Partial Differential Equations in Geometry – Lecture script

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Contents

1 Laplace's Equation

(1)
$$u = u(x_1, \dots, x_n) \quad \Delta u = 0 \quad \Delta = \sum_{\nu=1}^{n} \frac{\partial^2}{\partial x_\nu^2}$$

Definition: Let $\Omega \subset \mathbb{R}$ open set. A C^2-function $u : \Omega \to \mathbb{R}$ is called *harmonic* if it satisfies (1).

Theorem Main tool: Divergence Theorem (Gauss): $f : \overline{\Omega} \to \mathbb{R}^n$ C^1-function, $f \in C^1(\overline{\Omega}, \mathbb{R}^n)$, where $\Omega \subset \mathbb{R}^n$ bounded open (so the closure is compact) and $\partial\Omega = \overline{\Omega} \setminus \Omega$ a C^1-*submanifold* of $\mathbb{R}^n$.

$$\int_\Omega \mathrm{div} f \, \mathrm{d}x = \int_{\partial\Omega} \langle f, \nu \rangle \, \mathrm{d}S$$

$$|\nu(x)| = 1, \nu(x) \perp T_x \partial\Omega, \text{ outward}$$

$$\nu : \partial\Omega \to \mathbb{R}^n \text{ outward unit normal vector field}$$

$\mathrm{div} f := \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}$, $M \subset \mathbb{R}^n$ k-manifold, $V \subset \mathbb{R}^{kn}$ open set, $U \subset \mathbb{R}^n$ open.
$\psi : V \to U \cap M$ C^1-diffeo.
For $g : M \to \mathbb{R}$ define

$$\int_{M \cap U} g \, \mathrm{d}S = \int_V g(\psi(\xi)) \sqrt{\det(d\psi(\xi)^T d\psi(\xi))} \, \mathrm{d}\xi$$

Can also be interpreted as Lebesgue-integral, but we consider only continuous functions $\Rightarrow$ Riemann-integral.

Example: $\Omega = B_r = \{x \in \mathbb{R}^n \mid |x| < r\}$
$\partial\Omega = S_r = \{x \in \mathbb{R}^n \mid |x| = r\}$
$f(x) := x \qquad \frac{\partial f_i}{\partial x_i} = 1 \qquad \mathrm{div} f(x) = n$
$\nu(x) = \frac{x}{r} \qquad \langle f, \nu \rangle = r$

$$\int_\Omega \mathrm{div} f \, \mathrm{d}x = n \mathrm{Vol}(B_r) \qquad \int_{\partial\Omega} \langle f, \nu \rangle \, \mathrm{d}S = r \underbrace{\mathrm{Area}(S_r)}_{r^{n-1}\omega_n}$$

where $\mathrm{Area}(S_1) = \omega_n$

How to compute ω_n? Trick:

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{-|x|^2} \, \mathrm{d}x &= \int_0^\infty \int_{S_r} e^{-|x|^2} \, \mathrm{d}S \, \mathrm{d}r \\
&= \int_0^\infty e^{-r^2} \mathrm{Area}(S_r) \, \mathrm{d}r \\
&= \omega_n \int_0^\infty r^{n-1} e^{-r^2} \, \mathrm{d}r \\
&\overset{r^2 = s, 2r dr = ds}{=} \frac{\omega_n}{2} \int_0^\infty s^{\frac{n}{2}-1} e^{-s} \, \mathrm{d}s \\
&= \frac{\omega}{2} \Gamma(\frac{n}{2})
\end{aligned}$$

$$\omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} = \begin{cases} \frac{2\pi^{\frac{n}{2}}}{(\frac{n}{2}-1)!} & n \text{ even} \\ \frac{2\pi^{\frac{n}{2}}}{(\frac{n}{2}-1)(\frac{n}{2}-2)\cdots\frac{1}{2}} & n \text{ odd} \end{cases}$$

$$\omega_1 = 2 \quad \omega_2 = 2\pi \quad \omega_3 = 4\pi \quad \omega_4 = 2\pi^2$$

$\Omega \subset \mathbb{R}^n$ bounded open set with C^1-boundery. $u, v \in C^2(\overline{\Omega})$ and $f(x) := v(x) \bigtriangledown u(x)$ with $\bigtriangledown u = \begin{pmatrix} \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_n} \end{pmatrix} : \overline{\Omega} \to \mathbb{R}^n$

Then $f \in C^1(\overline{\Omega})$ and

$$\begin{aligned} \text{div} f &= \sum_{k=1}^n \frac{\partial f_k}{\partial x_k} \\ &= \sum_{k=1}^n \frac{\partial}{\partial x_k}\left(v \frac{\partial u}{\partial x_k}\right) \\ &= v\sum_{k=1}^n \frac{\partial^2 u}{\partial x_k} + \sum_{k=1}^n \frac{\partial v}{\partial x_k}\frac{\partial u}{\partial x_k} \end{aligned}$$

$$\text{div} f = v\Delta u + \langle \Delta v, \Delta u \rangle$$

$$\langle f, \nu \rangle = v \sum_{i=1}^n \frac{\partial u}{\partial x_i}\nu_i = v\langle \bigtriangledown u, \nu \rangle$$

$$\frac{\partial u}{\partial n} := \sum_{k=1}^n \frac{\partial u}{\partial x_i}\nu_i : \partial\Omega \to \mathbb{R}$$

Divergence Theorem ⇒ *Green's Identities*

(2)

$$\int_\Omega v\Delta u = -\int_\Omega \langle \bigtriangledown v, \bigtriangledown u \, \mathrm{d} + \int_{\partial\Omega} v\frac{\partial u}{\partial n}\,\mathrm{d}S$$

(3)

$$\int_\Omega (v\Delta u - u\Delta v) = \int_{\partial\Omega} v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\,\mathrm{d}S$$

Case 1 $v \equiv 1$ in (2) $\int_\Omega \Delta u = \int_{\partial\Omega} \frac{\partial u}{\partial n}\,\mathrm{d}S$

Case 2 $v = u$ in (2) $\int_\Omega u\Delta u + \int_\Omega |\bigtriangledown u|^2 = \int_{\partial\Omega} u\frac{\partial u}{\partial n}\,\mathrm{d}S$

Dirichlet Problem

D

$$\left.\begin{array}{l} \Delta u = f \text{ in } \Omega \\ u = g \text{ on } \partial\Omega \end{array}\right\} \exists \text{ at most one solution } u \in C^2(\overline{\Omega})$$

Neumann Problem

N

$$\left.\begin{array}{l} \Delta u = f \text{ in } \Omega \\ \frac{\partial u}{\partial n} = g \text{ on } \partial\Omega \end{array}\right\} \text{necessary condition: } \int_\Omega f = \int_{\partial\Omega} g \text{ uniqueness up to add. const.}$$

spherical symmetric

$$\Delta u(x) = 0 \qquad A \in O(n) \qquad A^T = A^{-1}$$

$$v(y) := u(Ay) \Rightarrow \Delta v(y) = 0 \quad \text{Exercise}$$

Are there a *spherically symmetric* solutions u of $\Delta u = 0$, ie $u(Ax) = u(x) \forall A \in O(n) \forall x$. Such a solution would only depend on $|x| = \sqrt{x_1^2 + \ldots + x_n^2}$.
Let $0 < a < b$ and $\psi : (a,b) \to \mathbb{R}$ a C^2-function.
Let $\Omega := \{x \in \mathbb{R}^n \mid a < |x| < b\}$ and define $u : \Omega \to \mathbb{R}$ by $u(x9) := \psi(|x|)$
What is Δu?

$$\begin{aligned} \frac{\partial u}{\partial x_i} &= \psi'(|x|) \\ \frac{\partial^2 u}{\partial x_i^2} &= \psi''(|x|)\frac{x_i^2}{|x|^2} + \frac{\psi'(|x|)}{|x|}\left(1 - \frac{x_i^2}{|x|^2}\right) \\ \Delta u(x) &= \psi''(|x|) + \psi'(|x|)\frac{n-1}{|x|} \end{aligned}$$

$$\Delta u = 0 \Leftrightarrow \psi''(r) + \psi'(r)\frac{n-1}{r} = 0$$

$$\begin{aligned} \psi'(r) &= cr^{1-n} \\ \psi''(r) &= c(1-n)r^{-n} \\ \psi(r) &= \begin{cases} \frac{c}{2-n} r^{2-n} & n > 2 \\ c\log(r) \end{cases} \end{aligned}$$

From now on choose

$$\psi(r) := \begin{cases} \frac{1}{\omega_{2-n}} r^{2-n} & n > 2 \\ \frac{\log(r)}{2\pi} & n = 2 \end{cases}$$

The function

$$K(x) := \psi(|x|) = \begin{cases} \frac{1}{\omega_{2-n}} |x|^{2-n} & n > 2 \\ \frac{\log(|x|)}{2\pi} & n = 2 \end{cases}$$

is called the *fundamental solution* of Laplace's equation.

Remark:

a) $\frac{\partial K}{\partial x_i} = \frac{x_i}{\omega_n |x|}$

b) $\frac{\partial^2 K}{\partial x_i \partial x_j} = -\frac{n}{\omega_n}\frac{x_i x_j}{|x|^{n+2}}$, $i \neq j$

c) $\frac{\partial K}{\partial x_i^2} = \frac{n}{\omega_n |x|^n}(\frac{1}{n} - \frac{x_i^2}{|x|^2})$
$\Delta K = 0$

Lemma 1: Let $K_\xi(x) := K(\xi - x)$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^1-boundery and $u \in C^2(\overline{\Omega})$.
Then for $\xi \in \Omega$

(i) $u(\xi) = \int_\Omega K_\xi \Delta u + \int\limits_{\partial\Omega} (u \frac{\partial K\xi}{\partial n} - K\xi \frac{\partial u}{\partial n} \, \mathrm{d}S$

(ii) if $\Omega = B_r(\xi) = \{x \in \mathbb{R}^n \mid |x - \xi| < r\}$ then
$u(\xi) = \int\limits_{B_r(\xi)} (\psi(|x-\xi|) - \psi(r))\Delta u(x) \, \mathrm{d}x + \frac{1}{\omega_n r^{n-1}} \int\limits_{\partial B_r(\xi)} u \, \mathrm{d}S$

Remark: The term $\frac{1}{\omega_n r^{n-1}} \int\limits_{\partial B_r(\xi)} u \, \mathrm{d}S$ is the mean value of u over the boundary of $B_r(\xi)$

Proof of Lemma 1:

Step 1 (i)⇒(ii):

$$\int_{\partial B_r(\xi)} (u\frac{\partial K_\xi}{\partial u} - K_\xi \frac{\partial u}{\partial r}) \overset{(\Delta)}{=} \frac{1}{\omega_n r^{n-1}} \int\limits_{\partial B_r(\xi)} u \,\mathrm{d}S - \psi(r) \int\limits_{\partial B_r} \frac{\partial u}{\partial n} \,\mathrm{d}S$$

$$(\Delta) \qquad \left.\begin{array}{l} |x-\xi| = r \quad \nu(x) = \frac{x-\xi}{|x-\xi|} \\ \bigtriangledown K(x) = \frac{1}{\omega_n} \frac{x-\xi}{|x-\xi|^n} \end{array}\right\} \frac{\partial K_\xi}{\partial n}(x) = \frac{1}{\omega_n} \frac{\langle x-\xi, x-\xi\rangle}{|x-\xi|^{n+1}} = \frac{1}{\omega_n r^{n-1}}$$

Proof of (ii) Key trick: $\Omega_r := \Omega \setminus \overline{B_r(\xi)}$

$$\overset{\text{Green},(2)}{\Rightarrow} \int_{\Omega_r} (K_\xi \Delta u - u \Delta K_\xi) = \int\limits_{\partial\Omega_r} (K_\xi \frac{\partial u}{\partial n} - u \frac{\partial K_\xi}{\partial n}) \,\mathrm{d}S$$

$$= \int\limits_{\partial\Omega} (K_\xi \frac{\partial u}{\partial n} - u \frac{\partial K_\xi}{\partial n}) \,\mathrm{d}S + \int\limits_{\partial B_r(\xi)} u \frac{\partial K_\xi}{\partial n} - K_\xi \frac{\partial u}{\partial n} \,\mathrm{d}S$$

$$\Rightarrow \int_{\Omega_r} K_\xi \Delta u + \int_{\partial\Omega} (u \frac{\partial K_\xi}{\partial n} - K_\xi \frac{\partial u}{\partial n})$$

$$= \int\limits_{\partial B_r(\xi)} (u \frac{\partial K_\xi}{\partial n} - K_\xi \frac{\partial u}{\partial n}) \,\mathrm{d}S$$

$$\overset{\text{Step 1}}{=} \underbrace{\frac{1}{\omega_n r^{n-1}} \int\limits_{\partial B_r(\xi)} u \,\mathrm{d}S}_{\overset{r\to 0}{\to} u(\xi)} + \underbrace{\psi(r) \int_{B_r(\xi)} \Delta u}_{\overset{r\to 0}{\to}}$$

Key fact:

$$K_\xi|_{\overline{\Omega}} \text{ is integrable, so } \int K_\xi \Delta u \text{ is well defined}$$

$$\text{and } \int_\Omega K_\xi \Delta u = \lim_{r\to\infty} \int_{\Omega_r} K_\xi \Delta u$$

□

We have proved:
$\Omega \subset \mathbb{R}^n$ bounded open C^1-boundary
$u \in C^2(\overline{\Omega}), \xi \in \Omega$
$u(\xi) = \int_\Omega K_\xi \Delta u + \int\limits_{\partial\Omega} (u\frac{\partial K_\xi}{\partial n} - K_\xi \frac{\partial u}{\partial n}) \,\mathrm{d}S$ (Lemma 1)

$$K_\xi(x) := K(\xi - x) \quad K(x) := \psi(|x|) = \begin{cases} \frac{1}{\omega_{2-n}} |x|^{2-n} & n > 2 \\ \frac{\log(|x|)}{2\pi} & n = 2 \end{cases}$$

Case 1: u harmonic $\Rightarrow u(\xi) = \int\limits_{\partial\Omega} (u\frac{\partial K_\xi}{\partial n} - K_\xi \frac{\partial u}{\partial n}) \,\mathrm{d}S$

Given $g_0, g_1 : \Omega \to \mathbb{R}$. Define

$$u(\xi) := \int\limits_{\partial\Omega} (g_0 \frac{\partial K_\xi}{\partial n} - g_1 K_\xi) \,\mathrm{d}S$$

Then u is harmonic, but $u|_{\partial\Omega}$ need not be equal to g_0 and $\frac{\partial u}{\partial n}|_{\partial\Omega}$ need not be g_1. The problem

$$\Delta u = 0 \text{ in } \Omega \quad u|_{\partial\Omega} = g_0 \quad \frac{\partial u}{\partial n}|_{\partial\Omega} = g_1$$

is overdetermined!
$C_0^l(\Omega) := \{u \in C^l(\Omega) \mid u \text{ vanishes near } \partial\Omega\}$, where u vanishes near $\partial\Omega \Leftrightarrow \text{supp}(u) := \overline{\{x \in \mathbb{R}^n \mid u(x) \neq 0\}} \subset \Omega$.

Case 2: $u \in C_0^2(\Omega)$ convolution $\Rightarrow u(\xi) = \int_\Omega K_\xi u = \int\limits_\Omega K(\xi - x)\Delta u(x)\,\mathrm{d}x$
extend u by defining $u(x) := 0$ for $x \notin \Omega$

$$\Rightarrow u = K_* \Delta u \quad \forall u \in C_0^2(\mathbb{R}^n)$$

Question: Given a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ with compact support, define

(*) $$u(x) := \int\limits_{\mathbb{R}^n} K(x-y)f(y)\,\mathrm{d}y$$

Is it true that $\Delta u = f$? Answer: YES and NO!

Lemma 2: $\Omega \subset \mathbb{R}^n$ bounded, open. $f \in C^2(\Omega)$ bounded. Define $u : \Omega \to \mathbb{R}$ by (*)
$\Rightarrow u \in C^2(\Omega) \quad \Delta u = f$

Remark: $K : \mathbb{R}^n \to \mathbb{R}$ is integrable near zero and hence on every compact set.

Proof:

Step 1: $g : \mathbb{R}^n \to \mathbb{R}$ locally integrable $\Rightarrow \quad \forall$ bounded Borel set $B \subset \mathbb{R}^n$:

$$\lim_{h\to 0} \int\limits_B |g(y+h) - g(y)|\,\mathrm{d}y = 0$$

Exercise:

a) true for continuous g

b) approximate g by continuous functions
(we proved that $C_c^0(\Omega)$ is dense in $L^1(\Omega)$)

Step 2: $f : \Omega \to \mathbb{R}$ bounded measurable $\Rightarrow \quad u$ is continuous.

Proof of Step 2:

$$u(x) = \int\limits_\Omega K(x-y)f(y)\,\mathrm{d}y$$

$$\begin{aligned}
|(u(x_{n+1}) - u(x_n)| &= |\int\limits_\Omega (K(x+h-y) - K(x-y))f(y)\,\mathrm{d}y| \\
&\leq \int\limits_\Omega |K(x+h-y) - K(x-y)|\,\mathrm{d}y \\
&\leq \sup_\Omega |f| \int\limits_\Omega |K(x+h-y) - K(x-y)|\,\mathrm{d}y \stackrel{h\to 0}{\to} 0
\end{aligned}$$

(by Step 1 with $g(y) := K(x-y)$)

Step 3: Assume $f \in C_0^2(\Omega) \Rightarrow u \in C^2(\Omega) \quad \Delta u = f$

Proof of Step 3:

$$\begin{aligned} u(x) &= \int\limits_{\mathbb{R}^n} K(y) f(x-y)\,\mathrm{d}y \\ \partial_i u(x) &= \lim_{t\to 0} \int\limits_{\mathbb{R}^n} K(y) \frac{f(x-te_i-y)-f(x-y)}{t}\,\mathrm{d}y \\ &= \int\limits_{\mathbb{R}^n} K(y)\partial_i f(x-y)\,\mathrm{d}y \end{aligned}$$

(continuous by Step 2, similarly $u \in C^2$)

$$\begin{aligned} \Rightarrow \Delta u(x) &= \int\limits_{\mathbb{R}} K(y)\Delta f(x-y)\,\mathrm{d}y \\ &= (K_* \Delta f)(x) \\ &= f(x) \text{ by Lemma 1} \Rightarrow u \in C^2(\Omega) \end{aligned}$$

Step 4: general case $f \in C^2(\Omega)$ bounded.

Proof of Step 4: Fix an element $\xi \in \Omega$. Choose $\varepsilon > 0$ such that $\overline{B_{2\varepsilon}} \subset \Omega$. Choose a smooth cutoff function

$$\rho : \mathbb{R}^n \to [0,1], \rho(x) = \begin{cases} 1 & |x-\xi| \leq \varepsilon \\ 0 & |x-\xi| \geq 2\varepsilon \end{cases}$$

$$f = f_0 + f_1 \quad f_0 := \rho f \in C_0^2(\Omega) \quad f_1 := (1-\rho) f \in C^2(\Omega)$$

$$u = u_0 + u_1 \quad u_i := K_* f_i \overset{\text{Step 3}}{\Rightarrow} u_0 \in C^2(\Omega) \quad \Delta u_0 = f_0$$

u_1 is harmonic in $B_\varepsilon(\xi)$!

$$u_1(x) = \int\limits_{\Omega \setminus B_\varepsilon(\xi)} K(x-y) f_1(y)\,\mathrm{d}y \text{ for } x \in B_\varepsilon(\xi)$$

(because $f_i(y) = 0$ for $y \in B_\varepsilon(\xi)$)
differentiate under integral sign to get

$$u_1|_{B_\varepsilon(\xi)} \in C^2(B_\varepsilon(\xi))$$

and

$$\Delta u_1(x) = \int\limits_{\Omega \setminus B_\varepsilon(\xi)} \Delta K(x-y) f(y)\,\mathrm{d}y = 0 \text{ for } x \in B_\varepsilon(\xi)$$

□

Case 3: $\Omega = B_r(\xi)$ in Lemma 1:

(**)
$$u(\xi) = \frac{1}{\omega_n r^{n-1}} \int\limits_{B_r(\xi)} \underbrace{(\psi(|x-\xi|) - \psi(r))}_{<0 \text{ for } 0<|x-\xi|<r} \Delta u(x)\,\mathrm{d}x$$

Definition: $\Omega \subset \mathbb{R}^n$ open set. A continuous function $u : \Omega \to \mathbb{R}$ is called *subharmonic* if

$$\forall \xi \in \mathbb{R}^n \forall r > 0 : \overline{B_r(\xi)} \subset \Omega \Rightarrow u(\xi) \leq \frac{1}{\omega_n r^{n-1}} \int\limits_{\partial B_r(\xi)} u\,\mathrm{d}S$$

Lemma 3: For $u \in C^2(\Omega)$ the following are equivalent:

i) u is subharmonic

ii) $\forall \xi \in \mathbb{R}^n \forall r > 0$ with $\overline{B_r(\xi)} \subset \Omega$:
$u(\xi) \leq \frac{n}{\omega_n r^n} \int\limits_{B_r(\xi)} u(x)\,\mathrm{d}x$

iii) $\Delta u(x) \geq 0 \forall x \in \Omega$

Proof: (iii)⇒(i): (**)
(i)⇒(ii):
If $\overline{B_r(\xi)} \subset \Omega$ then $\omega_n \rho^{n-1} u(\xi) \leq \int\limits_{\partial B_\rho} u\,\mathrm{d}S$ for $0 \leq \rho \leq r$
integrate $\int\limits_0^r \mathrm{d}\rho$
(ii)⇒(iii):
Assume, by contradiction, that $\Delta u(\xi) < 0$ for some $\xi \in \Omega$.
$\Rightarrow \exists r > 0 \forall x \in B_r(\xi) : \Delta u(x) < 0$
$\overset{(**)}{\Rightarrow} u(\xi) > \frac{1}{\omega_n \rho^{n-1}} \int\limits_{\partial B_\rho(\xi)} u\,\mathrm{d}S$ for $0 < \rho < r$
integrate to get $u(\xi) > \frac{n}{\omega_n r^n} \int_{B_r(\xi)} u$ □

Corollary: $u \in C^2(\Omega)$

$$u \text{ harmonic} \Leftrightarrow u(\xi) = \frac{1}{\omega_n r^{n-1} \int\limits_{\partial B_r(\xi)} u\,\mathrm{d}S}$$

$$\forall \xi \forall r \overline{B_r(\xi)} \subset \Omega$$

$$\Leftrightarrow u(\xi) = \frac{n}{\omega_n r^n} \int_{B_r(\xi)} u \quad \forall \xi \forall r \overline{B_r(\xi)} \subset \Omega$$

Lemma 4: $u : \Omega \to \mathbb{R}$ continuous. Equivalent are:

i) $\forall \xi \in \mathbb{R}^n, r > 0$ with $\overline{B_r(\xi)} \subset \Omega$:
$u(\xi) = \frac{1}{\omega_n r^{n-1} \int\limits_{\partial B_r(\xi)} u\,\mathrm{d}S}$

ii) $forall \xi \in \mathbb{R}^n, r > 0$ with $\overline{B_r(\xi)} \subset \Omega$:
$u(\xi) = \frac{n}{\omega_n r^n} \int\limits_{B_r(\xi)} u(x)\,\mathrm{d}x$

iii) $u \in C^2(\Omega)$ and $\Delta u(x) = 0 \quad \forall x \in \Omega$

Proof:

(iii)⇒(i): Lemma 3 for u and $-u$

(i)⇒(ii): Proof of Lemma 3.

(ii)⇒(i): $\frac{\omega_n r^n}{n} u(\xi) = \int\limits_0^r \left(\int\limits_{\partial B_\rho(\xi)} u\,\mathrm{d}S \right) \mathrm{d}S$ differentiate to set (i).

(i)⇒(iii):
1. Choose a smooth function $\phi : [0, \infty) \to [0, \infty)$ such that $\phi(r) = 0$ for $r \geq 0$ and $\phi(r) = \phi(0)$ for $r \leq \frac{1}{2}$ with $\int\limits_0^1 r^{n-1} \phi(r)\,\mathrm{d}r = \frac{1}{\omega_n}$
2. Define $\rho : \mathbb{R}^n \to [0, \infty)$ by $\rho(x) := \phi(|x|)$
$\Rightarrow \rho$ is smooth (ie C^∞) and $\int\limits_{\mathbb{R}^n} \rho(x)\,\mathrm{d}x = 1, \rho(x) = 0$ for $|x| \geq 1$.

3. Define $\rho_\varepsilon(x) := \frac{1}{\varepsilon^n}\phi(\frac{x}{\varepsilon}) \Rightarrow \phi_varepsilon$ is smooth.
   $\phi_varepsilon(x) = 0$ for $|x| \geq \varepsilon \quad \int_{\mathbb{R}^n} \rho_\varepsilon = 1$.
4. $\Omega_\varepsilon := \{\xi \in \Omega \mid \overline{B_\varepsilon(\xi)} \subset \Omega\}$. Define $u_\varepsilon : \Omega_\varepsilon \to \mathbb{R}$ by $u_\varepsilon(x) := \int\limits_{B_\varepsilon(x)} \rho_\varepsilon(x-y)u(y)\,\mathrm{d}y$
   $\Rightarrow u_\varepsilon \in C^\infty \quad \partial^\alpha u_\varepsilon(x) = \int\limits_{B_\varepsilon(x)} \partial^\alpha \rho_\varepsilon(x-y)u(y)\,\mathrm{d}y$
5. $u_\varepsilon = u|_{\Omega_\varepsilon} \quad \overline{B_\varepsilon(\xi)} \subset \Omega$

$$\begin{aligned}
u_\varepsilon(\xi) &= \int\limits_{B_\varepsilon(\xi)} \frac{1}{\varepsilon^n}\phi(\frac{|\xi-y|}{\varepsilon})u(y)\,\mathrm{d}y \\
&= \int_0^\varepsilon (\int\limits_{\partial B_r(\xi)} \frac{1}{\varepsilon^n}\phi(\frac{r}{\varepsilon})\,\mathrm{d}S)\,\mathrm{d}r \\
&= \int_0^\varepsilon \frac{1}{\varepsilon}\phi(\frac{r}{\varepsilon}u)r^{n-1}\omega_n u(\xi)\,\mathrm{d}r \\
&= \int_0^\varepsilon \frac{r^{n-1}}{\varepsilon^{n-1}}\phi(\frac{r}{\varepsilon})\,\mathrm{d}\frac{r}{\varepsilon}\omega_n u(\xi) \\
&= \int_0^1 r^{n-1}\phi(r)\,\mathrm{d}r\omega_n u(\xi) \\
&= \frac{1}{\omega_n}\omega_n u(\xi) \\
&= u(\xi) \quad \Delta u = 0 \text{ by Lemma 3}
\end{aligned}$$

□

Example:

$$\begin{aligned}
u(x) &= |x|^3 \\
\partial_i u(x) &= 3|x|x_i \\
\partial_i\partial_j u(x) &= 3\frac{x_i x_j}{|x|} \quad i \neq j \\
\partial_i^2 u(x) &= 3|x|(1+\frac{x_i^2}{|x|^2} \\
\Delta u(x) &= 3(n+1)|x|
\end{aligned}$$

Example:

(1)
$$u(x,y) := xy\rho(\underbrace{x^2+y^2}_{s}), \text{ where } \rho(s) = \sqrt{-\log(s)} \quad 0 < s \leq 1$$

$$\partial_x u = y \underbrace{\rho(s)}_{\to\infty \text{ for } s\to 0} + 2x^2y\rho'(s) = y\rho(s) + \frac{x^2y}{s}s\rho'(s) \text{ continuous at the origin}$$

*
$$\partial_x^2 u = 6xy\rho'(s) + 4x^3y\rho''(s) = 6\frac{xy}{s}\underbrace{s\rho'(s)}_{\text{cont's}} + 4\frac{x^3y}{s^2}\underbrace{s^2\rho''(s)}_{\text{cont's}} \text{ continuous}$$

$$s\rho'(s) = -\frac{1}{2\sqrt{-\log(s)}} \qquad s^2\rho''(s) + s\rho'(s) = -\frac{1}{4(-\log(s))^{3/2}}$$

*

$$\Rightarrow f := \Delta u = 12xy\rho'(s) + 4xys\rho''(s)$$
$$= \frac{xy}{x^2+y^2}\left(\frac{1}{\log(x^2+y^2)} - 4\right)\frac{1}{\sqrt{-\log(x^2+y^2)}} \text{ cont's}$$

This is the function we get for the Laplace. $\Rightarrow u \in C^1(\Omega), \Omega = \{(x,y) \in \mathbb{R}^2 \mid \sqrt{x^2+y^2} < \frac{1}{2}\}$ and $\Delta u \in C^0(\Omega)$.
$\partial_x \partial_y u = \underbrace{\rho(s)}_{notcont's} + 2s\rho'(s) + \underbrace{4x^2y^2\rho''(s)}_{cont's}$, ie. $u \notin C^2(\Omega)$.

Claim:

1) With $f \in C^0(\Omega)$ as above we have $K_* f \notin C^2(\Omega)$
2) $\nexists u \in C^2(\Omega)$ with $\Delta u = f$.

u given by (1). Then $u \in C^2(\Omega \backslash 0) \quad u + K \in C^2(\Omega\backslash 0)$ and $\Delta(u+K) = f \in \Omega\backslash 0$.

Basic observation: $\Omega \subset \mathbb{R}^n$ open, $u \in C^2(\Omega)$, $f \in C^0(\Omega)$. Equivalent are:

1. $\Delta u = f \in \Omega$
2. $\int_\Omega u\Delta\varphi = \int_\Omega f\varphi \quad \forall \varphi \in C_0^\infty(\Omega)$

Definition: A function $f : \Omega \to \mathbb{R}$ is called *locally integrable* if f is Lebesgue measurable and $\int_K |f| \, d\mu < \infty \quad \forall$ compact sets $K \subset \Omega$.

Notation:

$$\mathrm{L}^1_{\mathrm{loc}}(\Omega) := \{f : \Omega \to \mathbb{R} \mid f \text{ is locally integrable}/\sim\}$$
$$f \sim g \overset{\text{def.}}{\Leftrightarrow} f - g = 0 \text{ almost everywhere}$$

Definition: Let $u, f \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$. u is called a *weak solution* of $\Delta u = f$ if

$$\int_\Omega u\Delta\varphi = \int_\Omega f\phi \quad \forall \varphi \in C_0^\infty(\Omega)$$

FACT:

$$u \in C^2(\Omega), f \in C^0(\Omega) \quad u \text{ is a weak solution of } \Delta u = f$$
$$\Leftrightarrow$$
$$u \text{ is a strong solution}$$

Example 1: $\Omega \subset \mathbb{R}^n$ open, $\xi \in \Omega$. $u \in C^1(\Omega) \cap C^2(\Omega \setminus \{\xi\})$ and $f \in C^0(\Omega)$, $\Delta u = f \in \Omega \setminus \{\xi\}$
$\Rightarrow u$ is a weak solution of $\Delta u = f$.

Proof:

$$\begin{aligned}
\int_{\Omega\setminus B_\varepsilon(\xi)} (\varphi f - u\Delta\varphi) &= \int_{\Omega\setminus B_\varepsilon(\xi)} (\varphi\Delta u - u\Delta\varphi) \\
&= \int\limits_{\partial(\Omega\setminus B_\varepsilon(\xi))} (\varphi\frac{\partial u}{\partial n} - u\frac{\partial\varphi}{\partial n})\,\mathrm{d}S \\
&= \int\limits_{\partial B_\varepsilon(\xi)} \underbrace{(u\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial u}{\partial n})}_{|.|\le 2\|u\|_{C^1}\|\varphi\|_{C^1}}\,\mathrm{d}S \\
&= \varphi^{n-1}\omega_n c \overset{\varphi\to 0}{\to} 0
\end{aligned}$$

This gives

$$\int_\Omega (\varphi f - u\Delta\varphi) = \lim_{\varepsilon\to 0}\int_{\Omega\setminus B_\varepsilon(\xi)} (\varphi f - u\Delta\varphi) = 0$$

□

Example 2: $\Omega \subset \mathbb{R}^n$ open, $\xi \in \Omega$. Define $K_\xi := K(x-\xi)$
$\Rightarrow K_\xi \in C^2(\Omega\setminus\{\xi\}) \quad \Delta K_\xi = 0 \in \Omega\setminus\{\xi\}$
K_ξ is not a weak solution of $\Delta u = 0$.

$$\text{Green's formula: } \int_\Omega K_\xi\Delta\varphi = \varphi(\xi)$$

$$\text{so "}\Delta K_\xi = \delta_\xi\text{"}$$

Example 3: $f \in \mathrm{L}^1(\Omega) \Rightarrow u := K_* f \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ is aweak solution of $\Delta u = f$.

Proof: $\forall\varphi \in C_0^\infty(\Omega)$ we have $K_*\delta\varphi = \varphi$ (Lemma 2)

$$\begin{aligned}
\Rightarrow \int_\Omega (K_* f)\Delta\varphi &= \int\limits_{\mathbb{R}^n} (K_* f)(x)\Delta\varphi(x))\,\mathrm{d}x \\
&= \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} K(x-y)f(y)\Delta\varphi(x)\,\mathrm{d}y\,\mathrm{d}x \\
&\overset{\text{Fubini}}{=} \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} K(y-x)\Delta\varphi(x)f(y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int\limits_{\mathbb{R}^n} (K_*\Delta\varphi)(y)f(y)\,\mathrm{d}y \\
&= \int\limits_{\mathbb{R}^n} \varphi(y)f(y)\,\mathrm{d}y
\end{aligned}$$

□

Lemma 5: (Weyl's Lemma)
$u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ weak solution of $\Delta u = 0$
$\Rightarrow u \in C^\infty(\Omega)$ and u is harmonic.

