$ as initial condition and a smooth compactly supported inhomogeneous term $f : (0,\infty) \times \mathbb{R}^n \to \mathbb{C}$ has the form

$$\partial_t u = \Delta u + f, \qquad u(0,\cdot) = u_0. \tag{1.7}$$

This equation has a unique solution $u : [0,\infty) \times \mathbb{R}^n \to \mathbb{C}$ such that $u(t,\cdot)$ is square integrable for all t. This solution can be expressed in the form

$$u(t,x) = \int_{\mathbb{R}^n} K_t(x-y)u_0(y)\,dy + \int_0^t \int_{\mathbb{R}^n} K_{t-s}(x-y)f(s,y)\,dy\,ds \tag{1.8}$$

for $t \geq 0$, where $K : \mathbb{R}^{n+1} \to \mathbb{R}$ denotes the fundamental solution of the heat equation. It is given by

$$K(t,x) := K_t(x) := \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t} \tag{1.9}$$

for $t > 0$ and by $K(t,x) := K_t(x) := 0$ for $t \leq 0$.

Remark 1.4. **(i)** A simple computation shows that, for every $n \in \mathbb{N}$, there exists a constant $c = c(n) > 0$ such that

$$\|K_t\|_{L^1} = 1, \qquad \|\nabla K_t\|_{L^1} \leq \frac{c}{\sqrt{t}}, \qquad \|\partial_t K_t\|_{L^1} \leq \frac{c}{t}, \qquad \left\|\partial_t^2 K_t\right\|_{L^1} \leq \frac{c}{t^2}$$

for all $t > 0$.

(ii) Let $u : \mathbb{R}^{n+1} \to \mathbb{C}$ be a smooth function with compact support contained in $[0,\infty) \times \mathbb{R}^n$ and define $f_t(x) := f(t,x) := \partial_t u(t,x) - (\Delta u)(t,x)$ for $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$. Then u satisfies equation (1.7) with $u_0 = 0$ and hence is given by equation (1.8). Thus the gradien of u_t is given by $\nabla u_t = \int_0^t \nabla K_{t-s} * f_s\,ds$ for $t \geq 0$. Hence, by (i) and Young's inequality,

$$\begin{aligned}
\|\nabla u_t\|_{L^p} &\leq \int_0^t \|K_{t-s} * f_s\|_{L^p}\,ds \\
&\leq \int_0^t \|K_{t-s}\|_{L^1} \|f_s\|_{L^p}\,ds \leq \int_0^t \frac{c}{\sqrt{t-s}} \|f_s\|_{L^p}\,ds.
\end{aligned}$$

By Hölder's inequality this implies the estimate (1.6) with the exponent 2 on the right replaced by any number $q > 2$ (and a constant depending on T). To prove the estimate for $q = 2$ requires different arguments that will be spelled out in Section 2.

The next corollary asserts that the heat equation defines a strongly continuous semigroup on the Besov spaces. The condition $s = 2 - 2/q$ is not needed for this result but it suffices for our purposes.

Corollary 1.5. *Fix an integer $n \in \mathbb{N}$ and real numbers $p, q > 1$. Define $s := 2 - 2/q$. Then the solutions of the heat equation* (1.2) *on $\mathbb{R}^n$ define a strongly continuous semigroup $S(t)$ on the Besov space $B_q^{s,p}(\mathbb{R}^n, \mathbb{C})$ given by*

$$S(t)f := \begin{cases} K_t * f, & \text{for } t > 0, \\ f, & \text{for } t = 0, \end{cases} \qquad f \in B_q^{s,p}(\mathbb{R}^n, \mathbb{C}).$$

It is a contraction semigroup with respect to the norm

$$\|f\|_{p,q} := \|f\|_{L^p} + \left(\int_0^\infty \|\Delta(K_t * f)\|_{L^p}^q \, dt \right)^{1/q}. \tag{1.10}$$

Proof. By Theorem 1.2 the norm $\|\cdot\|_{p,q}$ in (1.10) is equivalent to the norm in (1.3). Hence the completion of $C_0^\infty(\mathbb{R}^n, \mathbb{C})$ with respect to the norm (1.10) is the Besov space $B_q^{s,p}(\mathbb{R}^n, \mathbb{C})$. Moreover, by definition,

$$\|S(t)f\|_{p,q} = \|K_t * f\|_{L^p} + \left(\int_t^\infty \|\Delta(K_\tau * f)\|_{L^p}^q \, d\tau \right)^{1/q} \le \|f\|_{p,q}$$

for all $t > 0$ and all $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$. This implies that the function

$$[0, \infty) \to B_q^{s,p}(\mathbb{R}^n, \mathbb{C}) : t \mapsto S(t)f \tag{1.11}$$

is continuous for every $f \in B_q^{s,p}(\mathbb{R}^n, \mathbb{C})$. (Choose a sequence $f_k \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$ that converges to f with respect to the norm (1.10); then the function $[0, \infty) \to B_q^{s,p}(\mathbb{R}^n, \mathbb{C}) : t \mapsto S(t)f_k$ is continuous for each k and converges uniformly to (1.11); so the latter is continuous.) This proves Corollary 1.5. □

The next corollary implies that the solution to the inhomogeneous heat equation with an inhomogeneous term in $L^q([0, T], L^p(\mathbb{R}^n, \mathbb{C}))$ is a continuous function with values in the appropriate Besov space.

Corollary 1.6. *Fix an integer $n \in \mathbb{N}$, real numbers $p, q > 1$, and a compact interval $I = [0, T]$. Define $s := 2 - 2/q$. Then there exists a constant $c > 0$ such that every smooth function $I \times \mathbb{R}^n \to \mathbb{C} : (t, x) \mapsto u(t, x) = u_t(x)$ with compact support satisfies the inequality*

$$\sup_{0 \le t \le T} \|u_t\|_{B_q^{s,p}} \le c \left(\int_0^T \left(\|u_t\|_{L^p}^q + \|\partial_t u_t\|_{L^p}^q + \|\Delta u_t\|_{L^p}^q \right) dt \right)^{1/q}. \tag{1.12}$$

Proof. Throughout denote by $c_1 = c_1(n,p,q)$ the constant of Theorem 1.1 and by $c_2 = c_2(n,p,q)$ the constant of Theorem 1.2. Fix a smooth function $u : [0,T] \times \mathbb{R}^n \to \mathbb{C}$ with compact support and abbreviate

$$u_t(x) := u(t,x), \qquad f_t(x) := \partial_t u_t(x) - \Delta u_t(x)$$

for $0 \le t \le T$ and $x \in \mathbb{R}^n$. Then

$$u_t = K_t * u_0 + \int_0^t K_{t-\tau} * f_\tau \, d\tau$$

for $0 < t \le T$ and hence

$$\|u_{t_0} - K_{t_0} * u_0\|_{L^p} \le \int_0^{t_0} \|f_t\|_{L^p} \, dt \le {t_0}^{1-1/q} \left(\int_0^{t_0} \|f_t\|_{L^p}^q \, dt \right)^{1/q} \tag{1.13}$$

for $t_0 \in [0,T]$ by Hölder's inequality. Fix a constant $0 < t_0 \le T$ and define

$$v_t := \begin{cases} u_t - K_t * u_0, & \text{for } 0 \le t \le t_0, \\ K_{t-t_0} * (u_{t_0} - K_{t_0} * u_0), & \text{for } t > t_0, \end{cases}$$
$$g_t := \begin{cases} f_t, & \text{for } 0 \le t \le t_0, \\ 0, & \text{for } t > t_0. \end{cases}$$

Then

$$v_t = \int_0^t K_{t-\tau} * g_\tau \, d\tau, \qquad \partial_t v_t - \Delta v_t = g_t$$

for all $t > 0$ and hence, by Theorem 1.1,

$$\left(\int_0^\infty \|\partial_t v_t\|_{L^p}^q \, dt \right)^{1/q} \le c_1 \left(\int_0^\infty \|g_t\|_{L^p}^q \, dt \right)^{1/q} = c_1 \left(\int_0^{t_0} \|f_t\|_{L^p}^q \, dt \right)^{1/q}.$$

Since $v_t = K_{t-t_0} * (u_{t_0} - K_{t_0} * u_0)$ for $t > t_0$, it follows from Theorem 1.2 that

$$\|u_{t_0} - K_{t_0} * u_0\|_{b_q^{s,p}} \le c_2 \left(\int_{t_0}^\infty \|\partial_t v_t\|_{L^p}^q \, dt \right)^{1/q} \le c_1 c_2 \left(\int_0^{t_0} \|f_t\|_{L^p}^q \, dt \right)^{1/q}.$$

Combine this with (1.13) to obtain

$$\sup_{0 \le t \le T} \|u_t - K_t * u_0\|_{B_q^{s,p}} \le c_3 \left(\int_0^T \|\partial_t u_t - \Delta u_t\|_{L^p}^q \, dt \right)^{1/q}. \tag{1.14}$$

where $c_3 := c_3(n,p,q,T) := c_1 c_2 + T^{1-1/q}$.

It remains to estimate u_0. First, since

$$u_0 = \frac{1}{T}\int_0^T (u_t + (t-T)\partial_t u_t)\,dt,$$

we have

$$\|u_0\|_{L^p} \le T^{-1/q}\|u\|_{L^q(I,L^p)} + T^{1-1/q}\|\partial_t u\|_{L^q(I,L^p)}. \tag{1.15}$$

Second, define

$$h_t := \partial_t u_t - \Delta u_t, \qquad w_t := \int_0^t K_{t-\tau} * h_\tau \, d\tau$$

for $0 \le t \le T$. Then, by Theorem 1.1,

$$\|\partial_t w\|_{L^q(I,L^p)} \le c_1 \|h\|_{L^q(I,L^p)} = c_1 \|\partial_t u - \Delta u\|_{L^q(I,L^p)}. \tag{1.16}$$

Third, $K_t * u_0 = u_t - w_t$ and so

$$\left(\int_0^T \|\Delta(K_t * u_0)\|_{L^p}^q \, dt\right)^{1/q} \le \|\partial_t u\|_{L^q(I,L^p)} + \|\partial_t w_t\|_{L^q(I,L^p)}. \tag{1.17}$$

Fourth, since $\|\Delta K_t\|_{L^1} \le C/t$ for some constant $C = C(n) > 0$, we have

$$\left(\int_T^\infty \|\Delta(K_t * u_0)\|_{L^p}^q \, dt\right)^{1/q} \le C\left(\frac{T^{q-1}}{q-1}\right)^{1/q} \|u_0\|_{L^p}. \tag{1.18}$$

By (1.15), (1.16), (1.17), (1.18), there is a constant $c_0 > 0$ such that every compactly supported smooth function $I \times \mathbb{R}^n \to \mathbb{C} : (t,x) \mapsto u(t,x) = u_t(x)$ satisfies the estimate

$$\|u_0\|_{p,q} \le c_0 \Big(\|u\|_{L^q(I,L^p)} + \|\partial_t u\|_{L^q(I,L^p)} + \|\Delta u\|_{L^q(I,L^p)}\Big). \tag{1.19}$$

By Corollary 1.5 the function $K_t * u_0$ satisfies the same estimate with the same constant. Moreover,

$$\|K_t * u_0\|_{B_q^{s,p}} \le c_2 \|K_t * u_0\|_{p,q}$$

by Theorem 1.2. This implies

$$\|K_t * u_0\|_{B_q^{s,p}} \le c_0 c_2 \Big(\|u\|_{L^q(I,L^p)} + \|\partial_t u\|_{L^q(I,L^p)} + \|\Delta u\|_{L^q(I,L^p)}\Big). \tag{1.20}$$

for all $t \in [0,T]$. The estimate (1.12) follows directly from (1.14) and (1.20). This proves Corollary 1.6. □

Corollary 1.7. *Let n, p, q, s and $I = [0, T]$ be as in Corollary 1.6. Then the identity on the space of complex valued smooth functions on $I \times \mathbb{R}^n$ with compact support extends to a bounded linear operator*

$$L^q(I, W^{2,p}(\mathbb{R}^n, \mathbb{C})) \cap W^{1,q}(I, L^p(\mathbb{R}^n, \mathbb{C})) \to C(I, B_q^{s,p}(\mathbb{R}^n, \mathbb{C})).$$

Proof. This follows directly from the estimate (1.12) in Corollary 1.6. □

The next corollary shows that the result of Corollary 1.7 is sharp.

Corollary 1.8. *Let n, p, q, s and $I = [0, T]$ be as in Corollary 1.6 and let $f \in L^p(\mathbb{R}^n, \mathbb{C})$. Then the following are equivalent.*

(i) $f \in B_q^{s,p}(\mathbb{R}^n, \mathbb{C})$

(ii) *There exists a function*

$$u \in L^q(I, W^{2,p}(\mathbb{R}^n, \mathbb{C})) \cap W^{1,q}(I, L^p(\mathbb{R}^n, \mathbb{C}))$$

such that $u(0, \cdot) = f$.

Proof. If $f \in B_q^{s,p}(\mathbb{R}^n, \mathbb{C})$ then the function $u_t := K_t * f$ satisfies the requirements of part (ii) with $\partial_t u = \Delta u$, by Theorem 1.2 and the Calderón–Zygmund inequality in Corollary 6.2. That (ii) implies (i) follows immediately from Corollary 1.7. This proves Corollary 1.8. □

Corollary 1.9. *Let n, p, q, s and $I = [0, T]$ be as in Corollary 1.6 and consider the Banach spaces*

$$\begin{aligned} \mathcal{W}^{q,p} &:= L^q(I, W^{2,p}(\mathbb{R}^n, \mathbb{C})) \cap W^{1,q}(I, L^p(\mathbb{R}^n, \mathbb{C})), \\ \mathcal{F}^{q,p} &:= B_q^{s,p}(\mathbb{R}^n, \mathbb{C}) \times L^q(I, L^p(\mathbb{R}^n, \mathbb{C})). \end{aligned} \tag{1.21}$$

Define the operators $\mathcal{D} : \mathcal{W}^{q,p} \to \mathcal{F}^{q,p}$ and $\mathcal{T} : \mathcal{F}^{q,p} \to \mathcal{W}^{q,p}$ by

$$\begin{aligned} \mathcal{D}u &:= (u(0, \cdot), \partial_t u - \Delta u), \\ (\mathcal{T}(f, g))(t, x) &:= (K_t * f)(x) + \int_0^t \int_{\mathbb{R}^n} K_{t-s}(x - y) g(s, y)\, dy\, ds \end{aligned} \tag{1.22}$$

Then $\mathcal{D}$ and $\mathcal{T}$ are bijective bounded linear operators and $\mathcal{T} = \mathcal{D}^{-1}$

Proof. That $\mathcal{D}$ is a bounded linear operator follows from Corollary 1.6 and that $\mathcal{T}$ is a bounded linear operator follows from Theorems 1.1 and 1.2. Moreover, it follows from the basic properties of the heat kernel K_t that $\mathcal{D}\mathcal{T}(f, g) = (f, g)$ for every pair of smooth functions $f : \mathbb{R}^n \to \mathbb{C}$ and $g : I \times \mathbb{R}^n \to \mathbb{C}$ with compact support. Hence $\mathcal{D} \circ \mathcal{T} = \mathrm{id}$ and so $\mathcal{D}$ is surjective. That $\mathcal{D}$ is injective follows from a standard uniqueness result for solutions of the heat equation. Thus $\mathcal{D}$ is a bijective bounded linear operator and $\mathcal{D}^{-1} = \mathcal{T}$. This proves Corollary 1.9. □

2 Proof of Theorem 1.3

For $p = q = 2$ the estimates of Theorem 1.1 and Theorem 1.3 follow from a straight forward integration by parts argument. An extension of this argument leads to a proof of Theorem 1.3. For a compactly supported smooth function $f : \mathbb{R}^n \to \mathbb{C}$ with mean value zero and a real number $p > 1$ define

$$\|f\|_{W^{-1,p}(\mathbb{R}^n)} := \sup_{\phi\not\equiv 0} \frac{|\int_{\mathbb{R}^n} \mathrm{Re}(\overline{\phi} f)|}{\|\nabla\phi\|_{L^q(\mathbb{R}^n)}}, \qquad q := \frac{p}{p-1}.$$

Here the supremum is understood over all nonvanishing smooth functions $\phi : \mathbb{R}^n \to \mathbb{C}$ with compact support.

Theorem 2.1. *Let $T > 0$ and let $u : [0,T] \times \mathbb{R}^n \to \mathbb{C}$ be a compactly supported smooth function. Write*

$$u_t(x) := u(t,x)$$

and suppose that

$$\int_{\mathbb{R}^n} u_t(x)\,dx = 0$$

for all $t \in [0,T]$. Then, for every $p \geq 2$,

$$\begin{aligned} &\|u_T\|^2_{L^p(\mathbb{R}^n)} + \int_0^T \frac{1}{\|u_t\|^{p-2}_{L^p(\mathbb{R}^n)}} \left(\int_{\mathbb{R}^n} |u_t|^{p-2}|\nabla u_t|^2\right) dt \\ &\leq \|u_0\|^2_{L^p(\mathbb{R}^n)} + (p-1)^2 \int_0^T \|\partial_t u_t - \Delta u_t\|^2_{W^{-1,p}(\mathbb{R}^n)}\,dt. \end{aligned} \tag{2.1}$$

The integrand on the left is taken to be zero for each t with $u_t \equiv 0$.

Proof. It suffices to prove the assertion under the assumption that $u_t \not\equiv 0$ for all $t \in [0,T]$. Define

$$f_t(x) := f(t,x) := \partial_t u(t,x) - (\Delta u)(t,x)$$

for $t \in [0,T]$ and $x \in \mathbb{R}^n$. Then $f_t : \mathbb{R}^n \to \mathbb{C}$ is a smooth function with compact support and mean value zero for all $t \in [0,T]$. Moreover,

$$\partial_t|u|^p = p|u|^{p-2}\mathrm{Re}(\overline{u}\partial_t u), \qquad \nabla(|u|^{p-2}\overline{u}) = \frac{p}{2}|u|^{p-2}\nabla\overline{u} + \frac{p-2}{2}|u|^{p-4}\overline{u}^2\nabla u.$$

Hence, by Hölder's inequality,

$$\begin{aligned}
\frac{1}{p-1}\left\|\nabla(|u|^{p-2}u)\right\|_{L^q(\mathbb{R}^n)} &= \left(\int_{\mathbb{R}^n}\left(|u|^{p-2}|\nabla u|\right)^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \\
&= \left(\int_{\mathbb{R}^n}\left(|u|^{p-2}|\nabla u|^2\right)^{\frac{p}{2p-2}}|u|^{p\frac{p-2}{2p-2}}\right)^{\frac{p-1}{p}} \\
&\leq \left(\int_{\mathbb{R}^n}|u|^{p-2}|\nabla u|^2\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^n}|u|^p\right)^{\frac{p-2}{2p}} \\
&= \left(\|u\|_{L^p(\mathbb{R}^n)}^{p-2}\int_{\mathbb{R}^n}|u|^{p-2}|\nabla u|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

This implies

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{p}\int_{\mathbb{R}^n}|u|^p = \int_{\mathbb{R}^n}|u|^{p-2}\mathrm{Re}(\overline{u}\partial_t u) = \int_{\mathbb{R}^n}|u|^{p-2}\mathrm{Re}(\overline{u}(f+\Delta u)) \\
&= \int_{\mathbb{R}^n}|u|^{p-2}\overline{u}f - \frac{p}{2}\int_{\mathbb{R}^n}|u|^{p-2}|\nabla u|^2 - \frac{p-2}{2}\int_{\mathbb{R}^n}|u|^{p-4}\sum_i \mathrm{Re}(\overline{u}^2(\partial_i u)^2) \\
&\leq \left\|\nabla(|u|^{p-2}u)\right\|_{L^q(\mathbb{R}^n)}\|f\|_{W^{-1,p}(\mathbb{R}^n)} - \int_{\mathbb{R}^n}|u|^{p-2}|\nabla u|^2 \\
&\leq (p-1)\left(\|u\|_{L^p(\mathbb{R}^n)}^{p-2}\int_{\mathbb{R}^n}|u|^{p-2}|\nabla u|^2\right)^{\frac{1}{2}}\|f\|_{W^{-1,p}(\mathbb{R}^n)} - \int_{\mathbb{R}^n}|u|^{p-2}|\nabla u|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{d}{dt}\frac{1}{2}\|u\|_{L^p(\mathbb{R}^n)}^2 &= \frac{1}{\|u\|_{L^p(\mathbb{R}^n)}^{p-2}}\frac{d}{dt}\frac{1}{p}\int_{\mathbb{R}^n}|u|^p \\
&\leq (p-1)\left(\frac{1}{\|u\|_{L^p(\mathbb{R}^n)}^{p-2}}\int_{\mathbb{R}^n}|u|^{p-2}|\nabla u|^2\right)^{\frac{1}{2}}\|f\|_{W^{-1,p}(\mathbb{R}^n)} \\
&\quad - \frac{1}{\|u\|_{L^p(\mathbb{R}^n)}^{p-2}}\int_{\mathbb{R}^n}|u|^{p-2}|\nabla u|^2 \\
&\leq \frac{(p-1)^2}{2}\|f\|_{W^{-1,p}(\mathbb{R}^n)}^2 - \frac{1}{2}\frac{1}{\|u\|_{L^p(\mathbb{R}^n)}^{p-2}}\int_{\mathbb{R}^n}|u|^{p-2}|\nabla u|^2.
\end{aligned}$$

The assertion of Theorem 2.1 follows by integrating this inequality over the interval $0 \leq t \leq T$. □

The next corollary shows that Theorem 1.3 holds (for $p \geq 2$) with the constant $c = \sqrt{n(p-1)}$.

Corollary 2.2. *Let $p \geq 2$ and $T \geq 0$. Then every compactly supported smooth function $u : \mathbb{R}^{n+1} \to \mathbb{C}$ satisfies the estimates*

$$\|\nabla u_T\|^2_{L^p(\mathbb{R}^n)} \leq n \|\nabla u_0\|^2_{L^p(\mathbb{R}^n)} + n(p-1)^2 \int_0^T \|\partial_t u - \Delta u\|^2_{L^p(\mathbb{R}^n)} \, dt$$

and

$$\|u_T\|_{L^p(\mathbb{R}^n)} \leq \|u_0\|_{L^p(\mathbb{R}^n)} + \int_0^T \|\partial_t u - \Delta u\|_{L^p(\mathbb{R}^n)} \, dt.$$

Proof. Let $u : \mathbb{R}^{n+1} \to \mathbb{C}$ be a compactly supported smooth function. Then the function $\partial_i u_t : \mathbb{R}^n \to \mathbb{C}$ has mean value zero for every t. Moreover, it follows directly from the definition and the Hölder inequality that

$$\|\partial_i f\|_{W^{-1,p}(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)}$$

for every compactly supported smooth function $f : \mathbb{R}^n \to \mathbb{C}$ and every index $i = 1, \ldots, n$. Hence it follows from Theorem 2.1 that

$$\begin{aligned}
\|\partial_i u_T\|^2_{L^p(\mathbb{R}^n)} &\leq \|\partial_i u_0\|^2_{L^p(\mathbb{R}^n)} + (p-1)^2 \int_0^T \|\partial_i(\partial_t u - \Delta u)\|^2_{W^{-1,p}(\mathbb{R}^n)} \, dt \\
&\leq \|\nabla u_0\|^2_{L^p(\mathbb{R}^n)} + (p-1)^2 \int_0^T \|\partial_t u - \Delta u\|^2_{L^p(\mathbb{R}^n)} \, dt
\end{aligned}$$

for all i. Since $\|\nabla f\|^2_{L^p} = \|\sum_i |\partial_i f|^2\|_{L^{p/2}} \leq \sum_i \||\partial_i f|^2\|_{L^{p/2}} = \sum_i \|\partial_i f\|^2_{L^p}$ for all $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$, the first inequality follows by taking the sum over all i. The second inequality follows from (1.8) with $f_t(x) := \partial_t u(t,x) - (\Delta u)(t,x)$. Namely,

$$\begin{aligned}
\|u_T\|_{L^p(\mathbb{R}^n)} &= \left\| K_T * u_0 + \int_0^T K_{T-t} * f_t \, dt \right\|_{L^p(\mathbb{R}^n)} \\
&\leq \|K_T * u_0\|_{L^p(\mathbb{R}^n)} + \int_0^T \|K_{T-t} * f_t\|_{L^p(\mathbb{R}^n)} \, dt \\
&\leq \|K_T\|_{L^1(\mathbb{R}^n)} \|u_0\|_{L^p(\mathbb{R}^n)} + \int_0^T \|K_{T-t}\|_{L^1(\mathbb{R}^n)} \|f_t\|_{L^p(\mathbb{R}^n)} \, dt \\
&\leq \|u_0\|_{L^p(\mathbb{R}^n)} + \int_0^T \|f_t\|_{L^p(\mathbb{R}^n)} \, dt.
\end{aligned}$$

Here the third step follows from Young's inequality and the last step follows from part (i) of Remark 1.4. This proves Corollary 2.2. □

Corollary 2.3. *For every $p \geq 2$ and every $T > 0$ the identity on the space of compactly supported smooth functions on $[0,T] \times \mathbb{R}^n$ extends to a continuous inclusion operator*

$$W^{1,2}([0,T], W^{2,p}(\mathbb{R}^n, \mathbb{C})) \cap L^2([0,T], L^p(\mathbb{R}^n, \mathbb{C})) \to C([0,T], W^{1,p}(\mathbb{R}^n, \mathbb{C})).$$

Proof. For $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$ define

$$\|f\|_{W^{1,p}(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |f|^p + |\nabla f|^p \right)^{1/p}$$

and

$$\|f\|_{W^{2,p}(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |f|^p + |\nabla f|^p + \sum_{i,j=1}^n |\partial_i \partial_j f|^p \right)^{1/p}.$$

Then, in particular,

$$\|\Delta f\|_{L^p(\mathbb{R}^n)} \leq n^{1/p} \|f\|_{W^{2,p}(\mathbb{R}^n)}.$$

By Corollary 2.2, every smooth function function $u : [0,T] \times \mathbb{R}^n \to \mathbb{C}$ with compact support satisfies the inequality

$$\begin{aligned}
&\|u_t\|^2_{W^{1,p}(\mathbb{R}^n)} \leq \|u_t\|^2_{L^p(\mathbb{R}^n)} + \|\nabla u_t\|^2_{L^p(\mathbb{R}^n)} \\
&\leq \|u_s\|^2_{L^p(\mathbb{R}^n)} + \left(\int_s^t \|\partial_t u - \Delta u\|_{L^p(\mathbb{R}^n)} \right)^2 \\
&\quad + n \|\nabla u_s\|^2_{L^p(\mathbb{R}^n)} + n(p-1)^2 \int_s^t \|\partial_t u - \Delta u\|^2_{L^p(\mathbb{R}^n)} \\
&\leq (n+1) \|u_s\|^2_{W^{1,p}(\mathbb{R}^n)} + (n(p-1)^2 + T) \int_0^T \|\partial_t u - \Delta u\|^2_{L^p(\mathbb{R}^n)} \\
&\leq (n+1) \|u_s\|^2_{W^{2,p}(\mathbb{R}^n)} \\
&\quad + 2n^{2/p}(n(p-1)^2 + T) \int_0^T \left(\|\partial_t u\|^2_{L^p(\mathbb{R}^n)} + \|u\|^2_{W^{2,p}(\mathbb{R}^n)} \right)
\end{aligned}$$

for $0 \leq s \leq t \leq T$. Replacing $u(t,x)$ with $u(T-t,x)$, we obtain the same inequality for $t \leq s \leq T$. Integrate the resulting inequality over the interval $0 \leq s \leq T$ to obtain

$$\sup_{0 \leq t \leq T} \|u_t\|^2_{W^{1,p}(\mathbb{R}^n)} \leq c \int_0^T \left(\|\partial_t u\|^2_{L^p(\mathbb{R}^n)} + \|u\|^2_{W^{2,p}(\mathbb{R}^n)} \right),$$

where $c := \frac{n+1}{T} + 2n^{2/p}(n(p-1)^2 + T)$. This proves Corollary 2.3. □

Corollary 2.4. *Theorems 1.1 and 1.3 hold for $p = q = 2$.*

Proof. For $p = 2$ Theorem 2.1 asserts that the inequality

$$\|u_T\|^2_{L^2(\mathbb{R}^n)} + \int_{-\infty}^T \|\nabla u_t\|^2_{L^2(\mathbb{R}^n)}\, dt \le \int_{-\infty}^T \|\partial_t u_t - \Delta u_t\|^2_{W^{-1,2}(\mathbb{R}^n)}\, dt$$

holds for every $T \in \mathbb{R}$ and every compactly supported smooth function $u : \mathbb{R}^{n+1} \to \mathbb{C}$ such that $\int_{\mathbb{R}^n} u_t = 0$ for all t. For T sufficiently large it follows under the same assumption on u that

$$\int_{-\infty}^\infty \|\nabla u_t\|^2_{L^2(\mathbb{R}^n)}\, dt \le \int_{-\infty}^\infty \|\partial_t u_t - \Delta u_t\|^2_{W^{-1,2}(\mathbb{R}^n)}\, dt.$$

Replace u by $\partial_i u$, use the inequality $\|\partial_i f\|_{W^{-1,2}(\mathbb{R}^n)} \le \|f\|_{L^2(\mathbb{R}^n)}$ for every smooth function $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$, and take the sum over all i to obtain

$$\begin{aligned}
&\int_{-\infty}^\infty \|\partial_t u_t\|^2_{L^2(\mathbb{R}^n)}\, dt \\
&\le (n+1)\int_{-\infty}^\infty \|\partial_t u - \Delta u\|^2_{L^2(\mathbb{R}^n)}\, dt + (n+1)\int_{-\infty}^\infty \|\partial_i\partial_i u\|^2_{L^2(\mathbb{R}^n)}\, dt \\
&\le (n+1)\int_{-\infty}^\infty \|\partial_t u - \Delta u\|^2_{L^2(\mathbb{R}^n)}\, dt + (n+1)\sum_{i=1}^n\int_{-\infty}^\infty \|\nabla\partial_i u_t\|^2_{L^2(\mathbb{R}^n)}\, dt \\
&\le (n+1)^2\int_{-\infty}^\infty \|\partial_t u_t - \Delta u_t\|^2_{L^2(\mathbb{R}^n)}\, dt.
\end{aligned}$$

This proves the assertion of Theorem 1.1 for $p = q = 2$ with $c = n+1$. Moreover, it follows from Corollary 2.2 that the assertion of Theorem 1.3 holds for $p = 2$ with $c = \sqrt{n}$. This proves Corollary 2.4. □

Corollary 2.5. *Assume $1 < p \le 2$, let $u_0 : \mathbb{R}^n \to \mathbb{C}$ be a compactly supported smooth function, and define*

$$u_t := K_t * u_0 : \mathbb{R}^n \to \mathbb{C}$$

for $t \ge 0$. Thus $u(t,x) := u_t(x)$ is the unique solution of (1.7) with $f = 0$ such that u_t is square integrable for all t. Then

$$\left(\int_0^\infty \|\nabla u_t\|^2_{L^p(\mathbb{R}^n)}\, dt\right)^{1/2} \le \frac{1}{p-1}\|u_0\|_{L^p(\mathbb{R}^n)}. \tag{2.2}$$

Proof. Define $q := p/(p-1) \geq 2$ and let $g : [0,\infty) \times \mathbb{R}^n \to \mathbb{C}$ be any smooth function with compact support such that $g_t := g(t,\cdot)$ has mean value zero for all t. Define $v_t(x) = v(t,x)$ by

$$v_t(x) := \int_t^\infty (K_{s-t} * g_s)(x)\,ds = \int_t^\infty \left(\int_{\mathbb{R}^n} K_{s-t}(x-y) * g_s(y)\,dy \right) ds$$

for $t \geq 0$ and $x \in \mathbb{R}^n$. Then $\partial_t v + \Delta v = g$, v_t is square integrable for all t, and Theorem 2.1 implies that

$$\|v_0\|^2_{L^q(\mathbb{R}^n)} \leq (q-1)^2 \int_0^\infty \|g_t\|^2_{W^{-1,q}(\mathbb{R}^n)}\,dt.$$

Moreover,

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^n} \mathrm{Re}(\overline{v}_t u_t) &= \int_{\mathbb{R}^n} \mathrm{Re}((\partial_t \overline{v}_t) u_t) + \int_{\mathbb{R}^n} \mathrm{Re}(\overline{v}_t(\partial_t u_t)) \\ &= \int_{\mathbb{R}^n} \mathrm{Re}((\partial_t \overline{v}_t) u_t) + \int_{\mathbb{R}^n} \mathrm{Re}(\overline{v}_t(\Delta u_t)) \\ &= \int_{\mathbb{R}^n} \mathrm{Re}((\partial_t \overline{v}_t + \Delta \overline{v}_t) u_t) \\ &= \int_{\mathbb{R}^n} \mathrm{Re}(\overline{g}_t u_t). \end{aligned}$$

Integrate this equation over the interval $0 \leq t < \infty$ to obtain

$$\begin{aligned} \int_0^\infty \left(\int_{\mathbb{R}^n} \mathrm{Re}(\overline{g}_t u_t) \right) dt = -\int_{\mathbb{R}^n} \mathrm{Re}(\overline{v}_0 u_0) &\leq \|u_0\|_{L^p(\mathbb{R}^n)} \|v_0\|_{L^q(\mathbb{R}^n)} \\ &\leq (q-1)\|u_0\|_{L^p(\mathbb{R}^n)} \left(\int_0^\infty \|g_t\|^2_{W^{-1,q}(\mathbb{R}^n)}\,dt \right)^{1/2}. \end{aligned}$$

Since

$$\left(\int_0^\infty \|\nabla u_t\|^2_{W^{1,p}(\mathbb{R}^n)}\,dt \right)^{1/2} = \sup_g \frac{\int_0^\infty \left(\int_{\mathbb{R}^n} \mathrm{Re}(\overline{g}_t u_t) \right) dt}{\left(\int_0^\infty \|g_t\|^2_{W^{-1,q}(\mathbb{R}^n)}\,dt \right)^{1/2}},$$

where the supremum on the right is over all nonvanishing compactly supported smooth functions $g : [0,\infty) \times \mathbb{R}^n \to \mathbb{C}$ such that $g_t := g(t,\cdot)$ has mean value zero for all t, it follows that

$$\left(\int_0^\infty \|\nabla u_t\|^2_{W^{1,p}(\mathbb{R}^n)}\,dt \right)^{1/2} \leq (q-1)\|u_0\|_{L^p(\mathbb{R}^n)}.$$

Since $(q-1) = (p-1)^{-1}$, this proves Corollary 2.5. □

Corollary 2.6. *Assume $1 < p \le 2$, let $u_0 : \mathbb{R}^n \to \mathbb{C}$ be a compactly supported smooth function, and define $u_t := K_t * u_0 : \mathbb{R}^n \to \mathbb{C}$ for $t \ge 0$ as in Corollary 2.5. Then*

$$\begin{aligned}\left(\int_0^\infty \|\partial_t u_t\|_{L^p(\mathbb{R}^n)}^2 \, dt\right)^{1/2} + \sum_{i=1}^n \left(\int_0^\infty \|\nabla \partial_i u_t\|_{L^p(\mathbb{R}^n)}^2 \, dt\right)^{1/2} \\ \le \frac{2n}{p-1} \|\nabla u_0\|_{L^p(\mathbb{R}^n)} .\end{aligned} \tag{2.3}$$

Proof. Apply the estimate (2.2) in Corollary 2.5 to the function $\partial_i u$ and take the sum over all i to obtain

$$\begin{aligned}\sum_{i=1}^n \left(\int_0^\infty \|\nabla \partial_i u_t\|_{L^p(\mathbb{R}^n)}^2 \, dt\right)^{1/2} &\le \sum_{i=1}^n \frac{1}{p-1} \|\partial_i u_0\|_{L^p(\mathbb{R}^n)} \\ &\le \frac{n}{p-1} \|\nabla u_0\|_{L^p(\mathbb{R}^n)} .\end{aligned}$$

Since

$$\partial_t u = \Delta u = \sum_{i=1}^n \partial_i \partial_i u$$

it follows that

$$\begin{aligned}\left(\int_0^\infty \|\partial_t u_t\|_{L^p(\mathbb{R}^n)}^2 \, dt\right)^{1/2} &\le \sum_{i=1}^n \left(\int_0^\infty \|\nabla \partial_i u_t\|_{L^p(\mathbb{R}^n)}^2 \, dt\right)^{1/2} \\ &\le \frac{n}{p-1} \|\nabla u_0\|_{L^p(\mathbb{R}^n)} .\end{aligned}$$

This proves Corollary 2.6. □

Corollary 2.6 is a kind of converse of Corollary 2.3. While Corollary 2.3 asserts (for $p \ge 2$) that every function in the space

$$\mathcal{W}^p := W^{1,2}([0,T], L^p(\mathbb{R}^n, \mathbb{C})) \cap L^2([0,T], W^{2,p}(\mathbb{R}^n, \mathbb{C}))$$

is a continuous function on the intervall $[0,T]$ with values in $W^{1,p}(\mathbb{R}^n, \mathbb{C})$, Corollary 2.6 asserts (for $p \le 2$) that every element $u_0 \in W^{1,p}(\mathbb{R}^n, \mathbb{C})$ extends to a function in $\mathcal{W}^p$ that agrees with u_0 at $t = 0$.

3 Riesz–Thorin and Stein interpolation

Assume throughout that $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are measure spaces. For $1 \le p \le \infty$ denote by $L^p(X,\mu)$ and $L^p(Y,\nu)$ the complex L^p-spaces. Also denote by $\mathscr{X}$ the set of all equivalence classes of $\mathcal{A}$-measurable step functions $f : X \to \mathbb{C}$ with support of finite measure and by $\mathscr{Y}$ the set of all equivalence classes of $\mathcal{B}$-measurable step functions $g : Y \to \mathbb{C}$ with support of finite measure. The equivalence relation in both cases is equality almost everywhere. Whenever convenient we abuse notation and denote by f either an equivalence class of measurable functions on X (respectively Y) or a representative of the corresponding equivalence class. We begin our exposition with the **Riesz–Thorin Interpolation Theorem** [32, 37].

Theorem 3.1 (**Riesz–Thorin**). *Let* $1 \le p_0, p_1, q_0, q_1 \le \infty$*, let*

$$T : L^{p_0}(X,\mu) \cap L^{p_1}(X,\mu) \to L^{q_0}(Y,\nu) \cap L^{q_1}(Y,\nu)$$

be a linear operator, and suppose that there exist positive real numbers c_0, c_1 *such that, for all* $f \in L^{p_0}(X,\mu) \cap L^{p_1}(X,\mu)$*,*

$$\|Tf\|_{L^{q_0}} \le c_0 \|f\|_{L^{p_0}}, \qquad \|Tf\|_{L^{q_1}} \le c_1 \|f\|_{L^{p_1}} \tag{3.1}$$

Fix a real number $0 < \lambda < 1$ *and define the numbers* $p_\lambda, q_\lambda, c_\lambda$ *by*

$$\frac{1}{p_\lambda} := \frac{1-\lambda}{p_0} + \frac{\lambda}{p_1}, \qquad \frac{1}{q_\lambda} := \frac{1-\lambda}{q_0} + \frac{\lambda}{q_1}, \qquad c_\lambda := c_0^{1-\lambda} c_1^{\lambda}. \tag{3.2}$$

If $q_\lambda = \infty$ *assume that* $(Y, \mathcal{B}, \nu)$ *is semi-finite. Then*

$$\|Tf\|_{L^{q_\lambda}} \le c_\lambda \|f\|_{L^{p_\lambda}} \tag{3.3}$$

for all $f \in L^{p_0}(X,\mu) \cap L^{p_1}(X,\mu) \subset L^{p_\lambda}(X,\mu)$*.*

Proof. See page 17. □

The proof requires Hadamard's Three Lines Theorem. Define

$$\mathbb{S} := \left\{ z \in \mathbb{C} \,\middle|\, 0 \le \mathrm{Re}(z) \le 1 \right\}. \tag{3.4}$$

Theorem 3.2 (**Hadamard Three Lines Theorem**). *Let* $\Phi : \mathbb{S} \to \mathbb{C}$ *be a bounded continuous function that is holomorphic in* $\mathrm{int}(\mathbb{S})$*. Then*

$$\sup_{\mathrm{Re}(z)=\lambda} |\Phi(z)| \le \left(\sup_{\mathrm{Re}(z)=0} |\Phi(z)| \right)^{1-\lambda} \left(\sup_{\mathrm{Re}(z)=1} |\Phi(z)| \right)^{\lambda}$$

for all $\lambda \in [0,1]$*.*

Proof. Define $c_0 := \sup_{\mathrm{Re}(z)=0}|\Phi(z)|$ and $c_1 := \sup_{\mathrm{Re}(z)=1}|\Phi(z)|$. Then the function $\Psi_n : \mathbb{S} \to \mathbb{C}$ defined by

$$\Psi_n(z) := \frac{\Phi(z)}{c_0^{1-z} c_1^z} e^{\frac{z^2-1}{n}}$$

is continuous, is holomorphic in $\mathrm{int}(\mathbb{S})$, and converges to zero as $|z|$ tends to infinity. Hence it attains its maximum on the boundary of $\mathbb{S}$. Since

$$|\Psi_n(z)| \le e^{\frac{\mathrm{Re}(z^2)-1}{n}} = e^{\frac{\mathrm{Re}(z)^2-\mathrm{Im}(z)^2-1}{n}} \le 1 \qquad \text{for } \mathrm{Re}(z) = 0, 1,$$

it follows that $|\Psi_n(z)| \le 1$ for all $z \in \mathbb{C}$ with $0 \le \mathbb{R}(z) \le 1$. Take the limit $n \to \infty$ to obtain the inequality

$$|\Phi(z)| \le |c_0^{1-z} c_1^z| = c_0^{1-\mathrm{Re}(z)} c_1^{\mathrm{Re}(z)}$$

for all $z \in \mathbb{C}$ with $0 \le \mathbb{R}(z) \le 1$. This proves Theorem 3.2. □

Proof of Theorem 3.1. The proof follows the exposition in [18].

Step 1. *The assertion holds when $p_0 = p_1 =: p$.*

In this case $p_\lambda = p$ and it follows from Hölder's inequality and equations (3.1) and (3.2) that

$$\|Tf\|_{L^{q_\lambda}} \le \|Tf\|_{L^{q_0}}^{1-\lambda} \|Tf\|_{L^{q_1}}^{\lambda} \le c_0^{1-\lambda} c_1^{\lambda} \|f\|_{L^p}$$

for all $f \in L^p(X, \mu)$. This proves Step 1.

Step 2. *Let $h \in L^{q_0}(Y, \nu) \cap L^{q_1}(Y, \nu)$. Thus $h \in L^{q_\lambda}(Y, \nu)$ and so gh is integrable for all $g \in \mathscr{Y}$. Define $r_\lambda \in [0, \infty]$ by*

$$\frac{1}{q_\lambda} + \frac{1}{r_\lambda} = 1, \tag{3.5}$$

let $c > 0$, and assume that

$$\left| \int_Y gh \, d\nu \right| \le c \|g\|_{L^{r_\lambda}} \tag{3.6}$$

for all $g \in \mathscr{Y}$. Then $\|h\|_{L^{q_\lambda}} \le c$.

Assume first that $1 < q_\lambda < \infty$ and so $1 < r_\lambda < \infty$. Hence $\mathscr{Y}$ is dense in $L^{r_\lambda}(Y, \nu)$ by [33, Lemma 4.12]. Thus the inequality (3.6) continues to hold for all $g \in L^{r_\lambda}(Y, \nu)$. Define $g : Y \to \mathbb{C}$ by $g(y) := |h(y)|^{q_\lambda - 2}\overline{h(y)}$ whenever $h(y) \ne 0$ and $g(y) := 0$ otherwise. Then $g \in L^{r_\lambda}(Y, \nu)$ and $\|g\|_{L^{r_\lambda}} = \|h\|_{L^{q_\lambda}}^{q_\lambda - 1}$. Hence $\|h\|_{L^{q_\lambda}}^{q_\lambda} = \left|\int_Y gh \, d\nu\right| \le c \|g\|_{L^{r_\lambda}} = c \|h\|_{L^{q_\lambda}}^{q_\lambda - 1}$ and so $\|h\|_{L^{q_\lambda}} \le c$.

Next assume $q_\lambda = \infty$ and so $r_\lambda = 1$. Then $\mathscr{Y}$ is dense in $L^1(Y, \nu)$ by [33, Lemma 4.12] and so (3.6) continues to hold for all $g \in L^1(Y, \nu)$. Assume, by contradiction, that $\|h\|_{L^\infty} > c$. Then there is a $\delta > 0$ such that the set $B := \{y \in Y \,|\, |h(y)| > c + \delta\}$ has positive measure. Since $(Y, \mathcal{B}, \nu)$ is semi-finite, there exists a measurable set $E \subset B$ such that $0 < \nu(E) < \infty$. Define g by $g(y) := |h(y)|^{-1}\overline{h(y)}$ for $y \in E$ and by $g(y) := 0$ for $y \in Y \setminus E$. Then $g \in L^1(Y, \nu)$ and

$$\left|\int_Y gh\, d\nu\right| = \int_E |h|\, d\nu \geq (c + \delta)\nu(E) > c\nu(E) = c\,\|g\|_{L^1}$$

in contradiction to (3.6).

Next assume $q_\lambda = 1$ and so $r_\lambda = \infty$. Suppose, by contradiction, that $\|h\|_{L^1} > c$. Since $\mathscr{Y}$ is dense in $L^1(Y, \nu)$ there is a $k \in \mathscr{Y}$ such that

$$\|h - k\|_{L^1} < \frac{\|h\|_{L^1} - c}{2}.$$

Define $g : Y \to \mathbb{C}$ by $g(y) := |k(y)|^{-1}\overline{k(y)}$ whenever $k(y) \neq 0$ and $g(y) := 0$ otherwise. Then $g \in \mathscr{Y}$, $\|g\|_{L^\infty} = 1$, and $\int_Y gk\, d\nu = \|k\|_{L^1}$. Hence

$$\left|\int_Y gh\, d\nu\right| \geq \|k\|_{L^1} - \|h - k\|_{L^1} \geq \|h\|_{L^1} - 2\,\|h - k\|_{L^1} > c = c\,\|g\|_{L^\infty}$$

in contradiction to (3.6). This proves Step 2.

Step 3. *Let r_λ be as in Step 2. Then the inequality*

$$\int_Y (Tf)g\, d\nu \leq c_\lambda\,\|f\|_{L^{p_\lambda}}\,\|g\|_{L^{r_\lambda}} \tag{3.7}$$

holds for all $f \in \mathscr{X}$ and all $g \in \mathscr{Y}$.

This is the heart of the proof of Theorem 3.1. Write

$$f = \sum_{i=1}^{k} a_i \chi_{A_i}, \qquad g = \sum_{j=1}^{\ell} b_j \chi_{B_j}, \tag{3.8}$$

where $a_1, \dots, a_k$ and $b_1, \dots, b_\ell$ are nonzero complex numbers, the A_i are pairwise disjoint measurable subsets of X with finite measure, and the B_j are pairwise disjoint measurable subsets of Y with finite measure. Here χ_A denotes the characteristic function of a set $A \subset X$ and χ_B denotes the characteristic function of a set $B \subset Y$. Choose $\phi_i, \psi_j \in \mathbb{R}$ such that

$$a_i = |a_i|e^{\mathbf{i}\phi_i}, \qquad b_j = |b_j|e^{\mathbf{i}\psi_j}$$

for $i = 1, \dots, k$ and $j = 1, \dots, \ell$.

For $z \in \mathbb{S}$ define $f_z : X \to \mathbb{C}$ and $g_z : Y \to \mathbb{C}$ by

$$\begin{aligned} f_z(x) &:= \sum_{i=1}^{k} |a_i|^{\frac{(1-z)p_1 + zp_0}{(1-\lambda)p_1+\lambda p_0}} e^{\mathbf{i}\phi_i} \chi_{A_i}, \\ g_z(y) &:= \sum_{j=1}^{\ell} |b_j|^{\frac{q_0q_1-(1-z)q_1-zq_0}{q_0q_1-(1-\lambda)q_1-\lambda q_0}} e^{\mathbf{i}\psi_j} \chi_{B_i} \end{aligned} \tag{3.9}$$

for $x \in X$ and $y \in Y$. Then $f_\lambda = f$ and $g_\lambda = g$. Moreover, the function $\Phi : \mathbb{S} \to \mathbb{C}$ defined by

$$\begin{aligned} \Phi(z) &:= \int_Y (Tf_z) g_z \, d\nu \\ &= \sum_{i,j} |a_i|^{\frac{(1-z)p_1+zp_0}{(1-\lambda)p_1+\lambda p_0}} |b_j|^{\frac{q_0q_1-(1-z)q_1-zq_0}{q_0q_1-(1-\lambda)q_1-\lambda q_0}} e^{\mathbf{i}(\phi_i+\psi_j)} \int_{B_j} (T\chi_{A_i}) \, d\nu \end{aligned} \tag{3.10}$$

for $z \in \mathbb{S}$ is holomorphic. Let $z \in \mathbb{C}$ with $\mathrm{Re}(z) = 0$. Then

$$|f_z(x)|^{p_0} = |a_i|^{\frac{p_0p_1}{(1-\lambda)p_1+\lambda p_0}} = |a_i|^{p_\lambda} = |f(x)|^{p_\lambda}$$

for $x \in A_i$ and

$$|g_z(y)|^{r_0} = |b_i|^{\frac{r_0(q_0-1)q_1}{q_0q_1-(1-\lambda)q_1-\lambda q_0}} = |b_i|^{\frac{q_0q_1}{q_0q_1-(1-\lambda)q_1-\lambda q_0}} = |b_j|^{\frac{q_\lambda}{q_\lambda-1}} = |g(x)|^{r_\lambda}$$

for $y \in B_j$. Hence

$$|\Phi(z)| \le \|Tf_z\|_{L^{q_0}} \|g_z\|_{L^{r_0}} \le c_0 \|f_z\|_{L^{p_0}} \|g_z\|_{L^{r_0}} = c_0 \|f\|_{L^{p_\lambda}}^{p_\lambda/p_0} \|g\|_{L^{r_\lambda}}^{r_\lambda/r_0}$$

for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) = 0$. A similar argument shows that

$$|\Phi(z)| \le \|Tf_z\|_{L^{q_0}} \|g_z\|_{L^{r_0}} \le c_1 \|f_z\|_{L^{p_1}} \|g_z\|_{L^{r_1}} = c_1 \|f\|_{L^{p_\lambda}}^{p_\lambda/p_1} \|g\|_{L^{r_\lambda}}^{r_\lambda/r_1}$$

for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) = 1$. Hence it follows from Hadamard's Three Lines Theorem 3.2 that

$$\begin{aligned} |\Phi(\lambda)| &\le \left(\sup_{\mathrm{Re}(z)=0} |\Phi(z)| \right)^{1-\lambda} \left(\sup_{\mathrm{Re}(z)=1} |\Phi(z)| \right)^{\lambda} \\ &\le \left(c_0 \|f\|_{L^{p_\lambda}}^{p_\lambda/p_0} \|g\|_{L^{r_\lambda}}^{r_\lambda/r_0} \right)^{1-\lambda} \left(c_1 \|f\|_{L^{p_\lambda}}^{p_\lambda/p_1} \|g\|_{L^{r_\lambda}}^{r_\lambda/r_1} \right)^{\lambda} \\ &= c_\lambda \|f\|_{L^{p_\lambda}} \|g\|_{L^{r_\lambda}} . \end{aligned}$$

The last equation uses the identities $c_\lambda = c_0^{1-\lambda} c_1^{\lambda}$ and $\frac{1}{p_\lambda} = \frac{1-\lambda}{p_0} + \frac{\lambda}{p_1}$ as well as $\frac{1}{r_\lambda} = 1 - \frac{1}{q_\lambda} = 1 - \frac{1-\lambda}{q_0} - \frac{\lambda}{q_1} = \frac{1-\lambda}{r_0} + \frac{\lambda}{r_1}$. This proves Step 3.

Step 4. $\|Tf\|_{L^{q_\lambda}} \le c_\lambda \|f\|_{L^{p_\lambda}}$ *for all* $f \in \mathscr{X}$.

Let $f \in \mathscr{X}$. Then Step 3 shows that $h := Tf$ satisfies the hypotheses of Step 2 with $c := c_\lambda \|f\|_{L^{p_\lambda}}$. Hence the assertion follows from Step 2.