Corollary: $\Omega \subset \mathbb{R}^n$ bounded open, $f : \Omega \to \mathbb{R}$ continuous and bounded. Equivalent are:

(i) $\exists$ weak solution $u \in C^2(\Omega)$ of $\Delta u = f$

(ii) Every weak solution of $\Delta u = f$ is in $C^2(\Omega)$

(iii) $K_* f \in C^2(\Omega)$

Proof:

(ii)$\Rightarrow$(i): obvious

(i)$\Rightarrow$(ii): Weyl's Lemma

(ii)$\Leftrightarrow$(iii): Example 3

$\square$

Example 4: $\Omega := \{(x,y) \in \mathbb{R}^2 \mid \sqrt{x^2+y^2} < \frac{1}{2}\}$ and

$$f(x,y) = \frac{xy}{x^2+y^2\sqrt{-\log(x^2+y^2)}}\left(\frac{1}{\log(x^2+y^2)} - 4\right)$$

$\overset{\text{Example}}{\Rightarrow} \exists$ weak solution of $\Delta = f$, $u \notin C^2$
$\Rightarrow K_* f \notin C^2(\Omega)$

Lemma 6: (APPROXIMATE DELTA-FUNCTIONS)
$\rho : \mathbb{R}^n \to [0,\infty)$ smooth. Assume $\rho(x) = 0$ for $|x| \geq 1$ and $\int\limits_{\mathbb{R}^n} \rho(x)\,\mathrm{d}x = 1$. Define $\rho_\varepsilon(x) := \frac{1}{\varepsilon^n}\phi(\frac{x}{\varepsilon})$ for $\varepsilon > 0$.

$$\Rightarrow \forall v \in L^p(\mathbb{R}^n) \quad \lim_{\varepsilon\to 0} \|\rho_{\varepsilon *} v - v\|_{L^p} = 0$$

Proof:

Lemma 6 $\Rightarrow$ Lemma 5: Denote $\Omega_\varepsilon := \{x \in \mathbb{R}^n \mid \overline{B_\varepsilon(x)} \subset \Omega\}$. Denote $u_\varepsilon := \rho_{\varepsilon *} u : \Omega_\varepsilon \to \mathbb{R}$ such that $u_\varepsilon(x) = \int\limits_{B_\varepsilon(x)} \rho_\varepsilon(x-y)u(y)\,\mathrm{d}y$

Step 1: u_ε smooth and $\Delta u_\varepsilon = 0$

Proof of Step 1:

$$\phi \in C_0'\infty(\Omega_\varepsilon) \quad \widehat{\rho}_\varepsilon(x) := \phi_\varepsilon(-x)$$

$$\begin{aligned}\int_{\Omega_\varepsilon} u_\varepsilon \Delta\varphi &= \int_{\Omega_\varepsilon} (\rho_{\epsilon *}\Delta\varphi) \\ &\overset{\text{Fubini}}{=} \int_\Omega u(\widehat{\rho}_\varepsilon * \Delta\varphi) \\ &= \int_\Omega u\Delta(\widehat{\rho}_\varepsilon * \varphi) = 0\end{aligned}$$

Step 2: $K \subset \Omega$ compact subset $\Rightarrow \lim_{\varepsilon\to 0} \|u_\varepsilon - u\|_{L^1(K)} = 0$

Proof of Step 2: Choose an open set $U \subset \mathbb{R}^n$ such that $K \subset U \subset \overline{U} \subset \Omega$. Define

$$v(x) := \begin{cases} u(x) & x \in U \\ 0 & x \notin U \end{cases}$$

$\Rightarrow v \in L^1(\mathbb{R}^n) \quad u_\varepsilon = \rho_{\varepsilon *} v$ in K provided that $\varepsilon > 0$ is sufficiently small
$\Rightarrow$ Step 2 follows from Lemma 6

Step 3: $u \in C^\infty(\Omega) \quad \Delta u = 0$

Proof of Step 3: By Step 1: $u_\varepsilon \in C^\infty \quad \Delta u_\varepsilon = 0$

$$\overset{\text{Lemma 3}}{\Rightarrow} u_\varepsilon(\xi) = \frac{n}{\omega_n r^n} \int_{B_r(\xi)} u_\varepsilon \text{ if } \overline{B_r(\xi)} \subset \Omega_\varepsilon$$

Define

$$v(\xi) := \frac{n}{\omega_n r^n} \int_{B_r(\xi)} u_\varepsilon \quad \xi \in \Omega_r$$

$$\Rightarrow |v(\xi) - u_\varepsilon(\xi)| \leq \frac{n}{\omega_n r^n} \int_{B_r(\xi)} |u_\varepsilon - u| \to 0 \text{ by Step 2}$$

$$\Rightarrow u = v \text{ cont's} \overset{\text{Lemma 4}}{\Rightarrow} u \text{ harmonic and } C^\infty$$

□

Lemma 7: (Young's inequality)
$f, g : \mathbb{R}^n \to \mathbb{R}$ continuous with compact support, $p \geq 1$

$$\Rightarrow \|f_* g\|_{L^p} \leq \|f\|_{L^p} \|g\|_{L^1}$$

Corollary: $f_* g$ well defined for $f \in L^p(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$ (continuous functions with compact support are dense in $L^p(\mathbb{R}^n)$).

Proof:

Lemma 7 ⇒ Lemma 6: Define $T_\varepsilon : L^p(\mathbb{R}^n) \circlearrowleft$ by $T_\varepsilon v := \rho_{\varepsilon *} v$.
Then $\|T_\varepsilon v\|_{L^p} \leq \|\rho_\varepsilon\|_{L^1} \|v\|_{L^p} = \|v\|_{L^p}$.
So $\|T_\varepsilon\| \leq 1$.

Claim: $\lim_{\varepsilon \to 0} T_\varepsilon v = v$ in $L^p(\mathbb{R}^n) \forall v \in L^p(\mathbb{R}^n)$.
By Banach-Steinhaus it suffices to prove that for continuous functions $v : \mathbb{R}^n \to \mathbb{R}$ with compact support

$$\begin{aligned}
\rho_{\varepsilon *} v(x) - v(x) &= \int_{\mathbb{R}^n} \rho_\varepsilon(x-y)(v(y) - v(x))\,\mathrm{d}y \\
&= \int_{y \in B_\varepsilon(x)} \rho_\varepsilon(x-y)(v(y) - v(x))\,\mathrm{d}y \\
&\leq \sup_{|y-x| \leq \varepsilon} |v(y) - v(x)| \underbrace{\int_{B_\varepsilon(x)} \rho_\varepsilon(x-y)\,\mathrm{d}y}_{=1} \\
&\leq \sup_{x,y \in \mathbb{R}^n} |v(y) - v(x)| \overset{\varepsilon \to 0}{\to} 0
\end{aligned}$$

so $\lim_{\varepsilon \to 0} \|\rho_{\varepsilon *} v - v\|_{L^\infty} = 0$ □

Lemma 8: (Hölder's inequality)

$$\|uv\|_{L^1} \leq \|u\|_{L^p} \|v\|_{L^q}$$

$u, v : \mathbb{R}^n \to \mathbb{R}$ continuous, compact support, $\frac{1}{p} + \frac{1}{q} = 1 \quad p, q \geq 1$.

Proof:

Lemma 8⇒ Lemma 7:

$$\begin{aligned}
\|f_*g\|_{L^p}^p &= \int_{\mathbb{R}^n} |f_*g(x)|^p \,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} |f_*g(x)|^{p-1} \int_{\mathbb{R}} f(x-y)g(y)\,\mathrm{d}y\,\mathrm{d}x \\
&\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \underbrace{|f_*g(x)|^{p-1}|f(x-y)|}_{(\Delta)}|g(y)|\,\mathrm{d}x\,\mathrm{d}y \\
&\overset{\text{Lemma 8}}{\leq} \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|f_*g(x)|^{(p-1)q}\,\mathrm{d}x\right)^{1/q}\left(\int_{\mathbb{R}^n}|f(x-y)|^p\,\mathrm{d}x\right)^{1/p}|g(y)|\,\mathrm{d}y \\
&\overset{(\Delta)}{=} \int_{\mathbb{R}^n}\|f_*g\|_{L^p}^{p-1}\quad\|f\|_{L^p}|g(y)|\,\mathrm{d}y \\
&= \|f_*g\|_{L^p}^{p-1}\|f\|_{L^p}\|g\|_{L^1}
\end{aligned}$$

$$(\Delta)\qquad \frac{1}{p}+\frac{1}{q}=1\quad \frac{1}{q}=1-\frac{1}{p}=\frac{p-1}{p}\quad p=q(p-1)$$

□

Lemma 9: $\Omega\mathbb{R}^n$ open, $u\in C^2(\Omega)$, $\Delta u=0$, $\overline{B_r(\xi)}\subset\Omega$

$$\triangledown u(\xi)=\frac{n}{\omega_n r}\int_{\partial B_1(0)} xu(\xi+rx)\,\mathrm{d}S(x)$$

Proof:

$$u(\xi)=\frac{u}{\omega_n r^n}\int_{B_r(\xi)}u(x)\,\mathrm{d}x=\frac{n}{\omega_n}\int_{B_1(0)}u(\xi+rx\,\mathrm{d}x$$

$$\begin{aligned}
\frac{\partial}{\partial\xi_i}u(\xi) &= \frac{\partial}{\partial\xi_i}\frac{n}{\omega n}\int_{B_1(0)}u(\xi+rx\,\mathrm{d}x \\
&= \frac{n}{\omega_n}\int_{B_1(0)}\underbrace{\frac{\partial}{\partial\xi_i}u(\xi+rx)}_{\frac{1}{r}\frac{\partial}{\partial x_i}u(\xi+rx)}\,\mathrm{d}x \\
&= \frac{n}{r\omega_n}\int_{B_1(0)}\frac{\partial}{\partial x_i}u(\xi+rx)\,\mathrm{d}x \\
&= \frac{n}{r\omega_n}\int_{\partial B_1(0)}\nu_i\cdot u(\xi+rx)\,\mathrm{d}S(x) \\
&\qquad \nu=x
\end{aligned}$$

□

Lemma 10: $u\in C^2(\Omega)$, $\Delta u=0$, bounded, $\Omega_\varepsilon=\{\xi\in\Omega\mid\overline{B_r(\xi)}\subset\Omega\}$

(i) $|\triangledown u(\xi)|\leq\frac{n}{\varepsilon}\sup_\Omega|u|\quad\forall\xi\in\Omega_\varepsilon$

(ii) $\xi_0, \xi_1 \in \Omega_\varepsilon$ such that

$$t\xi_1 + (1-t)\xi_0 \in \Omega_\varepsilon \quad \forall t \in [0,1]$$

$$|u(\xi_1) - u(\xi_0)| \leq \frac{n}{\varepsilon} \sup_\Omega |u| \cdot |\xi_1 - \xi_0$$

Proof:

(i) $\overline{B_\varepsilon(\xi)} \subset \Omega$ by Lemma 9,

$$\begin{aligned}
|\bigtriangledown u(\xi)| &\leq \frac{n}{\omega_n \varepsilon} | \int\limits_{\partial B_1(0)} x u(\xi + \varepsilon x \, \mathrm{d}S(x)| \\
&\leq \frac{n}{\varepsilon \omega_n} \int\limits_{\partial B_1(0)} |\underbrace{u(\xi + \varepsilon x}_{\leq \sup_\Omega u}| \, \mathrm{d}S(x) \\
&\leq \frac{n}{\varepsilon} \sup_\Omega |u|
\end{aligned}$$

(ii)

$$\begin{aligned}
u(\xi_1) - u(\xi_0) &= \int\limits_0^1 \frac{d}{dt} u(\xi_0 + t(\xi_1 - \xi_0)) \, \mathrm{d}t \\
&= \int\limits_0^1 \bigtriangledown u(\xi_0 + t(\xi_1 - \xi_0)) \cdot (\xi_1 - \xi_0) \, \mathrm{d}t \\
&\leq \int_0^1 |\bigtriangledown u(-)||\xi_1 - \xi_0| \\
&\leq \frac{n}{\varepsilon} \sup_\Omega |u| \cdot |\xi_1 - \xi_0|
\end{aligned}$$

□

Lemma 10 if $\Omega = \mathbb{R}^n$? $\Omega_\varepsilon = \mathbb{R}^n$.

Theorem: (Liouville's Theorem)
$u : \mathbb{R}^n \to \mathbb{R}$ harmonic, bounde. Then u is constant.

Proof: $M := \sup_{\mathbb{R}^n} |u| \quad \forall \varepsilon, r > 0, \overline{B_r(\xi)} \subset \mathbb{R}^n$, by Lemma 9,

$$|\bigtriangledown u(\xi)| \leq \frac{n}{r} \sup_{B_r(\xi)} |u| \leq \frac{M \cdot n}{r}$$

If $r \to \infty \Rightarrow |\bigtriangledown u(\xi)| = 0$. $\bigtriangledown u \equiv 0 \Rightarrow u$ constant. □

COMPACTNESS:

Theorem 1: $u_k \in C^2(\Omega)$ harmonic, $k \in \mathbb{N}$, $\sup_k \|u_k\|_{L^\infty} < \infty$.
$\exists u \in C^2$ harmonic, a subsequence u_{k_i} such that $u_{k_i} \to u$ uniformly on every $K \subset \Omega$ compact.

Proof: Define $M := \sup_k \|u_k\|_{L^\infty}$

Step 1: Fix $\overline{B_r(\xi)} \subset \Omega$. $\exists \varepsilon > 0$ such that $\overline{B_{r+\varepsilon}(\xi)} \subset \Omega$.
By Lemma 10, $\forall x, y \in \overline{B_r(\xi)}$, $(\Omega_\varepsilon = B_{r+\varepsilon}(\xi))$

$$|u_k(x) - u_k(y) \leq \frac{nM}{\varepsilon}|x - y|$$

$$\Rightarrow u_k\text{'s are equicontinuous on } \overline{B_r(\xi)}$$

$$\overset{\text{Arzela-Ascoli}}{\Rightarrow} \exists \{u_{k_i}\} \text{ such that } u_{k_i} \to u \text{ uniformly on } \overline{B_r(\xi)}$$

Step 2: $\Omega = \cup_{i \in \mathbb{N}} B_{r_i}(\xi_i) \quad \forall B_{r_i}(\xi)$. Apply 1 to obtain a subsequence, extract a diagonal sequence u_{k_j} such that

$$u_{k_j} \overset{\text{unif}}{\to} u \text{ on every } B_{r_i}(\xi_i)$$

$K \subset \Omega$ compact. Find $I \subset \mathbb{N}$, $\sharp I < \infty$ such that $K \subset \cup_{i \in I} B_{r_i}(\xi_i)$.
$u_{k_j} \to u$ uniformly on $B_{r_i} \quad i \in I$
$\Rightarrow u_{k_j} \to u$ uniformly on $\cup_{i \in I} B_{r_i}(\xi_i) \supset K$

Step 3: u is harmonic.
$\overline{B_r(\xi)} \subset \Omega$, $u_k \to u$ on $\overline{B_r \xi}$

$$u_k(\xi) = \frac{1}{r^{n-1}\omega_n} \int_{\partial B_r(\xi)} u_k(x) \, \mathrm{d}S(x)$$

$$u(\xi) = \frac{1}{r^{n-1}\omega_n} \int_{\partial B_r(\xi)} u(x) \, \mathrm{d}S(x)$$

by Lemma 4, u is harmonic. □

Lemma 11: $u \in C^\infty(\Omega)$ harmonic, $\overline{B_R(\xi)} \subset \Omega$.
$r < R$, $x \in B_r(\xi)$, $\alpha \in \mathbb{N}^n$, where $(\alpha = (\alpha_1, \ldots, \alpha_n), |\alpha| = \alpha_1 + \ldots + \alpha_n)$

$$|\partial^\alpha u(x)| \leq \frac{n^{|\alpha|} e^{|\alpha|-1} |\alpha|!}{(R-r)^{|\alpha|}} \sup_{B_R(\xi)} |u|$$

Proof: Induction on $|\alpha|$.
If $|\alpha| = 1$:

$$|\frac{\partial}{\partial x_i} u(x)| \leq \frac{n}{(R-r)} \sup_{B_R(\xi)} |u|$$

by Lemma 10, where $\Omega = B_R(\xi)$ and $B_r = \Omega_\varepsilon$.
Assume $\forall |\beta| < |\alpha|$ we proved the Lemma. Take β such that $\alpha = \beta + e_i =$

$(\beta_1, \ldots, \beta_i + 1, \ldots, \beta_n)$. $|\beta| + 1 = |\alpha|$

$$\begin{aligned}
\partial^\alpha u(x) &= \partial^\beta \frac{\partial}{\partial x^j} u(x) \overset{\text{fix } \rho, \rho + r < R}{=} \partial_x^\beta \frac{n}{\rho \omega_n} \int\limits_{\partial B_1(0)} y_j u(x + \rho y \, \mathrm{d}S(y) \\
&= \frac{n}{\rho \omega_n} \int\limits_{\partial B_1(0)} y_j \partial_x^\beta u(x + \rho y \, \mathrm{d}S(y) \\
&\leq \frac{n}{\rho} \sup_{B_{r+\rho}(\xi) \supset B_\rho(x)} |\partial^\beta u| \\
&\leq \frac{n}{\rho} \frac{n^{|\alpha|-1} e^{|\alpha|-2} (|\alpha| - 1)!}{(R - (r + \rho))^{|\alpha|-1)}} \sup_{B_{r+\rho}(\xi)} |u| \\
&\leq \frac{n^{|\alpha|} e^{|\alpha|-1} |\alpha|!}{(R - r)^{|\alpha|}} \sup_{B_R(\xi)} |u| \\
&\Leftrightarrow \frac{(R - r)^{|\alpha|}}{\rho (R - r - \rho)^{|\alpha|-1}} \\
&\overset{(*)}{\leq} e|\alpha| \quad \text{for some } \rho
\end{aligned}$$

Let $k := |\alpha| - 1$, $s = \frac{\rho}{R-r} \overset{(*)}{\Leftrightarrow} \frac{1}{\rho(1-\rho)^k} \leq e(k+1)$.
Choose $s = \frac{1}{k+1} \Leftrightarrow (1 - \frac{1}{k+1})^{-k} = (\frac{k+1}{k}^k = (1 + \frac{1}{k})^k \leq e$. □

Theorem 2: Every harmonic function is real analytic.

$$\forall \varepsilon \in \Omega \exists \rho > 0 \text{ such that } \forall |x| < \rho$$
$$u(\xi + x) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial^\alpha u(\xi) x^\alpha$$
$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \quad \alpha! = \alpha_1! \ldots \alpha_n!$$

Proof:

$$\overline{B_R(\xi)} \subset \Omega \quad r := \frac{R}{x} \quad |x| < \frac{R}{2}$$
$$u \in C^\infty \quad u(\xi + x) = \sum_{|\alpha| \leq R-1} \frac{1}{\alpha!} \partial^\alpha u(\xi) x^\alpha + R_k(\xi, x)$$
$$R_k(\xi, x) = \int\limits_0^1 k(1 - t)^{k-1} \sum_{|\alpha| = k} \frac{1}{\alpha!} \partial^\alpha u(\xi + tx) x^\alpha \, \mathrm{d}t$$

We want $R_k(\xi, x) \overset{k \to \infty}{\to} 0$ uniformly for $x \in B_\rho(0)$.
By Lemma 11:

$$|\partial^\alpha u(\xi + tx)| \leq \frac{n^k e^{k-1} k!}{(R/2)^k} \sup_{B_R(\xi)} |u| \quad k = |\alpha|$$

$$\int\limits_0^1 k(1 - t)^{k-1} \, \mathrm{d}t = 1 \quad |x^\alpha \leq |x|^k$$

$$\begin{aligned}
|R_k(\xi, x) &\leq n^k e^{k-1} (\frac{|x|}{R/2})^k \sum_{|\alpha| = k} \frac{k!}{\alpha!} \sup_{B_R(\xi)} |u| \\
&\leq (\underbrace{\frac{n^2 e |x|}{R/2}}_{<1})^k \sup_{B_R(\xi)} |u| \overset{k \to \infty}{\to} 0
\end{aligned}$$

if $|x| \le s_0 < \frac{R}{2n^2\rho}$ □

Lemma 12: (Maximum principle)
Ω bounded, compact, open. $u \in C^0(\overline{\Omega})$ subharmonic. Then

(i) $u(x) \le \max_{\partial\Omega} u \quad \forall x \in \Omega$

(ii) If u is non-constant, then $u(x) < \max_{\partial\Omega} u \quad \forall x \in \Omega$

Proof: $\Omega_0 := \{x \in \Omega \mid u(x) = \max_\Omega u\}$
$\Omega_1 := \{x \in \Omega \mid u(x) < \max_{\overline{\Omega}} u\}$
$\Omega = \Omega_0 \cup \Omega_1$. $\Omega 0 \cap \Omega_1 = \varnothing$, Ω_0 closed.
Assume $\xi \in \Omega_0$, $\exists r$ such that $\overline{B_r(\xi)} \subset \Omega$.

$$\max_{\overline{\Omega}} u = u(\xi) \le \frac{n}{\omega_n r^n} \int_{B_r(\xi)} u \le \max_{B_r(\xi)} u \le \max_{\overline{\Omega}} u$$

$$\frac{n}{\omega_n r^n} \int_{B_r(\xi)} \underbrace{\max u - u}_{\ge 0} = 0$$

$$u(x) = \max_{\overline{\Omega}} u \quad \forall x \in B_r(\xi)$$

$$\Rightarrow B_r(\xi) \subset \Omega_0 \Rightarrow \Omega_0 \text{ is open}$$

Either $\Omega_0 = \varnothing$ or $\Omega_0 = \Omega$.

(i) If $\exists x \in \Omega$ such that $u(x) > \max_{\partial(\Omega)} u$

$$\max_{\overline{\Omega}} u > \max_{\partial\Omega} u$$

$\Rightarrow \exists \in \Omega$ such that $u(\xi) = \max_{\overline{\Omega}} u$
$\Rightarrow \Omega_0 \ne \varnothing \Rightarrow \Omega_0 = 0 \Rightarrow u \equiv \max_{\overline{\Omega}} u$. Contradiction!

(ii) If $\exists x$ such that $u(x) = \max u$ with the same procedure $u \equiv \max_{\partial\overline{\Omega}} u$

□

Lemma 13: u subharmonic, Ω bounded

(i) $u(x) \ge \inf_{\partial\Omega} u \quad \forall x \in \Omega$

(ii) If u non-constant, $u(x) < \inf_{\partial\Omega} u$

Proof: Apply Lemma 12 to $-u$. □

Corollary: $u_k \in C^2(\Omega) \cap C^0(\overline{\Omega})$. Ω bounded open

$$u_k|_{\partial\Omega} \to g \text{ uniformly}$$

Theorem 3: $u_k \to u$ uniformly, u harmonic $\Rightarrow u|_{\partial\Omega}$

Proof: $u_k, -u_k$ subharmonic.

$$\sup_{\overline{\Omega}} |u_k - u_l| \overset{(*)}{\le} \sup_{\partial\Omega} |u_k - u_l| \overset{k,l\to\infty}{\to} 0$$

$$\Rightarrow u_k \to u \text{ unif} \quad u \in C^0(\overline{\Omega}) \quad u|_{\partial\Omega} = g$$

Lemma 4 $\Rightarrow u$ harmonic.

(*) $$\sup_{\Omega} \pm(u_k - u_l) \le \max_{\partial\Omega} \pm(u_k - u_l)$$

□

Theorem 4: $\Omega \subset \mathbb{R}^n$ open, bounded, $f \in C^0(\Omega)$, $g \in C^0(\partial\Omega)$.
The problem

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

has a unique solution!

Proof: Assume u, v are both solutions.

$$\begin{cases} \Delta(u-v) = \Delta u - \Delta v = f - f = 0 & \text{in } \Omega \\ u - v = g - g = 0 & \text{on } \partial\Omega \end{cases}$$

max principle

$$(u-v)(x) \leq \max_{\partial\Omega}(u-v) = 0 \quad \forall x \in \Omega$$

$$(v-u)(x) \leq \max_{\partial\Omega}(v-u) = 0 \quad \forall x \in \Omega$$

$$\Rightarrow u - v \equiv 0$$

□

To come: Poisson's formula for the ball $B_1(0)$ or $B_1(\xi)$
Poisson's method.
$\Omega \subset \mathbb{R}^n$ open bounded, $f : \partial\Omega \to \mathbb{R}$ continuous.
Dirichlet Problem: Find $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$, $\Delta u = 0$ in Ω with $u|_{\partial\Omega} = f$.

Proof: Uniqueness: ok
Existence:
Assume first: $\partial\Omega \subset \mathbb{R}^n$ smooth submanifold
By Lemma 1: if $u \in C^2(\overline{\Omega})$ is harmonic in Ω then

(*)
$$u(\xi) = \int_{\partial\Omega} (u \frac{\partial K_\xi}{\partial n} - K_\xi \frac{\partial u}{\partial n}) \, \mathrm{d}S \quad \xi \in \Omega$$

if $v \in C^2(\overline{\Omega})$ is harmonic

(*)
$$v = \int_{\partial\Omega} (u \frac{\partial u}{\partial n} - v \frac{\partial u}{\partial n}) \, \mathrm{d}S$$

if in addition $v_\xi(x) = K_\xi(x) \quad \forall x \in \partial\Omega$ then

$$(*) \Rightarrow u(\xi) = \int_{\partial\Omega} (u \frac{\partial(K_\xi - v_\xi)}{\partial n}) \, \mathrm{d}S$$

□

Definition: Let $K : \mathbb{R}^n \subset 0 \to \mathbb{R}$ be the fundamental solution of the Laplace equation and, for $\xi \in \Omega$, let $v_\xi : \overline{\Omega} \to \mathbb{R}$ be a harmonic function such that $v_\xi(x) = K_\xi(x) \forall x \in \partial\Omega$, where $K_\xi(x) = K(x - \xi)$.
Then $G_\xi := K_\xi - v_\xi$ is called a *Green's function* for Ω at ξ.

Remark: If $G_\xi : \Omega \subset \{\xi\} \to \mathbb{R}$ is a Green's function and $u \in C^2(\overline{\Omega})$ is harmonic then $u(\xi) = \int_{\partial\Omega} (u \frac{\partial G_\xi}{\partial n}) \, \mathrm{d}S$.
How do we find a formula for the Green's function?

Special Case: $\Omega = B_r = \{x \in \mathbb{R}^n \mid |x| < r\}$

Key Observation:

1. $K_{\xi^*}|_{\overline{\Omega}}$ is harmonic if $\xi^* \notin \overline{\Omega}$
2. Can we find a point $\xi^* \notin \overline{B_r}$ such that $K_{\xi^*}(x) = \text{const} \cdot K_\xi(x) \forall x \in \partial B_r$ (what is K_ξ?).
   Must find a point $\xi^* \in \mathbb{R}^n \setminus \overline{B_r}$ such that

$$\frac{K_{\xi^*}(x)}{K_\xi(x)} \equiv \underbrace{\text{const}}_{=(\frac{r}{|\xi|})^{2-n}} \quad \text{for } |x| = r$$

$$\frac{K_{\xi^*}(x)}{K_\xi(x)} = \left(\frac{|\xi^* - x|}{|\xi - x}\right)^{2-n} \quad \text{if } n > 2$$

$$\text{ie } \frac{|\xi^* - x}{\xi - x} \equiv \text{ const on } \partial B_r$$

Answer: $\xi^* = \frac{r_\xi^2}{|\xi|^2} \quad |\xi^*| = \frac{r^2}{|\xi|} > r$

$$\begin{aligned}
\frac{|\xi^* - x|}{|\xi - x|} &= \frac{|\xi^*|^2 - 2\langle \xi^*, x\rangle - |x|^2}{|\xi^*|^2 - 2\langle \xi, x\rangle + |x|^2} \\
&= \frac{\frac{r^4}{|\xi|^2} - 2\frac{r^2}{|\xi|^2\langle\xi,x\rangle + r^2}}{r^2 - 2\langle \xi, x\rangle + |\xi|^2} \\
&= \frac{r^2}{|\xi|^2} \quad (*)
\end{aligned}$$

The solution $v_\xi \in C^2(\overline{\Omega})$ of $\Delta v_\xi = 0$ in $\Omega (v_\xi - K_\xi)|_{\partial\Omega} = 0$ is

$$\begin{aligned}
v_\xi(x) &= \left(\frac{|\xi|}{r}\right)^{2-n} K_{\xi^*}(x) \\
&= \frac{1}{\omega_n(2-n)} \left(\frac{|\xi|}{r}\right)^{2-n} |\xi^* - x|^{2-n} \\
&= \frac{1}{\omega_n(2-n)} |\frac{r\xi}{|\xi|} - \frac{|\xi|x}{r}|^{2-n}
\end{aligned}$$

If $n > 2$ then the Green function for B_r at ξ is given by

$$G_\xi(x) = \frac{1}{\omega_n(2-n)} \left(|\xi - x|^{2-n} - |\frac{r\xi}{|\xi|} - \frac{|\xi|x}{r}|^{2-n}\right)$$

Exercise: For $n = 2$ we get

$$G_\xi(x) = \frac{1}{2\pi}\left(\log(|\xi - x|) - \log(|\frac{r\xi}{|\xi|} - \frac{|\xi|x}{r}|)\right)$$

Compute $\frac{\partial G_\xi}{\partial n}$ on ∂B_r.

$$\triangledown G_\xi(x) = \frac{x - \xi}{\omega|x - \xi|^n} - \frac{\frac{|\xi|x}{r} - \frac{r\xi}{|\xi|}}{\omega_n|\frac{|\xi|x}{r} - \frac{r\xi}{|\xi|}|^n} \cdot \frac{|\xi|}{r}$$

If $x \in \partial B_r$ then

$$|x - \xi| = |x - \xi^*|\frac{|\xi|}{r} = |\frac{|\xi|x}{r} - \frac{r\xi}{|\xi|}$$

$\Rightarrow$ for $x \in \partial B_r$ we have

$$\triangledown G_\xi(x) = \frac{1 - \frac{|\xi|^2}{r^2}}{\omega_n|x - \xi|^n} \quad \nu(x) = \frac{x}{r}$$

Poisson Kernel

$$\frac{\partial G_\xi}{\partial n}(x) = \frac{r^2 - |\xi|^2}{r\omega_n|x - \xi|^n} =: P(\xi, x)$$

Remark: If $u \in C^2(\overline{B_r})$ is a harmonic function then

$$u(\xi) = \int\limits_{\partial B_r} P(\xi, x) u(x) \,\mathrm{d}S(x) = \frac{1}{r\omega_n} \int\limits_{\partial B_r} \frac{r^2 - |\xi|^2}{|\xi - x|^n} \,\mathrm{d}S(x) \quad \forall \xi \in B_r$$

Theorem 5: Let $f : \partial B_r \to \mathbb{R}$ be continuous. Define $u : \overline{B_r} \to \mathbb{R}$ by

$$u(\xi) := \frac{1}{r\omega_n} \int\limits_{\partial B_r} \frac{r^2 - |\xi|^2}{|\xi - x|^n} \,\mathrm{d}S(x)$$

for $\xi \in B_r$ and $u(\xi) := f(\xi)$ for $\xi \in \partial B_r$.

Claim: Then $u \in C^0(\overline{B_r}) \cap C^2(B_r)$ and $\Delta u = 0$ in B_r.

Proof:

$$P(\xi, x) := \frac{1}{r\omega_n} \frac{r^2 - |\xi|^2}{|\xi - x|^n} \text{ for } |x| = r \text{ and } |\xi| < r$$

1. The map $(\xi, x) \mapsto P(\xi, x)$ is C^∞ for $|x| \leq r \quad |\xi| < r| \quad x \neq \xi$
2. For $|x| = 1$ the function $B_r \to \mathbb{R} : \xi \mapsto P(\xi, x)$ is harmonic.

   $$G_\xi(x) \overset{|\xi|<|x|=r}{=} \frac{1}{\omega_n(2-n)} \left(|\xi - x|^{2-n} - |\frac{|x|\xi}{|\xi|} - \frac{|\xi|x}{|x|}|^{2-n} \right) = G_x(\xi)$$

   $x \mapsto G_\xi(x)$ is harmonic in $|x| \neq |\xi|$ so $x \mapsto G_x(\xi)$ is harmonic in $\{\xi \mid |\xi| \neq |x|\}$.
   So $P(\xi, x) = \langle \nabla_x G(\xi, x), \nu(x) \rangle$. Hence u is harmonic in B_r.
3. $P(\xi, x) > 0$ for $|xi| < r$ and $|x| = r$
4. $\int\limits_{\partial B_r} P(\xi, x) \,\mathrm{d}S(x) = 1 \forall \xi \in B_r$ (by the Remark with $u(x) \equiv 1$)
5. $\lim_{\xi \to x_0 |\xi| < r} P(\xi, x) = 0$ for $x \neq x_0$
   Fix $x_0 \in \partial B_r$. We must prove that u is continuous at x_0

   $$u(\xi) - u(x_0) \overset{4.}{=} \int\limits_{\partial B_r} P(\xi, x)(f(x) - f(x_0)) \,\mathrm{d}S(x)$$

   $$= \underbrace{\int_{|x-x_0|<\delta |x|=r}}_{I_1} + \underbrace{\int_{|x-x_0|>\delta |x|=r}}_{I_2}$$

   Given $\varepsilon > 0$, choose $\delta > 0$ such that

   $$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

   Then

   $$|I_1| \leq \int\limits_{\partial B_r \cap \{|x - x_0| < \delta\}} P(\xi, x) |f(x) - f(x_0)| \,\mathrm{d}S(x) \leq \varepsilon$$

   Now choose $\delta' > 0$ such that

   $$|\xi - x_0| < \delta' \quad |x - x_0| > \delta \Rightarrow P(\xi, x) < \varepsilon$$

   $$\Rightarrow P(\xi, x) = \frac{1}{r\omega_n} \frac{r^2 - |\xi|^2}{|x - \xi|^n} \leq \frac{1}{r\omega_n} \frac{r^2 - |\xi|^2}{(\delta - \delta')^n}$$

Then

$$
\begin{aligned}
|I_2| &\le \underbrace{r^{n-1}\omega_n}_{\mathrm{Vol}^{n-1}(\partial B_r)} \sup_{|x-x_0|>\delta} P(\xi,x)|f(x)-f(x_0)| \\
&\le r^{n-1}\omega_n \|f\|_{L^\infty} \underbrace{\sup_{|x-x_0|>\delta} P(\xi,x)}_{<\varepsilon \text{ for } |\xi-x_0|<\delta'} \\
&\le 2r^{n-1}\omega_n\|f\|_{L^\infty}\varepsilon \text{ for } |\xi-x_0|<\delta'
\end{aligned}
$$

So $|u(\xi)-u(x_0)| < (1+2r^{n-1}\omega_n\|f\|_{L^\infty})\varepsilon$ for $|\delta - x_0| < \delta'$ (ie continuous).

□

Harnack inequality: $u \in C^2(\Omega)$ harmonic, $\overline{B_R(\xi)} \subset \Omega, r < R, u \ge 0$
$\Rightarrow$ For $|x-\xi| < r$:

$$\left(1-\frac{r^2}{R^2}\right)\left(\frac{R}{R+r}\right)^n u(\xi) \le u(x) \le \left(\frac{R}{R-r}\right)^n u(\xi)$$

Proof: Exercise ! :-) □

Theorem: Harnack's Monotone Convergence Theorem:
$u_k \in C^2(\Omega)$ harmonic bounded below on every compact subset of Ω. $u_k(x)$ nonincreasing for every $x \in \Omega$

$$u(x) := \lim_{k\to\infty} u_k(x) \quad x \in \Omega$$

$\Rightarrow$ u is harmonic and $\underline{u_k \text{ converges to } u \text{ uniformly, on every compact subset of } \Omega}$.

Proof: $u_k - u_{k+l} \ge 0$ in Ω.

$$\Rightarrow |u_k(x) - u_{k+l}(x)| \le (\frac{R}{R-r})^n |(u_k(\xi) - u_{k+l}(\xi)|$$

for $x \in \overline{B_r(\xi)}$ if $r < R$ and $\overline{B_R(\xi)} \subset \Omega$.
Hence for every $\xi \in \Omega \exists r > 0$ such that u_k converges uniformly in $\overline{B_r(\xi)}$. If $K \subset \Omega$ is compact, cover K by finitely many such balls to get uniform convergence on K. Use Lemma ... □

$\Omega \subset \mathbb{R}^n$ open, bounded

Dirichlet problem: Given $f \in C^0(\partial\Omega)$, find $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ satisfying

$$\Delta u = 0 \text{ in } \Omega$$

$$u = f \text{ on } \partial\omega$$

Perrons' method:

Recall:

1. $u \in C^0(\Omega)$ is called subharmonic if

$$u(\xi) = \frac{1}{r^{n-1}\omega_n} \int_{\partial B_r(\xi)} u \, \mathrm{d}S = M_u(\xi, r)$$

whenever $\overline{B_r(\xi)} \subset \Omega$.

2. $u \in C^2(\Omega)$ subharmonic $\Leftrightarrow \Delta u \geq 0$.
3. $u \in C^0(\overline{\Omega})$ subharmonic, $u \neq$ constant

$$\Rightarrow u(x) < \sup_{\partial\Omega} u \quad \forall x \in \Omega$$

4. u, v subharmonic $\Rightarrow u + v$ is subharmonic.

Let $u \in C^0(\overline{\Omega}) \quad B = B_r(\xi) \subset \Omega$.
Define

$$u_B(x) := \begin{cases} u(x) & x \in \overline{\Omega} \\ \frac{1}{r\omega_n} \int\limits_{\partial B_r(\xi)} \frac{r^2 - |x-\xi|^2}{|y-x|^n} u(y) \, \mathrm{d}S(y) \end{cases}$$

$u_B|_B$ is the harmonic extension of $u|_{\partial B}$ (Theorem 5).
This defines a bounded linear operator

$$H_B : C^0(\overline{\Omega}) \to C^0(\overline{\Omega}) \quad H_B u := u_B$$

- $(u + v)_B = u_B + v_B$
- $(\lambda u)_B = \lambda u_B$
- $\sup_{\overline{\Omega}} |u_B| \leq \sup_{\overline{\Omega}} |u|$

Lemma 1: Let $u \in C^0(\overline{\Omega})$. Then

$$u \text{ subharmonic} \Leftrightarrow u \leq u_B \quad \forall \text{ ball } B \subset \Omega$$

Proof:

$\Leftarrow$: $B = B_r(\xi) \quad \overline{B} \subset \Omega$

$$\Rightarrow u(\xi) \leq u_B(\xi) \overset{(*)}{=} M_u(\xi, r)$$

(*): because u_B is harmonic in B and $u_B|_{\partial B} = u|_{\partial B}$

$\Rightarrow$: u subharmonic and $B = B_r(\xi) \subset \Omega$
$\Rightarrow u - u_B$ subharmonic in $B \quad (u - u_B)|_{\partial B} \equiv 0$
$\overset{\text{Max-Principle}}{\Rightarrow} u - u_B \leq 0$ in B.

□

Example: $= 1$

harmonic $\Leftrightarrow$ linear and constant

subharmonic $\Leftrightarrow$ convex

Lemma 2: $u \leq v \Rightarrow u_B \leq v_B$

Proof: Maximum Principle:
$u_B - v_B$ harmonic in B
$u_B - v_B = u - v \leq 0$ on ∂B
$\Rightarrow u_B - v_B \leq 0$ ion B. □

Lemma 3: u subharmonic and $B \subset \Omega$ bounded $\Rightarrow u_B$ is subharmonic

Proof: Use Lemma 1:
Denot $v := u_B$. Let $\Lambda \subset \Omega$ be a ball.
To show: $v(x) \leq v_\Lambda(x) \forall x \in \Omega$

Case 1: $x \in \Omega \setminus \Lambda : v(x) = v_\Lambda(x) \quad \surd$

Case 2: $x \in \Lambda \setminus B : v(x) = u(x) \overset{L1}{\leq} u_\Lambda(x) \overset{L2}{\leq} v_\Lambda(x)$

Case 3: $x \in \Lambda \cap B : v - v_\Lambda$ harmonic in $\Lambda \cap B$
$v - v_\Lambda \leq 0$ on $\partial(\Lambda \cap B)$ (by cases 1 and 2)
$\overset{\text{Max Principle}}{\Rightarrow} v - v_\Lambda \leq 0$ in $\Lambda \cap B$.

□

Lemma 4: u, v subharmonic $\Rightarrow \max\{u, v\}$ is subharmonic

Proof: $w(x) := \max\{u(x), v(x)\} \Rightarrow w$ continuous, $u \leq w$, $v \leq w$
$\overset{\text{Lemma 1 and 2}}{\Rightarrow}$ for every ball $B \subset \Omega$
$u \leq u_B \leq w_B$ and $v \leq v_B \leq w_B$
$\Rightarrow w = \max\{u, v\} \leq w_B$
$\overset{\text{Lemma 1}}{\Rightarrow} w$ is subharmonic □

Fix a continuous function $f : \partial\Omega \to \mathbb{R}$. Denote

$$\mathcal{S}_f := \{u \in C^0(\overline{\Omega}) \mid u \text{ subharmonic, } u(x) \leq f(x) \forall x \in \partial\Omega\}$$

$\mathcal{S}$ like subharmonic.
Define $u_f : \overline{\Omega} \to \mathbb{R}$ by $u_f(x) := \sup_{u \in \mathcal{S}} u(x)$.

Proposition 1: u_f is harmonic in Ω and

(*)
$$\inf_{\partial\Omega} f \leq u_f(x) \leq \sup_{\partial\Omega} f \quad \forall x \in \overline{\Omega}$$

Proof: Basic facts:

(i) $\mathcal{S}_f \neq \varnothing \quad u_0(x) := \inf_{\partial\Omega} f \quad u_0 \in \mathcal{S}_f$

(ii) $u \in \mathcal{S}_f \Rightarrow u(x) \leq \sup_{\partial\Omega} f \quad \forall x \in \overline{\Omega}$

(iii) $u, v \in \mathcal{S}_f \quad B \subset \Omega$ ball $\Rightarrow \max\{u, v\}, u_B \in \mathcal{S}_f$

We prove that u_f is harmonic:
Fix an element $\xi \in \Omega$ and $r > 0$ such that $\overline{B_r(\xi)} \subset \Omega$. Denote $B := B_r(\xi)$.
Choose $u_1, u_2, u_3, \ldots, \in \mathcal{S}_f$ such that $u_f(\xi) = \lim_{k\to\infty} u_k(\xi)$.
w.l.o.g assume:

$$u_{k+1}(\xi) \geq u_k(\xi) \quad \forall \xi$$

Define $v_k := \max\{u_1, \ldots, u_k\}$. Then

$$v_k \in \mathcal{S}_f \quad v_{k+1} \geq v_k \quad u_f(\xi) = \lim_{k\to\infty} v_k(\xi)$$

Define $w_k := (v_k)_B$. Then

$$w_k \in \mathcal{S}_f \quad w_{k+1} \geq w_k \geq v_k \quad u_f(\xi) = \lim_{k\to\infty} w_k(\xi)$$

$$w_k \text{ is harmonic in } B$$

Because $w_k(x) \leq \sup_{\partial\Omega} f \forall k \forall x \in \Omega$ the limit $w(x) := \lim_{k\to\infty} w_k(x)$ exists for every $x \in \overline{\Omega}$.
By Hamack's monotone convergence theorem w is harmonic in B.

Claim: $u_f = w$ in B. By definition of u_f we have $w(x) \le u_f(x) \forall x \in B$.
Assume, by contradiction, that $\exists x^* \in B$ such that $w(x^*) < u_f(x^*)$

$$\Rightarrow \exists v \in \mathcal{S}_f : w(x^*) < v(x^*) \le u_f(x^*)$$

$$S := |x^* - \xi| > 0 \text{ because } w(\xi) = u_f(\xi) \quad B^* := B_\rho(\xi)$$

Let

$$w_K^* := \max\{w : Jmv\}_{B^*} \quad w^* := \max\{w, v\}_{B^*}$$

$$\Rightarrow w(x) \le \max\{w(x), v(x)\} = w^*(x) \quad \forall x \in \partial B^*$$

$$w(x^*) < v(x^*) = w^*(x^*) \overset{\text{Max Principle}}{\Rightarrow} w(x) < w^*(x) \forall x \in B^*$$

(because $w - w^*$ is harmonic in B^*, non-constant, and ≤ 0 on ∂B^*).

However:

$$w(\xi) 0 u_f(\xi) \overset{!}{\ge} w^*(\xi)$$

because

a) $w_k^* \in \mathcal{S}_f \quad w_k^* \le w_{k+1}^* \le u_f$

b) w_k^* converges to w^* uniformly on ∂B^*.
$\overset{\text{MP}}{\Rightarrow} w_k^*$ converges to w^* on B^*
$\Rightarrow w^* \le u_f$ in B^*
$\Rightarrow w^*(\xi) \le u_f(\xi)$

□

$$u \textit{ superharmonic} \overset{\text{DEF}}{\Leftrightarrow} -u \text{ subharmonic}$$

$$\mathcal{S}_f^* := \{u \in C^0(\overline{\omega}) \mid u \text{ superharmonic } u(x) \ge f(x) \forall x \in \partial\Omega\}$$

$$u_f^*(x) := \inf_{u \in \mathcal{S}_f^*} u(x)$$

Proposition 2:

(i) u_f^* is harmonic in Ω

(ii) $\inf_{\partial\Omega} f \le u_f^*(x) \le \sup_{\partial\Omega} f \quad \forall x \in \overline{\Omega}$

(iii) $u_f \le u_f^*$

Proof:

(i) and (ii) follows from Proposition 1 with f replaced by $-f$.

(iii) $u \in \mathcal{S}_f \quad u^* \in \mathcal{S}_f^*$
$\Rightarrow u - u^*$ is subharmonic and $u(x) - u^*(x) \le 0 \forall x \in \partial\Omega$
$\overset{\text{MP}}{\Rightarrow} u \le u^*$ in $\Omega \Rightarrow$ (iii).

□

Definition: $\Omega \subset \mathbb{R}^n$ open, $\xi \in \partial\Omega$
A *barrier function* for Ω at ξ is a subharmonic function $g \in C^0(\overline{\Omega})$ such that

$$g(\xi) = 0 \quad g(x) < 0 \quad \forall x \in \overline{\Omega} \setminus \{\xi\}$$

Proposition: $\xi \in \partial\Omega$. Assume that Ω admits a barrier function at ξ.
$\Rightarrow u_f, u_f^*$ continuous at ξ and $u_f(\xi) = f(\xi) = u_f^*(\xi)$.

Proof: Let $\varepsilon > 0$ and $u_\varepsilon(x) := f(\xi) - \varepsilon + cg(x)$.
Then if $c > 0$ is sufficiently large we have $u_\varepsilon(x) \leq f(x) \quad \forall x \in \partial\Omega$ and so $u_\varepsilon \in \mathcal{S}_f$

$$u^*_\varepsilon(x) := f(\xi) + \varepsilon - cg(x)$$

Likewise $u^*_\varepsilon \in \mathcal{S}^*_f$

$$\Rightarrow u_\varepsilon(x) \leq u_f(x) \leq u^*_f(x) \leq u^*_\varepsilon(x) \forall x \in \Omega$$

Given $\varepsilon > 0$ choose $\delta > 0$ such that

$$|x - \xi| < \delta \Rightarrow |g(x)| \leq \frac{\varepsilon}{c}$$

$$\Rightarrow f(\xi) - 2\varepsilon \leq u_f(x) \leq u^*_f(x) \leq f(\xi) + 2\varepsilon$$

for all x with $|x - \xi| \leq \delta$ □

Theorem 6: $\Omega \subset \mathbb{R}^n$ open and bounded. Equivalent are:

(i) $\forall f \in C^0(\partial\Omega) \exists$ solution $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ of the Dirichlet problem.

(ii) $\forall \xi \in \partial\Omega \exists$ a barrier function.

Proof:

(ii)⇒(i): Proposition 1-3.

(i)⇒(ii): Choose $f(x) := -|x - \xi| \quad x \in \partial\Omega$ and let $g \in C^0(\overline{\Omega})$ be the solution of Dirichlet Problem.

□

$\Delta u = 0$ in Ω
$u = f$ on $\partial\Omega$
Dirichlet integral is $\Phi(u) = \int_\Omega |\bigtriangledown u|^2$

Proposition 4: Suppose $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, $\partial\Omega$ is a C^1-submanifold of $\mathbb{R}^n$. $u|_{\partial\Omega} = f$. Equivalent are:

(i) u is harmonic in Ω

(ii) $\Phi(u) \leq \Phi(u + v) \quad \forall v \in C^1(\overline{\Omega})$ with $v|_{\partial\Omega} = 0$.

Proof:

$$\begin{aligned}
\Phi(u + v) &= \int_\Omega |\bigtriangledown u + \bigtriangledown v|^2 \\
&= \int_\Omega (|\bigtriangledown u|^2 + 2\langle \bigtriangledown u, \bigtriangledown v\rangle + |\bigtriangledown v|^2) \\
&= \Phi(u) + \Phi(v) - 2\int_\Omega (\Delta u) v + \underbrace{2\int_\Omega \frac{\partial u}{\partial n} \cdot v}_{=0} \\
&= \Phi(u) + \Phi(v) - 2\int_\Omega (\Delta u) \cdot v
\end{aligned}$$

(i)⇒(ii): $\Delta u = 0$ in Ω
$\Rightarrow \Phi(u + v) = \Phi(u) + \Phi(v) \geq \Phi(u)$
if " $=$ " then $v \equiv 0$

(ii)⇒(i): if $v \in C^1(\overline{\Omega})$ and $v|_{\partial\Omega} = 0$ then

$$\Phi(u) \leq \Phi(u + tv) \quad \forall t \in \mathbb{R}$$

$$= \Phi(u) - 2t \int_\Omega (\Delta u)v + t^2\Phi(v)$$

$$\Rightarrow 0 = \frac{d}{dt}|_{t=0}\Phi(u + tv)$$

$$= -2 \int_\Omega (\Delta u)v \quad \forall v \in C^1(\overline{\Omega}) \quad v|_{\partial\Omega} = 0$$

$$\Delta u = 0$$

□

Idea for a method to solve the Dirichlet-Problem: Minimize the functional Φ over the space $\mathcal{X} := \{u \in C^1(\overline{\Omega}) \mid u|_{\partial\Omega} = f\}$
Warning: We must choos f such that $\mathcal{X} \neq \emptyset$.
$f : \partial\Omega \to \mathbb{R}$ must be continuously differentiable.
Choose $u_i \in \mathcal{X}$ such that $\Phi(u_i) \to \inf_{u\in\mathcal{X}} \Phi(u)$.

Questions are:

- jDoes u_i converge in $\mathcal{X}$ (or in other words: is there a minimizer $u \in \mathcal{X}$ such that $\Phi(u) \leq \Phi(v) \quad \forall v \in \mathcal{X}$)?
- If u exists, is it true that $u \in C^2(\Omega)$?

2 Sobolev Spaces:

$\Omega \subset \mathbb{R}^n$ open. $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$
$\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ *multi-index*
$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$, where $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$.

Distributions: $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ topological vector space.
A sequence $\phi_i \in \mathcal{D}(\Omega)$ *converges* to $\phi \in \mathcal{D}(\Omega)$ if and only if $\exists K \subset \Omega$, K compact, such that

$$\operatorname{supp} \phi_i \subset K \quad \forall i \in \mathbb{N}$$

and

$$\lim_{i \to \infty} \|\phi_i - \phi\|_{C^l} = 0 \quad \forall l \in \mathbb{N}$$

$$\|\phi\|_{C^l} = \sum_{\alpha \in \mathbb{N}_0^n |\alpha| \leq l} \sup_{x \in \Omega} |\partial^\alpha \phi(x)|$$

The set $\mathcal{D}'(\Omega) := \{T : \mathcal{D}(\Omega) \to \mathbb{R} \mid T \text{ is continuous and linear}\}$ is the set of *distributions on* Ω.

Example: $u : \Omega \to \mathbb{R}$ continuous then the formula

$$Tu(\phi) := \int_\Omega u(x)\phi(x)\,\mathrm{d}x$$

defiones a distribution on Ω.
More generally, every locally integrable function $u : \Omega \to \mathbb{R}$ determines a distribution $T_u : \mathcal{D}(\Omega) \to \mathbb{R}$ by the same formula

$$T_u(\phi) := \int_\Omega u\phi \quad (\textit{Lebesgue integral})$$

Key fact: If $u : \Omega \to \mathbb{R}$ is locally integrable and

$$\int_\Omega u\phi = 0 \quad \forall \phi \in C_0^\infty(\Omega)$$

then $u = 0$ almost everywhere.

Remark: The map $L^1_{\mathrm{loc}}(\Omega) \to \mathcal{D}'(\Omega)$, $u \mapsto T_u$ is injective and linear.

Remark 1: $L^p(\Omega) \subset L^1_{\mathrm{loc}}(\Omega) \subset \mathcal{D}'(\Omega)$ for $1 \leq p \leq \infty$.

Remark 2: If $u : \Omega \to \mathbb{R}$ is continuously differentiable, then

$$\int_\Omega u \partial_i \phi = -\int_\Omega (\partial_i u)\phi$$

ie

$$T_{\partial_i u}(\phi) = -T_u(\partial_i \phi)$$

More generally if $u : \Omega \to \mathbb{R}$ is of class C^l, then

$$T_{\partial^\alpha u}(\phi) = (-1)^{|\alpha|} T_u(\partial^\alpha \phi) \forall \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq l$$

Definition: Let $T \in \mathcal{D}'(\Omega)$ be any distribution and $\alpha \in \mathbb{N}_0^n$. Then *partial derivative* $\partial^\alpha T$ is the distribution defined by

$$(\partial^\alpha T)(\phi) := (-1)^{|\alpha|} T(\partial^\alpha \phi)$$

Terminology: Let $T \in \mathcal{D}'(\Omega)$. We say "$T \in L^p(\Omega)$" if T belongs to the image of the inclusion

$$L^p(\omega) \hookrightarrow \mathcal{D}'(\Omega)$$

ie, if $\exists v \in L^p(\Omega)$ such that $T = T_v$. Then, for $u \in L^p(\Omega)$ and $\alpha \in \mathbb{N}_0^n$, we have "$\partial^\alpha T_u \in L^p(\Omega)$" if and only if

$$\exists! u_\alpha \in L^p(\Omega) \quad \forall \phi \in C_0^\infty(\Omega)$$

(*)
$$\int_\Omega u_\alpha \phi = (-1)^{|\alpha|} \int_\Omega u \partial^\alpha \phi$$

We also write "$\partial^\alpha u \in L^p(\Omega)$" instead of "$\partial^\alpha T_u \in L^p(\Omega)$".

Notation: The function $u_\alpha \in L^p(\Omega)$ in (*), if it exists, will be denoted by $\partial^\alpha u := u_\alpha$ and is called the *weak derivativ of u associated to α*

Definition: Let $k \in \mathbb{N}$ and $1 \le p \le \infty$.
The *Sobolev space* $W^{k,p}(\Omega)$ is defined by

$$\begin{aligned} W^{k,p}(\Omega) &:= \{u \in L^p(\Omega) \mid \forall \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \le k \exists u_\alpha \in L^p(\Omega) \text{ such that (*) holds}\} \\ &= \{u \in L^p(\Omega) \mid \partial^\alpha u \in L^p(\Omega) \forall \alpha \in \mathbb{N}_0^\infty \text{ with } |\alpha| \le k\} \end{aligned}$$

$$\|u\|_{k,p} = \|u\|_{W^{k,p}} := \left(\sum_{|\alpha| \le k} \int_\Omega |\partial^\alpha u|^p \right)^{1/p} \qquad 1 \le p < \infty$$

$$\|u\|_{k,\infty} := \max_{|\alpha| \le k} \|\partial^\alpha u\|_{L^\infty(\Omega)}$$

Remark: If $u \in C^k(\overline{\Omega})$ then the strong partial derivative $\partial^\alpha u$ agrees with the weak derivative $\forall \alpha \in \mathbb{N}_0^n$ with $|\alpha| \le k$.
Hence $\partial^\alpha u \in C^0(\overline{\Omega}) \subset L^p(\Omega) \quad \forall \alpha \in \mathbb{N}_0^n$ with $|\alpha| \le k$.

$$\text{So } C^k(\overline{\Omega}) \subset W^{k,p}(\Omega)$$

Hence also

$$C_0^\infty(\Omega) \subset C^\infty(\overline{\Omega}) \subset W^{k,p}(\Omega) \quad \forall k, p$$

Notation: $W_0^{k,p}(\Omega) :=$ closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$.

Lemma 1:

(i) $W^{k,p}(\Omega)$ is a Banach space

(ii) $W^{k,p}(\Omega)$ is reflexive for $1 < p < \infty$

(iii) $W^{k,p}(\Omega)$ is separable for $1 \le p < \infty$

(iv) $u \in Wk+1, p(\Omega) \Leftrightarrow u \in W^{1,p}(\Omega)$ and $\partial_i u \in W^{k,p}(\Omega)$ for $i = 1, \dots, n$

(v) $\forall k, p \exists c = c(k,p)$: If $u \in W^{k,p}(\Omega), v \in W^{k,\infty}(\Omega)$ then $uv \in W^{k,p}(\Omega)$ and $\|uv\|_{k,p} \le c \|u\|_{k,p} \|v\|_{k,\infty}$

Proof: We know $L^p(\Omega)$ is complete.

$$\mathcal{X} := \left\{ (u_\alpha)_{|\alpha| \le k \alpha \in \mathbb{N}_0^n} \mid u_\alpha \in L^p(\Omega) \right\}$$

$$\mathcal{X} \cong L^p(\Omega, \mathbb{R}^N) \quad \text{where}$$

$$N = \sharp\{\alpha \in \mathbb{N}_0^n \mid |\alpha| \le k\}$$

So $\mathcal{X}$ is a Banach space. There is an inclusion $W^{k,p} \to \mathcal{X} \quad u \mapsto (\partial^\alpha u)_\alpha$. This inclusion is isometric if $\|(u_\alpha)_\alpha\|_{\mathcal{X}} := \left(\sum_\alpha \|u_\alpha\|_{L^p(\Omega)}^p \right)^{1/p}$

Claim: The image of this inclusion is a closed subspace of $\mathcal{X}$.
Claim and known stuff from Measure and Integration and Functional Analysis $\Rightarrow$ i), ii), iii).
$(u_{i,\alpha})_\alpha \in \mathcal{X}$ sequence, $u_i \in W^{k,p}(\Omega)$. Assume $\partial^\alpha u = u_{i,\alpha}$. Assume $\exists (u_\alpha)_\alpha \in \mathcal{X}$ such that

$$\lim_{i\to\infty} \|u_{i,\alpha} - u_\alpha\|_{L^p} = 0 \quad \forall \alpha$$

Denote $u := u_{(0,0,\dots,0)}$. Then $u_\alpha = \partial^\alpha u$!

$$\int_\Omega u_\alpha \phi = \lim_{i\to\infty} \int_\Omega u_{i,\alpha}\phi = (-1)^{|\alpha|} \lim_{i\to\infty} \int_\Omega u_i \partial^\alpha \phi = (-1)^{|\alpha|} \int_\Omega u \partial^\alpha \phi$$

(iv) Exercise!
(v): $u \in W^{k,p}(\Omega), v \in W^{k,\infty}(\Omega) \Rightarrow uv \in W^{k,p}(\Omega$ and

(*)
$$\partial^\alpha(uv) = \sum_{\beta\le\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} \underbrace{\partial^{\alpha-\beta}u}_{\in L^p} \underbrace{\partial^\beta v}_{\in L^\infty} \quad \alpha \in \mathbb{N}_0^\infty, |\alpha| \le k$$

$\beta \le \alpha \Leftrightarrow \beta_i \le \alpha_i \quad i = 1,\dots,n$ and $\alpha! = \alpha_1! \cdots \alpha_n!$

Proof by induction:

Step 1: $k = 1$
$u \in W^{1,p}, v \in W^{1,\infty} \Rightarrow uv \in W^{1,p}$ and $\partial_i(uv) = (\partial_i u)v + u(\partial_i v) \quad i = 1,\dots,n$
(Leibnitz rule for weak partial derivatives).

Step 2: $k \ge 2$
Assume the result holds for $k-1$.
$u \in W^{k,p}, v \in W^{k,\infty} \overset{\text{Step}}{\Rightarrow} uv \in W^{k,p}$

$$\partial_i(uv) = \underbrace{(\partial_i u)}_{W^{k-1,p}} \underbrace{v}_{W^{k,\infty}} + \underbrace{u}_{W^{k,p}} \underbrace{(\partial_i v)}_{W^{k-1,\infty}}$$

$$\overset{\text{Induction Hypothesis}}{\Rightarrow} \partial_i(uv) \in W^{k-1,p} \quad i = 1,\dots,n$$

$$\overset{\text{Lemma 1, (iv)}}{\Rightarrow} uv \in W^{k,p}$$

Proof of (*): Same induction argument as in the case of strong derivatives. (Leibnitz rule and higher derivatives commute)

$$\partial^\alpha(\partial^\beta T) = \partial^\beta(\partial^\alpha T) = \partial^{\alpha+\beta)T}$$

□

Lemma 2: $\Omega \subset \mathbb{R}^n$ open. $u \in W^{k,p}(\Omega), 1 \le p < \infty$.
$\rho : \mathbb{R}^n \to \mathbb{R}$ smooth satisfying $\rho(x) = 0 \quad |x| \ge 1 \quad \int_{\mathbb{R}^n} \rho = 1$.
For $\delta > 0$ denote $\Omega_\delta := \{x \in \mathbb{R}^n \mid \overline{B_\delta(x)} \subset \Omega\}$ and

$$u_\delta(x) = (\rho_\delta * u)(x) = \int\limits_\Omega \rho_\delta(x-y)u(y)\,\mathrm{d}y \quad x \in \Omega_\delta$$

with $\rho_\delta(x) := \frac{1}{\delta^n}\rho(\frac{x}{\delta})$.
$\Rightarrow$ For every compact $K \subset \Omega$ we have

$$\lim_{\delta\to 0} \|u - u_\delta\|_{W^{k,p}(K)} = 0$$

Lemma 2 $\Rightarrow$ Lemma 1 (v) for $k = 1$: $u \in W^{1,p}, v \in W^{1,\infty}$. Choose $u_\delta : \Omega_\delta \to \mathbb{R}$ as in Lemma 2.
Pick $\phi \in C_0^\infty(\Omega)$. Let $K = \operatorname{supp}\phi \subset \Omega$. To show:

(**)
$$\int_\Omega (\partial_i \phi)(uv) = -\int_\Omega \phi((\partial_i u)v + u(\partial_i v))$$

If δ so smoll that $K \subset \Omega_\delta$ then $\phi_{u_\delta} \in C_0^\infty(\Omega_\delta)$. Hence

$$\int_{\Omega_\delta} ((\partial_i \phi)u_\delta + \phi \partial u_\delta)v = \int_{\Omega_\delta} (\partial_i(\phi u_\delta)) \cdot v = -\int_{\Omega_\delta} \phi u_\delta \partial_i v$$

$$\Rightarrow \int_K (\partial_i \phi) u_\delta v = -\int_K (u_d elta \partial_i v + (\partial_i u_\delta) v$$

By Lemma 2 we have

$$\lim_{\delta \to 0} \|u_\delta - u\|_{L^p(K)} = 0$$

$$\lim_{\delta \to 0} \|\partial_i u_\delta - \partial_i u\|_{L^p(K)} = 0$$

$$\stackrel{\delta \to 0}{\Rightarrow} \int_K (\partial_i \phi) uv = -\int_K \phi(u \partial_i v + (\partial_i u)v) \Rightarrow (**)$$

Proof: of Lemma 2: Assume $u \in W^{k,p}(\Omega)$. We know:

- $u_\delta := \rho_\delta * u : \Omega_\delta \to \mathbb{R}$ is smooth $C^\infty \forall \delta$.
- For $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le k$

(***)
$$\underbrace{\partial^\alpha u_\delta}_{\text{strongderivative}} = \rho_\delta * \underbrace{\partial^\alpha u}_{\text{weakderivative}}$$

Note: (***) $\Rightarrow$ Lemma 2, because

$$\lim_{\delta \to 0} \|\rho_\delta * u - u\|_{L^p(K)} = 0$$

so by (***)

$$\lim_{\delta \to 0} \|\partial^\alpha(\rho_\delta * u) - \partial^\alpha u\|_{L^p(K)} = 0$$

Proof of (***): Fix a point $x \in \Omega_\delta$. Define $\phi \in C_0^\infty(\Omega)$ by $\phi(y) := \rho_\delta(x - y)$. Then $\operatorname{supp}\phi \subset \overline{B_\delta(x)} \subset \Omega$. Hence

$$\begin{aligned}
\partial^\alpha(\rho_\delta * u)(x) &= \int_\Omega (\partial^\alpha \rho_\delta)(x-y)u(y)\,\mathrm{d}y \\
&= (-1)^{|\alpha|} \int_\Omega \partial^\alpha \phi(y) u(y)\,\mathrm{d}y \\
&= \int_\Omega \phi(y) \partial^\alpha u(y)\,\mathrm{d}y \\
&= (\rho_\delta * \partial^\alpha u)(x)
\end{aligned}$$

□

Definition: Let $k \in \mathbb{N}_0$. An open set $\Omega \subset \mathbb{R}^n$ is called a *C^k-domain* if for every $x_0 \in \partial\Omega$ there is

- an open neighbourhood $u \in \mathbb{R}^n$ of x_0.
- a unit vector $\xi \in S^{n-1}$
- a constant $\delta > 0$
- a C^k-function $f : \xi^\perp \to \mathbb{R}$ such that $f(0) = 0$ and

$$\Omega \cap U = \{x_0 + \eta + t\xi \mid \eta \perp \xi, |\eta| < \delta, f(\eta) < t < \delta\}$$
$$(\text{wlog } U = \{x_0 + \eta + t\xi \mid \eta \perp \xi, |\eta| < \delta, |t| < \delta\})$$

Example: $Q = [0,1]^n$ is a Lipschitz-domain.

Lemma 3: Let $\Omega \subset \mathbb{R}^n$ be a C^0-domain, Ω bounded.
Let $u \in W^{k,p}(\Omega)$ $\quad 1 \leq p < \infty$.
$\Rightarrow \exists$ sequence of open sets $\Omega_j \subset \mathbb{R}^n$
$\exists$ sequence $u_j \in W^{k,p}(\Omega_j)$ such that

$$\overline{\Omega} \subset \Omega_j \quad \forall j$$

and

$$\lim_{j\to\infty} \|u - u_j\|_{W^{k,p}(\Omega} = 0$$

Theorem 1: Let $\Omega \subset \mathbb{R}^n$ be a bounded C^0-domain. Then $C^\infty(\overline{\Omega})$ is dense in $W^{k,p}(\Omega)$ for all k and $1 \leq p < \infty$.

Proof: Let $\varepsilon > 0 \overset{\text{Lemm 3}}{\Rightarrow} \exists \Omega' \subset \mathbb{R}^n$ open
$\exists u' \in W^{k,p}(\Omega')$ such that $\overline{\Omega} \subset \Omega'$ with $\|u - u'\|_{W^{k,p}(\Omega) < \varepsilon/2}$
Now choose ρ_δ as in Lemma 2. Then $\rho_\delta * u' : \Omega'_\delta \to \mathbb{R}$ is smooth and

$$\lim_{\delta\to 0} \|\rho_\delta * u' - u'\|_{W^{k,p}(\Omega)} = 0$$

For $\delta > 0$ sufficiently small we have $\overline{\Omega} \subset \Omega'_\delta$ and

$$\|\rho_\delta * u' - u'\|_{W^{k,p}(\Omega)} < \varepsilon/2$$

Then

$$\rho_\delta * u'|_{\overline{\Omega}} \in C^\infty(\overline{\Omega})$$

and

$$\|\rho_\delta * u' - u\|_{W^{k,p}(\Omega)} < \varepsilon$$

□

Proof of Lemma 3:

Case 1: u as in definition of C^0-domain.

$$\Omega \cap U = \{x_0 + \eta + t\xi \mid \eta \perp \xi, |\eta| < \delta, f(\eta) < t < \delta\}$$

$$f : \xi^\perp \to \mathbb{R} \text{ continuous, } f(0) = 0$$

$$\Omega \cap V := \{x_0 + \eta + t\xi \mid \eta \perp \xi, |\eta| < \varepsilon, f(\eta) < t < \delta/2\}$$

where $\varepsilon > 0$ is such that $|\eta| < \varepsilon \Rightarrow |f(\eta)| < \delta/4, \varepsilon < \delta/2$.
Suppose that $u(x) = 0 \quad \forall x \notin V$. Define $u_\lambda(x) := u(x + \lambda\varepsilon)$
$x = x_0 + \eta + t\xi \quad |\eta| < \varepsilon \quad f(\eta) - \lambda < t < \delta/2$.
Choose Ω_λ such that $\overline{\Omega} \subset \Omega_\lambda$ and

$$\Omega_\lambda \cap U = \{x_0 + \eta + t\xi \mid \eta \perp \xi, |\eta| < \delta, f(\eta) - \lambda < t < \delta\}$$

Then

$$u_\lambda : \Omega_\lambda \to \mathbb{R} \qquad \lim_{\lambda\to 0} \|u_\lambda - u\|_{W^{k,p}(\Omega)} = 0$$

because

$$\partial^\alpha u_\lambda(x) = (\partial^\alpha u)(x + \lambda\xi) \quad x \in \Omega_\lambda \cap U$$

Case 2: Cover $\partial\Omega$ by finitely many nbhds. $V_1, \dots, V_N$ as in Case 1.
Choose (next) $V_0 \subset \mathbb{R}^n$ open such that $\overline{V_0} \subset \Omega$ and

$$\overline{\Omega} \subset V_0 \cap V_1 \cap \dots \cap V_N$$

Choose partition of unity $\rho_j : \mathbb{R}^n \to [0, \infty)$, ρ_j smooth, $j = 0, \dots, N$, $\operatorname{supp} \rho_j \subset V_j$, $\sum_{j=0}^{N} \rho_j(x) = 1 \forall x \in \overline{\Omega}$. Then

$$u = \sum_{j=0}^{N} \rho_j u$$

so $\rho_0 u$ extends (by zero) to an element of $C_0^\infty(\mathbb{R}^n)$. Apply Case 1 to $\rho_j u \quad j = 1, \dots, n$. Get sequence:

$$u_{j,i} \overset{i\to\infty}{\to} \rho_j u \quad \text{in } W^{k,p}(\Omega_{j,i}$$

$$u_{j,i} \in W^{k,p}(\Omega_{j,i}) \quad \overline{\Omega} \subset \Omega_{j,i}$$

Define

$$\Omega_i := \bigcap_{j=1}^{N} \Omega_{j,i}$$

$$U_i := \sum_{j=1}^{N} u_{j,i} + \rho_0 u \in W^{k,p}(\Omega_i)$$

$$\lim_{i\to\infty} \|u_i - u\|_{W^{k,p}(\Omega)} = 0$$

$$u \in C^1(\mathbb{R}^n), \triangledown u \equiv 0 \Rightarrow u \text{ constant}$$

(*) This remains true if weak derivatives vanish.