Step 5. *We prove the theorem.*

For $p_0 = p_1$ the assertion holds by Step 1. Hence assume $p_0 \ne p_1$ and, without loss of generality, that $p_0 < \infty$. Then $p_\lambda < \infty$. Fix a function

$$f \in L^{p_0}(X,\mu) \cap L^{p_1}(X,\mu).$$

We prove that there exists a sequence $f_n \in \mathscr{X}$ such that

$$\lim_{n\to\infty} \|f - f_n\|_{L^{p_\lambda}} = 0, \qquad \lim_{n\to\infty} (Tf_n)(y) = (Tf)(y) \tag{3.11}$$

for almost every $y \in Y$. To see this assume first that $f \ge 0$. Then there exists a monotone sequence of measurable step functions $f_n : X \to [0,\infty)$ such that $0 \le s_1(x) \le s_2(x) \le \cdots$ and $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in X$ (see [33, Theorem 1.26]). Since $p_0, p_\lambda < \infty$ and $f \in L^{p_\lambda}(X,\mu) \cap L^{p_0}(X,\mu)$, the functions f^{p_λ} and f^{p_0} are integrable. Since $|f_n(x) - f(x)|^{p_\lambda} \le f(x)^{p_\lambda}$ and $|f_n(x) - f(x)|^{p_0} \le f(x)^{p_0}$ for all $x \in X$, it follows from the Lebesgue Dominated Convergence Theorem (see [33, Theorem 1.45]) that

$$\lim_{n\to\infty} \|f_n - f\|_{L^{p_\lambda}} = 0, \qquad \lim_{n\to\infty} \|f_n - f\|_{L^{p_0}} = 0.$$

Since $\|Tf_n - Tf\|_{L^{q_0}} \le c_0 \|f_n - f\|_{L^{p_0}}$, we have $\lim_{n\to\infty} \|Tf_n - Tf\|_{L^{q_0}} = 0$. Hence there exists a subsequence, still denoted by f_n, such that Tf_n converges almost everywhere to Tf (see [33, Corollary 4.10]). This is the required sequence in the case $f \ge 0$. To obtain the result in general apply this argument to the positive and negative parts of the real and imaginary parts of an arbitrary function $f \in L^{p_0}(X,\mu) \cap L^{p_1}(X,\mu)$. This proves the existence of a sequence $f_n \in \mathscr{X}$ that satisfies (3.11).

Since $\|Tf_n - Tf_m\|_{L^{q_\lambda}} \le c_0 \|f_n - f_m\|_{L^{p_\lambda}}$ for all $n, m \in \mathbb{N}$, by Step 4, it follows from (3.11) that Tf_n is a Cauchy sequence in $L^{q_\lambda}(Y,\nu)$ and hence converges in $L^{q_\lambda}(Y,\nu)$. Since Tf_n converges to Tf almost everywhere, its limit in $L^{q_\lambda}(Y,\nu)$ agrees with Tf. Hence $Tf \in L^{q_\lambda}(Y,\nu)$ and

$$\lim_{n\to\infty} \|Tf_n\|_{L^{q_\lambda}} = \|Tf\|_{L^{q_\lambda}}. \tag{3.12}$$

By (3.11), (3.12), and Step 4 we have

$$\|Tf\|_{L^{q_\lambda}} = \lim_{n\to\infty} \|Tf_n\|_{L^{q_\lambda}} \le c_\lambda \lim_{n\to\infty} \|f_n\|_{L^{p_\lambda}} = c_\lambda \|f\|_{L^{p_\lambda}}.$$

This proves Step 5 and Theorem 3.1. □

The **Stein Interpolation Theorem** in [36] is an extension of the Riesz–Thorin Interpolation Theorem, where the operator T is replaced by a holomorphic operator family $\{T_z\}_{z\in\mathbb{S}}$, parametrized by the elements of the strip $\mathbb{S} = \{z \in \mathbb{C} \,|\, 0 \le \mathrm{Re}(z) \le 1\}$ in (3.4). Denote by $L^1_{\mathrm{loc}}(Y,\nu)$ the space of all equivalence classes of measurable functions $g : Y \to \mathbb{C}$ such that the restriction of g to every measurable subset of Y with finite measure is integrable. Recall that $\mathscr{X}$ denotes the set of all equivalence classes of $\mathcal{A}$-measurable step function $f : X \to \mathbb{C}$ with support of finite measure.

Theorem 3.3 (**Stein**). *Let* $1 \le p_0, p_1, q_0, q_1 \le \infty$ *and let*

$$T_z : \mathscr{X} \to L^1_{\mathrm{loc}}(Y,\nu), \qquad z \in \mathbb{S},$$

be a family of linear operators satisfying the following two conditions.
(a) *For all* $f \in \mathscr{X}$ *and all* $g \in \mathscr{Y}$ *the function*

$$\mathbb{S} \to \mathbb{C} : z \mapsto \int_Y g(T_z f)\, d\nu$$

is continuous and is holomorphic in $\mathrm{int}(\mathbb{S})$.
(b) *There exist positive real numbers* c_0, c_1 *such that*

$$\|T_{\mathbf{i}t} f\|_{L^{q_0}} \le c_0 \|f\|_{L^{p_0}}, \qquad \|T_{1+\mathbf{i}t} f\|_{L^{q_1}} \le c_1 \|f\|_{L^{p_1}} \tag{3.13}$$

for all $f \in \mathscr{X}$ *and all* $t \in \mathbb{R}$.
Let $0 < \lambda < 1$ *and define* $p_\lambda, q_\lambda, c_\lambda$ *by*

$$\frac{1}{p_\lambda} := \frac{1-\lambda}{p_0} + \frac{\lambda}{p_1}, \qquad \frac{1}{q_\lambda} := \frac{1-\lambda}{q_0} + \frac{\lambda}{q_1}, \qquad c_\lambda := c_0^{1-\lambda} c_1^{\lambda}. \tag{3.14}$$

If $q_\lambda = \infty$ *assume that* $(Y, \mathcal{B}, \nu)$ *is semi-finite. Then*

$$\|T_\lambda f\|_{L^{q_\lambda}} \le c_\lambda \|f\|_{L^{p_\lambda}} \tag{3.15}$$

for all $f \in \mathscr{X}$.

Proof. The proof is a straight forward extension of the proof of Theorem 3.1. Namely, let $f \in \mathscr{X}$ and $g \in \mathscr{Y}$, and define the function $\Phi : \mathbb{S} \to \mathbb{C}$ by

$$\Phi(z) := \int_Y g_z(T_z f_z)\, d\nu$$

for $z \in \mathbb{S}$, where $f_z : X \to \mathbb{C}$ and $g_z : Y \to \mathbb{C}$ are given by (3.9) (with f and g given by (3.8)). Then it follows as in Step 3 in the proof of Theorem 3.1 that $|\int_Y g(T_\lambda f)\, d\nu| = |\Phi(\lambda)| \le c_\lambda \|f\|_{L^{p_\lambda}} \|g\|_{L^{q_\lambda}}$. By Step 2 in the proof of Theorem 3.1 this implies the assertion of Theorem 3.3. □

4 Marcinkiewicz interpolation

The Marcinkiewicz interpolation theorem provides a criterion for a linear operator on $L^2(\mu)$ to induce a linear operator on L^p for $1 < p < 2$. Let $(X, \mathcal{A}, \mu)$ be a measure space. For a measurable function $f : X \to \mathbb{R}$ define the (Borel measurable) function $\kappa_f : [0, \infty) \to [0, \infty]$ by

$$\kappa_f(t) := \mu(A(t, f)), \quad A(t, f) := \left\{x \in X \,\middle|\, |f(x)| > t\right\}, \quad \text{for } t \geq 0. \tag{4.1}$$

Lemma 4.1. *Let $1 \leq p < \infty$ and let $f, g : X \to \mathbb{R}$ be measurable functions. Then, for all $t > 0$,*

$$\kappa_{f+g}(t) \leq \kappa_f(t/2) + \kappa_g(t/2), \tag{4.2}$$

$$t^p \kappa_f(t) \leq \int_X |f|^p \, d\mu = p \int_0^\infty s^{p-1} \kappa_f(s) \, ds. \tag{4.3}$$

Proof. The inequality (4.2) holds because $A(t, f + g) \subset A(t/2, f) \cup A(t/2, g)$. We prove (4.3) in four steps.

Step 1. *$t^p \kappa_f(t) \leq \int_X |f|^p \, d\mu$ for all $t \geq 0$.*

Since $t^p \chi_{A(t,f)} \leq |f|^p$ it follows that $t^p \kappa_f(t) = \int_X t^p \chi_{(A(t,f)} \, d\mu \leq \int_X |f|^p \, d\mu$ for all $t \geq 0$. This proves Step 1.

Step 2. *If $\kappa_f(t) = \infty$ for some $t > 0$ then $\int_X |f|^p \, d\mu = \infty = \int_0^\infty t^{p-1} \kappa_f(t) \, dt$.*

By Step 1, we have $\int_X |f|^p \, d\mu = \infty$. Moreover, $t^{p-1} \kappa_f(t) = \infty$ for $t > 0$ sufficiently small and hence $\int_0^\infty t^{p-1} \kappa_f(t) \, dt = \infty$. This proves Step 2.

Step 3. *Assume $(X, \mathcal{A}, \mu)$ is σ-finite and $\kappa_f(t) < \infty$ for all $t > 0$. Then equation (4.3) holds.*

Let $\mathcal{B} \subset 2^{[0,\infty)}$ be the Borel σ-algebra and denote by $m : \mathcal{B} \to [0, \infty]$ the restriction of the Lebesgue measure to $\mathcal{B}$. Let $(X \times [0, \infty), \mathcal{A} \otimes \mathcal{B}, \mu \otimes m)$ be the product measure space in [33, Def 7.10]. We prove that

$$Q(f) := \left\{(x, t) \in X \times [0, \infty) \,\middle|\, 0 \leq t < |f(x)|\right\} \in \mathcal{A} \otimes \mathcal{B}.$$

To see this, assume first that f is an $\mathcal{A}$-measurable step-function. Then there exist finitely many pairwise disjoint measurable sets $A_1, \ldots, A_\ell \in \mathcal{A}$ and positive real numbers $\alpha_1, \ldots, \alpha_\ell$ such that $|f| = \sum_{i=1}^{\ell} \alpha_i \chi_{A_i}$. In this case $Q(f) = \bigcup_{i=1}^{\ell} A_i \times [0, \alpha_i) \in \mathcal{A} \otimes \mathcal{B}$. Now consider the general case. Then there is a sequence of $\mathcal{A}$-measurable step-functions $f_i : X \to [0, \infty)$ such that $0 \leq f_1 \leq f_2 \leq \cdots$ and f_i converges pointwise to $|f|$ (see [33, Thm 1.26]). Since $Q(f_i) \in \mathcal{A} \otimes \mathcal{B}$ for all i, we have $Q(f) = \bigcup_{i=1}^{\infty} Q(f_i) \in \mathcal{A} \otimes \mathcal{B}$.

Now define $h : X \times [0,\infty) \to [0,\infty)$ by $h(x,t) := pt^{p-1}$. This function is $\mathcal{A} \otimes \mathcal{B}$-measurable and so is $h\chi_{Q(f)}$. Hence, by Fubini's Theorem,

$$\begin{aligned}\int_X |f|^p \, d\mu &= \int_X \left(\int_0^{|f(x)|} pt^{p-1} \, dt \right) d\mu(x) \\ &= \int_X \left(\int_0^\infty (h\chi_{Q(f)})(x,t) \, dm(t) \right) d\mu(x) \\ &= \int_0^\infty \left(\int_X (h\chi_{Q(f)})(x,t) \, d\mu(x) \right) dm(t) \\ &= \int_0^\infty pt^{p-1} \mu(A(t,f)) \, dt.\end{aligned}$$

This proves Step 3.

Step 4. *Assume $\kappa_f(t) < \infty$ for all $t > 0$. Then* (4.3) *holds.*

Define $X_0 := \{x \in X \mid f(x) \neq 0\}$, $\mathcal{A}_0 := \{A \in \mathcal{A} \,\big|\, A \subset X_0\}$, and $\mu_0 := \mu|_{\mathcal{A}_0}$. Then the measure space $(X_0, \mathcal{A}_0, \mu_0)$ is σ-finite because $X_n := A(1/n, f)$ is a sequence of $\mathcal{A}_n$-measurable sets such that $\mu_0(X_n) = \kappa_f(1/n) < \infty$ for all n and $X_0 = \bigcup_{n=1}^\infty X_n$. Moreover, $f_0 := f|_{X_0} : X_0 \to \mathbb{R}$ is $\mathcal{A}_0$-measurable and $\kappa_f = \kappa_{f_0}$. Hence it follows from Step 3 that

$$\int_X |f|^p \, d\mu = \int_{X_0} |f_0|^p \, d\mu_0 = \int_0^\infty t^{p-1} \kappa_{f_0}(t) \, dt = \int_0^\infty t^{p-1} \kappa_f(t) \, dt.$$

This proves Step 4 and Lemma 4.1. □

Fix real numbers $1 \leq p \leq q$. Then Hölder's inequality implies

$$\|f\|_p \leq \|f\|_1^{\frac{q-p}{p(q-1)}} \|f\|_q^{\frac{q(p-1)}{p(q-1)}} \tag{4.4}$$

for every measurable function $f : X \to \mathbb{R}$ and hence

$$L^1(\mu) \cap L^q(\mu) \subset L^p(\mu).$$

Since the intersection $L^1(\mu) \cap L^q(\mu)$ contains (the equivalences classes of) all characteristic functions of measurable sets with finite measure, it is dense in $L^p(\mu)$ (see [33, Lem 4.12]). The following theorem was proved in 1939 by Józef Marcinkiewicz (a PhD student of Antoni Zygmund). To formulate the result it will be convenient to slightly abuse notation and use the same letter f to denote an element of $\mathcal{L}^p(\mu)$ and its equivalence class in $L^p(\mu)$.

For a measurable function $f : X \to \mathbb{C}$ define

$$\|f\|_{1,\infty} := \sup_{t>0} t\kappa_f(t) \le \|f\|_{L^1}\,. \tag{4.5}$$

We emphasize that the map $f \mapsto \|f\|_{1,\infty}$ is not a norm because it only satisfies the *weak triangle inequality*

$$\|f+g\|_{1,\infty}^{1/2} \le \|f\|_{1,\infty}^{1/2} + \|g\|_{1,\infty}^{1/2}.$$

However the formula $d_{1,\infty}(f,g) := \|f-g\|_{1,\infty}^{1/2}$ defines a metric on $L^1(\mathbb{R}^n,\mathbb{C})$ and the completion of $L^1(\mathbb{R}^n,\mathbb{C})$ with respect to this metric is the topological vector space $L^{1,\infty}(\mathbb{R}^n,\mathbb{C})$ of *weakly integrable* functions (see [33, Section 6.1]).

Theorem 4.2 (**Marcinkiewicz**)**.** *Let $q > 1$ and let $T : L^q(\mu) \to L^q(\mu)$ be a linear operator. Suppose there are constants $c_1 > 0$ and $c_q > 0$ such that*

$$\|Tf\|_{1,\infty} \le c_1 \|f\|_1\,, \qquad \|Tf\|_q \le c_q \|f\|_q \tag{4.6}$$

for all $f \in L^1(\mu) \cap L^q(\mu)$. Fix a constant $1 < p < q$. Then

$$\|Tf\|_p \le c_p \|f\|_p\,, \qquad c_p := 2\left(\frac{p(q-1)}{(q-p)(p-1)}\right)^{1/p} c_1^{\frac{q-p}{p(q-1)}} c_q^{\frac{q(p-1)}{p(q-1)}}, \tag{4.7}$$

for all $f \in L^1(\mu) \cap L^q(\mu)$. Thus the restriction of T to $L^1(\mu) \cap L^q(\mu)$ extends (uniquely) to a bounded linear operator from $L^p(\mu)$ to itself for $1 < p < q$.

Proof. Let $c > 0$ and let $f \in \mathcal{L}^1(\mu) \cap \mathcal{L}^q(\mu)$. For $t \ge 0$ define

$$f_t(x) := \begin{cases} f(x), & \text{if } |f(x)| > ct, \\ 0, & \text{if } |f(x)| \le ct, \end{cases} \qquad g_t(x) := \begin{cases} 0, & \text{if } |f(x)| > ct, \\ f(x), & \text{if } |f(x)| \le ct. \end{cases}$$

Then

$$A(s,f_t) = \begin{cases} A(s,f), & \text{if } s > ct, \\ A(ct,f), & \text{if } s \le ct, \end{cases} \quad A(s,g_t) = \begin{cases} \emptyset, & \text{if } s \ge ct, \\ A(s,f) \setminus A(ct,f), & \text{if } s < ct, \end{cases}$$

$$\kappa_{f_t}(s) = \begin{cases} \kappa_f(s), & \text{if } s > ct, \\ \kappa_f(ct), & \text{if } s \le ct, \end{cases} \quad \kappa_{g_t}(s) = \begin{cases} 0, & \text{if } s \ge ct, \\ \kappa_f(s) - \kappa_f(ct), & \text{if } s < ct. \end{cases}$$

By Lemma 4.1 and Fubini's Theorem, this implies

$$\begin{aligned}
\int_0^\infty t^{p-2}\left\|f_t\right\|_1\,dt &= \int_0^\infty t^{p-2}\left(\int_0^\infty \kappa_{f_t}(s)\,ds\right)\,dt \\
&= \int_0^\infty t^{p-2}\left(ct\kappa_f(ct)+\int_{ct}^\infty \kappa_f(s)\,ds\right)\,dt \\
&= c^{1-p}\int_0^\infty t^{p-1}\kappa_f(t)\,dt + \int_0^\infty\int_0^{s/c} t^{p-2}\,dt\,\kappa_f(s)\,ds \\
&= c^{1-p}\int_0^\infty t^{p-1}\kappa_f(t)\,dt + \int_0^\infty \frac{(s/c)^{p-1}}{p-1}\kappa_f(s)\,ds \\
&= \frac{c^{1-p}p}{p-1}\int_0^\infty t^{p-1}\kappa_f(t)\,dt \\
&= \frac{c^{1-p}}{p-1}\int_X |f|^p\,d\mu
\end{aligned}$$

and

$$\begin{aligned}
\int_0^\infty t^{p-q-1}\left\|g_t\right\|_q^q\,dt &= \int_0^\infty t^{p-q-1}\left(\int_0^\infty qs^{q-1}\kappa_{g_t}(s)\,ds\right)\,dt \\
&= \int_0^\infty t^{p-q-1}\left(\int_0^{ct} qs^{q-1}(\kappa_f(s)-\kappa_f(ct))\,ds\right)\,dt \\
&= q\int_0^\infty\int_{s/c}^\infty t^{p-q-1}\,dt\,s^{q-1}\kappa_f(s)\,ds - c^q\int_0^\infty t^{p-1}\kappa_f(ct)\,dt \\
&= q\int_0^\infty \frac{s^{p-1}c^{q-p}}{q-p}\kappa_f(s)\,ds - c^{q-p}\int_0^\infty t^{p-1}\kappa_f(t)\,dt \\
&= \frac{c^{q-p}p}{q-p}\int_0^\infty t^{p-1}\kappa_f(t)\,dt \\
&= \frac{c^{q-p}}{q-p}\int_X |f|^p\,d\mu.
\end{aligned}$$

Moreover, $f = f_t + g_t$ for all $t \geq 0$. Hence, by Lemma 4.1 and (4.6),

$$\begin{aligned}
\kappa_{Tf}(t) &\leq \kappa_{Tf_t}(t/2) + \kappa_{Tg_t}(t/2) \\
&\leq \frac{2}{t}\left\|Tf_t\right\|_{1,\infty} + \frac{2^q}{t^q}\left\|Tg_t\right\|_q^q \\
&\leq \frac{2c_1}{t}\left\|f_t\right\|_1 + \frac{(2c_q)^q}{t^q}\left\|g_t\right\|_q^q.
\end{aligned}$$

Hence, by Lemma 4.1 and the identities on page 25,

$$\begin{aligned}\int_X |Tf|^p\,d\mu &= p\int_0^\infty t^{p-1}\kappa_{Tf}(t)\,dt \\ &\le p2c_1\int_0^\infty t^{p-2}\left\|f_t\right\|_1\,dt + p(2c_q)^q\int_0^\infty t^{p-q-1}\left\|g_t\right\|_q^q\,dt \\ &= \left(\frac{p2c_1c^{1-p}}{p-1}+\frac{p(2c_q)^qc^{q-p}}{q-p}\right)\int_X |f|^p\,d\mu \\ &= \frac{p(q-1)2^pc_1^{(q-p)/(q-1)}c_q^{(qp-q)/(q-1)}}{(q-p)(p-1)}\int_X |f|^p\,d\mu\end{aligned}$$

Here the last equation follows with the choice of $c := (2c_1)^{1/(q-1)}/(2c_q)^{q/(q-1)}$. This proves Theorem 4.2. □

Theorem 4.2 extends to Banach space valued functions. Here is an example for such an extension that is used in Section 9. Consider the positive real axis equipped with the Lebesgue measure. For a strongly Lebesgue measurable function $f : [0,\infty) \to X$ with values in a Banach space X define the function $\kappa_f : (0,\infty) \to [0,\infty]$ by

$$\kappa_f(r) := \mu\left(\left\{t \ge 0 \,\middle|\, \|f(t)\| > r\right\}\right) \qquad \text{for } r > 0.$$

This function depends only on the equivalence class of f up to equality almost everywhere.

Corollary 4.3 (Marcinkiewicz). *Fix a real number $1 < q < \infty$. Let X be a Banach space and let*

$$\mathcal{T} : L^q([0,\infty),X) \to L^q([0,\infty),X)$$

be a linear operator. Suppose there exist positive constants c_1, c_q such that

$$\left\|\mathcal{T}f\right\|_{L^q} \le c_q\left\|f\right\|_{L^q}, \qquad \sup_{r>0} r\kappa_{\mathcal{T}f}(r) \le c_1\left\|f\right\|_{L^1}$$

for all $f \in L^q([0,\infty),X) \cap L^1([0,\infty),X)$. Then

$$\left\|\mathcal{T}f\right\|_{L^p} \le c_p\left\|f\right\|_{L^p}, \qquad c_p := 2\left(\frac{p(q-1)}{(q-p)(p-1)}\right)^{1/p} c_1^{\frac{q-p}{p(q-1)}} c_q^{\frac{q(p-1)}{p(q-1)}}$$

for $1 < p < q$ and $f \in L^q([0,\infty),X) \cap L^1([0,\infty),X) \subset L^p([0,\infty),X)$.

Proof. The proofs of Lemma 4.1 and Theorem 4.2 carry over verbatim to Banach space valued functions. □

5 The Calderón–Zygmund inequality

The next definition is taken from the exposition in Parissis [29].

Definition 5.1. *Let $n \in \mathbb{N}$ and let $\Delta_n := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \,|\, x = y\}$ be the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$. Fix two constants $C > 0$ and $0 < \sigma \le 1$. A* **Calderón–Zygmund pair on $\mathbb{R}^n$ with constants** C **and** σ *is a pair (T, K), consisting of a bounded linear operator $T : L^2(\mathbb{R}^n, \mathbb{C}) \to L^2(\mathbb{R}^n, \mathbb{C})$ and a continuous function $K : (\mathbb{R}^n \times \mathbb{R}^n) \setminus \Delta_n \to \mathbb{C}$, satisfying the following axioms.*

(CZ1) $\|Tf\|_{L^2} \le C \|f\|_{L^2}$ *for all $f \in L^2(\mathbb{R}^n, \mathbb{C})$.*

(CZ2) *If $f : \mathbb{R}^n \to \mathbb{C}$ is a continuous function with compact support then the restriction of Tf to the open set $\mathbb{R}^n \setminus \operatorname{supp}(f)$ is continuous and*

$$(Tf)(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\, dy \qquad \text{for all } x \in \mathbb{R}^n \setminus \operatorname{supp}(f). \tag{5.1}$$

(CZ3) *Let $x, y \in \mathbb{R}^n$ such that $x \ne y$. Then*

$$|K(x,y)| \le \frac{C}{|x-y|^n}. \tag{5.2}$$

(CZ4) *Let $x, x', y, y' \in \mathbb{R}^n$ such that $x \ne y$, $x \ne y'$, and $x' \ne y$. Then*

$$\begin{aligned} |y - y'| < \frac{1}{2}|x-y| \quad &\Longrightarrow \quad |K(x,y) - K(x,y')| \le \frac{C\,|y-y'|^\sigma}{|x-y|^{n+\sigma}}, \\ |x - x'| < \frac{1}{2}|x-y| \quad &\Longrightarrow \quad |K(x,y) - K(x',y)| \le \frac{C\,|x-x'|^\sigma}{|x-y|^{n+\sigma}}. \end{aligned} \tag{5.3}$$

We remark that if (T, K) is a Calderón–Zygmund pair then so is (T', K), where the operator $T' : L^2(\mathbb{R}^n, \mathbb{C}) \to L^2(\mathbb{R}^n, \mathbb{C})$ is given by $T'f = Tf + bf$ for all $f \in L^2(\mathbb{R}^n, \mathbb{C})$ and some bounded measurable function $b : \mathbb{R}^n \to \mathbb{C}$. Thus the operator T is not uniquely determined by K. However, it is easy to see that the function K is uniquely determined by the operator T. (Exercise!)

Theorem 5.2 (Calderón–Zygmund). *Fix an integer $n \in \mathbb{N}$, a real number $1 < p < \infty$, and two constants $C > 0$ and $0 < \sigma \le 1$. Then there exists a constant $c = c(n, p, \sigma, C) > 0$ such that every Calderón–Zygmund pair (T, K) on $\mathbb{R}^n$ with constants C and σ satisfies the inequality*

$$\|Tf\|_{L^p} \le c \|f\|_{L^p} \tag{5.4}$$

for all $f \in L^2(\mathbb{R}^n, \mathbb{C}) \cap L^p(\mathbb{R}^n, \mathbb{C})$.

Proof. The proof has four steps. Denote by μ the Lebesgue measure on $\mathbb{R}^n$.

Step 1. *There is a constant $c = c(n, \sigma, C) \geq 1$ with the following significance. Let (T, K) be a Calderón–Zygmund pair on $\mathbb{R}^n$ with the constants C and σ, let $B \subset \mathbb{R}^n$ be a countable union of closed cubes $Q_i \subset \mathbb{R}^n$ with pairwise disjoint interiors, and let $h \in L^2(\mathbb{R}^n, \mathbb{C}) \cap L^1(\mathbb{R}^n, \mathbb{C})$ such that*

$$h|_{\mathbb{R}^n \setminus B} \equiv 0, \qquad \int_{Q_i} h(x)\, dx = 0 \qquad \textit{for all } i \in \mathbb{N}. \tag{5.5}$$

Then

$$\kappa_{Th}(t) \leq c \left(\mu(B) + \frac{1}{t} \|h\|_1 \right) \qquad \textit{for all } t > 0. \tag{5.6}$$

For $i \in \mathbb{N}$ define $h_i : \mathbb{R}^n \to \mathbb{R}$ by $h_i(x) := h(x)$ for $x \in Q_i$ and by $h_i(x) := 0$ for $x \in \mathbb{R} \setminus Q_i$. Denote by $q_i \in Q_i$ the center of the cube Q_i and by $2r_i > 0$ its side length. Then $|x - q_i| \leq \sqrt{n} r_i$ for all $x \in Q_i$. Fix an element $x \in \mathbb{R}^n \setminus Q_i$. Then it follows from (5.1) that

$$\begin{aligned} (Th_i)(x) &= \int_{Q_i} K(x,y) h_i(y)\, dy \\ &= \int_{Q_i} \big(K(x,y) - K(x,q_i) \big) h_i(y)\, dy. \end{aligned} \tag{5.7}$$

The function h_i need not be continuous. Since $x \notin Q_i$ one can approximate h_i in $L^2(\mathbb{R}^n, \mathbb{C})$ by a sequence of compactly supported continuous functions that vanish near x. For the approximating sequence the first equation in (5.7) holds by (5.1); now take the limit. The second equation follows from (5.5).

Now choose $x \in \mathbb{R}^n$ such that $|x - q_i| > 3\sqrt{n} r_i$. Then

$$d(x, Q_i) := \inf_{y \in Q_i} |x - y| > 2\sqrt{n} r_i$$

and so $|y - q_i| \leq \sqrt{n} r_i < \frac{1}{2}|x - y|$ for all $y \in Q_i$. Hence, by (5.3) and (5.7),

$$\begin{aligned} |(Th_i)(x)| &\leq \int_{Q_i} |K(x,y) - K(x,q_i)| |h_i(y)|\, dy \\ &\leq \sup_{y \in Q_i} |K(x,y) - K(x,q_i)| \, \|h_i\|_1 \\ &\leq \sup_{y \in Q_i} \frac{c|y - q_i|^\sigma}{|x - y|^{n+\sigma}} \|h_i\|_1 \\ &\leq \frac{c_1 r_i^\sigma}{d(x, Q_i)^{n+\sigma}} \|h_i\|_1 . \end{aligned}$$

Here C is the constant in (5.3) and $c_1 := C n^{\sigma/2}$.

Define

$$P_i := \left\{ x \in \mathbb{R}^n \,\middle|\, |x - q_i| \le 3\sqrt{n} r_i \right\}.$$

Then $d(x, Q_i) \ge |x - q_i| - \sqrt{n} r_i$ for all $x \in \mathbb{R}^n \setminus P_i$. Hence

$$\begin{aligned}
\int_{\mathbb{R}^n \setminus P_i} |(Th_i)(x)|\, dx &\le c_1 r_i^\sigma \int_{\mathbb{R}^n \setminus P_i} \frac{1}{\left(|x - q_i| - \sqrt{n} r_i\right)^{n+\sigma}}\, dx\, \|h_i\|_1 \\
&= c_1 r_i^\sigma \int_{|y| > 3\sqrt{n} r_i} \frac{1}{\left(|y| - \sqrt{n} r_i\right)^{n+1}}\, dy\, \|h_i\|_1 \\
&= c_1 r_i^\sigma \int_{3\sqrt{n} r_i}^\infty \frac{\omega_n r^{n-1} dr}{\left(r - \sqrt{n} r_i\right)^{n+\sigma}}\, \|h_i\|_1 \\
&= c_1 \omega_n r_i^\sigma \int_{2\sqrt{n} r_i}^\infty \frac{\left(s + \sqrt{n} r_i\right)^{n-1} ds}{s^{n+\sigma}}\, \|h_i\|_1 \\
&\le c_1 2^{n-1} \omega_n r_i^\sigma \int_{2\sqrt{n} r_i}^\infty \frac{ds}{s^{1+\sigma}}\, \|h_i\|_1 \\
&= c_2 \|h_i\|_1 .
\end{aligned}$$

Here $c_2 := c_1 2^{n-1-\sigma} n^{-\sigma/2} \sigma^{-1} \omega_n$ and $\omega_n := \mathrm{Vol}_{n-1}(S^{n-1})$. The third step in this computation follows from Fubini's Theorem in polar coordinates. Thus we have proved that

$$\int_{\mathbb{R}^n \setminus P_i} |(Th_i)(x)|\, dx \le c_2 \|h_i\|_1 \qquad \text{for all } i \in \mathbb{N}. \tag{5.8}$$

Recall that Th and Th_i are only equivalence classes in $L^2(\mathbb{R}^n)$. Choose square integrable functions on $\mathbb{R}^n$ representing these equivalence classes and denote them by the same letters Th, Th_i. We prove that there exists a Lebesgue null set $E \subset \mathbb{R}^n$ such that

$$|(Th)(x)| \le \sum_{i=1}^\infty |(Th_i)(x)| \qquad \text{for all } x \in \mathbb{R}^n \setminus E. \tag{5.9}$$

To see this, note that the sequence $\sum_{i=1}^\ell h_i$ converges to h in $L^2(\mathbb{R}^n)$ as ℓ tends to infinity and so the sequence $\sum_{i=1}^\ell Th_i$ converges to Th in $L^2(\mathbb{R}^n)$. By [33, Cor 4.10] a subsequence converges almost everywhere. Hence there exists a Lebesgue null set $E \subset \mathbb{R}^n$ and a sequence of integers $0 < \ell_1 < \ell_2 < \ell_3 < \cdots$ such that the sequence $\sum_{i=1}^{\ell_\nu} (Th_i)(x)$ coverges to $(Th)(x)$ as ν tends to infinity for all $x \in \mathbb{R}^n \setminus E$. Since $|\sum_{i=1}^{\ell_\nu} (Th_i)(x)| \le \sum_{i=1}^\infty |(Th_i)(x)|$ for all $x \in \mathbb{R}^n$, this proves (5.9).

Now define

$$A := \bigcup_{i=1}^{\infty} P_i.$$

Then it follows from (5.8) and (5.9) that

$$\begin{aligned}
\int_{\mathbb{R}^n \setminus A} |(Th)(x)|\, dx &\leq \int_{\mathbb{R}^n \setminus A} \sum_{i=1}^{\infty} |(Th_i)(x)|\, dx \\
&= \sum_{i=1}^{\infty} \int_{\mathbb{R}^n \setminus A} |(Th_i)(x)|\, dx \\
&\leq \sum_{i=1}^{\infty} \int_{\mathbb{R}^n \setminus P_i} |(Th_i)(x)|\, dx \\
&\leq c_2 \sum_{i=1}^{\infty} \|h_i\|_1 \\
&= c_2 \|h\|_1 .
\end{aligned}$$

Moreover,

$$\mu(A) \leq \sum_{i=1}^{\infty} \mu(P_i) = c_3 \sum_{i=1}^{\infty} \mu(Q_i) = c_3 \mu(B),$$

where

$$c_3 = c_3(n) := \frac{\mu(B_{3\sqrt{n}})}{\mu([-1,1]^n)} = \mu(B_{3\sqrt{n}/2}) = \frac{\omega_n 3^n n^{n/2}}{2^n n}.$$

Hence

$$\begin{aligned}
t\kappa_{Th}(t) &\leq t\mu(A) + t\mu\left(\left\{x \in \mathbb{R}^n \setminus A \,\middle|\, |(Th)(x)| > t\right\}\right) \\
&\leq t\mu(A) + \int_{\mathbb{R}^n \setminus A} |(Th)(x)|\, dx \\
&\leq c_3 t\mu(B) + c_2 \|h\|_1 \\
&\leq c_4 \big(t\mu(B) + \|h\|_1\big)
\end{aligned}$$

for all $t > 0$, where

$$c_4 := \max\{c_2, c_3\}.$$

This proves Step 1.

Step 2 (Calderón–Zygmund Decomposition).
Let $f \in L^2(\mathbb{R}^n, \mathbb{C}) \cap L^1(\mathbb{R}^n, \mathbb{C})$ and $t > 0$. Then there exists a countable collection of closed cubes $Q_i \subset \mathbb{R}^n$ with pairwise disjoint interiors such that

$$\mu(Q_i) < \frac{1}{t}\int_{Q_i} |f(x)|\,dx \le 2^n \mu(Q_i) \qquad \textit{for all } i \in \mathbb{N} \tag{5.10}$$

and

$$|f(x)| \le t \qquad \textit{for almost all } x \in \mathbb{R}^n \setminus B, \tag{5.11}$$

where $B := \bigcup_{i=1}^\infty Q_i$.

For $\xi \in \mathbb{Z}^n$ and $\ell \in \mathbb{Z}$ define

$$Q(\xi, \ell) := \left\{ x \in \mathbb{R}^n \,\middle|\, 2^{-\ell}\xi_i \le x_i \le 2^{-\ell}(\xi_i + 1) \right\}.$$

Let

$$\mathscr{Q} := \left\{ Q(\xi, \ell) \,\middle|\, \xi \in \mathbb{Z}^n,\ \ell \in \mathbb{Z} \right\}$$

and define the subset $\mathscr{Q}_0 \subset \mathscr{Q}$ by

$$\mathscr{Q}_0 := \left\{ Q \in \mathscr{Q} \,\middle|\, \begin{array}{l} t\mu(Q) < \int_Q |f(x)|\,dx \text{ and, for all } Q' \in \mathscr{Q}, \\ Q \subsetneq Q' \quad \Longrightarrow \quad \int_{Q'} |f(x)|\,dx \le t\mu(Q') \end{array} \right\}.$$

Then every decreasing sequence of cubes in $\mathscr{Q}$ contains at most one element of $\mathscr{Q}_0$. Hence every element of $\mathscr{Q}_0$ satisfies (5.10) and any two cubes in $\mathscr{Q}_0$ have disjoint interiors. Define $B := \bigcup_{Q \in \mathscr{Q}_0} Q$. We prove that

$$x \in \mathbb{R}^n \setminus B, \quad x \in Q \in \mathscr{Q} \qquad \Longrightarrow \qquad \frac{1}{\mu(Q)}\int_Q |f(x)|\,dx \le t. \tag{5.12}$$

Suppose, by contradiction, that there exists an element $x \in \mathbb{R}^n \setminus B$ and a cube $Q \in \mathscr{Q}$ such that $x \in Q$ and $t\mu(Q) < \int_Q |f(x)|\,dx$. Then, since $\|f\|_1 < \infty$, there exists a maximal cube $Q \in \mathscr{Q}$ such that $x \in Q$ and $t\mu(Q) < \int_Q |f(y)|\,dy$. Such a maximal cube would be an element of $\mathscr{Q}_0$ and hence $x \in B$, a contradiction. This proves (5.12). Now the Lebesgue Differentiation Theorem [33, Thm 6.14] asserts that there exists a Lebesgue null set $E \subset \mathbb{R}^n \setminus B$ such that every element of $\mathbb{R}^n \setminus (B \cup E)$ is a Lebesgue point of f. By (5.12), every point $x \in \mathbb{R}^n \setminus (B \cup E)$ is the intersection point of a decreasing sequence of cubes over which $|f|$ has mean value at most t. Hence it follows from [33, Thm 6.16] (a corollary of the Lebesgue Differentiation Theorem) that $|f(x)| \le t$ for all $x \in \mathbb{R}^n \setminus (B \cup E)$. This proves Step 2.

Step 3. *Let $c = c(n, \sigma, C) \geq 1$ be the constant in Step 1 and let (T, K) be a Calderón–Zygmund pair on $\mathbb{R}^n$ with the constants C and σ. Then*

$$\|Tf\|_{1,\infty} \leq \left(2^{n+1} + 6c\right) \|f\|_1 \tag{5.13}$$

for all $f \in L^2(\mathbb{R}^n, \mathbb{C}) \cap L^1(\mathbb{R}^n, \mathbb{C})$.

Fix a function $f \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n, \mathbb{C})$ and a constant $t > 0$. Let the Q_i be as in Step 2 and define

$$B := \bigcup_i Q_i.$$

Then $\mu(Q_i) < \frac{1}{t} \int_{Q_i} |f(x)|\,dx$ for all i by Step 2 and hence

$$\mu(B) = \sum_i \mu(Q_i) \leq \frac{1}{t} \sum_i \int_{Q_i} |f(x)|\,dx \leq \frac{1}{t} \|f\|_1 .$$

Define $g, h : \mathbb{R}^n \to \mathbb{R}$ by

$$g := f\chi_{\mathbb{R}^n \setminus B} + \sum_i \frac{\int_{Q_i} f(x)\,dx}{\mu(Q_i)} \chi_{Q_i}, \qquad h := f - g.$$

Then

$$\|g\|_1 \leq \|f\|_1 , \qquad \|h\|_1 \leq 2\|f\|_1 .$$

Moreover, h vanishes on $\mathbb{R}^n \setminus B$ and $\int_{Q_i} h(x)\,dx = 0$ for all i. Hence it follows from Step 1 that

$$\kappa_{Th}(t) \leq c\left(\mu(B) + \frac{1}{t}\|h\|_1\right) \leq \frac{3c}{t}\|f\|_1 . \tag{5.14}$$

Moreover, it follows from Step 2 that $|g(x)| \leq t$ for almost every $x \in \mathbb{R}^n \setminus B$ and $|g(x)| \leq 2^n t$ for every $x \in \mathrm{int}(Q_i)$. Thus $|g| \leq 2^n t$ almost everywhere. Hence it follows from [33, Lemma 7.36] that

$$\kappa_{Tg}(t) \leq \frac{1}{t^2} \int_{\mathbb{R}^n} |g(x)|^2\,dx \leq \frac{2^n}{t} \int_{\mathbb{R}^n} |g(x)|\,dx \leq \frac{2^n}{t}\|f\|_1 . \tag{5.15}$$

Now combine (5.14) and (5.15) with the inequality (4.2) to obtain

$$\kappa_{Tf}(2t) \leq \kappa_{Tg}(t) + \kappa_{Th}(t) \leq \frac{2^{n+1} + 6c}{2t}\|f\|_1 .$$

Here the splitting $f = g + h$ depends on t but the constant c does not. Multiply the inequality by $2t$ and take the supremum over all t to obtain (5.13). This proves Step 3.

Step 4. *Fix a real number* $1 < p < \infty$ *as well as* $n \in \mathbb{N}$ *and* $C > 0$ *and* $0 < \sigma \le 1$. *Then there exists a constant* $c = c(n, p, \sigma, C) > 0$ *such that* $\|Tf\|_{L^p} \le c\,\|f\|_{L^p}$ *for all* $f \in L^2(\mathbb{R}^n, \mathbb{C}) \cap L^p(\mathbb{R}^n, \mathbb{C})$ *and every Calderón–Zygmund pair* (T, K) *on* $\mathbb{R}^n$ *with constants* C *and* σ.

For $p = 2$ this holds by assumption, and for $1 < p < 2$ it follows from Step 3 and the Marcinkiewicz Interpolation Theorem 4.2 with $q = 2$.

Now assume $2 < p < \infty$ and choose $1 < q < 2$ such that $1/p + 1/q = 1$. Define the function $K^* : (\mathbb{R}^n \times \mathbb{R}^n) \setminus \Delta_n \to \mathbb{R}$ by

$$K^*(x, y) := \overline{K(y, x)}$$

and let $T^* : L^2(\mathbb{R}^n, \mathbb{C}) \to L^2(\mathbb{R}^n, \mathbb{C})$ be the adjoint operator of T. Then (T^*, K^*) is again a Calderón–Zygmund pair on $\mathbb{R}^n$ with constants C and σ. To see this, note that

$$\begin{aligned} \langle T^*g, f\rangle_{L^2} &= \langle g, Tf\rangle_{L^2} \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \overline{g(x)} K(x, y) f(y)\, dy\, dx \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \overline{K^*(y, x) g(x)} f(y)\, dx\, dy \end{aligned}$$

for any two continuous functions $f, g : \mathbb{R}^n \to \mathbb{C}$ with disjoint compact supports. This implies $(T^*g)(y) = \int_{\mathbb{R}^n} K^*(y, x) g(x)\, dx$ for every continuous function $g : \mathbb{R}^n \to \mathbb{C}$ with compact support and every $y \in \mathbb{R}^n \setminus \mathrm{supp}(g)$. Moreover, the function K^* evidently satisfies (5.2) and (5.3) and T^* has the same operator norm as T. Now define $c := c(n, q, \sigma, C)$. Then, by what we have already proved, $\|T^*g\|_{L^q} \le c\,\|g\|_{L^q}$ for all $g \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$. Hence

$$\begin{aligned} \|Tf\|_{L^p} &= \sup_{0 \ne g \in C_0^\infty(\mathbb{R}^n, \mathbb{C})} \frac{\langle g, Tf\rangle_{L^2}}{\|g\|_{L^q}} \\ &= \sup_{0 \ne g \in C_0^\infty(\mathbb{R}^n, \mathbb{C})} \frac{\langle T^*g, f\rangle_{L^2}}{\|g\|_{L^q}} \\ &\le \sup_{0 \ne g \in C_0^\infty(\mathbb{R}^n, \mathbb{C})} \frac{\|T^*g\|_{L^q}\,\|f\|_{L^p}}{\|g\|_{L^q}} \\ &\le c_q\,\|f\|_{L^p} \end{aligned}$$

for all $f \in L^2(\mathbb{R}^n, \mathbb{C}) \cap L^p(\mathbb{R}^n, \mathbb{C})$. Here the first equality follows from [33, Thm 4.33] and the fact that $C_0^\infty(\mathbb{R}^n, \mathbb{C})$ is a dense subspace of $L^q(\mathbb{R}^n, \mathbb{C})$. This proves Theorem 5.2. □

6 The Mikhlin multiplier theorem

The Fourier transform on $\mathbb{R}^n$ is the unique bounded linear operator

$$\mathscr{F} : L^2(\mathbb{R}^n, \mathbb{C}) \to L^2(\mathbb{R}^n, \mathbb{C})$$

given by

$$(\mathscr{F}(u))(\xi) := \widehat{u}(\xi) := \int_{R^n} e^{-\mathbf{i}\langle \xi, x\rangle} u(x)\, dx \tag{6.1}$$

for $\xi \in \mathbb{R}^n$ and $u \in L^2(\mathbb{R}^n, \mathbb{C}) \cap L^1(\mathbb{R}^n, \mathbb{C})$. Its inverse is

$$(\mathscr{F}^{-1}(\widehat{u}))(x) = u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{\mathbf{i}\langle \xi, x\rangle} \widehat{u}(\xi)\, d\xi \tag{6.2}$$

for $x \in \mathbb{R}^n$ and $\widehat{u} \in L^2(\mathbb{R}^n, \mathbb{C}) \cap L^1(\mathbb{R}^n, \mathbb{C})$. If $m : \mathbb{R}^n \to \mathbb{C}$ is a bounded measurable function, it determines a bounded linear operator

$$T_m : L^2(\mathbb{R}^n, \mathbb{C}) \to L^2(\mathbb{R}^n, \mathbb{C})$$

given by

$$T_m u := \mathscr{F}^{-1}(m \mathscr{F}(u))$$

for $u \in L^2(\mathbb{R}^n, \mathbb{C})$. The **Mikhlin Multiplier Theorem** gives conditions on m under which this operator extends to a (unique) bounded linear operator from $L^p(\mathbb{R}^n, \mathbb{C})$, still denoted by T_m, which agrees with the original operator on the intersection $L^2(\mathbb{R}^n, \mathbb{C}) \cap L^p(\mathbb{R}^n, \mathbb{C})$. We state and prove this result in a slightly weaker form than in Mikhlin [25] and Hörmander [16]. This version suffices for the purposes of the present exposition.

Theorem 6.1 (Mikhlin). *For every integer $n \in \mathbb{N}$, every constant $C > 0$, and every real number $1 < p < \infty$ there exists a constant $c = c(n, p, C) > 0$ with the following significance. Let $m : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ be a C^{n+2} function that satisfies the inequality*

$$|\partial^\alpha m(\xi)| \le \frac{C}{|\xi|^{|\alpha|}} \tag{6.3}$$

for every $\xi \in \mathbb{R}^n \setminus \{0\}$ and every multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ with $|\alpha| = \alpha_1 + \cdots + \alpha_n \le n + 2$. Then

$$\|T_m f\|_{L^p} \le c \|f\|_{L^p}. \tag{6.4}$$

for all $f \in L^2(\mathbb{R}^n, \mathbb{C}) \cap L^p(\mathbb{R}^n, \mathbb{C})$.

Proof. The proof is based on the generalized Calderón–Zygmund inequality in Theorem 5.2 and follows the argument in Parissis [29]. The main idea is to show that there exists a function $K_m : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ such that T_m and the function

$$\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta_m \to \mathbb{C} : (x, y) \mapsto K_m(x - y)$$

form a Calderón–Zygmund pair as in Definition 5.1. One would like to choose K_m such that m is the Fourier transform of K_m. The difficulty is that, in the interesting cases, m is not the Fourier transform of any integrable function. To overcome this problem one can use the Littlewood–Paley decomposition (Section 8). More precisely, let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a Schwartz function such that $\phi(x) = \phi(-x)$ for all $x \in \mathbb{R}^n$ so that its Fourier transform $\widehat{\phi} := \mathscr{F}(\phi)$ is a smooth real valued function that satisfies $\widehat{\phi}(\xi) = \widehat{\phi}(-\xi)$ for all $\xi \in \mathbb{R}^n$. Assume in addition that $\widehat{\phi}$ satisfies the conditions

$$\begin{aligned} \widehat{\phi}(\xi) &> 0 && \text{for } 1/\sqrt{2} \le |\xi| \le \sqrt{2}, \\ \widehat{\phi}(\xi) &\ge 0 && \text{for } 1/2 \le |\xi| \le 2, \\ \widehat{\phi}(\xi/2) + \widehat{\phi}(\xi) &= 1 && \text{for } 1 \le |\xi| \le 2, \\ \widehat{\phi}(\xi) &= 0 && \text{for } |\xi| \notin [1/2, 2]. \end{aligned} \tag{6.5}$$

In Definition 8.1 below a function ϕ with these properties is called a *Littlewood–Paley function* and that it exists is shown in Example 8.2. For $j \in \mathbb{Z}$ define the function $\phi_j : \mathbb{R}^n \to \mathbb{R}$ by

$$\phi_j(x) := 2^{nj}\phi(2^j x), \qquad \widehat{\phi}_j(\xi) := \widehat{\phi}(2^{-j}\xi) \tag{6.6}$$

for $x, \xi \in \mathbb{R}^n$. Then it follows from (6.5) that $\sum_{j=-\infty}^{\infty} \widehat{\phi}_k(\xi) = 1$ and hence

$$\sum_{j=-\infty}^{\infty} \widehat{\phi}_j(\xi) m(\xi) = m(\xi) \qquad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}. \tag{6.7}$$

For $j \in \mathbb{Z}$ the function $\widehat{\phi}_j m : \mathbb{R}^n \to \mathbb{C}$ is of class C^{n+2} and has compact support and we denote its inverse Fourier transform by

$$K_j := \mathscr{F}^{-1}(\widehat{\phi}_j m).$$

We prove in three steps that the series $K_m := \sum_{j=-\infty}^{\infty} K_j$ defines a continuous function on $\mathbb{R}^n \setminus \{0\}$ and that the pair $(T_m, (x, y) \mapsto K_m(x - y))$ satisfies the requirements of Definition 5.1.

Step 1. *There exists a constant $c_1 = c_1(n, C) > 0$ with the following significance. Let $m : \mathbb{R}^n \to \mathbb{C}$ be a C^{n+2} function that satisfies (6.3) for $|\alpha| \le n+2$. Then, for all $x \in \mathbb{R}^n \setminus \{0\}$, the limit*

$$\begin{aligned} K_m(x) &:= \sum_{j=-\infty}^{\infty} K_j(x) \\ &= \lim_{N\to\infty} \sum_{j=-N}^{N} \frac{1}{(2\pi)^n} \int_{2^{j-1}\le|\xi|\le 2^{j+1}} e^{\mathbf{i}\langle\xi,x\rangle} \widehat{\phi}(2^{-j}\xi) m(\xi)\, d\xi \end{aligned} \tag{6.8}$$

exists, the resulting function $K_m : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ is C^1, and

$$|K_m(x)| + |x||\nabla K_m(x))| \le \frac{c_1}{|x|^n} \qquad \text{for all } x \in \mathbb{R}^n \setminus \{0\}. \tag{6.9}$$

Each function K_j has a C^{n+2} Fourier transform $\widehat{K}_j = \widehat{\phi}_j m$ with support in the compact set $\{\xi \in \mathbb{R}^n \,|\, 2^{j-1} \le |\xi| \le 2^{j+1}\}$. Hence K_j is smooth and, for every $\alpha \in \mathbb{N}_0^n$, every integer $0 \le k \le n+2$, and every $x \in \mathbb{R}^n$, we have

$$\begin{aligned} \partial^\alpha K_j(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (\mathbf{i}\xi)^\alpha \widehat{\phi}(2^{-j}\xi) m(\xi) e^{\mathbf{i}\langle\xi,x\rangle}\, d\xi \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (\mathbf{i}\xi)^\alpha \widehat{\phi}(2^{-j}\xi) m(\xi) \left(\sum_{i=1}^n \frac{x_i}{\mathbf{i}|x|^2} \frac{\partial}{\partial \xi_i} \right)^k e^{\mathbf{i}\langle\xi,x\rangle}\, d\xi \\ &= \frac{(-1)^k}{(2\pi)^n} \int_{\mathbb{R}^n} e^{\mathbf{i}\langle\xi,x\rangle} \left(\sum_{i=1}^n \frac{x_i}{\mathbf{i}|x|^2} \frac{\partial}{\partial \xi_i} \right)^k (\mathbf{i}\xi)^\alpha \widehat{\phi}(2^{-j}\xi) m(\xi)\, d\xi \\ &= \frac{(-1)^k}{(2\pi)^n} \int_{\mathbb{R}^n} e^{\mathbf{i}\langle\xi,x\rangle} \sum_{|\beta|=k} \frac{k!}{\beta!} \left(\frac{x}{\mathbf{i}|x|^2} \right)^\beta \partial_\xi^\beta \big((\mathbf{i}\xi)^\alpha \widehat{\phi}(2^{-j}\xi) m(\xi) \big)\, d\xi \end{aligned}$$

The integrand is supported in the domain $\{\xi \in \mathbb{R}^n \,|\, 2^{j-1} \le |\xi| \le 2^{j+1}\}$. Hence there exists a constant $c_2 > 0$, depending only on n, ϕ, α, and the constant C in (6.3), such that $|\partial_\xi^\beta((\mathbf{i}\xi)^\alpha \widehat{\phi}(2^{-j}\xi) m(\xi))| \le c_2 2^{j(|\alpha|-k)}$ for all $\xi \in \mathbb{R}^n$, all $j \in \mathbb{Z}$, and all $\beta \in \mathbb{N}_0^n$ with $|\beta| = k \le n+2$. This implies

$$|\partial^\alpha K_j(x)| \le \frac{c_2}{(2\pi)^n} \sum_{|\beta|=k} \frac{k!}{\beta!} \frac{2^{j(|\alpha|-k)}}{|x|^k} \int_{|\xi|\le 2^{j+1}} d\xi = c_3 \frac{2^{j(n+|\alpha|-k)}}{|x|^k} \tag{6.10}$$

for all $\alpha \in \mathbb{N}_0^n$, all $j \in \mathbb{Z}$, all $k \in \{0, 1, \ldots, n+2\}$, and all $x \in \mathbb{R}^n \setminus \{0\}$. Here $c_3 := c_2 \pi^{-n} n^{k-1} \omega_n$ and $\omega_n := \mathrm{Vol}_{n-1}(S^{n-1})$, so ω_n/n is the volume of the unit ball in $\mathbb{R}^n$, and we have used the identity $\sum_{|\beta|=k} k!/\beta! = n^k$.