$\Omega \subset \mathbb{R}^n$ open, bounded, $\operatorname{diam}(\Omega) := \sup\{|x - y| \mid x, y \in \Omega\}$

Lemma 4: (Poincaré-inequality):

(i) For $u \in W^{k,p}(\Omega)$ we have

$$\|u\|_{L^p(\Omega)} \leq \operatorname{diam}(\Omega) \| \triangledown u\|_{L^p(\Omega)}$$

(ii) If $\Omega = Q^n = (0,1)^n$ and $u \in W^{1,p}(\Omega)$ then

$$\int_Q u = 0 \Rightarrow \|u\|_{L^p(Q) \leq n \|\triangledown u\|_{L^p(Q)}}$$

Exercise: $Q = (a,b)^n \quad \int_Q u = 0$

$$\Rightarrow \|u\|_{L^p(Q)} \leq n(b-a) \| \triangledown u\|_{L^p(Q)}$$

Proof:

(i): Assume $\Omega \subset \{x \in \mathbb{R}^n \mid x_n > 0\}$ and $0 \in \partial\Omega$. $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega)$ suffices to assume $u \in C_0^\infty(\Omega)$. Extend u to a smooth function on $\mathbb{R}^n$ by defining

$$u(x) := 0 \text{ for } x \notin \Omega$$

Then:

$$u(x) = \int_0^{x_n} \partial_n u(x_1, \dots, x_{n-1}, t \, \mathrm{d}t \quad \forall x \in \mathbb{R}^n$$

$$\partial_n u = \frac{\partial u}{\partial x_n}$$

$$\begin{aligned}\Rightarrow |u(x)| &\leq \int_0^{x_n} |\partial_n u(x_1, \ldots, x_{n-1}, t)| \,\mathrm{d}t \\ &\leq \int_0^{\operatorname{diam}(\Omega)} |\partial_n (x_1, \ldots, x_{n-1}, t)| \,\mathrm{d}t \quad x \in \Omega\end{aligned}$$

Define $d := \operatorname{diam}(\Omega)$

$$\overset{\text{Hölder}}{\Rightarrow} |u(x)| \leq d^{1-1/p} \left(\int_0^3 |\partial_n u(x_1, \ldots, x_{n-1}, t)|^p \,\mathrm{d}t \right)^{1/p}$$

$$|u(x_1, \ldots, x_n)|^p \leq d^{p-1} \int_0^d |\partial_n u(x_1, \ldots, x_{n-1}, t)|^p \,\mathrm{d}t$$

$$\Rightarrow \int_0^d |u(x_1, \ldots, x_n)|^p \,\mathrm{d}x_n \leq d^p \int_0^d |\partial_n u(x_1, \ldots, x_{n-1}, t)|^p \,\mathrm{d}t$$

$$\|u\|^p_{L^p(\Omega)} \leq d^p \|\partial_n u\|^p_{L^p(\Omega)} \leq d^p \| \bigtriangledown u\|^p_{L^p(\Omega)}$$

(ii): Q is a C^0-domain. So, by Theorem 1, $C^\infty(\overline{Q})$ is dense in $W^{1,p}(Q)$.

Remark: $\forall u \in W^{1,p}(Q)$ with $\int_Q u = 0 \quad \exists$ sequence $u_\nu \in C^\infty(\overline{Q})$ with $\int_Q u_\nu = 0$ and $\lim_{\nu \to 0} \|u - u_\nu\|_{W^{1,p}(Q)} = 1$. So, it suffices to prove (ii) for $u \in C^\infty(\overline{Q})$.

Induction:

$n = 1$: $Q = (0, 1)$. For $x, y \in [0, 1]$:

$$|u(y) - u(x)| = |\int_x^y u'(t) \,\mathrm{d}t| \leq \int_0^1 |u'(t)| \,\mathrm{d}t \overset{\text{Hölder}}{\leq} \|u'\|_{L^p}$$

$$\Rightarrow -\|u'\|_{L^p} \leq u(x) - u(y) \leq \|u'\|_{L^p}$$

$$\Rightarrow -\|u'\|_{L^p} \leq u(x) \leq \|u'\|_{L^p}$$

$$|u(x) \leq \|u'\|_{L^p} \quad \forall x \in [0, 1]$$

$$\|u\|_{L^p} \leq \sup_{0 \leq x \leq 1} |u(x)| \leq \|u'\|_{L^p}$$

$n \leq 2$: Assume the result holds for $n - 1$. Let $u \in C^\infty(\overline{Q}^n), \int_Q u = 0$ and define

$$v(t) := \int_{Q^{n-1}} \partial_n u(\widehat{x}, t) \,\mathrm{d}t \quad \widehat{x} := (x_1, \ldots, x_{n-1})$$

$$\Rightarrow \int_0^1 v(t) \,\mathrm{d}t = 0 \quad v'(t) = \int_{Q^{n-1}} \partial_n u(\widehat{x}, t) \,\mathrm{d}t$$

$$\overset{n=1}{\Rightarrow} \int_0^1 |v(t)|^p \,\mathrm{d}t \leq \int_0^1 |v'(t)|^p \,\mathrm{d}t$$

$$|v'(t) \leq \int\limits_{Q^{n-1}} |\partial_n u(\widehat{x}, t) \,\mathrm{d}\widehat{x}$$

$$\stackrel{\text{Hölder}}{\Rightarrow} |v'(t)|^p \leq \int\limits_{Q^{n-1}} |\partial_n u(\widehat{x}, t)|^p \,\mathrm{d}\widehat{x}$$

$$\int\limits_0^1 |v'(t)|^p \,\mathrm{d}t \leq \int\limits_{Q^n} |\partial_n u|^p \,\mathrm{d}x$$

Moreover; by the induction hypothesis for $\widehat{x} \mapsto u(\widehat{x}, t) - v(t)$

$$\int\limits_{Q^{n-1}} |u(\widehat{x}, t) - v(t)|^p \,\mathrm{d}\widehat{x} \leq (n-1)^p \int\limits_{Q^{n-1}} |Nabla_{\widehat{x}} u(\widehat{x}, t)|^p \,\mathrm{d}\widehat{x}$$

$$\Rightarrow \int\limits_{Q^n} |u(x) - v(x)|^p \,\mathrm{d}x \leq (n-1)^p \int\limits_{Q^p} | \bigtriangledown u(x)|^p \,\mathrm{d}x$$

$$\Rightarrow \|u - v\|_{L^p(Q^n)} \leq (n-1) \| \bigtriangledown u\|_{L^p(Q^n)}$$

$$\|v\|_{L^p(Q^n)} \leq \|\partial_n u\|_{L^p(Q^n)} \leq \| \bigtriangledown u\|_{L^p(Q^n)}$$

$$\Rightarrow \|u\|_{L^p(Q^n) \leq n \|\bigtriangledown u\|_{L^p(Q^n)}}$$

□

Corollary: $\Omega \subset \mathbb{R}^n$ open, connected. $u \in W^{1,p}_{\text{loc}}(\Omega), \bigtriangledown u = 0$ almost everywhere. $\Rightarrow u$ is constant (ie. $\exists c \in \mathbb{R}$ such that $u(x) = c$ for almost every $x \in \Omega$). Moreover, if $u \in W^{1,p}_0(\Omega)$ and $\bigtriangledown u \equiv 0$ then $u \equiv 0$.

Proof:

Claim: u is locally constant, ie. $\forall x \in \Omega \exists$ open set $U \subset \Omega, \exists c \in \mathbb{R}$ such that $x \in U$ and $u(y) = c$ for almost every $y \in U$.
Note: c in this claim does not depend on u, so $c = c(x)$.
For $a \in \mathbb{R}$ the claim shows that the set $A := \{x \in \Omega \mid c(x) = a\}$ is open and closed rel. Ω. So either $A = \Omega$ or $A = \varnothing$.
$u \in W^{1,p}_0(\Omega)$ extend u to $\mathbb{R}^n$ by $u(x) := 0$ for $x \notin \Omega$, $c = 0$.

Proof of Claim: Choose $U = Q = (a, b)^n$ with $x \in U$.
$\stackrel{\text{Thm 1}}{\Rightarrow} \exists u_\nu \in C^\infty(\overline{Q})$ such that $u_\nu \to u$ in $W^{1,p}(Q)$

$$\Rightarrow c_\nu := \frac{1}{\text{Vol}(Q)} \int_Q u_\nu \to c := \frac{1}{\text{Vol}(Q)} \int_Q u$$

$$\Rightarrow u_\nu - c_\nu \to u - c \text{ in } W^{1,p}(Q), \text{ by Lemma 4}$$

$$\|u_\nu - v_\nu\|_{L^p} \leq n(b-a) \| \bigtriangledown u_\nu\|_{L^p} \to 0$$

$$u - c = 0 \text{ in } L^p(Q)$$

ie. $u(x) = c$ almost everywhere on Q. □

Question: Is every function $u \in W^{1,p}_{\text{loc}}(\Omega)$ continuous? No!

Example: $\Omega = \mathbb{R}^n \quad u(x) = \frac{1}{|x|^\alpha} \quad \alpha > 0$
$\triangledown u(x) = \frac{\alpha}{|x|^{\alpha+2}} x \quad | \triangledown u(x)| = \frac{\alpha}{|x|^{\alpha+1}}$ So $| \triangledown u(x)|^p = \frac{\alpha^p}{|x|^{\alpha p + p}}$. When ist this function integrable?
integrable near $x = 0 \Leftrightarrow \alpha p + p < n$ (Exercise: prove it!). So if $p < n, 0 < \alpha < \frac{n}{p} - 1$ then $u \in W^{1,p}_{\text{loc}}$ but u is not continuous.
Exercise: strong derivative = weak derivative.

- $u : \Omega \to \mathbb{R}$ is called *Hölder continuous with exponent* μ if $0 < \mu \leq 1$:

$$|u(x) - u(y)| \leq c|x - y|^\mu \quad \forall x, y \in \Omega$$

- *Hölder norm*:

$$\|u\|_{C^{0,\mu}} := \sup_{x \neq y x, y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^\mu} + \sup_{x \in \Omega} |u(x)|$$

-

$$C^{0,\mu}(\Omega) := \{u : \Omega \to \mathbb{R} \mid u \text{ continuous}, \|u\|_{C^{0,\mu}} < \infty\}$$
$$C^{k,\mu}(\Omega) := \{u \in C^k(\Omega) \mid \partial^\alpha u \in C^{0,\mu}(\Omega \forall \alpha \in \mathbb{N}^n_0 \text{ with } |\alpha| \leq k\}$$
$$\|u\|_{C^{k\mu}} := \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{C^{0,\mu}}$$

Theorem 2: $\Omega \subset \mathbb{R}^n$ open, $k \in \mathbb{N}$, $1 \leq p < \infty$.

(i) If $kp > n$ and $0 < \mu := k - \frac{n}{p} < 1$ then

$$\exists c = c(k,p) \forall u \in C^\infty_0(\Omega) :$$
$$\|u\|_{C^{0,\mu}(\Omega} \leq c \|u\|_{W^{k,p}(\Omega}$$

Hence there is an embedding

$$W^{k,p}_0(\Omega) \hookrightarrow C^{0,k-n/p}(\Omega)$$

(ii) If $kp < n$ then $\exists c = c(k,p) > 0$. $\forall u \in C^\infty_0(\Omega)$:

$$\|u\|_{L^{\frac{np}{n-kp}}(\Omega)} \leq c \|u\|_{W^{k,p}(\Omega}$$

Hence there is an embedding

$$W^{k,p}_0(\Omega) \hookrightarrow L^{\frac{np}{n-kp}}(\Omega)$$

Sobolev Embedding Theorems

Lemma 5: For $p > n$ and $u \in C^\infty_0(\Omega)$ we have

(1)

$$\sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\mu} \leq c \| \triangledown u\|_{L^p}$$

(2)

$$\sup_x |u(x)| \leq c(\|u\|_{L^p} + \| \triangledown u\|_{L^p}$$

where $\mu = 1 - \frac{n}{p}$ and $c = \frac{2^{n+1}}{\omega_n^{1/p}} \left(\frac{p-1}{p-n}\right)^{1-1/p}$

Proof: (of the Lemma)

Claim: $B \subset \mathbb{R}^n$ bounded, convex. $u : B \to \mathbb{R}$ smooth, $\int_B u = 0$

$$\Rightarrow u(x) \leq \frac{d^n \omega_n}{nV\omega_n^{1/p}} \left(\frac{p-1}{p-n}\right)^{1-1/p} d^{1-n/p} \| \bigtriangledown u\|_{L^p(B)}$$

$$\forall x \in B \text{ where } d := \operatorname{diam}(B), V := \operatorname{Vol}(B)$$

Claim⇒(1):

$$B := B_r(\frac{x+y}{2}) \quad r := \frac{|x-y|}{2}$$

then $d := |x-y| \quad V = (\frac{d}{2})^n \frac{\omega_n}{n}$. So $\frac{d^n \omega_n}{nV} = 2^n$. Define $u_B := \frac{1}{V}\int_B u$. Then

$$|u(x) - u_B| \leq \frac{2^n}{\omega_n^{1/p}} \left(\frac{p-1}{p-n}\right)^{1-1/p} |x-y|^{1-n/p} \| \bigtriangledown u\|_{L^p}$$

$$\Rightarrow \frac{|u(x)-u(y)|}{|x-y|^{1-n/p}} \leq \frac{2^{n+1}}{\omega_n^{1/p}} \left(\frac{p-1}{p-n}\right)^{1-1/p} \| \bigtriangledown u\|_{L^p}$$

That's exactly the Claim of (1).

Claim⇒(2):

$$B = B_1(x) \quad d = 2 \quad \frac{d^n\omega_n}{nV} = 2^n$$

$$\Rightarrow |u(x) - u_B| \leq \frac{2^n}{\omega_n^{1/p}} \left(\frac{p-1}{p-n}\right)^{1-1/p} 2^{1-n/p} \| \bigtriangledown u\|_{L^p}$$

$$\leq C \| \bigtriangledown u\|_{L^p}$$

$$|u_B| \leq \frac{1}{V}\int \|u\| \leq \frac{1}{V^{1/p}}\|u\|_{L^p} = (\frac{n}{\omega_n})^{1/p}\|u\|_{L^p}$$

Proof of Claim:

$$\begin{aligned} u(x) - u(y) &= -\int_0^1 \frac{d}{dt} u(x + t(y-x))\, \mathrm{d}t \\ &= -\int_0^1 \langle \bigtriangledown u(x + t(y-x)), y - x\rangle\, \mathrm{d}t \\ &= \int_0^1 \langle \bigtriangledown u(x + t(y-x), x-y\rangle\, \mathrm{d}t \\ \Rightarrow u(x) &= \frac{1}{V}\int_B \int_0^1 \langle \bigtriangledown u(x+t(y-x), x-y\rangle\, \mathrm{d}t\, \mathrm{d}y \end{aligned}$$

Define: $\bigtriangledown u(z) := 0$ for $z \notin B$

$$\begin{aligned} \Rightarrow |u(x)| &= \frac{1}{V}|\int_{B-x}\int_0^1 \langle \bigtriangledown u(x+ty), y\rangle\, \mathrm{d}t\, \mathrm{d}y|, B - x \subset \{y \in \mathbb{R}^n \mid |y| \leq d\} \\ &\leq \frac{1}{V}\int_{|y|\leq d}\int | \bigtriangledown u(x+ty)||y|\, \mathrm{d}t\, \mathrm{d}y, y = r\eta, |\eta| = 1 |y| = r \\ &= \frac{1}{V}\int_0^d r^{n-1}\int_{|\eta|=1}\int_0^1 | \bigtriangledown u(x+tr_\eta)|\, \mathrm{d}tr\, \mathrm{d}S(\eta)\, \mathrm{d}r, \rho := tr \\ &= \frac{1}{V}\int_0^d \int_{|\eta|=1}\int_0^r | \bigtriangledown u(x + \rho\eta\, \mathrm{d}\rho\, \mathrm{d}S(\eta)\, \mathrm{d}r \quad (*) \end{aligned}$$

$$
\begin{aligned}
(*) &= \frac{1}{V}\int_0^d\int_0^1\int_{|\eta|=\rho} |\bigtriangledown u(x+\eta)|\rho^{1-n}\,\mathrm{d}S(\eta)\,\mathrm{d}\rho\,\mathrm{d}r \\
&= \frac{1}{V}\int_0^d r^{n-1}\Big(\int_{B_r(0)} |\bigtriangledown u(x+y)||y|^{1-n}\,\mathrm{d}y\Big)\,\mathrm{d}r \\
&= \frac{1}{V}\int_0^d r^{n-1}\int_{B_r(x)} |\bigtriangledown u(y)||y-x|^{1-n}\,\mathrm{d}y\,\mathrm{d}r \\
&\leq \frac{1}{V}\frac{d^n}{n}\|\bigtriangledown u\|_{L^p(B)}\left(\int_B |y-x|^{q-nq}\,\mathrm{d}y\right)^{1/q} \\
&= \frac{1}{V}\frac{d^n}{n}\|\bigtriangledown u\|_{L^p(B)}\frac{\omega_n}{\omega_n^{1/p}}d^{1-n/p}\left(\frac{p-1}{p-n}\right)^{1-1/p}
\end{aligned}
$$

□

Lemma 6:

$$p < n \quad u \in C_0^\infty(\mathbb{R}^n) \text{ then}$$

$$\|u\|_{L^{\frac{np}{n-p}}} \leq \frac{p}{\sqrt{n}}\frac{n-1}{n-p}\|\bigtriangledown u\|_{L^p}$$

Proof: (L. Nirenberg) $p = 1$

$$
\begin{aligned}
u(x) &= \int_{-\infty}^{x^i} \partial_i u(x_1,\dots,x_{i-1},t,x_{i+1},\dots,x_n)\,\mathrm{d}t \\
|u(x)| &\leq \int_{-\infty}^{\infty} |\partial_i u(x)|\,\mathrm{d}x^i \\
|u(x)|^{\frac{n}{n-1}} &\leq \Pi_{i=1}^n\left(\int |\partial_i u|\,\mathrm{d}x^i\right)^{\frac{1}{n-1}}
\end{aligned}
$$

Hölder: Marginpar(*)

$$\|v_1\cdots v_m\|_{L^1} \leq \|v_1\|_{L^m}\cdots\|v_m\|_{L^m}$$

$$(\|v_1 v_2\|_{L^1} \leq \|v_1\|_{L^2}\|v_2\|_{L^2})$$

Apply (*) with $m = n-1$.

$$
\begin{aligned}
\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}}\,\mathrm{d}x^1 &\leq \int_{-\infty}^{\infty} \Pi_{i=1}^n\left(\int |\partial_i u|\,\mathrm{d}x^i\right)^{\frac{1}{n-1}}\mathrm{d}x^1 \\
&= \left(\int |\partial_1 u|\,\mathrm{d}x^1\right)^{\frac{1}{n-1}}\int_{-\infty}^{\infty}\Pi_{i=2}^n\left(\int |\partial_i u|\,\mathrm{d}x^i\right)^{\frac{1}{n-1}}\mathrm{d}x^1 \\
&\overset{(*)}{\leq} \left(\int |\partial_1 u|\,\mathrm{d}x^1\right)^{\frac{1}{n-1}}\Pi_{i=2}^n\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\partial_i u|\,\mathrm{d}x^i\,\mathrm{d}x^1\right)^{\frac{1}{n-1}}
\end{aligned}
$$

$$\int\int |u|^{\frac{n}{n-1}}\,\mathrm{d}x^1\,\mathrm{d}x^2 \leq \int_{-\infty}^{\infty} [\Pi]\,\mathrm{d}x^2$$

$$(**) \text{ Hölder with } p = n-1, q = p' = \frac{n-1}{n-2}$$

$$(*) \overset{(**)}{\leq} \left(\int\int |\partial_1 u| \, \mathrm{d}x^1 \, \mathrm{d}x^2\right)^{\frac{1}{n-1}} \left[\int_{-\infty}^{\infty} \Pi_{i=2}^n \left(\int\int |\partial_i u| \, \mathrm{d}x^i \, \mathrm{d}x^1\right)^{\frac{1}{n-2}} \mathrm{d}\right]^{\frac{n}{n-1}}$$

$$\leq \left(\int\int |\partial_1 u| \, \mathrm{d}x^1 \, \mathrm{d}x^2\right)^{\frac{1}{n-1}} \left(\int\int |\partial_2 u| \, \mathrm{d}x^2 \, \mathrm{d}x^1\right)^{\frac{1}{n-2}}$$

$$\left[\int_{-\infty}^{\infty} \Pi_{i=3}^n \left(\int\int |\partial_i u| \, \mathrm{d}x^i \, \mathrm{d}x^1\right)^{\frac{1}{n-2}} \mathrm{d}\right]^{\frac{n-2}{n-1}}$$

$$\leq \left(\int\int |\partial_1 u| \, \mathrm{d}x^1 \, \mathrm{d}x^2\right)^{\frac{1}{n-1}} \left(\int\int |\partial_2 u| \, \mathrm{d}x^2 \, \mathrm{d}x^1\right)^{\frac{1}{n-2}}$$

$$\Pi_{i=3}^n \left[\int\int\int |\partial_i u| \, \mathrm{d}x^i \, \mathrm{d}x^1 \, \mathrm{d}x^2\right]^{\frac{1}{n-1}}$$

$\forall 1 \leq k \leq n$

$$\int\int \ldots \int |u|^{\frac{n}{n-1}} \, \mathrm{d}x^1 \ldots \mathrm{d}x^k \leq \Pi_{i=1}^k \left(\int\int \ldots \int |\partial_i u| \, \mathrm{d}x^1 \ldots \mathrm{d}x^k\right)^{\frac{1}{n-1}}$$

$$\Pi_{i=k+1}^n \left(\int\int \ldots \int |\partial_i u| \, \mathrm{d}x^1 \ldots \mathrm{d}x^k \, \mathrm{d}x^i\right)^{\frac{1}{n}}$$

$$\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} \, \mathrm{d}x \leq \Pi_{i=1}^n \left(\int_{\mathbb{R}^n} |\partial_i u| \, \mathrm{d}x\right)^{\frac{1}{n-1}} \text{ exp to } \frac{n-1}{n}$$

$$\|u\|_{L^{\frac{n}{n-q}}(\mathbb{R}^n)} \leq \Pi_{i=1}^n \|\partial_i u\|_{L^1}^{\frac{1}{n}} \overset{(1)}{\leq} \frac{1}{n} \sum \|\partial_i u\|_{L^1} \overset{(2)}{\leq} \frac{1}{\sqrt{n}} \| \bigtriangledown u\|_{L^1}$$

(1): $a_1, \ldots, a_n \geq 0 \quad a_i = \|\partial_i u\|_{L^1}$

$$(a_1 \cdots a_n)^{\frac{1}{n}} \leq \frac{a_1 + \cdots + a_n}{n}$$

(2):

$$\frac{a_1 + \cdots + a_n}{n} \leq \left(\frac{a_1^2 + \cdots + a_n^2}{n}\right)^{\frac{1}{2}}$$

Case $u \in W^{1,1}$ done.
Case $u \in W^{1,p}$ $\quad (W^{k,p} \hookrightarrow W^{1,p})$:
Consider $v := |u|^\alpha \quad \alpha := p\frac{n-1}{n-p}$. Set $q = p'$; $(\frac{1}{p} + \frac{1}{q}) = 1$, where $q = \frac{p}{p-1}$.

$$\alpha \cdot \frac{n}{n-1} = \frac{np}{n-p} = (\alpha - 1) \cdot q$$

$$| \bigtriangledown v| = \alpha |u|^{\alpha-1} | \bigtriangledown |$$

$$\left(\int |u|^{\frac{np}{n-p}}\right)^{1-1/n} = \left(\int |u|^{\alpha\frac{n}{n-1}}\right)^{\frac{n-1}{n}} = \|v\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n}$$

$$\begin{aligned}
\|v\|_{L^{\frac{n}{n-1}}} &\leq \frac{1}{\sqrt{n}}\|\nabla v\|_{L^1} \\
&\leq \frac{\alpha}{\sqrt{n}}\int |u|^{\alpha-1}|\nabla u| \\
&\overset{\text{Hölder}}{\leq} \frac{\alpha}{\sqrt{n}}\left(\int |u|^{(\alpha-1)q}\right)^{1/q}\left(\int |\nabla u|^p\right)^{1/p} \\
&= \frac{\alpha}{\sqrt{n}}\left(\int |u|^{\frac{np}{n-p}}\right)^{1-1/p}\|\nabla u\|_{L^p}
\end{aligned}$$

$$1-\frac{1}{n}-(1-\frac{1}{p}) = \frac{1}{p}-\frac{1}{n} = \frac{n-p}{np}$$

$$\|u\|_{L^{\frac{np}{n-p}}} \leq \frac{\alpha}{\sqrt{n}}\|\nabla u\|_{L^p} = \frac{p}{\sqrt{n}}\frac{n-1}{n-p}\|\nabla u\|_{L^p}$$

□

Theorem 2: $\Omega \subset \mathbb{R}^n$ bounded open, $k \in \mathbb{N}$, $1 \leq p < \infty$.

(i) If $kp > n$, $0 < \mu = k - \frac{n}{p} < 1$

$$\exists c = c(k,n,p) > 0 \text{ such that } \forall u \in C_0^\infty(\Omega)$$

$$\|u\|_{C^{0,\mu}} \leq c\|u\|_{W^{k,p}}$$

$$\exists \text{ constant inclusion } W_0^{k,p}(\Omega) \hookrightarrow C^{0,\mu}(\Omega)$$

(ii) $kp < n$, $\exists c = c(k,p,n)$ such that $\forall u \in C_0^\infty(\Omega)$

$$\|u\|_{L^{\frac{np}{n-p}}} \leq c\|u\|_{W^{k,p}}$$

$$\exists \text{ inclusion } W_0^{k,p}(\Omega) \hookrightarrow L^{\frac{np}{n-p}}(\Omega)$$

Proof: Induction:

k=1: is Lemma 5 and 6.
Assume we proved the statement for $k-1$.
Given k, set $k' = k-1$ and $p' = \frac{np}{n-p}$

$$k' - \frac{n}{p'} = k - \frac{n}{p} = \mu$$

If $kp > n \Rightarrow k'p' > n$ by induction hypothesis.

$$W_0^{k',p'}(\Omega) \hookrightarrow C^{0,\mu}$$

$$W_0^{k,p}(\Omega) \hookrightarrow W_0^{k-1,\frac{np}{n-p}}(\Omega) = W_0^{k',p'}$$

(Lemma 6, applied $u, \partial \underbrace{u, \ldots, \partial^{(k-1)}u}_{\in W^{1,p}}$)

$$W_0^{k,p}(\Omega) \hookrightarrow W_0^{k',p'} \hookrightarrow C^{0,\mu}(\Omega)$$

$$i : W_0^{1,p} \to L^{\frac{np}{n-p}} \quad u \in C_0^\infty$$

$$i|_{C_0^\infty} : C_0^\infty(\Omega) \subset W_0^{1,p}(\Omega) \to L^{\frac{np}{n-p}} \text{ bounded}$$

$\Rightarrow i$ can be extended to $\overline{C_0^\infty}(\Omega) = W_0^{1,p}(\Omega)$
$kp < n \Rightarrow k'p' < n \quad \frac{np}{n-kp} = \frac{np'}{n-k'p'} = q$.
By induction hypothesis $W_0^{k'p'}(\Omega) \hookrightarrow L^q(\Omega)$

$$W_0^{k,p}(\Omega) \hookrightarrow W_0^{k',p'}(\Omega) \hookrightarrow L^q(\Omega)$$

□

Remark: The theorem is true if Ω is Lipschitz.

Lemma 7: $\Omega \subset \mathbb{R}^n$ bounded open, C^1-domain,$\Omega' \subset \mathbb{R}^n$ bounded open with $\Omega' \subset \Omega$.

$$\exists \text{ a bounded linear operator}$$
$$E : W^{k,p}(\Omega) \to W_0^{k,p}(\Omega') \text{ such that}$$
$$Eu|_\Omega = u \quad \forall u \in W^{k,p}(\Omega$$

Lemma 8: $\Omega, \Omega' \subset \mathbb{R}^n$ C^1-bounded domains

$$k \in \mathbb{N}, 1 \leq p < \infty, \psi : \overline{\Omega'} \to \overline{\Omega} \quad C^k\text{-diffeomorphism}$$
$$\exists c > 0 \text{ such that } \forall u \in W^{k,p}(\Omega) u \circ \psi \in W^{k,p}(\Omega') \text{ and}$$

(*)
$$\|u \circ \psi\|_{W^{k,p}(\Omega')} \leq c\|u\|_{W^{k,p}(\Omega)}$$

Proof: By theorem 1 it is enough to prove the lemma for $u \in C^k(\Omega)$.

$$\bigtriangledown(u \circ \psi) = d\psi^T \cdot \bigtriangledown u \circ \psi$$
$$\|d\psi^T\|_{L^\infty} \leq c \qquad \|\text{det} d\psi(y)\| \geq \delta > 0$$

$$\begin{aligned}
\| \bigtriangledown (u \circ \psi)\|_{L^p} &\leq \left(\int c^p | \bigtriangledown u(\psi(y))|^p \, \mathrm{d}y \right)^{1/p} \\
&\leq \left(\int_{\Omega'} \frac{c^p}{\delta} | \bigtriangledown u(\psi(y))|^p |\text{det}\psi(y)| \, \mathrm{d}y \right)^{1/p} \\
&= \frac{c}{\delta^{1/p}} \left(\int_{\Omega} | \bigtriangledown u(x)|^p \, \mathrm{d}x \right)^{1/p} \\
&= \frac{c}{\delta^{1/p}} \| \bigtriangledown u\|_{L^p}
\end{aligned}$$

General Case: $\bigtriangledown u \in W^{k-1,p}$

$$\begin{aligned}
\| \bigtriangledown u \circ \psi\|_{W^{k-1,p}} &\leq c_{k-1}\| \bigtriangledown u\|_{W^{k-1,p}(\Omega)} \\
\| \bigtriangledown (u \circ \psi)\|_{W^{k-1,p}} &= \|d\psi^T \cdot (\bigtriangledown u \circ \psi)\|_{W^{k-1,p}} \\
&\leq c\| \bigtriangledown u \circ \psi\|_{W^{k-1,p}} \\
&\leq c'\|u\|_{W^{k,p}(\Omega)}
\end{aligned}$$

where

$$\| \underbrace{\underbrace{d\psi^T}_{C^{k-1}} \cdot \underbrace{(\bigtriangledown u \circ \psi)}_{W^{k-1,p}}}_{\text{Lemma 1}} \|_{W^{k-1,p}}$$

□

Proof of Lemma 7: Consider $\mathbb{H}^n = \{x \in \mathbb{R}^n \mid x^n > 0\}$
$E_0 : C^k(\mathbb{H}^n) \to C^k(\mathbb{R}^n)$
$E_0(u)(x_1, \ldots, x_n) = \sum_{i=1}^{k+1} c_i u(x_1, \ldots, x_{n-1}, -\frac{x_n}{i})$ if $x_n \leq 0$, where

(*)
$$\sum_{i=1}^{n-1} c_i (\frac{-1}{i})^m = 1 \quad \forall m = 0, \ldots, k$$

Facts:

a) $(*) \Rightarrow E_0 u \in C^k(\mathbb{R}^n)$, ie. all derivatives up to order k math on $\partial\mathbb{H}^n$

b) $u(x) = 0$ for $|x| \geq R$

$$\Rightarrow E_0 u(x) = 0 \text{ for } |x| \geq (k+1)R$$

c)

$$\|E_0 u\|_{W^{k,p}(\mathbb{R}^n)} \leq c\|u\|_{W^{k,p}(\mathbb{H}^n)} \leq c(k,p,n)\|u\|_{W^{k,p}(\mathbb{H}^n)}$$

In general: $\overline{\Omega} \subset U_0 \cap \ldots \cap U_n$
$\overline{U_0} \subset \Omega$ such that $\overline{U_j} \subset U_j' \subset \Omega'$

$$\psi_j : U_j' \to B_R \quad \psi_j(U_j \cap \Omega) = B_R \cap \mathbb{H}^n$$

$$\beta_j : \mathbb{R}^n \to [0,1] \quad \operatorname{supp} \beta_j \subset U_j \quad \sum \beta_J = 1 \text{ in } \Omega$$

$$Eu = \beta_0 u + \sum_{j=1}^n E_0(\underbrace{\beta_j u \circ \psi_j^{-1}}_{\text{on } B_R^+}) \circ \psi_j$$

$$\psi_j \in C^k\text{-diffeomorphism } (\Omega \text{ is } C^k\text{-domain})$$

$$\beta_j u \circ \circ \psi_j^{-1} \in W^{k,p}(B_R^+) \quad \text{Lemma 8}$$

$$E_0(\beta_j u \circ \psi_j) \in W^{k,p}(\mathbb{R}^n)$$

Compose with ψ_j + Lemma 8. □

Proof of Thm 3:

$k = 1$, $p < n$:

$$W^{1,p}(\Omega) \overset{E}{\hookrightarrow} W_0^{1,p}(\Omega') \text{ we choose } \Omega' \supset \overline{\Omega}$$

$$\overset{\text{Thm 2}}{\Rightarrow} L^{\frac{np}{n-p}}(\Omega') \overset{restr.}{\hookrightarrow} L^{\frac{np}{n-p}}(\Omega)$$

$p > n$:

$$W^{1,p}(\Omega) \overset{E}{\hookrightarrow} W_0^{1,p}(\Omega) \overset{\text{Thm 2}}{\hookrightarrow} C^{0,\mu}(\Omega') \hookrightarrow C^{0,\mu}(\Omega)$$

$k \geq 2$: Induction.

$p < n$:

$$W^{k,p}(\Omega) \hookrightarrow W^{k',p'}(\Omega')$$

$$k' = k-1 \quad p' = \frac{np}{n-p}$$

(Case $k = 1$ applied to $u, \ldots, \partial^{k-1}u$).

$$k' - \frac{n}{p'} = k - \frac{n}{p} = \mu \text{ if } k'p' = kp > n$$

$$W^{k,p}(\Omega) \hookrightarrow W^{k',p'}(\Omega) \hookrightarrow C^{0,\mu}(\Omega)$$

Primarly, if $kp = k'p' < n$

$$W^{k,p}(\Omega) \hookrightarrow W^{k',p'}(\Omega) \hookrightarrow L^{\frac{np'}{n-kp'}} = L^{\frac{np}{n-kp}}(\Omega)$$

□

Theorem 4: (Rellich)
$\Omega \subset \mathbb{R}^n$ open, **bounded**

(i) If $kp > n$ then the embedding

$$W_0^{k,p}(\Omega) \hookrightarrow C^0(\overline{\Omega})$$

is compact

(ii) If $kp < n$ and $1 \leq q < \frac{np}{n-kp}$ then the embedding

$$W_0^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$$

is compact.

If Ω is a *Lipschitz domain*, then both assertions remain valid with $W_0^{k,p}$ replaced by $W^{k,p}$.

Remark 1: In the proof we'll assume Ω is a C^1-domain.

Remark 2: The inclusion

$$W_0^{1,p}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$$

is not compact. (The condition that Ω is bounded cannot be removed in theorem 4).

Remark 3: $\Omega\mathbb{R}^2$

$$\Omega := \{(x,y) \in \mathbb{R}^2 \mid 0 < x < 1, 0 < y < \frac{1}{2}\} \cup \bigcup_{m \geq 0} \{x, y \mid \frac{1}{2} \leq y < 1, \frac{1}{2^{2n+1}} < x < \frac{1}{2^n}\}$$

In this case the inclusion

$$W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$$

is not compact. (The condition that Ω is a Lipschitz domain cannot be removed in theorem 4). Also

$$W^{1,2}(\Omega) \not\subseteq L^q \text{ for } q > 2$$

Remark 4:

$$W^{1,2}(\mathbb{R}^2) \hookrightarrow C^0(\mathbb{R}^2)$$

Exercise: Find a sequence of Lipschitz continuous functions

$$u_\nu : \mathbb{R}^2 \to [0,1] \quad \operatorname{supp} u_\nu \subset B_1$$

$$u_\nu(0) = 1 \quad \|u_\nu\|_{W^{1,2}} \to 0$$

Proof: (of theorem 4)

The case $kp > n$: Assume $0 < \mu = k - \frac{n}{p} < 1$. Then $W_0^{k,p}(\Omega) \subset C^{0,\mu}(\Omega)$ (Thm 2) and

(*) $$\|u\|_{C^{0,\mu}} \leq c\|u\|_{W^{k,p}} \quad \forall u \in W_0^{k,p}(\Omega$$

Let

$$\mathcal{F} := \{u \in W_0^{k,p}(\Omega) \mid \|u\|_{W^{k,p}} \leq 1\} \subset C^0(\overline{\Omega})$$

To show: $\mathcal{F}$ has a compact closure. By Arzelà-Ascoli this means:

- $\mathcal{F}$ is bounded
- $\mathcal{F}$ is equicontinuous

We know that for every $u \in \mathcal{F}$

$$\|u\|_{C^{0,\mu}} = \sup_{x\in\overline{\Omega}} |u(x)| + \sup_{x,y\in\overline{\Omega} x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\mu} \leq c$$

$\Rightarrow \mathcal{F}$ is bounded
Pick $\varepsilon > 0$. Let $\delta := (\frac{\varepsilon}{c})^{\frac{1}{\mu}}$. Then for $|x-y| < \delta$ and $u \in \mathcal{F}$ we get

$$|u(x)-u(y)| \leq c \underbrace{|x-y|^\mu}_{\leq \frac{\varepsilon}{c}} \leq \varepsilon$$

so $\mathcal{F}$ is equicontinuous.
If Ω is a C^1-domain, then (*) holds for all $u \in W^{k,p}(\Omega)$ by Thm 3. So the same argument shows that the inclusion

$$W^{k,p}(\Omega) \to C^0(\overline{\Omega})$$

is compact.

The case $kp < n$: $q < \frac{np}{n-kp}$

Step 1: Let $\rho : \mathbb{R}^n \to \mathbb{R}$ be smooth, $\mathrm{supp}(\rho) \subset B_1$, $\int_{\mathbb{R}^n} \rho = 1$. Define $\rho_\delta(x) = \frac{1}{\delta^n}\rho(\frac{x}{\delta})$ and

$$(\rho_\delta u)(x) := \int_\Omega \rho_\delta(x-y)u(y)\,\mathrm{d}y = \rho_\delta * u(x)$$

$\Rightarrow$ The operator $\rho_\delta : L^1(\Omega) \to L^1(\Omega)$ is compact.

Proof of Step 1: ρ_δ is the composition

$$L^1(\Omega) \overset{u\mapsto\rho_\delta * u \text{ cpct}}{\rightarrow} C^0(\overline{(\Omega)} \overset{\text{inclusion, bded lin operator}}{\rightarrow} L^1(\Omega)$$

Claim: The set

$$\mathcal{F} := \{\rho_\delta * u \mid u \in L^1(\Omega), \|u\|_{L^1} = 1\} \subset C^0(\overline{\Omega})$$

has a compact closure. <u>$\mathcal{F}$ is bounded:</u>

$$|\rho_\delta * u(x)| \leq \sup|\rho_\delta| \circ \|u\|_{L^1} \leq \sup|\rho_\delta|$$

<u>$\mathcal{F}$ is equicontinuous:</u>
Let $\varepsilon > 0$. Choose $\lambda > 0$ such that $\forall \xi, \xi' \in \mathbb{R}^n$

$$|\xi - \xi'| < \lambda \Rightarrow |\rho_\delta(\xi) - \rho_\delta(\xi')| < \varepsilon$$

Suppose $x, x' \in \Omega$ with $|x-x'| < \lambda$ and $u \in L^1(\Omega), \|u\|_{L^1} \leq 1$. Define $u_\delta := \rho_\delta * u$, then

$$\begin{aligned} |u_\delta(x) - u_\delta(x')| &\leq \int_\Omega |\rho_\delta(x-y) - \rho_\delta(x'-y)||u(y)|\,\mathrm{d}y \\ &\leq \varepsilon \int_\Omega |u(y)|\,\mathrm{d}y \\ &= \varepsilon\|u\|_{L^1} \\ &\leq \varepsilon \end{aligned}$$

Step 2: $u \in W_0^{1,p}(\Omega), u_\delta := \rho_\delta * u$. Assume $\rho(x) \geq 0 \forall x$, then the claim is:

$$\|u - u_\delta\|_{L^1(\Omega)} \leq \delta \mathrm{Vol}(\Omega)^{1-\frac{1}{p}} \| \nabla u\|_{L^p(\Omega)}$$

Proof of Step 2: Suffices to assume $u \in C_0^\infty(\Omega)$. Extend u to $\mathbb{R}^n$ by $u(x) := 0$ for $x \notin \Omega$, then $u \in C_0^\infty(\mathbb{R}^n)$. Now:

$$\begin{aligned}
u(x) - u_\delta(x) &= u(x) - \int\limits_{\mathbb{R}^n} \frac{1}{\delta^n}\rho(\frac{y}{\delta}u(x-y)\,\mathrm{d}y \\
&= u(x) - \int\limits_{\mathbb{R}^n} \rho(y)u(x-\delta y)\,\mathrm{d}y \\
&= \int\limits_{\mathbb{R}^n} \rho(y)(u(x) - u(x-\delta y))\,\mathrm{d}y \\
&= -\int\limits_{\mathbb{R}^n} \rho(y)\left(\int\limits_0^\delta \frac{d}{dt}u(x-ty)\,\mathrm{d}t\right)\mathrm{d}y \\
&= \int\limits_{\mathbb{R}^n} \rho(y)\int\limits_0^\delta \langle \nabla u(x-ty), y\rangle\,\mathrm{d}t\,\mathrm{d}y \\
&= \int\limits_{|y|\leq 1}\int\limits_0^\delta \langle \nabla u(x-ty), y\rangle\,\mathrm{d}t\,\mathrm{d}y \\
|u(x) - u_\delta(x)| &\leq \int\limits_{|y|\leq 1}\int\limits_0^\delta |\langle \nabla u(x-ty), y\rangle|\,\mathrm{d}t\,\mathrm{d}y \\
&\leq \int\limits_{|y|\leq 1}\int\limits_0^\delta | \nabla u(x-ty)|\,\mathrm{d}t\,\mathrm{d}y
\end{aligned}$$

Integrate over x:

$$\begin{aligned}
\|u - u_\delta\|_{L^1} &= \int\limits_\Omega |u(x) - u_\delta(x)|\,\mathrm{d}x \\
&\leq \int\limits_{\mathbb{R}^n}\int\limits_{|y|\leq 1} \rho(y)\int\limits_0^\delta | \nabla u(x-ty)|\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}x \\
&= \int\limits_{|y|\leq 1} \rho(y)\int\limits_0^\delta\int\limits_{\mathbb{R}^n} | \nabla u(x-ty)|\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y \\
&= \int\limits_{|y|\leq 1} \rho(y)\int\limits_0^\delta \| \nabla \|_{L^1}\,\mathrm{d}t\,\mathrm{d}y \quad (\Delta)
\end{aligned}$$

$$(\Delta) = \delta \| \nabla u\|_{L^1(\Omega)} \overset{\text{Hölder}}{\leq} \delta \mathrm{Vol}(\Omega)^{1-1/p} \| \nabla \|_{L^p(\Omega)}$$

Note:

$$\|fg\|_{L^1} \leq \|f\|_{L^p}\|g\|_{L^q}$$

$$\frac{1}{p} + \frac{1}{q} = 1 \quad f = \nabla u \quad g = 1$$

$$\|g\| = \left(\int_\Omega 1^q\right)^{1/q} = \mathrm{Vol}(\Omega)^{1/q} = \mathrm{Vol}(\Omega)^{1-1/p}$$

Step 3: The inclusion

$$\iota_p : W_0^{1,p}(\Omega) \to L^1(\Omega)$$

is compact.

Proof of Step 3: By Step 2

$$\begin{aligned}
\|\rho_\delta \circ \iota_p(u) - \iota_p(u)\|_{L^1(\Omega)} &= \|u - u_\delta\| \\
&\leq \delta \mathrm{Vol}(\Omega)^{1-\frac{1}{p}} \\
\|\rho_\delta \circ \iota_p - \iota_p\| &= \sup_{u \in W_0^(1,p} \frac{\|u - u_\delta\|}{\|u\|_{W^{1,p}}} \\
&\leq \delta \mathrm{Vol}(\Omega)^{1-\frac{1}{p}} \to 0
\end{aligned}$$

By Step 1, $\rho_\delta \circ \iota_p : W_0^{1,p}(\Omega) \to L^1(\Omega)$ is compact for every $\delta > 0 \Rightarrow \iota_p$ compact!

Step 4: For $1 \leq q < \frac{np}{n-p}$ the inclusion $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact.

Proof of Step 4: Claim: For $1 \leq q < \frac{np}{n-p}$

(*) $$\|u\|_{L^q} \leq \|u\|_{L^1}^\lambda \|u\|_{L^{\frac{np}{n-p}}}^{1-\lambda}$$

$$\frac{1}{q} = \lambda + (1-\lambda)\frac{n-p}{np} \quad 0 < \lambda \leq 1$$

Let $u_\nu \in W_0^{1,p}$ be a bounded sequence $\overset{\text{Step 3}}{\Rightarrow} \exists$ subsequence u_{ν_i} which converges in $L^1(\Omega)$. By Thm 2, the sequence u_{ν_i} is bounded in $L^{\frac{np}{n-p}}(\Omega)$.

$$\overset{(*)}{\Rightarrow} \|u_{\nu_i} - u_{\nu_j}\|_{L^q} \leq \underbrace{\|u_{\nu_i} - u_{\nu_j}\|_{L^1}^\lambda}_{\to 0 \text{ for } i,j\to\infty} \underbrace{\left(\|u_{v_i}\|_{L^{\frac{np}{n-p}}} + \|u_{\nu_j}\|_{L^{\frac{np}{n-p}}}\right)^{1-\lambda}}_{\text{bounded}}$$

$\Rightarrow u_{\nu_i}$ is a Cauchy sequence in $L^q(\Omega)$
$\Rightarrow u_{\nu_i}$ converges in $L^q(\Omega)$

Step 5: For $1 \leq q < \frac{np}{n-kp}$ the inclusion

$$W_0^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$$

is compact.

Proof of Step 5: <u>Induction:</u> By Thm 2

$$W_0^{k,p}(\Omega) \hookrightarrow W_0^{k-1,\frac{np}{n-p}}(\Omega)$$

$$k' := k-1, p' := \frac{np}{n-p} \Rightarrow \frac{np'}{n-k'p'} = \frac{np}{n-kp} > q$$

$\overset{\text{Ind. hyp.}}{\Rightarrow}$ The inclusion

$$W_0^{k',p'}(\Omega) \hookrightarrow L^q(\Omega)$$

is compact $\Rightarrow$ Step 5.

Step 6: ΩC^1-domain. $1 \leq q < \frac{np}{n-p} \Rightarrow$ the inclusion

$$W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$$

is compact.

Proof of Step 6: By Lemma 7 $\exists$ extension operator

$$E : W^{1,p}(\Omega) \to W_0^{1,p}(\Omega')$$

for any (bounded) open set $\Omega' \subset \mathbb{R}^n$ with $\overline{\Omega} \subset \Omega'$, satisfying

$$(Eu)|_\Omega = u$$

$$W^{1,p}(\Omega) \overset{E \text{bounded}}{\to} W_0^{1,p}(\Omega') \overset{\text{cpct by Step 4}}{\to} L^q(\Omega') \overset{bdedrest.}{\to} L^q(\Omega)$$

Step 7: $\Omega \subset \mathbb{R}^n$ bounded, open, C^1-domain. $k \in \mathbb{N}$ with $1 \leq q < \frac{np}{n-kp}$, where $kp < n$. $\Rightarrow$ The inclusion

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$$

is compact.

Proof of Step 7:

$k = 1$: Step 6

$k \geq 2$: Inclusion as in Step 5 with Thm 3 instead of Thm 2.

□

Lemma 9: $1 \leq p \leq q \leq r \leq \infty$, $0 \leq \lambda \leq 1$, $\frac{1}{q} = \lambda \frac{1}{p} + (1-\lambda)\frac{1}{p}$

$$\Rightarrow \|u\|_{L^q} \leq \|u\|_{L^p}^{\lambda} \|u\|_{L^p}^{1-\lambda}$$

Proof:

$$1 = \frac{\lambda q}{p} + \frac{(1-\lambda)q}{r} \quad p' := \frac{p}{\lambda q} \quad r' := \frac{r}{(1-\lambda)q}$$

$$\Rightarrow \frac{1}{p'} + \frac{1}{r'} = 1$$

$$\begin{aligned}
\Rightarrow \int |u|^q &= \int |u|^{\lambda q} |u|^{(1-\lambda)q} \\
&\overset{\text{Hölder}}{\leq} \left(\int |u|^{\lambda q p'} \right)^{\frac{1}{p'}} \left(\int |u|^{(1-\lambda)qr'} \right)^{\frac{1}{p'}} \\
&= \left(\int |u|' p \right)^{\frac{\lambda q}{p}} \left(\int |u|^p \right)^{\frac{(1-\lambda)q}{r}} \\
&= \|u\|_{L^p}^{\lambda q} \|u\|_{L^p}^{(1-\lambda)q}
\end{aligned}$$

□

Interpolation:

Remark:

$$W^{j,q}(\mathbb{R}^n \supset W^{k,p}(\mathbb{R}^n) \Leftrightarrow j - \frac{n}{q} \leq k - \frac{n}{p} \quad j \leq k$$

(except when $k - \frac{n}{p} = j \quad q = \infty$).

Note:

$$L^q \supset W^{k-j,p} \Leftrightarrow q \leq \frac{np}{n-(k-j)p} \text{ if } (k-j)p < n$$

(or ofr $q < \infty$ if $(k-j)p = n$ or for any $q \leq \infty$ if $(k-j)p > n$).

$$W^{k,p} \subset W^{j,q} \subset L^r$$

Want:

$$\|u\|_{j,q} \leq c\|u\|_{k,p}^{\lambda}\|u\|_{L^r}^{1-\lambda}$$

What's the "right" value of λ?

Theorem 5: (Gagliardo-Nirenberg)
$\Omega \subset \mathbb{R}^n$ bounded open C^k-domain with $1 \leq p, q, r \leq \infty$ and $j, k \in \mathbb{N}_0$.
Assume $0 \leq j < k$

$$-\frac{n}{r} \leq j - \frac{n}{q} \leq k - \frac{n}{p}$$

and, if $(k-j)p = n$ assume also that $\lambda \neq 1$.

$$j - \frac{n}{q} = \lambda(k - \frac{n}{p}) + (1-\lambda)(\frac{n}{r} \quad \frac{j}{k} \leq \lambda \leq 1$$

$$\Rightarrow \exists c > 0 \forall u \in W^{k,p}(\Omega)$$

$$\|u\|_{j,p} \leq c\|u\|_{k,p}^{\lambda}\|u\|_{0,r}^{1-\lambda}$$

Proof: Arnos Friedman, PDEs, Halt-Rinenart-Winston 1969 □

Example 1:

$$j = 0 \quad kp < n \quad r \leq q \leq \frac{np}{n-kp}$$

$$\lambda = \frac{\frac{1}{r} - \frac{1}{q}}{\frac{1}{r} - \frac{n-kp}{np}}$$

$$\frac{1}{q} = \lambda\frac{n-kp}{np} + (1-\lambda)\frac{1}{r}$$

$$\overset{\text{Lemma 9}}{\Rightarrow} \|u\|_{L^1} \leq \|u\|_{L^{\frac{np}{n-kp}}}^{\lambda}\|u\|_{L^r}^{1-\lambda}$$

By Thm 3 $\|u\|_{L^{\frac{np}{n-kp}}} \leq c\|u\|_{W^{k,p}}$
Remains true for $kp > n$ and any q. Proof in this case more difficult.

Example 2: $p = q = r \quad \lambda = \frac{j}{k}$

$$\|u\|_{W^{j,p}} \leq \|u\|_{W^{k,p}}^{\frac{j}{k}}\|u\|_{L^p}^{1-\frac{j}{k}}$$

Example 3: $jq = kp > n \quad r = \infty$

$$\|u\|_{j,\frac{kp}{j}} \leq \|u\|_{k,p}^{\frac{j}{k}}\|u\|_{L^\infty}^{1-\frac{j}{k}}$$

Theorem 6: (product estimate)
$\Omega \subset \mathbb{R}^n$ bounded open C^k-domain, $kp > n$

$$\Rightarrow \exists c > 0 \quad \forall u, v \in W^{k,p}(\Omega)$$

$$\|uv\|_{k,p} \leq c\left(\|u\|_{k,p}\|v\|_{L^\infty} + \|u\|_{L^\infty}\|v\|_{k,p}\right)$$

Proof: Let $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = k$, $u, v \in C^\infty(\overline{\Omega})$

$$\partial^\alpha(uv) = \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\beta-\alpha)!} \partial^\beta u \partial^{\alpha-\beta} v$$

$$|\beta| = i \quad |\alpha - \beta| = j \quad i + j = k$$

$$\|\partial^\beta u \partial^{\alpha-\beta} v\|_{L^p} \leq \|\partial^\beta u\|_{L^q} \|\partial^{\alpha-\beta} v\|_{L^r}$$

$$\frac{1}{q} + \frac{1}{r} = \frac{1}{p}$$

choose $q = \frac{kp}{i}$ and $r = \frac{kp}{j}$

$$\begin{aligned}
\|\partial^\beta u \partial^{\alpha-\beta} v\|_{L^p} &\leq \|u\|_{i,\frac{kp}{i}} \|v\|_{j,\frac{kp}{j}} \\
&\overset{\text{Example 3}}{\leq} c\|u\|_{k,p}^{i/k} \|u\|_{L^\infty}^{j/k} \|v\|_{k,p}^{j/k} \|v\|_{L^\infty}^{i/k} \\
&\leq c(\|u\|_{k,p}\|v\|_{L^\infty})^{i/k} (\|v\|_{k,p}\|u\|_{L^\infty})^{j/k}
\end{aligned}$$

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q \quad p = \frac{k}{i} \quad q = \frac{k}{j}$$

□

Proof: (of Thm 5 in the case $p = q = r$:)

(I) $$u : [0,1] \to \mathbb{R} \text{ smooth}$$

$$\Rightarrow |u'(0)|^p \leq 2^{p-1} 9^p \int_0^1 (|u(t)|^p + |u''(t)|^p) \, \mathrm{d}t$$

$$u'(0) = u'(t) - \int_0^t u''(s) \, \mathrm{d}s \text{ integrate from x to y}$$

$$(y-x)u'(0) = u(y) - u(x) - \int \int_x^y u''(s) \, \mathrm{d}s \, \mathrm{d}t$$

$$\Rightarrow \frac{1}{3}|u'(0)| \leq |u(x)| + |u(y)| + \frac{1}{3} \int_0^1 |u''(s)| \, \mathrm{d}s$$

integrate over x and y

$$\Rightarrow \frac{1}{9}|u'(0)| \leq \int_0^{1/3} |u(x)| \, \mathrm{d}x + \int_{2/3}^1 |u(y)| \, \mathrm{d}y + \frac{1}{9} \int_0^1 |u''(s)| \, \mathrm{d}s$$

$$|u'(0)| \leq 9 \left(\int_0^1 |u(t)| \, \mathrm{d}t + \int_0^1 |u''(t)| \, \mathrm{d}t \right)$$

use $(a+b) \leq 2^{p-1}(a^p + b^p)$

(II) $$\forall \varepsilon > 0 \quad \forall u \in C_0^\infty(\mathbb{R})$$

$$\int_{\mathbb{R}} |u'|^p \leq 2^{p-1} 9^p \int_{\mathbb{R}} \left(\varepsilon^{-p}|u|^p + \varepsilon^p |u''|^p \right)$$

Proof of (II): By (I), we have for every $x \in \mathbb{R}$

$$|u'(x)|^p \leq 2^{p-1}9^p \int_0^1 (|u(x+t)|^p + |u''(x+t)|^p)\, \mathrm{d}t$$

integrate over x.

$$\Rightarrow \int_{\mathbb{R}} |u'(x)|^p \, \mathrm{d}x \leq 2^{p-1}9^p \int \int_0^1 (|u(x+t)|^p + |u''(x+t)|^p \, \mathrm{d}x \, \mathrm{d}t$$

$$= 2^{p-1}9^p \int_{\mathbb{R}} (|u(x)|^p + |u''(x)|^p)\, \mathrm{d}x$$

Let $v(x) := u(\varepsilon x) \Rightarrow v'(x) = \varepsilon u'(\varepsilon x)$ and $v''(x) = \varepsilon^2 u''(\varepsilon x)$

$$\int |v|^p = \varepsilon^{-1} \int |u|^p$$

$$\int |v'|^p = \varepsilon^{p-1} \int |u'|^p$$

$$\int |v''|^p = \varepsilon^{2p-1} \int |u''|^p$$

we know

$$\int |v'|^p \leq 2^{p-1}9^p \int (|v|^p + |v''|^p)$$

$$\int |v'|^p = \varepsilon^{p-1} \int |u'|^p \leq 2^{p-1}9^p \int (\varepsilon^{-1}|u|^p + \varepsilon^{2p-1}|u'|^p) \Rightarrow (II)$$

(III)

$$\forall u \in C_0^\infty(\mathbb{R}^n)$$

$$\|u\|_{1,p} \leq \left((1 + n2^{p-1}9^p)2\right)^{1/p} \|u\|_{2,p}^{1/2} \|u\|_{L^p}^{1/2}$$

By (II), for $t \mapsto u(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)$ we get

$$\int_{\mathbb{R}} |\partial_i u|^p \, \mathrm{d}x_i \leq 2^{p-1}9^p \int_{\mathbb{R}} (\varepsilon^{-p}|u|^p + \varepsilon^p |\partial_i^2 u|^p)\, \mathrm{d}x_i$$

integrate over $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$:

$$\Rightarrow \int_{\mathbb{R}^n} |\partial_i u|^p \leq 2^{p-1}9^p \int_{\mathbb{R}^n} (\varepsilon^{-p}|u|^p + \varepsilon^p |\partial_i^2 u|^p)$$

$$\int_{\mathbb{R}^n} |u|^p \leq \int_{\mathbb{R}^n} \varepsilon^{-p}|u|^p \quad 0 < \varepsilon \leq 1$$

$$\begin{aligned}
\Rightarrow \|u\|_{1,p}^p &= \int_{\mathbb{R}^n} (|u|^p + \sum |\partial_i u|^p) \\
&\leq (1 + n2^{p-1}9^p) \int_{\mathbb{R}^n (\varepsilon^{-p}|u|^p + \varepsilon^p \sum_{i=1}^n |\partial_i^2 u|^p} \\
&\leq (1 + n2^{p-1}9^p)(\varepsilon^{-p}\|u\|_{L^p}^p + \varepsilon^p \|u\|_{2,p}^p) \\
&\leq (1 + n2^{p-1}9^p) 2 \|u\|_{L^p}^{p/2} \|u\|_{k,p}^{p/2}
\end{aligned}$$

where ε is such that $\varepsilon := \left(\frac{\|u\|_{L^p}}{\|u\|_{k,p}}\right)^{\frac{1}{2}}$
$\Rightarrow$ (III)

(IV)

$$\forall k \geq 2 \forall p \in [0, \infty] \forall n \geq 1$$

$$\exists c = c(k,p,n) > 0 \quad \forall u \in C_0^\infty(\mathbb{R}^n)$$

$$\|u\|_{j,p} \leq c \|u\|_{k,p}^{j/k} \|u\|_{0,p}^{1-j/k}$$

$k = 2$: (III)

$k \geq 3$: Induction:
By induction hypothesis $\exists c \forall u \in C_0^\infty(\mathbb{R}^n) \forall 0 \leq j \leq l < k$

$$\|u\|_{j,p} \leq c\|u\|_{l,p}^{j/l}\|u\|_{0,p}^{1-j/l}$$

$\Rightarrow$ for $0 \leq j \leq k-1$

$$\begin{aligned} \|\partial_i u\|_{j,p} &\leq c\|\partial_i u\|^{\frac{j}{k-1}}\|\partial_i u\|_{1-\frac{j}{k-1}} \\ &\leq c\|u\|_{k,p}^{\frac{j}{k-1}}\|u\|_{1,p}^{\frac{k-1-j}{k-1}} \\ \Rightarrow \|u\|_{j+1,p} &\leq (n+1)c\|u\|_{k,p}^{\frac{j}{k-1}}\|u\|_{1,p}^{\frac{k-1-j}{k-1}} \end{aligned}$$

Also:

$$\|u\|_{1,p} \leq c\|u\|_{j+1,p}^{\frac{1}{j+1}}\|u\|_{0,p}^{\frac{j}{j+1}}$$

$$\Rightarrow \|u\|_{j+1,p} \leq c_1\|u\|_{k,p}^{\frac{j}{k-1}}\|u\|_{j+1,p}^{\frac{k-1-j}{(k-1)(j+1)}}\|u\|_{0,p}^{\frac{j(k-1-j)}{(k-1)(j+1)}}$$

$$\Rightarrow \|u\|_{j+1,p}^{1-\frac{k-1-j}{(k-1)(j+1)}} \leq c_1\|u\|_{k,p}^{\frac{j}{k-1}}\|u\|_{0,p}^{\frac{j(k-1-j)}{(k-1)(j+1)}}$$

$$1 - \frac{k-1-j}{(k-1)(j+1)} = \frac{(k-1)(j+1)-k+1+j}{(k-1)(j+1)} = \frac{jk}{(j+1)(k-1)}$$

$$\Rightarrow \|u\|_{j+1,p} \leq c_2\|u\|_{k,p}^{\frac{j+1}{k}}\|u\|_{0,p}^{\frac{k-1-j}{k}} \quad 0 \leq j \leq k-1$$

$\Rightarrow$ (IV)

This proves Thm 5 for $p = q = r < \infty$ and $u \in C_0^\infty(\mathbb{R}^n)$.
For Ω a bounded C^k-domain use the Extension result Lemma 7. □

Theorem 7: (Trace theorem)
Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary $1 < p < \infty$. Then:

(i) $\exists c = c(n,p,\Omega) > 0 \forall u \in C^\infty(\overline{\Omega})$

$$\|u\|_{L^p(\partial\Omega)} \leq c\|u\|_{W^{1,p}(\Omega)}^{1/p}\|u\|_{L^p(\Omega)}^{1-1/p}$$

(ii) Let $u \in W^{k,p}(\Omega)$. Then:

$$u \in W_0^{k,p}(\Omega) \Leftrightarrow \partial^\alpha u|_{\partial\Omega} = 0 \forall \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq k-1$$

Proof:

(i) $\nu : \partial\Omega \to S^{n-1}$ outward unit normal. Choose $f : \mathbb{R}^n \to \mathbb{R}^n$ smooth such that $f|_{\partial\Omega} = \nu$

$$\begin{aligned} \int_{\partial\Omega} |u|^p \, \mathrm{d}S &\overset{(*)}{=} \int_\Omega \mathrm{div}(|u|^p f) \\ &\overset{(\Delta)}{=} \int_\Omega ((\mathrm{div} f)|u|^p + p\langle f, \nabla u\rangle u|u|^{p-2}) \\ &\leq c\int_\Omega |u|^p + |\nabla u||u|^{p-1} \end{aligned}$$

(*), because $\langle |u|^p f, \nu\rangle|_{\partial\Omega} = |u|^p$
(Δ), because $\nabla|u|^p = p|u|^{p-1}\mathrm{sign}(u)\nabla u$
$c = \sup|\mathrm{div} f| + p \cdot \sup|f|$

$$\begin{aligned} \int_{\partial\Omega} |u|^p \, \mathrm{d}S &\leq c\int_\Omega |u|^{p-1}(|u| + |\nabla u|) \\ &\leq \left(\int_\Omega (|u| + |\nabla u|)^p\right)^{1/p}\left(\int_\Omega |u|^{(p-1)q}\right)^{1/q} \\ &= c\|u\|_{1,p}^{1/p}\|u\|_{L^p}^{1-1/p} \end{aligned}$$

□

ATTENTION: 2 pages of information are lacking!!!
$\Omega \subset \mathbb{R}^n$ bounded open (set), smooth boundary. $1 < p < \infty$.
We proved:

$$\exists c > 0 \quad \forall u \in C^\infty(\overline{\Omega})$$

$$\|u|_{\partial\Omega}\|_{L^p} \leq c\|u\|^{1/p}_{W^{1,p}(\Omega)}\|u\|^{1-1/p}_{L^p(\Omega)}$$

(*)
$$(\leq c\|u\|_{W^{1,p}(\Omega)}$$

Consider the linear operator

$$\begin{array}{ccc} C^\infty(\overline{\Omega}) & \to & C^\infty(\partial\Omega) \\ \bigcap & u \mapsto u|_{\partial\Omega} & \bigcap \\ W^{1,p}(\Omega) & & L^p(\partial\Omega) \end{array}$$

This operator is bounded wrt. the $W^{1,p}$-norm on $C^\infty(\overline{\Omega})$ and the L^p-norm on $C^\infty(\partial\Omega)$. Hence the operator $u \mapsto u|_{\partial\Omega}$ extends uniquely to a bounded linear operator

$$T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

This extended operator is called the *trace operator*.

What is T_u? Given u, choose a sequence $u_i \in C^\infty(\overline{\Omega})$ converging to u in the $W^{1,p}$-norm. Then u_i is Cauchy wrt. $W^{1,p}$-norm. By the inequalities (*) the sequence $u_i|_{\partial\Omega}$ is Cauchy wrt. the L^p-norm on $C^\infty(\partial\Omega)$. So $u_i|_{\partial\Omega}$ converges in $L^p(\partial\Omega)$ and we define

$$T_u := \lim_{i\to\infty} u_i|_{\partial\Omega}$$

Notation:

$$u|_{\partial\Omega} := T_u$$

$$\ker T = \{u \in W^{1,p} \mid \forall u_i \in C^\infty(\Omega), u_i \overset{W^{1,p}}{\to} u, \lim_{i\to\infty} \|u_i|_{\partial\Omega}\|_{L^p} = 0\}$$

Thm 7, (ii) says:

$$\ker T = W^{1,p}_0(\Omega)(= \text{ closure of } C^\infty_0(\Omega) \text{ in } W^{1,p}(\Omega)$$

Lemma 10: $\Omega \subset \mathbb{R}^n$ open, bounded, smooth boundary, $1 < p < \infty$.
Let $u \in W^{1,p}(\Omega)$ with $u|_{\partial\Omega} = 0$, ie. $u \in \ker T$. Define

$$\widehat{u} : \mathbb{R}^n \to \mathbb{R}$$

by

$$\widehat{u}(x) := \begin{cases} u(x) & x \in \Omega \\ 0 & x \notin \Omega \end{cases}$$

$$\Rightarrow \widehat{u} \in W^{1,p}_0(\mathbb{R}^n)$$

Proof: Define $\widehat{u}_j : \mathbb{R}^n \to \mathbb{R}$ by

$$\widehat{u}_j(x) := \begin{cases} \partial_j u(x) & x \in \Omega \\ 0 & x \notin \Omega \end{cases}$$

Claim:

$$\int_{\mathbb{R}^n} ((\partial_j\phi)\widehat{u} + \phi\widehat{u}_j) = 0 \quad \forall \phi \in C_0^\infty(\mathbb{R}^n)$$

true by definition if $\operatorname{supp}\phi \subset \Omega$. To show:

$$\int_\Omega ((\partial_j\phi)u + \phi(\partial_j u)) = 0 \quad \forall \phi \in C^\infty(\overline{\Omega})$$

Take a sequence $u_i \overset{W^{1,p}}{\to} u \quad u_i \in C^\infty(\overline{\Omega})$

$$\begin{aligned}\Rightarrow \int_\Omega ((\partial_j\phi)u + \phi(\partial_j u)) &= \lim_{i\to\infty} \int_\Omega ((\partial_j\phi)u_i + \phi\partial_j u_i) \\ &= \lim_{i\to\infty} \int_\Omega \partial_j(\phi u_i) \\ &= \lim_{i\to\infty} \int_{\partial\Omega} \phi u_i \nu_j = 0\end{aligned}$$

where $\nu : \partial\Omega \to S^{n-1} \quad \nu(x) \perp T_x\partial\Omega$ Let $u \in W^{1,p}(\Omega) \quad u|_{\partial\Omega} = 0$

To show: $\exists$ sequence $u_i \in C_0^\infty(\Omega)$ such that $\lim_{i\to\infty} \|u - u_i\|_{W^{1,p}(\Omega)} = 0$
Let $x_0 \in \partial\Omega$. Since $\partial\Omega$ is smooth $\exists U \subset \mathbb{R}^n$ open, $x_0 \in U$ $\exists \psi : U \to (-1,1)^n$ diffeomorphism such that

$$\psi(x_0) = 0$$
$$\psi(U \cap \partial\Omega) = \{y_n = 0\}$$
$$\psi(U \cap \Omega) = \{y_n > 0\}$$

Choose open neighbourhood W of x_0 such that $\overline{W} \subset U$. Suppose $u(x) = 0 \forall x \in \Omega \setminus W$. Denote $e_n := (0,\dots,0,1) \in \mathbb{R}^n$ and define

$$u^h : \Omega \to \mathbb{R}$$

by

$$u^h(x) := u(\psi^{-1}(\psi(x) - he_n)) \quad x \in U \cap \Omega$$

where $u(x) := 0$ for $x \notin \Omega$ and $u^h(x) := 0$ for $x \in \Omega \setminus U$.
$\overset{\text{Observations}}{\Rightarrow}$ $u^h \in W^{1,p}(\Omega)$ and u^h vanishes near $\partial\Omega$. So $u^h \in W_0^{1,p}(\Omega)$. Moreover

$$u^h \circ \psi^{-1}(y) = u \circ \psi^{-1}(y - he_n)$$

$$\Rightarrow \lim_{h\to\infty} \|u^h \circ \psi^{-1} - u \circ \psi^{-1}\|_{W^{1,p}((-1,1)^n)} = 0$$

$$\Rightarrow \lim_{h\to\infty} \|u^h - u\|_{W^{1,p}} = 0$$

$\Rightarrow$ because $W_0^{1,p}(\Omega)$ is a closed subspace of $W^{1,p}(\Omega)$ we have $u \in W_0^{1,p}(\Omega)$. This proves Thm 7 for $k = 1$ and u supported in a small neighbourhood of a boundary-point.