Now fix a nonzero vector $x \in \mathbb{R}^n$ and let $j_0 \in \mathbb{Z}$ be the largest integer such that $2^{j_0} \le |x|^{-1}$. Then $2^{j_0} \le |x|^{-1}$ and $2^{-(j_0+1)} < |x|$. Hence it follows from (6.10) with $k = 0$ that

$$\begin{aligned}\sum_{j=-\infty}^{j_0} |\partial^\alpha K_j(x)| &\le c_3 \sum_{j=-\infty}^{j_0} 2^{j(n+|\alpha|)} \\ &= \frac{c_3}{1-2^{-n-|\alpha|}} 2^{j_0(n+|\alpha|)} \\ &\le \frac{2c_3}{|x|^{n+|\alpha|}},\end{aligned}$$

because $\frac{c_3}{1-2^{-n-|\alpha|}} \le 2c_3$ and $2^{j_0(n+|\alpha|)} \le |x|^{-(n+|\alpha|)}$. This holds for all $\alpha \in \mathbb{N}_0^n$. Now assume $|\alpha| \in \{0,1\}$ and use (6.10) with $k = n+2 > n+|\alpha|$ to obtain

$$\begin{aligned}\sum_{j=j_0+1}^{\infty} |\partial^\alpha K_j(x)| &\le c_3 \sum_{j=j_0+1}^{\infty} \frac{2^{j(|\alpha|-2)}}{|x|^{n+2}} \\ &= \frac{c_3}{1-2^{|\alpha|-2}} \frac{2^{(j_0+1)(|\alpha|-2)}}{|x|^{n+2}} \\ &< \frac{2c_3}{|x|^{n+|\alpha|}},\end{aligned}$$

because $\frac{c_3}{1-2^{|\alpha|-2}} \le 2c_3$ and $2^{(j_0+1)(|\alpha|-2)} < |x|^{2-|\alpha|}$. This proves Step 1 with the constant $c_1 = 4(n+1)c_3$.

Step 2. *Let c_1, m, K_m be as in Step 1. Then the function*

$$\mathbb{R}^n \times \mathbb{R}^n \setminus \{\Delta_n) \to \mathbb{C} : (x,y) \mapsto K_m(x-y) \tag{6.11}$$

satisfies conditions (5.2) *and* (5.3) *in Definition 5.1 with $\sigma = 1$ and with C replaced by $2^{n+1}c_1$.*

The estimate (5.2) follows directly from (6.9). To prove (5.3), fix a vector $x \in \mathbb{R}^n \setminus \{0\}$ and let $y \in \mathbb{R}^n$ such that $|y| \le |x|/2$. Then

$$|\nabla K_m(x-ty)| \le \frac{c_1}{|x-ty|^{n+1}} \le \frac{2^{n+1}c_1}{|x|^{n+1}}$$

for $0 \le t \le 1$. Hence it follows from the mean value inequality that

$$|K_m(x) - K_m(x-y)| \le \frac{2^{n+1}c_1|y|}{|x|^{n+1}}.$$

Hence the function (6.11) satisfies (5.3) with $\sigma = 1$ and this proves Step 2.

Step 3. *Let c_1, m, K_m be as in Step 1. Then the function* (6.11) *satisfies condition* (5.1) *in Definition 5.1 with $T = T_m$.*

Let $f, g : \mathbb{R}^n \to \mathbb{C}$ be continuous functions with compact support and assume $\mathrm{supp}(f) \cap \mathrm{supp}(g) = \emptyset$. Define the function $h : \mathbb{R}^n \to \mathbb{C}$ by

$$h(x) := \int_{\mathbb{R}^n} \overline{g(y)} f(x+y)\,dy \qquad \text{for } x \in \mathbb{R}^n$$

Then h vanishes near the origin and $\widehat{h} = \overline{\widehat{g}}\widehat{f}$. Hence

$$\begin{aligned}
\langle g, T_m f\rangle_{L^2} &= \frac{1}{(2\pi)^n}\left\langle \widehat{g}, \widehat{T_m f}\right\rangle_{L^2} \\
&= \frac{1}{(2\pi)^n}\left\langle \widehat{g}, m\widehat{f}\right\rangle_{L^2} \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} m(\xi)\widehat{h}(\xi)\,d\xi \\
&= \lim_{N\to\infty}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\sum_{j=-N}^{N}\widehat{\phi}_j(\xi)m(\xi)\widehat{h}(\xi)\,d\xi \\
&= \lim_{N\to\infty}\int_{\mathbb{R}^n}\sum_{j=-N}^{N} K_j(x)h(-x)\,dx \\
&= \int_{\mathbb{R}^n} K_m(x)h(-x)\,dx \\
&= \int_{\mathbb{R}^n}\int_{\mathrm{supp}(g)} K_m(x)\overline{g(y)}f(y-x)\,dy\,dx \\
&= \langle g, K_m * f\rangle_{L^2}
\end{aligned}$$

Here the first equality follows from Plancherel's Theorem, the second equality follows from the definition of the operator T_m, the third equality uses the formula $\widehat{h} = \overline{\widehat{g}}\widehat{f}$, the fourth equality uses Lebesgue dominated convergence, the fifth equality follows again from Plancherel's Theorem, the sixth equality follows from Lebesgue dominated convergence and the fact that h has compact support and vanishes near the origin, the seventh equality follows from the definition of h, and the last equality follows from Fubini's theorem. It follows that $(T_m f)(x) = \int_{\mathbb{R}^n} K(x)f(x-y)\,dx$ for all $x \in \mathbb{R}^n \setminus \mathrm{supp}(f)$. This proves Step 3. By Step 2 and Step 3 the operator T_m and the function (6.11) form a Calderón–Zygmund pair on $\mathbb{R}^n$ with constants $2^{n+1}c_1$ and $\sigma = 1$. Hence the assertion follows from Theorem 5.2. This proves Theorem 6.1. □

Corollary 6.2 (Calderón–Zygmund). *For every integer $n \in \mathbb{N}$ and every real number $1 < p < \infty$ there exists a constant $c = c(n,p) > 0$ such that*

$$\sum_{i,j=1}^{n} \left\|\partial_i\partial_j u\right\|_{L^p} \le c\left\|\Delta u\right\|_{L^p} \tag{6.12}$$

for all $u \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$.

Proof. For $i,j \in \{1,\dots,n\}$ define

$$T_{ij}f := \partial_i(K_j * f), \qquad K_j(x) := \frac{x_j}{\omega_n |x|^n}, \tag{6.13}$$

for $f \in C_0^\infty(\mathbb{R}^n)$. Then Poisson's identity asserts that

$$T_{ij}\Delta u = \partial_i\partial_j u, \qquad \Delta(K_j * f) = \partial_j f \tag{6.14}$$

for all $u, f \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$ (e.g. [33, Thm 7.41]). The second equation in (6.14) implies that $\widehat{K_j * f}(\xi) = -\mathbf{i}\xi_j|\xi|^{-2}\widehat{f}(\xi)$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$, and hence

$$\widehat{T_{ij}f} = m_{ij}\widehat{f}, \qquad m_{ij}(\xi) = \frac{\xi_i\xi_j}{|\xi|^2}. \tag{6.15}$$

This implies that T_{ij} extends to a bounded linear operator from $L^2(\mathbb{R}^n, \mathbb{C})$ to itself (see also [33, Lem 7.44]). Since the function m_{ij} satisfies the requirements of Theorem 6.1, there is a constant $c > 0$ such that $\|T_{ij}f\|_{L^p} \le c\|f\|_{L^p}$ for all $f \in C_0^\infty(\mathbb{R}^n)$. Take $f := \Delta u$ and use the first equation in (6.14) to obtain the estimate (6.12). This proves Corollary 6.2. □

Corollary 6.3. *For every real number $1 < p < \infty$ and every $C > 0$ there exists a constant $c = c(p, C) > 0$ with the following significance. Let $m : \mathbb{R} \setminus \{0\} \to \mathbb{C}$ be a C^3 function such that $|m^{(k)}(\xi)| \le C|\xi|^{-k}$ for $k = 0,1,2,3$ and $\xi \in \mathbb{R} \setminus \{0\}$. If $(X, \mathcal{A}, \mu)$ is a σ-finite measure space, $L^p(X,\mu)$ is the complex L^p-space, $f \in L^p(\mathbb{R}, L^p(X,\mu)) \cap L^2(\mathbb{R}, L^2(X,\mu))$, and the function $\mathcal{T}_m f \in L^2(\mathbb{R}, L^2(X,\mu))$ is defined by*

$$\widehat{\mathcal{T}_m f}(\xi) := m(\xi)\widehat{f}(\xi)$$

for $\xi \in \mathbb{R}$, then $\mathcal{T}_m f \in L^p(\mathbb{R}, L^p(X,\mu))$ and

$$\left\|\mathcal{T}_m f\right\|_{L^p(\mathbb{R}, L^p(X,\mu))} \le c\left\|f\right\|_{L^p(\mathbb{R}, L^p(X,\mu))}. \tag{6.16}$$

Proof. For functions $f : \mathbb{R} \to \mathbb{C}$, where X is a singleton, the assertion follows from Theorem 6.1. The general case then follows from Fubini's theorem. □

7 The Khinchin inequality

The present section of preparatory nature. It is used in the proof of the Littlewood–Paley inequality in Section 8, which in turn is useful for understanding Besov spaces. Neither this nor the next section are needed for the proof of Theorem 1.1. They are placed here because the Mikhlin Multiplier Theorem in Section 6 and the Khinchin inequality play central roles in the proof of the Littlewood–Paley inequality in Section 8.

The **Khinchin Inequality** was discovered by Alexandr Khinchin [19] in 1923. It is an estimate for the L^p norms of linear combinations of the **Rademacher functions** $\rho_k : [0,1] \to \mathbb{R}$, defined by

$$\rho_k(t) := \begin{cases} 1, & \text{if } \sin(2^k \pi t) \geq 0, \\ -1, & \text{if } \sin(2^k \pi t) < 0 \end{cases} \tag{7.1}$$

for $0 \leq t \leq 1$ and $k \in \mathbb{N}$. These functions form an orthonormal sequence in the Hilbert space $L^2([0,1])$. In the language of probability theory they are independent random variables with values ± 1, each with probability $1/2$.

Theorem 7.1 (Khinchin Inequality). *Fix a real number $0 < p < \infty$. Then there exist constants $A_p > 0$ and $B_p > 0$ such that*

$$A_p \left(\sum_{k=1}^n |\lambda_k|^2 \right)^{1/2} \leq \left(\int_0^1 \left| \sum_{k=1}^n \lambda_k \rho_k(t) \right|^p dt \right)^{1/p} \leq B_p \left(\sum_{k=1}^n |\lambda_k|^2 \right)^{1/2} \tag{7.2}$$

for all $n \in \mathbb{N}$ and all n-tuples of complex numbers $\lambda_1, \dots, \lambda_n \in \mathbb{C}$.

Proof. If $0 < p \leq 2$ then, by Hölder's inequality with exponent $2/p$, every Lebesgue measurable function $f : [0,1] \to \mathbb{C}$ satisfies $\|f\|_{L^p} \leq \|f\|_{L^2}$, so the second inequality in (7.2) holds with $B_p = 1$, because the ρ_k form an orthonormal sequence in $L^2([0,1])$. To prove the inequality for $p > 2$, define

$$\mathcal{E}_n := \{\pm 1\}^n = \left\{ \varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \,\middle|\, \varepsilon_i \in \{-1, +1\} \text{ for } i = 1, \dots, n \right\}.$$

For $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ define the function $f_\lambda : [0,1] \to \mathbb{C}$ by

$$f_\lambda(t) := \sum_{i=1}^n \lambda_i \rho_i(t)\, dt \qquad \text{for } 0 \leq t \leq 1.$$

Then

$$\int_0^1 |f_\lambda(t)|^p \, dt = \frac{1}{2^n} \sum_{\varepsilon \in \mathcal{E}_n} \left| \sum_{i=1}^n \varepsilon_i \lambda_i \right|^p$$

for all $p > 0$.

Fix an even integer $p = 2m \geq 2$ and let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$. Then

$$
\begin{aligned}
\|f_\lambda\|_{L^{2m}}^{2m} &= \frac{1}{2^n} \sum_{\varepsilon \in \mathcal{E}_n} \left(\sum_{i=1}^n \varepsilon_i \lambda_i \right)^{2m} \\
&= \frac{1}{2^n} \sum_{\varepsilon \in \mathcal{E}_n} \sum_{|\beta|=2m} \frac{(2m)!}{\beta_1! \cdots \beta_n!} \varepsilon^\beta \lambda^\beta \\
&= \frac{1}{2^n} \sum_{|\beta|=2m} \frac{(2m)!}{\beta_1! \cdots \beta_n!} \left(\sum_{\varepsilon \in \mathcal{E}_n} \varepsilon^\beta \right) \lambda^\beta \\
&= \sum_{|\alpha|=m} \frac{(2m)!}{(2\alpha_1)! \cdots (2\alpha_n)!} (\lambda_1^2)^{\alpha_1} \cdots (\lambda_n^2)^{\alpha_n}.
\end{aligned}
$$

Here the β-sums are over all multi-indices $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}_0^n$ that satisfy $|\beta| = \beta_1 + \cdots + \beta_n = 2m$. The last step uses the identities $\sum_{\varepsilon \in \mathcal{E}_n} \varepsilon^\beta = 0$ for every multi-index $\beta \in \mathbb{N}_0^n$ such that one of the numbers β_i is odd and $\sum_{\varepsilon \in \mathcal{E}_n} \varepsilon^\beta = 2^n$ when all the β_i are even. Since $\ell! 2^\ell \leq (2\ell)! \leq \ell! (2\ell)^\ell$ for every integer $\ell \geq 0$ (with equality for $\ell = 0$ and $\ell = 1$), we obtain

$$
\begin{aligned}
\|f_\lambda\|_{L^{2m}}^{2m} &\leq \sum_{|\alpha|=m} \frac{m!(2m)^m}{2^{\alpha_1 + \cdots + \alpha_n} \alpha_1! \cdots \alpha_n!} (\lambda_1^2)^{\alpha_1} \cdots (\lambda_n^2)^{\alpha_n} \\
&= m^m \sum_{|\alpha|=m} \frac{m!}{\alpha_1! \cdots \alpha_n!} (\lambda_1^2)^{\alpha_1} \cdots (\lambda_n^2)^{\alpha_n} \\
&= m^m \left(\sum_{i=1}^n \lambda_i^2 \right)^m.
\end{aligned}
$$

This implies

$$
\|f_\lambda\|_{L^{2m}}^2 \leq m \sum_{i=1}^n \lambda_i^2 \tag{7.3}
$$

for all $n \in \mathbb{N}$ all $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ and all $m \in \mathbb{N}$. For $m = 2$ a slightly better estimate is $4!/(2\alpha)! \leq 3 \cdot (2!/\alpha!)$ for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = 2$, and so

$$
\|f_\lambda\|_{L^4}^2 \leq \sqrt{3} \sum_{i=1}^n \lambda_i^2 \tag{7.4}
$$

for all $n \in \mathbb{N}$ and all $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$.

Now let $p > 2$ and choose an integer $m \geq 2$ such that $2m - 2 \leq p < 2m$. Use Hölder's inequality with exponents

$$q := \frac{2m-2}{2m-p}, \qquad r := \frac{2m-2}{p-2}, \qquad \frac{1}{q} + \frac{1}{r} = 1$$

to obtain

$$\begin{aligned}\int_0^1 |f_\lambda|^p &= \int_0^1 |f_\lambda|^{\frac{2m-p}{m-1}} |f_\lambda|^{\frac{pm-2m}{m-1}} \\ &\leq \left(\int_0^1 |f_\lambda|^2\right)^{\frac{2m-p}{2m-2}} \left(\int_0^1 |f_\lambda|^{2m}\right)^{\frac{p-2}{2m-2}} = \|f_\lambda\|_{L^2}^{\frac{2m-p}{m-1}} \|f_\lambda\|_{L^{2m}}^{\frac{pm-2m}{m-1}}\end{aligned}$$

and hence

$$\begin{aligned}\|f_\lambda\|_{L^p} &\leq \|f_\lambda\|_{L^2}^{\frac{2m-p}{pm-p}} \|f_\lambda\|_{L^{2m}}^{\frac{pm-2m}{pm-p}} \leq \|\lambda\|_2^{\frac{2m-p}{pm-p}} \left(m^{1/2} \|\lambda\|_2\right)^{\frac{pm-2m}{pm-p}} \\ &= m^{\frac{pm-2m}{2pm-2p}} \|\lambda\|_2 \leq \sqrt{\frac{p}{2}+1}\, \|\lambda\|_2 .\end{aligned}$$

This proves the second inequality in (7.2) for $\lambda_i \in \mathbb{R}$ with $B_p := \sqrt{p/2+1}$. Hence it holds for $\lambda_i \in \mathbb{C}$ with $B_p := 2\sqrt{p/2+1} = \sqrt{2p+4}$.

The first inequality in (7.2) holds for $p \geq 2$ with $A_p = 1$ by Hölder's inequality. To prove it for $1 \leq p < 2$, we use Young's inequality $ab \leq \frac{1}{q}a^q + \frac{1}{r}b^r$ with $q := 3/2$, $r := 3$, $a := |\sum_i \varepsilon_i \lambda_i|^{2/3}$, and $b := t|\sum_i \varepsilon_i \lambda_i|^{4/3}$ to obtain

$$t\left|\sum_i \varepsilon_i \lambda_i\right|^2 \leq \frac{2}{3}\left|\sum_i \varepsilon_i \lambda_i\right| + \frac{t^3}{3}\left|\sum_i \varepsilon_i \lambda_i\right|^4 \qquad \text{for all } t \geq 0.$$

Multiply by $3/2$ and take take the average over all $\varepsilon \in \mathcal{E}_n$. Then

$$\|f_\lambda\|_{L^1} \geq \frac{3t}{2}\|f_\lambda\|_{L^2}^2 - \frac{t^3}{2}\|f_\lambda\|_{L^4}^4 \geq \frac{3t}{2}\|\lambda\|_2^2 - \frac{3t^3}{2}\|\lambda\|_2^4 = t\|\lambda\|_2^2 = 3^{-1/2}\|\lambda\|_2 .$$

Here the second step uses (7.4) and the last two steps use $t := 3^{-1/2}\|\lambda\|_2^{-1}$. This proves the first inequality in (7.2) for $p = 1$ with $A_1 = 3^{-1/2}$. For $1 < p < 2$ use the Riesz–Thorin Interpolation Theorem 3.1 qith $q_0 = q_1 = 2$, $p_0 = 1$, $p_1 = 2$, $c_0 = \sqrt{3}$, $c_1 = 1$, $(2/p - 1)/p_0 + (2 - 2/p)/p_1 = 1/p$. Then

$$\|\lambda\|_2 \leq (\sqrt{3})^{\frac{2}{p}-1}\|f_\lambda\|_{L^p} = 3^{\frac{1}{p}-\frac{1}{2}}\|f_\lambda\|_{L^p} .$$

This proves the first inequality in (7.2) for $1 \leq p < 2$ and $\lambda_i \in \mathbb{R}$ with $A_p := 3^{\frac{1}{2}-\frac{1}{p}}$. Hence it holds for $\lambda_i \in \mathbb{C}$ with $A_p := 3^{\frac{1}{2}-\frac{1}{p}}/2$. This proves Theorem 7.1 for $p \geq 1$. □

Remark 7.2. Sharp constants for the Khinchin inequalities in (7.2) (for tuples of real numbers λ_i) were found by Haagerup [14]. They are

$$A_p = 2^{\frac{1}{2}-\frac{1}{p}} \min\left\{1, \left(\frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{3}{2})}\right)^{\frac{1}{p}}\right\},$$
$$B_p = 2^{\frac{1}{2}-\frac{1}{p}} \max\left\{1, \left(\frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{3}{2})}\right)^{\frac{1}{p}}\right\}. \tag{7.5}$$

Another proof that this number A_p is the sharp constant for the first inequality in (7.2) was given by Nazarov–Potkorytov [27].

Proof of Theorem 7.1 for $p > 0$. The second inequality in (7.2) was proved above for all $p > 0$. We prove the first inequality in (7.2) for $p > 0$ following the argument in [27]. If $p \geq 2$ then, by Hölder's inequality with exponent $p/2$, every Lebesgue measurable function $f : [0,1] \to \mathbb{C}$ satisfies $\|f\|_{L^2} \leq \|f\|_{L^p}$, so the first inequality in (7.2) holds with $A_p = 1$, because the ρ_k form an orthonormal sequence in $L^2([0,1])$. Now assume $0 < p < 2$ and define the number $c_p > 0$ and the function $\kappa_p : (0,1] \to (0,\infty)$ by

$$c_p := \int_0^\infty \frac{1-\cos(r)}{r^{p+1}}\,dr \tag{7.6}$$

and

$$\kappa_p(\lambda) := \int_0^\infty \frac{1-|\cos(\lambda r)|^{1/\lambda^2}}{r^{p+1}}\,dr \qquad \text{for } 0 < \lambda \leq 1. \tag{7.7}$$

The function $\kappa_p : (0,1] \to (0,\infty)$ is continuous. We prove that it satisfies

$$\lim_{\lambda\to 0} \kappa_p(\lambda) = \int_0^\infty \frac{1-e^{-r^2/2}}{r^{p+1}}\,dr > 0. \tag{7.8}$$

To see this, choose $\varepsilon > 0$ such that $\log(1-x^2/2) > -x^2$ for $0 < x < \varepsilon$. Since $1-x^2/2 \leq |\cos(x)| \leq 1$ for all $x \in \mathbb{R}$, this implies

$$0 = \log\left(|\cos(\lambda r)|^{1/\lambda^2}\right) \geq \frac{1}{\lambda^2}\log\left(1-\frac{(\lambda r)^2}{2}\right) > -r^2$$

for all $\lambda, r > 0$ such that $0 < r < \varepsilon/\lambda$.

This implies $|\cos(\lambda r)|^{1/\lambda^2} \geq e^{-r^2}$ and hence

$$1 - |\cos(\lambda r)|^{1/\lambda^2} \leq 1 - e^{-r^2} \leq r^2 \qquad \text{for } 0 < r < \varepsilon \leq \frac{\varepsilon}{\lambda}.$$

Thus $(1 - |\cos(\lambda r)|^{1/\lambda^2})/r^{p+1} \leq r^{1-p}$ for $0 < r < \varepsilon$ and the function $r \mapsto r^{1-p}$ is integrable on the interval $(0, \varepsilon)$ because $p < 2$. Second, the inequality $(1 - |\cos(\lambda r)|^{1/\lambda^2})/r^{p+1} \leq r^{p+1}$ holds for all r and the function $r \mapsto r^{p+1}$ is integrable on the interval $[\varepsilon, \infty)$ because $p > 0$. Third, take the logarithm and use l'Hospital's rule to obtain

$$\lim_{\lambda \to 0} |\cos(\lambda r)|^{1/\lambda^2} = e^{-r^2/2} \qquad \text{for all } r > 0.$$

Hence (7.8) follows from the Lebesgue Dominated Convergence Theorem. By (7.8) and continuity, we have

$$\delta_p := \inf_{0<\lambda\leq 1} \int_0^\infty \frac{1 - |\cos(\lambda r)|^{1/\lambda^2}}{r^{p+1}}\, dr > 0. \tag{7.9}$$

Moreover, by definition of the constant $c_p > 0$ in (7.6), we have

$$c_p\, |s|^p = \int_0^\infty \frac{1 - \cos(rs)}{r^{p+1}}\, dr \tag{7.10}$$

for all $s \in \mathbb{R}$. Choose $\lambda_1, \ldots, \lambda_n \in \mathbb{R} \setminus \{0\}$ such that

$$\sum_{k=1}^n |\lambda_k|^2 = 1.$$

Now take $s := \sum_{k=1}^n \lambda_k \rho_k(t)$ in (7.10) to obtain

$$\begin{aligned} c_p \left| \sum_{k=1}^n \lambda_k \rho_k(t) \right|^p &= \int_0^\infty \frac{1 - \cos(r \sum_{k=1}^n \lambda_k \rho_k(t))}{r^{p+1}}\, dr \\ &= \operatorname{Re} \int_0^\infty \frac{1 - \exp(\mathbf{i} r \sum_{k=1}^n \lambda_k \rho_k(t))}{r^{p+1}}\, dr \\ &= \operatorname{Re} \int_0^\infty \frac{1 - \prod_{k=1}^n \exp(\mathbf{i} r \lambda_k \rho_k(t))}{r^{p+1}}\, dr. \end{aligned}$$

Integrate this identity over the interval $0 \le t \le 1$. Then

$$\begin{aligned} c_p \int_0^1 \left| \sum_{k=1}^n \lambda_k \rho_k(t) \right|^p dt &= \mathrm{Re} \int_0^\infty \frac{1 - \int_0^1 \prod_{k=1}^n \exp(\mathbf{i} r \lambda_k \rho_k(t))\, dt}{r^{p+1}}\, dr \\ &= \mathrm{Re} \int_0^\infty \frac{1 - 2^{-n} \sum_{\varepsilon_k = \pm 1} \prod_{k=1}^n \exp(\mathbf{i} r \varepsilon_k \lambda_k)}{r^{p+1}}\, dr \\ &= \int_0^\infty \frac{1 - \prod_{k=1}^n \cos(r\lambda_k)}{r^{p+1}}\, dr. \end{aligned}$$

This formula is called **Haagerup's Integral Representation**. Young's inequality asserts that

$$\prod_{k=1}^n a_k \le \sum_{k=1}^n \frac{1}{p_k} a_k^{p_k}$$

for $a_k \ge 0$ and $1 \le p_k < \infty$ such that $\sum_{k=1}^n 1/p_k = 1$. Take

$$p_k := \frac{1}{\lambda_k^2}, \qquad a_k := |\cos(r\lambda_k)|$$

to obtain

$$\prod_{k=1}^n |\cos(r\lambda_k)| \le \sum_{k=1}^n \lambda_k^2 \, |\cos(r\lambda_k)|^{1/\lambda_k^2} = 1 - \sum_{k=1}^n \lambda_k^2 \left(1 - |\cos(r\lambda_k)|^{1/\lambda_k^2} \right)$$

for all $r > 0$ and hence

$$\begin{aligned} c_p \int_0^1 \left| \sum_{k=1}^n \lambda_k \rho_k(t) \right|^p dt &\ge \int_0^\infty \frac{1 - \prod_{k=1}^n |\cos(r\lambda_k)|}{r^{p+1}}\, dr \\ &\ge \sum_{k=1}^n \lambda_k^2 \int_0^\infty \frac{1 - |\cos(r\lambda_k)|^{1/\lambda_k^2}}{r^{p+1}}\, dr \\ &\ge \delta_p. \end{aligned}$$

Here the last step follows from (7.9). This proves the first inequality in (7.2) with the constant

$$A_p := (\delta_p / c_p)^{1/p}$$

for n-tuples of nonzero real numbers $\lambda_1, \dots, \lambda_n$ with $\sum_{k=1}^n \lambda_k^2 = 1$. Hence it holds for all n-tuples of real numbers by linearity and continuity. This completes the second proof of Theorem 7.1. □

It is shown in Nazarov–Potkorytov [27] that the number $A_p = (\delta_p / c_p)^{1/p}$ in the above proof is the sharp constant in (7.5).

8 The Littlewood–Paley inequality

The Littlewood–Paley decomposition expresses a function $f : \mathbb{R}^n \to \mathbb{C}$ as a sum of functions f_k whose Fourier transforms are supported in the domain $2^{k-1} \le |\xi| \le 2^{k+1}$. Such a decomposition is a powerful tool for obtaining L^p estimates via the Fourier transform. It was already used in the proof of the Mikhlin Multiplier Theorem 6.1.

Definition 8.1. *Fix an integer $n \in \mathbb{N}$ and a smooth function $\phi : \mathbb{R}^n \to \mathbb{R}$ in the Schwartz space $\mathscr{S}(\mathbb{R}^n)$. Assume*

$$\phi(x) = \phi(-x) \qquad \text{for all } x \in \mathbb{R}^n, \tag{8.1}$$

so that the Fourier transform (6.1) *is real valued and satisfies $\widehat{\phi}(\xi) = \widehat{\phi}(-\xi)$ for all $\xi \in \mathbb{R}^n$. The function ϕ is called a* **Littlewood–Paley function** *if it satisfies the conditions*

$$\begin{aligned} \widehat{\phi}(\xi) &> 0, && \text{for } 1/\sqrt{2} \le |\xi| \le \sqrt{2}, \\ \widehat{\phi}(\xi) &\ge 0, && \text{for } 1/2 \le |\xi| \le 2, \\ \widehat{\phi}(\xi) &= 0, && \text{for } |\xi| \notin [1/2, 2], \\ \widehat{\phi}(\xi/2) + \widehat{\phi}(\xi) &= 1, && \text{for } 1 \le |\xi| \le 2, \end{aligned} \tag{8.2}$$

and hence

$$\sum_{k=-\infty}^{\infty} \widehat{\phi}(2^{-k}\xi) = 1 \qquad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}. \tag{8.3}$$

Example 8.2. Let $\beta_0 : [0, \infty) \to [0, 1]$ be a smooth function such that

$$\begin{aligned} &\beta_0(r) = 1, && \text{for } r \le 1, \\ &\beta_0(r) > 0, && \text{for } 1 \le r \le \sqrt{2}, \\ &\beta_0(r) < 1, && \text{for } \sqrt{2} \le r \le 2, \\ &\beta_0(r) = 0, && \text{for } r \ge 2. \end{aligned}$$

Define the function $\beta : \mathbb{R}^n \to \mathbb{R}$ by

$$\beta(\xi) := \beta_0(|\xi|) - \beta_0(2|\xi|) \qquad \text{for } \xi \in \mathbb{R}^n.$$

Then β satisfies the conditions in (8.2) and hence its inverse Fourier transform ϕ is a Littlewood–Paley function.

Theorem 8.3 (Littlewood–Paley). *Fix an integer $n \in \mathbb{N}$, a real number $1 < p < \infty$, and a Littlewood–Paley function $\phi : \mathbb{R}^n \to \mathbb{R}$. Then there exists a constant $c = c(n, p, \phi) \geq 1$ with the following significance. For $k \in \mathbb{Z}$ define the function $\phi_k : \mathbb{R}^n \to \mathbb{R}$ by $\phi_k(x) := 2^{nk}\phi(2^k x)$. Let $f \in L^p(\mathbb{R}^n, \mathbb{C})$ and define the function $S_\phi(f) : \mathbb{R}^n \to [0, \infty]$ by*

$$(S_\phi(f))(x) := \left(\sum_{k=-\infty}^{\infty} |(\phi_k * f)(x)|^2 \right)^{1/2} \qquad \text{for } x \in \mathbb{R}^n. \tag{8.4}$$

Then $(S_\phi f)(x) < \infty$ for almost every $x \in \mathbb{R}^n$ and

$$c^{-1} \|f\|_{L^p} \leq \|S_\phi(f)\|_{L^p} \leq c \|f\|_{L^p}. \tag{8.5}$$

Proof. See page 48. □

Lemma 8.4. *Let $\theta : \mathbb{R}^n \to \mathbb{R}$ be a function in the Schwartz space $\mathscr{S}(\mathbb{R}^n)$, whose Fourier transform has compact support, and assume $\int_{\mathbb{R}^n} \theta(x)\, dx = 1$. For $k \in \mathbb{Z}$ define the function $\theta_k : \mathbb{R}^n \to \mathbb{R}$ by*

$$\theta_k(x) := 2^{kn}\theta(2^k x), \qquad \widehat{\theta}_k(\xi) = \widehat{\theta}(2^{-k}\xi) \qquad \text{for } x, \xi \in \mathbb{R}^n. \tag{8.6}$$

Let $p > 1$. Then

$$\lim_{k\to\infty} \|f - \theta_k * f\|_{L^p} = 0, \qquad \lim_{k\to-\infty} \|\theta_k * f\|_{L^p} = 0. \tag{8.7}$$

for all $f \in L^p(\mathbb{R}^n)$. The convergence is in L^∞ whenever $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and has compact support.

Proof. Since $\|\theta_k\|_{L^1} = \|\theta\|_{L^1} < \infty$ for all $k \in \mathbb{Z}$, the family of convolution operators $L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n) : f \mapsto \theta_k * f$ is uniformly bounded. Hence it suffices to prove (8.7) for smooth functions with compact support.

Let $f \in C_0^\infty(\mathbb{R}^n)$ and $\varepsilon > 0$. For $r > 0$ denote $B_r := \{x \in \mathbb{R}^n \,|\, |x| < r\}$ and choose $R > 0$ such that $\mathrm{supp}(f) \subset B_R$. Second, choose $r > 0$ such that, for all $x, y \in \mathbb{R}^n$,

$$|y| \leq r \qquad \Longrightarrow \qquad |f(x-y) - f(x)| \leq \frac{\varepsilon}{3\|\theta\|_{L^1} \mathrm{Vol}(B_{R+r})^{1/p}}.$$

Third, choose $k_0 \in \mathbb{N}$ so large that

$$\int_{\mathbb{R}^n \setminus B_{2^{k_0} r}} |\theta(y)|\, dy < \frac{\varepsilon}{3\|f\|_{L^p}}.$$

Now let $k \in \mathbb{N}$ such that $k \geq k_0$. Then $\theta_k * f - f = f_k + g_k - c_k f$, where

$$\begin{aligned} f_k(x) &:= \int_{B_r} \theta_k(y)(f(x-y) - f(x))\,dy, \\ g_k(x) &:= \int_{\mathbb{R}^n \setminus B_r} \theta_k(y) f(x-y)\,dy, \\ c_k &:= \int_{\mathbb{R}^n \setminus B_r} \theta_k(y)\,dy. \end{aligned}$$

The function f_k is supported in B_{R+r} and satisfies the inequality

$$|f_k(x)| \leq \|\theta\|_{L^1} \sup_{|y| \leq r} |f(x-y) - f(x)| \leq \frac{\varepsilon}{3\mathrm{Vol}(B_{R+r})^{1/p}}$$

for all $x \in B_{R+r}$. Hence $\|f_k\|_{L^p} \leq \varepsilon/3$. Moreover, it follows from Young's inequality that

$$\|g_k\| \leq \|\theta_k\|_{L^1(\mathbb{R}^n \setminus B_r)} \|f\|_{L^p} = \|\theta\|_{L^1(\mathbb{R}^n \setminus B_{2^k r})} \|f\|_{L^p} < \varepsilon/3,$$

and we have $\|c_k f\|_{L^p} \leq \|\theta_k\|_{L^1(\mathbb{R}^n \setminus B_r)} \|f\|_{L^p} < \varepsilon/3$. Hence $\|\theta_k * f - f\|_{L^p} < \varepsilon$ for every integer $k \geq k_0$ and this proves the first assertion in (8.7). To prove the second assertion, observe that $\|\theta_k\|_{L^p} = 2^{kn(1-1/p)} \|\theta\|_{L^p}$ and so, by Young's inequality, $\|\theta_k * f\|_{L^p} \leq 2^{kn(1-1/p)} \|\theta\|_{L^p} \|f\|_{L^1}$. Since $p > 1$, this shows that

$$\lim_{k \to -\infty} \|\theta_k * f\|_{L^p} = 0.$$

The verification of the assertion about uniform convergence will be omitted. This proves Lemma 8.4. □

Proof of Theorem 8.3. The proof follows the exposition of Machedon [24]. Fix a constant $1 < p < \infty$, let $A_p > 0$ be the constant in the Khinchin inequality (7.2), and, for $k \in \mathbb{N}$, let $\rho_k : [0,1] \to \mathbb{R}$ be the Rademacher function in (7.1). First observe that ϕ_k is a Schwartz function and hence belongs to $L^q(\mathbb{R}^n, \mathbb{C})$ for $q := p/(p-1)$. Hence $\phi_k * f : \mathbb{R}^n \to \mathbb{C}$ is continuous for each k. For $f \in L^p(\mathbb{R}^n, \mathbb{C})$ and $N \in \mathbb{N}$ define the function $S_\phi^N(f) : \mathbb{R}^n \to [0, \infty)$ by

$$(S_\phi^N(f))(x) := \left(\sum_{k=-N}^{N} |(\phi_k * f)(x)|^2 \right)^{1/2} \qquad \text{for } x \in \mathbb{R}^n. \tag{8.8}$$

Define the bijection $\kappa : \mathbb{N} \to \mathbb{Z}$ by $\kappa(2j) = j$ for $j \in \mathbb{N}$ and $\kappa(2j+1) := -j$ for $j \in \mathbb{N}_0$. Then κ restricts to a bijection from $\{1, \dots, 2N+1\}$ to $\{-N, \dots, N\}$ for each $N \in \mathbb{N}$. Hence Theorem 7.1 asserts that

$$\left|(S_\phi^N f)(x)\right| = \left(\sum_{k=-N}^{N} |(\phi_k * f)(x)|^2\right)^{1/2} \le \frac{1}{A_p}\left\|\sum_{j=1}^{2N+1}(\phi_{\kappa(j)} * f)(x)\rho_j\right\|_{L^p([0,1])}$$

for all $x \in \mathbb{R}^n$ and so

$$\begin{aligned}\left\|S_\phi^N f\right\|_{L^p}^p &= \int_{\mathbb{R}^n}\left(\sum_{k=-N}^{N}|(\phi_k * f)(x)|^2\right)^{p/2} dx \\ &\le \left(\frac{1}{A_p}\right)^p \int_0^1\int_{\mathbb{R}^n}\left|\sum_{j=1}^{2N+1}\rho_j(t)(\phi_{\kappa(j)} * f)(x)\right|^p dx\, dt\end{aligned} \tag{8.9}$$

for $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$ and $N \in \mathbb{N}$. For $0 \le t \le 1$ and $N \in \mathbb{N}$ define the function $m_t^N : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ and the operator $T_t^N : L^2(\mathbb{R}^n, \mathbb{C}) \to L^2(\mathbb{R}^n, \mathbb{C})$ by

$$m_t^N := \sum_{j=1}^{2N+1}\rho_j(t)\widehat{\phi}_{\kappa(j)}, \qquad T_t^N f := \sum_{j=1}^{2N+1}\rho_j(t)\phi_{\kappa(j)} * f$$

for $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$. Then $\widehat{T_t^N f} = m_t^N \widehat{f}$ for all $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$ and all t. We prove that m_t^N satisfies condition (6.3) in the Mikhlin Multiplies Theorem 6.1 with a constant $C > 0$ that is independent of t and N. To see this, define

$$C_\alpha := \sup_{\xi\in\mathbb{R}^n}\left|\partial^\alpha\widehat{\phi}(\xi)\right| \qquad \text{for } \alpha \in \mathbb{N}_0^n.$$

Fix an element $\xi \in \mathbb{R}^n \setminus \{0\}$ and choose $k \in \mathbb{Z}$ such that $2^k \le |\xi| \le 2^{k+1}$. Then $\widehat{\phi}_{\kappa(j)}(\xi) = \widehat{\phi}(2^{-\kappa(j)}\xi) = 0$ for $\kappa(j) \ne k, k+1$. Hence

$$\begin{aligned}\left|\partial^\alpha m_t^N(\xi)\right| &\le \left|\rho_{\kappa^{-1}(k)}(t)(\partial^\alpha\widehat{\phi}_k)(\xi) + \rho_{\kappa^{-1}(k+1)}(t)(\partial^\alpha\widehat{\phi}_{k+1})(\xi)\right| \\ &\le \left|(\partial^\alpha\widehat{\phi}_k)(\xi)\right| + \left|(\partial^\alpha\widehat{\phi}_{k+1})(\xi)\right| \\ &= 2^{-|\alpha|k}\left|(\partial^\alpha\widehat{\phi})(2^{-k}\xi)\right| + 2^{-|\alpha|(k+1)}\left|(\partial^\alpha\widehat{\phi})(2^{-k-1}\xi)\right| \\ &\le 2C_\alpha 2^{-|\alpha|(k+1)} \\ &\le 2C_\alpha|\xi|^{-|\alpha|}\end{aligned}$$

for all $\alpha \in \mathbb{N}_0^n$, $\xi \in \mathbb{R}^n \setminus \{0\}$, and $0 \le t \le 1$.

By Theorem 6.1 there exists a constant $c = c(n, p, \phi) > 0$ such that

$$\left\|T_t^N f\right\|_{L^p} \leq c \left\|f\right\|_{L^p} \tag{8.10}$$

for all $f \in L^p(\mathbb{R}^n, \mathbb{C})$, $N \in \mathbb{N}$, and $0 \leq t \leq 1$. By (8.9) and (8.10), we have

$$\left\|S_\phi^N f\right\|_{L^p}^p \leq \left(\frac{1}{A_p}\right)^p \int_0^1 \left\|T_t^N f\right\|_{L^p}^p \, dt \leq \left(\frac{c}{A_p}\right)^p \left\|f\right\|_{L^p}^p \tag{8.11}$$

for all $f \in L^p(\mathbb{R}^n, \mathbb{C})$ and $N \in \mathbb{N}$. For each $f \in L^p(\mathbb{R}^n, \mathbb{C})$ this implies that the monotonically increasing sequence $\big((S_\phi^N f)(x)\big)_{N \in \mathbb{N}}$ is bounded for almost every $x \in \mathbb{R}^n$ (see [33, Lem 1.47]). Moreover, the second inequality in (8.5) follows from (8.11) and the Lebesgue Monotone Convergence Theorem by taking the limit $N \to \infty$.

It suffices to prove the first inequality in (8.5) for real valued functions. Consider the real Hilbert space $H := \ell^2(\mathbb{Z})$ and, for $N \in \mathbb{N}$, define the linear operator $\mathscr{S}^N : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n, H)$ by

$$(\mathscr{S}^N f)(x) := \Big((\phi_k * f)(x)\Big)_{k=-N}^N$$

for $f \in L^p(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. Then

$$\left\|\mathscr{S}^N f\right\|_{L^p} = \left\|S_\phi^N(f)\right\|_{L^p} \leq \frac{c}{A_p} \left\|f\right\|_{L^p}$$

for all $f \in L^p(\mathbb{R}^n, \mathbb{C})$ by (8.11), and so the operator sequence $\mathscr{S}^N$ is uniformly bounded. A function $g \in L^p(\mathbb{R}^n, H)$ assigns to (almost) every $x \in \mathbb{R}^n$ a square summable sequence $(g_k(x))_{k \in \mathbb{Z}}$ of real numbers, such that each function g_k is p-integrable. Thus g is a bi-infinite sequence of L^p functions such that

$$\left\|g\right\|_{L^p(\mathbb{R}^n, H)} = \left\|\left(\sum_{k=-\infty}^{\infty} |g_k|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^n)} < \infty.$$

Define the linear operator $\mathscr{T}^N : L^p(\mathbb{R}^n, H) \to L^p(\mathbb{R}^n)$ by

$$\mathscr{T}^N g := \sum_{k=-N}^{N} \phi_k * g_k$$

for $g \in L^p(\mathbb{R}^n, H)$. Since $\phi_k(-x) = \phi_k(x)$ for all $k \in \mathbb{N}$ and all $x \in \mathbb{R}^n$, $\mathscr{T}^N$ is the dual of the operator $\mathscr{S}^N : L^q(\mathbb{R}^n) \to L^q(\mathbb{R}^n, H)$ with $q := p/(p-1)$.

Hence the operator sequence $\mathscr{T}^N$ is uniformly bounded and so is the operator sequence $\mathscr{U}^N : L^p(\mathbb{R}^n, H) \to L^p(\mathbb{R}^n)$ defined by

$$\mathscr{U}^N g := \sum_{k=-N}^{N} \left(\phi_{k-1} + \phi_k + \phi_{k+1}\right) * g_k \qquad \text{for } g \in L^p(\mathbb{R}^n, H).$$

Thus there exists a constant $C = C(n, p, \phi) > 0$ such that

$$\left\|\mathscr{U}^N g\right\|_{L^p(\mathbb{R}^n)} \leq C \left\|g\right\|_{L^p(\mathbb{R}^n, H)} \tag{8.12}$$

for all $g \in L^p(\mathbb{R}^n, H)$.

Now let $f \in L^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. Then $\|\phi_k * f\|_{L^1} \leq \|\phi\|_{L^1}\|f\|_{L^1}$, so $\phi_k * f$ is integrable for all $k \in \mathbb{Z}$. Moreover, since ϕ is a Littlewood–Paley function, we have $\widehat{\phi}_{k-1}(\xi)+\widehat{\phi}_k(\xi)+\widehat{\phi}_{k+1}(\xi) = 1$ for all $\xi \in \mathbb{R}^n$ that satisfy $2^{k-1} \leq |\xi| \leq 2^{k+1}$ and thus for all $\xi \in \mathrm{supp}(\widehat{\phi}_k)$. This implies

$$\widehat{\mathscr{U}^N \mathscr{S}^{N+1} f} = \sum_{k=-N}^{N} \left(\widehat{\phi}_{k-1} + \widehat{\phi}_k + \widehat{\phi}_{k+1}\right)\widehat{\phi}_k \widehat{f} = \sum_{k=-N}^{N} \widehat{\phi}_k \widehat{f}$$

and hence

$$\mathscr{U}^N \mathscr{S}^{N+1} f = \sum_{k=-N}^{N} \phi_k * f \tag{8.13}$$

for all $N \in \mathbb{N}$ and all $f \in L^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. Since both sides of this equation depend continuously on f in $L^p(\mathbb{R}^n)$, and $L^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, equation (8.13) continues to hold for all $f \in L^p(\mathbb{R}^n)$. Hence

$$\left\|\sum_{k=-N}^{N} \phi_k * f\right\|_{L^p} = \left\|\mathscr{U}^N \mathscr{S}^{N+1} f\right\|_{L^p} \leq C \left\|\mathscr{S}^{N+1} f\right\|_{L^p} = C \left\|S_\phi^{N+1}(f)\right\|_{L^p}$$

for all $f \in L^p(\mathbb{R}^n)$, where C is the constant in (8.12). Take the limit $N \to \infty$ and use Lemma 8.4 to obtain the inequality

$$\|f\|_{L^p} \leq C \left\|S_\phi(f)\right\|_{L^p}$$

for all $f \in L^p(\mathbb{R}^n)$. This proves Theorem 8.3. □

The Littlewood–Paley inequality is used in Corollary 13.9 to relate the Besov space $B_2^{1,p}(\mathbb{R}^n, \mathbb{C})$ to the Sobolev space $W^{1,p}(\mathbb{R}^n, \mathbb{C})$.

9 Maximal regularity for semigroups

Let X be a complex Banach space and let $\{S(t)\}_{t\geq 0}$ be a strongly continuous semigroup of operators, i.e. the map $[0,\infty)\to X : t\mapsto S(t)x$ is continuous for every $x\in X$, $S(0)=\mathrm{id}$, and $S(s+t)=S(s)S(t)$ for all $s,t\geq 0$. Its *infinitesimal generator* is the linear operator $A:\mathrm{dom}(A)\to X$ defined by

$$Ax := \lim_{t\to 0}\frac{1}{t}\big(S(t)x-x\big)$$

for $x\in\mathrm{dom}(A)$ (the linear subspace of all elements $x\in X$ such that the limit exists); the domain is dense and the operator A has a closed graph.

Example 9.1. The main example in the present setting is the Banach space $X := L^p(\mathbb{R}^n,\mathbb{C})$ and the semigroup associated to the heat equation. Its infinitesimal generator is the Laplace operator

$$A=\Delta : W^{2,p}(\mathbb{R}^n,\mathbb{C})\to L^p(\mathbb{R}^n,\mathbb{C}) \tag{9.1}$$

and the semigroup $S(t):L^p(\mathbb{R}^n,\mathbb{C})\to L^p(\mathbb{R}^n,\mathbb{C})$ generated by A is given by

$$S(t)u_0 = K_t * u_0 \tag{9.2}$$

for $t\geq 0$ and $u_0\in L^p(\mathbb{R}^n,\mathbb{C})$, where $K_t:\mathbb{R}^n\to\mathbb{R}$ is the fundamental solution (1.9). That the Laplace operator (9.1) is closed is a consequence of the Calderón–Zygmund inequality in Corollary 6.2. Namely, if two functions $u,f\in L^p(\mathbb{R}^n,\mathbb{C})$ satisfy

$$\int_{\mathbb{R}^n} u\Delta\phi = \int_{\mathbb{R}^n} f\phi$$

for all $\phi\in C_0^\infty(\mathbb{R}^n)$, then the standard local regularity theory, based on the Calderón–Zygmund inequality, asserts that $u\in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{C})$ and $\Delta u=f$. Moreover, using a suitable smooth cutoff function one obtains an inequality

$$\|u\|^p_{W^{2,p}(Q)}\leq c\left(\|u\|^p_{L^p(\Omega)}+\|\Delta u\|^p_{L^p(\Omega)}\right)$$

for $Q:=[0,1]^n$ and $\Omega=(-1,2)^n$. Take a countable sum of such inequalities over appropriately shifted domains with the same constant c to obtain

$$\|u\|^p_{W^{2,p}(\mathbb{R}^n)}\leq 3^n c\left(\|u\|^p_{L^p(\mathbb{R}^n)}+\|\Delta u\|^p_{L^p(\mathbb{R}^n)}\right).$$

This shows that $u\in W^{2,p}(\mathbb{R}^n,\mathbb{C})$. It follows that the operator (9.1) has a closed graph and that the subspace $W^{2,p}(\mathbb{R}^n,\mathbb{C})$ is indeed the domain of the infinitesimal generator of the semigroup (9.2).

If $f : [0,T] \to X$ is continuously differentiable, then the general theory of semigroups asserts that the function $u : [0,T] \to X$, defined by

$$u(t) := \int_0^t S(t-s) f(s)\, ds \qquad \text{for } 0 \le t \le T, \tag{9.3}$$

is continuously differentiable, takes values in the domain of A, and satisfies

$$\dot{u} = Au + f. \tag{9.4}$$

Thus $Au : [0,T] \to X$ is continuous whenever $f : [0,T] \to X$ is continuously differentiable.

Definition 9.2. *Fix a constant $p > 1$. A strongly continuous semigroup $[0,\infty) \to \mathcal{L}(X) : t \mapsto S(t)$ with infinitesimal generator A is called* **maximal p-regular** *if, for every $T > 0$, there exists a constant $c_T > 0$, such that every continuously differentiable function $f : [0,T] \to X$ satisfies the inequality*

$$\left(\int_0^T \left\| A \int_0^t S(t-s) f(s)\, ds \right\|^p dt \right)^{1/p} \le c_T \left(\int_0^T \|f(t)\|^p \, dt \right)^{1/p}. \tag{9.5}$$

This condition is independent of T. The semigroup S is called **uniformly maximal p-regular** *if it is maximal p-regular and the constant in* (9.5) *can be chosen independent of T.*

The next theorem is due to Benedek–Calderón–Panzone [2].

Theorem 9.3 (Benedek–Calderón–Panzone). *Let X be a complex reflexive Banach space and let $S(t)$ be a strongly continuous semigroup on X with infinitesimal generator A. Suppose that*

$$\operatorname{im} S(t) \subset \operatorname{dom}(A^2) \qquad \textit{for all } t > 0$$

and that there exists a constant $c > 0$ such that

$$\left\| A^2 S(t) x \right\| \le \frac{c}{t^2} \|x\| \tag{9.6}$$

for all $t > 0$ and all $x \in X$. If S is (uniformly) maximal p-regular for some $p > 1$ then S is (uniformly) maximal p-regular for every $p > 1$.

Proof. See page 60. □

We reproduce the proof in [2]. It relies on the following three lemmas.

Lemma 9.4. *Let $S(t)$ be a strongly continuous semigroup of operators with infinitesimal generator A that satisfies* (9.6) *for some constant $c > 0$. Then*

$$\int_{2|s|}^{\infty} \|AS(t-s) - AS(t)\|_{\mathcal{L}(X)}\, dt \leq C \tag{9.7}$$

for all $s \in \mathbb{R}$, where $C := c\log(2)$.