General Case: (still $k = 1$)
Cover $\partial\Omega$ by finitely many such neighbourhoods $W_1, \dots, W_N$. Choose a partition of unity subordinate to $W_1, \dots, W_N$

$$\sum_i \rho_i = 1 \quad \rho_i : \mathbb{R}^n \to [0,1]$$

near $\partial\Omega \quad \operatorname{supp}\rho_i < W_i$

$$\Rightarrow u = \underbrace{(u - \sum_i \rho_i u)}_{W_0^{1,p}(\Omega)} + \sum_i \underbrace{\rho_i u}_{W_0^{1,p}(\Omega)} \in W_0^{1,p}(\Omega)$$

□

Exercise: Extend the proof to the case $k \geq 2$.

Key point: in Lemma 10:
Assume $u \in W^{k,p}(\Omega)$ and

$$\underbrace{\partial^\alpha u}_{\in W^{1,p}} |_{\partial\Omega} = 0 \quad \forall \alpha \in \mathbb{N}_0^n, |\alpha| \leq k-1$$

Prove: $\widehat{u} \in W_0^{k,p}(\mathbb{R}^n)$.
Back to Thm 5.

Interpolation: $0 \leq j \leq k \quad p \geq 2$ and $\Omega \subset \mathbb{R}^n$ bounded open.

$$\Rightarrow \exists c > 0 \forall u \in C_0^\infty(\Omega)$$

(*)
$$\|u\|_{W^{j,\frac{kp}{j}}} \leq c\|u\|_{k,p}^{j/k} \|u\|_{L^\infty}^{j/k}$$

(special case of Thm 5, $q = \frac{kp}{j}, r = \infty, \lambda = \frac{j}{k}$).

Notation:

$$\partial^k u(x) := (\partial^\alpha u(x))_{\alpha \in \mathbb{N}_0^n |\alpha| = k}$$

$$\partial^k u(x) := \sqrt{\sum_{|\alpha|=k} |\partial^\alpha u(x)|^2}$$

$$\|\partial^k u\|_{L^p} := \left(\int_\Omega |\partial^k u|^p\right)^{1/p}$$

Lemma 11:

$$\forall p \geq 2 \forall k \in \mathbb{N} \exists c > 0 \forall j \in \{0, 1, \ldots, k\} \forall u \in C_0^\infty(\mathbb{R}^n):$$

$$\|\partial^j u\|_{L^{\frac{kp}{j}}} \leq c\|\partial^k u\|_{L^p}^{\frac{j}{k}} \|u\|_{L^\infty}^{1-\frac{j}{k}}$$

Remark 1: Lemma 11 and Poincaré inequality $\Rightarrow$ (*) for $p \geq 2$.

Remark 2: Lemma 11 is obvious for $j = 0, k$. First nontrivial case is $k = 2, j = 1$.

Lemma 12: $\forall p \geq 1 \forall u \in C_0^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} |\partial_i^1 u|^{2p} \leq n^p (2p-1)^p \int_{\mathbb{R}^n} |u|^p |\partial^2 u|^p$$

Proof: Let $\alpha \geq 0$ and define

$$f : \mathbb{R} \to \mathbb{R} \quad f(y) := y|y|^\alpha$$

$\Rightarrow f$ is C^1 and $f'(y) = (\alpha+1)|y|^\alpha$
$\overset{\alpha=2p-2}{\Rightarrow}$ The function $uf(\partial_i u)$ is C^1 and

$$\begin{aligned}\partial_i(uf(\partial_i u)) &= \partial_i u f(\partial_i u) + uf'(\partial_i u)\partial_i^2 u \\ &= |\partial_i u|^{2^p} + (2p-1)u\partial_i^2 u|\partial_i u|^{2p-2}\end{aligned}$$

$$\Rightarrow \int_{\mathbb{R}^n} |\partial_i u|^{2p} = -(2p-1) \cdot \int_{\mathbb{R}^n} u \partial_i^2 u |\partial_i u|^{2p-2}$$

$$\begin{aligned}
\Rightarrow \int_{\mathbb{R}^n} |\partial_i u|^{2p} &\leq (2p-1) \int_{\mathbb{R}^n} |u\partial_i^2 u||\partial_i u|^{2p-2} \\
&\overset{\text{Holder}}{\leq} (2p-1) \left(\int_{\mathbb{R}^n |u\partial_i^2 u|^p} \right)^{1/p} \left(\int_{\mathbb{R}^n |\partial_i u|^{2p}} \right)^{\frac{p-1}{p}} \\
\Rightarrow \left(\int_{\mathbb{R}^n} |\partial_i u|^{2p} \right)^{1/p} &\leq (2p-1) \left(\int_{\mathbb{R}^n} |u \partial_i^2 u|^p \right)^{1/p} \\
\Rightarrow \int |\partial_i u|^{2p} &\leq (2p-1)^p \int |u \partial_i u|^p \quad (\Delta)
\end{aligned}$$

Now use

$$\begin{aligned}
|\partial^1 u|^{2p} &= \left(\sum |\partial_i u|^2 \right)^p \\
&\leq \left(h \cdot \max |partial_i u|^2 \right)^p \\
&= h^p \max |\partial_i u|^{2p}
\end{aligned}$$

$$\Rightarrow (\Delta) \leq (2p-1)^p \int |u|^p |\partial^2 u|^p$$

$$\Rightarrow \int |\partial^1 u|^{2p} \leq h^p (2p-1)^p \int |u|^p |\partial^2 u|^p$$

□

Proof: (of Lemma 11):
Induction on k

$k = 1$: nothing to prove

$k = 2$: Lemma 12

$k \geq 3$: Assume the result holds with k replaced by $k-1$.
Must prove result holds for k and $j = 1, \ldots, k-1$.

Case 1: $j = k - 1$**:** Apply Lemma 12 to $\partial^\alpha u$ with $|\alpha| = k - 1$.

$$\Rightarrow \exists c_1 > 0 \quad \forall u \in C_0^\infty(\mathbb{R}^n)$$

$$\int_{\mathbb{R}^n} |\partial^{k-1} u|^{\frac{kp}{k-1}} \leq c_1 \int_{\mathbb{R}^n} |\partial^{k-2} u|^{\frac{kp}{2k-2}} |\partial^k u|^{\frac{kp}{2k-2}}$$

Use Hölder with $p_1 = \frac{2k-2}{k-2}$ $p_2 = \frac{2k-2}{k}$. So $\frac{1}{p_1} + \frac{1}{p_2} = 1$

(1)
$$\int |\partial^{k-1} u|^{\frac{kp}{k-1}} \leq c_1 \left(\int |\partial^{k-1} u|^{\frac{kp}{k-2}} \right)^{\frac{k-2}{2k-2}} \left(\int |\partial^k u|^p \right)^{\frac{k}{2k-2}}$$

use induction hypothesis

$$\int |\partial^{k-2} u|^{\frac{(k-1)q}{k-2}} \leq c_2 \sup |u|^{\frac{1}{k-2}} \int |\partial^{k-1} u|^q$$

choose $q := \frac{kp}{k-1} \geq 2$

(2)
$$\int |\partial^{k-2} u|^{\frac{kp}{k-2}} \leq c_2 \sup |u|^{\frac{kp}{(k-1)(k-2)}} \int |\partial^{k-1} u|^{\frac{kp}{k-1}}$$

$$\overset{(1)\text{ and }(2)}{\Rightarrow} \int |\partial^{k-1}|^{\frac{kp}{k-1}} \leq c_1 c_2^{\frac{k-2}{2k-2}} \sup |u|^{\frac{kp}{(k-1)(2k-2)}} \cdot \left(\int |\partial^{k-1} u|^{\frac{kp}{k-1}}\right)^{\frac{k-2}{2k-2}} \cdot \left(\int |\partial^k u|^p\right)^{\frac{k}{2k-2}}$$

$$\Rightarrow \left(\int |\partial^{k-1} u|^{\frac{kp}{k-1}}\right)^{\frac{k}{2k-2}} \leq c_1 c_2^{\frac{k-2}{2k-2}} \sup |u|^{\frac{p}{k-1}\frac{k}{2k-2}} \left(\int |\partial^k u|^p\right)^{\frac{k}{2k-2}}$$

(3) $$\int |\partial^{k-1} u|^{\frac{kp}{k-1}} \leq c_3 \sup |u|^{\frac{p}{k-1}} \int |\partial^k u|^p$$

Case 2: $1 \leq j \leq k-2$**:** use induction hypothesis with p replaced by $\frac{kp}{k-1}$

$$\begin{aligned}\int |\partial^j u|^{\frac{kp}{j}} &= \int |\partial^j u|^{\frac{k-1}{j}(\frac{kp}{k-1})} \\ &\leq c_4 \sup |u|^{\frac{(k-1-j)}{j}(\frac{kp}{k-1})} \int |\partial^{k-1}|^{\frac{kp}{k-1}} \\ &\overset{(3)}{\leq} c_3 c_4 \sup |u|^{\frac{k-1-j}{j}\frac{kp}{k-1}\frac{p}{k-1}} \int |\partial^k u|^p\end{aligned}$$

where we use that

$$\frac{k-1-j}{j}\frac{kp}{k-1}\frac{p}{k-1} = \frac{(k-1)kp-(k-1)jp}{j(k-1)} = \frac{(k-j)p}{j}$$

□

3 The Calderon-Zygmund Inequality:

Question: $\Omega \subset \mathbb{R}^n$ open, $f \in L^p_{\text{loc}}(\Omega)$. Suppose $u \in \mathrm{L}^1_{\text{loc}}(\Omega)$ is a weak solution of $\Delta u = f$

$$\overset{?}{\Rightarrow} u \in W^{2,p}_{\text{loc}}(\Omega)$$

Answer: Yes, if $1 < p < \infty$.

RECAP:

1. $u, f \in \mathrm{L}^1_{\text{loc}}(\Omega)$, u is a *weak solution* of $\Delta u = f$ if

$$\int_\Omega (\Delta\varphi) u = \int_\Omega \varphi f \quad \forall \varphi \in C_0^\infty(\Omega)$$

2. *Weyl's Lemma:* $u \in \mathrm{L}^1_{\text{loc}}(\Omega)$ weak solution of $\Delta u = 0$
$\Rightarrow u$ is smooth, harmonic.

3. $K(x) := \begin{cases} \frac{1}{\omega_n(2-n)}|x|^{2-n} & n > 2 \\ \frac{1}{2\pi}\log(|x|) & n = 2 \end{cases}$ fundamental solution.
$f \in L^1(\Omega)$, Ω bounded $\Rightarrow u := K * f \in L^1(\Omega)$ is a weak solution of $\Delta u = f$.
2. and 3. show: for $f \in L^p(\Omega)$, Ω bounded, we have

$$K * f \in W^{2,p}_{\text{loc}}(\Omega) \Leftrightarrow \text{every weak solution } u \text{ of } \Delta u = f \text{ is in } W^{2,p}_{\text{loc}}(\Omega)$$

4. The function $K_j(x) = \partial_j K(x) = \frac{x_j}{\omega_n |x|^n}$ is integrable near 0 and hence integrable on every compact set.

Lemma 1: $\Omega \subset \mathbb{R}^n$ bounded, $f \in C_0^2(\mathbb{R})$, $u := K * f \Rightarrow \partial_i u = K_i * f$.

Proof: Key observation:

(1)
$$\int_{\mathbb{R}^n} \partial_i(\varphi K) = 0 \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n)$$

$$\begin{aligned}
\int_{\mathbb{R}^n} \partial_i(\varphi K) &= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n\setminus B_\varepsilon} \partial_i(\varphi K) \\
&\overset{\text{Gauss}}{=} -\lim_{\varepsilon\to 0} \int_{\partial B_\varepsilon} -\frac{x_i}{\varepsilon}\varphi(x)K(x)dS \\
&= 0,
\end{aligned}$$

because, if $|x| = \varepsilon$ we have

$$\begin{aligned}
|-\frac{x_i}{\varepsilon} &\leq 1 \\
|\varphi(x)| &\leq \|\varphi\|_{L^\infty}
\end{aligned}$$

$|K(x)| = \begin{cases} \frac{c}{\varepsilon^{n-2}} & n \leq 3 \\ c|\log(\varepsilon)| & n = 2 \end{cases}$ Choose $\varphi(y) := f(x-y)$, then

$$\begin{aligned}
0 &= \int_{\mathbb{R}^n} f(x-y)K_i(y)\,\mathrm{d}y - \int_{\mathbb{R}^n} \partial_i f(x-y)K(y)\,\mathrm{d}y \\
&= (K_i * f)(x) - \partial_i u(x)
\end{aligned}$$

□

Corollary:

$$f \in L^p(\Omega) \quad u := K * f : \Omega \to \mathbb{R} \quad \Omega \text{ bounded}$$
$$\Rightarrow u \in W^{1,p}(\Omega) \quad \partial_i u = K_i * f$$

Proof: Choose a sequence $f_k \in C_0^\infty(\Omega)$ such that $\lim_{k\to\infty} \|f_k - f\|_{L^p(\Omega)} = 0$. Define $u_k(x) := (K * f_k)(x)$, $x \in \Omega$.

$$\overset{\text{Lemma 1}}{\Rightarrow} \partial_i u_k = K_i * f_k$$
$$\overset{\text{Young's ineq}}{\Rightarrow} \|\partial_i u_k - K_i * f\|_p = \|K_i * (f_k - f)\|_p$$
$$\leq \underbrace{\|K_i\|_{L^1(U)} \|f_k - f\|_p}_{\to 0}$$

$$K * f(x) = \int_\Omega K(x-y) f(y)\, \mathrm{d}y = (K|_U * f)(x) \quad x \in \Omega$$

where $U := \{x - y | x, y \in \Omega\}$ bounded

so $K|_U \in L^1(U)$ and

$$\left.\begin{array}{l} u_k \to K * f \\ \partial_i u_k \to K_i * f \end{array}\right\}$$

in $L^p(\Omega)$. □

<u>To show:</u> $f \in L^p(\Omega)$

$$\Rightarrow K_i * f \in W^{1,p}_{\text{loc}}(\Omega)$$

Theorem 1: (Calderon-Zygmund)
Let $n \in \mathbb{N}$ and $1 < p < \infty$

$$\Rightarrow \exists c > 0 \quad \forall f \in C_0^\infty(\mathbb{R}^n)$$

$$\| \bigtriangledown (K_i * f)\|_{L^p} \leq c\|f\|_{L^p} \quad (i = 1, \dots, n)$$

(Proof will follow later)

Remark: Let $u, f \in C_0^2(\Omega)$. Then $\Delta u = f \Leftrightarrow u = K * f$
"$\Rightarrow$" Chapter I, Lemma 1
"$\Leftarrow$" Chapter II, Lemma 2
Hence. for $u \in C_0^2(\Omega)$ we have $u = K * \Delta u$ and so $\partial_j u = K_j * \Delta u$. So

$$\partial_i \partial_j u = \partial_i (K_j * Deltau)$$

<u>Corollary of Thm. 1:</u>

$$u \in C_0^2(\mathbb{R}^n \Rightarrow \|\partial_i \partial_j u\|_p \leq c\|\Delta u\|_p \quad \forall i, j \in \{1, \dots, n\}$$

Theorem 2: (elliptic regularity for the Laplace equation)
$\Omega \subset \mathbb{R}^n$ open, $1 < p < \infty$, $f \in L^p_{\text{loc}}(\Omega)$. Let $u \in \mathrm{L}^1_{\text{loc}}(\Omega)$ be a weak solution of $\Delta u = f \Rightarrow u \in W^{2,p}_{\text{loc}}(\Omega)$ Thm. 1 $\Rightarrow$ Thm. 2: $C \subset \Omega$ compact!
Choose a smooth cutoff function $\rho : \Omega \to [0,1]$ with compact support such that

$$\rho(x) = 1 \text{ for } x \in C \quad v := K * \rho f$$

Claim: $v \in W^{2,p}(\text{supp}(\rho))$ then $u - v$ is a weak solution of

$$\Delta(u - v) = f - \rho f = 0 \text{ in } C$$

$$\overset{\text{Weyl's Lemma}}{\Rightarrow} u - v \text{ harmonic in } \text{int}(C).$$

So $u|_{\text{int}(C)} \in W^{2,p}(\text{int}(C))$. This holds for every compact set $C \subset \Omega$ and so $u \in W^{2,p}_{\text{loc}}(\Omega)$.
Claim: $g := \rho f \in L^p(\mathbb{R}^n)$

$$g = 0 \text{ on } \mathbb{R}^n \setminus B \quad B = \text{supp}\, \rho \subset \Omega \text{ compact}$$

$$\Rightarrow v := K * g|_\Omega \in W^{2,p}(\Omega)$$

Proof of Claim:
Choose $g_k \in C^\infty_0(\mathbb{R}^n)$ and $\text{supp}\, g_k \subset B$ such that $\lim_{k\to\infty} \|g_k - g\|_p = 0$
We prove that $v_k := K * g_k|_\Omega \in C^\infty(\overline{\Omega})$ is uniformly bounde in $W^{2,p}(\Omega)$.

(Δ)

$$\|v_k\|_{L^p(\Omega)} \overset{\text{Young}}{\leq} \|U\|_{L^1(K)} \|g_k\|_{L^p(\Omega)} \quad U := \{x - y | x, y \in \Omega\}$$

$$\|\partial_i v_k\|_{L^p(\Omega)} \overset{\text{Young, L1}}{\leq} \|K_i\|_{L^1(U)} \|g_k\|_{L^p(\Omega}$$

(*)

$$\|\partial_i \partial_j\|_{L^p(\Omega)} \overset{L1,Thm1}{=} \|partial_i(K_j * g_k)\|_{L^p(\mathbb{R}^n)} \leq c\|g_k\|_{L^p(\mathbb{R}^n)}$$

$$(*) \Rightarrow (\Delta)$$

By Banach-Alaoglu and *Rellich*'s theorem v_k has a subsequence v_{k_i} converging weakly in $W^{2,p}(\Omega)$ and strongly in $L^p(\Omega)$. But

$$\lim_{k\to\infty} \|v_k - v\|_{L^p(\Omega)} = 0.$$

So by uniqueness of limites we have

$$v_{k_i} \overset{W^{2,p}}{\to} v \quad \Rightarrow \quad v \in W^{2,p}(\Omega)$$

Lemma 2: Theorem 1 holds for $p = 2$ with $c = 1$.

Proof: $u := K_j * f = K * \partial_j f$ (Lemma 1). So by Chapter I, we have

(*)

$$\Delta u = \partial_j f$$

WARNING: $f, \partial_j f$ have compact support, but u need not have compact support.
By (1) we have

$$\int_{B_R} |\bigtriangledown u|^2 = -\int_{B_R} u\Delta u + \int_{\partial B_R} u\frac{\partial u}{\partial \nu} \text{ Green's formula}$$

Key observation: $\lim_{R\to\infty}\int_{\partial B_R} u\frac{\partial u}{\partial\nu} = 0$. $\operatorname{supp} f \subset B_r$.

$$\begin{aligned}
|u(x)| &= |\int_{B_r} K_i(x-y)f(y)\,\mathrm{d}y| \\
&= |\int_{B_r} \frac{x_i-y_i}{\omega_n|x-y|^n} f(y)\,\mathrm{d}y| \\
&\le \int_{B_r} \frac{1}{\omega_n|x-y|^{n-1}}|f(y)|\,\mathrm{d}y \\
&\le \mathrm{Vol}(B_r)\|f\|_{L^\infty} \sup_{y\in B_r}\frac{1}{\omega_n|x-y|^{n-1}} \\
&\le \mathrm{Vol}(B_r)\|f\|_{L^\infty}\frac{1}{\omega_n(|x|-r)^{n-1}} \\
|\langle\nabla u(x),\frac{x}{|x|}\rangle| &\le \mathrm{Vol}(B_r)\|\nabla f\|_{L^\infty}\frac{1}{\omega_n(|x|-r)^{n-1}}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow |\int_{\partial B_R} u\frac{\partial u}{\partial\nu}| &\le \mathrm{Area}(\partial B_R)\mathrm{Vol}(B_r)^2\|f\|_{L^\infty}\|\nabla f\|_{L^\infty}\frac{1}{\omega_n^2(R-r)^{2n-2}} \\
&= \frac{\mathrm{Vol}(B_r)^2}{\omega_n}\|f\|_{L^\infty}\|\nabla f\|_{L^\infty}\frac{R^{n-1}}{(R-r)^{2n-2}} \to 0
\end{aligned}$$

$$\begin{aligned}
\overset{R\to\infty}{\Rightarrow} \int_{\mathbb{R}^n} |\nabla u|^2 &= -\int_{\mathbb{R}^n} u\Delta u \\
&= -\int_{\mathbb{R}^n} u\partial_j f \\
&= \int_{\mathbb{R}^n} (\partial_j u) f \\
&\le \|partial_j u\|_{L^2}\|f\|_{L^2}
\end{aligned}$$

$$\Rightarrow \|\nabla u\|_2^2 \le \|\partial_j u\|_2\|f\|_2 \le \|\nabla u\|_2\|f\|_2$$

$$\Rightarrow \|\nabla u\|_2 \le \|f\|_2$$

$$\Rightarrow \|\nabla u\|_2 = \|\nabla(K_j * f)\|_2 \le \|f\|_2$$

□

Marcinkiewicz Interpolation:

$f:\mathbb{R}^n\to\mathbb{R}$ measurable and m Lebesgue measure

$$\mu(t,f) := m(\{x\in\mathbb{R}^n \mid |f(x)|>t\}) \quad t>0$$

Lemma 3:

(i) f, g, measurable

$$\Rightarrow \mu(t, f+g) \le \mu(\frac{t}{2}, f) + \mu(\frac{t}{2}, g)$$

(ii) $f\in L^p(\mathbb{R}^n)$

$$\Rightarrow t^p\mu(t,f) \le \int_{\mathbb{R}^n}|f|^p = p\int_0^\infty s^{p-1}\mu(s,f)\,\mathrm{d}s$$

Proof:

(i) $\{x \mid |f(x)+g(x)| > t\} \subset \{x \mid |f(x)| > \frac{t}{2}\} \cup \{x \mid |g(x)| > \frac{t}{2}\}$

(ii) $t^p \mu(t,f) = \int_{\{x||f(x)|>t\}} t^p \mathrm{d}m(x) \leq \int |f|^p \mathrm{d}m \leq \int_{\mathbb{R}^n} |f|^p \mathrm{d}m$. Define $F(x,t) := \begin{cases} pt^{p-1} & 0 \leq t \leq |f(x)| \\ 0 & t > |f(x)| \end{cases}$

$$\overset{\text{Fubini}}{\Rightarrow} \int |f|^p = \int\limits_{\mathbb{R}^n} \left(\int\limits_0^\infty F(x,s)\,\mathrm{d}s \right) \mathrm{d}x$$

$$= \int\limits_0^\infty \left(\int\limits_{\mathbb{R}^n} F(x,s)\,\mathrm{d}x \right) \mathrm{d}s = \int\limits_0^\infty ps^{p-1}\mu(s,t)\,\mathrm{d}s$$

□

Lemma 4: (Marcinkiewicz)
Let $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be a bounded linear operator. Suppose that $\exists C > 0 \forall f \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$

$$t\mu(t,Tf) \leq C\|f\|_{L^1}$$

$$\Rightarrow \forall p \in \mathbb{R} \quad 1 < p < 2$$

$$\exists c = c(\|T\|, C, p) > 0 \quad \forall f \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n):$$

$$\|Tf\|_{L^p} \leq c\|f\|_{L^p}$$

Remark 1: $L^2 \cap L^1 \subset L^p \quad 1 < p < 2$, because

$$\|f\|_p \leq \|f\|_1^{\frac{2}{p}-1} \|f\|_2^{2-\frac{2}{p}}$$

(Chapter II, Lemma 9)

Remark 2: $C_0^\infty(\mathbb{R}^n) \subset L^2(\mathbb{R}) \cap L^1(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$. So T extends uniquely to a bounded linear operator from $L^p(\mathbb{R}^n)$ to itself for $1 < p < 2$ (not necessary for $p = 1$).

Remark 3: $\|T\| := \sup_{\|f\|_2} = 1\|Tf\|_2$ (see Functional Analysis).

Proof: of Lemma 4 (Marcinkiewicz interpolation):
Assumptions:

$$T : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2) \text{ bounded and linear}$$

$$\|T\| := \sup_{\|f\|_2=1} \|Tf\|_2$$

$$t\mu(t,Tf) \leq C\|f\|_1 \quad \forall t > 0 \quad \forall f \in L^2(\mathbb{R}^n \cap L^1(\mathbb{R}^n)$$

(1)

Let's pick a function $f \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and $1 < p < p2$. For $t > 0$ define $f_t, g_t : \mathbb{R}^n \to \mathbb{R}$ by

$f_t(x) := \begin{cases} f(x) & |f(x)| > t \\ 0 & |f(x)| \leq t \end{cases}$

$g_t(x) := \begin{cases} 0 & |f(x)| > t \\ f(x) & |f(x)| \leq t \end{cases}$

$$\Rightarrow f_t + g_t = f$$

$$\frac{t}{2}\mu(\frac{t}{2}, Tf_t) \leq C\|f_t\|_1$$

$$(\frac{t}{2})^2\mu(\frac{t}{2},Tg_t) \le \|Tg_t\|_2^2 \le \|T\|^2\|g_t\|_2^2$$

$$\overset{L3}{\Rightarrow} \mu(t,Tf) \le \mu(\frac{t}{2},Tf_t) + \mu(\frac{t}{2},Tg_t)$$

(2) $$\le \frac{2C}{t}\|f_t\|_1 + \frac{4\|T\|^2}{t^2}\|g_t\|_2^2$$

$$\begin{aligned}
\int_{\mathbb{R}^n} ||Tf(x)|^p &= p\int_0^\infty t^{p-1}\mu(t,Tf)\,\mathrm{d}t \\
&\overset{(2)}{\le} 2pC\int_0^\infty t^{p-2}\|f_t\|_1\,\mathrm{d}t + 4p\|T\|^2\int_0^\infty t^{p-3}\|g_t\|_2^2\,\mathrm{d}t \\
&\overset{1<p<2}{=} \left(\frac{2pC}{p-1} + \frac{4p\|T\|^2}{2-p}\right)\int_{\mathbb{R}^n}|f|^p
\end{aligned}$$

<u>Proof of the last line</u>

$$\begin{aligned}
\int_0^\infty t^{p-2}\|f_t\|_1\,\mathrm{d}t &= \int_0^\infty t^{p-2}\int_{\{|f(x)|>t\}}|f(x)|\,\mathrm{d}x\,\mathrm{d}t \\
&\overset{\text{Fubini}}{=} \int_{\mathbb{R}^n}\int_0^{|f(x)|} t^{p-2}\,\mathrm{d}t|f(x)|\,\mathrm{d}x \\
&= \int_{\mathbb{R}^n}\frac{1}{p-1}|f(x)|^p\,\mathrm{d}x
\end{aligned}$$

$$\begin{aligned}
\int_0^\infty t^{p-3}\|g_t\|_2^2\,\mathrm{d}t &= \int_0^\infty t^{p-3}\int_{\{|f(x)|\le t\}}|f(x)|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\overset{\text{Fubini}}{=} \int_{\mathbb{R}^n}\int_{|f(x)|}^\infty t^{p-3}\,\mathrm{d}t|f(x)|^2\,\mathrm{d}t \\
&= \int_{\mathbb{R}^n}\frac{1}{2-p}|f(x)|^p\,\mathrm{d}x
\end{aligned}$$

□

Lemma 5: Let $n \in \mathbb{N}$. Then

$$\exists c = c(n) > 0 \quad \forall f \in L^2(\mathbb{R}^n)\cap L^1(\mathbb{R}^n):$$
$$\forall k,j \in \{1,\dots,n\}:$$
$$\Rightarrow \mu(t,\partial_k(K_j * f)) \le \frac{c}{t}\int_{\mathbb{R}^n}|f|$$

Remark: What ist mean by $\partial_k(K_j * f)$, when f is only in $L^2(\mathbb{R}^n)$?

Fact: For $f \in C_0^\infty(\mathbb{R}^n)$ we have:

$$K_j * f : \mathbb{R}^n \to \mathbb{R}$$

is smooth and

$$\|\partial_k(K_j * f)\|_2 \le \|f\|_2$$

(Lemma 2)

This means: $\exists$ unique bounded linear operator $T : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ such that

$$Tf = \partial_k(K_j * f) \quad \forall f \in C_0^\infty(\mathbb{R}^n)$$

Proof: of theorem 1:
T as in remark satisfies, by lemma 5, the assumptions of lemma 4

$$\overset{1<p<2}{\Rightarrow} \exists c_p = c_p(n) > 0 \quad \forall f \in C_0^\infty(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$$

$$\|Tf\|_p \leq c_p(n)\|f\|_p$$

□

Proof: of lemma 5 (important):

$$Tf := \partial_k(K_j * f) \quad f \in C_0^\infty(\mathbb{R}^n) \quad T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

We prove in 3 steps that

$$\exists c \quad \forall f \in L^2 \cap L^1 : \quad t\mu(t, Tf) \leq c\|f\|_1$$

Step 1: $\exists c = c(n)$ with the following property:
If $Q = \bigcup_i Q_i$ is a countable union of closed cubes $Q_i \subset \mathbb{R}^n$ with disjoint interiors and $h \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ satisfying

$$\int_{Q_i} h = 0 \quad \forall i \quad h|_{\mathbb{R}^n \setminus Q} \equiv 0$$

then

$$\mu(t, Th) \leq c(\mathrm{Vol}(Q) + \frac{1}{t}\|h\|_1$$

. Step 2: $f \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \quad t > 0$:
$\Rightarrow \exists$ countably many closed cubes $Q_i \subset \mathbb{R}^n$ with disjoint interiors such that

(i) $t \cdot \mathrm{Vol}(Q_i) \leq \|f\|_{L^1(Q_i) \leq 2^n \cdot t \cdot \mathrm{Vol}(Q_i)}$

(ii) $|f(x)| \leq t$ for almost every $x \in \mathbb{R}^n \setminus Q \quad Q := \bigcup_i Q_i$

Step 3: Steps 1 and 2 $\Rightarrow$ lemma 5
Proof of Step 3: Fix a $t > 0$, $Q = \bigcup_i Q_i$ as in Step 2. Define g, h by $g(x) := \begin{cases} f(x) & x \notin Q \\ \frac{1}{\mathrm{Vol}(Q_i)} \int_{Q_i} f & x \in Q_i \text{ a.e.} \end{cases}$

$$h := f - g$$

$\Rightarrow h$ is as in Step 1 and

$$\|g\|_1 \leq \|f\|_1 \quad \|h\|_1 \leq 2\|f\|_1$$

$$\begin{aligned} \Rightarrow \mu(t, Th) &\overset{\text{Step 1}}{\leq} c(\mathrm{Vol}(Q) + \frac{1}{t}\|h\|_1) \\ &\overset{\text{Step 2}}{\leq} c(\frac{1}{t}\|f\|_1 + \frac{2}{t}\|f\|_1) \\ &= \frac{3c}{t}\|f\|_1 \end{aligned}$$

Also, by Step 2,

$$|g(x)| = |f(x)| \leq t \quad \text{in } \mathbb{R}^n \setminus Q \text{ a.e.}$$

$$|g(x)| \leq \frac{1}{\mathrm{Vol}(Q_i)}\|f\|_{L^1(Q_i)} \leq 2^n t \quad \text{in } Q_i$$

$$\overset{\text{Lemma 3}}{\Rightarrow} \mu(t, Tg) \leq \frac{\|Tg\|_2^2}{t^2} \leq \frac{\|g\|_2^2}{t^2}$$

$$= \frac{\int |g|^2}{t^2} \leq \sup \frac{|g|}{t} \int \frac{|g|}{t} \leq 2^n \int \frac{|g|}{t} \leq \frac{2^n}{t} \|f\|_1$$

$$\Rightarrow \mu(2t, Tf) \overset{\text{Lemma 3}}{\leq} \mu(t, Tg) + \mu(t, Th) \leq \frac{3c + 2^n}{t} \|f\|_1$$

For $k \in \mathbb{Z}^n, l \in \mathbb{Z}$, denote

$$Q(k,l) := \{x \in \mathbb{R}^n \mid \frac{k_i}{2^l} \leq x_i \leq \frac{k_i + 1}{2^l}\}$$

$$\mathcal{Q} := \{Q(k,l) \mid k \in \mathbb{Z}^n, l \in \mathbb{Z}\}$$

Define

$$\mathcal{Q}_0 := \{Q \in \mathcal{Q} \mid \frac{1}{\text{Vol}(Q) \|f\|_{L^1(Q)}} > t, \forall Q' \in \mathcal{Q} : Q \subset Q' : \frac{1}{\text{Vol}(Q')} \|f\|_{L^1(Q')} \leq t\}$$

Note:

1. Each decreasing sequence $Q_1 \supset Q_2 \supset \ldots$ in $\mathcal{Q}$ contains at most one element of $\mathcal{Q}_0$.
   Any two cubes $Q, Q' \in \mathcal{Q}$ satisfy:
   either $Q \subset Q'$ or $Q' \subset Q$ or $\text{int}(Q) \cap \text{int}(Q') = \varnothing$.
   Hence two distinct elements of $\mathcal{Q}$, have disjoint interiors.

2. $Q \in \mathcal{Q}_0$, $Q' \in \mathcal{Q}$, $Q \subset Q'$
   Q' is the next highest element of $\mathcal{Q}$. So sidelength$(Q') = 2 \cdot$sidelength(Q). So $\text{Vol}(Q') = 2^n \text{Vol}(Q)$. Hence

   $$t\text{Vol}(Q) < \|f\|_{L^1(Q)} \leq \|f\|_{L^1(Q')} \leq t\text{Vol}(Q') = 2^n t \text{Vol}(Q) \quad \forall Q \in \mathcal{Q}_0$$

   $\Rightarrow$ each $Q \in \mathcal{Q}_0$ satisfies (i) in Step 2!

3. $Q_0 := \bigcup_{Q \in \mathcal{Q}_0} Q$. Let $x \in \mathbb{R}^n$. Then

   (*)
   $$x \in Q_0 \Leftrightarrow \exists Q \in \mathcal{Q} \text{ such that } x \in Q, \frac{1}{\text{Vol}(Q)} \|f\|_{L^1(Q)} > t$$

   Proof of (*):

   $\Rightarrow$: obvious

   $\Leftarrow$: If $x \in Q \in \mathcal{Q}$ with $\frac{1}{\text{Vol}(Q)} \|f\|_{L^1(Q)} > t$ consider the sequence $Q_0 \subset Q_1 \subset Q_2 \subset \ldots$ with $Q_0 = Q, Q_i \in \mathcal{Q}$ and $\text{Vol}(Q_{i+1}) = 2^n \text{Vol}(Q_i)$. One of the Q_i is an element of $\mathcal{Q}_0$.

4. *Lebesgue differentiation:* $f : \mathbb{R}^n \to \mathbb{R}$ integrable $\Rightarrow \exists E \subset \mathbb{R}^n$ of measure zero

   $$\forall x \in \mathbb{R}^n \setminus E \quad \forall \varepsilon > 0 \quad \exists \delta > 0 :$$

   $$\left.\begin{array}{l} U \text{ measurable} \\ x \in U \end{array}\right\} \Rightarrow |f(x) - \tfrac{1}{m(U)} \int_u f| < \varepsilon$$

   ie.

   $$\lim_{m(U) \to 0} \frac{1}{m(U)} \int_U f = f(x) \text{ for a.e } x \in \mathbb{R}^n$$

5. By 3. and 4., we have $|f(x)| \leq t$ almost everywhere in $\mathbb{R}^n \setminus Q_0$. Denn $x \in \mathbb{R}^n \setminus (Q_0 \cup E)$ Choose sequence $Q_i \in \mathcal{Q}$ with $\text{diam}(Q_i) \to 0, x \in Q_i$

   $$\overset{3.}{\Rightarrow} \frac{1}{\text{Vol}(Q_i)} \int_{Q_i} |f| \leq t$$

   $$\overset{4.}{\Rightarrow} |f(x)| = \lim_{i \to \infty} \frac{1}{\text{Vol}(Q_i)} \int_{Q_i} |f| \leq t$$

<u>Proof of Step 1:</u> Define $h_i : \mathbb{R}^n \to \mathbb{R}$ by $h_i(x) := \begin{cases} h(x) & x \in Q_i \\ 0 & x \notin Q_i \end{cases}$ Then $\sum h_i = h$ (almost everywhere).
$q_i :=$ center of Q_i
$2r_i :=$ sidelength of Q_i
$\max_{x\in Q_i} |x - q_i| = \sqrt{n} r_i \Rightarrow \forall x \notin Q : Th_i(x) = \int\limits_{Q_i} \partial_k K_j(x-y)h(y)\,\mathrm{d}y$

$$\begin{aligned} |Th_i(x)| &= |\int\limits_{Q_i} (\partial_k K_j(x-y) - \partial_k K_j(x-q_i))h(y)\,\mathrm{d}y \\ &\leq \max_{y\in Q_i} |\partial_k K_j(x-y) - \partial_k K_j(x-q_i)| \cdot \|h\|_{L^1(Q_i)} \\ &\leq \sqrt{n} r_i \sup_{y\in Q_i} |\bigtriangledown \partial_k K_j(x-y)| \|h\|_{L^1(Q_i)} \\ &\leq c_1 r_i \max_{y\in Q_i} |\frac{1}{|x-y|^{n+1}}| \|h\|_{L^1}(Q_i) \\ &\leq \frac{c_1 r_i}{d(x,Q_i)^{n+1}} \|h\|_{L^1(Q_i)} \end{aligned}$$

Define $B_i := \{x \in \mathbb{R}^n \mid |x - q_i| < 2\sqrt{n} r_i\}$

$$\Rightarrow \forall x \in \mathbb{R}^n \setminus B_i : d(x, Q_i) \geq |x - q_i| - \sqrt{n} r_i$$

$$\begin{aligned} \Rightarrow \int_{\mathbb{R}^n\setminus B_i} |Th_i| &\leq c_1 r_i \int_{\mathbb{R}^n\setminus B_i} \frac{\mathrm{d}s}{(|x-q_i| - \sqrt{n} r_i)^{n+1} \|h\|_{L^1(Q_i}} \\ &= c_1 r_i \int_{|x|\geq 2\sqrt{n} r_i} \frac{\mathrm{d}x}{(|x| - \sqrt{n} r_i)^{n+1}} \|h\|_{L^1(Q_i)} \\ &= c_1 r_i \int\limits_{2\sqrt{n} r_i}^{\infty} \frac{\omega_n \rho^{n-1}}{(\rho - \sqrt{n} r_i}^{n+1} \mathrm{d}S \|h\|_{L^1(Q_i)} \\ &= c_1 r_i \int\limits_{\sqrt{n} r_i}^{\infty} \frac{\omega_n (\rho + \sqrt{n} r_i)^{n-1}}{\rho^{n+1}} \mathrm{d}\rho \|h\|_{L^1(Q_i)} \\ &\leq 2^{n-1} c_1 \omega_n r_i \int\limits_{\sqrt{n} r_i}^{\infty} \frac{1}{\rho^2} \mathrm{d}\rho \|h\|_{L^1(Q_i)} \\ &\leq c_2 \|h\|_{L^1(Q_i)} \end{aligned}$$

<u>Need:</u> $\sup(r_i) < \infty$ (very important!!!)
$B := \bigcup B_i$

$$\begin{aligned} \Rightarrow \int_{\mathbb{R}^n\setminus B} |Th| &\leq \sum_i \int_{\mathbb{R}^n\setminus B} |Th_i| \\ &\leq \sum_i \int_{\mathbb{R}^n\setminus B_i} |Th_i| \\ &\leq c_2 \sum_i \|h\|_{L^1(Q_i)} \\ &= c_2 \|h\|_{L^1} \end{aligned}$$

<u>Moreover:</u> (what we have):

$$\begin{aligned} \mathrm{Vol}(B) &\leq \sum_i \mathrm{Vol}(B_i) \\ &= c_3 \sum_i \mathrm{Vol}(Q_i) \\ &= c_3 \mathrm{Vol}(Q) \end{aligned}$$

Hence

$$\begin{aligned} t\mu(t,Th) &= tm(\{x \in \mathbb{R}^n \mid |Th(x)| > t\}) \\ &= tm(\{x \in B \mid |Th(X)| = t\}) + tm(\{x \notin B \mid |Th(x)| > t\}) \\ &\leq t\mathrm{Vol}(B) + \int_{\mathbb{R}^n \setminus B} |Th| \\ &\leq tc_3\mathrm{Vol}(Q) + c_2\|h\|_{L^1} \end{aligned}$$

□

4 Second Order Elliptic Operators:

(1)
$$Lu := -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right)$$

$\Omega \subset \mathbb{R}^n$ bounded, open set with smooth boundary.
$a_{ij} : \overline{\Omega} \to \mathbb{R}$ continuously differentiable.

Example: $a_{ji} = a_{ij} = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ Then $L = -\Delta$.

Definition: The operator L is called *elliptic* if $\exists \delta > 0$ such that

(2)
$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \delta|\xi| \quad \forall x \in \overline{\Omega}\forall \xi \in \mathbb{R}^n$$

Note: $A(X) = (a_{ij}(x))^n$ is a symmetric matrix
(2) $\Leftrightarrow$ $A(x)$ is positive definite for every $x \in \overline{\Omega}$.

Dirichlet Problem: Given a function $f : \Omega \to \mathbb{R}$ find a solution $u : \overline{\Omega} \to \mathbb{R}$ of the equation

(3)
$$Lu = f \text{ in } \Omega \quad u|_{\partial\Omega} = 0$$

Theorem 3: Let $p > 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary. Let $a_{ij} = a_{ji} \in C^1(\overline{\Omega})$ satisfy (2)
$\Rightarrow$ the operator $L : W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \to L^p(\Omega)$ given by (1) is bijective, ie. for every $f \in L^p(\Omega)$ there is a unique solution $u \in W^{2,p}(\Omega) \cap W^{1,p}(\Omega)$ of equation (3).
In particular

$$\exists c > 0 \quad \forall u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$$
$$\|u\|_{W^{2,p}} \leq c\|Lu\|_{L^p}$$

Definition: A function $u \in W_0^{1,p}(\Omega)$ is called a *weak solution of (3)* if for every test function $\phi \in C_0^\infty(\Omega)$ we have:

(4)
$$\sum_{i,j=1}^{n} \int_\Omega \frac{\partial \phi}{\partial x_i} a_{ij} \frac{\partial u}{\partial x_j} = \int_\Omega \phi f$$

Remark 1: The condition $u \in W_0^{1,p}(\Omega)$ means in particular, that the boundary condition $u|_{\partial\Omega} = 0$ is already satisfied!

Remark 2: If $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ is a *strong solution of (3)* then u is a weak solution.

Remark 3: The *Poincaré inequality* says

$$\|u\|_{L^p} \leq \text{diam}(\Omega) \cdot \| \bigtriangledown u\|_{L^p}$$

for every $u \in W_0^{1,p}(\Omega)$. Hence

$$\| \bigtriangledown u\|_{L^p} \leq \|u\|_{W^{1,p}} \leq (1 + \text{diam}(\Omega))\| \bigtriangledown u\|_{L^p}$$

Hence the function

$$W_0^{1,p}(\Omega) \to [0, \infty) : u \mapsto \| \bigtriangledown u\|_{L^p}$$

is a norm, equivalent to the usual $W^{1,p}$-norm.

Remark 4: Given $u \in W_0^{1,p}(\Omega)$ and $q > 1$ with $p^{-1} + q^{-1} = 1$, the left had side of (4) defines a bounded linear functional

$$W_0^{1,q}(\Omega) \to \mathbb{R}$$

$$\phi \mapsto \sum_{i,j=1}^{n} \inf_{\Omega} \underbrace{\partial_i \phi}_{\in L^q} a_{ij} \underbrace{\partial_j u}_{\in L^p} =: B(\phi, u)$$

Hence equation (4) is equivalent to

$$B(\phi, u) = \Phi_f(\phi) \quad \forall \phi \in C_0^\infty(\Omega)$$

where the bounded linear functional $\Phi_f : W_0^{1,q}(\Omega) \to \mathbb{R}$ is given by

$$\Phi_f(\phi) := \int_\Omega f\phi$$

Remark 5: Let us denote by

$$W^{-1,q}(\Omega) := (W_0^{1,q}(\Omega))^*$$

the dual space (of all bounded linear functionals $\Phi : W_0^{1,q}(\Omega) \to \mathbb{R}$) with the norm

$$\begin{aligned} \|\Phi\| &:= \sup_{v \in W_0^{1,q}} \frac{|\Phi(v)|}{\| \bigtriangledown v\|_{L^q}} \\ &= \sup_{\phi \in C_0^\infty(\Omega)} \frac{|\Phi(\phi)|}{\| \bigtriangledown \phi\|_{L^q}} \end{aligned}$$

Remark 6: The map

$$\begin{cases} L^p(\Omega) \to W^{-1,p}(\Omega) \\ f \mapsto \Phi_f \end{cases}$$

(with Φ_f as in remark 4) is a dual to the inclusion

$$W_0^{1,q}(\Omega) \hookrightarrow L^q(\Omega)$$

This operator is injective, compact and has a dense image (→ FA). Hence the map

$$L^p(\Omega) \to W^{-1,p}(\Omega)$$

is also injective, compact and ahas a dense image

$$W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow W^{-1,p}(\Omega)$$

dual to

$$W^{-1,q}(\Omega) \hookleftarrow L^q(\Omega) \hookleftarrow W^{1,q}(\Omega$$

Remark 7: There's a commuting diagram

$$W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \xrightarrow{L} L^p(\Omega)$$

$$W_0^{1,p}(\Omega) \xrightarrow{L} W^{-1,p}(\Omega)$$

where the operator

$$L : W_0^{1,p}(\Omega) \to W^{-1,p}(\Omega)$$

is given by

$$Lu := B(u, .)$$

Proof: (of Calderon-Zygmund for $p > 2$)
$f \in C_0^\infty(\mathbb{R}^n)$ $\phi \in C_0^\infty(\mathbb{R}^n)$, $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < q < 2$.

$$\begin{aligned}
|\int_{\mathbb{R}^n} \phi \partial_j (K_i * f)| &= |-\int_{\mathbb{R}^n} (\partial_j \phi)(K_i * f)| \\
&= |\int_{\mathbb{R}^n} (K_i * \partial_j \phi) f| \\
&= |\int_{\mathbb{R}^n} (\partial_j (K_i * \phi)) f| \\
&\overset{\text{Hölder}}{\leq} \|\partial_j (K_i * \phi)\|_q \|f\|_p \\
&\leq c\|\phi\|_q \|f\|_p \\
\Rightarrow \|\partial_j (K_i * f)\|_p &= \sup_{\phi \neq 0} \frac{|\int \phi \partial_j (K_i * f)|}{\|\phi\|_q} \leq c\|f\|_p
\end{aligned}$$

□

Theorem 4: Ω, a_{ij} as in theorem 3. Let $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$
$\Rightarrow$ the operator $L : W_0^{1,p}(\Omega) \to W^{-1,p}(\Omega)$ defined in remark 7 is bijective, ie. for every bounded linear functional $\Phi : W_0^{1,q}(\Omega) \to \mathbb{R}$ there is a unique solution of the equation

(4') $$\sum_{i,j} \int_\Omega \partial_i \phi a_{ij} \partial_j u = \Phi(\phi) \quad \forall \phi \in C_0^\infty(\Omega)$$

In particular $\exists c > 0 \quad \forall u \in W_0^{1,p}(\Omega)$

$$\begin{aligned}
&\|\nabla u\|_p \leq c\|Lu\|_{W^{-1,p}} \\
&= \sup_{0 \neq \phi \in C_0'\infty} \frac{|\sum \int_\Omega \partial_i \phi a_{ij} \partial_j u|}{\|\nabla \phi\|_q}
\end{aligned}$$

Warm up: $p = 2$

Lemma 6: (*Lax-Milgram*)
V real Hilbert space, $\|v\| = \sqrt{\langle v, v \rangle}$
$B : V \times V \to \mathbb{R}$ is a bilinear map satisfying

$$\begin{gathered}
|B(u, v)| \leq c\|u\|\|v\| \quad \forall u, v \in V \\
B(u, u) \geq \delta\|u\|^2 \quad \forall u \in V \\
\Rightarrow \forall \Phi \in V^* \exists! u \in V \text{ such that} \\
B(u, \phi) = \Phi(\phi) \quad \forall \phi \in V
\end{gathered}$$

Proof: B is a inner product on V with $\|u\|_B := \sqrt{B(u, u)}$. The norms $\|.\|_B$ and $\|.\|$ are equivalent:

$$\sqrt{\delta}\|u\| \leq \|u\|_B \leq \sqrt{c}\|u\|$$

use the *Riesz representation theorem.* □

Proof: (of theorem 4 for $p = 2$)
$V := W_0^{1,2}(\Omega)$

$$\begin{gathered}
B(u, v) := \sum_{i,j} \int_\Omega \partial_i u a_{ij} \partial_j v \\
\Rightarrow B(u, u) = \sum_{i,j} \int_\Omega \partial_i u a_{ij} \partial_j u \\
\overset{(2)}{\geq} \delta \int_\Omega |\nabla u|^2 = \delta\|\nabla u\|_2^2
\end{gathered}$$

□

Lemma 7: $\Omega \subset \mathbb{R}^n$ bounded, open. Let $p, q > 1$ with $p^{-1} + q^{-1} = 1$
$\Rightarrow \exists c > 0 \quad \forall u \in C_0^\infty(\Omega)$

(5)
$$\| \nabla u\|_p \leq c \sup_{0 \neq \phi \in C_0^\infty} \frac{|\int_\Omega \langle \nabla \phi, \nabla u \rangle|}{\| \nabla \phi\|_q}$$

Proof: Define
$$\Phi : W_0^{1,q}(\Omega) \to \mathbb{R}$$
by
$$\Phi(\phi) := \int_\Omega \langle \nabla u, \nabla \phi \rangle$$
Observe
$$X := L^q(\Omega, \mathbb{R}^n)$$
$$Y := \{\nabla v \mid v \in W_0^{1,q}(\Omega)\} \xrightarrow{\Phi} \mathbb{R}$$
$\overset{Hahn\text{-}Banach}{\Rightarrow} \exists f \in L^p(\Omega, \mathbb{R}^n) = X^*$ such that

(6)
$$\int \langle f, \nabla \phi \rangle = \Phi(\phi) = \int_\Omega \langle \nabla u, \nabla \phi \rangle$$
$$\|f\|_p = \|\Phi\| = \sup_{\phi \neq 0} \frac{\int_\Omega \langle \nabla u, \nabla \phi \rangle}{\| \nabla \phi\|_q}$$
so, by (6), we have
$$(5) \Leftrightarrow \| \nabla u\|_p \leq c\|f\|_p$$
whenever
$$\int_\Omega \langle \nabla u - f, \nabla \phi \rangle = 0 \quad \forall \phi \in C_0^\infty$$
Define
$$v := \sum_{i=1}^n K_i * f_i \in W^{1,p}(\mathbb{R}^n) \quad f = (f_1, \ldots, f_n) : \Omega \to \mathbb{R}^n$$
$$K_i = \partial_i K = \frac{x_i}{\omega_n |x|^n}$$
$\Rightarrow$ By Calderon-Zygmund

(7)
$$\| \nabla v\|_p \leq \sum_{i=1}^n \| \nabla (K_i * f_i)\|_p \leq c \sum_{i=1}^n \|f_i\|_p$$

Moreover v is a weak solution of $\nabla v = \sum_i \partial_i f_i$, ie.
$$\int_\Omega \langle \nabla v - f, \nabla \phi \rangle = 0 \quad \forall \phi \in C_0^\infty$$
$\Rightarrow v - u$ is harmonic

(8)
$$| \nabla (u - v)(x)| \leq \frac{n+1}{R} \frac{1}{\mathrm{Vol}(B_R)} \int_{B_R(x)} |u - v|$$

Choose $R > 0$ such that

(9)
$$\Omega \subset B_R \quad \frac{(n+1)\mathrm{diam}(\Omega)}{R} < \frac{1}{2}$$

Put the things together:

$$
\begin{aligned}
& \overset{(8)}{\Rightarrow} \forall x \in \Omega \\
|\bigtriangledown (u-v)(x)| &\leq \frac{n+1}{R}\left(\frac{1}{\mathrm{Vol}(B_R)}\right)^{1/p}\|u-v\|_{L^p(B_R(x))} \\
&\leq \frac{n+1}{R}\left(\frac{1}{\mathrm{Vol}(B_R)}\right)^{1/p}\|u-v\|_{L^p(B_{2R}} \\
\Rightarrow \|\bigtriangledown (u-v)\|_p &\leq \frac{n+1}{R}\left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(B_R)}\right)^{1/p}\|u-v\|_{L^p(B_{2R})} \\
&\leq \frac{n+1}{R}\|u-v\|_{L^p(B_R)} \\
&\leq \frac{n+1}{R}(\|u\|_p + \|v\|_{L^p(B_{2R})})v|_{B_{2R}} - \sum_i (K_i|_{B_{4R}} * f) \\
&\overset{\text{Young}}{\leq} \frac{n+1}{R}(\|u\|_p + \sum_i \|K_i\|_{L^1(B_{4R})}\|f_i\|_p) \\
&\overset{\text{Poincare}}{\leq} \frac{(n+1)\mathrm{diam}(\Omega)}{R}\|\bigtriangledown\|_p + c'\|f\|_p \\
&\leq \frac{1}{2}\|\bigtriangledown u\|_p + c'\|f\|_p \\
\Rightarrow \|\bigtriangledown u\|_p &\leq \|\bigtriangledown v\|_p + \|\bigtriangledown (u-v)\|_p \\
&\overset{(7)}{\leq} c''\|f\|_p + \frac{1}{2}\|\bigtriangledown u\|_p
\end{aligned}
$$

□

<u>Attention:</u> One page is missing!!!

Recap: $\Omega \subset \mathbb{R}^n$ open, bounded, $\partial\Omega$ smooth. $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. $a_{ij} = a_{ji} \in C^1(\overline{\Omega})$

$$\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq \delta|\xi|^2 \quad \forall x \in \overline{\Omega} \forall \xi \in \mathbb{R}^n$$

$$L = -\sum_{i,j}\int_\Omega \partial_i u a_{ij} \partial_j v$$

$$\|Lu\|_{W^{-1,p}} := \sup_{0 \neq \phi \in C_0^\infty(\Omega)} \frac{|B(u,\phi)|}{\|\bigtriangledown \phi\|_q} \quad u \in W_0^{1,p}(\Omega)$$

Goal:

$$L : W_0^{1,p} \to W^{-1,p} := (W_0^{1,q})^* \text{ is bijective}$$

Lemma 8: $\exists c_0, c_1 > 0 \quad \forall u \in C_0^\infty(\Omega)$:

(*) $$\|\bigtriangledown u\|_p \leq c_1\|Lu\|_{-1,p} + c_0\|u\|_p$$

Proof: (has 4 steps!)

Step 1: $a_{ij} = \delta_{ij} \quad L = \Delta$

$$\|Lu\| = \sup_\phi \frac{\int_\Omega \langle \bigtriangledown u, \bigtriangledown \phi\rangle}{\|\bigtriangledown \phi\|_q}$$

(1) follows from Lemma 7 with $c_0 = 0$.

Step 2: $a_{ij} \equiv$ const $\Rightarrow$ (1) holds with $c_0 = 0$
Exercise with hints:
Denote $A := (a_{ij})_{i,j=1}^N = A^T \in \mathbb{R}^{n\times n}$
ellipticity $\Leftrightarrow$ A positive definite, ie. $\langle \xi, A\xi\rangle > 0 \quad \forall \xi \neq 0$
$\Rightarrow \exists B = B^T \in \mathbb{R}^{n\times n} \quad B > 0 \quad B^2 = A.$
Denote $C := B^{-1}$. $B = (b_{ij}) \quad C = (c_{ij})$.
$\Rightarrow \sum_j b_{ij} b_{jk} = a_{ik}$
$\Rightarrow \sum_j b_{ij} c_{jk} = \delta_{ik}$
Define $\widehat{\Omega} := B^{-1}\Omega$ and $\widehat{u}(y) := u(By)$
Prove that

$$\nabla \widehat{u}(y) = B^T \nabla u(By)$$

$$\| \nabla \widehat{u}\|_p = \frac{1}{(\det B)^{1/p}} \|B^T \nabla u\|_p$$

$$\sum_{i,j} \int_\Omega \partial_i u a_{ij} \partial_j v = \frac{1}{\det B} \int_{\widehat{\Omega}} \langle \nabla \widehat{u}, \nabla \widehat{v}\rangle$$

Then Step 2 follows from Step 1 for $\widehat{\Omega}$.

Step 3: Let $x_0 \in \Omega$ and define

$$L^0 u := \sum_{i,j} a_{ij}(x_0) \partial_i \partial_j u$$

By Step 2 $\exists c > 0$ such that

$$\| \nabla u\|_p \leq c \|L^0 u\|_{-1,p}$$

Choose $\delta > 0$ so small that

$$\sum_{i,j} \sup_{|x-x_0|<\delta} |a_{ij}(x) - a_{ij}(x_0)| < \frac{1}{2c}$$

$\overset{\text{Claim}}{\Rightarrow}$ If $\operatorname{supp} u \subset B_\delta(x_0)$ then

$$\| \nabla u\|_p \leq 2c \|Lu\|_{-1,p}$$

Proof of Step 3: Let $\widehat{a}_{ij}(x) := a_{ij}(x) - a_{ij}(x_0)$ then

$$\begin{aligned}
\|(L - L^0)u\|_{-1,p} &= \sup_{0\neq\phi\in C_0^\infty} \frac{|\sum \int_\Omega \partial_i u \widehat{a}_{ij} \partial_j \phi|}{\| \nabla \phi\|_q} \\
&\overset{\text{Holder}}{\leq} \sup_{0\neq\phi\in C_0^\infty} \left(\frac{\sum \|\partial_i u\|_p \|\widehat{a}_{ij}\|_{L^\infty(B_\delta(x_0))} \|\partial_j \phi\|_q}{\| \nabla \phi\|_q} \right) \\
&\leq \sum \|\partial_i u\|_p \|\widehat{a}_{ij}\|_{L^\infty(B_\delta(x_0))} \\
&\leq \| \nabla u\|_p \sum_{i,j} \|\widehat{a}_{ij}\|_{L^\infty(B_\delta(x_0))} \\
&\leq \frac{\| \nabla u\|_p}{2c}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \| \nabla u\|_p &\leq c\|L^0 u\|_{-1,p} \\
&\leq c\|L^0 u\|_{-1,p} \\
&\leq c\|Lu\|_{-1,p} + c\|L^0 u - Lu\|_{-1,p} \\
&\leq c\|Lu\|_{-1,p} + \frac{1}{2}\| \nabla u\|_p
\end{aligned}$$

Step 4: general case:
Choose a finite open cover
$$\overline{\Omega} \subset \bigcup_k U_k$$
such that Step 3 holds for each u with $\operatorname{supp} u \subset \Omega \cap U_k$ and constant c_k. Choose a partition of unity $\rho_k : \mathbb{R}^n \to [0,1]$ with $\operatorname{supp} \rho_k \subset U_k$ and $\operatorname{supp} \rho_k$ compact.
$$\sum_k \rho_k(x) = 1 \forall x \in \overline{\Omega} \Rightarrow \forall u \in C_0^\infty(\Omega):$$
$$\begin{aligned} \| \nabla u \|_p &= \| \nabla (\sum_k \rho_k u) \|_p \\ &\leq \sum_k \| \nabla (\rho_k u) \|_p \\ &\overset{\text{Step 3}}{\leq} \sum_k c_k \| L(\rho_k u) \|_{-1,p} \end{aligned}$$

Claim: $\forall \rho \in C_0^\infty(\mathbb{R}^n) \quad \exists c > 0 \quad \forall u \in C_0^\infty(\Omega):$

(2)
$$\| L(\rho u) \|_{-1,p} \leq c \left(\| Lu \|_{-1,p} + \| u \|_p \right)$$
equivalently

(3)
$$B(\rho u, \phi) \leq c \left(\| Lu \|_{-1,p} + \| u \|_p \right) \| \nabla \phi \|_q \quad \forall \phi \in C_0^\infty(\Omega)$$

Proof of (3):
$$\begin{aligned} B(\phi, \rho u) &= \sum_{i,j} \int_\Omega \partial_i \phi a_{ij} \partial_j (\rho u) \\ &= \underbrace{\sum \int_\Omega \partial_i \phi a_{ij} (\partial_j \rho) u}_{I} + \sum \int_\Omega \partial_i \phi a_{ij} \rho \partial_j u \\ &= I + \underbrace{\sum \int \partial_i (\rho \phi) a_{ij} \partial_j u}_{II} - \sum \int \phi (\partial_i \rho) a_{ij} \partial_j u \\ &= I + II + \underbrace{\sum \int \partial_j (\phi (\partial_i \rho) a_{ij}) \cdot u}_{III} \end{aligned}$$

$$\begin{aligned} |I| &\leq \sum \| \partial_i \phi \|_q \| a_{ij} \partial_j \rho \|_\infty \| u \|_p \\ |II| &= |B(\rho \phi, u)| \\ &\leq \| Lu \|_{-1,p} \| \nabla (\rho \phi) \|_q \\ &\leq \| Lu \|_{-1,p} \left(\| \nabla \rho \|_\infty \| \phi \|_q + \| \rho \|_p \| \nabla \phi \|_q \right) \\ &\overset{\text{Poincare}}{\leq} c \| Lu \|_{-1,p} \| \nabla \phi \|_q \\ |III| &\leq \| u \|_p \sum_{i,j} \| \partial_j (\phi (\partial_i \rho) a_{ij}) \|_q \\ &\leq c' \| u \|_p \| \nabla \phi \|_q \end{aligned}$$

□

Theorem 5: Ω, a_{ij}, L, B as in theorem 3 and 4.
Let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.
Let $u \in W_0^{1,p}(\Omega)$ and $c > 0$ such that
$$|B(u, \phi)| \leq c \| \phi \|_q \quad \forall \phi \in C_0^\infty(\Omega)$$
$$\Rightarrow u \in W^{2,p}(\Omega)$$

Proof: (Thm. 5 and Lemma 8 $\Rightarrow$ Thm. 4:) We know, by Lemma 8:

$$\| \nabla u\|_p \leq c(\|Lu\|_{-1,p} + \|u\|_p) \quad \forall u \in W_0^{1,p}(\Omega)$$

and by Rellich's Thm, the inclusion

$$W_0^{1,p}(\Omega) \to L^p(\Omega) \text{ is compact}$$

$\Rightarrow$ By a result in Functional Analysis (Fredholm theory) we obtain that the operator

$$L : W_0^{1,p}(\Omega) \to W^{-1,p}(\Omega)$$

has a closed image and a finite dimensional kernel.

Claim: L is injective.
Obvious for $p = 2$. For $p \geq 2$ we have

$$W_0^{1,p}(\Omega) \subset W_0^{1,2}(\Omega)$$

so L is injective for $p \geq 2$.

Proof of claim for $1 < p < 2$: Let $u \in W_0^{1,p}(\Omega) \quad 1 < p < 2$ and $Lu = 0$, ie.

$$B(u, \phi) = 0 \quad \forall \phi \in C_0^\infty(\Omega)$$

$$\overset{\text{Thm. 5}}{\Rightarrow} u \in W^{2,p}(\Omega) \subset W^{1,\frac{np}{n-p}}(\Omega) \quad u|_{\partial\Omega} = 0$$

$$\overset{\text{Induction}}{\Rightarrow} u \in W_0^{1,p_2}(\Omega) \quad p_2 := \frac{np_1}{n - p_1} > p_1$$

$$u \in W_0^{1,p_k}(\Omega) \quad p_k := \frac{np_{k-1}}{n - p_{k-1}}$$

continue until $p_k > n \geq 2$.

Exercise: $p_k > \frac{n}{n-k}$ at most $k = \frac{n}{2}$

$$\Rightarrow u \in W_0^{1,2}(\Omega) \quad Lu = 0 \Rightarrow u = 0$$

Why is $L^{(p)} : W_0^{1,p}(\Omega) \to W^{-1,p}(\Omega)$ also surjective?
Consider the dual operator

$$L^{(q)} = (L^{(p)})^* : W_0^{1,q}(\Omega) \to W^{-1,q}(\Omega)$$

$$L^{(q)} \text{ is injective } \Rightarrow L^{(p)} \text{ has a dense image} \Rightarrow L^{(p)} \text{ onto}$$

□

Proof: (Thms 4 and 5 $\Rightarrow$ Thm 3:)

Claim:

$$L : W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \to L^p(\Omega) \text{ is bijective!}$$

Let $f \in L^p(\Omega)$. Define

$$\Phi_f \in W^{-1,p}(\Omega)$$

by

$$\Phi_f(\phi) := \int_\Omega f\phi \quad \phi \in W_0^{1,q}(\Omega)$$

$$\overset{\text{Thm 4}}{\Rightarrow} \exists! u \in W_0^{1,p}(\Omega) \text{such that}$$

$$B(u, \phi) = \Phi_f(\phi) \quad (\text{ie.} Lu = \Phi_f)$$

$$|B(u, \phi)| = |\Phi_f(\phi)| \overset{\text{Holder}}{\leq} \|f\|_p \|\phi\|_q$$

$$\overset{\text{Thm 5}}{\Rightarrow} u \in W^{2,p}(\Omega)$$

$$u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \text{ and } Lu = f$$

□

Proof: of Thm 5: $1 < p < \infty$ $u \in W_0^{1,p}(\Omega)$. Assume:

$$B(u, \phi) \le c\|\phi\|_q \quad \forall \phi \in C_0^\infty(\Omega)$$

We prove first that $u \in W^{2,p}_{\text{loc}}(\Omega)$ ("interior regularity"), ie.

$$v := \rho u \in W^{2,p}(\Omega) \quad \forall \rho \in C_0^\infty(\Omega)$$

Proof: *Difference quotient method*:
Fix an integer $k \in \{1, \dots, n\}$
Let $e_k = (0, \dots, 0, 1, 0, \dots, 0)$
Define, for $h \neq 0$ small,

$$v^h(x) : 0 \frac{v(x + he_k) - v(x)}{h}$$

Fact 1:

$$\|v^h\|_p \le \|\partial_k v\|_p \quad \text{Example Sheet 10.2.}$$

Fact 2:

$$v \in L^p(\mathbb{R}^n)$$

Assume $\sup_{h\neq 0} \|v^h\|_p < \infty$ then $\partial_k v \in L^p$, ie.

$$\exists v_k \in L^p(\mathbb{R}^n) \quad \forall \phi \in C_0^\infty(\mathbb{R}^n):$$

$$\int_{\mathbb{R}^n} (\partial_k \phi) v = -\int_{\mathbb{R}^n} \phi v_k$$

Fact 3: $\partial_i(\phi^h) = (\partial_i \phi)^h$

Fact 4:

a) $\|v^h\|_{1,p} \le \|\partial_k v\|_{1,p}$

b) $v \in W^{1,p}(\mathbb{R}^n)$ $\sup_{h\neq 0} \|v^h\|_{1,p} < \infty$
$\Rightarrow \partial_k v \in W^{1,p}$

Claim: u as in hypothesis of theorem 5

$$\rho \in C_0^\infty(\Omega) \quad k \in \{1, \dots, n\} \quad v := \rho u$$
$$\Rightarrow \exists c > 0 \quad \forall \phi \in C_0^\infty(\Omega) \forall h \neq 0:$$
$$|B(\phi, v^h)| \le c\| \nabla \phi\|_q$$

Proof of "claim $\Rightarrow$ Thm 5": (interior regularity).
The claim means

$$\|Lv^h\|_{-1,p} \le c \overset{\text{Lemma 8}}{\Rightarrow} v^h \in W_0^{1,p}(\Omega)$$

and

$$\begin{aligned} \|v^h\|_{1,p} &\le c'(\|Lv^h\|_{1,p} + \|v^h\|_p) \\ &\le c'(c + \|v^h\|_p) \\ &\overset{\text{Fact 1}}{\le} c'(c + \|\partial_k v\|_p) \end{aligned}$$

$$\overset{\text{Fact 4}}{\Rightarrow} \partial_k v \in W^{1,p} \forall k$$
$$\Rightarrow v = \rho u \in W^{2,p}$$

Fact 5:

$$(fg)^h(x) = f^h(x)g(x) + f(x)g^h(x) \quad x_h := x + he_k$$

$$B(\phi, v^h) = \sum_{i,j} \int_\Omega \partial_i \phi a_{ij} \partial_j (\rho u)^h$$

Now the previous Claim implies interior regularity. ($x_h := x + he_k$)

$$\begin{aligned} B(\phi, v^h) &= \sum \int_\Omega \partial_i \phi a_{ij} (\partial_j(\rho u))^h \\ &= \sum \int_\Omega \partial_i \phi a_{ij} ((\partial_j \rho)u + \rho \partial_j u)^h \\ &= \sum \partial_i \phi a_{ij} (\partial_j \rho)^h u(x_h) + \sum \partial_i \phi a_{ij} (\partial_j \rho) u^h \\ &\quad + \underbrace{\sum \int_\Omega \partial_i \phi a_{ij} (\rho \partial_j u)^h}_{I} \end{aligned}$$

$$\begin{aligned} I &= \underbrace{\sum_{i,j} \int_\Omega \partial_i \phi (a_{ij} \rho \partial_j u)^h}_{II} - \sum_{i,j} \int_\Omega \partial_i \phi a_{ij}^h \rho(x_h) \partial_j u(x_h) \\ II &= -\sum \int_\Omega (\partial_i \phi)^{-h} a_{ij} \rho \partial_j u \\ &= -\underbrace{\sum \int \partial_i (\phi^{-h} \rho) a_{ij} \partial_j u}_{III} + \sum \int \phi^{-h} \partial_i \rho a_{ij} \partial_j u \\ III &= -B(\phi^{-h} \rho, u) \\ |III| &\leq c \|\phi^{-h} \rho\|_q \\ &\leq c \sup |\rho| \cdot \|\phi^{-h}\|_q \\ &\leq c \sup |\rho| \cdot \|\partial_k \phi\|_q \end{aligned}$$

Boundary Regularity: Case 1: Assume $\Omega \subset \{x \in \mathbb{R}^n \mid x_n > 0\}$ and

$$Q := \{x \in \mathbb{R}^n \mid |x_i| \leq 1 \text{ for } i = 1, \ldots, n \quad 0 < x_n \leq 1\} \subset \Omega$$

Denote

$$U := \{x \in \mathbb{R}^n \mid |x_i| < 1 \quad \forall i\}$$

Choose $\rho \in C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp} \rho \subset U$

$$\rho|_{U_\varepsilon} \equiv 1 \quad U_\varepsilon := \{x \mid |x_i| < 1 - \varepsilon \forall i\}$$

Let $v := \rho u|_Q \in W^{1,p}(\Omega) \quad v|_{\partial Q} = 0$ and

$$v^h(x) = \frac{v(x + he_k) - v(x)}{h} \qquad x + he_k, x \in \Omega, k = 1, \ldots, n-1$$

The same argument as before shows

$$\sup_{h \neq 0} \|v^h\|_{1,p} < \infty \quad \Rightarrow \partial_k v \in W^{1,p}(\Omega), k = 1, \ldots, n-1$$

$$\Rightarrow \partial_i \partial_j v \in L^p \quad \forall (i,j) \neq (n,n)$$

$$\Rightarrow \partial_n \partial_n v \in L^p$$

$$\begin{aligned} Lu &= -\sum_{i,j} \partial_i (a_{ij} \partial_j u) \\ &= a_{nn} \partial_n \partial_n u + f \end{aligned}$$

$$f := a_{nn} \partial_n \partial_n u + Lu \in L^p$$

$$\overset{a_{nn} > \delta}{\Rightarrow} \partial_n \partial_n u = \frac{f - Lu}{a_{nn}} \in L^p$$

Case 2: (general case):
"Coordinate change* near any point in $\partial\Omega$ $\square$

Higher Regularity: $\Omega \subset \mathbb{R}^n$ bounded, open, $\partial\Omega$ smooth. $k \geq 0$ integer and $1 < p < \infty$.