Proof. If $t > 2|s|$ then $t > 0$ and $t - s > 0$ and hence

$$\begin{aligned}
\|AS(t-s) - AS(t)\| &= \left\|\int_t^{t-s} A^2 S(r)\, dr\right\| \\
&\leq \left|\int_t^{t-s} \left\|A^2 S(r)\right\|\, dr\right| \\
&\leq c\left|\int_t^{t-s} \frac{1}{r^2}\, dr\right| \\
&= c\left|\frac{1}{t} - \frac{1}{t-s}\right| \\
&= \frac{c|s|}{t(t-s)} \\
&\leq \frac{c|s|}{t(t-|s|)}.
\end{aligned}$$

Hence

$$\begin{aligned}
\int_{2|s|}^{\infty} \|AS(t-s) - AS(t)\|\, dt &\leq \int_{2|s|}^{\infty} \frac{c|s|}{t(t-|s|)}\, dt \\
&= \int_{|s|}^{\infty} \frac{c|s|}{(t+|s|)t}\, dt \\
&= c\int_{|s|}^{\infty} \left(\frac{1}{t} - \frac{1}{t+|s|}\right)\, dt \\
&= c\int_{|s|}^{\infty} \frac{d}{dt}\Big(\log(t) - \log(t+|s|)\Big)\, dt \\
&= c\Big(\log(2|s|) - \log(|s|)\Big) \\
&= c\log(2).
\end{aligned}$$

This proves Lemma 9.4. ∎

Lemma 9.5. *Let $S(t)$ be a strongly continuous semigroup of operators with infinitesimal generator A that satisfies* (9.7) *for some constant $C > 0$. Then the following holds. If $t_0 \ge 0$ and $\varepsilon > 0$ and $f : [0,\infty) \to X$ is a bounded function whose restriction to the interval $(t_0-\varepsilon, t_0+\varepsilon)\cap[0,\infty)$ is continuous and which satisfies*

$$\mathrm{supp}(f) \subset [t_0-\varepsilon, t_0+\varepsilon], \qquad \int_{t_0-\varepsilon}^{t_0+\varepsilon} f(t)\,dt = 0, \tag{9.8}$$

then

$$\int_{t_0+2\varepsilon}^{\infty} \left\| A\int_0^t S(t-s)f(s)\,ds \right\| dt \le C\int_{t_0-\varepsilon}^{t_0+\varepsilon} \|f(t)\|\,dt. \tag{9.9}$$

Proof. Define the function $u : [0,\infty) \to X$ by

$$u(t) := \int_0^t S(t-s)f(s)\,ds \qquad \text{for } t \ge 0.$$

Since $f(t) = 0$ for $t \notin [t_0-\varepsilon, t_0+\varepsilon]$ we have $u(t) = 0$ for $t \le t_0 - \varepsilon$ and

$$u(t) := S(t-t_0-\varepsilon)\int_{t_0-\varepsilon}^{t_0+\varepsilon} S(t_0+\varepsilon-s)f(s)\,ds \in \mathrm{dom}(A) \quad \text{for } t > t_0+\varepsilon.$$

Since f has mean value zero (the fourth equality below) it follows that

$$\begin{aligned}
\int_{t_0+2\varepsilon}^{\infty} \|Au(t)\|\,dt &= \int_{2\varepsilon}^{\infty} \|Au(t_0+t)\|\,dt \\
&= \int_{2\varepsilon}^{\infty} \left\| A\int_{t_0-\varepsilon}^{t_0+\varepsilon} S(t_0+t-s)f(s)\,ds \right\| dt \\
&= \int_{2\varepsilon}^{\infty} \left\| A\int_{-\varepsilon}^{\varepsilon} S(t-s)f(t_0+s)\,ds \right\| dt \\
&= \int_{2\varepsilon}^{\infty} \left\| A\int_{-\varepsilon}^{\varepsilon} (S(t-s)-S(t))f(t_0+s)\,ds \right\| dt \\
&\le \int_{2\varepsilon}^{\infty}\int_{-\varepsilon}^{\varepsilon} \|AS(t-s)-AS(t)\|\,\|f(t_0+s)\|\,ds\,dt \\
&\le \int_{-\varepsilon}^{\varepsilon}\int_{2|s|}^{\infty} \|AS(t-s)-AS(t)\|\,dt\,\|f(t_0+s)\|\,ds \\
&\le C\int_{-\varepsilon}^{\varepsilon} \|f(t_0+s)\|\,ds.
\end{aligned}$$

This proves Lemma 9.5. $\square$

The next lemma is the heart of the proof of Theorem 9.3. It can be stated in two versions, namely for a finite interval $[0, T]$ or for the half infinite interval $[0, \infty)$. We choose the latter version because it is relevant for the Laplace operator on $\mathbb{R}^n$. More precisely, we shall assume that our semigroup is uniformly maximal q-regular for some $q > 1$. This means that there exists a constant $c_q > 0$ such that the inequality

$$\left(\int_0^\infty \left\| A \int_0^t S(t-s) f(s)\, ds \right\|^q dt\right)^{1/q} \leq c_q \left(\int_0^\infty \|f(t)\|^q \, dt\right)^{1/q} \tag{9.10}$$

holds for every continuously differentiable function $f : [0, \infty) \to X$ with compact support. Denote by μ the Lebesgue measure on $[0, \infty)$.

Lemma 9.6. *Let $S(t)$ be a strongly continuous semigroup of operators with infinitesimal generator A that satisfies (9.9) for some constant $C > 0$. Assume also that there exist constants $q > 1$ and $c_q > 0$ such that (9.10) holds for every continuously differentiable function $f : [0, \infty) \to X$ with compact support. Then every continuously differentiable function $f : [0, \infty) \to X$ with compact support satisfies the inequality*

$$\mu\left(\left\{ t \geq 0 \,\middle|\, \left\| A \int_0^t S(t-s) f(s)\, ds \right\| > r \right\}\right) \leq \frac{c_1}{r} \int_0^\infty \|f(t)\| \, dt \tag{9.11}$$

for all $r > 0$, where $c_1 := 4 + 4C + (2c_q)^q$.

Proof. By assumption there exists a unique bounded linear operator

$$\mathcal{T} : L^q([0, \infty), X) \to L^q([0, \infty), X)$$

such that

$$(\mathcal{T} f)(t) = A \int_0^t S(t-s) f(s)\, ds \tag{9.12}$$

for every continuously differentiable function $f : [0, \infty) \to X$ with compact support. For $h \in L^q([0, \infty), X)$ define the function $\kappa_{\mathcal{T}h} : (0, \infty) \to [0, \infty)$ by

$$\kappa_{\mathcal{T}h}(r) := \mu\left(\left\{ t \geq 0 \,\middle|\, \|(\mathcal{T}h)(t)\| > r \right\}\right).$$

Although $\mathcal{T}h$ is only an equivalence class of measurable functions from $[0, \infty)$ to X, the number $\kappa_{\mathcal{T}h}(r)$ is independent of the choice of the representative of this equivalence class and it is finite because $\mathcal{T}h$ is q-integrable. We prove in three steps that the operator $\mathcal{T}$ satisfies (9.11) for some constant $c_1 > 0$.

Step 1. *Let $h \in L^q([0,\infty), X) \cap L^1([0,\infty), X)$ and suppose that there exists a countable collection of compact intervals $I_i \subset \mathbb{R}$ with pairwise disjoint interiors such that h is continuous in the interior of I_i, satisfies*

$$\int_{I_i} h(t)\,dt = 0 \qquad \text{for all } i \in \mathbb{N},$$

and vanishes on the complement of the set $B := \bigcup_{i=1}^\infty I_i$. Then

$$\kappa_{\mathcal{T}h}(r) \le 2\mu(B) + \frac{C}{r}\,\|h\|_{L^1} \tag{9.13}$$

for every $r > 0$.

For $i \in \mathbb{N}$ denote by t_i the center of the interval I_i, let $\varepsilon_i := \frac{1}{2}|I_i|$ be half its length, so $I_i = [t_i - \varepsilon_i, t_i + \varepsilon_i]$, and define the function $h_i : [0,\infty) \to X$ by

$$h_i(t) := \begin{cases} h(t), & \text{if } t \in I_i, \\ 0, & \text{if } t \notin I_i. \end{cases}$$

Then it follows from Lemma 9.5 that

$$\int_{t_i+2\varepsilon_i} \|(\mathcal{T}h_i)(t)\|\;dt \le C\int_{t_i-\varepsilon_i}^{t_i+\varepsilon_i} \|h_i(t)\|\;dt \tag{9.14}$$

for all $i \in \mathbb{N}$. Now recall that the functions $\mathcal{T}h_i$ and $\mathcal{T}h$ are only defined as equivalence classes in $L^q([0,\infty), X)$. Choose representatives of these equivalence classes and denote them by the same letters $\mathcal{T}h_i$ and $\mathcal{T}h$. We claim that there exists a Lebesgue null set $E \subset [0,\infty)$ such that

$$\|(\mathcal{T}h)(t)\| \le \sum_{i=1}^\infty \|(\mathcal{T}h_i)(t)\| \qquad \text{for all } t \in [0,\infty) \setminus E. \tag{9.15}$$

To see this, note that the sequence $\sum_{i=1}^\ell h_i$ converges to h in L^q as ℓ tends to infinity. Hence the sequence $\sum_{i=1}^\ell \mathcal{T}h_i$ converges to $\mathcal{T}h$ in L^q. Hence a subsequence converges to $\mathcal{T}h$ almost everywhere. This means that there exists a Lebesque null set $E \subset \mathbb{R}$ and a sequence of integers $0 < \ell_1 < \ell_2 < \cdots$ such that the sequence $\sum_{i=1}^{\ell_\nu}(\mathcal{T}h_i)(t)$ converges to $(\mathcal{T}h)(t)$ as ν tends to infinity for all $t \in [0,\infty) \setminus E$. Since $\|\sum_{i=1}^{\ell_\nu}(\mathcal{T}h_i)(t)\| \le \sum_{i=1}^\infty \|(\mathcal{T}h_i)(t)\|$ for all t and all ν, this proves (9.15).

Now define the set

$$A := \left(\bigcup_{i=1}^{\infty}[t_i - 2\varepsilon_i, t_i + 2\varepsilon_i]\right) \cap [0,\infty).$$

Then it follows from (9.14) and (9.15) that

$$\begin{aligned}
\int_{\mathbb{R}\setminus A} \|(\mathcal{T}h)(t)\| \, dt &\leq \sum_{i=1}^{\infty} \int_{\mathbb{R}\setminus A} \|(\mathcal{T}h_i)(t)\| \, dt \\
&\leq \sum_{i=1}^{\infty} \int_{t_i+2\varepsilon_i}^{\infty} \|(\mathcal{T}h_i)(t)\| \, dt \\
&\leq C \sum_{i=1}^{\infty} \int_{t_i-\varepsilon_i}^{t_i+\varepsilon_i} \|h_i(t)\| \, dt \\
&= C \int_0^{\infty} \|h(t)\| \, dt.
\end{aligned}$$

Moreover,

$$\mu(A) \leq 2\sum_{i=1}^{\infty} \mu(I_i) = 2\mu(B).$$

Hence

$$\begin{aligned}
\kappa_{\mathcal{T}h}(r) &\leq \mu(A) + \mu\left(\left\{t \in [0,\infty) \setminus A \,\middle|\, \|(\mathcal{T}h)(t)\| > r\right\}\right) \\
&\leq \mu(A) + \frac{1}{r}\int_{[0,\infty)\setminus A} \|(\mathcal{T}h)(t)\| \, dt \\
&\leq 2\mu(B) + \frac{C}{r}\int_0^{\infty} \|h(t)\| \, dt.
\end{aligned}$$

This proves Step 1.

Step 2. *Fix a continuously differentiable function $f : [0,\infty) \to X$ with compact support and a constant $r > 0$. Then there exists a countable collection of compact intervals $I_i \subset [0,\infty)$ with pairwise disjoint interiors such that*

$$|I_i| < \frac{1}{r}\int_{I_i} \|f(t)\| \, dt \leq 2|I_i| \qquad \textit{for all } i \in \mathbb{N} \tag{9.16}$$

and $\|f(t)\| \leq r$ for almost all $t \in [0,\infty) \setminus B$, where $B := \bigcup_{i=1}^{\infty} I_i$. Here $|I_i| = \mu(I_i)$ denotes the length of the interval I_i.

For $k, \ell \in \mathbb{Z}$ define

$$I(k,\ell) := [2^{-\ell}k, 2^{-\ell}(k+1)].$$

Let $\mathscr{I} := \{ I(k,\ell) \,|\, k \in \mathbb{N}_0,\, \ell \in \mathbb{Z} \}$ and define the subset $\mathscr{I}_0 \subset \mathscr{I}$ by

$$\mathscr{I}_0 := \left\{ I \in \mathscr{I} \,\middle|\, \begin{array}{l} r|I| < \int_I \|f\| \text{ and, for all } I' \in \mathscr{I}, \\ I \subsetneq I' \implies \int_{I'} \|f\| \le r|I'| \end{array} \right\}.$$

Then every decreasing sequence of intervals in $\mathscr{I}$ contains at most one element of $\mathscr{I}_0$. Hence every element of $\mathscr{I}_0$ satisfies (9.16) and any two intervals in $\mathscr{I}_0$ have disjoint interiors. Define $B := \bigcup_{I \in \mathscr{I}_0} I$. We prove that

$$t \in [0,\infty) \setminus B, \quad t \in I \in \mathscr{I} \qquad \implies \qquad \frac{1}{|I|}\int_I \|f\| \le r. \tag{9.17}$$

Suppose, by contradiction, that there is a $t \in [0,\infty) \setminus B$ and an interval $I \in \mathscr{I}$ such that $t \in I$ and $r|I| < \int_I \|f\|$. Then, since $\|f\|_{L^1} < \infty$, there is a maximal interval $I \in \mathscr{I}$ such that $t \in I$ and $r|I| < \int_I \|f\|$. Such a maximal interval would be an element of $\mathscr{I}_0$ and hence $t \in B$, a contradiction. This proves (9.17). Now the Lebesgue differentiation theorem [33, Theorem 6.14] asserts that there exists a Lebesgue null set $E \subset [0,\infty) \setminus B$ such that every element of $[0,\infty) \setminus (B \cup E)$ is a Lebesgue point of the integrable function $\|f\|$. By (9.17), every $t \in [0,\infty) \setminus (B \cup E)$ is the intersection point of a decreasing sequence of intervals over which $\|f\|$ has mean value at most r. Hence $\|f(t)\| \le r$ for all $r \in [0,\infty) \setminus (B \cup E)$ by Lebesgue differentiation. This proves Step 2.

Step 3. *We prove the lemma.*

Fix a continuously differentiable function $f : [0,\infty) \to X$ with compact support and a constant $r > 0$. Let I_i be as in Step 2 and define $B := \bigcup_{i=1}^\infty I_i$. Then the set B has Lebesgue measure

$$\mu(B) = \sum_{i=1}^\infty |I_i| \le \frac{1}{r}\sum_{i=1}^\infty \int_{I_i} \|f(t)\|\, dt \le \frac{1}{r}\int_0^\infty \|f(t)\|\, dt.$$

For each subset $A \subset [0,\infty)$ denote by $\chi_A : [0,\infty) \to \{0,1\}$ the characteristic function of A. Define $g, h : [0,\infty) \to X$ by

$$g := f\chi_{[0,\infty)\setminus B} + \sum_{i=1}^\infty \frac{1}{|I_i|}\left(\int_{I_i} f(t)\, dt\right)\chi_{I_i}, \qquad h := f - g.$$

Then $f = g + h$ and $\|g\|_{L^1} \le \|f\|_{L^1}$ and $\|h\|_{L^1} \le 2\|f\|_{L^1}$. Since h vanishes on $\mathbb{R} \setminus B$ and has mean value zero on I_i for all i it follows from Step 1 that

$$\kappa_{\mathcal{T}h}(r) \le 2\mu(B) + \frac{C}{r}\|h\|_{L^1} \le \frac{2+2C}{r}\|f\|_{L^1}.$$

Moreover, it follows from Step 2 that $\|g(t)\| \le 2r$ almost everywhere. Hence it follows from [33, Lemma 7.36] that

$$\begin{aligned}\kappa_{\mathcal{T}g}(r) &\le \frac{1}{r^q}\int_0^\infty \|(\mathcal{T}g)(t)\|^q\, dt \le \frac{c_q^q}{r^q}\int_0^\infty \|g(t)\|^q\, dt \\ &\le \frac{c_q^q}{r^q}\int_0^\infty (2r)^{q-1}\|g(t)\|\, dt \le \frac{2^{q-1}c_q^q}{r}\|f\|_{L^1}.\end{aligned}$$

Hence

$$\kappa_{\mathcal{T}f}(2r) \le \kappa_{\mathcal{T}g}(r) + \kappa_{\mathcal{T}h}(r) \le \frac{2+2C+2^{q-1}c_q^q}{r}\|f\|_{L^1}.$$

This proves Step 3 and Lemma 9.6. □

Proof of Theorem 9.3. Let $q > 1$ and assume that S is uniformly maximal q-regular and satisfies (9.6). Then, by Lemma 9.6, there exist constants $c_1 > 0$ and $c_q > 0$ such that the operator

$$\mathcal{T} : L^q([0,\infty), X) \to L^q([0,\infty), X)$$

in (9.12) satisfies the inequalities

$$\|\mathcal{T}f\|_{L^q} \le c_q\|f\|_{L^q}, \qquad \sup_{r>0} r\kappa_{\mathcal{T}f}(r) \le c_1\|f\|_{L^1}$$

for all $f \in L^q([0,\infty), X) \cap L^1([0,\infty), X)$. Hence, by Corollary 4.3,

$$\|\mathcal{T}f\|_{L^p} \le c_p\|f\|_{L^p}, \qquad c_p := 2\left(\frac{p(q-1)}{(q-p)(p-1)}\right)^{1/p} c_1^{\frac{q-p}{p(q-1)}} c_q^{\frac{q(p-1)}{p(q-1)}}$$

for $1 < p < q$ and $f \in L^q([0,\infty), X) \cap L^1([0,\infty), X) \subset L^p([0,\infty), X)$. This shows that S is uniformly maximal p-regular for $1 < p < q$. Moreover the dual semigroup S^* also satisfies (9.6) and is uniformly maximal q^*-regular, where $1/q + 1/q^* = 1$. Hence, by what we have just proved, it is uniformly maximal p^*-regular for $1 < p^* < q^*$. By duality this implies that S is uniformly maximal p-regular for $1/p + 1/p^* = 1$ and hence for all p with $q < p < \infty$. This proves Theorem 9.3. □

10 Coifman–Weiss Transference

Fix a positive integer n and a constant $1 \le p < \infty$. For an integrable function $\phi \in L^1(\mathbb{R}^n, \mathbb{C})$ and a complex Banach space X define

$$N_p^X(\phi) := \sup_{f \in L^p(\mathbb{R}^n, X)\setminus\{0\}} \frac{\|\phi * f\|_{L^p}}{\|f\|_{L^p}}.$$

For $X = \mathbb{C}$ abbreviate

$$N_p(\phi) := N_p^{\mathbb{C}}(\phi) = \sup_{f \in L^p(\mathbb{R}^n, \mathbb{C})\setminus\{0\}} \frac{\|\phi * f\|_{L^p}}{\|f\|_{L^p}}. \tag{10.1}$$

The map $N_p : L^1(\mathbb{R}^n, \mathbb{C}) \to [0, \infty)$ is a norm and $N_p(\phi) \le \|\phi\|_{L^1}$ for all $\phi \in L^1(\mathbb{R}^n, \mathbb{C})$, by Young's inequality. For many functions this inequality is strict and it may then be interesting to obtain estimates in terms of the number $N_p(\phi)$ instead of the L^1-norm. A specific instance of this is Theorem 10.4 below, which is an example of the **Coifman–Weiss transference principle**.

Lemma 10.1. *Fix a constant $1 \le p < \infty$, let $(Y, \mathcal{B}, \nu)$ be a measure space, and denote by $L^p(Y, \nu)$ the Banach space of complex valued L^p functions on $(Y, \mathcal{B}, \nu)$. Then $N_p^{L^p(Y,\nu)}(\phi) = N_p(\phi)$ for all $\phi \in L^1(\mathbb{R}^n, \mathbb{C})$.*

Proof. Denote by μ the Lebesgue measure on $\mathbb{R}^n$ and fix a Lebesgue integrable function $\phi : \mathbb{R}^n \to \mathbb{C}$. Let $\mathscr{Y} \subset L^p(Y, \nu)$ be the space of (equivalence classes of) $\mathcal{B}$-measurable step functions on Y with ν-finite support (up to equality ν-almost everywhere). Then $\mathscr{Y}$ is dense in $L^p(Y, \nu)$ by [33, Lem 4.12]. Hence the space of (equivalence classes of) Lebesgue measurable step functions on $\mathbb{R}^n$ with values in $\mathscr{Y}$ (up to equality almost everywhere) is dense in $L^p(\mathbb{R}^n, L^p(Y, \nu))$. Now fix a Lebesgue measurable step function $f : \mathbb{R}^n \to Y$. Then there exists a finite sequence of pairwise disjoint Lebesgue measurable sets $A_1, \dots, A_\ell \subset \mathbb{R}^n$ with $\mu(A_i) < \infty$ for all i, and a finite sequence $g_1, \dots, g_\ell \in \mathscr{Y}$ such that

$$f(t) = \sum_{i=1}^{\ell} \chi_{A_i}(t) g_i$$

for $t \in \mathbb{R}^n$. With this notation, we have

$$\|f\|^p_{L^p(\mathbb{R}^n, L^p(Y,\nu))} = \sum_{i=1}^{\ell} \mu(A_i) \int_Y |g_i|^p \, d\nu.$$

Now abbreviate

$$f^y(t) := \big(f(t)\big)(y) = \sum_{i=1}^{\ell} \chi_{A_i}(t) g_i(y)$$

for $t \in \mathbb{R}^n$ and $y \in Y$. Then

$$\big((\phi * f)(t)\big)(y) = \sum_{i=1}^{\ell} \left(\int_{A_i} \phi(t-s)\, d\mu(s) \right) g_i(y) = (\phi * f^y)(t)$$

for all $t \in \mathbb{R}^n$ and all $y \in Y$. Since each of the functions $g_i : Y \to \mathbb{C}$ is supported on a subset of finite measure, it follows from Fubini's Theorem for σ-finite measure spaces (see [33, Thm 7.20]) that

$$\|\phi * f\|_{L^p(\mathbb{R}^n, L^p(Y,\nu))}^p = \int_Y \left(\int_{\mathbb{R}^n} |(\phi * f^y)(t)|^p \, d\mu(t) \right) d\nu(y)$$

Now it follows from the definition of $N_p(\phi)$ that

$$\begin{aligned} \int_{\mathbb{R}^n} |(\phi * f^y)(t)|^p \, d\mu(t) &\leq N_p(\phi)^p \int_{\mathbb{R}^n} |f^y(t)|^p \, d\mu(t) \\ &= N_p(\phi)^p \sum_{i=1}^{\ell} \mu(A_i) \, |g_i(y)|^p \end{aligned}$$

for all $y \in Y$. Integrate this inequality over Y to obtain

$$\begin{aligned} \|\phi * f\|_{L^p(\mathbb{R}^n, L^p(Y,\nu))}^p &\leq N_p(\phi)^p \sum_{i=1}^{\ell} \mu(A_i) \int_Y |g_i|^p \, d\nu \\ &= N_p(\phi)^p \, \|f\|_{L^p(\mathbb{R}^n, L^p(Y,\nu))}^p . \end{aligned}$$

Since the space of measurable step functions $f : \mathbb{R}^n \to \mathscr{Y}$ with support of finite Lebesgue measure is dense in $L^p(\mathbb{R}^n, L^p(Y,\nu))$, this implies

$$\|\phi * f\|_{L^p(\mathbb{R}^n, L^p(Y,\nu))} \leq N_p(\phi) \, \|f\|_{L^p(\mathbb{R}^n, L^p(Y,\nu))}$$

for all $f \in L^p(\mathbb{R}^n, L^p(Y,\nu))$. Thus

$$N_p^{L^p(Y,\nu)}(\phi) \leq N_p(\phi).$$

The converse inequality is obvious and this proves Lemma 10.1. $\square$

The next lemma is a special case of the transference principle in Coifman–Weiss [7, 8]. Let X be a complex Banach space and denote by $\mathrm{Aut}(X)$ the group of invertible bounded complex linear operators on X.

Lemma 10.2 (Coifman–Weiss). *Let $U : \mathbb{R}^n \to \mathrm{Aut}(X)$ be a strongly continuous group homomorphisms and suppose that there is an $M \geq 1$ such that*

$$\|U(s)\| \leq M \qquad \text{for all } s \in \mathbb{R}^n.$$

Then

$$\left\| \int_{\mathbb{R}^n} \phi(s) U(s) x \, ds \right\| \leq M^2 N_p^X(\phi) \, \|x\| \tag{10.2}$$

for all $x \in X$, all $p \geq 1$, and all $\phi \in L^1(\mathbb{R}^n, \mathbb{C})$.

Proof. Fix a constant $p \geq 1$ and an element $x \in X$. Since both sides of the inequality (10.2) depend continuously on $\phi \in L^1(\mathbb{R}^n, \mathbb{C})$, and $C_0^\infty(\mathbb{R}^n, \mathbb{C})$ is dense in $L^1(\mathbb{R}^n, \mathbb{C})$, it suffices to prove the estimate for $\phi \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$. Fix a smooth function $\phi : \mathbb{R}^n \to \mathbb{C}$ with compact support and choose vectors $a, b \in \mathbb{R}^n$ such that $a_i < b_i$ for all i and $\mathrm{supp}(\phi) \subset [a_1, b_1] \times \cdot \times [a_n, b_n]$. For $T = (T_1, \dots, T_n) \in \mathbb{R}^n$ with $T_i > b_i$ define the function $f_T : \mathbb{R}^n \to X$ by

$$f_T(s) := \begin{cases} U(T-s)x, & \text{if } 0 \leq s_i \leq T_i - a_i \text{ for all } i, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\|f_T\|_{L^p} \leq \prod_i (T_i - a_i)^{1/p} M \|x\|$. Let $t = (t_1, \dots, t_n) \in \mathbb{R}^n$ such that $b_i \leq t_i \leq T_i$ for all i. Then $0 \leq t_i - s_i \leq T_i - a_i$ for all $s_i \in [a_i, b_i]$ and hence

$$(\phi * f_T)(t) = \int_{\mathbb{R}^n} \phi(s) f_T(t-s) \, ds = U(T-t) \int_{\mathbb{R}^n} \phi(s) U(s) x \, ds.$$

Thus $\|\int_{\mathbb{R}^n} \phi(s) U(s) x \, ds\| = \|U(t-T)(\phi * f_T)(t)\| \leq M \, \|(\phi * f_T)(t)\|$ when $b_i \leq t_i \leq T_i$. Take the L^p norm over the product of these intervals to obtain

$$\begin{aligned} \left\| \int_{\mathbb{R}^n} \phi(s) U(s) x \, ds \right\| &\leq \frac{M \, \|\phi * f_T\|_{L^p}}{\prod_i (T_i - b_i)^{1/p}} \leq \frac{M N_p^X(\phi) \, \|f_T\|_{L^p}}{\prod_i (T_i - b_i)^{1/p}} \\ &\leq \prod_i \left(\frac{T_i - a_i}{T_i - b_i} \right)^{1/p} M^2 N_p^X(\phi) \|x\|. \end{aligned}$$

The inequality (10.2) follows by taking the limit $T_i \to \infty$. This completes the proof of Lemma 10.2. □

The Coifman–Weiss transference principle extends to strongly continuous positive contraction semigroups on L^p spaces. Fix a number $1 < p < \infty$ and any measure space $(Y, \mathcal{B}, \nu)$. Let $L^p(Y, \nu)$ be the Banach space of complex valued L^p functions on $(Y, \mathcal{B}, \nu)$. A strongly continuous semigroup $S(t)$ of bounded linear operators on $L^p(Y, \nu)$ is called a **contraction semigroup** if $\|S(t)\| \le 1$ for all $t \ge 0$. It is called a **positive semigroup** if it is the complexification of a strongly continuous semigroup on the real L^p space and

$$f \ge 0 \qquad \Longrightarrow \qquad S(t)f \ge 0$$

for all $t \ge 0$ and every real valued L^p function $f : Y \to \mathbb{R}$.

Theorem 10.3. *Let $S(t)$ be a strongly continuous positive contraction semigroup on $L^p(Y, \nu)$. Then there exist*

- *a measure space $(\widetilde{Y}, \widetilde{\mathcal{B}}, \widetilde{\nu})$,*
- *linear operators $\iota : L^p(Y, \nu) \to L^p(\widetilde{Y}, \widetilde{\nu})$ and $\pi : L^p(\widetilde{Y}, \widetilde{\nu}) \to L^p(Y, \nu)$,*
- *and a strongly continuous group of linear operators $\{\widetilde{S}(t)\}_{t\in\mathbb{R}}$ on $L^p(\widetilde{Y}, \widetilde{\nu})$,*

satisfying the following conditions.

(i) *For all $t \ge 0$ we have*

$$\pi \circ \widetilde{S}(t) \circ \iota = S(t) \tag{10.3}$$

and, in particular, $\pi \circ \iota = \mathrm{id}$.

(ii) *ι is an isometric embedding, i.e.*

$$\|\iota(f)\|_{L^p(\widetilde{Y},\widetilde{\nu})} = \|f\|_{L^p(Y,\nu)} \qquad \text{for all } f \in L^p(Y, \nu).$$

(iii) *π is a contraction, i.e.*

$$\|\pi(\widetilde{f})\|_{L^p(Y,\nu)} \le \|\widetilde{f}\|_{L^p(\widetilde{Y},\widetilde{\nu})} \qquad \text{for all } \widetilde{f} \in L^p(\widetilde{Y}, \widetilde{\nu}).$$

(iv) *$\widetilde{S}(t) : L^p(\widetilde{Y}, \widetilde{\nu}) \to L^p(\widetilde{Y}, \widetilde{\nu})$ is a group of isometries, i.e.*

$$\|\widetilde{S}(t)\widetilde{f}\|_{L^p(\widetilde{Y},\widetilde{\nu})} = \|\widetilde{f}\|_{L^p(\widetilde{Y},\widetilde{\nu})} \qquad \text{for all } t \in \mathbb{R} \text{ and all } \widetilde{f} \in L^p(\widetilde{Y}, \widetilde{\nu}).$$

(v) *The operator $\widetilde{S}(t)$ is positive for every $t \in \mathbb{R}$, i.e. it is the complexification of a bounded linear operator on the real L^p space and*

$$\widetilde{f} \ge 0 \implies \widetilde{S}(t)\widetilde{f} \ge 0$$

for every L^p-funtion $\widetilde{f} : \widetilde{Y} \to \mathbb{R}$.

Proof. The proof of Theorem 10.3 is a rather lengthy construction. The starting point is the case where Y is a finite set and the semigroup $S(t)$ is replaced by the iterates of a single operator S. In this situation the assertion is a theorem of Akcoglu–Sucheston [1]. We explain their result, following the exposition in Fendler [11, Theorem 2.2.1].

Assume Y is a finite set. Then the space $L^p(Y,\nu)$ of complex valued L^p functions can be identified with the vector space $\mathbb{C}^n$ and it suffices to consider the standard ℓ^p-norm $\|y\|_p := \left(\sum_{i=1}^n |y_i|^p\right)^{1/p}$. (Otherwise conjugate first by the diagonal matrix whose diagonal entries are the pth roots of the weights.) A positive linear operator is then a matrix $S = (S_{ji})_{i,j=1}^n \in \mathbb{R}^{n\times n}$ with nonnegative entries. Given such a matrix with ℓ^p operator norm $\|S\| \le 1$, the **Akcoglu–Sucheston Theorem** asserts that there is a σ-finite measure space $(\widetilde{Y}, \widetilde{\mathcal{B}}, \widetilde{\nu})$, a Banach space isometry $\widetilde{S} : L^p(\widetilde{Y},\widetilde{\nu}) \to L^p(\widetilde{Y},\widetilde{\nu})$, an isometric embedding $\iota : \mathbb{C}^n \to L^p(\widetilde{Y},\widetilde{\nu})$, and a contracting projection $\pi : L^p(\widetilde{Y},\widetilde{\nu}) \to \mathbb{C}^n$ such that $\pi \circ \widetilde{S}^k \circ \iota = S^k$ for every integer $k \ge 0$.

Let $S \in \mathbb{R}^{n\times n}$ be a matrix with nonnegative entries and $\|S\| \le 1$. Denote the transposed matrix by $S^* = (S_{ij})_{i,j=1}^n$ and think of it as as the dual operator on $\mathbb{C}^n$, equipped with the norm $\|\cdot\|_q$, where $1/p + 1/q = 1$. Denote by $\mathbb{R}^n_+ \subset \mathbb{R}^n$ the subset of vectors with nonnegative entries and, for $y = (y_1,\dots,y_n) \in \mathbb{R}^n_+$ and $r > 0$, abbreviate $y^r := (y_1^r,\dots,y_n^r)$.

For $I \subset \{1,\dots,n\}$ denote by $\iota_I : \mathbb{R}^I \to \mathbb{R}^n$ the obvious injection and by $\pi_I : \mathbb{R}^n \to \mathbb{R}^I$ the obvious projection. Define $S_I := S \circ \iota_I : \mathbb{R}^I \to \mathbb{R}^n$. Then

$$y \in \mathbb{R}^I_+, \quad \|S_I y\|_p = \|S_I\|\,\|y\|_p \quad \Longrightarrow \quad S_I^*(S_I y)^{p-1} = \|S_I\|^p\, y^{p-1}. \qquad (10.4)$$

To see this, let $y \in \mathbb{R}^I_+$ with $\|S_I y\|_p = \|S_I\|\,\|y\|_p$ and define $x := (S_I y)^{p-1}$. Then $\|x\|_q^q = \|S_I y\|_p^p$, hence $\|x\|_q = \|S_I y\|_p^{p-1}$, and therefore

$$\langle y, S_I^* x\rangle \le \|y\|_p \|S_I^* x\|_q \le \|S_I\| \|y\|_p \|x\|_q = \|S_I y\|_p^p = \langle S_I y, x\rangle = \langle y, S_I^* x\rangle .$$

This implies $\langle y, S_I^* x\rangle = \|y\|_p \|S_I^* x\|_q$, hence $S_I^* x$ is a positive multiple of y^{p-1}, and so $S_I^* x = \|S_I\|^p\, y^{p-1}$. This proves (10.4).

Next observe that any two vectors $y, z \in \mathbb{R}^n_+$ satisfy

$$\left.\begin{array}{c} \langle y, z\rangle = 0, \\ S^*(Sy)^{p-1} \le y^{p-1} \end{array}\right\} \quad\Longrightarrow\quad \left\{\begin{array}{c} \langle Sy, Sz\rangle = 0, \\ S(y+z)^{p-1} = (Sy)^{p-1} + (Sz)^{p-1}. \end{array}\right. \qquad (10.5)$$

To see this, note that $0 \le \langle Sz, (Sy)^{p-1}\rangle = \langle z, S^*(Sy)^{p-1}\rangle \le \langle z, y^{p-1}\rangle = 0$ and hence $\langle Sz, (Sy)^{p-1}\rangle = 0$. This shows that when $(Sy)_i \ne 0$ we must have $(Sz)_i = 0$ and vice versa. Hence $(S(y+z))^{p-1} = (Sy)^{p-1} + (Sz)^{p-1}$ and this proves (10.5).

Next observe that, since $\|S\| \leq 1$, there exists a vector $u \in \mathbb{R}_+^n$ with strictly positive entries such that

$$S^*(Su)^{p-1} \leq u^{p-1}. \tag{10.6}$$

To see this, choose a vector $y \in \mathbb{R}_+^n$ such that $\|y\|_p = 1$ and $\|Sy\|_p = \|S\|$. Then $S^*(Sy)^{p-1} = \|S\|^p y^{p-1} \leq y^{p-1}$ by (10.4). Assume that the set

$$I := \{i \in \{1, \dots, n\} \,|\, y_i = 0\}$$

is nonempty and choose $z \in \mathbb{R}_+^I$ such that $\|z\|_p = 1$ and $\|S_I z\|_p = \|S_I\|$. Then $S_I^*(S_I z)^{p-1} = \|S_I\|^p z^{p-1} \leq z^{p-1}$ by (10.4) and $\langle y, \iota_I(z)\rangle = 0$ by definition. Hence $\langle Sy, S\iota_I(z)\rangle = 0$ by (10.5) and so

$$S^*(S\iota_I(z))^{p-1} = \iota_I(S_I^*(S\iota_I(z))^{p-1}) \leq \iota_I(z)^{p-1}.$$

With this understood, it follows also from (10.5) that

$$S^*(Sx)^{p-1} = S^*(Sy)^{p-1} + S^*(S\iota_I(z))^{p-1} \leq y^{p-1} + \iota_I(z)^{p-1} = x^{p-1}.$$

If $\{i \in \{1, \dots, n\} \,|\, x_i = 0\} \neq \emptyset$, continue by induction to obtain a vector $u \in \mathbb{R}^n$ with positive entries that satisfies (10.6).

The measure space $(\widetilde{Y}, \widetilde{\mathcal{B}}, \widetilde{\nu})$ will be constructed as a Borel subset of $\mathbb{R}^2$ equipped with the Borel σ-algebra $\widetilde{\mathcal{B}}$ and the restriction of the Lebesgue measure to $\widetilde{\mathcal{B}}$. Choose two n-tuples of pairwise disjoint compact intervals of length one, denoted by $I_1, \dots, I_n$ and $J_1, \dots, J_n$, and define

$$Z_0 := \bigcup_{i=1}^n I_i \times J_i.$$

Now choose a bi-infinite sequence of pairwise disjoint compact rectangles

$$Z_k \subset \mathbb{R}^2, \qquad k \in \mathbb{Z} \setminus \{0\},$$

such that each Z_k has positive Lebesgue measure and $Z_k \cap Z_0 = \emptyset$ for all k. By (10.6) there exists a vector $u = (u_1, \dots, u_n) \in \mathbb{R}_+^n$ such that

$$S^*(Su)^{p-1} \leq u^{p-1} \qquad \text{and } u_i > 0 \text{ for all } i.$$

Let $v := Su$ and observe that $v_j = 0$ if and only if $S_{ji} = 0$ for all i. Define

$$I := \{1, \dots, n\}, \qquad J := \{j \in I \,|\, v_j \neq 0\}.$$

Moreover, define

$$\begin{aligned} \xi_{ij} &:= S_{ji}\frac{u_i}{v_j} \quad \text{for } (i,j) \in I \times J, \\ \eta_{ij} &:= S_{ji}\left(\frac{v_j}{u_i}\right)^{p-1} \quad \text{for } (i,j) \in I \times I. \end{aligned} \tag{10.7}$$

Since $v_j = (Su)_j = \sum_{i=1}^n S_{ji}u_i$ for all j, we have

$$\sum_{i=1}^n \xi_{ij} = 1 \qquad \text{for all } j \in J.$$

Since $\sum_{j=1}^n S_{ji}v_j^{p-1} = (S^*v^{p-1})_i = (S^*(Su)^{p-1})_i \le u_i^{p-1}$ for all i, we have

$$\sum_{j=1}^n \eta_{ij} \le 1 \qquad \text{for all } i \in I.$$

For $j \in J$ divide the interval I_j into n compact subintervals I_{ij} of length ξ_{ij} whose interiors are disjoint. For $i \in I$ choose n compact subintervals $J_{ij} \subset J_i$ of length η_{ij} whose interiors are disjoint. (Thus J_{ij} is a point when $j \notin J$.) Define

$$\begin{aligned} Q &:= \bigcup_{i,j \in I \times J} Q_{ij}, \qquad Q_{ij} := I_{ij} \times J_j \subset I_j \times J_j, \\ R &:= \bigcup_{i,j \in I \times J} R_{ij}, \qquad R_{ij} := I_i \times J_{ij} \subset I_i \times J_i. \end{aligned}$$

For $i \in I$ and $j \in J$ there is a unique affine diffeomorphism

$$\tau_{ij} : R_{ij} \to Q_{ij}, \quad \tau_{ij}(x,y) = (a_{ij}x + b_{ij}, c_{ij}y + d_{ij}), \quad a_{i,j}, c_{ij} > 0.$$

Now define a transformation $\tau : \bigcup_{k\in\mathbb{Z}} Z_k \to \bigcup_{k\in\mathbb{Z}} Z_k$ as follows.

(a) The restriction of τ to R is the transformation from R to Q given by $\tau|_{R_{ij}} := \tau_{ij} : R_{ij} \to Q_{ij}$ (up to a set of measure zero).

(b) If $R \neq Z_0$, define τ as a piecewise affine bijection (up to a set of measure zero) from $Z_0 \setminus R$ onto Z_1 and from Z_k onto Z_{k+1} for $k \in \mathbb{N}$. If $R = Z_0$ define τ as the identity on $\bigcup_{k=1}^\infty Z_k$.

(c) If $Q \neq Z_0$, define τ as a piecewise affine bijection (up to a set of measure zero) from Z_{-1} onto $Z_0 \setminus Q$ and from Z_{-k-1} onto Z_{-k} for $k \in \mathbb{N}$. If $Q = Z_0$ define τ as the identity on $\bigcup_{k=1}^\infty Z_{-k}$.

With this definition the map

$$\tau : \bigcup_{k\in\mathbb{Z}} Z_k \to \bigcup_{k\in\mathbb{Z}} Z_k$$

is a diffeomorphism on the complement of a set of measure zero. Now define

$$\widetilde{Y} := \bigcup_{k\in\mathbb{Z}} Z_k,$$

let $\widetilde{\mathcal{B}}$ be the Borel σ-algebra on $\widetilde{Y}$, and denote by $\widetilde{\nu} = \mu$ the restriction of the Lebesgue measure to $\widetilde{\mathcal{B}}$. Then the pushforward measure $\tau_*\mu$ is absolutely continuous with respect to the Lebesgue measure and vice versa, by construction. Thus there exists a measurable function

$$\rho : \widetilde{Y} \to (0, \infty),$$

unique almost everywhere, such that

$$(\tau_*\mu)(B) = \mu(\tau^{-1}(B)) = \int_B \rho \, d\mu$$

for every Borel set $B \subset \widetilde{Y}$. Since $\tau^{-1}(Q_{ij}) = R_{ij}$ and the restriction of τ to R_{ij} is an affine diffeomorphism, the restriction $\rho|_{Q_{ij}} : Q_{ij} \to (0, \infty)$ is the constant function

$$\rho|_{Q_{ij}} = \frac{\mu(R_{ij})}{\mu(Q_{ij})} = \frac{\eta_{ij}}{\xi_{ij}} = \left(\frac{v_j}{u_i}\right)^p \qquad \text{for } (i, j) \in I \times J. \tag{10.8}$$

(See equation (10.7).) Define the linear map $\widetilde{S} : L^p(\widetilde{Y}, \widetilde{\nu}) \to L^p(\widetilde{Y}, \widetilde{\nu})$ by

$$\widetilde{S}\widetilde{f} := \rho^{1/p}(f \circ \tau^{-1}) \tag{10.9}$$

for $\widetilde{f} \in L^p(\widetilde{Y}, \widetilde{\nu})$, define the projection $\pi : L^p(\widetilde{Y}, \widetilde{\nu}) \to \mathbb{C}^n$ by

$$\pi(\widetilde{f})_j := \int_{I_j\times J_j} \widetilde{f} \, d\mu \tag{10.10}$$

for $j = 1, \ldots, n$, and define the injection $\iota : \mathbb{C}^n \to L^p(\widetilde{Y}, \widetilde{\nu})$ by

$$\iota(y) := \sum_{i=1}^n y_i \chi_{I_i\times J_i} \tag{10.11}$$

for $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Then $\widetilde{S}$ is an isometry, ι is an isometric embedding, π is a contracting projection, and $\pi \circ \iota = \mathrm{id}$.

To prove that $\pi \circ \widetilde{S}^k \circ \iota = S^k$, call a function $\widetilde{f} \in L^p(\widetilde{Y}, \widetilde{\nu})$ **admissible** if it depends only on the first variable in $\mathbb{R}^2$ and vanishes on $\bigcup_{k=1}^{\infty} Z_{-k}$.

Claim. *For every $y \in \mathbb{C}^n$ the function $\iota(y)$ is admissible. If $\widetilde{f} \in L^p(\widetilde{Y}, \widetilde{\nu})$ is admissible then $\widetilde{S}\widetilde{f}$ is admissible and $\pi(\widetilde{S}\widetilde{f}) = S\pi(\widetilde{f})$.*

The first assertion follows directly from the definition. Hence assume that $\widetilde{f} \in L^p(\widetilde{Y}, \widetilde{\nu})$ is admissible. Then $\widetilde{S}\widetilde{f}$ depends only on the first variable, because τ is piecewise affine and τ^{-1} maps the vertical rectangle Q_{ij} onto the horizontal rectangle R_{ij}. And $\widetilde{S}\widetilde{f}$ vanishes on Z_{-k} for each $k \in \mathbb{N}$ because

$$\tau^{-1}(Z_{-k}) \subset Z_{-k} \cup Z_{-k-1}.$$

So $\widetilde{S}\widetilde{f}$ is admissible. If $j \notin J$ then

$$I_j \times J_j \subset Z_0 \setminus Q,$$

hence $\widetilde{S}\widetilde{f}$ vanishes on $I_j \times J_j$, because $\tau^{-1}(Z_0 \setminus Q) \subset Z_{-1}$, and therefore

$$\pi(\widetilde{S}\widetilde{f})_j = 0 = (S\pi(\widetilde{f}))_j.$$

If $j \in J$ then, by (10.7), (10.8), (10.9), and (10.10),

$$\begin{aligned}
\pi(\widetilde{S}\widetilde{f})_j &= \int_{I_j \times J_j} \rho^{1/p}(\widetilde{f} \circ \tau^{-1}) d\mu = \sum_{i=1}^n \int_{Q_{ij}} \rho^{1/p-1}(\widetilde{f} \circ \tau^{-1}) d(\tau_* \mu) \\
&= \sum_{i=1}^n \left(\frac{v_j}{u_i} \right)^{1-p} \int_{R_{ij}} \widetilde{f} d\tau = \sum_{i=1}^n S_{ji} \frac{1}{\eta_{ij}} \int_{R_{ij}} \widetilde{f} d\mu \\
&= \sum_{i=1}^n S_{ji} \int_{I_i \times J_i} \widetilde{f} d\mu = (S\pi(\widetilde{f}))_j.
\end{aligned}$$

The last but one equation holds because $\widetilde{f}$ depends only on the first variable. This proves the claim. It follows directly from the claim that $\pi \circ \widetilde{S}^k \circ \iota = S^k$ for every integer $k \geq 0$ and this proves the Akcoglu–Sucheston theorem [1].

The Akcoglu–Sucheston theorem was carried over to positive contractions on general L^p spaces by Coifman–Rochberg–Weiss [7]. The extension to strongly continuous positive contraction semigroups on general L^p spaces as stated above can be found in the manuscript of Fendler [11, Theorem 4.2.1], to which we also refer for further details of the proof of Theorem 10.3. □

Theorem 10.4 (Coifman–Weiss). *Let $(Y, \mathcal{B}, \nu)$ be a measure space and let $S(t)$ be a strongly continuous positive contraction semigroup on $L^p(Y, \nu)$. Then*

$$\left\| \int_0^\infty \phi(s) S(s) f \, ds \right\|_{L^p(Y,\nu)} \le N_p(\phi) \, \|f\|_{L^p(Y,\nu)} \tag{10.12}$$

for all $f \in L^p(Y, \nu)$, all $p > 1$, and all $\phi \in L^1([0, \infty), \mathbb{C})$.

Proof. Let

$$\widetilde{X} := L^p(\widetilde{Y}, \widetilde{\nu})$$

and the unitary group

$$\mathbb{R} \to \mathcal{L}(\widetilde{X}) : t \mapsto \widetilde{S}(t)$$

and the linear operators

$$\iota : L^p(Y, \nu) \to L^p(\widetilde{Y}, \widetilde{\nu}), \qquad \pi : L^p(\widetilde{Y}, \widetilde{\nu}) \to L^p(Y, \nu),$$

be as in Theorem 10.3. Thus

$$S(t) = \pi \circ \widetilde{S}(t) \circ \iota$$

for all $t \ge 0$, the map $\iota : L^p(Y, \nu) \to L^p(\widetilde{Y}, \widetilde{\nu})$ is an isometric embedding, and

$$\left\| \pi \left(\widetilde{f} \right) \right\|_{L^p(Y,\nu)} \le \left\| \widetilde{f} \right\|_{L^p(\widetilde{Y},\widetilde{\nu})}$$

for all $\widetilde{f} \in L^p(\widetilde{Y}, \widetilde{\nu})$. Then

$$N_p^{\widetilde{X}}(\phi) = N_p(\phi)$$

by Lemma 10.1. Since $\widetilde{S}(t)$ is a unitary group on $\widetilde{X}$, it then follows from Lemma 10.2 with $M = 1$ that

$$\begin{aligned}
\left\| \int_0^\infty \phi(s) S(s) f \, ds \right\|_{L^p(Y,\nu)} &= \left\| \pi \left(\int_0^\infty \phi(s) \widetilde{S}(s) \iota(f) \, ds \right) \right\|_{L^p(Y,\nu)} \\
&\le \left\| \int_0^\infty \phi(s) \widetilde{S}(s) \iota(f) \, ds \right\|_{L^p(\widetilde{Y},\widetilde{\nu})} \\
&\le N_p(\phi) \, \|\iota(f)\|_{L^p(\widetilde{Y},\widetilde{\nu})} \\
&= N_p(\phi) \, \|f\|_{L^p(Y,\nu)}
\end{aligned}$$

for all $f \in L^p(Y, \nu)$. This proves Theorem 10.4. □

11 Proof of Theorem 1.1

In view of Theorem 9.3 it suffices to prove Theorem 1.1 for $p = q$. For this case expositions based on different methods can be found in the paper [20] by Lamberton and in the book [22] by Lieberman. The proof in [22] is close in spirit to the proof of the Calderón–Zygmund inequality in Theorem 5.2. In our proof of Theorem 1.1 for $p = q$ we follow the approach of Lamberton [20] which is based on semigroup theory, Stein interpolation, the Mikhlin and Marcinkiewicz multiplier theorems, and Coifman–Weiss transference.

Lemma 11.1. *Let $\zeta \in \mathbb{C} \setminus (-\infty, 0]$. Then the linear operator*

$$\zeta - \Delta : W^{2,2}(\mathbb{R}^n, \mathbb{C}) \to L^2(\mathbb{R}^n, \mathbb{C})$$

is bijective and satisfies the following estimates.
(i) *If $\mathrm{Re}(\zeta) \geq 0$ and $\zeta \neq 0$ then, for all $f \in L^2(\mathbb{R}^n, \mathbb{C})$*

$$\left\|(\zeta - \Delta)^{-1} f\right\|_{L^2} \leq \frac{1}{|\zeta|} \|f\|_{L^2}.$$

(ii) *If $\mathrm{Re}(\zeta) < 0$ and $\mathrm{Im}(\zeta) \neq 0$ then, for all $f \in L^2(\mathbb{R}^n, \mathbb{C})$,*

$$\left\|(\zeta - \Delta)^{-1} f\right\|_{L^2} \leq \frac{1}{|\mathrm{Im}(\zeta)|} \|f\|_{L^2}.$$

Proof. The proof has three steps. Throughout the proof all norms are L^2 norms and all inner products are L^2 inner products for compactly supported functions on $\mathbb{R}^n$ with values in $\mathbb{R}$, $\mathbb{R}^n$, or $\mathbb{C}$.

Step 1. *Let $u, v \in C_0^\infty(\mathbb{R}^n)$ and let $\zeta = \xi + \mathbf{i}\eta \in \mathbb{C}$ with $\xi \geq 0$. Define $f, g \in C_0^\infty(\mathbb{R}^n)$ by $\zeta(f + \mathbf{i}g) := (\zeta - \Delta)(u + \mathbf{i}v)$ so that*

$$\begin{aligned} \Delta u &= \xi u - \eta v - \xi f + \eta g, \\ \Delta v &= \eta u + \xi v - \eta f - \xi g. \end{aligned} \tag{11.1}$$

Then $\|u\|_{L^2}^2 + \|v\|_{L^2}^2 \leq \|f\|_{L^2}^2 + \|g\|_{L^2}^2$.

Integration by parts shows that

$$\begin{aligned} -\|\nabla u\|^2 &= \langle u, \Delta u \rangle = \xi \|u\|^2 - \eta \langle u, v \rangle - \xi \langle u, f \rangle + \eta \langle u, g \rangle, \\ -\|\nabla v\|^2 &= \langle v, \Delta v \rangle = \eta \langle u, v \rangle + \xi \|v\|^2 - \eta \langle v, f \rangle - \xi \langle v, g \rangle, \\ -\langle \nabla v, \nabla u \rangle &= \langle v, \Delta u \rangle = \xi \langle u, v \rangle - \eta \|v\|^2 - \xi \langle v, f \rangle + \eta \langle v, g \rangle, \\ -\langle \nabla u, \nabla v \rangle &= \langle u, \Delta v \rangle = \eta \|u\|^2 + \xi \langle u, v \rangle - \eta \langle u, f \rangle - \xi \langle u, g \rangle. \end{aligned} \tag{11.2}$$

Add the first two inequalities in (11.2) and subtract the last two identities in (11.2) to obtain

$$\begin{aligned} \xi\left(\|u\|^2+\|v\|^2\right) &\le \xi\big(\langle u,f\rangle+\langle v,g\rangle\big)+\eta\big(\langle v,f\rangle-\langle u,g\rangle\big) \\ \eta\left(\|u\|^2+\|v\|^2\right) &= \eta\big(\langle u,f\rangle+\langle v,g\rangle\big)-\xi\big(\langle v,f\rangle-\langle u,g\rangle\big). \end{aligned} \tag{11.3}$$

Multiply the (first) inequality in (11.3) by $\xi \ge 0$ and multiply the (second) equation in (11.3) by η and take the sum to obtain

$$(\xi^2+\eta^2)\big(\|u\|^2+\|v\|^2\big) \le (\xi^2+\eta^2)\big(\langle u,f\rangle+\langle v,g\rangle\big).$$

Since $\xi^2+\eta^2>0$ it follows that

$$\|u\|^2+\|v\|^2 \le \langle u,f\rangle+\langle v,g\rangle \le \frac{1}{2}\big(\|u\|^2+\|v\|^2+\|f\|^2+\|g\|^2\big).$$

This proves Step 1.

Step 2. *Let $u,v \in C_0^\infty(\mathbb{R}^n)$ and let $\zeta=\xi+\mathbf{i}\eta \in \mathbb{C}$ with $\xi<0$ and $\eta \ne 0$. Define $f,g \in C_0^\infty(\mathbb{R}^n)$ by $\zeta(f+\mathbf{i}g) := (\zeta-\Delta)(u+\mathbf{i}v)$ so that* (11.1) *holds. Then*

$$\|u\|_{L^2}^2+\|v\|_{L^2}^2 \le \left|\eta^{-1}\zeta\right|^2\left(\|f\|_{L^2}^2+\|g\|_{L^2}^2\right).$$

We argue as in the proof of Step 1 to obtain (11.3). Since $\eta \ne 0$ the (second) equation in (11.3) can be written in the form

$$\begin{aligned} \|u\|^2+\|v\|^2 &= \left\langle u, f+\eta^{-1}\xi g\right\rangle+\left\langle v, g-\eta^{-1}\xi f\right\rangle \\ &\le \|u\|\left\|f+\eta^{-1}\xi g\right\|+\|v\|\left\|g-\eta^{-1}\xi f\right\| \\ &\le \frac{1}{2}\left(\|u\|^2+\|v\|^2\right)+\frac{1}{2}\left(\left\|f+\eta^{-1}\xi g\right\|^2+\left\|g-\eta^{-1}\xi f\right\|^2\right) \\ &= \frac{1}{2}\left(\|u\|^2+\|v\|^2\right)+\frac{1}{2}\big(1+|\eta^{-1}\xi|^2\big)\left(\|f\|^2+\|g\|^2\right). \end{aligned}$$

This proves Step 2.

Step 3. *We prove the lemma.*

If $u \in C_0^\infty(\mathbb{R}^n,\mathbb{C})$ then, by Step 1 and Step 2, we have $\|u\| \le |\zeta|^{-1}\|(\zeta-\Delta)u\|$ for $\mathrm{Re}(\zeta)>0$ and $\|u\| \le |\mathrm{Im}(\zeta)|^{-1}\|(\zeta-\Delta)u\|$ for $\mathrm{Re}(\zeta)\le 0$, $\mathrm{Im}(\zeta)\ne 0$. These inequalities continue to hold for all $u \in W^{2,2}(\mathbb{R}^n,\mathbb{C})$ and show that the linear operator $\zeta-\Delta : W^{2,2}(\mathbb{R}^n,\mathbb{C}) \to L^2(\mathbb{R}^n,\mathbb{C})$ is injective. By the regularity argument outlined in Example 9.1 it is also surjective. This proves Lemma 11.1. □

It follows from part (i) of Lemma 11.1 and the Hille–Yoshida–Phillips Theorem that the operator Δ generates a strongly continuous contraction semigroup $S(t)$ on $L^2(\mathbb{R}^n, \mathbb{C})$. In Example 9.1 we have seen that this semigroup is given by

$$S(t)u_0 = K_t * u_0 = \int_{\mathbb{R}^n} K_t(x-y)u_0(y)\,dy \tag{11.4}$$

for $u_0 \in L^2(\mathbb{R}^n, \mathbb{C})$ and $t \geq 0$, where $K_t : \mathbb{R}^n \to \mathbb{R}$ is the fundamental solution (1.9) of the heat equation on $\mathbb{R}^n$. Now fix a constant $1 < p < \infty$. Since $\|K_t\|_{L^1} = 1$ for all $t > 0$ it follows from Young's inequality that

$$\|S(t)u_0\|_{L^p} \leq \|u_0\|_{L^p} \tag{11.5}$$

for all $t \geq 0$ and all $u_0 \in L^2(\mathbb{R}^n, \mathbb{C}) \cap L^p(\mathbb{R}^n, \mathbb{C})$. In [20] Lamberton proved a general L^p regularity theorem for strongly continuous semigroups on L^2 that satisfy the estimates of Lemma 11.1 and (11.5). Here is his result. As before all L^p spaces are understood as complex L^p spaces.

Theorem 11.2 (**Lamberton**). *Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and let $A : \mathrm{dom}(A) \to L^2(X, \mu)$ be the infinitesimal generator of a strongly continuous semigroup $S(t)$ on $L^2(X, \mu)$. Assume A and S satisfy the following.*

(I) *If $\zeta \in \mathbb{C} \setminus (-\infty, 0]$ then the operator $\zeta - A : \mathrm{dom}(A) \to L^2(X, \mu)$ is bijective and, for all $f \in L^2(X, \mu)$,*

$$\left\|(\zeta - A)^{-1} f\right\|_{L^2} \leq \begin{cases} |\zeta|^{-1} \|f\|_{L^2}, & \text{if } \mathrm{Re}(\zeta) > 0, \\ |\mathrm{Im}(\zeta)|^{-1} \|f\|_{L^2}, & \text{if } \mathrm{Re}(\zeta) \leq 0,\ \mathrm{Im}(\zeta) \neq 0. \end{cases} \tag{11.6}$$

(II) *Let $1 < p < \infty$. Then $S(t)$ defines a strongly continuous positive contraction semigroup on L^p, i.e. if $t \geq 0$ and $u_0 \in L^2(X, \mu) \cap L^p(X, \mu)$ then*

$$\|S(t)u_0\|_{L^p} \leq \|u_0\|_{L^p}, \tag{11.7}$$

and, if u_0 is real valued, then so is $S(t)u_0$ and $u_0 \geq 0$ implies $S(t)u_0 \geq 0$.

Then, for every real number $1 < p < \infty$, there exists a constant $C_p > 0$ such that every continuously differentiable function $f : \mathbb{R} \to L^2(X, \mu)$ with compact support satisfies the inequality

$$\left(\int_{-\infty}^{\infty} \left\|A \int_{-\infty}^{t} S(t-s)f(s)\,ds\right\|_{L^p}^p dt\right)^{1/p} \leq C_p \left(\int_{-\infty}^{\infty} \|f(t)\|_{L^p}^p\,dt\right)^{1/p}. \tag{11.8}$$

Proof. See page 90 □

Condition (I) in Theorem 11.2 implies that A generates what in semigroup theory is called an **analytic semigroup**. Lamberton's theorem in [20] is more general in that he merely assumes that A generates an analytic semigroup. This corresponds to a more general estimate than (11.6) and, in particular, the spectrum of A is then not required to be contained in the negative real axis but only in a cone of angle less than π around the negative real axis. The discussion in this section is restricted to the case where (11.6) holds because that suffices for our intended application.

A key ingredient in Lamberton's proof is the **Fourier transform** for L^2 functions on $\mathbb{R}$ with values in $L^2(X,\mu)$. It is the unique bounded linear operator $\mathscr{F}: L^2(\mathbb{R}, L^2(X,\mu)) \to L^2(\mathbb{R}, L^2(X,\mu))$ given by

$$(\mathscr{F}(u))(\xi) := \widehat{u}(\xi) := \int_{-\infty}^{\infty} e^{-\mathbf{i}\xi t} u(t)\, dt \tag{11.9}$$

for $\xi \in \mathbb{R}$ and $u \in L^2(\mathbb{R}, L^2(X,\mu)) \cap L^1(\mathbb{R}, L^2(X,\mu))$. Its inverse is

$$(\mathscr{F}^{-1}(\widehat{u}))(t) = u(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\mathbf{i}\xi t} \widehat{u}(\xi)\, d\xi \tag{11.10}$$

for $t \in \mathbb{R}$ and $\widehat{u} \in L^2(\mathbb{R}, L^2(X,\mu)) \cap L^1(\mathbb{R}, L^2(X,\mu))$. Using the Fourier transform and the operator A one can define an operator family on $L^2(\mathbb{R}, L^2(X,\mu))$ as follows. As before we use the notation $\mathbb{S} = \{z \in \mathbb{C} \,|\, 0 \le \mathrm{Re}(z) \le 1\}$.

Definition 11.3. *Fix two real numbers θ_0, θ_1 such that*

$$0 < \theta_0 < \frac{\pi}{2} < \theta_1 < \pi$$

and define the function $\rho : \mathbb{S} \times \mathbb{R} \to \mathbb{C}$ by

$$\rho(z,\xi) := |\xi| e^{\mathbf{i}((1-z)\theta_0 + z\theta_1)\mathrm{sign}(\xi)} \tag{11.11}$$

for $z \in \mathbb{S}$ and $\xi \in \mathbb{R}$, where $\mathrm{sign}(\xi) := 1$ for $\xi \ge 0$ and $\mathrm{sign}(\xi) := 0$ for $\xi < 0$. For $z \in \mathbb{S}$ define the operator $U_z : L^2(\mathbb{R}, L^2(X,\mu)) \to L^2(\mathbb{R}, L^2(X,\mu))$ by

$$U_z := \mathscr{F}^{-1} \circ \widehat{U}_z \circ \mathscr{F}, \qquad (\widehat{U}_z \widehat{f})(\xi) := \rho(z,\xi)\big(\rho(z,\xi) - A\big)^{-1} \widehat{f}(x) \tag{11.12}$$

for $\widehat{f} \in L^2(\mathbb{R}, L^2(X,\mu))$ and $\xi \in \mathbb{R}$. It is well defined because the operator norm of the bounded linear operator $\rho(z,\xi)\big(\rho(z,\xi)-A\big)^{-1}$ on $L^2(X,\mu)$ satisfies the inequality

$$\left\|\rho(z,\xi)\big(\rho(z,\xi) - A\big)^{-1}\right\|_{\mathcal{L}(L^2)} \le \begin{cases} 1, & \text{if } |(1-\lambda)\theta_0 + \lambda\theta_1| \le \frac{\pi}{2}, \\ \frac{1}{\sin(\theta_1)}, & \text{if } |(1-\lambda)\theta_0 + \lambda\theta_1| > \frac{\pi}{2}, \end{cases} \tag{11.13}$$

for all $z = \lambda + \mathbf{i}\tau \in \mathbb{S}$ and all $\xi \in \mathbb{R}$, by (11.6).

With this definition in place, Lamberton's proof of Theorem 11.2 requires the following seven lemmas. The first lemma shows that for a suitable choice of λ the operator U_λ agrees with the operator for which we intend to prove an L^p estimate.

Lemma 11.4. *Choose a constant* $0 < \lambda < 1$ *such that*

$$(1-\lambda)\theta_0 + \lambda\theta_1 = \frac{\pi}{2} \tag{11.14}$$

and let $f : \mathbb{R} \to L^2(X,\mu)$ *be a continuously differentiable function with compact support. Then*

$$(U_\lambda f)(t) = A\int_{-\infty}^t S(t-s)f(s)\,ds + f(t) \tag{11.15}$$

for $t \geq 0$ *and* $(U_\lambda f)(t) = 0$ *for* $t < 0$.

Proof. Define the function $v : \mathbb{R} \to L^2(X,\mu)$ by

$$v(t) := A\int_{-\infty}^t S(t-s)f(s)\,ds + f(t) \qquad \text{for } t \in \mathbb{R}.$$

For $\delta \geq 0$ define $u_\delta, v_\delta : \mathbb{R} \to L^2(X,\mu)$ by

$$u_\delta(t) := \int_{-\infty}^t e^{-\delta s}S(s)f(t-s)\,ds, \qquad v_\delta(t) := \dot{u}_\delta(t) = (A-\delta)u_\delta(t) + f(t)$$

for $t \in \mathbb{R}$. (Here we use the fact that u_δ takes values in the domain of A and is continuously differentiable.) Then $v_0 = v$. With $g := \dot{f}$ the function $v_\delta = \dot{u}_\delta$ can also be expressed in the form

$$v_\delta(t) = \int_{-\infty}^t e^{-\delta(t-s)}S(t-s)g(s)\,ds \qquad \text{for } t \geq 0. \tag{11.16}$$

For $\delta > 0$ the function v_δ belongs to $L^2(\mathbb{R}, L^2(X,\mu))$. Denote its Fourier transform by $\widehat{v}_\delta := \mathscr{F}(v_\delta)$. The Fourier transform of the operator valued function $[0,\infty) \to \mathcal{L}(L^2(X,\mu)) : t \mapsto e^{-\delta t}S(t)$ is given by $(\mathbf{i}\xi + \delta - A)^{-1}$. Hence $\widehat{v}_\delta(\xi) = (\mathbf{i}\xi + \delta - A)^{-1}\widehat{g}(\xi) = \mathbf{i}\xi(\mathbf{i}\xi + \delta - A)^{-1}\widehat{f}(\xi)$ by (11.16). Take the limit $\delta \to 0$ to obtain $v \in L^2(\mathbb{R}, L^2(X,\mu))$ and

$$\widehat{v}(\xi) := (\mathscr{F}(v))(\xi) = \mathbf{i}\xi(\mathbf{i}\xi - A)^{-1}\widehat{f}(\xi).$$

Now it follows from equations (11.11) and (11.14) that $\rho(\lambda,\xi) = \mathbf{i}\xi$ for all $\xi \in \mathbb{R}$. This implies $\widehat{U_\lambda f}(\xi) = \mathbf{i}\xi(\mathbf{i}\xi - A)^{-1}\widehat{f}(\xi) = \widehat{v}(\xi)$ for all ξ and hence $U_\lambda f = v$. This proves Lemma 11.4. □

The main task is now to prove a uniform L^q estimate for the operators $U_{\mathbf{i}\tau}$ for all $\tau \in \mathbb{R}$. This task will be accomplished in Lemma 11.10 below. The first step is to express the operator U_0 in Definition 11.3 in terms of the semigroup. The proof will take up the next four pages.

Lemma 11.5. *Define the function $\phi_0 : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ by*

$$\phi_0(r, s) := \frac{(s^2 - r^2)\cos(\theta_0)}{\pi(r^2 + s^2 - 2rs\sin(\theta_0))^2} \qquad \text{for } (r, s) \in \mathbb{R} \times (0, \infty). \tag{11.17}$$

This function is integrable over $I \times [\varepsilon, \infty)$ for every $\varepsilon > 0$ and every compact interval $I \subset \mathbb{R}$, and the operator $U_0 : L^2(\mathbb{R}, L^2(X, \mu)) \to L^2(\mathbb{R}, L^2(X, \mu))$ is given by

$$(U_0 f)(t) = \lim_{\varepsilon \to 0} \int_\varepsilon^\infty \int_{-\infty}^\infty \phi_0(r, s) S(s) f(t - r)\, dr ds \tag{11.18}$$

for every $t \in \mathbb{R}$ and every function $f : \mathbb{R} \to L^2(X, \mu)$ that is twice continuously differentiable and has compact support.

Proof. The proof has six steps.

Step 1. *Define the functions $\widehat{K}, \widehat{K}_\varepsilon : \mathbb{R} \to \mathcal{L}(L^2(X, \mu))$ by*

$$\widehat{K}(\xi) := \rho(0, \xi)\big(\rho(0, \xi) - A\big)^{-1}, \qquad \widehat{K}_\varepsilon(\xi) := e^{-\rho(0,\xi)\varepsilon} S(\varepsilon)\widehat{K}(\xi).$$

Then $\|\widehat{K}_\varepsilon(\xi)\|_{\mathcal{L}(L^2)} \le \|\widehat{K}(\xi)\|_{\mathcal{L}(L^2)} \le 1$ and

$$\lim_{\varepsilon \to 0} \|\widehat{K}_\varepsilon(\xi) f - \widehat{K}(\xi) f\|_{L^2} = 0$$

for all $\xi \in \mathbb{R}$ and all $f \in L^2(X, \mu)$.

By (11.11),

$$\rho(0, \xi) = |\xi| e^{\mathbf{i}\theta_0 \operatorname{sign}(\xi)} = |\xi| \cos(\theta_0) + \mathbf{i}\xi \sin(\xi_0).$$

Since $0 < \theta_0 < \pi/2$ the number $\rho(0, \xi)$ has nonnegative real part and hence $|e^{-\rho(0,\xi)\varepsilon}| \le 1$ for all $\xi \in \mathbb{R}$ and all $\varepsilon > 0$. Moreover, the operator norm of $S(t) : L^2(X, \mu) \to L^2(X, \mu)$ is at most one for all t, by (11.7). Hence

$$\|\widehat{K}_\varepsilon(\xi)\|_{\mathcal{L}(L^2)} \le \|\widehat{K}(\xi)\|_{\mathcal{L}(L^2)} \le 1,$$

where the last inequality follows from (11.6). This proves the first assertion. Since the operator $\widehat{K}(\xi)$ commutes with the semigroup, it follows from the definitions that $\|\widehat{K}_\varepsilon(\xi) f - \widehat{K}_\varepsilon(\xi) f\|_{L^2} \le \|e^{-\rho(0,\xi)\varepsilon} S(\varepsilon) f - f\|_{L^2}$ for $\xi \in \mathbb{R}$, $\varepsilon > 0$, and $f \in L^2(X, \mu)$. Hence the second assertion follows from the fact that S is a strongly continuous semigroup. This proves Step 1.

Step 2. *Let $f : \mathbb{R} \to L^2(X, \mu)$ be a twice continuously differentiable function with compact support. Then $\widehat{f} : \mathbb{R} \to L^2(X, \mu)$ is integrable and*

$$(U_0 f)(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\mathbf{i}t\xi} \widehat{K}(\xi) \widehat{f}(\xi)\, d\xi \tag{11.19}$$

for all $t \in \mathbb{R}$.

Define $g := \ddot{f}$. Then $\widehat{g}(\xi) = -\xi^2 \widehat{f}(\xi)$ and hence

$$\|\widehat{f}(\xi)\|_{L^2} = \frac{1}{\xi^2} \|\widehat{g}(\xi)\|_{L^2} \le \frac{1}{\xi^2} \|\ddot{f}\|_{L^1(\mathbb{R}, L^2(X,\mu))}$$

for all $\xi \in \mathbb{R} \setminus \{0\}$. Since $\widehat{f}$ is continuous (near $\xi = 0$) it follows that the function $\widehat{f} : \mathbb{R} \to L^2(X, \mu)$ is integrable. Hence Step 2 follows from the fact that the Fourier transform of $U_0 f$ is the function $\widehat{U_0 f} = \widehat{K} \widehat{f} : \mathbb{R} \to L^2(X, \mu)$, by definition, and hence is integrable as well.

Step 3. *Let f be as in Step 2 and define the function $U_{0,\varepsilon} f : \mathbb{R} \to L^2(X, \mu)$ by*

$$(U_{0,\varepsilon} f)(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\mathbf{i}t\xi} \widehat{K}_\varepsilon(\xi) \widehat{f}(\xi)\, d\xi. \tag{11.20}$$

for $t \in \mathbb{R}$ and $\varepsilon > 0$. Then

$$\lim_{\varepsilon \to 0} \|(U_{0,\varepsilon} f)(t) - (U_0 f)(t)\|_{L^2} = 0$$

for all $t \in \mathbb{R}$.

By Step 2,

$$\begin{aligned} \|(U_0 f)(t) - (U_{0,\varepsilon} f)(t)\|_{L^2} &= \frac{1}{2\pi} \left\| \int_{-\infty}^{\infty} e^{\mathbf{i}t\xi} \left(\widehat{K}(\xi)\widehat{f}(\xi) - \widehat{K}_\varepsilon(\xi)\widehat{f}(\xi) \right) d\xi \right\|_{L^2} \\ &\le \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\| \widehat{K}(\xi)\widehat{f}(\xi) - \widehat{K}_\varepsilon(\xi)\widehat{f}(\xi) \right\|_{L^2} d\xi \end{aligned}$$

for all $\xi \in \mathbb{R}$. Moreover,

$$\begin{aligned} \left\| \widehat{K}(\xi)\widehat{f}(\xi) - \widehat{K}_\varepsilon(\xi)\widehat{f}(\xi) \right\|_{L^2} &\le 2 \left\| \widehat{f}(\xi) \right\|_{L^2}, \\ \lim_{\varepsilon \to 0} \left\| \widehat{K}(\xi)\widehat{f}(\xi) - \widehat{K}_\varepsilon(\xi)\widehat{f}(\xi) \right\|_{L^2} &= 0 \end{aligned}$$

for all $\xi \in \mathbb{R}$ by Step 2. Since the function $\mathbb{R} \to \mathbb{R} : \xi \mapsto \|\widehat{f}(\xi)\|_{L^2}$ is integrable, the assertion of Step 3 follows from the Lebesgue dominated convergence theorem.

Step 4. *The function ϕ_0 in* (11.17) *is integrable over $I \times [\varepsilon, \infty)$ for every $\varepsilon > 0$ and every compact interval $I \subset \mathbb{R}$.*

Since $0 < \theta_0 < \pi/2$ we have $\sin(\theta_0) < 1$. Since $2rs \le r^2 + s^2$ it follows that

$$|\phi_0(r,s)| \le \frac{\cos(\theta_0)|s^2 - r^2|}{\pi(r^2+s^2)^2(1-\sin(\theta_0))^2} \le \frac{\cos(\theta_0)}{\pi(1-\sin(\theta_0))^2}\frac{1}{r^2+s^2}. \qquad (11.21)$$

Since

$$\int_\varepsilon^\infty \frac{1}{r^2+s^2}\,ds = \frac{1}{|r|}\int_{\varepsilon/r}^\infty \frac{ds}{1+s^2} \le \frac{1}{|r|}\int_{\varepsilon/|r|}^\infty \frac{ds}{s^2} = \frac{1}{\varepsilon}$$

the function $(r,s) \mapsto (r^2+s^2)^{-1}$ is integrable over $I \times [\varepsilon, \infty)$ for every $\varepsilon > 0$ and every compact interval $I \subset \mathbb{R}$, and hence so is ϕ_0. This proves Step 4.

Step 5. *Let ϕ_0 be given by* (11.17)*. Then*

$$\phi_0(r,s) = \frac{1}{2\pi}\int_{-\infty}^\infty \rho(0,\xi)e^{-\rho(0,\xi)s + \mathbf{i}r\xi}\,d\xi \qquad (11.22)$$

for all $(r,s) \in \mathbb{R}^2 \setminus \{(0,0)\}$.

Fix a pair $(r,s) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Since

$$\rho(0,\xi) = |\xi|e^{\mathbf{i}\theta_0 \mathrm{sign}(\xi)} = |\xi|\cos(\theta_0) + \mathbf{i}\xi\sin(\theta_0),$$

the right hand side in (11.22) is the sum $z^+ + z^-$, where

$$z^+ := \int_0^\infty \frac{\rho(0,\xi)e^{-\rho(0,\xi)s+\mathbf{i}r\xi}}{2\pi}\,d\xi = \int_0^\infty \frac{e^{\mathbf{i}\theta_0}\xi e^{-\xi(e^{\mathbf{i}\theta_0}s + \mathbf{i}r)}}{2\pi}\,d\xi = \frac{e^{\mathbf{i}\theta_0}}{2\pi(e^{\mathbf{i}\theta_0}s - \mathbf{i}r)^2}$$

and

$$z^- := \int_{-\infty}^0 \frac{\rho(0,\xi)e^{-\rho(0,\xi)s+\mathbf{i}r\xi}}{2\pi}\,d\xi = \int_0^\infty \frac{\rho(0,-\xi)e^{-\rho(0,-\xi)s-\mathbf{i}r\xi}}{2\pi}\,d\xi = \overline{z}^+.$$

Hence

$$\begin{aligned} z^+ + z^- &= \mathrm{Re}\left(\frac{e^{\mathbf{i}\theta_0}}{\pi(e^{\mathbf{i}\theta_0}s - \mathbf{i}r)^2}\right) = \frac{\mathrm{Re}(e^{\mathbf{i}\theta_0}(e^{-\mathbf{i}\theta_0}s + \mathbf{i}r)^2)}{\pi|e^{\mathbf{i}\theta_0}s - \mathbf{i}r|^4} \\ &= \frac{\mathrm{Re}(e^{-\mathbf{i}\theta_0}s^2 + 2\mathbf{i}rs - e^{\mathbf{i}\theta_0}r^2)}{\pi(\cos^2(\theta_0)s^2 + (\sin(\theta_0)s - r)^2)^2} = \frac{\cos(\theta_0)(s^2 - r^2)}{\pi(r^2 + s^2 - 2rs\sin(\theta_0))^2}. \end{aligned}$$

This proves Step 5.

Step 6. *Let f be as in Step 2. Then*

$$(U_{0,\varepsilon}f)(t) = \int_\varepsilon^\infty \int_{-\infty}^\infty \phi_0(r,s)S(s)f(t-r)\,drds$$

for all $\varepsilon > 0$ and all $t \in \mathbb{R}$.

For all $\zeta \in \mathbb{C}$ with $\mathrm{Re}(\zeta) > 0$,

$$(\zeta - A)^{-1} = \int_0^\infty e^{-\zeta s} S(s)\,ds.$$

Since $\mathrm{Re}(\rho(0,\xi)) > 0$ for all $\xi \in \mathbb{R}\setminus\{0\}$ it follows from the definition of $\widehat{K}_\varepsilon(\xi)$ in Step 1 that

$$\begin{aligned}
\widehat{K}_\varepsilon(\xi) &= \rho(0,\xi)e^{-\rho(0,\xi)\varepsilon}S(\varepsilon)\big(\rho(0,\xi) - A\big)^{-1} \\
&= \rho(0,\xi)e^{-\rho(0,\xi)\varepsilon}S(\varepsilon)\int_0^\infty e^{-\rho(0,\xi)s}S(s)\,ds \\
&= \rho(0,\xi)\int_\varepsilon^\infty e^{-\rho(0,\xi)s}S(s)\,ds
\end{aligned}$$

for all $\varepsilon > 0$ and all $\xi \in \mathbb{R}$. Hence it follows from the definition of $U_{0,\varepsilon}f$ in Step 3 that

$$\begin{aligned}
(U_{0,\varepsilon}f)(t) &= \frac{1}{2\pi}\int_{-\infty}^\infty e^{\mathbf{i}t\xi}\widehat{K}_\varepsilon(\xi)\widehat{f}(\xi)\,d\xi \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty \int_\varepsilon^\infty \rho(0,\xi)e^{-\rho(0,\xi)s+\mathbf{i}t\xi}S(s)\widehat{f}(\xi)\,dsd\xi \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty \int_\varepsilon^\infty \int_{-\infty}^\infty \rho(0,\xi)e^{-\rho(0,\xi)s+\mathbf{i}(t-r)\xi}S(s)f(r)\,drdsd\xi \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty \left(\int_\varepsilon^\infty \int_{-\infty}^\infty \rho(0,\xi)e^{-\rho(0,\xi)s+\mathbf{i}r\xi}S(s)f(t-r)\,drds\right)d\xi.
\end{aligned}$$

Since $\|\rho(0,\xi)e^{-\rho(0,\xi)s+\mathbf{i}(t-r)\xi}S(s)f(r)\|_{L^2} \le |\xi|e^{-|\xi|\cos(\theta_0)\varepsilon}e^{-\delta s}\|f(r)\|_{L^2}$ whenever $s \ge \varepsilon$, the function $(r,s,\xi) \mapsto \rho(0,\xi)e^{-\rho(0,\xi)s+\mathbf{i}r\xi}S(s)f(t-r)$ is integrable over $\mathbb{R}\times[\varepsilon,\infty)\times\mathbb{R}$ for all $\varepsilon > 0$. Hence, by Fubini's theorem,

$$\begin{aligned}
(U_{0,\varepsilon}f)(t) &= \int_\varepsilon^\infty \int_{-\infty}^\infty \left(\frac{1}{2\pi}\int_{-\infty}^\infty \rho(0,\xi)e^{-\rho(0,\xi)s+\mathbf{i}r\xi}\,d\xi\right)S(s)f(t-r)\,drds \\
&= \int_\varepsilon^\infty \int_{-\infty}^\infty \phi_0(r,s)S(s)f(t-r)\,drds.
\end{aligned}$$

Here the last equation follows from Step 5 and this proves Step 6. The assertions of Lemma 11.5 follow immediately from Steps 3, 4, and 6. □

The next lemma (whose proof takes up three pages) examines the properties of the function $\phi_0 : \mathbb{R} \times (0,\infty) \to \mathbb{R}$ in equation (11.17) in Lemma 11.5. It makes use of the Fourier transform on $\mathbb{R}^2$, given by

$$(\mathscr{F}(v))(\xi,\eta) := \widehat{v}(\xi,\eta) := \int_{\mathbb{R}^2} e^{-\mathbf{i}(\xi r+\eta s)} v(r,s)\,drds$$

for $v \in L^2(\mathbb{R}^2,\mathbb{C}) \cap L^1(\mathbb{R}^2,\mathbb{C})$ and $\xi,\eta \in \mathbb{R}$. The inverse is

$$(\mathscr{F}^{-1}(\widehat{v}))(r,s) = v(r,s) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{\mathbf{i}(\xi r+\eta s)}\widehat{v}(\xi,\eta)\,d\xi d\eta$$

for $\widehat{v} \in L^2(\mathbb{R}^2,\mathbb{C}) \cap L^1(\mathbb{R}^2,\mathbb{C})$ and $r,s \in \mathbb{R}$. A function $v : \mathbb{R}^2 \to \mathbb{C}$ is called a **Schwartz test function**, if the functions $(r,s) \mapsto r^k s^\ell \partial_r^m v(r,s)\partial_s^n v(r,s)$ is bounded for all quadruples of nonnegative integers k,ℓ,m,n. The (topological vector) space of Schwartz test functions is denoted by $\mathscr{S}(\mathbb{R}^2,\mathbb{C})$ and the subspace of real valued Schwartz test functions is denoted by $\mathscr{S}(\mathbb{R}^2)$. The space $\mathscr{S}(\mathbb{R}^2,\mathbb{C})$ is invariant under the Fourier transform. A **tempered distribution** on $\mathbb{R}^2$ is a continuous linear functional $\mathscr{S}(\mathbb{R}^2,\mathbb{C}) \to \mathbb{C}$.

Lemma 11.6. *Let $\phi_0 : \mathbb{R} \times (0,\infty) \to \mathbb{R}$ be the function defined by* (11.17) *and define the linear map $\Phi_0 : C_0^\infty(\mathbb{R}^2,\mathbb{C}) \to \mathbb{C}$ by*

$$\Phi_0(v) := -\int_0^\infty \int_{-\infty}^\infty \frac{r\cos(\theta_0)}{\pi(r^2+s^2-2rs\sin(\theta_0))}\frac{\partial v}{\partial r}(r,s)\,drds \tag{11.23}$$

for $v \in C_0^\infty(\mathbb{R}^2,\mathbb{C})$. Then the following holds.

(i) *Φ_0 extends to a tempered distribution.*

(ii) *For all $v \in C_0^\infty(\mathbb{R}^2,\mathbb{C})$*

$$\Phi_0(v) = \lim_{\varepsilon\to 0}\int_\varepsilon^\infty \int_{-\infty}^\infty \phi_0(r,s)v(r,s)\,ds. \tag{11.24}$$

(iii) *The formula $(T_0 v)(r,s) := \Phi_0(v(r-\cdot,s-\cdot))$ for $v \in \mathscr{S}(\mathbb{R}^2,\mathbb{C})$ defines a continuous linear operator $T_0 : \mathscr{S}(\mathbb{R}^2,\mathbb{C}) \to \mathscr{S}(\mathbb{R}^2,\mathbb{C})$. Moreover,*

$$\widehat{T_0 v}(\xi,\eta) = m(\xi,\eta)\widehat{v}(\xi,\eta), \qquad m(\xi,\eta) := \frac{|\xi|e^{\mathbf{i}\theta_0\mathrm{sign}(\xi)}}{|\xi|e^{\mathbf{i}\theta_0\mathrm{sign}(\xi)}+\mathbf{i}\eta}, \tag{11.25}$$

for $v \in \mathscr{S}(\mathbb{R}^2,\mathbb{C})$ and $|m(\xi,\eta)| \le (1-\sin(\theta_0))^{-1/2}$ for all $(\xi,\eta) \in \mathbb{R}^2\setminus\{(0,0)\}$.

(iv) *If $1 < q < \infty$ then T_0 extends to a bounded linear operator from $L^q(\mathbb{R}^2,\mathbb{C})$ to itself.*

Proof. We prove (i). Define the function $K_0 : \mathbb{R}^2 \to \mathbb{R}$ by $K_0(r,s) := 0$ whenever $s < 0$ or $r = s = 0$, and by

$$K_0(r,s) := \frac{r\cos(\theta_0)}{\pi(r^2 + s^2 - 2rs\sin(\theta_0))}$$

whenever $s \geq 0$ and $(r,s) \neq (0,0)$. Then K_0 is locally integrable and a simple computation shows that

$$\frac{\partial K_0}{\partial r}(r,s) = \phi_0(r,s) \qquad \text{for } s > 0.$$

Hence the right hand side of (11.23) is well defined and it follows directly from the definition that Φ_0 is a tempered distribution. This proves (i).

We prove (ii). Fix a function $v \in C_0^\infty(\mathbb{R}^2, \mathbb{C})$ and define

$$\Phi_\varepsilon(v) := -\int_\varepsilon^\infty \int_{-\infty}^\infty K_0(r,s) \frac{\partial v}{\partial r}(r,s)\, drds$$

for $\varepsilon > 0$. Then $\lim_{\varepsilon \to 0} \Phi_\varepsilon(v) = \Phi_0(v)$ and integration by parts shows that

$$\Phi_\varepsilon(v) = \int_\varepsilon^\infty \int_{-\infty}^\infty \frac{\partial K_0}{\partial r}(r,s) v(r,s)\, drds = \int_\varepsilon^\infty \int_{-\infty}^\infty \phi_0(r,s) v(r,s)\, drds.$$

This proves part (ii).

We prove (iii). Since

$$\rho(0,\xi) = |\xi| e^{\mathbf{i}\theta_0 \mathrm{sign}(\xi)} = |\xi|\cos(\theta_0) + \mathbf{i}\xi\sin(\theta_0),$$

we have $|\rho(0,\xi)|^2 = |\xi|^2$ and

$$|\rho(0,\xi) + \mathbf{i}\eta|^2 = \xi^2 + \eta^2 - 2\xi\eta\sin(\theta_0) \geq (\xi^2 + \eta^2)(1 - \sin(\theta_0)).$$

Since $0 < \theta_0 < \pi/2$ it follows that $\sin(\theta_0) < 1$ and

$$|m(\xi,\eta)| \leq \frac{1}{\sqrt{1 - \sin(\theta_0)}} \qquad \text{for all } (\xi,\eta) \in \mathbb{R}^2 \setminus \{(0,0)\}.$$

Since $\mathrm{Re}(\rho(0,\xi)) > 0$ for $\xi \in \mathbb{R} \setminus \{0\}$, we have

$$\int_0^\infty \rho(0,\xi) e^{-\rho(0,\xi)s - \mathbf{i}s\eta}\, ds = \frac{\rho(0,\xi)}{\rho(0,\xi) + \mathbf{i}\eta} = m(\xi,\eta). \tag{11.26}$$

for all nonzero pairs $(\xi,\eta) \in \mathbb{R}^2$. Moreover,

$$\phi_0(r,s) = \frac{1}{2\pi} \int_{-\infty}^\infty \rho(0,\xi) e^{-\rho(0,\xi)s + \mathbf{i}r\xi}\, d\xi \tag{11.27}$$

for all nonzero pairs $(r,s) \in \mathbb{R} \times [0,\infty)$, by Step 5 in the proof of Lemma 11.5.

Next define

$$m_\varepsilon(\xi,\eta) := e^{-\rho(0,\xi)\varepsilon-\mathbf{i}\varepsilon\eta}m(\xi,\eta) = \int_\varepsilon^\infty \rho(0,\xi)e^{-\rho(0,\xi)s-\mathbf{i}s\eta}\,ds \tag{11.28}$$

for $(\xi,\eta)\in\mathbb{R}^2\setminus\{(0,0)\}$. Here the second equation follows from (11.26). Fix a function $v\in C_0^\infty(\mathbb{R}^2,\mathbb{C})$ and denote its Fourier transform by $\widehat{v} := \mathscr{F}(v)$. Then $\widehat{v}$ is integrable and hence so is $m_\varepsilon\widehat{v}$. Thus the inverse Fourier transform of $m_\varepsilon\widehat{v}$ is given by

$$\begin{aligned}
&(\mathscr{F}^{-1}(m_\varepsilon\widehat{v}))(r,s)\\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{\mathbf{i}r\xi+\mathbf{i}s\eta}m_\varepsilon(\xi,\eta)\widehat{v}(\xi,\eta)\,d\xi d\eta\\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\int_\varepsilon^\infty \rho(0,\xi)e^{-\rho(0,\xi)s'+\mathbf{i}r\xi+\mathbf{i}(s-s')\eta}\widehat{v}(\xi,\eta)\,ds'd\xi d\eta.
\end{aligned}$$

The integrand satisfies the inequality

$$\left|\rho(0,\xi)e^{-\rho(0,\xi)s'+\mathbf{i}r\xi+\mathbf{i}(s-s')\eta}\widehat{v}(\xi,\eta)\right| \le |\xi|e^{-|\xi|\cos(\theta_0)\varepsilon}|\widehat{v}(\xi,\eta)|$$

for $s'\ge\varepsilon$ and so the function is integrable on $\mathbb{R}^2\times[\varepsilon,\infty)$. Hence, by Fubini's theorem,

$$\begin{aligned}
&(\mathscr{F}^{-1}(m_\varepsilon\widehat{v}))(r,s)\\
&= \int_\varepsilon^\infty\left(\frac{1}{2\pi}\int_{-\infty}^\infty \rho(0,\xi)e^{-\rho(0,\xi)s'+\mathbf{i}r\xi}\left(\frac{1}{2\pi}\int_{-\infty}^\infty e^{\mathbf{i}(s-s')\eta}\widehat{v}(\xi,\eta)\,d\eta\right)d\xi\right)ds'\\
&= \int_\varepsilon^\infty\left(\frac{1}{2\pi}\int_{-\infty}^\infty \rho(0,\xi)e^{-\rho(0,\xi)s'+\mathbf{i}r\xi}\left(\int_{-\infty}^\infty e^{-\mathbf{i}r'\xi}v(r',s-s')\,dr'\right)d\xi\right)ds'\\
&= \int_\varepsilon^\infty\left(\frac{1}{2\pi}\int_{-\infty}^\infty\left(\int_{-\infty}^\infty \rho(0,\xi)e^{-\rho(0,\xi)s'+\mathbf{i}(r-r')\xi}v(r',s-s')\,dr'\right)d\xi\right)ds'\\
&= \int_\varepsilon^\infty\left(\frac{1}{2\pi}\int_{-\infty}^\infty\left(\int_{-\infty}^\infty \rho(0,\xi)e^{-\rho(0,\xi)s'+\mathbf{i}r'\xi}v(r-r',s-s')\,dr'\right)d\xi\right)ds'\\
&= \int_\varepsilon^\infty\left(\int_{-\infty}^\infty\left(\frac{1}{2\pi}\int_{-\infty}^\infty \rho(0,\xi)e^{-\rho(0,\xi)s'+\mathbf{i}r'\xi}d\xi\right)v(r-r',s-s')\,dr'\right)ds'\\
&= \int_\varepsilon^\infty\int_{-\infty}^\infty \phi_0(r',s')v(r-r',s-s')\,dr'ds' = \Phi_\varepsilon(v(r-\cdot,s-\cdot)).
\end{aligned}$$

Here the last but one equation follows from (11.27).

Now define the function $\phi_\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ by

$$\phi_\varepsilon(r,s) := \begin{cases} \phi_0(r,s), & \text{if } s \geq \varepsilon, \\ 0, & \text{if } s < \varepsilon. \end{cases} \tag{11.29}$$

This function is locally integrable, by Lemma 11.5, and we have just proved that

$$\widehat{\phi_\varepsilon * v} = m_\varepsilon \widehat{v}$$

for every smooth compactly supported function $v : \mathbb{R}^2 \to \mathbb{C}$. Moreover, $|m_\varepsilon(\xi,\eta)| \leq |m(\xi,\eta)| \leq \cos(\theta_0)^{-1}$ for all ξ and η. Thus

$$|m_\varepsilon(\xi,\eta)\widehat{v}(\xi,\eta) - m(\xi,\eta)\widehat{v}(\xi,\eta)| \leq \frac{2}{\cos(\theta_0)}|\widehat{v}(\xi,\eta)|$$

$$\lim_{\varepsilon\to 0}|m_\varepsilon(\xi,\eta)\widehat{v}(\xi,\eta) - m(\xi,\eta)\widehat{v}(\xi,\eta)| = 0$$

for all $(\xi,\eta) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Since the function $\mathbb{R}^2 \to \mathbb{R} : (\xi,\eta) \mapsto |\widehat{v}(\xi,\eta)|^2$ is integrable, it follows from the Lebesgue dominated convergence theorem that $\lim_{\varepsilon\to 0} \|m_\varepsilon\widehat{v} - m\widehat{v}\|_{L^2} = 0$. This implies that the functions $\phi_\varepsilon * v = \mathscr{F}^{-1}(m_\varepsilon\widehat{v})$ converge in L^2 as ε tends to zero. Since the pointwise limit of this family is the function $T_0 v$, by part (ii) of the lemma (already proved), it follows that

$$\lim_{\varepsilon\to 0} \|\phi_\varepsilon * v - T_0 v\|_{L^2} = 0. \tag{11.30}$$

Hence

$$\mathscr{F}(T_0 v) = \lim_{\varepsilon\to 0} \mathscr{F}(\phi_\varepsilon * v) = \lim_{\varepsilon\to 0} m_\varepsilon\widehat{v} = m\widehat{v},$$

where the convergence is in $L^2(\mathbb{R}^2)$. This proves part (iii).

We prove (iv). The real and imaginary parts of m are given by

$$\mathrm{Re}(m(\xi,\eta)) = \frac{\xi^2 + \xi\eta\sin(\theta_0)}{\xi^2+\eta^2+2\xi\eta\sin(\theta_0)}, \quad \mathrm{Im}(m(\xi,\eta)) = \frac{|\xi|\eta\cos(\theta_0)}{\xi^2+\eta^2+2\xi\eta\sin(\theta_0)}.$$

A calculation shows that there exist positive constants c_1 and c_2 such that

$$\left|\frac{\partial m}{\partial\xi}(\xi,\eta)\right| + \left|\frac{\partial m}{\partial\eta}(\xi,\eta)\right| \leq \frac{c_1}{\sqrt{\xi^2+\eta^2}}$$

and

$$\left|\frac{\partial^2 m}{\partial\xi^2}(\xi,\eta)\right| + \left|\frac{\partial^2 m}{\partial\xi\partial\eta}(\xi,\eta)\right| + \left|\frac{\partial^2 m}{\partial\eta^2}(\xi,\eta)\right| \leq \frac{c_2}{\xi^2+\eta^2}$$

for all $(\xi,\eta) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Hence the assertion of part (iv) follows from the Mikhlin Multiplier Theorem 6.1. This proves Lemma 11.6 □

To prove the desired estimate for the operator U_0 we must replace ϕ_0 by integrable functions ϕ. The next lemma introduces a class of operators that will be used to approximate U_0. This is where the transference estimates in Section 10 are used.

Lemma 11.7. *Fix a number $1 < q < \infty$ and let $\phi \in L^1(\mathbb{R}^2, \mathbb{C})$ be a complex valued Lebesgue integrable function such that $\phi(r,s) = 0$ for $s < 0$. Let $f : \mathbb{R} \to L^q(X,\mu)$ be a continuous function with compact support and define the function $\mathcal{T}_\phi f : \mathbb{R} \to L^q(X,\mu)$ by*

$$(\mathcal{T}_\phi f)(t) := \int_0^\infty \int_{-\infty}^\infty \phi(r,s)S(s)f(t-r)\,drds \tag{11.31}$$

for $t \in \mathbb{R}$. Then $\mathcal{T}_\phi f$ is q-integrable and

$$\|\mathcal{T}_\phi f\|_{L^q(\mathbb{R},L^q(X,\mu))} \le N_q(\phi)\,\|f\|_{L^q(\mathbb{R},L^q(X,\mu))}\,. \tag{11.32}$$

(See equation (10.1) *for the definition of $N_q(\phi)$.)*

Proof. By Theorem 10.3 and assumption (II) in Theorem 11.2 there is a σ-finite measure space $(Y, \mathcal{B}, \nu)$, an isometric embedding $\iota : L^q(X,\mu) \to L^q(Y,\nu)$, a contracting projection $\pi : L^q(Y,\nu) \to L^q(X,\mu)$, and a strongly continuous group of positive isometries $U(t) \in \mathrm{Aut}(L^q(Y,\nu))$, such that

$$\pi \circ U(s) \circ \iota = S(s) \qquad \text{for all } s \ge 0.$$

For $r, s \in \mathbb{R}$ define the operator $\mathscr{U}(r,s)$ on $L^q(\mathbb{R}, L^q(Y,\nu)) \cong L^q(\mathbb{R}\times Y, \sigma\otimes\nu)$ (σ the Lebesgue measure on $\mathbb{R}$) by

$$(\mathscr{U}(r,s)f)(t) := U(s)f(t-r)$$

for $r, s, t \in \mathbb{R}$ and $f \in L^q(\mathbb{R}, L^q(Y,\nu))$. Then $\mathscr{U} : \mathbb{R}^2 \to \mathrm{Aut}(L^q(\mathbb{R}, L^q(Y,\nu)))$ is a strongly continuous group of isometries. Hence, by Lemma 10.2,

$$\begin{aligned}
\|\mathcal{T}_\phi f\|_{L^q(\mathbb{R},L^q(X,\mu))} &= \left\|\int_0^\infty \int_{-\infty}^\infty \phi(r,s)S(s)f(\cdot - r)\,drds\right\|_{L^q(\mathbb{R},L^q(X,\mu))} \\
&= \left\|\int_0^\infty \int_{-\infty}^\infty \phi(r,s)\pi(U(s)\iota(f(\cdot - r)))\,drds\right\|_{L^q(\mathbb{R},L^q(X,\mu))} \\
&\le \left\|\int_0^\infty \int_{-\infty}^\infty \phi(r,s)\mathscr{U}(s,r)(\iota\circ f)\,drds\right\|_{L^q(\mathbb{R},L^q(Y,\nu))} \\
&\le N_q(\phi)\,\|\iota\circ f\|_{L^q(\mathbb{R},L^q(Y,\nu))} \\
&= N_q(\phi)\,\|f\|_{L^q(\mathbb{R},L^q(X,\mu))}
\end{aligned}$$

for all $f \in L^q(\mathbb{R}, L^q(X,\mu))$. This proves Lemma 11.7 $\square$

The next lemma introduces the required sequence of integrable functions that converge to ϕ_0 in the distributional sense.

Lemma 11.8. *Let $\phi_0 : \mathbb{R} \times (0,\infty) \to \mathbb{R}$ be the function in (11.17) and let $T_0 : L^q(\mathbb{R}^2,\mathbb{C}) \to L^q(\mathbb{R}^2,\mathbb{C})$ be the operator in Lemma 11.6. Then there is a sequence of smooth functions $\phi_n \in \mathscr{S}(\mathbb{R}^2)$ that satisfies the following.*

(i) *For all $n \in \mathbb{N}$ and all $(r,s) \in \mathbb{R}^2$ with $s \le 0$, we have $\phi_n(r,s) = 0$.*

(ii) *For all $n \in \mathbb{N}$ and all real numbers $1 < q < \infty$, we have*

$$N_q(\phi_n) \le \|T_0\|_{\mathcal{L}(L^q)}. \tag{11.33}$$

(iii) *For all $v \in \mathscr{S}(\mathbb{R}^2,\mathbb{C})$, we have*

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} \phi_n(r,s)v(r,s)\,drds = \lim_{\varepsilon\to 0} \int_\varepsilon^\infty \int_{-\infty}^\infty \phi_0(r,s)v(r,s)\,drds. \tag{11.34}$$

Proof. Choose a sequence of smooth functions $\psi_n : \mathbb{R}^2 \to [0,\infty)$ such that

$$\mathrm{supp}(\psi_n) \subset [-1/n, 1/n] \times [0, 1/n], \qquad \int_{R^2} \psi_n(r,s)\,drds = 1$$

for all $n \in \mathbb{N}$ and define $\phi_n := T_0\psi_n$. Then the Fourier transform of ϕ_n is

$$\widehat{\phi}_n(\xi,\eta) = m(\xi,\eta)\widehat{\psi}_n(\xi,\eta), \qquad m(\xi,\eta) := \frac{|\xi|e^{\mathbf{i}\theta_0\mathrm{sign}(\xi)}}{|\xi|e^{\mathbf{i}\theta_0\mathrm{sign}(\xi)} + \mathbf{i}\eta}.$$

(See equation (11.25) in Lemma 11.6.)

We prove that ϕ_n satisfies (i). An integrable function is supported in the half plane $\mathbb{R}\times[0,\infty)$ if and only if its Fourier transform extends to a bounded holomorphic function in the domain $\{\eta \in \mathbb{C} \,|\, \mathrm{Im}(\eta) < 0\}$ for every $\xi \in \mathbb{R}$. Since the summand $|\xi|e^{\mathbf{i}\theta_0\mathrm{sign}(\xi)}$ has a positive real part the multiplier m has this property. Since ψ_n is supported in $\mathbb{R} \times [0,\infty)$, its Fourier transform has this property as well, and hence so does $\widehat{\phi}_n = m\widehat{\psi}_n$. This shows that ϕ_n is supported in $\mathbb{R} \times [0,\infty)$.

We prove that ϕ_n satisfies (ii). Namely,

$$\begin{aligned} \|\phi_n * v\|_{L^q} &= \|(T\psi_n) * v\|_{L^q} \\ &= \|T_0(\psi_n * v)\|_{L^q} \\ &\le \|T_0\|_{\mathcal{L}(L^q)} \|\psi_n * v\|_{L^q} \\ &\le \|T_0\|_{\mathcal{L}(L^q)} \|v\|_{L^q} \end{aligned}$$

for all $v \in \mathscr{S}(\mathbb{R}^2)$ and so $N_p(\phi_n) \le \|T_0\|_{\mathcal{L}(L^q)}$. This proves (ii).

We prove that ϕ_n satisfies (iii). Let $v \in \mathscr{S}(\mathbb{R}^n, \mathbb{C})$ be a Schwartz test function and recall that

$$\Phi_0(v) = \lim_{\varepsilon \to 0} \int_\varepsilon^\infty \int_{-\infty}^\infty \phi_0(r,s) v(r,s)\, drds$$

by (11.24), that

$$(T_0 v)(r,s) = (\Phi_0 v)(r - \cdot, s - \cdot)$$

in Lemma 11.6, and that the Fourier transform of $T_0 v$ is given by $\widehat{T_0 v} = m\widehat{v}$, where $m : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{C}$ is the bounded function in (11.25). Define

$$w(r,s) := v(-r,-s)$$

so that $\widehat{w}(\xi,\eta) = \widehat{v}(-\xi,-\eta)$. Then

$$\Phi_0(v) = (T_0 w)(0,0) = \frac{1}{2\pi} \int_{\mathbb{R}^2} m(\xi,\eta) \widehat{v}(-\xi,-\eta)\, d\xi d\eta$$

and

$$\begin{aligned} \int_{\mathbb{R}^2} \phi_n(r,s) v(r,s)\, drds &= (\phi_n * w)(0,0) \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \widehat{\phi}_n(\xi,\eta) \widehat{v}(-\xi,-\eta)\, d\xi d\eta \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} m(\xi,\eta) \widehat{\psi}_n(\xi,\eta) \widehat{v}(-\xi,-\eta)\, d\xi d\eta. \end{aligned}$$

Since $\psi_n \in \mathscr{S}(\mathbb{R}^2)$ is a sequence of approximate Dirac δ-functions, its Fourier transforms satisfy $|\widehat{\psi}_n(\xi,\eta)| \le 1$ for all $n \in \mathbb{N}$ and all $(\xi,\eta) \in \mathbb{R}^n$, and the sequence $\widehat{\psi}_n$ converges to 1, uniformly on every compact subset of $\mathbb{R}^2$. Moreover, since v is a Schwartz test function, so is $\widehat{v}$, and so $\widehat{v}$ is integrable. (In fact, a sufficient condition for the integrability of $\widehat{v}$ is that v is three times continuously differentiable and all derviatives of v up to order three are integrable.) Now let $\kappa > 0$. Then there exists an $R > 0$ such that

$$\frac{1}{2\pi\sqrt{1 - \sin(\theta_0)}} \int_{\mathbb{R}^2 \setminus B_R} |\widehat{v}| < \frac{\kappa}{3}.$$

Choose $n_0 \in \mathbb{N}$ such that $(1 - \sin(\theta_0))^{-1/2} R^2 \sup_{B_R} |\widehat{v}| \sup_{B_R} |\widehat{\psi}_n - 1| < \kappa/3$ for every integer $n \ge n_0$. Then $|\Phi_0(v) - \int_{\mathbb{R}^2} \phi_n v| < \kappa$ for every integer $n \ge n_0$. This proves Lemma 11.8. $\square$

The next lemma is a convergence result that plays a central role in Lamberton's proof of Theorem 11.2 (see [20, Lemma 3.4]). We reproduce his proof below.

Lemma 11.9. *Let $f_0, g_0 \in \mathscr{S}(\mathbb{R}, \mathbb{C})$, $v_0 \in L^2(X, \mu)$, $u_0 \in \bigcap_{k=1}^{\infty} \operatorname{dom}(A^k)$, and $t_0 > 0$. Define the functions $f, g : \mathbb{R} \to L^2(X, \mu)$ by*

$$f(t) := f_0(t) S(t_0) u_0, \qquad g(t) := g_0(t) v_0. \tag{11.35}$$

Fix a constant $\delta > 0$ and define the functions $U_0^\delta f, \mathcal{T}_{\phi_n}^\delta f : \mathbb{R} \to L^2(X, \mu)$ by

$$\begin{aligned} (U_0^\delta f)(t) &:= \lim_{\varepsilon \to 0} \int_\varepsilon^\infty \int_{-\infty}^\infty \phi_0(r, s) e^{-\delta s} S(s) f(t - r)\, dr ds, \\ (\mathcal{T}_{\phi_n}^\delta f)(t) &:= \int_0^\infty \int_{-\infty}^\infty \phi_n(r, s) e^{-\delta s} S(s) f(t - r)\, dr ds, \end{aligned} \tag{11.36}$$

where the ϕ_n are as in Lemma 11.8. Then

$$\lim_{n \to \infty} \left\langle g, \mathcal{T}_{\phi_n}^\delta f \right\rangle_{L^2(\mathbb{R}, L^2(X, \mu))} = \left\langle g, U_0^\delta f \right\rangle_{L^2(\mathbb{R}, L^2(X, \mu))}. \tag{11.37}$$

Proof. Choose a smooth cutoff function $\beta : \mathbb{R} \to \mathbb{R}$ such that $\beta(s) = 0$ for $s \le -t_0/2$ and $\beta(s) = 1$ for $s \ge 0$. Define the functions $a_0, b_0 : \mathbb{R} \to \mathbb{C}$ by

$$a_0(r) := \int_{-\infty}^\infty \overline{g}_0(t) f_0(t - r)\, dt, \quad b_0(s) := \beta(s) e^{-\delta s} \int_X \overline{v}_0(S(t_0 + s) u_0)\, d\mu$$

for $r \in \mathbb{R}$ and $s \ge -t_0$ and by $b_0(s) := 0$ for $s < -t_0$. Then a_0 and b_0 are Schwartz test functions and hence so is the function $v : \mathbb{R}^2 \to \mathbb{C}$ given by $v(r, s) := a_0(r) b_0(s)$. Moreover, it follows from (11.35) that

$$\begin{aligned} &\left\langle g, \mathcal{T}_{\phi_n}^\delta f \right\rangle_{L^2(\mathbb{R}, L^2(X, \mu))} \\ &= \int_{-\infty}^\infty \int_0^\infty \int_{-\infty}^\infty \phi_n(r, s) \overline{g}_0(t) f_0(t - r) e^{-\delta s} \int_X \overline{v}_0(S(t_0 + s) u_0)\, d\mu\, dr ds dt \\ &= \int_{-\infty}^\infty \int_{-\infty}^\infty \phi_n(r, s) a_0(r) b_0(s)\, dr ds \end{aligned}$$

for all $n \in \mathbb{N}$. Here we have used the fact that $\phi_n(r, s) = 0$ for $s \le 0$. Similarly,

$$\left\langle g, U_0^\delta f \right\rangle_{L^2(\mathbb{R}, L^2(X, \mu))} = \lim_{\varepsilon \to 0} \int_\varepsilon^\infty \int_{-\infty}^\infty \phi_0(r, s) a_0(r) b_0(s)\, dr ds.$$

Thus (11.37) follows from (11.34) in Lemma 11.8 with $v(r, s) = a_0(r) b_0(s)$. This proves Lemma 11.9. □

The final preparatory lemma for the proof of Theorem 11.2 puts things together and establishes the required L^q estimate for the operators $U_{\mathbf{i}\tau}$ introduced in Definition 11.3.

Lemma 11.10. *For every real number $1 < q < \infty$ there exists a constant $C_q > 0$ such that, for all $\tau \in \mathbb{R}$ and all $f \in L^2(\mathbb{R}, L^2(X,\mu)) \cap L^q(\mathbb{R}, L^q(X,\mu))$,*

$$\left(\int_{-\infty}^{\infty} \|(U_{\mathbf{i}\tau} f)(t)\|_{L^q(X,\mu)}^q \, dt\right)^{1/q} \le C_q \left(\int_{-\infty}^{\infty} \|f(t)\|_{L^q(X,\mu)}^q \, dt\right)^{1/q}. \quad (11.38)$$

Proof. The proof has two steps.

Step 1. *The inequality* (11.38) *holds for $\tau = 0$ with $C_q = \|T_0\|_{\mathcal{L}(L^q)}$. Here T_0 is the linear operator introduced in Lemma 11.6.*

Choose $1 < q' < \infty$ such that $1/q + 1/q' = 1$. Assume first that f and g are given by (11.35) with $f_0, g_0 \in \mathscr{S}(\mathbb{R})$, $t_0 > 0$, $v_0 \in L^2(X,\mu) \cap L^{q'}(X,\mu)$, and $u_0 \in L^q(X,\mu) \cap \bigcap_{k\in\mathbb{N}} \mathrm{dom}(A^k)$. Then

$$\begin{aligned}
\left\langle g, U_0^\delta f\right\rangle_{L^2(\mathbb{R},L^2(X,\mu))} &= \lim_{n\to\infty} \left\langle g, \mathcal{T}_{\phi_n}^\delta f\right\rangle_{L^2(\mathbb{R},L^2(X,\mu))} \\
&\le \lim_{n\to\infty} \|g\|_{L^{q'}(\mathbb{R},L^{q'}(X,\mu))} \left\|\mathcal{T}_{\phi_n}^\delta f\right\|_{L^q(\mathbb{R},L^q(X,\mu))} \\
&\le \lim_{n\to\infty} N_q(\phi_n) \|g\|_{L^{q'}(\mathbb{R},L^{q'}(X,\mu))} \|f\|_{L^q(\mathbb{R},L^q(X,\mu))} \\
&\le \|T_0\|_{\mathcal{L}(L^q)} \|g\|_{L^{q'}(\mathbb{R},L^{q'}(X,\mu))} \|f\|_{L^q(\mathbb{R},L^q(X,\mu))}.
\end{aligned}$$

Here the first step follows from Lemma 11.9, the second step is Hölder's inequality, the third step follows from Lemma 11.7, and the last step follows from (11.33) in Lemma 11.8. The estimate extends to linear combinations of functions g as in (11.35). These form a dense subspace of $L^{q'}(\mathbb{R}, L^{q'}(X,\mu))$, so this inequality continues to hold for all $g \in L^{q'}(\mathbb{R}, L^{q'}(X,\mu))$. Divide by $\|g\|_{L^{q'}(\mathbb{R},L^{q'}(X,\mu))}$ and take the supremum over all nonzero g to obtain

$$\left\|U_0^\delta f\right\|_{L^q(\mathbb{R},L^q(X,\mu))} \le \|T_0\|_{\mathcal{L}(L^q)} \|f\|_{L^q(\mathbb{R},L^q(X,\mu))}.$$

for f as in (11.35) and their linear combinations. Take the limit $\delta \to 0$. Then

$$\|U_0 f\|_{L^q(\mathbb{R},L^q(X,\mu))} \le \|T_0\|_{\mathcal{L}(L^q)} \|f\|_{L^q(\mathbb{R},L^q(X,\mu))}. \quad (11.39)$$

for all f as in (11.35), by Lemma 11.5. Now take the limit $t_0 \to 0$ in (11.35) to obtain that (11.39) continues to hold for all $f : \mathbb{R} \to L^q(\mathbb{R}, L^q(X,\mu))$ of the form $f(t) = f_0(t)u_0$, where $f_0 \in \mathscr{S}(\mathbb{R})$ is a Schwartz test function and $u_0 \in L^q(X,\mu) \cap \bigcap_{k\in\mathbb{N}} \mathrm{dom}(A^k)$, as well as their linear combinations. These form a dense linear subspace of $L^q(\mathbb{R}, L^q(X,\mu))$, so (11.39) continues to hold for all $f \in L^2(\mathbb{R}, L^2(X,\mu)) \cap L^q(\mathbb{R}, L^q(X,\mu))$. This proves Step 1.

Step 2. *There exists a constant $C_q > 0$ such that the inequality* (11.38) *holds for all $\tau \in \mathbb{R}$ and all $f \in L^2(\mathbb{R}, L^2(X,\mu)) \cap L^q(\mathbb{R}, L^q(X,\mu))$.*

For all $\tau \in \mathbb{R}$ we have

$$\begin{aligned}
\rho(\mathbf{i}\tau, \xi) &= |\xi| e^{\mathbf{i}((1-\mathbf{i}\tau)\theta_0 + \mathbf{i}\tau\theta_1)\mathrm{sign}(\xi)} \\
&= |\xi| e^{\mathbf{i}\theta_0 \mathrm{sign}(\xi)} e^{-\tau(\theta_1-\theta_0)\mathrm{sign}(\xi)} \\
&= \rho(0, e^{-\tau(\theta_1-\theta_0)\mathrm{sign}(\xi)}\xi) \\
&= \begin{cases} \rho(0, e^{-\tau(\theta_1-\theta_0)}\xi), & \text{if } \xi \geq 0, \\ \rho(0, e^{\tau(\theta_1-\theta_0)}\xi), & \text{if } \xi < 0. \end{cases}
\end{aligned}$$

(see equation (11.11)). Thus it follows from the definition of the operator $U_{\mathbf{i}\tau}$ in equation (11.12) that

$$\widehat{U_{\mathbf{i}\tau} f}(\xi) = \widehat{K}(e^{-\tau(\theta_1-\theta_0)}\xi)\widehat{f}_+(\xi) + \widehat{K}(e^{\tau(\theta_1-\theta_0)}\xi)\widehat{f}_-(\xi) \tag{11.40}$$

where $\widehat{K}(\xi) = \rho(0,\xi)(\rho(0,\xi) - A)^{-1}$ as in the proof of Lemma 11.5, and

$$\widehat{f}_+(\xi) := \begin{cases} \widehat{f}(\xi), & \text{for } \xi \geq 0, \\ 0, & \text{for } \xi < 0, \end{cases} \qquad \widehat{f}_-(\xi) := \begin{cases} 0, & \text{for } \xi \geq 0, \\ \widehat{f}(\xi), & \text{for } \xi < 0. \end{cases}$$

For $\lambda > 0$ consider the scaled operator $U_{0,\lambda} : L^q(\mathbb{R}, L^q(X,\mu))$ defined by

$$U_{0,\lambda} f := (U_0 f_\lambda)_{\lambda^{-1}}, \qquad f_\lambda(t) := \lambda f(\lambda t).$$

Then $\widehat{f}_\lambda(\lambda\xi) = \widehat{f}(\xi)$, hence

$$\widehat{U_{0,\lambda} f}(\xi) = \widehat{K}(\lambda\xi)\widehat{f}(\xi),$$

and $U_{0,\lambda}$ has the same operator norm as U_0 in $\mathcal{L}(L^q(\mathbb{R}, L^q(X,\mu)))$. Now define the linear operators $\Pi_\pm : L^2(\mathbb{R}, L^2(X,\mu)) \to L^2(\mathbb{R}, L^2(X,\mu))$ by

$$\widehat{\Pi_\pm f} := \widehat{f}_\pm.$$

These extend to bounded linear operators from $L^q(\mathbb{R}, L^q(X,\mu))$ to itself, still denoted by $\Pi_\pm$, by Corollary 6.3 with $m := \chi_{[0,\infty)}$. The formula (11.40) shows that

$$U_{\mathbf{i}\tau} = U_{0,e^{-\tau(\theta_1-\theta_0)}}\Pi_+ + U_{0,e^{\tau(\theta_1-\theta_0)}}\Pi_-$$

for all $\tau \in \mathbb{R}$. Hence

$$\|U_{\mathbf{i}\tau}\|_{\mathcal{L}(L^q(\mathbb{R},L^q(X,\mu)))} \leq \Big(\|\Pi_+\|_{\mathcal{L}(L^q(\mathbb{R},L^q(X,\mu)))} + \|\Pi_-\|_{\mathcal{L}(L^q(\mathbb{R},L^q(X,\mu)))}\Big) \|T_0\|_{\mathcal{L}(L^q)}$$

by Step 1. This proves Step 2 and Lemma 11.10. □

Proof of Theorem 11.2. Fix a constant $1 < p < \infty$, and choose constants

$$0 < \theta_0 < \frac{\pi}{2} < \theta_1 < \pi, \qquad 0 < \lambda < 1, \qquad 1 < q < \infty$$

such that

$$(1-\lambda)\theta_0 + \lambda\theta_1 = \frac{\pi}{2}, \qquad \frac{1}{p} = \frac{1-\lambda}{q} + \frac{\lambda}{2}. \tag{11.41}$$

For $z \in \mathbb{S} = \{z \in \mathbb{C} \,|\, 0 \le \mathrm{Re}(z) \le 1\}$ let

$$U_z : L^2(\mathbb{R}, L^2(X,\mu)) \to L^2(\mathbb{R}, L^2(X,\mu))$$

be the operator introduced in equation (11.12) in Definition 11.3. By (11.13),

$$\|U_{1+\mathbf{i}\tau}\|_{\mathcal{L}(L^2(\mathbb{R},L^2(X,\mu))} \le \frac{1}{\sin(\theta_1)}$$

for all $\tau \in \mathbb{R}$ and, by Lemma 11.10, there exists a constant $C_q > 0$ such that

$$\|U_{\mathbf{i}\tau}\|_{\mathcal{L}(L^q(\mathbb{R},L^q(X,\mu))} \le C_q$$

for all $\tau \in \mathbb{R}$. Since the operator family $\{U_z\}_{z\in\mathbb{S}}$ is holomorphic, it satisfies the hypotheses of the Stein Interpolation Theorem 3.3 with $(Y,\mathcal{B},\nu) = (X,\mathcal{A},\mu)$ and $p_0 = q_0 = q$ and $p_1 = q_1 = 2$. Since $1/p = (1-\lambda)/q + \lambda/2$ by (11.41), it follows from Theorem 3.3 that

$$\|U_\lambda f\|_{L^p(\mathbb{R},L^p(X,\mu))} \le c_p \|f\|_{L^p(\mathbb{R},L^p(X,\mu))}, \qquad c_p := \frac{C_q^{1-\lambda}}{\sin(\theta_1)^\lambda},$$

for every measurable step function $f : \mathbb{R} \to L^2(X,\mu) \cap L^p(X,\mu)$ with compact support, and hence also for every continuously differentiable function $f : \mathbb{R} \to L^p(X,\mu)$ with compact support. Since $(1-\lambda)\theta_0 + \lambda\theta_1 = \pi/2$, by (11.41), it follows from Lemma 11.4 that

$$(U_\lambda f)(t) = A\int_0^t S(t-s)f(s)\,ds + f(t)$$

and hence

$$\left(\int_0^\infty \left\| A\int_0^t S(t-s)f(s)\,ds\right\|_{L^p(X)}^p dt\right)^{1/p} \le C_p\left(\int_0^\infty \|f(t)\|_{L^p(X)}^p\,dt\right)^{1/p}$$

for every continuously differentiable function $f : \mathbb{R} \to L^p(X,\mu)$ with compact support, where $C_p := c_p + 1$. This proves Theorem 11.2. □

Proof of Theorem 1.1. By Theorem 11.2 the assertion of Theorem 1.1 holds for $p = q$. Hence it holds for all p and q by Theorem 9.3. □

12 Besov spaces

Throughout $|h|$ denotes the Euclidean norm of $h \in \mathbb{R}^n$, the closed ball in $\mathbb{R}^n$ of radius r, centered at the origin, is denoted by

$$B_r := \{h \in \mathbb{R}^n \,|\, |h| \le r\},$$

and μ denotes the Lebesgue measure on $\mathbb{R}^n$. For $f \in L^p(\mathbb{R}^n, \mathbb{C}^m)$ and $h \in \mathbb{R}^n$ define the functions $\Delta_h f, \Delta_h^2 f \in L^p(\mathbb{R}^n, \mathbb{C}^m)$ by

$$\begin{aligned}(\Delta_h f)(x) &:= f(x+h) - f(x),\\ (\Delta_h^2 f)(x) &:= f(x+2h) - 2f(x+h) + f(x).\end{aligned} \tag{12.1}$$

For $f \in L^p(\mathbb{R}^n, \mathbb{C}^m)$ and $r > 0$ define

$$\omega_0(r,f)_p := \left(\frac{1}{\mu(B_r)} \int_{B_r} \|\Delta_h f\|_{L^p}^p \, dh\right)^{1/p} \tag{12.2}$$

and

$$\omega_1(r,f)_p := \sup_{|h|\le r} \|\Delta_h f\|_{L^p}, \qquad \omega_2(r,f)_p := \sup_{|h|\le r} \left\|\Delta_h^2 f\right\|_{L^p}. \tag{12.3}$$

Theorem 12.1. *Fix integers $n, m \in \mathbb{N}$ and real numbers $p, q \ge 1$, $0 < s < 1$. Then the following four norms on $C_0^\infty(\mathbb{R}^n, \mathbb{C}^m)$ are equivalent:*

$$\|f\|_{b^{s,p}_{q,0}} := \left(\int_0^\infty \left(\frac{\omega_0(r,f)_p}{r^s}\right)^q \frac{dr}{r}\right)^{1/q}, \tag{12.4}$$

$$\|f\|_{b^{s,p}_{q,1}} := \left(\int_0^\infty \left(\frac{\omega_1(r,f)_p}{r^s}\right)^q \frac{dr}{r}\right)^{1/q}, \tag{12.5}$$

$$\|f\|_{b^{s,p}_{q,2}} := \left(\int_0^\infty \left(\frac{\omega_2(r,f)_p}{r^s}\right)^q \frac{dr}{r}\right)^{1/q}, \tag{12.6}$$

$$\|f\|_{b^{s,p}_{q,3}} := \left(\sum_{k\in\mathbb{Z}} \left(\frac{\omega_2(2^k,f)_p}{2^{ks}}\right)^q\right)^{1/q}. \tag{12.7}$$

The equivalence of the norms (12.6) *and* (12.7) *continues to hold for $1 \le s < 2$. All these equivalences extend to the case $q = \infty$ (where the L^q norm with respect to the measure dr/r is replaced by the supremum).*

Proof. See page 94. □

Definition 12.2. *Let $n, m \in \mathbb{N}$ and $1 \le p, q < \infty$ and $0 < s < 2$. The completion of $C_0^\infty(\mathbb{R}^n, \mathbb{C}^m)$ with respect to the norm in* (12.6) *is called the* **homogeneous Besov space** *and is denoted by $b_q^{s,p}(\mathbb{R}^n, \mathbb{C}^m)$. The completion of $C_0^\infty(\mathbb{R}^n, \mathbb{C}^m)$ with respect to the norm*

$$\|f\|_{B^{s,p}_{q,2}} := \|f\|_{L^p} + \|f\|_{b^{s,p}_{q,2}}$$

is called the **Besov space** *and is denoted by $B_q^{s,p}(\mathbb{R}^n, \mathbb{C}^m)$. These spaces were introduced in 1959 by Besov [3]. The definition extends to $q = \infty$, with the L^q-norm in* (12.6) *with respect to dr/r replaced by the supremum.*

We emphasize that Definition 12.2 allows for $s \ge 1$, while Theorem 12.1 is restricted to the case $s < 1$. The norms (12.4) and (12.5) are infinite for nonconstant functions when $s \ge 1$ and thus cannot be used directly to define the Besov spaces $B_q^{s,p}$ for $s \ge 1$.

Lemma 12.3. *Fix an integer $n \in \mathbb{N}$ and a real number $p \ge 1$, and define $c := c(n,p) := 2^{1/p} + 2^{1+(n+1)/p}$. Then, for all $f \in L^p(\mathbb{R}^n, \mathbb{C}^m)$ and all $r > 0$,*

$$\omega_0(r,f)_p \le \omega_1(r,f)_p \le c\,\omega_0(r,f)_p. \tag{12.8}$$

Proof. The first inequality in (12.8) follows directly from the definitions. To prove the second inequality, fix an element $f \in L^p(\mathbb{R}^n, \mathbb{C}^m)$ and abbreviate $\phi(r) := \omega_0(f,r)_p$ for $r > 0$. Since $\|\Delta_{2h} f\|_{L^p} \le 2\|\Delta_h f\|_{L^p}$ for all h, we have

$$\phi(2r)^p = \frac{1}{\mu(B_r)} \int_{B_r} \|\Delta_{2h} f\|_{L^p}^p \, dh \le \frac{1}{\mu(B_r)} \int_{B_{2r}} 2^p \|\Delta_h f\|_{L^p}^p \, dh = 2^p \phi(r)^p.$$

Now let $r > 0$ and suppose, by contradiction, that there is an element $h_0 \in B_r$ such that $\|\Delta_{h_0} f\|_{L^p} > c\phi(r)$. Define $A_r := \{h \in B_r \mid \|\Delta_h f\|_{L^p} \le 2^{1/p}\phi(r)\}$. Then $\|\Delta_h f\|_{L^p}^p > 2\phi(r)^p$ for all $h \in B_r \setminus A_r$ and so

$$\begin{aligned} \frac{\mu(B_r \setminus A_r)}{\mu(B_r)} 2\phi(r)^p &< \frac{1}{\mu(B_r)} \int_{B_r \setminus A_r} \|\Delta_h f\|_{L^p}^p \, dh \\ &\le \frac{1}{\mu(B_r)} \int_{B_r} \|\Delta_h f\|_{L^p}^p \, dh = \phi(r)^p. \end{aligned}$$

Since $\phi(r) \ne 0$, it follows that $\mu(B_r \setminus A_r) < \mu(B_r)/2$ and so $\mu(A_r) > \mu(B_r)/2$. Moreover, for all $h \in A_r$,

$$\|\Delta_{h_0 - h} f\|_{L^p} \ge \|\Delta_{h_0} f\|_{L^p} - \|\Delta_h f\|_{L^p} > \left(c - 2^{1/p}\right)\phi(r) = 2^{1+(n+1)/p}\phi(r).$$

Since $h_0 - A_r \subset B_{2r}$, it follows that

$$\begin{aligned}\phi(2r)^p &\geq \frac{1}{\mu(B_{2r})}\int_{h_0-A_r}\|\Delta_h f\|_{L^p}^p\,dh \\ &> \frac{\mu(A_r)}{\mu(B_{2r})}2^{p+n+1}\phi(r)^p > \frac{\mu(B_r)}{\mu(B_{2r})}2^{p+n}\phi(r)^p = 2^p\phi(r)^p,\end{aligned}$$

contradicting to the inequality $\phi(2r) \leq 2\phi(r)$. This proves Lemma 12.3. □

Lemma 12.4 (**Marchaud inequality).** *Fix two integers $n, m \in \mathbb{N}$ and a real number $p \geq 1$. Then*

$$\frac{\omega_1(r,f)_p}{r} \leq 2\int_r^\infty \frac{\omega_2(\rho,f)_p}{\rho^2}\,d\rho \tag{12.9}$$

for all $f \in L^p(\mathbb{R}^n, \mathbb{C}^m)$ and all $r > 0$.

Proof. Since $\Delta_h^2 f = \Delta_{2h}f - 2\Delta_h f$, we have

$$\left\|\Delta_h f - 2^{-1}\Delta_{2h}f\right\|_{L^p} = 2^{-1}\left\|\Delta_h^2 f\right\|_{L^p}$$

for all $h \in \mathbb{R}^n$. Replace h by $2^k h$ to obtain

$$\left\|2^{-k}\Delta_{2^k h}f - 2^{-k-1}\Delta_{2^{k+1}h}f\right\|_{L^p} = 2^{-k-1}\left\|\Delta_{2^k h}^2 f\right\|_{L^p}$$

for $k \in \mathbb{Z}$ and $h \in \mathbb{R}^n$. Take the sum over $k = 0, 1, \ldots, m-1$ to obtain

$$\begin{aligned}\left\|\Delta_h f - 2^{-m}\Delta_{2^m h}f\right\|_{L^p} &\leq \sum_{k=0}^{m-1} 2^{-k-1}\left\|\Delta_{2^k h}^2 f\right\|_{L^p} \\ &\leq \sum_{k=0}^{m-1} 2^{-k-1}\omega_2(2^k r, f)_p) \\ &= 2r\sum_{k=0}^{m-1}\frac{\omega_2(2^k r, f)_p}{(2^{k+1}r)^2}2^k r \\ &\leq 2r\int_r^{2^m r}\frac{\omega_2(\rho,f)_p}{\rho^2}\,d\rho\end{aligned}$$

for every $h \in \mathbb{R}^n$ with $|h| \leq r$ and every $m \in \mathbb{N}$. This implies

$$\omega_1(r,f)_p \leq 2r\int_r^{2^m r}\frac{\omega_2(\rho,f)_p}{\rho^2}\,d\rho + 2^{-m}\omega_1(2^m r, f)_p.$$

Take the limit $m \to \infty$ and use the inequality $\omega_1(2^m r, f)_p \leq 2\|f\|_{L^p}$ to obtain the estimate (12.9). This proves Lemma 12.4. □

Proof of Theorem 12.1. By definition and Lemma 12.3, we have

$$\frac{1}{\mu(B_r)}\int_{B_r}\|\Delta_h f\|_{L^p}^p\,dh \le \omega_1(r,f)_p^p \le \frac{c^p}{\mu(B_r)}\int_{B_r}\|\Delta_h f\|_{L^p}^p\,dh$$

with $c := 2^{1/p} + 2^{1+(n+1)/p}$. Hence

$$\|f\|_{b^{s,p}_{q,0}} \le \|f\|_{b^{s,p}_{q,1}} \le \left(2^{1/p} + 2^{1+(n+1)/p}\right)\|f\|_{b^{s,p}_{q,0}}.$$

This shows that the norms (12.4) and (12.5) are equivalent. Second, it follows directly from equation (12.1) that $\|\Delta_h^2 f\|_{L^p} \le 2\|\Delta_h f\|_{L^p}$ and hence

$$\|f\|_{b^{s,p}_{q,2}} \le 2\|f\|_{b^{s,p}_{q,1}}.$$

Third, the Hardy inequality asserts that every Lebesgue measurable function $\phi : (0,\infty) \to [0,\infty)$ satisfies the inequality

$$\left(\int_0^\infty r^{a-1}\left(\int_r^\infty \phi(\rho)\,d\rho\right)^q dr\right)^{1/q} \le \frac{q}{a}\left(\int_0^\infty r^{q+a-1}\phi(r)^q\,dr\right)^{1/q}. \quad (12.10)$$

for any two real numbers $q \ge 1$ and $a > 0$ (see for example [33, Exercise 4.52]). Apply this inequality with $a := q(1-s)$ to the function $\phi(r) := r^{-2}\omega_2(r,f)_p$ and use the Marchaud inequality

$$\omega_1(r,f)_p \le 2r\int_r^\infty \frac{\omega_2(\rho,f)_p}{\rho^2}\,d\rho$$

in Lemma 12.4 to obtain the estimate

$$\begin{aligned}
\|f\|_{b^{s,p}_{q,1}} &= \left(\int_0^\infty r^{-sq-1}\omega_1(r,f)_p^q\,dr\right)^{1/q} \\
&\le 2\left(\int_0^\infty r^{q(1-s)-1}\left(\int_r^\infty \frac{\omega_2(\rho,f)_p}{\rho^2}\,d\rho\right)^q dr\right)^{1/q} \\
&\le \frac{2}{1-s}\left(\int_0^\infty r^{q+q(1-s)-1}\left(\frac{\omega_2(r,f)_p}{r^2}\right)^q dr\right)^{1/q} \\
&= \frac{2}{1-s}\left(\int_0^\infty r^{-qs-1}\omega_2(r,f)_p^q\,dr\right)^{1/q} \\
&= \frac{2}{1-s}\|f\|_{b^{s,p}_{q,2}}.
\end{aligned}$$

This shows that the norms (12.5) and (12.6) are equivalent.

Now assume $0 < s < 2$ and observe that

$$\begin{aligned}
\int_0^\infty \left(\frac{\omega_2(r,f)_p}{r^s}\right)^q \frac{dr}{r} &= \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \left(\frac{\omega_2(r,f)_p}{r^s}\right)^q \frac{dr}{r} \\
&\le \sum_{k\in\mathbb{Z}} 2^k \sup_{2^k\le r\le 2^{k+1}} \left(\frac{\omega_2(r,f)_p}{r^s}\right)^q \frac{1}{r} \\
&\le \sum_{k\in\mathbb{Z}} 2^k \left(\frac{\omega_2(2^{k+1},f)_p}{2^{ks}}\right)^q \frac{1}{2^k} \\
&= 2^{qs} \sum_{k\in\mathbb{Z}} \left(2^{-ks}\omega_2(2^k,f)_p\right)^q, \\
\sum_{k\in\mathbb{Z}} \left(2^{-ks}\omega_2(2^k,f)_p\right)^q &= 2^{1+qs} \sum_{k\in\mathbb{Z}} 2^k \left(\frac{\omega_2(2^k,f)_p}{2^{(k+1)s}}\right)^q \frac{1}{2^{k+1}} \\
&\le 2^{1+qs} \sum_{k\in\mathbb{Z}} 2^k \inf_{2^k\le r\le 2^{k+1}} \left(\frac{\omega_2(r,f)_p}{r^s}\right)^q \frac{1}{r} \\
&\le 2^{1+qs} \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \left(\frac{\omega_2(r,f)_p}{r^s}\right)^q \frac{dr}{r} \\
&= 2^{1+qs} \int_0^\infty \left(\frac{\omega_2(r,f)_p}{r^s}\right)^q \frac{dr}{r}.
\end{aligned}$$

Hence $2^{-s}\|f\|_{b^{s,p}_{q,2}} \le \|f\|_{b^{s,p}_{q,3}} \le 2^{s+1/q}\|f\|_{b^{s,p}_{q,2}}$, so the norms (12.6) and (12.7) are equivalent. This proves Theorem 12.1. □

Corollary 12.5. **(i)** *If $1 \le q_1 \le q_2 \le \infty$ then $B^{s,p}_{q_1} \subset B^{s,p}_{q_2}$ for all p and s.*
(ii) *If $0 < s_1 < s_2 < 2$ then $B^{s_2,p}_{q_2} \subset B^{s_1,p}_{q_1}$ for all p, q_1, q_2.*

Proof. $B^{s,p}_q$ is the completion of $C_0^\infty(\mathbb{R}^n,\mathbb{C}^m)$ with respect to the norm

$$\|f\|_{B^{s,p}_q} := \|f\|_{L^p} + \left(\sum_{k=0}^\infty \left(2^{ks}\omega_2(2^{-k},f)_p\right)^q\right)^{1/q},$$

which is nondecreasing in s and nonincreasing in q. This implies (i) and $B^{s_2,p}_q \subset B^{s_1,p}_q$ for $s_1 < s_2$. Moreover, for $0 < s < s+\varepsilon < 2$, we have $\sum_{k=0}^\infty 2^{ks}\omega_2(2^{-k},f)_p \le c_\varepsilon \sup_{k\in\mathbb{N}_0} 2^{k(s+\varepsilon}\omega_2(2^{-k},f)_p$, where $c_\varepsilon := 1/(1-2^{-\varepsilon})$. If $\varepsilon := s_2 - s_1$, this yields $\|f\|_{B^{s_1,p}_{q_1}} \le \|f\|_{B^{s_1,p}_1} \le c_\varepsilon \|f\|_{B^{s_2,p}_\infty} \le c_\varepsilon \|f\|_{B^{s_2,p}_{q_2}}$. This proves Corollary 12.5. □

Definition 12.6. *Let $n \in \mathbb{N}$ and fix real numbers $p \geq 1$ and $0 < s < 1$. The completion of $C_0^\infty(\mathbb{R}^n, \mathbb{C})$ with respect to the norm*

$$\|f\|_{w^{s,p}} := \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \, dy \, dx \right)^{1/p} \tag{12.11}$$

is called the **homogeneous Sobolev-Slobodeckij space** *and is denoted by $w^{s,p}(\mathbb{R}^n, \mathbb{C})$. The completion of $C_0^\infty(\mathbb{R}^n, \mathbb{C})$ with respect to the norm*

$$\|f\|_{W^{s,p}} := \|f\|_{L^p} + \|f\|_{w^{s,p}}$$

is called the **Sobolev-Slobodeckij space** *and is denoted by $W^{s,p}(\mathbb{R}^n, \mathbb{C})$. These are refinements of the Sobolev spaces $w^{1,p}(\mathbb{R}^n, \mathbb{C})$ and $W^{1,p}(\mathbb{R}^n, \mathbb{C})$ in Definition 12.8 below. They were introduced in 1958 by Slobodeckij [34].*

Lemma 12.7. *Fix an integer $n \in \mathbb{N}$ and real numbers $p \geq 1$ and $0 < s < 1$. Then, for every $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$,*

$$\|f\|_{b^{s,p}_{p,0}} = \left(\frac{1}{(n+sp)\mu(B_1)} \right)^{1/p} \|f\|_{w^{s,p}} .$$

Hence $b^{s,p}_p(\mathbb{R}^n, \mathbb{C}) = w^{s,p}(\mathbb{R}^n, \mathbb{C})$ and $B^{s,p}_p(\mathbb{R}^n, \mathbb{C}) = W^{s,p}(\mathbb{R}^n, \mathbb{C})$.

Proof. For $x \in \mathbb{R}^n$ and $r > 0$ define $S_r(x) := \{y \in \mathbb{R}^n \,|\, |y - x| = r\}$. Then

$$\begin{aligned}
\|f\|^p_{b^{s,p}_{p,0}} &= \int_0^\infty \frac{1}{r^{sp+1}} \int_{\mathbb{R}^n} \frac{1}{\mu(B_r)} \int_{B_r} |f(x) - f(x+h)|^p \, dh \, dx \, dr \\
&= \frac{1}{\mu(B_1)} \int_{\mathbb{R}^n} \int_0^\infty \frac{1}{r^{n+sp+1}} \int_{B_r(x)} |f(x) - f(y)|^p \, dy \, dr \, dx \\
&= \frac{1}{\mu(B_1)} \int_{\mathbb{R}^n} \int_0^\infty \int_0^r \frac{1}{r^{n+sp+1}} \int_{S_\rho(x)} |f(x) - f(y)|^p \, dS(y) \, d\rho \, dr \, dx \\
&= \frac{1}{\mu(B_1)} \int_{\mathbb{R}^n} \int_0^\infty \int_\rho^\infty \frac{dr}{r^{n+sp+1}} \int_{S_\rho(x)} |f(x) - f(y)|^p \, dS(y) \, d\rho \, dx \\
&= \frac{1}{(n+sp)\mu(B_1)} \int_{\mathbb{R}^n} \int_0^\infty \frac{1}{\rho^{n+sp}} \int_{S_\rho(x)} |f(x) - f(y)|^p \, dS(y) \, d\rho \, dx \\
&= \frac{1}{(n+sp)\mu(B_1)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \, dy \, dx \\
&= \frac{1}{(n+sp)\mu(B_1)} \|f\|^p_{w^{s,p}} .
\end{aligned}$$

This proves Lemma 12.7. □

Definition 12.8. *Fix an integer $n \in \mathbb{N}$ and a real number $p \geq 1$. The completion of $C_0^\infty(\mathbb{R}^n, \mathbb{C})$ with respect to the norm*

$$\|\nabla f\|_{L^p} = \left(\int_{\mathbb{R}^n} |\nabla f(x)|^p \, dx \right)^{1/p} \tag{12.12}$$

is called the **homogeneous Sobolev space** *and is denoted by $w^{1,p}(\mathbb{R}^n, \mathbb{C})$. The completion of $C_0^\infty(\mathbb{R}^n, \mathbb{C})$ with respect to the norm*

$$\|f\|_{W^{1,p}} := \|f\|_{L^p} + \|\nabla f\|_{L^p}$$

is called the **Sobolev space** *and is denoted by $W^{1,p}(\mathbb{R}^n, \mathbb{C})$.*

Lemma 12.9. *Fix an integer $n \in \mathbb{N}$ and a real number $p \geq 1$. Then every function $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$ satisfies the inequalities*

$$\sup_{r>0} \frac{\omega_1(r,f)_p}{r} \leq \|\nabla f\|_{L^p} \leq n \liminf_{r \to 0} \frac{\omega_1(r,f)_p}{r} \tag{12.13}$$

and

$$\frac{1}{2} \sup_{r>0} \frac{\omega_2(r,f)_p}{r} \leq \|\nabla f\|_{L^p} \leq 2n \int_0^\infty \frac{\omega_2(r,f)_p}{r^2} \, dr. \tag{12.14}$$

Hence

$$\begin{aligned} b_1^{1,p}(\mathbb{R}^n, \mathbb{C}) \subset w^{1,p}(\mathbb{R}^n, \mathbb{C}) \subset b_\infty^{1,p}(\mathbb{R}^n, \mathbb{C}), \\ B_1^{1,p}(\mathbb{R}^n, \mathbb{C}) \subset W^{1,p}(\mathbb{R}^n, \mathbb{C}) \subset B_\infty^{1,p}(\mathbb{R}^n, \mathbb{C}). \end{aligned} \tag{12.15}$$

Proof. The inequalities in (12.14) follow from (12.13) by Marchaud's inequality (12.9) in Lemma 12.4, and because $\omega_2(r,f)_p \leq 2\omega_1(r,f)_p$. To prove (12.13), abbreviate $\langle \nabla f(x), h \rangle := \sum_{i=1}^n \partial_i f(x) h_i$ for $x, h \in \mathbb{R}^n$. Then, by the fundamental theorem of calculus, $(\Delta_h f)(x) = \int_0^1 \langle \nabla f(x+th), h \rangle \, dt$ and hence

$$|(\Delta_h f)(x)| \leq |h| \int_0^1 |\nabla f(x+th)| \, dt. \tag{12.16}$$

By Minkowsky's inequality (e.g. [33, Theorem 7.19]), this implies

$$\begin{aligned} \|\Delta_h f\|_{L^p} &\leq |h| \left(\int_{\mathbb{R}^n} \left(\int_0^1 |\nabla f(x+th)| \, dt \right)^p dx \right)^{1/p} \\ &\leq |h| \int_0^1 \left(\int_{\mathbb{R}^n} |\nabla f(x+th)|^p \, dx \right)^{1/p} dt \\ &= |h| \, \|\nabla f\|_{L^p}. \end{aligned} \tag{12.17}$$

Take the supremum over all $h \in B_r$ to obtain $\omega_1(r,f)_p \leq r \|\nabla f\|_{L^p}$ for all $r > 0$. This proves the first inequality in (12.13).

Next we observe that

$$f(x+h) - f(x) - \langle \nabla f(x), h \rangle = \int_0^1 \langle \nabla f(x+th) - \nabla f(x), h \rangle \, dt$$

and hence

$$|(\Delta_h f)(x) - \langle \nabla f(x), h \rangle| \le |h| \int_0^1 |\nabla f(x+th) - \nabla f(x)| \, dt$$

for all $x, h \in \mathbb{R}^n$. Fix $h \in \mathbb{R}^n$, integrate the pth power of this inequality, and use Minkowsky's inequality, to obtain

$$\begin{aligned}
\|\Delta_h f - \langle \nabla f, h \rangle\|_{L^p} &\le |h| \left(\int_{\mathbb{R}^n} \left(\int_0^1 |\nabla f(x+th) - \nabla f(x)| \, dt \right)^p dx \right)^{1/p} \\
&\le |h| \int_0^1 \left(\int_{\mathbb{R}^n} |\nabla f(x+th) - \nabla f(x)|^p \, dx \right)^{1/p} dt \\
&= |h| \int_0^1 \|\Delta_{th} \nabla f\|_{L^p} \, dt.
\end{aligned}$$

For $r > 0$ and $h \in \mathbb{R}^n$ with $|h| \le r$ this implies

$$\|\langle \nabla f, h \rangle\|_{L^p} \le \|\Delta_h f\|_{L^p} + |h| \int_0^1 \|\Delta_{th} \nabla f\|_{L^p} \, dt \le \omega_1(r, f)_p + r\omega_1(r, \nabla f)_p.$$

Take $h := re_i$ to obtain

$$\|\partial_i f\|_{L^p} \le \frac{\omega_1(r, f)_p}{r} + \omega_1(r, \nabla f)_p$$

and hence

$$\|\partial_i f\|_{L^p} \le \inf_{0<\rho\le r} \left(\frac{\omega_1(\rho, f)_p}{\rho} + \omega_1(\rho, \nabla f)_p \right)$$

for all $r > 0$. Take the limit $r \to 0$ and use the fact that $\lim_{r\to 0} \omega_1(r, \nabla f)_p = 0$ to obtain

$$\|\partial_i f\|_{L^p} \le \liminf_{r\to 0} \frac{\omega_1(r, f)_p}{r} \qquad \text{for } i = 1, \ldots, n.$$

Hence

$$\|\nabla f\|_{L^p} \le \sum_{i=1}^n \|\partial_i f\|_{L^p} \le n \liminf_{r\to 0} \frac{\omega_1(r, f)_p}{r}.$$

This proves the second inequality in (12.13) and Lemma 12.9. □

Remark 12.10. The second inequality in (12.13) implies that the right hand side of (12.5) is infinite for $s \ge 1$ unless f is constant. In contrast, equation (12.6) still defines a meaningful norm for $1 \le s < 2$.

13 Besov, Littlewood–Paley, Peetre, Triebel

This section is devoted to the *Littlewood–Paley characterization* of the Besov spaces (Theorem 13.5). The starting point is the choice of a suitable smooth function $\phi : \mathbb{R}^n \to \mathbb{R}$ in the Schwartz space. The crucial point is that the characterization of the Besov space in Theorem 13.5 is independent of the choice of the function ϕ, as long as it satisfies the conditions (13.1) and (13.2) below. The connection between Besov spaces and Littlewood–Paley theory was first noted by Peetre [31].

Definition 13.1. *Fix an integer $n \in \mathbb{N}$ and a smooth function $\phi : \mathbb{R}^n \to \mathbb{R}$ in the Schwartz space $\mathscr{S}(\mathbb{R}^n)$. Assume*

$$\phi(x) = \phi(-x) \qquad \text{for all } x \in \mathbb{R}^n, \tag{13.1}$$

so that the Fourier transform

$$\widehat{\phi}(\xi) := \int_{\mathbb{R}^n} e^{-\mathbf{i}\langle x,\xi\rangle}\phi(x)\,dx$$

is real valued and satisfies

$$\widehat{\phi}(\xi) = \widehat{\phi}(-\xi) \qquad \text{for all } \xi \in \mathbb{R}^n.$$

The function ϕ is called a **Triebel function** *if there is a $0 < \delta \leq 1/2$ such that*

$$\begin{aligned} \widehat{\phi}(\xi) &> 0, && \text{for } 1/\sqrt{2} \leq |\xi| \leq \sqrt{2}, \\ \widehat{\phi}(\xi) &\geq 0, && \text{for } 1/2 \leq |\xi| \leq 2, \\ \widehat{\phi}(\xi) &= 0, && \text{for } |\xi| \notin [\delta, 1/\delta]. \end{aligned} \tag{13.2}$$

It follows from (13.1) and (13.2) that

$$\int_{\mathbb{R}^n} \phi(x)\,dx = 0, \qquad \int_{\mathbb{R}^n} x_i\phi(x)\,dx = 0 \qquad \text{for } i = 1,\dots,n. \tag{13.3}$$

In particular, every Littlewood-Paley function is a Triebel function (see Definition 8.1).

Lemma 13.2. (i) *If $\phi, \psi : \mathbb{R}^n \to \mathbb{R}$ are Triebel functions then so is $\phi * \psi$.*
(ii) *For every Triebel function $\phi : \mathbb{R}^n \to \mathbb{R}$ there exists a Triebel function $\psi : \mathbb{R}^n \to \mathbb{R}$ such that $\operatorname{supp}(\widehat{\psi}) \subset \{\xi \in \mathbb{R}^n \,|\, 1/2 \leq |\xi| \leq 2\}$ and $\phi * \psi$ is a Littlewood–Paley function.*

Proof. Part (i) follows directly from the definition and the fact that the Fourier transform of the convolution $\phi * \psi$ is the product $\widehat{\phi}\widehat{\psi}$ of the Fourier transforms. To prove (ii) fix a Triebel function $\phi : \mathbb{R}^n \to \mathbb{R}$. Then there exists a constant $\sqrt{2} < \alpha < 2$ such that $\widehat{\phi}(\xi) > 0$ for all $\xi \in \mathbb{R}^n$ such that $1/\alpha \le |\xi| \le \alpha$. Choose a smooth function $\beta : [1,2] \to [0,1]$ such that

$$\begin{array}{ll} \beta(r) = 1, & \text{for } 1 \le r \le 2/\alpha, \\ \beta(r) > 0, & \text{for } 2/\alpha \le r \le \sqrt{2}, \\ \beta(r) < 1, & \text{for } \sqrt{2} \le r \le \alpha, \\ \beta(r) = 0, & \text{for } \alpha \le r \le 2, \end{array}$$

and extend it to $[0,\infty)$ by setting $\beta(r) := 0$ for $r \in [0,1/2] \cup [2,\infty)$ and

$$\beta(r) := 1 - \beta(2r) \qquad \text{for } 1/2 \le r \le 1.$$

Then $\mathrm{supp}(\beta) \subset [1/\alpha, \alpha]$ and $\beta(r) > 0$ for $1/\sqrt{2} \le r \le \sqrt{2}$.

Now let $\psi : \mathbb{R}^n \to \mathbb{C}$ be the unique function in the Schwartz space $\mathscr{S}(\mathbb{R}^n, \mathbb{C})$ whose Fourier transform is given by

$$\widehat{\psi}(\xi) = \begin{cases} \beta(|\xi|)/\widehat{\phi}(\xi), & \text{if } 1/\alpha \le |\xi| \le \alpha, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\widehat{\psi}$ is a real valued function and $\widehat{\psi}(\xi) = \widehat{\psi}(-\xi) \ge 0$ for all $\xi \in \mathbb{R}^n$. Hence ψ is real valued and

$$\psi(x) = \psi(-x) \qquad \text{for all } x \in \mathbb{R}^n.$$

Second,

$$\mathrm{supp}(\widehat{\psi}) \subset \{\xi \in \mathbb{R}^n \,|\, 1/\alpha \le |\xi| \le \alpha\} \subset \{\xi \in \mathbb{R}^n \,|\, 1/2 \le |\xi| \le 2\}$$

and

$$\widehat{\psi}(\xi) = \beta(|\xi|)/\widehat{\phi}(\xi) > 0 \qquad \text{for all } \xi \in \mathbb{R}^n \text{ such that } 1/\sqrt{2} \le |\xi| \le \sqrt{2}.$$

Hence ψ is a Triebel function. Third,

$$\widehat{\phi}(\xi/2)\widehat{\psi}(\xi/2) + \widehat{\phi}(\xi)\widehat{\psi}(\xi) = \beta(|\xi|/2) + \beta(|\xi|) = 1$$

for all $\xi \in \mathbb{R}^n$ with $1 \le |\xi| \le 2$ and so $\phi * \psi$ is a Littlewood–Paley function. This proves Lemma 13.2. □

Definition 13.3 (Peetre Maximal Functions).
Let $n \in \mathbb{N}$ and let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a Triebel function. For $k \in \mathbb{Z}$ and $t > 0$ define the functions $\phi_k, \varphi_t : \mathbb{R}^n \to \mathbb{R}$ by

$$\phi_k(x) := 2^{kn}\phi(2^k x), \qquad \varphi_t(x) = t^{-n}\phi(t^{-1}x) \qquad \text{for } x \in \mathbb{R}^n. \tag{13.4}$$

Thus $\widehat{\phi}_k(\xi) = \widehat{\phi}(2^{-k}\xi)$ and $\widehat{\varphi}_t(\xi) = \widehat{\phi}(t\xi)$ for all $\xi \in \mathbb{R}^n$ and

$$\begin{aligned} \mathrm{supp}(\widehat{\phi}_k) &\subset \{\xi \in \mathbb{R}^n \,|\, \delta 2^k \le |\xi| \le 2^k/\delta\}, \\ \mathrm{supp}(\widehat{\varphi}_t) &\subset \{\xi \in \mathbb{R}^n \,|\, \delta/t \le |\xi| \le 1/\delta t\}, \end{aligned}$$

where $\delta > 0$ is as in (13.2). Now fix a real number $\lambda > 0$. For $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$ denote by $\phi_k * f$ and $\varphi_t * f$ the convolution and define the **Peetre maximal functions** *$\phi^*_{k,\lambda} f, \varphi^*_{t,\lambda} f : \mathbb{R}^n \to [0, \infty)$ by*

$$\begin{aligned} (\phi^*_{k,\lambda} f)(x) &:= \sup_{z \in \mathbb{R}^n} \frac{|(\phi_k * f)(x+z)|}{(1 + 2^k|z|)^\lambda}, \\ (\varphi^*_{t,\lambda} f)(x) &:= \sup_{z \in \mathbb{R}^n} \frac{|(\varphi_t * f)(x+z)|}{(1 + |z|/t)^\lambda}. \end{aligned} \tag{13.5}$$

for $x \in \mathbb{R}^n$, $k \in \mathbb{Z}$, and $t > 0$.

Lemma 13.4. *Let $n \in \mathbb{N}$ and $\lambda > 0$ and let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a Triebel function. Then the following holds.*
(i) *The Peetre maximal functions satisfy the inequalities*

$$\begin{aligned} (\phi^*_{k,\lambda} f)(x+y) &\le (1 + 2^k|y|)^\lambda (\phi^*_{k,\lambda} f)(x), \\ (\varphi^*_{t,\lambda} f)(x+y) &\le (1 + |y|/t)^\lambda (\varphi^*_{t,\lambda} f)(x) \end{aligned} \tag{13.6}$$

for all $k \in \mathbb{Z}$, $t > 0$, $x, y \in \mathbb{R}^n$, and $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$.
(ii) *For every $\ell \in \mathbb{N}_0$ there exists a constant $c = c(n, \ell, \lambda, \phi) > 0$ such that*

$$\begin{aligned} \sum_{|\alpha| \le \ell} |\partial^\alpha(\phi_k * f)(x)| &\le c 2^{k\ell} \sup_{z \in \mathbb{R}^n} \frac{|f(x+z)|}{(1 + 2^k|z|)^\lambda}, \\ \sum_{|\alpha| \le \ell} |\partial^\alpha(\varphi_t * f)(x)| &\le c t^{-\ell} \sup_{z \in \mathbb{R}^n} \frac{|f(x+z)|}{(1 + |z|/t)^\lambda} \end{aligned} \tag{13.7}$$

for all $k \in \mathbb{Z}$, $t > 0$, $x \in \mathbb{R}^n$, and $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$. Here the sum runs over all multi-indices $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ such that $|\alpha| = \alpha_1 + \cdots + \alpha_n \le \ell$.

Proof. For all $t > 0$ and all $x, y, z \in \mathbb{R}^n$, we have

$$\frac{1}{1 + |z - x - y|/t} \le \frac{1 + |y|/t}{1 + |z - x|/t}.$$

Hence

$$\begin{aligned}
(\varphi^*_{t,\lambda} f)(x + y) &= \sup_{z \in \mathbb{R}^n} \frac{|(\varphi_t * f)(z)|}{(1 + |z - x - y|/t)^\lambda} \\
&\le (1 + |y|/t)^\lambda \sup_{z \in \mathbb{R}^n} \frac{|(\varphi_t * f)(z)|}{(1 + |z - x|/t)^\lambda} \\
&= (1 + |y|/t)^\lambda (\varphi^*_{t,\lambda} f)(x).
\end{aligned}$$

This proves the second inequality in (13.6). The first inequality follows by taking $t := 2^{-k}$. This proves part (i).

To prove part (ii), define the constant $c_\alpha := c_\alpha(n, \lambda, \phi)$ by

$$c_\alpha := \int_{\mathbb{R}^n} (1 + |y|)^\lambda \, |\partial^\alpha \phi(y)| \, dy$$

for every multi-index $\alpha \in \mathbb{N}_0^n$. This number is finite because the function ϕ and all its derivatives belong to the Schwartz space $\mathscr{S}(\mathbb{R}^n)$. Now fix a number $t > 0$. Since $\varphi_t(x) = t^{-n}\phi(t^{-1}x)$ for all $x \in \mathbb{R}^n$, we have

$$\begin{aligned}
(\varphi_t * f)(x) &= \int_{\mathbb{R}^n} \phi(t^{-1}x - y) f(ty) \, dy, \\
\partial^\alpha(\varphi_t * f)(x) &= t^{-|\alpha|} \int_{\mathbb{R}^n} (\partial^\alpha \phi)(z) f(x - tz) \, dz.
\end{aligned}$$

Hence

$$\begin{aligned}
|\partial^\alpha(\varphi_t * f)(x)| &= t^{-|\alpha|} \left| \int_{\mathbb{R}^n} (\partial^\alpha \phi)(z) f(x - tz) \, dz \right| \\
&= t^{-|\alpha|} \left| \int_{\mathbb{R}^n} (1 + |z|)^\lambda (\partial^\alpha \phi)(z) \frac{f(x - tz)}{(1 + |z|)^\lambda} \, dz \right| \\
&\le t^{-|\alpha|} \int_{\mathbb{R}^n} (1 + |z|)^\lambda \, |(\partial^\alpha \phi)(z)| \, \frac{|f(x - tz)|}{(1 + |z|)^\lambda} \, dz \\
&\le c_\alpha t^{-|\alpha|} \sup_{z \in \mathbb{R}^n} \frac{|f(x - tz)|}{(1 + |z|)^\lambda} \\
&\le c_\alpha t^{-|\alpha|} \sup_{z \in \mathbb{R}^n} \frac{|f(x + z)|}{(1 + |z|/t)^\lambda}.
\end{aligned}$$

This proves the second inequality in (13.7) with

$$c = c(n, \ell, \lambda, \phi) := \sum_{|\alpha| \le \ell} c_\alpha(n, \lambda, \phi).$$

The first inequality follows by taking $t = 2^{-k}$. This proves Lemma 13.4. $\square$

The next theorem is the main result of this section. It is stated in this form in Ullrich [40, Theorem 2.9]. The proof given below is based on the proof in [40] of the analogous, but technically more difficult, result about the (in)homogeneous Triebel–Lizorkin spaces, and on the paper by Besov [4] about weighted Besov and Triebel–Lizorkin spaces.

Theorem 13.5 **(Peetre/Triebel).** *Fix an integer* $n \in \mathbb{N}$ *and real numbers* $p, q \ge 1$*,* $s \ge 0$*, and* $\lambda > n/p$*.*

(i) *Let* $\phi : \mathbb{R}^n \to \mathbb{R}$ *be a Triebel function, so it satisfies* (13.1) *and* (13.2)*. For* $k \in \mathbb{Z}$ *and* $t > 0$ *define the Peetre maximal functions* $\phi_{k,\lambda}$ *and* $\varphi_{t,\lambda}$ *by* (13.4)*. Then the formulas*

$$\|f\|_{b^{s,p}_{q,4}} := \|f\|_{b^{s,p}_{q,4;\phi}} := \left(\sum_{k=-\infty}^{\infty} \left(2^{ks} \|\phi_k * f\|_{L^p} \right)^q \right)^{1/q}, \tag{13.8}$$

$$\|f\|_{b^{s,p}_{q,5}} := \|f\|_{b^{s,p}_{q,5;\phi,\lambda}} := \left(\sum_{k=-\infty}^{\infty} \left(2^{ks} \left\|\phi^*_{k,\lambda} f\right\|_{L^p} \right)^q \right)^{1/q}, \tag{13.9}$$

$$\|f\|_{b^{s,p}_{q,6}} := \|f\|_{b^{s,p}_{q,6;\phi}} := \left(\int_0^\infty \left(\frac{\|\varphi_t * f\|_{L^p}}{t^s} \right)^q \frac{dt}{t} \right)^{1/q}, \tag{13.10}$$

$$\|f\|_{b^{s,p}_{q,7}} := \|f\|_{b^{s,p}_{q,7;\phi,\lambda}} := \left(\int_0^\infty \left(\frac{\left\|\varphi^*_{t,\lambda} f\right\|_{L^p}}{t^s} \right)^q \frac{dt}{t} \right)^{1/q} \tag{13.11}$$

(for $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$*) define equivalent norms on* $C_0^\infty(\mathbb{R}^n, \mathbb{C})$*. The equivalence class of these norms is independent of the choice of the Triebel function* ϕ*.*

(ii) *Assume* $p > 1$ *and* $0 < s < 2$*. Then the norms in part (i) are equivalent to the norm* $\|\cdot\|_{b^{s,p}_{q,2}}$ *in* (12.6)*.*

Proof. The proof has six steps. The first four steps prove part (i) and the last two steps prove part (ii). Throughout $n \in \mathbb{N}$ is fixed.

Step 1 (Peetre). *Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a Triebel function and let $s \geq 0$, $p, q \geq 1$, and $\lambda > n/p$. Then there exists a constant $c = c(s, p, q, \phi, \lambda) > 0$ such that*

$$\|f\|_{b^{s,p}_{q,4;\phi}} \leq \|f\|_{b^{s,p}_{q,5;\phi,\lambda}} \leq c \|f\|_{b^{s,p}_{q,4;\phi}}$$

for all $f \in C_0^\infty(\mathbb{R}^n)$.

It follows directly from the definitions that $|(\phi_k * f)(x)| \leq (\phi^*_{k,\lambda} f)(x)$ for all $x \in \mathbb{R}^n$ all $f \in C_0^\infty(\mathbb{R}^n)$, and all $\lambda > 0$. This proves the first inequality. The second inequality in Step 3 is a theorem of Peetre [30]. The proof below follows the argument in Ullrich [40, Step 1] with the appropriate modifications.

Let $0 < \delta \leq 1/2$ be as in (13.2) and choose $m \in \mathbb{N}$ such that $2^{-m} < \delta$. We prove that, for each $\lambda > 0$, there is a $c = c(n, p, \lambda, \phi) > 0$ such that

$$|(\phi_k * f)(x)|^p \leq c^p \sum_{j=-m}^{m} \int_{\mathbb{R}^n} \frac{2^{kn} |(\phi_{j+k} * f)(y)|^p}{(1 + 2^k |x - y|)^{\lambda p}} \, dy \tag{13.12}$$

for all $k \in \mathbb{N}$, $x \in \mathbb{R}^n$, and $f \in C_0^\infty(\mathbb{R}^n)$. Choose a Triebel function ψ such that $\psi * \phi$ is a Littlewood–Paley function (Lemma 13.2). Then

$$\sum_{j=-m}^{m} \widehat{\psi}_j(\xi) \widehat{\phi}_j(\xi) = \sum_{j=-m}^{m} \widehat{\psi}(2^{-j}\xi) \widehat{\phi}(2^{-j}\xi) = 1 \qquad \text{for } 2^{-m} \leq |\xi| \leq 2^m.$$

This implies $\sum_{j=-m}^{m} \widehat{\psi}_j \widehat{\phi}_j \widehat{\phi} = \widehat{\phi}$, hence $\sum_{j=-m}^{m} \psi_j * \phi_j * \phi = \phi$, and hence

$$\sum_{j=-m}^{m} \psi_{j+k} * \phi_{j+k} * \phi_k = \phi_k \qquad \text{for all } k \in \mathbb{Z}.$$

Define $r := (n+1)(1 - 1/p)$. Then, for all $f \in C_0^\infty(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, $k \in \mathbb{Z}$,

$$\begin{aligned}
|(\phi_k * f)(x)| &\leq \sum_{j=-m}^{m} |(\psi_{j+k} * \phi_{j+k} * \phi_k * f)(x)| \\
&\leq \sum_{j=-m}^{m} \int_{\mathbb{R}^n} |(\psi_{j+k} * \phi_k)(x - y)| \, |(\phi_{j+k} * f)(y)| \, dy \\
&\leq \sum_{j=-m}^{m} c_{j,k,n,p,\lambda} \int_{\mathbb{R}^n} \frac{|(\phi_{j+k} * f)(y)|}{(1 + 2^k |x - y|)^{\lambda + r}} \, dy,
\end{aligned}$$

where $c_{j,k,n,p,\lambda} := \sup_{z \in \mathbb{R}^n} (1 + 2^k |z|)^{\lambda + r} \, |(\psi_{j+k} * \phi_k)(z)|$.

We estimate the constant $c_{j,k,n,p,\lambda}$. For $x \in \mathbb{R}^n$ and $|j| \le m$,

$$
\begin{aligned}
&(1+|z|)^{\lambda+r}\left|(\psi_{j+k}*\phi_k)(2^{-k}z)\right| \\
&\quad\le (1+|z|)^{\lambda+r}\int_{\mathbb{R}^n}\left|\psi_{j+k}(2^{-k}z-y)\right||\phi_k(y)|\,dy \\
&\quad= (1+|z|)^{\lambda+r}2^{kn}\int_{\mathbb{R}^n}\left|\psi(2^jz-y))\right||\phi(2^{-j}y)|\,dy \\
&\quad\le C_{n,p,\lambda}2^{kn}\int_{\mathbb{R}^n}\frac{(1+|z|)^{\lambda+r}}{(1+|2^jz-y|)^{\lambda+r}(1+2^{-j}|y|)^{n+1+\lambda+r}}\,dy \\
&\quad\le C_{n,p,\lambda}2^{kn}\int_{\mathbb{R}^n}\frac{2^{m(n+1+2\lambda+2r)}(1+|2^jz|)^{\lambda+r}}{(1+|2^jz-y|)^{\lambda+r}(1+|y|)^{n+1+\lambda+r}}\,dy \\
&\quad\le C_{n,p,\lambda}2^{m(n+1+2\lambda+2r)}2^{kn}\int_{\mathbb{R}^n}\frac{1}{(1+|y|)^{n+1}}\,dy.
\end{aligned}
$$

Here $C_{n,p,\lambda} > 0$ has been chosen such that $|\phi(y)| \le \sqrt{C_{n,p,\lambda}}(1+|y|)^{-n-1-\lambda-r}$ and $|\psi(y)| \le \sqrt{C_{n,p,\lambda}}(1+|y|)^{-\lambda-r}$ for all $y \in \mathbb{R}^n$. Now take the supremum over all $z \in \mathbb{R}^n$ to obtain the inequality

$$
c_{j,k,n,p,\lambda} \le C2^{kn}, \qquad C := C_{n,p,\lambda}2^{m(n+1+2\lambda+2r)}\int_{\mathbb{R}^n}\frac{1}{(1+|y|)^{n+1}}\,dy.
$$

Thus we have proved the estimate

$$
|(\phi_k * f)(x)| \le C\sum_{j=-m}^{m}\int_{\mathbb{R}^n}\frac{2^{kn}\,|(\phi_{j+k}*f)(y)|}{(1+2^k|x-y|)^{\lambda+r}}\,dy \tag{13.13}
$$

for all $k \in \mathbb{N}$, $x \in \mathbb{R}^n$, and $f \in C_0^\infty(\mathbb{R}^n)$. If $p = 1$ then $r = 0$ and so (13.13) is equivalent to (13.12). For $p > 1$ use Hölder's inequality with the exponents p, p' such that $1/p + 1/p' = 1$ to obtain $rp' = n+1$ and

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\frac{2^{kn}\,|(\phi_{j+k}*f)(y)|}{(1+2^k|x-y|)^{\lambda+r}}\,dy \\
&\le \left(\int_{\mathbb{R}^n}\frac{2^{kn}}{(1+|2^ky|)^{rp'}}\,dy\right)^{1/p'}\left(\int_{\mathbb{R}^n}\frac{2^{kn}\,|(\phi_{j+k}*f)(y)|^p}{(1+2^k|x-y|)^{\lambda p}}\,dy\right)^{1/p} \\
&= \left(\int_{\mathbb{R}^n}\frac{1}{(1+|y|)^{n+1}}\,dy\right)^{1-1/p}\left(\int_{\mathbb{R}^n}\frac{2^{kn}\,|(\phi_{j+k}*f)(y)|^p}{(1+2^k|x-y|)^{\lambda p}}\,dy\right)^{1/p}.
\end{aligned}
$$

Hence (13.12) holds with $c := C(\int_{\mathbb{R}^n}(1+|y|)^{-n-1}\,dy)^{1-1/p}(2m+1)^{1-1/p}$.

Thus we have proved the estimate (13.12). This implies

$$\begin{aligned}\frac{|(\phi_k * f)(x+z)|^p}{(1+2^k|z|)^{\lambda p}} &\leq c^p \sum_{j=-m}^{m} \int_{\mathbb{R}^n} \frac{2^{kn}\,|(\phi_{j+k} * f)(y)|^p}{(1+2^k|z|)^{\lambda p}(1+2^k|x+z-y|)^{\lambda p}}\,dy \\ &\leq c^p \sum_{j=-m}^{m} \int_{\mathbb{R}^n} \frac{2^{kn}\,|(\phi_{j+k} * f)(y)|^p}{(1+2^k|x-y|)^{\lambda p}}\,dy.\end{aligned}$$

Take the supremum over all $z \in \mathbb{R}^n$ to obtain the inequality

$$(\phi^*_{k,\lambda} f)(x)^p \leq c^p \sum_{j=-m}^{m} \int_{\mathbb{R}^n} \frac{2^{kn}\,|(\phi_{j+k} * f)(y)|^p}{(1+2^k|x-y|)^{\lambda p}}\,dy \tag{13.14}$$

for all $x \in \mathbb{R}^n$, $k \in \mathbb{Z}$, and $f \in C_0^\infty(\mathbb{R}^n)$. Assume $\lambda p > n$, integrate over $\mathbb{R}^n$, and use Fubini's Theorem to obtain

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{2^{kn}\,|(\phi_{j+k} * f)(y)|^p}{(1+2^k|x-y|)^{\lambda p}}\,dy\,dx &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\left|2^{kn}(\phi_{j+k} * f)(y)\right|^p}{(1+2^k|x-y|)^{\lambda p}}\,dx\,dy \\ &= \int_{\mathbb{R}^n} \frac{2^{kn}}{(1+|2^k x|)^{\lambda p}}\,dx\,\|\phi_{j+k} * f\|_{L^p}^p \\ &= \int_{\mathbb{R}^n} \frac{1}{(1+|x|)^{\lambda p}}\,dx\,\|\phi_{j+k} * f\|_{L^p}^p .\end{aligned}$$

Hence it follows from (13.14) that

$$\left\|\phi^*_{k,\lambda} f\right\|_{L^p} \leq c \left(\int_{\mathbb{R}^n} \frac{1}{(1+|x|)^{\lambda p}}\,dx\right)^{1/p} \sum_{j=-m}^{m} \|\phi_{j+k} * f\|_{L^p}, \tag{13.15}$$

where c is the constant in (13.12). Let $c' := c(\int_{\mathbb{R}^n}(1+|x|)^{-\lambda p}\,dx)^{1/p}$. Then

$$\begin{aligned}\|f\|_{b^{s,p}_{q,5;\phi,\lambda}} &= \left(\sum_{k=-\infty}^{\infty}\left(2^{ks}\left\|\phi^*_{k,\lambda} f\right\|_{L^p}\right)^q\right)^{1/q} \\ &\leq c'\left(\sum_{k=-\infty}^{\infty}\left(2^{ks}\sum_{j=-m}^{m}\|\phi_{j+k} * f\|_{L^p}\right)^q\right)^{1/q} \\ &\leq 2^{ms}c'\sum_{j=-m}^{m}\left(\sum_{k=-\infty}^{\infty}\left(2^{(j+k)s}\|\phi_{j+k} * f\|_{L^p}\right)^q\right)^{1/q} \\ &= 2^{ms}c'(2m+1)\,\|f\|_{b^{s,p}_{q,4;\phi}}\end{aligned}$$

This proves Step 1.

Step 2 (Peetre). *Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a Triebel function and let $s \geq 0$, $p, q \geq 1$, and $\lambda > n/p$. Then there exists a constant $c = c(s, p, q, \phi, \lambda) > 0$ such that*

$$\|f\|_{b^{s,p}_{q,6;\phi}} \leq \|f\|_{b^{s,p}_{q,7;\phi,\lambda}} \leq c\,\|f\|_{b^{s,p}_{q,6;\phi}}$$

for all $f \in C_0^\infty(\mathbb{R}^n)$.

This is the continuous time analogue of Step 1 and is proved by the same arguments. First choose $m \in \mathbb{N}$ such that $2^{-m} \leq \delta$. Then, for each $p > 1$ and each $\lambda > 0$, there is a constant $c = c(n, p, \lambda, \phi) > 0$, such that

$$|(\varphi_t * f)(x)|^p \leq c^p \sum_{k=-m}^{m} \int_{\mathbb{R}^n} \frac{t^{-n}\,|(\varphi_{2^k t} * f)(y)|^p}{(1 + |x-y|/t)^{\lambda p}}\,dy \tag{13.16}$$

for all $t > 0$, $x \in \mathbb{R}^n$, $f \in C_0^\infty(\mathbb{R}^n)$. This is proved by the same argument as (13.12), with 2^k replaced by t^{-1}. As before, the estimate (13.16) implies

$$(\varphi^*_{t,\lambda} f)(x)^p \leq c^p \sum_{j=-m}^{m} \int_{\mathbb{R}^n} \frac{t^{-n}\,|(\varphi_{2^j t} * f)(y)|^p}{(1 + |x-y|/t)^{\lambda p}}\,dy \tag{13.17}$$

and, for $\lambda p > n$, Fubini's Theorem gives

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{t^{-n}\,|(\varphi_{2^j t} * f)(y)|^p}{(1 + |x-y|/t)^{\lambda p}}\,dy\,dx = \int_{\mathbb{R}^n} \frac{1}{(1+|x|)^{\lambda p}}\,dx\,\|\varphi_{2^j t} * f\|^p_{L^p}\,.$$

Hence it follows from (13.17) that

$$\left\|\varphi^*_{t,\lambda} f\right\|_{L^p} \leq c' \sum_{j=-m}^{m} \|\varphi_{2^j t} * f\|_{L^p}\,, \tag{13.18}$$

where $c' := c(\int_{\mathbb{R}^n} (1+|x|)^{-\lambda p}\,dx)^{1/p}$, and so

$$\begin{aligned}
\|f\|_{b^{s,p}_{q,7;\phi,\lambda}} &= \left(\int_0^\infty \left(t^{-s}\left\|\varphi^*_{t,\lambda} f\right\|_{L^p}\right)^q \frac{dt}{t}\right)^{1/q} \\
&\leq c' \left(\int_0^\infty \left(t^{-s} \sum_{j=-m}^{m} \|\varphi_{2^j t} * f\|_{L^p}\right)^q \frac{dt}{t}\right)^{1/q} \\
&= 2^{ms} c'(2m+1)\,\|f\|_{b^{s,p}_{q,6;\phi}}\,.
\end{aligned}$$

This proves Step 2.

Step 3. *Let $\phi, \psi : \mathbb{R}^n \to \mathbb{R}$ be Triebel functions and let $s \geq 0$ and $p, q \geq 1$. Then there exists a constant $c = c(s, p, q, \phi) > 0$ such that*

$$\|f\|_{b^{s,p}_{q,6;\phi}} \leq c \|f\|_{b^{s,p}_{q,4;\psi}}$$

for all $f \in C_0^\infty(\mathbb{R}^n)$.

Choose $\delta > 0$ such that

$$\mathrm{supp}(\widehat{\phi}) \subset \{\xi \in \mathbb{R}^n \,|\, \delta \leq |\xi| \leq 1/\delta\}$$

as in (13.2) and choose $m \in \mathbb{N}$ such that $2^{-m} \leq \delta/2$. Choose a Triebel function θ such that $\psi * \theta$ is a Littlewood–Paley function (Lemma 13.2). Then

$$\sum_{j=-m}^{m} \widehat{\theta}_{j+k}(\xi)\widehat{\psi}_{j+k}(\xi) = \sum_{j=-m}^{m} \widehat{\theta}(2^{-j-k}\xi)\widehat{\psi}(2^{-j-k}\xi) = 1$$

for $2^{k-1}\delta \leq |\xi| \leq 2^{k+1}/\delta$. Since

$$\mathrm{supp}(\widehat{\varphi}_{2^{-k}t}) \subset \{\xi \in \mathbb{R}^n \,|\, 2^{k-1}\delta \leq |\xi| \leq 2^k/\delta\}$$

for $1 \leq t \leq 2$ and $k \in \mathbb{Z}$, this implies

$$\sum_{j=-m}^{m} \widehat{\varphi}_{2^{-k}t}\widehat{\theta}_{j+k}\widehat{\psi}_{j+k} = \widehat{\varphi}_{2^{-k}t}$$

and so

$$\sum_{j=-m}^{m} \varphi_{2^{-k}t} * \theta_{j+k} * \psi_{j+k} = \varphi_{2^{-k}t} \qquad \text{for } 1 \leq t \leq 2 \text{ and } k \in \mathbb{Z}. \tag{13.19}$$

Hence, by Young's inequality,

$$\begin{aligned}\|\varphi_{2^{-k}t} * f\|_{L^p} &\leq \sum_{j=-m}^{m} \|\varphi_{2^{-k}t} * \theta_{j+k} * \psi_{j+k} * f\|_{L^p} \\ &\leq \|\phi\|_{L^1} \|\theta\|_{L^1} \sum_{j=-m}^{m} \|\psi_{j+k} * f\|_{L^p}\end{aligned}$$

for all $k \in \mathbb{Z}$, $1 \leq t < 2$, and $f \in C_0^\infty(\mathbb{R}^n)$.

This implies

$$\begin{aligned}
\|f\|_{b^{s,p}_{q,6;\phi}} &= \left(\sum_{k=-\infty}^{\infty}\int_1^2 \left(\frac{\|\varphi_{2^{-k}t}*f\|_{L^p}}{(2^{-k}t)^s}\right)^q \frac{dt}{t}\right)^{1/q} \\
&\le \left(\sum_{k=-\infty}^{\infty}\left(2^{ks}\sup_{1\le t\le 2}\|\varphi_{2^{-k}t}*f\|_{L^p}\right)^q\right)^{1/q} \\
&\le \|\phi\|_{L^1}\|\theta\|_{L^1}\left(\sum_{k=-\infty}^{\infty}\left(2^{ks}\sum_{j=-m}^{m}\|\psi_{j+k}*f\|_{L^p}\right)^q\right)^{1/q} \\
&\le \|\phi\|_{L^1}\|\theta\|_{L^1}\sum_{j=-m}^{m}\left(\sum_{k=-\infty}^{\infty}\left(2^{ks}\|\psi_{j+k}*f\|_{L^p}\right)^q\right)^{1/q} \\
&\le 2^{ms}(2m+1)\|\phi\|_{L^1}\|\theta\|_{L^1}\|f\|_{b^{s,p}_{q,4;\psi}}.
\end{aligned}$$

This proves Step 3.

Step 4. *Let $\phi,\psi:\mathbb{R}^n\to\mathbb{R}$ be Triebel functions and let $s\ge 0$ and $p,q\ge 1$. Then there exists a constant $c=c(s,p,q,\phi)>0$ such that*

$$\|f\|_{b^{s,p}_{q,4;\psi}} \le c\,\|f\|_{b^{s,p}_{q,6;\phi}}$$

for all $f\in C_0^\infty(\mathbb{R}^n)$.

The proof is an adaptation of the argument in Ullrich [40, Substep 2.1]. Choose $\delta>0$ such that $\mathrm{supp}(\widehat{\psi})\subset\{\xi\in\mathbb{R}^n\,|\,\delta\le|\xi|\le 1/\delta\}$ as in (13.2) and choose $m\in\mathbb{N}$ such that $2^{-m}\le\delta/2$. Choose a Triebel function θ such that $\theta*\phi$ is a Littlewood–Paley function (Lemma 13.2). Then

$$\sum_{j=-m}^{m}\widehat{\psi}_{j+k}(t\xi)\widehat{\theta}_{j+k}(t\xi)\widehat{\varphi}_{2^{-j-k}t}(\xi) = \sum_{j=-m}^{m}\widehat{\psi}(2^{-j-k}t\xi)\widehat{\theta}(2^{-j-k}t\xi)\widehat{\phi}(2^{-j-k}t\xi)=1$$

for $2^{-m}\le 2^{-k}t|\xi|\le 2^m$ or, equivalently, for $2^{k-m}/t\le|\xi|\le 2^{k+m}/t$. Since $\mathrm{supp}(\widehat{\psi}_k)\subset\{\xi\in\mathbb{R}^n\,|\,2^{k-m+1}\le|\xi|\le 2^{k+m-1}\}$, this implies

$$\widehat{\psi}_k(\xi)=\sum_{j=-2}^{2}\widehat{\psi}_k(\xi)\widehat{\psi}_{k+j}(t\xi)\widehat{\theta}_{k+j}(t\xi)\widehat{\varphi}_{2^{-j-k}t}(\xi) \tag{13.20}$$

for all $k\in\mathbb{Z}$, $1\le t\le 2$, and $\xi\in\mathbb{R}^n$.

For $t > 0$ let us temporarily define the function $\Psi_t : \mathbb{R}^n \to \mathbb{R}$ by

$$\Psi_t(x) := t^{-n}(\psi + \theta)(t^{-1}x) \qquad \text{for } x \in \mathbb{R}^n,$$

so its Fourier transform is $\xi \mapsto \widehat{\psi}(t\xi)\widehat{\theta}(t\xi)$ and its L^1 norm agrees with that of $\psi * \theta$. Then, by (13.20),

$$\psi_k = \sum_{j=-2}^{2} \psi_k * \Psi_{2^{-j-k}t} * \varphi_{2^{-j-k}t} \qquad \text{for } k \in \mathbb{Z} \text{ and } 1 \le t \le 2.$$

By Young's inequality, this implies

$$\begin{aligned}
\|\psi_k * f\|_{L^p} &\le \|\psi * \theta\|_{L^1}^2 \sum_{j=-2}^{2} \|\varphi_{2^{-j-k}t} * f\|_{L^p} \\
&\le 2^{3s} \|\psi * \theta\|_{L^1}^2 \sum_{j=-2}^{2} \frac{\|\varphi_{2^{-j-k}t} * f\|_{L^p}}{(2^{-j}t)^s}
\end{aligned}$$

for $k \in \mathbb{Z}$, $1 \le t \le 2$, and $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$. Exponentiate by q and integrate the resulting inequality over the interval $1 \le t \le 2$ to obtain

$$\left(2^{ks} \|\psi_k * f\|_{L^p}\right)^q \le 2^{3qs+1} \|\psi * \theta\|_{L^1}^{2q} \int_1^2 \left(\sum_{j=-2}^{2} \frac{\|\varphi_{2^{-j-k}t} * f\|_{L^p}}{(2^{-j-k}t)^s} \right)^q \frac{dt}{t}$$

for all $k \in \mathbb{Z}$ and all $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$. With this understood, it follows from the definition of the norms that

$$\begin{aligned}
\|f\|_{b^{s,p}_{q,4;\psi}} &= \left(\sum_{k=-\infty}^{\infty} \left(2^{ks} \|\psi_k * f\|_{L^p}\right)^q \right)^{1/q} \\
&\le 2^{3s+1/q} \|\psi * \theta\|_{L^1}^2 \left(\sum_{k=-\infty}^{\infty} \int_1^2 \left(\sum_{j=-2}^{2} \frac{\|\varphi_{2^{-j-k}t} * f\|_{L^p}}{(2^{-j-k}t)^s} \right)^q \frac{dt}{t} \right)^{1/q} \\
&\le 2^{3s+1/q} \|\psi * \theta\|_{L^1}^2 \sum_{j=-2}^{2} \left(\sum_{k=-\infty}^{\infty} \int_1^2 \left(\frac{\|\varphi_{2^{-j-k}t} * f\|_{L^p}}{(2^{-j-k}t)^s} \right)^q \frac{dt}{t} \right)^{1/q} \\
&= 2^{3s+1/q} \|\psi * \theta\|_{L^1}^2\, 5 \left(\int_0^\infty \left(\frac{\|\varphi_t * f\|_{L^p}}{t^s} \right)^q \frac{dt}{t} \right)^{1/q} \\
&= 2^{3s+1/q} \|\psi * \theta\|_{L^1}^2\, 5\, \|f\|_{b^{s,p}_{q,6;\phi}} .
\end{aligned}$$

This proves Step 4 and part (i) of Theorem 13.5.

Step 5. *Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a Triebel function and let $0 < s < 2$, $p > 1$, and $q \geq 1$. Then there exists a constant $c = c(s, p, q, \phi) > 0$ such that*

$$\|f\|_{b^{s,p}_{q,4;\phi}} \leq c\,\|f\|_{b^{s,p}_{q,2}}$$

for all $f \in C_0^\infty(\mathbb{R}^n)$.

The proof follows the argument in [4, Section 5]. Since ϕ is a Triebel function, its Fourier transform satisfies (13.2) for some constant $0 < \delta \leq 1/2$. Choose a function $\theta : \mathbb{R}^n \to \mathbb{R}$ in the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ such that $\theta(x) = \theta(-x)$ for all $x \in \mathbb{R}^n$ and

$$\widehat{\theta}(\xi) = \begin{cases} 1, & \text{if } |\xi| \leq \delta/2, \\ 0, & \text{if } |\xi| \geq \delta. \end{cases}$$

For $k \in \mathbb{Z}$ define $\theta_k(x) := 2^{kn}\theta(2^k x)$ so that $\widehat{\theta}_k(\xi) = \theta(2^{-k}\xi)$ and define the function $f_k : \mathbb{R}^n \to \mathbb{R}$ by

$$\begin{aligned} f_k(x) &:= f(x) - \int_{\mathbb{R}^n} \theta(y)\Delta^2_{2^{-k}y} f(x)\,dy \\ &= f(x) - \int_{\mathbb{R}^n} \theta(y)\big(f(x) - 2f(x - 2^{-k}y) + f(x - 2^{1-k}y)\big)\,dy \\ &= 2(\theta_k * f)(x) - (\theta_{k-1} * f)(x). \end{aligned} \tag{13.21}$$

Here the second equation uses the fact that $\theta(y) = \theta(-y)$ for all $y \in \mathbb{R}^n$ and the last equation uses the identity $\int_{\mathbb{R}^n} \theta = \widehat{\theta}(0) = 1$. Since $p > 1$, it follows from equation (13.21) and Lemma 8.4 that

$$f = \lim_{N\to\infty}(f_{N+1} - f_{-N}) = \lim_{N\to\infty}\sum_{j=-N}^{N}(f_{j+1} - f_j) = \sum_{j=-\infty}^{\infty}(f_{j+1} - f_j),$$

where the convergence is in $L^p(\mathbb{R}^n)$. Thus

$$\phi_k * f = \sum_{j=-\infty}^{\infty} \phi_k * (f_{j+1} - f_j) = \sum_{j=-\infty}^{\infty} \phi_k * (2\theta_{j+1} - 3\theta_j + \theta_{j-1}) * f.$$

Since $\operatorname{supp}(\widehat{\theta}_{j+1}) \subset B_{2^{j+1}\delta}$ and $\widehat{\phi}_k$ vanishes on $B_{2^k\delta}$, we have $\widehat{\phi}_k\widehat{\theta}_{j+1} = 0$ for all $j < k$ and so

$$\phi_k * f = \sum_{j=k}^{\infty} \phi_k * (f_{j+1} - f_j) \qquad \text{for all } k \in \mathbb{Z}.$$

Hence

$$\|\phi_k * f\|_{L^p} \leq \sum_{j=k}^{\infty} \|\phi_k * (f_{j+1} - f_j)\|_{L^p} \leq \sum_{j=k}^{\infty} \|\phi\|_{L^1} \|f_{j+1} - f_j\|_{L^p}.$$

Multiply this inequality by 2^{ks} to obtain

$$2^{ks} \|\phi_k * f\|_{L^p} \leq \|\phi\|_{L^1} \sum_{j=k}^{\infty} 2^{-(j-k)s} 2^{js} \|f_{j+1} - f_j\|_{L^p}.$$

The right hand side is the convolution of the summable sequence $(2^{-ks})_{k\in\mathbb{N}_0}$ with a bi-infinite sequence in $\ell^q(\mathbb{Z})$. Hence, by Young's inequality,

$$\begin{aligned}
\|f\|_{b^{s,p}_{q,4;\phi}} &= \left(\sum_{k=-\infty}^{\infty} \left(2^{ks} \|\phi_k * f\|_{L^p}\right)^q\right)^{1/q} \\
&\leq \|\phi\|_{L^1} \left(\sum_{k=0}^{\infty} 2^{-ks}\right) \left(\sum_{j=-\infty}^{\infty} \left(2^{js} \|f_{j+1} - f_j\|_{L^p}\right)^q\right)^{1/q} \\
&= \frac{\|\phi\|_{L^1}}{1 - 2^{-s}} \left(\sum_{j=-\infty}^{\infty} \left(2^{js} \|f_{j+1} - f_j\|_{L^p}\right)^q\right)^{1/q}.
\end{aligned}$$

Now it follows from the definition of f_j in (13.21) and Minkowsky's inequality in [33, Thm 7.19] that

$$\begin{aligned}
\|f_{j+1} - f_j\|_{L^p} &= \left(\int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} \theta(y) \left(\Delta^2_{2^{-j-1}y} f(x) - \Delta^2_{2^{-j}y} f(x)\right) dy\right|^p dx\right)^{1/p} \\
&\leq \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\theta(y)| \left|\Delta^2_{2^{-j-1}y} f(x) - \Delta^2_{2^{-j}y} f(x)\right| dy\right)^p dx\right)^{1/p} \\
&\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\theta(y)|^p \left|\Delta^2_{2^{-j-1}y} f(x) - \Delta^2_{2^{-j}y} f(x)\right|^p dx\right)^{1/p} dy \\
&= \int_{\mathbb{R}^n} |\theta(y)| \left\|\Delta^2_{2^{-j-1}y} f - \Delta^2_{2^{-j}y} f\right\|_{L^p} dy \\
&\leq \int_{\mathbb{R}^n} |\theta(y)| \left(\left\|\Delta^2_{2^{-j-1}y} f\right\|_{L^p} + \left\|\Delta^2_{2^{-j}y} f\right\|_{L^p}\right) dy.
\end{aligned}$$

Abbreviate $c := \|\phi\|_{L^1} / (1 - 2^{-s})$. Then the last two estimates yield

$$\begin{aligned}
\|f\|_{b^{s,p}_{q,4;\phi}} &\le c \left(\sum_{j=-\infty}^{\infty} \left(2^{js} \|f_{j+1} - f_j\|_{L^p} \right)^q \right)^{1/q} \\
&\le c \left(\sum_{j=-\infty}^{\infty} \left(2^{js} \int_{\mathbb{R}^n} |\theta(y)| \left(\left\|\Delta^2_{2^{-j-1}y} f\right\|_{L^p} + \left\|\Delta^2_{2^{-j}y} f\right\|_{L^p} \right) dy \right)^q \right)^{1/q} \\
&\le c \int_{\mathbb{R}^n} |\theta(y)| \left(\sum_{j=-\infty}^{\infty} 2^{jqs} \left(\left\|\Delta^2_{2^{-j-1}y} f\right\|_{L^p} + \left\|\Delta^2_{2^{-j}y} f\right\|_{L^p} \right)^q \right)^{1/q} dy \\
&\le 2c \int_{\mathbb{R}^n} |\theta(y)| \left(\sum_{j=-\infty}^{\infty} 2^{jqs} \left(\left\|\Delta^2_{2^{-j-1}y} f\right\|_{L^p}^q + \left\|\Delta^2_{2^{-j}y} f\right\|_{L^p}^q \right) \right)^{1/q} dy \\
&\le 2(1 + 2^{-qs})^{1/q} c \int_{\mathbb{R}^n} |\theta(y)| \left(\sum_{j=-\infty}^{\infty} 2^{jqs} \left\|\Delta^2_{2^{-j}y} f\right\|_{L^p}^q \right)^{1/q} dy.
\end{aligned}$$

Here the third step follows from Minkowsky's inequality in [33, Thm 7.19]. Now let $N > n + s$ and choose a constant $C > 0$ such that

$$2(1 + 2^{-qs})^{1/q} c |\theta(y)| \le \frac{C}{(1 + |y|)^N} \qquad \text{for all } y \in \mathbb{R}^n.$$

Then

$$\begin{aligned}
\|f\|_{b^{s,p}_{q,4;\phi}} &\le \int_{B_1} \frac{C}{(1 + |y|)^N} \left(\sum_{j=-\infty}^{\infty} 2^{jqs} \left\|\Delta^2_{2^{-j}y} f\right\|_{L^p}^q \right)^{1/q} dy \\
&\qquad + \sum_{\ell=1}^{\infty} \int_{B_{2^\ell} \setminus B_{2^{\ell-1}}} \frac{C}{(1 + |y|)^N} \left(\sum_{j=-\infty}^{\infty} 2^{jqs} \left\|\Delta^2_{2^{-j}y} f\right\|_{L^p}^q \right)^{1/q} dy \\
&\le \sum_{\ell=0}^{\infty} \frac{2^N C \mathrm{Vol}(B_{2^\ell})}{2^{\ell N}} \left(\sum_{j=-\infty}^{\infty} 2^{jqs} \sup_{|h| \le 2^{\ell-j}} \left\|\Delta^2_h f\right\|_{L^p}^q \right)^{1/q} \\
&= \sum_{\ell=0}^{\infty} \frac{2^N C \mathrm{Vol}(B_1)}{2^{\ell(N-n-s)}} \left(\sum_{j=-\infty}^{\infty} 2^{(j-\ell)qs} \sup_{|h| \le 2^{\ell-j}} \left\|\Delta^2_h f\right\|_{L^p}^q \right)^{1/q} \\
&= \frac{2^N C \mathrm{Vol}(B_1)}{1 - 2^{-(N-n-s)}} \|f\|_{b^{s,p}_{q,3}}.
\end{aligned}$$

Hence Step 5 follows from Theorem 12.1.

Step 6. *Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a Triebel function and let $0 < s < 2$, $p > 1$, $q \geq 1$, and $\lambda > 0$. Then there is a constant $c = c(s, p, q, \phi, \lambda) > 0$ such that*

$$\|f\|_{b^{s,p}_{q,2}} \leq c \|f\|_{b^{s,p}_{q,5;\phi,\lambda}}$$

for all $f \in C_0^\infty(\mathbb{R}^n)$.

The proof follows the argument in [4, Section 4] with a modification suggested by the discussion in [40, Substep 1.1]. By Lemma 13.2 there exists a Triebel function $\psi : \mathbb{R}^n \to \mathbb{R}$ such that $\psi * \phi$ is a Littlewood–Paley function. This means that $\sum_{j\in\mathbb{Z}} \widehat{\psi}_j(x)\widehat{\phi}_j(x) = 1$ for all $x \in \mathbb{R}^n \setminus \{0\}$, and so it follows from Lemma 8.4 with $\theta_k := \sum_{j=-\infty}^{k} \psi_j * \phi_j$ that

$$f = \lim_{N\to\infty} \sum_{j=-N}^{N} \psi_j * \phi_j * f = \sum_{j=-\infty}^{\infty} \psi_j * \phi_j * f$$

for all $f \in C_0^\infty(\mathbb{R}^n)$, where the convergence is both in L^p and in L^∞. Now let $c = c(n, 2, \lambda, \psi) > 0$ be the constant in part (ii) of Lemma 13.4 for $\ell = 2$, fix a function $f \in C_0^\infty(\mathbb{R}^n)$, and let $k \in \mathbb{Z}$ and $h \in \mathbb{R}^n$ such that $|h| \leq 2^{-k}$. Then $\Delta_h^2 f = \sum_{j=-\infty}^{\infty} \Delta_h^2(\psi_j * \phi_j * f)$ and hence

$$\left\|\Delta_h^2 f\right\|_{L^p} \leq \sum_{j=-\infty}^{\infty} \left\|\Delta_h^2(\psi_j * \phi_j * f)\right\|_{L^p}. \tag{13.22}$$

For $j < k$ we use the inequality

$$\begin{aligned}
\left|\Delta_h^2(\psi_j * \phi_j * f)(x)\right| &\leq \sup_{|y|\leq 2|h|} \left|\nabla^2(\psi_j * \phi_j * f)(x+y)\right| |h|^2 \\
&\leq 2^{-2k} \sup_{|y|\leq 2^{1-k}} \left|\nabla^2(\psi_j * \phi_j * f)(x+y)\right| \\
&\leq c2^{2(j-k)} \sup_{|y|\leq 2^{1-k}} (\phi_{j,\lambda}^* f)(x+y) \\
&\leq c2^{2(j-k)}(1 + 2^{1-k+j})^\lambda (\phi_{j,\lambda}^* f)(x).
\end{aligned}$$

Here the first step follows from (12.16) by replacing f with $\Delta_h f$ and repeating the estimate. The last two steps follow from Lemma 13.4. (The matrix $\nabla^2 g(x) \in \mathbb{C}^{n\times n}$ is the Hessian of a smooth function $g : \mathbb{R}^n \to \mathbb{C}$ at $x \in \mathbb{R}^n$, and $|\nabla^2 g(x)|$ is the Euclidean matrix norm.) Now take L^p norms to obtain

$$\sup_{|h|\leq 2^{-k}} \left\|\Delta_h^2(\psi_j * \phi_j * f)\right\|_{L^p} \leq C2^{-2(k-j)} \left\|\phi_{j,\lambda}^* f\right\|_{L^p} \qquad \text{for } j < k, \tag{13.23}$$

where $C := 2^\lambda c$.

For $j \geq k$ we use the inequality

$$\left\|\Delta_h^2(\psi_j * \phi_j * f)\right\|_{L^p} \leq 4\left\|\psi_j * \phi_j * f\right\|_{L^p} \leq 4\left\|\psi\right\|_{L^1}\left\|\phi_j * f\right\|_{L^p}. \qquad (13.24)$$

It follows from (13.22), (13.23), and (13.24) that

$$\begin{aligned}
2^{ks}\sup_{|h|\leq 2^{-k}}\left\|\Delta_h^2 f\right\|_{L^p} &\leq \sum_{j=-\infty}^{k-1} 2^{ks}\sup_{|h|\leq 2^{-k}}\left\|\Delta_h^2(\psi_j * \phi_j * f)\right\|_{L^p} \\
&\quad + \sum_{j=k}^{\infty} 2^{ks}\sup_{|h|\leq 2^{-k}}\left\|\Delta_h^2(\psi_j * \phi_j * f)\right\|_{L^p} \\
&\leq C\sum_{j=-\infty}^{k-1} 2^{-(k-j)(2-s)}2^{js}\left\|\phi_{j,\lambda}^* f\right\|_{L^p} \\
&\quad + 4\left\|\psi\right\|_{L^1}\sum_{j=k}^{\infty} 2^{-(j-k)s}2^{js}\left\|\phi_j * f\right\|_{L^p}.
\end{aligned}$$

Now use Young's inequality for the convolution of bi-infinite sequences to obtain

$$\begin{aligned}
\|f\|_{b^{s,p}_{q,3}} &= \left(\sum_{k=-\infty}^{\infty}\left(2^{ks}\sup_{|h|\leq 2^{-k}}\left\|\Delta_h^2 f\right\|_{L^p}\right)^q\right)^{1/q} \\
&\leq C\left(\sum_{k=1}^{\infty} 2^{-k(2-s)}\right)\left(\sum_{j=-\infty}^{\infty}\left(2^{js}\left\|\phi_{j,\lambda}^* f\right\|_{L^p}\right)^q\right)^{1/q} \\
&\quad + 4\left\|\psi\right\|_{L^1}\left(\sum_{k=0}^{\infty} 2^{-ks}\right)\left(\sum_{j=-\infty}^{\infty}\left(2^{js}\left\|\phi_j * f\right\|_{L^p}\right)^q\right)^{1/q} \\
&= \frac{C2^{-(2-s)}}{1-2^{-(2-s)}}\|f\|_{b^{s,p}_{q,5;\phi,\lambda}} + \frac{4\left\|\psi\right\|_{L^1}}{1-2^{-s}}\|f\|_{b^{s,p}_{q,4;\phi}} \\
&\leq \left(\frac{C2^{-(2-s)}}{1-2^{-(2-s)}} + \frac{4\left\|\psi\right\|_{L^1}}{1-2^{-s}}\right)\|f\|_{b^{s,p}_{q,5;\phi,\lambda}}.
\end{aligned}$$

Here the last step uses the inequality $\|f\|_{b^{s,p}_{q,4;\phi}} \leq \|f\|_{b^{s,p}_{q,5;\phi,\lambda}}$ in Step 1. Thus Step 6 follows from Theorem 12.1. This proves part (ii) of Theorem 13.5. □

Remark 13.6. **(i)** Theorem 13.5 is formulated in terms of norms on the space $C_0^\infty(\mathbb{R}^n, \mathbb{C})$ and the homogeneous Besov space $b_q^{s,p}(\mathbb{R}^n, \mathbb{C})$ is the completion of $C_0^\infty(\mathbb{R}^n, \mathbb{C})$ with respect to any of the norms in Theorem 12.1 and Theorem 13.5.

(ii) The norms in Theorem 13.5 extend naturally to the space of Schwartz distributions. The convolution of a Schwartz function and a Schwartz distribution is a function and so the right hand sides of equations (13.8-13.11) remain meaningful when f is a Schwartz distribution. However, the resulting maps $\mathscr{S}'(\mathbb{R}^n, \mathbb{C}) \to [0, \infty]$ are no longer norms. They vanish on the subspace $\mathscr{P}(\mathbb{R}^n, \mathbb{C}) \subset \mathscr{S}'(\mathbb{R}^n, \mathbb{C})$ of all polynomials and may take the value infinity. The Besov space embeds into the quotient space $\mathscr{S}'(\mathbb{R}^n, \mathbb{C})/\mathscr{P}(\mathbb{R}^n, \mathbb{C})$ and can be identified with the subspace of this quotient space on which the quasi-norms in Theorem 13.5 are finite. To prove this, one is confronted with the additional difficulty of showing that if one of the quasi-norms in Theorem 13.5 is finite for some Schwartz distribution then so are the others. This difficulty is carefully adressed in the paper by Ullrich [40], which also contains many further results as well as copious references to the existing literature.

Theorem 13.7. *Fix a positive integer n and real numbers $p > 1$ and $q \geq 1$.*
(i) *For every real number $0 < s < 1$ there is a $c = c(n, p, q, s) \geq 1$ such that*

$$\|f\|_{b_{q,2}^{1+s,p}} \leq \|\nabla f\|_{b_{q,1}^{s,p}} \leq c \|f\|_{b_{q,2}^{1+s,p}} \tag{13.25}$$

for all $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$.

(ii) *For every Triebel function $\phi : \mathbb{R}^n \to \mathbb{R}$ there is a $c = c(n, p, q, \phi) \geq 1$ such that*

$$c^{-1} \|f\|_{b_{q,2}^{1,p}} \leq \left(\int_0^\infty \|\varphi_t * \nabla f\|_{L^p}^q \frac{dt}{t} \right)^{1/q} \leq c \|f\|_{b_{q,2}^{1,p}} \tag{13.26}$$

for all $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$.

Proof. The proof has three steps.

Step 1. *We prove the first inequality in* (13.25).

Let $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$. Then $\|\Delta_h^2 f\|_{L^p} \leq |h| \|\Delta_h \nabla f\|_{L^p}$ for $h \in \mathbb{R}^n$ by (12.17). Take the supremum over all $h \in \mathbb{R}^n$ with $|h| \leq r$ to obtain the estimate $\omega_2(r, f)_p \leq r\omega_1(r, \nabla f)_p$ for all $r > 0$ and hence

$$\|f\|_{b_{q,2}^{1+s,p}}^q = \int_0^\infty \left(\frac{\omega_2(r, f)_p}{r^{1+s}} \right)^q \frac{dr}{r} \leq \int_0^\infty \left(\frac{\omega_1(r, \nabla f)_p}{r^s} \right)^q \frac{dr}{r} = \|\nabla f\|_{b_{q,1}^{s,p}}^q$$

for all $p, q \geq 1$. This proves the first inequality in (13.25).

Step 2. *For every real number $p > 1$ there exists a constant $c = c(n, p) \geq 1$ with the following significance. If $f : \mathbb{R}^n \to \mathbb{C}$ is a function in the Schwartz space and $g : \mathbb{R}^n \to \mathbb{C}$ is the unique function in the Schwartz space whose Fourier transform is given by*

$$\widehat{g}(\xi) = |\xi|\widehat{f}(\xi) \qquad \textit{for } \xi \in \mathbb{R}^n$$

then

$$c^{-1} \|g\|_{L^p} \leq \|\nabla f\|_{L^p} \leq c \|g\|_{L^p} .$$

For $i = 1, \dots, n$ define the operator $T_i : L^2(\mathbb{R}^n, \mathbb{C}) \to L^2(\mathbb{R}^n, \mathbb{C})$ by

$$\widehat{T_i g}(\xi) := \frac{\mathbf{i}\xi_i}{|\xi|}\widehat{g}(\xi).$$

Let $p > 1$. Then, by the Mikhlin Multiplier Theorem 6.1, there exists a constant a $C = C(n, p) > 0$ such that

$$\|T_i f\|_{L^p} \leq C\|f\|_{L^p}$$

for all $f \in L^2(\mathbb{R}^n, \mathbb{C}) \cap L^p(\mathbb{R}^n, \mathbb{C})$ and all i. Now let f and g be as in Step 2. Since

$$\widehat{\partial_i f}(\xi) = \mathbf{i}\xi_i \widehat{f}(\xi)$$

for $i = 1, \dots, n$ and $\xi \in \mathbb{R}^n$, we have

$$\partial_i f = T_i g \qquad \text{for } i = 1, \dots, n$$

and

$$g = -\sum_{i=1}^{n} T_i \partial_i f.$$

Thus the estimates of Step 2 hold with $c := nC$.

Step 3. *We prove* (13.26) *and the second inequality in* (13.25)*.*

Let ϕ be a Triebel functions and define $\varphi_t(x) := t^{-n}\phi(t^{-1}x)$ for $t > 0$ and $x \in \mathbb{R}^n$ as in Definition 13.5. Define the functions $\psi, \psi_t : \mathbb{R}^n \to \mathbb{R}$ by

$$\widehat{\psi}(\xi) := |\xi|\widehat{\phi}(\xi), \qquad \psi_t(x) := t^{-n}\psi(t^{-1}x)$$

for $t > 0$ and $x, \xi \in \mathbb{R}^n$. Then ψ satisfies (13.1) and (13.2) and hence is also a Triebel function. Now let $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$ and define $g : \mathbb{R}^n \to \mathbb{C}$ by

$$\widehat{g}(\xi) := |\xi|\widehat{f}(\xi) \qquad \text{for } \xi \in \mathbb{R}^n. \tag{13.27}$$

Then, for all $t > 0$ and all $\xi \in \mathbb{R}^n$, we have

$$\widehat{\varphi_t * g}(\xi) = \widehat{\varphi}_t(\xi)\widehat{g}(\xi) = |\xi|\widehat{\phi}(t\xi)\widehat{f}(\xi) = |\xi|\widehat{\varphi}_t(\xi)\widehat{f}(\xi) = |\xi|\widehat{\varphi_t * f}(\xi). \tag{13.28}$$

Since $|\xi|\widehat{\phi}(t\xi) = t^{-1}\widehat{\psi}(t\xi)$, this implies

$$\widehat{\varphi_t * g}(\xi) = t^{-1}\widehat{\psi}(t\xi)\widehat{f}(\xi) = t^{-1}\widehat{\psi}_t(\xi)\widehat{f}(\xi) = t^{-1}\widehat{\psi_t * f}(\xi). \tag{13.29}$$

It follows from (13.29) that

$$\|\varphi_t * g\|_{L^p} = t^{-1}\|\psi_t * f\|_{L^p}$$

for all $t > 0$. Now let $c = c(n, p) \geq 1$ be the constant of Step 2. Then it follows from (13.28) that

$$c^{-1}\|\varphi_t * g\|_{L^p} \leq \|\varphi_t * \nabla f\|_{L^p} \leq c\|\varphi_t * g\|_{L^p}$$

for all $t > 0$. Combining these two assertions we obtain

$$c^{-1}\frac{\|\psi_t * f\|_{L^p}}{t} \leq \|\varphi_t * \nabla f\|_{L^p} \leq c\frac{\|\psi_t * f\|_{L^p}}{t} \qquad \text{for all } t > 0.$$

and hence

$$\begin{aligned}
c^{-1}\|f\|_{b^{s+1,p}_{q,6;\psi}} &= c^{-1}\left(\int_0^\infty \left(\frac{\|\psi_t * f\|_{L^p}}{t^{1+s}}\right)^q \frac{dt}{t}\right)^{1/q} \\
&\leq \left(\int_0^\infty \left(\frac{\|\varphi_t * \nabla f\|_{L^p}}{t^s}\right)^q \frac{dt}{t}\right)^{1/q} \\
&\leq c\left(\int_0^\infty \left(\frac{\|\psi_t * f\|_{L^p}}{t^{1+s}}\right)^q \frac{dt}{t}\right)^{1/q} \\
&= c\|f\|_{b^{s+1,p}_{q,6;\psi}}
\end{aligned}$$

for $0 \leq s < 1$. For $s = 0$ this implies the inequality (13.26) by Theorem 13.5. For $0 < s < 1$ this estimate can be written in the form

$$c^{-1}\|f\|_{b^{s+1,p}_{q,6;\psi}} \leq \|\nabla f\|_{b^{s,p}_{q,6;\phi}} \leq c\|f\|_{b^{s+1,p}_{q,6;\psi}}$$

and so the second inequality in (13.25) follows from Theorem 12.1 and Theorem 13.5. This proves Step 3 and Theorem 13.7. □

For $s = 0$ the equivalent norms of Theorem 13.5 can be used to define a space $b_q^{0,p}(\mathbb{R}^n, \mathbb{C})$ as the completion of $C_0^\infty(\mathbb{R}^n, \mathbb{C})$ with respect to these norms. These spaces are increasing in q and it follows directly from the definitions that $b_1^{0,p}(\mathbb{R}^n, \mathbb{C}) \subset L^p(\mathbb{R}^n, \mathbb{C}) \subset b_\infty^{0,p}(\mathbb{R}^n, \mathbb{C})$. The next corollary refines this assertion. It shows that $b_2^{0,p}(\mathbb{R}^n, \mathbb{C}) \subset L^p(\mathbb{R}^n, \mathbb{C})$ for $1 \le p \le 2$ and $L^p(\mathbb{R}^n, \mathbb{C}) \subset b_2^{0,p}(\mathbb{R}^n, \mathbb{C})$ for $2 \le p \le \infty$.

Corollary 13.8. *Let $n \in \mathbb{N}$ and $1 < p < \infty$ and let ϕ be a Triebel function. Then the following holds.*

(i) *If $p \le 2$ then there exists a constant $c > 0$ such that*

$$\left(\int_0^\infty \|\varphi_t * f\|_{L^p}^2 \frac{dt}{t} \right)^{1/2} \le c \|f\|_{L^p} \tag{13.30}$$

for all $f \in C_0^\infty(\mathbb{R}^n)$.

(ii) *If $p \ge 2$ then there exists a constant $c > 0$ such that*

$$\|f\|_{L^p} \le c \left(\int_0^\infty \|\varphi_t * f\|_{L^p}^2 \frac{dt}{t} \right)^{1/2} \tag{13.31}$$

for all $f \in C_0^\infty(\mathbb{R}^n)$.

Proof. We prove part (i). Assume $1 < p \le 2$, let ψ be a Littlewood–Paley function, and let $c = c(n, p, \psi)$ be the constant of Theorem 8.3. Then

$$\begin{aligned}
\|f\|_{b^{0,p}_{2,3;\psi}}^p &= \left(\sum_{k=-\infty}^\infty \|\psi_k * f\|_{L^p}^2 \right)^{p/2} \\
&= \left(\sum_{k=-\infty}^\infty \left(\int_{\mathbb{R}^n} |(\psi_k * f)(x)|^p \, dx \right)^{2/p} \right)^{p/2} \\
&\le \int_{\mathbb{R}^n} \left(\sum_{k=-\infty}^\infty |(\psi_k * f)(x)|^2 \right)^{p/2} dx \\
&= \|S_\psi(f)\|_{L^p}^p \\
&\le c^p \|f\|_{L^p}^p
\end{aligned}$$

for every $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$. Here the third step uses Minkowski's inequality in [33, Thm 7.19] with the exponent $2/p$ and the last step follows from Theorem 8.3. Now part (i) follows from the equivalence of the norms $\|\cdot\|_{b^{s,p}_{q,4;\psi}}$ and $\|\cdot\|_{b^{s,p}_{q,6;\phi}}$ in Theorem 13.5 with $s = 0$ and $q = 2$.

We prove part (ii). Thus assume $2 \le p < \infty$, let ψ be a Littlewood–Paley function, and let $c = c(n,p,\psi)$ be the constant of Theorem 8.3. Then

$$\begin{aligned}
c^{-p}\left\|f\right\|_{L^p}^2 &\le \left\|S_\psi(f)\right\|_{L^p}^2 \\
&= \left(\int_{\mathbb{R}^n}\left(\sum_{k=-\infty}^{\infty}\left|(\psi_k * f)(x)\right|^2\right)^{p/2} dx\right)^{2/p} \\
&\le \sum_{k=-\infty}^{\infty}\left(\int_{\mathbb{R}^n}\left|(\psi_k * f)(x)\right|^p\, dx\right)^{2/p} \\
&= \sum_{k=-\infty}^{\infty}\left\|\psi_k * f\right\|_{L^p}^2 \\
&= \left\|f\right\|_{b^{0,p}_{2,3;\psi}}^2
\end{aligned}$$

for every $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$. Here the first step uses Theorem 8.3 and the third step uses Minkowski's inequality in [33, Thm 7.19] with exponent $p/2$. Thus part (ii) follows from the equivalence of the norms $\left\|\cdot\right\|_{b^{s,p}_{q,4;\psi}}$ and $\left\|\cdot\right\|_{b^{s,p}_{q,6;\phi}}$ in Theorem 13.5 with $s = 0$ and $q = 2$. This proves Corollary 13.8. □

Corollary 13.9. *Let $n \in \mathbb{N}$ and $1 < p < \infty$.*
(i) *If $p \le 2$ then $w^{1,p}(\mathbb{R}^n,\mathbb{C}) \subset b_2^{1,p}(\mathbb{R}^n,\mathbb{C})$ and $W^{1,p}(\mathbb{R}^n,\mathbb{C}) \subset B_2^{1,p}(\mathbb{R}^n,\mathbb{C})$.*
(ii) *If $p \ge 2$ then $b_2^{1,p}(\mathbb{R}^n,\mathbb{C}) \subset w^{1,p}(\mathbb{R}^n,\mathbb{C})$ and $B_2^{1,p}(\mathbb{R}^n,\mathbb{C}) \subset W^{1,p}(\mathbb{R}^n,\mathbb{C})$.*
(iii) *If $p = 2$ then $b_2^{1,2}(\mathbb{R}^n,\mathbb{C}) = w^{1,2}(\mathbb{R}^n,\mathbb{C})$ and $B_2^{1,2}(\mathbb{R}^n,\mathbb{C}) = W^{1,2}(\mathbb{R}^n,\mathbb{C})$.*

Proof. Assume $p \le 2$. Then, by Theorem 13.7 and part (i) of Corollary 13.8, there exists a constant $c \ge 1$ such that, for all $f \in C_0^\infty(\mathbb{R}^n)$,

$$c^{-1}\left\|f\right\|_{b^{1,p}_{2,2}} \le \left(\int_0^\infty \left\|\phi_t * \nabla f\right\|_{L^p}^2 \frac{dt}{t}\right)^{1/2} \le c\left\|\nabla f\right\|_{L^p}.$$

Thus $w^{1,p}(\mathbb{R}^n,\mathbb{C}) \subset b_2^{1,p}(\mathbb{R}^n,\mathbb{C})$. The inclusion $W^{1,p}(\mathbb{R}^n,\mathbb{C}) \subset B_2^{1,p}(\mathbb{R}^n,\mathbb{C})$ follows by adding $\left\|f\right\|_{L^p}$ on both sides of the inequality.

Assume $p \ge 2$. Then, by Theorem 13.7 and part (ii) of Corollary 13.8, there exists a constant $c \ge 1$ such that, for all $f \in C_0^\infty(\mathbb{R}^n)$,

$$c^{-1}\left\|\nabla f\right\|_{L^p} \le \left(\int_0^\infty \left\|\phi_t * \nabla f\right\|_{L^p}^2 \frac{dt}{t}\right)^{1/2} \le c\left\|f\right\|_{b^{1,p}_{2,2}}.$$

Thus $b_2^{1,p}(\mathbb{R}^n,\mathbb{C}) \subset w^{1,p}(\mathbb{R}^n,\mathbb{C})$. The inclusion $B_2^{1,p}(\mathbb{R}^n,\mathbb{C}) \subset W^{1,p}(\mathbb{R}^n,\mathbb{C})$ follows by adding $\left\|f\right\|_{L^p}$ on both sides of the inequality. □

14 Besov spaces and heat kernels

The following theorem is due to Triebel [39] for the standard heat equation on $\mathbb{R}^n$ and for $0 < s < 1$. It was extended by Grigor'yan and Liu [13] to a large class of parabolic equations on general metric measure spaces.

Recall the notation $\Delta = \sum_i \partial_i^2$ for the standard Laplace operator on $\mathbb{R}^n$ and $K_t : \mathbb{R}^n \to \mathbb{R}$ for the fundamental solution of the heat equation in (1.9). For $k > 0$ define the linear operator $(-\Delta)^k : \mathscr{S}(\mathbb{R}^n, \mathbb{C}) \to \mathscr{S}(\mathbb{R}^n, \mathbb{C})$ by

$$\widehat{(-\Delta)^k f}(\xi) := |\xi|^{2k} \widehat{f}(\xi) \qquad \text{for } f \in \mathscr{S}(\mathbb{R}^n, \mathbb{C}) \text{ and } \xi \in \mathbb{R}^n. \tag{14.1}$$

For $p, q \geq 1$, $0 < s/2 < k$, and $f \in \mathscr{S}(\mathbb{R}^n, \mathbb{C})$ define

$$\|f\|_{s,p,q;k} := \left(\int_0^\infty \left(\frac{\left\| (-t\Delta)^k (K_t * f) \right\|_{L^p}}{t^{s/2}} \right)^q \frac{dt}{t} \right)^{1/q}. \tag{14.2}$$

Then, for $0 < m < s$ and $k > (s - m)/2$,

$$\|f\|_{s,p,q;k+m/2} = \|(-\Delta)^{m/2} f\|_{s-m,p,q;k}. \tag{14.3}$$

Theorem 14.1 (Triebel/Grigor'yan–Liu). *Let $n \in \mathbb{N}$ and $p, q > 1$.*
(i) *Let $0 < s < 1$. Then there exists a constant $c \geq 1$ such that every Schwartz test function $f \in \mathscr{S}(\mathbb{R}^n, \mathbb{C})$ satisfies the inequalities*

$$\frac{1}{c} \|f\|_{b^{s,p}_{q,2}} \leq \left(\int_0^\infty \left(\frac{\|t\Delta(K_t * f)\|_{L^p}}{t^{s/2}} \right)^q \frac{dt}{t} \right)^{1/q} \leq c \|f\|_{b^{s,p}_{q,2}}.$$

The first inequality continues to hold for $1 \leq s < 2$.
(ii) *Let $0 < s < 2$ and $k > s/2$. Then there exists a constant $c \geq 1$ such that every Schwartz test function $f \in \mathscr{S}(\mathbb{R}^n, \mathbb{C})$ satisfies the inequalities*

$$\frac{1}{c} \|f\|_{B^{s,p}_{q,2}} \leq \|f\|_{L^p} + \left(\int_0^\infty \left(\frac{\left\| (-t\Delta)^k (K_t * f) \right\|_{L^p}}{t^{s/2}} \right)^q \frac{dt}{t} \right)^{1/q} \leq c \|f\|_{B^{s,p}_{q,2}}.$$

(iii) *Let $0 < m < s < 2$. Then there exists a constant $c > 0$ such that every Schwartz test function $f \in \mathscr{S}(\mathbb{R}^n, \mathbb{C})$ satisfies the inequalities*

$$\frac{1}{c} \|f\|_{b^{s,p}_{q,2}} \leq \left\| (-\Delta)^{m/2} f \right\|_{b^{s-m,p}_{q,2}} \leq c \|f\|_{b^{s,p}_{q,2}}.$$

Proof. Part (i) can be found in Triebel [39, Thm 1.7.3 & Thm 1.8.3]. Parts (ii) and (iii) are due to Grigor'yan–Liu [13, Thm 1.5]. The proof given below follows the argument in [13] for the standard Laplace operator on $\mathbb{R}^n$.

Assume $0 < s < 2$ and $k > s/2$. We prove in seven steps that the norm

$$\mathscr{S}(\mathbb{R}^n, \mathbb{C}) \to [0, \infty) : f \mapsto \|f\|_{L^p} + \|f\|_{s,p,q;k} \tag{14.4}$$

defined by (14.2) is equivalent to the norm

$$\mathscr{S}(\mathbb{R}^n, \mathbb{C}) \to [0, \infty) : f \mapsto \|f\|_{L^p} + \|f\|_{b^{s,p}_{q,2}}$$

in (12.6). The first two steps establish the inequality $\|f\|_{b^{s,p}_{q,2}} \le c\|f\|_{s,p,q;1}$. The heart of the proof is Step 3, which shows that $\|f\|_{s,p,q;1} \le c\|f\|_{b^{s,p}_{q,0}}$ for $0 < s < 1$, and hence proves part (i). Steps 4 and 5 show that the norms in (14.4) are equivalent for any two values of $k > s/2$. The last two steps establish part (ii).

Step 1. *Define the function $\Phi : \mathbb{R}^n \to \mathbb{R}$ by*

$$\Phi(x) := \frac{1}{(4\pi)^{n/2}} \left(\frac{n}{2} - \frac{|x|^2}{4} \right) e^{-|x|^2/4}, \qquad \widehat{\Phi}(\xi) = |\xi|^2 e^{-|\xi|^2}.$$

For $t > 0$ define $\Phi_t(x) := t^{-n}\Phi(t^{-1}x)$ so that $\widehat{\Phi}_t(\xi) = \widehat{\Phi}(t\xi)$. Then

$$\|f\|_{s,p,q;1} = 2^{1/q} \left(\int_0^\infty \left(\frac{\|\Phi_t * f\|_{L^p}}{t^s} \right)^q \frac{dt}{t} \right)^{1/q} \tag{14.5}$$

for all $f \in \mathscr{S}(\mathbb{R}^n, \mathbb{C})$ and all p, q, s such that $p, q \ge 1$ and $0 < s < 2$.

By definition, we have

$$\Phi_{\sqrt{t}} = -t\partial_t K_t = -t\Delta K_t, \qquad \widehat{\Phi}_{\sqrt{t}} = -t\widehat{\Delta K_t}.$$

Hence

$$\begin{aligned}
\|f\|_{s,p,q;1} &= \left(\int_0^\infty \left(\frac{\|t\Delta(K_t * f)\|_{L^p}}{t^{s/2}} \right)^q \frac{dt}{t} \right)^{1/q} \\
&= \left(\int_0^\infty \left(\frac{\left\|\Phi_{\sqrt{t}} * f\right\|_{L^p}}{t^{s/2}} \right)^q \frac{dt}{t} \right)^{1/q} \\
&= \left(2 \int_0^\infty \left(\frac{\|\Phi_t * f\|_{L^p}}{t^s} \right)^q \frac{dt}{t} \right)^{1/q}
\end{aligned}$$

for all $f \in \mathscr{S}(\mathbb{R}^n, \mathbb{C})$. This proves Step 1.

Step 2. *Let $p, q \geq 1$ and $0 < s < 2$. Then there exists a constant $c > 0$ such that*

$$\|f\|_{b^{s,p}_{q,2}} \leq c \|f\|_{s,p,q;1}$$

for all $f \in \mathscr{S}(\mathbb{R}^n, \mathbb{C})$.

The function $\widehat{\Phi}$ satisfies the first condition in (13.2), however, it does not satisfy the second condition in (13.2). To obtain a function that does, choose a smooth function $\beta : \mathbb{R}^n \to \mathbb{R}$ in the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ such that $\beta(x) = \beta(-x)$ for all $x \in \mathbb{R}^n$ and whose Fourier transform satisfies

$$\mathrm{supp}(\widehat{\beta}) = \{\xi \in \mathbb{R}^n \,|\, 1/2 \leq |\xi| \leq 2\}, \qquad \widehat{\beta}(\xi) > 0 \quad \text{for } 1/2 < |\xi| < 2.$$

Then β is integrable. Define

$$\phi := \beta * \Phi.$$

Then the Fourier transform of ϕ is given by $\widehat{\phi}(\xi) = \widehat{\beta}(\xi)\widehat{\Phi}(\xi)$ and so ϕ satisfies (13.1) and (13.2), i.e. ϕ is a Triebel function. Moreover,

$$\phi_t = \beta_t * \Phi_t$$

for all $t > 0$ and hence, by Young's inequality,

$$\|\phi_t * f\|_{L^p} = \|\beta_t * \Phi_t * f\|_{L^p} \leq \|\beta_t\|_{L^1} \|\Phi_t * f\|_{L^p} = \|\beta\|_{L^1} \|\Phi_t * f\|_{L^p}.$$

This implies

$$\left(\int_0^\infty \left(\frac{\|\phi_t * f\|_{L^p}}{t^s}\right)^q \frac{dt}{t}\right)^{1/q} \leq \|\beta\|_{L^1} \left(\int_0^\infty \left(\frac{\|\Phi_t * f\|_{L^p}}{t^s}\right)^q \frac{dt}{t}\right)^{1/q}$$

for all $f \in \mathscr{S}(\mathbb{R}^n, \mathbb{C})$. Since the norm on the left is equivalent to $\|\cdot\|_{b^{s,p}_{q,2}}$ by Theorem 13.5, and the second factor on the right is $2^{-1/q}\|f\|_{s,p,q;1}$ by Step 1, this proves Step 2.

Step 3. *Let $p > 1$, $q \geq 1$, and $0 < s < 1$. Then there exists a constant $c > 0$ such that*

$$\|f\|_{s,p,q;1} \leq c \|f\|_{b^{s,p}_{q,0}}$$

for all $f \in \mathscr{S}(\mathbb{R}^n, \mathbb{C})$.

This is the argument in [13, Section 3.1]. There is a constant $c_1 = c_1(n) > 0$ such that, for all $t > 0$ and all $x \in \mathbb{R}^n$,

$$|\Phi_t(x)| = \frac{1}{(4\pi)^{n/2}t^n}\left|\frac{|x|^2}{4t^2} - \frac{n}{2}\right| e^{-\frac{|x|^2}{4t^2}} \le \frac{c_1}{t^n}\left(1 + \frac{|x|}{t}\right)^2 e^{-\frac{1}{8}\left(1+\frac{|x|}{t}\right)^2}.$$

The last step uses the inequality $\frac{1}{8}(1+\frac{|x|}{t})^2 \le \frac{1}{4} + \frac{|x|^2}{4t^2}$. Since Φ_t has mean value zero, we have

$$(\Phi_t * f)(x) = \int_{\mathbb{R}^n} \Phi_t(h)\big(f(x-h) - f(x)\big)\, dh$$

and hence

$$\begin{aligned} |(\Phi_t * f)(x)| &\le \int_{\mathbb{R}^n} |\Phi_t(h)||f(x) - f(x-h)|\, dh \\ &\le \frac{c_1}{t^n}\int_{\mathbb{R}^n}\left(1 + \frac{|h|}{t}\right)^2 e^{-\frac{1}{8}\left(1+\frac{|h|}{t}\right)^2}|f(x) - f(x-h)|\, dh \end{aligned} \tag{14.6}$$

for all $t > 0$ and all $x \in \mathbb{R}^n$. Choose a real number $\alpha > s$. Then there is a constant $c_2 > 0$ such that $(1+r)^2 e^{-\frac{1}{8}(1+r)^2} \le c_2(1+r)^{-(n+\alpha p)}$ for all $r > 0$. Define $c_3 := c_1c_2$. Then it follows from (14.6) that

$$\begin{aligned} |(\Phi_t * f)(x)| &\le \frac{c_1c_2}{t^n}\int_{\mathbb{R}^n}\left(1 + \frac{|h|}{t}\right)^{-n-\alpha p}|f(x) - f(x-h)|\, dh \\ &= c_3 t^{\alpha p}\int_{\mathbb{R}^n}\frac{|f(x) - f(x-h)|}{(t+|h|)^{n+\alpha p}}\, dh \\ &= c_3 t^{\alpha p}\int_{|h|\le t}\frac{|f(x) - f(x-h)|}{(t+|h|)^{n+\alpha p}}\, dh \\ &\quad + c_3 t^{\alpha p}\sum_{i=1}^{\infty}\int_{2^{i-1}t\le|h|\le 2^i t}\frac{|f(x) - f(x-h)|}{(t+|h|)^{n+\alpha p}}\, dh \\ &\le \frac{c_3}{t^n}\int_{|h|\le t}|f(x) - f(x-h)|\, dh \\ &\quad + \frac{c_3}{t^n}\sum_{i=1}^{\infty}\frac{1}{2^{(i-1)(n+\alpha p)}}\int_{2^{i-1}t\le|h|\le 2^i t}|f(x) - f(x-h)|\, dh \\ &\le \frac{c_4}{t^n}\sum_{i=0}^{\infty}\frac{1}{2^{i(n+\alpha p)}}\int_{B_{2^i t}}|f(x) - f(x-h)|\, dh, \end{aligned}$$

where $c_4 := 2^{n+\alpha p}c_3$.

Now use Hölder's inequality to obtain

$$\begin{aligned}
|(\Phi_t * f)(x)| &\le \frac{c_4}{t^n}\sum_{i=0}^\infty \frac{\mathrm{Vol}(B_{2^it})}{2^{i(n+\alpha p)}}\left(\frac{1}{\mathrm{Vol}(B_{2^it})}\int_{B_{2^it}}|f(x)-f(x-h)|^p\,dh\right)^{1/p} \\
&= c_5\sum_{i=0}^\infty 2^{-i\alpha p}\left(\frac{1}{\mathrm{Vol}(B_{2^it})}\int_{B_{2^it}}|f(x)-f(x-h)|^p\,dh\right)^{1/p} \\
&= c_5\sum_{i=0}^\infty 2^{-i\alpha(p-1)}\left(\frac{2^{-i\alpha p}}{\mathrm{Vol}(B_{2^it})}\int_{B_{2^it}}|f(x)-f(x-h)|^p\,dh\right)^{1/p} \\
&\le c_6\left(\sum_{i=0}^\infty \frac{2^{-i\alpha p}}{\mathrm{Vol}(B_{2^it})}\int_{B_{2^it}}|f(x)-f(x-h)|^p\,dh\right)^{1/p}.
\end{aligned}$$

Here $c_5 := c_4\mathrm{Vol}(B_1)$. The last step uses Hölder's inequality again and holds with $c_6 := c_5(\sum_{i=0}^\infty 2^{-i\alpha p})^{(p-1)/p} = c_5(1-2^{-\alpha p})^{-(p-1)/p}$. Integrate the pth power of the last estimate over $x \in \mathbb{R}^n$. This gives

$$\begin{aligned}
\|\Phi_t * f\|_{L^p} &\le c_6\left(\sum_{i=0}^\infty 2^{-i\alpha p}\frac{1}{\mathrm{Vol}(B_{2^it})}\int_{\mathbb{R}^n}\int_{B_{2^it}}|f(x)-f(x-h)|^p\,dh\,dx\right)^{1/p} \\
&= c_6\left(\sum_{i=0}^\infty 2^{-i\alpha p}\frac{1}{\mathrm{Vol}(B_{2^it})}\int_{B_{2^it}}\|\Delta_h f\|_{L^p}^p\;dh\right)^{1/p}
\end{aligned}$$

and hence

$$\frac{\|\Phi_t * f\|_{L^p}}{t^s} \le c_6\left(\sum_{i=0}^\infty 2^{-i(\alpha-s)p}\frac{(2^it)^{-sp}}{\mathrm{Vol}(B_{2^it})}\int_{B_{2^it}}\|\Delta_h f\|_{L^p}^p\;dh\right)^{1/p}. \tag{14.7}$$

Now assume $q > p$, raise both sides of equation (14.7) to the power q, and use Hölders inequality with the exponent $q/p > 1$ and the dual exponent $q/(q-p)$. Then, since $\sum_{i=0}^\infty 2^{-i(\alpha-s)p} < \infty$, we have

$$\begin{aligned}
\left(\frac{\|\Phi_t * f\|_{L^p}}{t^s}\right)^q &\le c_6^q\left(\sum_{i=0}^\infty 2^{-i(\alpha-s)p}\frac{(2^it)^{-sp}}{\mathrm{Vol}(B_{2^it})}\int_{B_{2^it}}\|\Delta_h f\|_{L^p}^p\;dh\right)^{q/p} \\
&\le c_7\sum_{i=0}^\infty 2^{-i(\alpha-s)p}\left(\frac{(2^it)^{-sp}}{\mathrm{Vol}(B_{2^it})}\int_{B_{2^it}}\|\Delta_h f\|_{L^p}^p\;dh\right)^{q/p}.
\end{aligned}$$

By Step 1 this implies

$$\begin{aligned}
\|f\|_{s,p,q;1}^q &= 2\int_0^\infty \left(\frac{\|\Phi_t * f\|_{L^p}}{t^s}\right)^q \frac{dt}{t} \\
&\le 2c_7 \sum_{i=0}^\infty 2^{-i(\alpha-s)p} \int_0^\infty \left(\frac{(2^i t)^{-sp}}{\mathrm{Vol}(B_{2^i t})}\int_{B_{2^i t}} \|\Delta_h f\|_{L^p}^p \, dh\right)^{q/p} \frac{dt}{t} \\
&= 2c_7 \sum_{i=0}^\infty 2^{-i(\alpha-s)p} \int_0^\infty \left(\frac{t^{-sp}}{\mathrm{Vol}(B_t)}\int_{B_t} \|\Delta_h f\|_{L^p}^p \, dh\right)^{q/p} \frac{dt}{t} \\
&= \frac{2c_7}{1-2^{-(\alpha-s)p}} \int_0^\infty \frac{1}{t^{sq}} \left(\frac{1}{\mathrm{Vol}(B_t)}\int_{B_t} \|\Delta_h f\|_{L^p}^p \, dh\right)^{q/p} \frac{dt}{t} \\
&= \left(c_8 \|f\|_{b^{s,p}_{q,0}}\right)^q.
\end{aligned}$$

This proves the estimate in Step 3 for $q > p$ with $c = c_8$.

Now assume $1 \le q \le p$. Then $(\sum_{i=1}^\infty a_i)^{q/p} \le \sum_{i=1}^\infty a_i^{q/p}$ for every sequence of nonnegative real numbers $a_i \ge 0$, by the triangle inequality in $\ell^{p/q}$. Hence it follows from (14.7) that

$$\begin{aligned}
\left(\frac{\|\Phi_t * f\|_{L^p}}{t^s}\right)^q &\le c_6^q \left(\sum_{i=0}^\infty 2^{-i(\alpha-s)p} \frac{(2^i t)^{-sp}}{\mathrm{Vol}(B_{2^i t})}\int_{B_{2^i t}} \|\Delta_h f\|_{L^p}^p \, dh\right)^{q/p} \\
&\le c_6^q \sum_{i=0}^\infty 2^{-i(\alpha-s)q} \left(\frac{(2^i t)^{-sp}}{\mathrm{Vol}(B_{2^i t})}\int_{B_{2^i t}} \|\Delta_h f\|_{L^p}^p \, dh\right)^{q/p}.
\end{aligned}$$

This implies

$$\begin{aligned}
\|f\|_{s,p,q;1}^q &= 2\int_0^\infty \left(\frac{\|\Phi_t * f\|_{L^p}}{t^s}\right)^q \frac{dt}{t} \\
&\le 2c_6^q \sum_{i=0}^\infty 2^{-i(\alpha-s)q} \int_0^\infty \left(\frac{(2^i t)^{-sp}}{\mathrm{Vol}(B_{2^i t})}\int_{B_{2^i t}} \|\Delta_h f\|_{L^p}^p \, dh\right)^{q/p} \frac{dt}{t} \\
&= \frac{2c_6^q}{1-2^{-(\alpha-s)q}} \int_0^\infty \frac{1}{t^{sq}} \left(\frac{1}{\mathrm{Vol}(B_t)}\int_{B_t} \|\Delta_h f\|_{L^p}^p \, dh\right)^{q/p} \frac{dt}{t} \\
&= \left(c_9 \|f\|_{b^{s,p}_{q,0}}\right)^q.
\end{aligned}$$

Thus the estimate in Step 3 also holds for $1 \le q \le p$ with $c = c_9$.

Step 4. *Fix real numbers s,p,q,k,m such that $p,q \geq 1$ and $0 < s/2 < k$ and $m > 0$. Then there exists a constant $c > 0$ such that*

$$\|f\|_{s,p,q;k+m/2} \leq c\,\|f\|_{s,p,q;k} \tag{14.8}$$

for all $f \in \mathscr{S}(\mathbb{R}^n, \mathbb{C})$.

This is the easy direction of [13, Proposition 2.9], which asserts that, for every fixed triple of real numbers $s > 0$ and $p,q \geq 1$, the norms (14.4) for different values of $k > s/2$ (not necessarily integers) are all equivalent. To prove Step 2 we return to the notation

$$S(t)f := K_t * f$$

for the strongly continuous semigroup $S(t)$ on $L^p(\mathbb{R}^n, \mathbb{C})$ generated by the Laplace operator $A := \Delta$. Since the operator $(-A)^{m/2}$ commutes with $S(t)$ for all m and t, we have

$$\begin{aligned}\left\|(-tA)^{k+m/2}S(t)f\right\|_{L^p} &= \left\|(-tA)^{m/2}S(t/2)(-tA)^kS(t/2)f\right\|_{L^p} \\ &\leq \left\|(-tA)^{m/2}S(t/2)\right\|_{\mathcal{L}(L^p)}\left\|(-tA)^kS(t/2)f\right\|_{L^p} \\ &\leq 2^{m/2}C\left\|(-tA)^kS(t/2)f\right\|_{L^p},\end{aligned}$$

where $C := C(n,m,p) := \sup_{t>0}\|(-tA)^{m/2}S(t)\|_{\mathcal{L}(L^p)} < \infty$. This number is finite because $(-tA)^{m/2}S(t)$ is given by convolution with the function $K_{m,t} := (-t\Delta)^{m/2}K_t$. Its Fourier transform is $\widehat{K}_{m,t}(\xi) = (t|\xi|^2)^{m/2}e^{-t|\xi|^2}$. The function $K_{m,1}$ belongs to the Schwartz space and hence is integrable. Moreover, $\widehat{K}_{m,t}(\xi) = \widehat{K}_{m,1}(t^{1/2}\xi)$ and so $K_{m,t}(x) = t^{-n/2}K_{m,1}(t^{-1/2}x)$. This implies that the L^1 norm of $K_{m,t}$ is independent of t and therefore $C(n,m,p) < \infty$. With this understood, we obtain

$$\begin{aligned}\|f\|_{s,p,q;k+m/2} &= \left(\int_0^\infty\left(\frac{\left\|(-tA)^{k+m/2}S(t)f\right\|_{L^p}}{t^{s/2}}\right)^q\frac{dt}{t}\right)^{1/q} \\ &\leq 2^{m/2}C\left(\int_0^\infty\left(\frac{\left\|(-tA)^kS(t/2)f\right\|_{L^p}}{t^{s/2}}\right)^q\frac{dt}{t}\right)^{1/q} \\ &= 2^{m/2+k-s/2}C\left(\int_0^\infty\left(\frac{\left\|(-(t/2)A)^kS(t/2)f\right\|_{L^p}}{(t/2)^{s/2}}\right)^q\frac{dt}{t}\right)^{1/q} \\ &= c\,\|f\|_{s,p,q;k}\end{aligned}$$

with $c := 2^{k-s/2}2^{m/2}C$. This proves Step 4.

Step 5. *Fix real numbers s, p, q, k, m such that $p, q > 1$, $0 < s < 2k$, and $m > 0$. Then there exists a constant $c > 0$ such that*

$$\|f\|_{s,p,q;k} \le c\left(\|f\|_{L^p} + \|f\|_{s,p,q;k+m/2}\right) \tag{14.9}$$

for all $f \in \mathscr{S}(\mathbb{R}^n, \mathbb{C})$.

This is the nontrivial part of [13, Prop 2.9]. Here is the argument in [13]. Choose an integer $N > \ell := k + m/2$. The proof is based on the identity

$$f = \sum_{j=0}^{N-1} \frac{1}{j!}(-A)^j S(1) f + \frac{1}{(N-1)!}\int_0^1 (-\lambda A)^N S(\lambda) f \, \frac{d\lambda}{\lambda} \tag{14.10}$$

for $f \in \mathscr{S}(\mathbb{R}^n, \mathbb{C})$ (sometimes called the *Calderón Reproduction Formula*). The last summand in (14.10) is the remainder term in the Taylor expansion of the function $t \mapsto S(t)f$ at $t = 1$ and this proves equation (14.10) for all $f \in W^{2N,p}(\mathbb{R}^n, \mathbb{C})$. Replace f by $(-tA)^k S(t) f$ in (14.10) to obtain

$$\begin{aligned}(-tA)^k S(t) f &= \sum_{j=0}^{N-1} \frac{t^k}{j!}(-A)^{k+j} S(t+1) f \\ &\quad + \frac{1}{(N-1)!}\int_0^1 t^k \lambda^{N-\ell}(-A)^{k+N-\ell} S(t+\lambda/2)(-\lambda A)^\ell S(\lambda/2) f \, \frac{d\lambda}{\lambda}.\end{aligned}$$

Since $\sup_{t>0} \|(-tA)^{k+j} S(t)\|_{\mathcal{L}(L^p)} < \infty$ for all $j \ge 0$, this shows that there exists a constant $C = C(n, N, p) > 0$ such that

$$\begin{aligned}\left\|(-tA)^k S(t) f\right\|_{L^p} &\le C t^k \|f\|_{L^p} \\ &\quad + C\int_0^{1/2} \frac{t^k \lambda^{N-\ell}}{(t+\lambda)^{k+N-\ell}} \left\|(-\lambda A)^\ell S(\lambda) f\right\|_{L^p} \frac{d\lambda}{\lambda}\end{aligned} \tag{14.11}$$

for all $f \in \mathscr{S}(\mathbb{R}^n, \mathbb{C})$ and all $t > 0$. Following [13] we denote the two terms on the right in (14.11) by

$$\begin{aligned}J_1(t) &:= t^k \|f\|_{L^p}, \\ J_2(t) &:= \int_0^{1/2} \frac{t^k \lambda^{N-\ell}}{(t+\lambda)^{k+N-\ell}} \left\|(-\lambda A)^\ell S(\lambda) f\right\|_{L^p} \frac{d\lambda}{\lambda}.\end{aligned}$$

Hence, with $C_1 := (q(k - \frac{s}{2}))^{-1/q}$,

$$\left(\int_0^1 \left(\frac{J_1(t)}{t^{\frac{s}{2}}}\right)^q \frac{dt}{t}\right)^{1/q} = \left(\int_0^1 t^{q(k-\frac{s}{2})-1} dt\right)^{1/q} \|f\|_{L^p} = C_1 \|f\|_{L^p}. \tag{14.12}$$

Moreover, since $k > s/2$, we have

$$\begin{aligned}\int_0^\infty \frac{t^{k-\frac{s}{2}}\lambda^{N-\ell+\frac{s}{2}}}{(t+\lambda)^{k+N-\ell}}\,\frac{d\lambda}{\lambda} &\le \int_0^t \frac{\lambda^{N-\ell+\frac{s}{2}-1}}{t^{N-\ell+\frac{s}{2}}}\,d\lambda + \int_t^\infty \frac{t^{k-\frac{s}{2}}}{\lambda^{k-\frac{s}{2}+1}}\,d\lambda \\ &= \frac{1}{N-\ell+\frac{s}{2}} + \frac{1}{k-\frac{s}{2}} =: C_2.\end{aligned} \tag{14.13}$$

Hence, by Hölder's inequality,

$$\begin{aligned}\left(\frac{J_2(t)}{t^{\frac{s}{2}}}\right)^q &= \left(\int_0^{1/2} \frac{t^{k-\frac{s}{2}}\lambda^{N-\ell+\frac{s}{2}}}{(t+\lambda)^{k+N-\ell}}\frac{\left\|(-\lambda A)^\ell S(\lambda) f\right\|_{L^p}}{\lambda^{\frac{s}{2}}}\,\frac{d\lambda}{\lambda}\right)^q \\ &\le C_2^{q-1}\int_0^{1/2} \frac{t^{k-\frac{s}{2}}\lambda^{N-\ell+\frac{s}{2}}}{(t+\lambda)^{k+N-\ell}}\left(\frac{\left\|(-\lambda A)^\ell S(\lambda) f\right\|_{L^p}}{\lambda^{\frac{s}{2}}}\right)^q \frac{d\lambda}{\lambda}.\end{aligned}$$

By Fubini's Theorem, this implies

$$\begin{aligned}&\int_0^1 \left(\frac{J_2(t)}{t^{\frac{s}{2}}}\right)^q \frac{dt}{t} \\ &\le C_2^{q-1}\int_0^1\int_0^{1/2} \frac{t^{k-\frac{s}{2}}\lambda^{N-\ell+\frac{s}{2}}}{(t+\lambda)^{k+N-\ell}}\frac{\left\|(-\lambda A)^\ell S(\lambda) f\right\|_{L^p}^q}{\lambda^{\frac{sq}{2}}}\,\frac{d\lambda}{\lambda}\,\frac{dt}{t} \\ &\le C_2^{q-1}\int_0^{1/2}\left(\int_0^1 \frac{t^{k-\frac{s}{2}}\lambda^{N-\ell+\frac{s}{2}}}{(t+\lambda)^{k+N-\ell}}\,\frac{dt}{t}\right)\frac{\left\|(-\lambda A)^\ell S(\lambda) f\right\|_{L^p}^q}{\lambda^{\frac{sq}{2}}}\,\frac{d\lambda}{\lambda} \\ &\le C_2^q\int_0^{1/2}\frac{\left\|(-\lambda A)^\ell S(\lambda) f\right\|_{L^p}^q}{\lambda^{\frac{sq}{2}}}\,\frac{d\lambda}{\lambda}.\end{aligned} \tag{14.14}$$

Here the last inequality follows from (14.13) with $d\lambda/\lambda$ replaced by dt/t. By (14.11), (14.12), and (14.14) we have

$$\begin{aligned}\|f\|_{s,p,q;k} &\le \left(\int_1^\infty \frac{\left\|(-tA)^k S(t) f\right\|_{L^p}^q}{t^{sq/2}}\,\frac{dt}{t}\right)^{1/q} + \left(\int_0^1 \frac{\left\|(-tA)^k S(t) f\right\|_{L^p}^q}{t^{sq/2}}\,\frac{dt}{t}\right)^{1/q} \\ &\le (C_0 + CC_1)\|f\|_{L^p} + CC_2\left(\int_0^\infty \left(\frac{\left\|(-tA)^\ell S(t) f\right\|_{L^p}}{t^{s/2}}\right)^q \frac{dt}{t}\right)^{1/q}\end{aligned}$$

for all $f \in \mathscr{S}(\mathbb{R}^n,\mathbb{C})$, where $C_0 := \sup_{t\ge1}\left\|(-tA)^k S(t)\right\|_{\mathcal{L}(L^p)}(\frac{2}{sq})^{1/q}$. This proves the estimate (14.9) and Step 5.

Step 6. *We prove part (iii) of Theorem 14.1. Let $p > 1$, $q \geq 1$, and $0 < m < s < 2$. Then there is a $c > 0$ such that every $f \in \mathscr{S}(\mathbb{R}^n, \mathbb{C})$ satisfies*

$$\frac{1}{c}\|f\|_{b^{s,p}_{q,2}} \leq \left\|(-\Delta)^{m/2} f\right\|_{b^{s-m,p}_{q,2}} \leq c\|f\|_{b^{s,p}_{q,2}}. \tag{14.15}$$

Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a Triebel function and define $\phi_t(x) := t^{-n}\phi(t^{-1}x)$ for $x \in \mathbb{R}^n$ and $t > 0$. Define the functions $\psi, \psi_t : \mathbb{R}^n \to \mathbb{R}$ by $\widehat{\psi}(\xi) := |\xi|^m\widehat{\phi}(\xi)$ and $\psi_t(x) := t^{-n}\psi(t^{-1}x)$ for $t > 0$ and $x, \xi \in \mathbb{R}^n$, so ψ is also a Triebel function. Now let $f \in \mathscr{S}(\mathbb{R}^n, \mathbb{C})$ and $g := (-\Delta)^{m/2} f$. Then $\widehat{g}(\xi) = |\xi|^m\widehat{f}(\xi)$ for all $\xi \in \mathbb{R}^n$. Since $\widehat{\psi}_t(\xi) = t^m|\xi|^m\widehat{\phi}_t(\xi)$, this implies $\widehat{\psi}_t\widehat{f} = t^m\widehat{\phi}_t\widehat{g}$, therefore $\psi_t * f = t^m\phi_t * g$ for all $t > 0$, and so

$$\begin{aligned}
\left\|(-\Delta)^{m/2} f\right\|_{b^{s-m,p}_{q,6;\phi}} &= \left(\int_0^\infty \left(\frac{\|\phi_t * g\|_{L^p}}{t^{s-m}}\right)^q \frac{dt}{t}\right)^{1/q} \\
&= \left(\int_0^\infty \left(\frac{\|\psi_t * f\|_{L^p}}{t^{s}}\right)^q \frac{dt}{t}\right)^{1/q} \\
&= \|f\|_{b^{s,p}_{q,6;\psi}}
\end{aligned}$$

for $m < s < 2$. Hence Step 6 follows from Theorem 13.5.

Step 7. *We prove parts (i) and (ii) of Theorem 14.1.*

Part (i) follows from Step 2 and Step 3. Moreover, the first inequality in part (ii) follows from Step 2 and Step 4 for $s/2 < k \leq 1$ and from Step 2 and Step 5 for $k > 1$. If $0 < s < 1$ then the second inequality in part (ii) follows from Step 3 and Step 5 for $s/2 < k < 1$ and from Step 3 and Step 4 for $k \geq 1$. To prove the second inequality in part (ii) for $1 \leq s < 2$, choose m such that $0 < m < s < m + 1 \leq 2$. By Step 4 (in the case $k \geq 1 + m/2$) and Step 5 (in the case $k < 1 + m/2$), there is a constant $c_{45} > 0$ such that

$$\begin{aligned}
\|f\|_{s,p,q;k} &\leq c_{45}\left(\|f\|_{L^p} + \|f\|_{s,p,q;1+m/2}\right) \\
&= c_{45}\left(\|f\|_{L^p} + \left\|(-\Delta)^{m/2} f\right\|_{s-m,p,q;1}\right)
\end{aligned}$$

for all $f \in \mathscr{S}(\mathbb{R}^n, \mathbb{C})$ (see (14.3)). Since $0 < s - m < 1$, Step 3 implies

$$\left\|(-\Delta)^{m/2} f\right\|_{s-m,p,q;1} \leq c_3\left\|(-\Delta)^{m/2} f\right\|_{b^{s-m,p}_{q,2}}$$

with $c_3 \geq 1$. By Step 6, $\|(-\Delta)^{m/2} f\|_{b^{s-m,p}_{q,2}} \leq c_6\|f\|_{b^{s,p}_{q,2}}$ with $c_6 \geq 1$. Combine these estimates to obtain $\|f\|_{s,p,q;k} \leq c_7(\|f\|_{L^p} + \|f\|_{b^{s,p}_{q,2}})$ for all $f \in \mathscr{S}(\mathbb{R}^n, \mathbb{C})$ with $c_7 := c_3c_{45}c_6$. This proves Step 7 and Theorem 14.1. □

15 Proof of Theorem 1.2

Proof of Theorem 1.2. Take $k = 1$ and $s = 2 - 2/q$ in Theorem 14.1. Then

$$\frac{1}{c}\|f\|_{b^{s,p}_{q,2}} \leq \left(\int_0^\infty \|\Delta(K_t * f)\|^q_{L^p}\,dt\right)^{1/q} \leq c\,\|f\|_{B^{s,p}_{q,2}} \tag{15.1}$$

for $f \in \mathscr{S}(\mathbb{R}^n, \mathbb{C})$ (with $\|f\|_{B^{s,p}_{q,2}}$ on the right replaced by $\|f\|_{b^{s,p}_{q,2}}$ when $s < 1$). Since $u_t := K_t * f$ is the solution of the heat equation with $u_0 = f$ such that $u_t \in L^2$ for all t, the estimate (15.1) is equivalent to (1.5). □

The next corollary restates the estimates of Theorem 1.2 for $p = q$.

Corollary 15.1. *Let $n \in \mathbb{N}$ and $p > 1$.*
(i) *If $1 < p < 2$ then there exists a constant $c \geq 1$ such that every Schwartz test function $f \in \mathscr{S}(\mathbb{R}^n, \mathbb{C})$ satisfies the inequalities*

$$\begin{aligned}
&\frac{1}{c}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|^p}{|x-y|^{n+2p-2}}\,dy\,dx\right)^{1/p} \\
&\leq \left(\int_0^\infty \|\Delta(K_t * f\|^p_{L^p}\,dt\right)^{1/p} \\
&\leq c\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|^p}{|x-y|^{n+2p-2}}\,dy\,dx\right)^{1/p}.
\end{aligned} \tag{15.2}$$

(ii) *If $p > 2$ then there exists a constant $c \geq 1$ such that every Schwartz test function $f \in \mathscr{S}(\mathbb{R}^n, \mathbb{C})$ satisfies the inequalities*

$$\begin{aligned}
&\frac{1}{c}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\nabla f(x)-\nabla f(y)|^p}{|x-y|^{n+p-2}}\,dy\,dx\right)^{1/p} \\
&\leq \left(\int_0^\infty \|\Delta(K_t * f\|^p_{L^p}\,dt\right)^{1/p} \\
&\leq c\left(\|f\|_{L^p} + \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\nabla f(x)-\nabla f(y)|^p}{|x-y|^{n+p-2}}\,dy\,dx\right)^{1/p}\right).
\end{aligned} \tag{15.3}$$

Proof. If $0 < s < 1$ then, by Lemma 12.7, the norm $\|\cdot\|_{w^{s,p}}$ in (12.11) differs from the norm $\|\cdot\|_{b^{s,p}_{p,0}}$ in (12.4) by a constant factor and so is equivalent to the norm $\|\cdot\|_{b^{s,p}_{p,2}}$ in (12.6) by Theorem 12.1. Hence part (i) follows from

Theorem 1.2 with $p = q$ and $0 < s = 2 - 2/p < 1$. If $1 < s < 2$ then, by Lemma 12.7 and Theorems 12.1 and 13.7, the norm $f \mapsto \|\nabla f\|_{w^{s-1,p}}$ in (12.11) is equivalent the norm $\|\cdot\|_{b^{s,p}_{p,2}}$. Hence part (ii) follows from Theorem 1.2 with $p = q$ and $1 < s = 2 - 2/p < 2$. This proves Corollary 15.1 $\square$

We remark that for $p = q = 2$ the energy identity asserts that

$$\frac{1}{2}\|\nabla f\|_{L^2} = \left(\int_0^\infty \|\Delta(K_t * f)\|_{L^2}^2 \, dt\right)^{1/2} = \|f\|_{1,2,2;1}. \tag{15.4}$$

Thus $\|\cdot\|_{1,2,2;1}$ is the norm of the homogeneous Sobolev space $w^{1,2}(\mathbb{R}^n, \mathbb{C})$. The next corollary restates the estimates of Theorem 1.2 in general.

Corollary 15.2. *Let $n \in \mathbb{N}$ and $p, q > 1$.*
(i) *If $1 < q < 2$ then there exists a constant $c \geq 1$ such that every Schwartz test function $f \in \mathscr{S}(\mathbb{R}^n, \mathbb{C})$ satisfies the inequalities*

$$\begin{aligned}
&\frac{1}{c}\left(\int_0^\infty \left(\int_{\mathbb{R}^n}\int_{B_r} \frac{|f(x+h) - f(x)|^p}{r^{n+(2-2/q)p}} dh dx\right)^{q/p} \frac{dr}{r}\right)^{1/q} \\
&\leq \left(\int_0^\infty \|\Delta(K_t * f\|_{L^p}^q \, dt\right)^{1/q} \\
&\leq c\left(\int_0^\infty \left(\int_{\mathbb{R}^n}\int_{B_r} \frac{|f(x+h) - f(x)|^p}{r^{n+(2-2/q)p}} dh dx\right)^{q/p} \frac{dr}{r}\right)^{1/q}.
\end{aligned}$$

(ii) *Assume $q \geq 2$ and choose m such that $0 < m < 2 - 2/q < m + 1 \leq 2$. Then there exists a constant $c \geq 1$ such that every Schwartz test function $f \in \mathscr{S}(\mathbb{R}^n, \mathbb{C})$ satisfies the inequalities*

$$\begin{aligned}
&\frac{1}{c}\left(\int_0^\infty \left(\int_{\mathbb{R}^n}\int_{B_r} \frac{|((-\Delta)^{m/2} f)(x+h) - ((-\Delta)^{m/2} f)(x)|^p}{r^{n+(2-2/q-m)p}} dh dx\right)^{q/p} \frac{dr}{r}\right)^{1/q} \\
&\leq \left(\int_0^\infty \|\Delta(K_t * f\|_{L^p}^q \, dt\right)^{1/q} \leq c\,\|f\|_{L^p} + \\
&c\left(\int_0^\infty \left(\int_{\mathbb{R}^n}\int_{B_r} \frac{|((-\Delta)^{m/2} f)(x+h) - ((-\Delta)^{m/2} f)(x)|^p}{r^{n+(2-2/q-m)p}} dh dx\right)^{q/p} \frac{dr}{r}\right)^{1/q}.
\end{aligned}$$

When $q > 2$ and $m = 1$, the function $(-\Delta)^{m/2} f$ can be replaced by ∇f.

Proof. This is just a restatement of Theorem 1.2 in more explicit terms. Part (i) uses the equivalence of the norms $\|\cdot\|_{b^{s,p}_{q,2}}$ and $\|\cdot\|_{b^{s,p}_{q,0}}$ established in Theorem 12.1. Part (ii) uses the fact that the norm $\|\cdot\|_{b^{s,p}_{q,2}}$ in (12.6) is equivalent to the norm $f \mapsto \|(-\Delta)^{m/2} f\|_{b^{s-m,p}_{q,0}}$ by Step 6 in the proof of Theorem 14.1. The last assertion follows from Theorem 13.7. □

Remark 15.3. The assertions of Corollary 13.9 about the inhomogeneous Besov spaces $B^{1,p}_2(\mathbb{R}^n, \mathbb{C})$ can also be derived from Theorems 1.2 and 2.1. (See also Lemma 12.9 for the weaker inclusions relating the Besov spaces $B^{1,p}_1(\mathbb{R}^n, \mathbb{C})$ and $B^{1,p}_\infty(\mathbb{R}^n, \mathbb{C})$ to $W^{1,p}(\mathbb{R}^n, \mathbb{C})$.) Here are the details.

(i) Let $n \in \mathbb{N}$ and $p \leq 2$. By Theorem 1.2 with $q = 2$ and $s = 2 - 2/q = 1$, there exists a constant $c > 0$ such that

$$\frac{1}{c} \|f\|_{b^{1,p}_2} \leq \left(\int_0^\infty \|\Delta(K_t * f)\|^2_{L^p} \, dt \right)^{1/2} \leq \frac{2n}{p-1} \|\nabla f\|_{L^p} \tag{15.5}$$

for all $f \in C^\infty_0(\mathbb{R}^n, \mathbb{C})$. Here the second inequality follows from Corollary 2.6. Hence $w^{1,p}(\mathbb{R}^n, \mathbb{C}) \subset b^{1,p}_2(\mathbb{R}^n, \mathbb{C})$ and $W^{1,p}(\mathbb{R}^n, \mathbb{C}) \subset B^{1,p}_2(\mathbb{R}^n, \mathbb{C})$ for $p \leq 2$ and the inclusions are bounded linear operators.

(ii) Let $n \in \mathbb{N}$ and $p \geq 2$. Let $f \in C^\infty_0(\mathbb{R}^n)$ and define $u(t, x) := (K_t * f)(x)$ for $t \geq 0$ and $x \in \mathbb{R}^n$. Then the proof of Theorem 2.1 shows that

$$\frac{d}{dt} \frac{1}{p} \int_{\mathbb{R}^n} |u|^p = -(p-1) \int_{\mathbb{R}^n} |u|^{p-2} |\nabla u|^2 \geq -(p-1) \|u\|^{p-2}_{L^p} \|\nabla u\|^2_{L^p}.$$

The last step follows from Hölders inequality. This implies

$$\frac{d}{dt} \frac{1}{2} \|u_t\|^2_{L^p} = \frac{1}{\|u_t\|^{p-2}_{L^p}} \frac{d}{dt} \frac{1}{p} \|u_t\|^p_{L^p} \geq -(p-1) \|\nabla u\|^2_{L^p}$$

Integrate this inequality to obtain $\|f\|^2_{L^p} \leq (2p-2) \int_0^\infty \|\nabla u_t\|^2_{L^p} \, dt$. Now replace f by $\partial_i f$ for $i = 1, \ldots, n$. Then, by the Calderón–Zygmund inequality in Corollary 6.2,

$$\|\nabla f\|_{L^p} \leq c_1 \left(\int_0^\infty \|\Delta u_t\|^2_{L^p} \, dt \right)^{1/2} \leq c_2 \|f\|_{B^{1,p}_2}$$

for all $f \in C^\infty_0(\mathbb{R}^n, \mathbb{C})$ with suitable constants $c_1 > 0$ and $c_2 > 0$, dependng only on n and p. Here the last step follows from Theorem 1.2 with $q = 2$ and $s = 2 - 2/q = 1$. Hence $B^{1,p}_2(\mathbb{R}^n, \mathbb{C}) \subset W^{1,p}(\mathbb{R}^n, \mathbb{C})$ for $p \geq 2$ and the inclusion is a bounded linear operator.

(iii) It follows from (i) and (ii) that $B^{1,2}_2(\mathbb{R}^n, \mathbb{C}) = W^{1,2}(\mathbb{R}^n, \mathbb{C})$.

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