(1)
$$Lu = \sum_{i,j} a_{ij}\partial_i\partial_j u + \sum_i b_i\partial_i u + cu$$

where $a_{ji} = a_{ij}, b_i, c \in C^k(\overline{\Omega})$ (if $k = 0$ assume that $a_{ij} \subset C^1(\overline{\Omega})$).

$$\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq \delta|\xi|^2 \quad \forall x \in \overline{\Omega}\forall\xi \in \mathbb{R}^n$$

Definition: Let $f \in L^p(\Omega)$ and $u \in W_0^{1,p}(\Omega)$. u is called a *weak solution* of the Dirichlet problem

(2)
$$Lu = f$$
$$u|_{\partial\Omega} = 0$$

if

(3)
$$-\sum_{i,j}\int_\Omega \partial_i(\phi a_{ij})\partial_j u + \int_\Omega \phi(\sum_i b_i\partial_i u + cu) = \int_\Omega \phi f \quad \forall\phi \in C_0^\infty(\Omega)$$

Remark: Every strong solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ of (2) is a weak solution.

Theorem 6: Ω, L, k, p as above. Let $f \in W^{k,p}(\Omega)$ and suppose that $u \in W_0^{1,p}(\Omega)$ is a weak solution of $Lu = f \quad u|_{\partial\Omega} = 0$

$$\Rightarrow u \in W^{k+2,p}(\Omega)$$

Moreover $\exists c > 0 \forall u \in W^{k+2,p}(\Omega) \cap W_0^{1,p}(\Omega)$:

$$\|u\|_{k+2,p} \leq c(\|Lu\|_{k,p} + \|u\|_p$$

Proof:

Induction: Step 1: $k = 0$. Denote

$$L_0 u := -\sum_{i,j}\partial_i(a_{ij}\partial_j u)$$

1st order operator
$$P_u := Lu + L_0 u = \sum_j (b_j - \sum_i \partial_i a_{ij})\partial_j u + cu$$

$$\Rightarrow \exists c_1 > 0 \forall u \in W_0^{1,p}(\Omega) :$$
$$\|Pu\|_{1,p} \leq c_1\| \bigtriangledown u\|_p$$

Denote
$$B_0(u, v) = \sum_{i,j}\int_\Omega \partial_i u a_{ij}\partial_j v$$

$$\begin{aligned}\Rightarrow B_0(\phi,u)+\int_\Omega \phi f &\overset{(3)}{=} \sum_{i,j}\int_\Omega \partial_i\phi a_{ij}\partial_j u\\ &-\sum\int_\Omega \partial_i(\phi a_{ij})\partial_j u+\int_\Omega \phi(\sum_i b_i\partial_i u+cu)\\ &= \int_\Omega \phi(-\sum(\partial_i a_{ij})\partial_j u+\sum_i b_i\partial_i u+cu)\\ &= \sum_\Omega \phi P_u\end{aligned}$$

$$\Rightarrow B_0(\phi,u)=\int_\Omega)\phi(P_u-f)$$

$$\Rightarrow |B_0(\phi,u)|\le \|\phi\|_q\|P_u-f\|_p$$

$$\overset{\text{Thm 5}}{\Rightarrow} u\in W^{2,p}\cap W_0^{1,p}$$

Also:

$$\begin{aligned}\|u\|_{2,p} &\le c_0\|L_0u\|_p\\ &= c_0\|P_u-Lu\|_p\\ &\le c_0\|P_u\|_p+c_0\|Lu\|_p\\ &\le c_0c_1\|\bigtriangledown u\|_p+c_0\|Lu\|_p\\ &\le c_0c_1\|u\|_{1,p}+c_0\|Lu\|_p\end{aligned}$$

Interpolation estimate of Chapter II:

$$\|u\|_{1,p}\le c_2\|u\|_{1/2}\|u\|_{2,p}^{1/2}$$

$$\Rightarrow \|u\|_{2,p}\le c_0\|Lu\|_p+\underbrace{c_0c_1c_2\|u\|_p^{1/2}}_{a}\underbrace{\|u\|_{2,p}^2}_{b}$$

$$\le c_0\|Lu\|_p+\frac{(c_0c_1c_2)^2}{2}\|u\|_p+\frac{1}{2}\|u\|_{2,p}$$

$$\Rightarrow \|u\|_{2,p}\le 2c_0\|Lu\|_p+(c_0c_1c_2)^2\|u\|_p$$

Step 2: (Induction argument)
Let $k\ge 1$: Assume the result holds for $k-1$. $u\in W_0^{1,p}(\Omega)$ weak solution of (2), $f\in W^{k,p}(\Omega)\subset W^{k-1,p}(\Omega)$
$\Rightarrow$ By the induction hypothesis we have

$$u\in W^{k+1,p}(\Omega)$$

Fix a smooth cutoff function $\rho\in C_0^\infty(\Omega)$. Let $v:=\rho u$. Then

$$\begin{aligned}L\partial_i v &= \partial_i Lv+(L\partial_i-\partial_i L)v\\ &= \partial_i L(\rho u)+(L\partial_i-\partial_i L)v\\ &= \partial_i(\rho f)+\partial_i(L(\rho u)-\rho Lu)+(L\partial_i-\partial_i L)v\in W^{k-1,p}\end{aligned}$$

1.
$$\partial_\nu Lv-L\partial_\nu v=\sum_{i,j}(\underbrace{\partial_\nu a_{ij}}_{C^{k-1}}\underbrace{\partial_i\partial_j v}_{W^{k-1,p}}$$

$$+\sum_i(\underbrace{\partial_\nu b_i}_{C^{k-1}})\underbrace{\partial_i v}_{W^{k,p}}+(\underbrace{\partial_\nu c}_{C^{k-1}})\underbrace{v}_{W^{k+1,p}}\in W^{k-1,p}$$

2nd order operator, $v\in W^{k+1,p}\in W^{k-1,p}$ and

$$\|partial_\nu Lv-L\partial_\nu v\|_{k-1,p}\le c_1\|v\|_{k+1,p}$$

2.
$$L(\rho u) - \rho Lu = \sum_{i,j} 2a_{ij}\partial_i\rho\partial_j u + \sum_{i,j} a_{ij}\partial_i\partial_j\rho + \sum_i b_i(\partial_i\rho)u$$

1st order operator, $u \in W^{k+1,p}$, so $\partial_i(L\rho u - \rho Lu) \in W^{k-1,p}$

$$\|\partial_i(L\rho u - \rho Lu)\|_{k-1,p} \leq c_2\|u\|_{k+1,p}$$

3.
$$\|\partial_i\rho f\|_{k-1,p} \leq c\|f\|_{k,p}$$
$$\overset{\text{ind.hyp.}}{\Rightarrow} \partial_i v \in W^{k+1,p}$$

$$\begin{aligned}\|partial_i v\|_{k+1,p} &\leq c_0(\|L\partial_i v\|_{k-1,p} + \|v\|_p) \\ &\leq c_0(c_3\|f\|_{k,p} + c_2\|u\|_{k+1,p}(1+c)\|v\|_{k+1,p} \\ &\leq c(\|f\|_{k,p} + \|u\|_{k+1,p})\end{aligned}$$

Boundary Regularity:

$$\Rightarrow \partial_i(\rho u) \in W^{k+1,p} \quad i = 1,\dots,n-1$$
$$\Rightarrow \partial_n\partial_n(\rho u) \in W^{k,p}$$

local coordination

$$\Rightarrow \partial_i u \in W^{k+1,p} \quad \forall i \text{ and so } u \in W^{k+2,p}(\Omega)$$

Our estimate is:

$$\begin{aligned}\|u\|_{k+2,p} &\leq c(\|f\|_{k,p} + \|u\|_{k+1,p}) \\ &\overset{\text{interpolation}}{\leq} c\|f\|_{k,p} + c'\|u\|_p^{\frac{1}{k+2}}\|u\|_{k+2,p}^{\frac{k+1}{k+2}} \\ &\leq c\|f\|_{k,p} + \frac{1}{k+2}c'^{k+2}\|u\|_p + \frac{k+1}{k+2}\|u\|_{k+2,p}\end{aligned}$$

$\Rightarrow$ ESTIMATE! □

Lemma 9: $\Omega \subset \mathbb{R}^n$ bounded open, $\partial\Omega$ smooth. $1 < p < \infty$
$\Rightarrow$ The subspace
$$\{\phi \in C_0^\infty(\overline{\Omega} \mid \phi|_{\partial\Omega} = 0\}$$
is dense in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$

Proof: Exercise. Hint:

$$u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$$
$$\Delta u = f \quad (+ \text{ Thm } 6)$$

Corollary of Lemma 6 □

$$Lu = \sum_{i,j} a_{ij}\partial_i\partial_j u + \sum_i b_i u + cu$$
$$L^*v = \sum_{i,j} \partial_i\partial_j(a_{ij}v) - \sum_i (b_i v) + cv$$
$$a_{ij} = a_{ji} \in C^2(\overline{\Omega}), b_i \in C^1(\overline{\Omega}), c \in C^0(\overline{\Omega})$$
$$\Omega \subset \mathbb{R}^n \text{ bounded, open} \quad \partial\Omega \text{ smooth}$$

Lemma 10: $p, q > 1 \quad \frac{1}{p} + \frac{1}{q} = 1$

$$u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$$
$$v \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$$
$$\Rightarrow \int_\Omega (Lu)v = \int_\Omega u(L^* v)$$

Proof: <u>Exercise with hints:</u> Exercise for $u, v \in C^2(\overline{\Omega})$

$$u|_{\partial\Omega} = v|_{\partial\Omega} = 0$$

Then use Lemma 9. □

Definition: Let $f \in L^p(\Omega)$ and $u \in L^p(\Omega)$. u is called a *weak solution* of the Dirichlet problem

(1) $$Lu = f \quad u|_{\partial\Omega} = 0$$

if

(2) $$\int_\Omega u(L^*\phi) = \int_\Omega f\phi \quad \forall \phi \in C^\infty(\overline{\Omega}) \quad \phi|_{\partial\Omega} = 0$$

Remark 1: We can replace in (1), (2) $f \in L^p(\Omega)$ by a functional $\Phi \in W^{-1,p}(\Omega) = W_0^{1,q}(\Omega)^*$. Special case: $\Phi(\phi) = \int_\Omega f\phi$.

Remark 2: For $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$

$$u \text{ strong solution } \Leftrightarrow u \text{ weak solution}$$

Theorem 7: $\Omega, a_{ij}, b_i, c, L, p, q$ as above.

(i) Let $u \in L^p(\Omega), \Phi \in W^{-1,p}(\Omega)$ such that

$$\int_\Omega u(L^*\phi) = \Phi(\phi) \quad \forall \phi \in C^\infty(\overline{\Omega})$$

with

$$\phi|_{\partial\Omega} = 0$$
$$\Rightarrow u \in W_0^{1,p}(\Omega)$$

(ii) $\exists c > 0 \quad \forall u \in W_0^{1,p}(\Omega)$:

$$\|u\|_{1,p} \le c(\|Lu\|_{-1,p} + \|u\|_p)$$

(iii) $u, f \in L^p(\Omega)$
If u is a weak solution of $Lu = f$, $u|_{\partial\Omega} = 0$, ie. if (2) holds, then

$$u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$$

Remark 3: By (iii), the operator

$$L^* : W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \to L^q(\Omega)$$

is the functional analytic adjoint operator of

$$L : W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \to L^p(\Omega)$$

(as an unbounded operator).

$$L : \mathrm{dom}(L) \to X = L^p(\Omega)$$
$$\mathrm{dom}(L) \subset X \text{ dense} \quad \mathrm{dom}(L) = W^{2,p} \cap W_0^{1,p}$$

Let $v \in X^* = L^q(\Omega)$:

$$v \in \mathrm{dom}(L^*) \Leftrightarrow \exists c > 0 \forall u \in \mathrm{dom}(L) : |\langle v, Lu\rangle| \le c\|u\|_X$$

Exercise: Show that

$$\mathrm{dom}(L^*) = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$$

Corollary: If $L^* = L$ formally $\Rightarrow$ The operator:

$$L : W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \to L^2(\Omega)$$

is self-adjoint.

Proof: of Theorem 7

(i)$\Rightarrow$(iii): u satisfies the hypothesis of (i) with $\Phi(\phi) = \int_\Omega f\phi$

$$\overset{(i)}{\Rightarrow} u \in W_0^{1,p}(\Omega)$$

$$\overset{\text{Thm. 6, k¿0}}{\Rightarrow} u \in W^{2,p}(\Omega)$$

$$\overset{Lemma 10}{\Rightarrow} \int_\Omega (Lu - f)\phi = 0 \quad \forall \phi \in C^\infty(\Omega) \quad \phi|_{\partial_\Omega} = 0$$

$$\Rightarrow Lu = f$$

Proof of (i) and (ii):

$$L_0 := \sum_{i,j} \partial_i (a_{ij} \partial_j) = L_0^*$$

$$B_0(u, v) := \sum_{i,j} \int_\Omega (\partial_i u) a_{ij} (\partial_j v)$$

Observations:

1. $L = -L_0 + P$ and $L^* = -L_0 + P^*$, where

$$P^*\phi = \sum_i \partial_i((\sum_j \partial_j a_{ij} - b_i)\phi) + c\phi$$

   1st order operator

$$\Rightarrow \exists c > 0 \forall \phi \in C^\infty(\overline{\Omega}) \quad \phi|_{\partial\Omega} = 0$$

$$\|P^*\phi\|_q \leq c \| \bigtriangledown \phi\|_q$$

2. Define $\Psi \in W^{-1,p}(\Omega)$ by

$$\Psi(\phi) := \int_\Omega u P^* \phi$$

$$\Rightarrow \|\Psi\| = \sup_{0 \neq \phi \in C_0^\infty} \frac{|\Psi(\phi)|}{\| \bigtriangledown \phi\|_q} \leq c\|u\|_p$$

3. By Thm. 4, the operator $L_0 : W_0^{1,p}(\Omega) \to W^{-1,p}(\Omega)$ is bijective

$$\Rightarrow \exists! v \in W_0^{1,p}(\Omega) : \quad L_0 v = \Psi - \Phi$$

4. For $\phi \in C^\infty(\overline{\Omega})$ with $\phi|_{\partial\Omega} = 0$ we have

$$\begin{aligned}\int_\Omega (u-v)L_0\phi &\overset{1.}{=} \int_\Omega u(P^*\phi - L^*\phi) - \int_\Omega vL_0\phi \\ &= \int_\Omega u(P^*\phi - L^*\phi) - B_0(v,\phi) \\ &= \Psi(\phi) - \Phi(\phi) - B_0(v,\phi) \\ &\overset{3.}{=} 0\end{aligned}$$

$$\overset{\text{Lemma 9}}{\Rightarrow} \int_\Omega (u-v)L_0\phi = 0 \quad \forall \phi \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$$

By Thm. 3 the operator

$$L_0 : W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \to L^q(\Omega)$$

is bijective

$$\Rightarrow \int_\Omega (u-v)f = 0 \quad \forall f \in L^q(\Omega)$$

$$\Rightarrow u = v \in W_0^{1,p}(\Omega)$$

5. Estimate in (ii):

$$\begin{aligned}\|u\|_{1,p} &\le c_0\|L_0 v\|_{-1,p} \\ &\le c_0(\|\Phi\|_{-1,p} + \|\Psi\|_{-1,p}) \\ &\le c_0(\|Lu\|_{-1,p} + c\|u\|_p)\end{aligned}$$

□

Theorem 8: (*local regularity*):
Ω, L, L^*, p, q as in Thm. 7.
$a_{ij} \in C^{km}(\overline{\Omega}), b_i, c \in C^k(\overline{\Omega})$
(if $k = 0$ then assume $a_{ij} \in C^2(\overline{\Omega})$)

(i) if $u, f \in L^p_{\text{loc}}(\Omega)$ such that

$$\int_\Omega u(L^*\phi) = \int_\Omega f\phi \quad \phi \in C_0^\infty(\Omega)$$

Then

$$f \in W^{k,p}_{\text{loc}}(\Omega) \Rightarrow u \in W^{k+2,p}_{\text{loc}}(\Omega)$$

(ii) $\forall$ compact subsets $K \subset \Omega$

$$\exists c > 0 \quad \forall u \in W^{k+2,p}_{\text{loc}}(\Omega)$$

$$\|u\|_W^{k+2,p}(K) \le c(\|Lu\|_{k,p} + \|u\|_p)$$

Proof: of Thm. 8:

Proof of (i): Induction: $k = 0$:
Choose $\rho \in C_0^\infty(\Omega)$ such that $\rho|_K \equiv 1$
Choose $\beta \in C_0^\infty(\Omega)$ such that $\beta|_{\operatorname{supp}\rho} \equiv 1$
Define $\Phi \in W^{-1,p}(\Omega)$ by

$$\Phi(\phi) := \int_\Omega u(\rho L^*\phi - L^*(\rho\phi)) + \int_\Omega f\rho\phi$$

Note:

1.
$$\|\rho L^*\phi - L^*(\rho\phi)\|_q \le c\|\bigtriangledown\phi\|_q \quad \forall \phi \in C_0^\infty(\Omega)$$
.

2.
$$\begin{aligned}\int_\Omega (\rho u)L^*\phi &= \int_\Omega \rho u L^*\phi - \int_\Omega u L^*(\rho\phi) + \int_\Omega f\rho\phi \\ &= \Phi(\phi) \quad \forall\phi \in C_0^\infty(\Omega)\end{aligned}$$

3. Let $\phi \in C^\infty(\overline{\Omega}) \quad \phi|_{\partial\Omega} = 0$. Then $\beta\rho \in C_0^\infty(\Omega)$
$$\overset{2.}{\Rightarrow} \int_\Omega \rho u L^*\phi = \int_\Omega \rho u L^*(\beta\phi) = \Phi(\beta\phi) = \Phi(\phi)$$
$$\overset{\text{Thm. 7 (i)}}{\Rightarrow} \rho u \in W_0^{1,p}$$
$$\Phi(\phi) = \int_\Omega f\rho\phi + \int_\Omega (\underbrace{L(u\rho) - \rho L u}_{\text{1st order operator}})\phi$$
$$\Rightarrow \|L(u\rho) - \rho L u\|_{L^p}(\operatorname{supp}\rho) \le c\|\bigtriangledown u\|_{L^p(\operatorname{supp}\rho)}$$
$$\Rightarrow |\Phi(\phi)| \le c'\|\phi\|_q$$
$$\Rightarrow \exists g \in L^p(\Omega) \text{ such that } \Phi(\phi) = \int_\Omega g\phi$$
$$\overset{\text{Thm. 7 (iii)}}{\Rightarrow} \rho u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \quad \forall \rho \in C_0^\infty(\Omega)$$
$$\Rightarrow u \in W_{\text{loc}}^{2,p}(\Omega)$$

Induction on Step:
Let $k \ge 1$. Suppose the result holds for $k-1$.
Let $u \in L^p_{\text{loc}}(\Omega)$ be a weak solution of
$$Lu = f \in W_{\text{loc}}^{k,p}(\Omega) \subset W_{\text{loc}}^{k-1,p}(\Omega)$$
$$\overset{\text{ind. hyp.}}{\Rightarrow} u \in W_{\text{loc}}^{k+1,p}(\Omega) \text{ and}$$
$$L(\partial_\nu u) = \underbrace{(L\partial_\nu - \partial_\nu L)u + \partial_\nu f}_{\in W_{\text{loc}}{}^{k-1,p}(\Omega)}$$

in particular,
$$\begin{aligned}(L\partial_\nu - \partial_\nu L)u &= \sum_{i,j}(\partial_\nu a_{ij})\partial_i\partial_j u \\ &\quad + \sum_i (\partial_\nu b_i)\partial_i u + (\partial_\nu c)u \in W_{\text{loc}}^{k-1,p}(\Omega)\end{aligned}$$
$$\overset{\text{ind.hyp.}}{\Rightarrow} \partial_\nu u \in W_{\text{loc}}^{k+1,p}(\Omega) \quad \nu = 1, \ldots, n$$
$$\Rightarrow u \in W_{\text{loc}}^{k+2,p}(\Omega)$$

(enough to assume $a_{ij}, b_i, c \in C^k$).
Proof of the estimate in (ii):
Let $u \in W_{\text{loc}}^{k+2,p}$, $K \subset \Omega$ compact.
Choose $\rho \in C_0^\infty(\Omega) \quad \rho|_K \equiv 1$. $K' := \operatorname{supp}\rho$.
$$\begin{aligned}\Rightarrow \|u\|_{W^{k+2,p}(K)} &\le \|\rho u\|_{k+2,p} \\ &\overset{\text{Thm. 6}}{\le} c(\|L(\rho u)\|_{k,p} + \|\rho u\|_p) \\ &\le c(\|\rho L u\|_{k,p} + \|L(\rho u) - \rho L u\|_{k,p} + \|\rho u\|_p) \\ &\le c'(\|Lu\|_{k,p} + \|u\|_{W^{k-1,p}(K')} + \|u\|_{L^p}(K')) \\ &\overset{\text{ind.hyp for K'}}{\le} c''(\|Lu\|_{k,p} + \|u\|_{W^{1,p}(K')}\end{aligned}$$

$$\underline{k=0}: \|u\|_{W^{2,p}(K)} \leq c\|Lu\|_p + \|u\|_{W^{1,p}(K')}$$

$$\underline{k=-1}: u \in W^{1,p}_{\text{loc}}(\Omega): \quad \|u\|_{W^{1,p}(K)} \leq$$

$$\leq c(\|L(\rho u)\|_{-1,p} + \|\rho u\|_p) \leq c'(\|Lu\|_{-1,p} + \|u\|u_p)$$

Exercise:

$$\|L(\rho u)\|_{-1,p} \leq c(\|lu\|_{-1,p} + \|u\|_p)$$

□

We have proved that the operator

$$L_0 = -\sum \partial_i(a_{ij}\partial_j)$$

is bijective.

Maximum Principle: $\Omega \subset \mathbb{R}^n$ open, connected
$L := \sum_{i,j} a_{ij}\partial_i\partial_j + \sum_i b_i\partial_i$
$a_{ij} = a_{ji} \quad b_i \in C^0(\Omega)$
$A(x) := (a_{ij}(x))^n_{i,j=1}$ pos. def. $\forall x \in \Omega$

Theorem 9: (E. Hopf, 1927):
Let $u \in C^2(\Omega)$ such that $Lu \geq 0$. Assume $x_0 \in \Omega$

$$u(x_0) = \max_{x\in\Omega} u(x) =: M$$

$$\Rightarrow u(x) = M \quad \forall x \in \Omega$$

Corollary: If in addition $a_{ij} \in C^2(\overline{\Omega}), b_i \in C^1(\overline{\Omega})$:

$$L: W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \to L^p(\Omega) \text{ is bijective}$$

Proof: Exercise (Sheet 11) □

Proof: (of Thm. 9): 5 steps!
Assume $u \neq M$

Step 1: If $Lu(x_0) > 0$ then this is impossible (because $\exists \xi \in \mathbb{R}^n \setminus \{0\}$ such that $u(x_0 + t\xi) < M$ for $t \neq 0$ sufficiently small). Exercise!

Step 2: $\exists \xi, \eta \in \Omega \exists \rho > 0$ such that

$$\overline{B_\rho(\xi)} \subset \Omega \quad \eta \in \partial B_\rho(\xi)$$

$$u(\eta) = M \quad u < M \text{ in } \overline{B_\rho(\xi)} \setminus \{\eta\}$$

Proof of Step 2: Ω connected

$$\Rightarrow \exists \gamma : [0,1] \to \Omega$$

$$\gamma(0) = x_0 \quad u(\gamma(1)) < M$$
$$t_1 := \max\{t > 0 \mid u(\gamma(s)) = M \quad \forall s \in [0,t]\}$$
$$x_1 := \gamma(t_1) \Rightarrow 0 \le t_1 \le 1 : u(x_1) = M$$
$$\exists \varepsilon_1 > 0 \quad \overline{B_{2\varepsilon_1}(x_1)} \subset \Omega$$

Choose

$$t_2 > t_1 : u(\gamma(t_2)) < M_1 |\gamma(t_2) - x_1| \le \varepsilon_1$$
$$x_2 := \gamma(t_2) \quad \varepsilon_2 := \sup\{\varepsilon > 0 \mid u < M \text{ in } B_{\varepsilon_2}(x_2)\}$$
$$\Rightarrow \varepsilon_2 < \varepsilon_1$$
$$\Rightarrow \overline{B_{\varepsilon_2}(x_2)} \subset \overline{B_{2\varepsilon_1}(x_1} \subset \Omega$$
$$u < M \text{ in } B_{\varepsilon_2}(x_2)$$
$$\exists \eta \in \partial B_{\varepsilon_2}(x_2) \quad u(\eta) = M$$
$$\xi := \frac{\eta + x_2}{2} \quad \rho := \frac{\varepsilon_2}{2}$$

Choose $0 < r < \frac{\rho}{2}$ such that $\overline{B_r(\eta)} \subset \Omega$

$$\alpha := \partial B_r(\eta) \cap \overline{B_\rho(\xi)} \text{ closed}$$

$$\beta := \partial B_r(\eta) \setminus \overline{B_\rho(\xi)}$$
$$\Rightarrow u < M \text{ on } \alpha \quad u \le M \text{ on } \beta$$
$$\overset{\alpha \text{ cpct}}{\Rightarrow} \exists \delta > 0 \text{ such that } u \le M - \delta \text{ on } \alpha$$
$$\exists \mu > 0 : \sum_{i,j=1}^{n} a_{ij}(x)\zeta_i\zeta_j \ge |\zeta|^2 \quad \forall x \in \overline{B_r(\eta)} \quad \forall \zeta \in \mathbb{R}^n$$

Step 4: $\exists v : \mathbb{R}^n \to \mathbb{R}$ smooth such that

(i)
$$v > 0 \text{ on } B_\rho(\xi)$$
$$v = 0 \text{ on } \partial B_\rho(\xi)$$
$$v < 0 \text{ on } \mathbb{R}^n \setminus \overline{B_\rho(\xi)}$$

(ii) $Lv > 0$ on $\overline{B_r(\eta)}$

Proof of Step 4: $v(x) := \exp(-\theta|x-\xi|^2) - \exp(-\theta\rho^2)$ satisfies (i) for any θ and satisfies (ii) for θ large.

$$\partial_i v = -2\theta(x_i - \xi_i)\exp(-\theta|x-\xi|^2)$$
$$\partial_i\partial_j v = (4\theta^2(x_i-\xi_i)(x_j-\xi_j) - 2\theta\delta_{ij})\exp(-\theta|x-\xi|^2)$$

$$\begin{aligned} Lv &= \exp(-\theta|x-\xi|^2)(4\theta^2 \sum a_{ij}(x)(x_i-\xi_i)(x_j-\xi_j) \\ &\quad - \sum 2\theta a_{ij}(x) - 2\theta \sum b_i(x)(x_i - \xi_i) \\ &\ge \exp(-\theta|x-\xi|^2)(4\theta^2\mu|x-\xi|^2 - \ldots) \\ &\ge \theta\exp(-\theta|x-\xi|^2)(4\theta\mu|x-\xi|^2 - const) \\ &\ge \theta\exp(-\theta|x-\xi|^2)(4\theta\mu\frac{\rho}{2} - const) \end{aligned}$$

Step 5: Proof of Thm. 9:
Let $\omega : 0u + \varepsilon v$, where $0 < \varepsilon < \frac{\delta}{2c}$ and $c := \sup_{\overline{B_\rho(\xi)}} v$

$$\Rightarrow \varepsilon v < \frac{\delta}{2} \text{ in } \overline{B_\rho(\xi)}$$

Recall: $u \leq M - \delta$ on α and $v < 0$ on β

$$\Rightarrow \omega = u + \varepsilon v \leq M - \frac{\delta}{2} \text{ on } \alpha$$

$$\omega = u + \varepsilon v < M \text{ on } \beta$$

$$\omega < M \text{ on } \partial B_r(\eta)$$

$$L\omega = \underbrace{Lu}_{\geq 0} + \varepsilon \underbrace{Lv}_{>0} \text{ in } \overline{B_r(\eta)} \quad \omega(\eta) = M$$

and this contradicts Step 1. □

5 The Laplace Beltrami operator and Uniformization:

Diff. Top.: smooth manifolds and smooth maps
Diff. Geo.: Riemannian manifolds isometries
M^m smooth m-manifold, where M top. space. $\{U_\alpha\}$ open cover.
Coordinate charts $\phi_\alpha : U_\alpha \to \mathbb{R}^m$ homeomorphism onto open set.
Transition maps:

$$\phi_{\beta\alpha} := \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta) \text{ Diffeo}$$

$\{U_\alpha, \phi_\alpha\}$ *atlas*

Definition: $\{U_\alpha, \phi_\alpha\}_\alpha$ *oriented atlas* if

$$\det(d\phi_{\beta\alpha}(x)) > 0 \quad \forall \alpha, \beta \forall x$$

smooth maps: $f : M^m \to N^n$, $\{U_\alpha, \phi_\alpha\}$, $\{V_\beta, \psi_\beta\}$. Then

- f continuous
- $$f_{\beta_\alpha} := \psi_\beta \circ f \circ \phi_a lpha^{-1} : \phi_\alpha(U_\alpha \cap f^{-1}(V_\beta)) \to \psi_\beta(V_\beta) \text{ smooth } \forall \beta, \alpha$$

tangent space:

$$T_pM = \bigcup_{p\in U_\alpha} \{\alpha\} \times \mathbb{R}^m \backslash \sim$$

$$(\alpha, \xi) \sim (\alpha', \xi') \Leftrightarrow \xi' = d(\phi_{\alpha'} \circ \phi_\alpha^{-1})(x)\xi \quad x := \phi_\alpha(p)$$

derivative of f at p:

$$df(p) : T_pM \to T_{f(p)}N \quad v = [\alpha, \xi] \in T_pM$$

$$df(p)v := [\beta, df_{\beta\alpha}(x)\xi]$$

$$p \in U_\alpha \quad f(p) \in V_\beta \quad x := \phi_\alpha(p)$$

Notation for local coordinates: Fix a coordinate chart $\phi_\alpha : U_\alpha \to \mathbb{R}^m$

$$\mathbb{R}^m \ni \phi_\alpha(p) = (x^1(p), \ldots, x^m(p))$$

$$(x^1, \ldots, x^m) : U_\alpha \to \phi_\alpha(U_\alpha) \subset \mathbb{R}^m$$

$$x^i : U_\alpha \to \mathbb{R} \quad dx^i(p) : T_pM \to \mathbb{R} \text{ basis of } T_p^*M$$

Any linear map $\omega : T_pM \to \mathbb{R}$ can be written as $\omega = \sum \omega_i dx^i$
dual basis of T_pM is $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}$ and

$$dx_i(\frac{\partial}{\partial x^j}) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Exercise:

$$\frac{\partial}{\partial x^i}(p) = [\alpha, e_i] \quad e_i := (0, \ldots, \underbrace{1}_{i}, 0, \ldots, 0)$$

$$p \in U_\alpha \quad v := \frac{\partial}{\partial x_i}(p) = [\alpha, e_i]$$

$$f : M \to \mathbb{R} \quad f_\alpha := f \circ \phi_\alpha^{-1}$$

$$df(p)v = df_\alpha(x)e_i = \frac{\partial f_\alpha}{\partial x_i}$$

$$df(p)\frac{\partial}{\partial x_i}(p) = \frac{\partial f_\alpha}{\partial x_i}(x) \quad x := \phi_\alpha(p)$$

tangent bundle:

$$TM = \{(p,v) \mid p \in M, v \in T_pM\} = \sqcup_{q\in M} T_pM$$

T_pM is a $2m$-dimensional manifold
Riemannian metric: inner product on each tangent space

$$\begin{cases} T_pM \times T_pM \to \mathbb{R} \\ (v,w) \mapsto \langle v,w\rangle \end{cases}$$

"depending smoothly on p", ie.

$$R^m \supset \phi_\alpha(U_\alpha) \to \mathbb{R}$$

$$x \mapsto g_{ij}(x) := \langle \frac{\partial}{\partial x^i}(\phi_\alpha^{-1}(x)), \frac{\partial}{\partial x^j}(\phi_\alpha^{-1}(x))\rangle$$

is smooth for all i,j. $g = (g_{ij}) : \phi_\alpha(U_\alpha) \to \mathbb{R}^{m\times m}$.
Note: If $v = [\alpha, \xi]$ and $w = [\alpha, \eta]$

$$\xi = (\xi^1, \ldots, \xi^m) \quad \eta = (\eta^1, \ldots, \eta^m) \text{ and } x = \phi_\alpha(p)$$

then

$$\langle v,w\rangle = \sum_{i,j=1}^m \xi^i g_{ij}(x)\eta^j$$

$g_\alpha : \phi_\alpha(U_\alpha) \to \mathbb{R}^{m\times m}$
transition maps:

$$g_\alpha(x) = d\phi_{\beta\alpha}(x)^T g_\beta(\phi_{\beta\alpha}(x)) d\phi_{\beta\alpha}(x)$$
$$\forall \alpha, \beta \quad \forall x \in \phi_\alpha(U_\alpha \cap U_\beta)$$

Example: $f : M \to \mathbb{R}$ determines functions $f_\alpha : \phi_\alpha(U_\alpha) \to \mathbb{R}$ such that

$$f_\alpha|_{\phi_\alpha(U_\alpha\cap U_\beta)} = f_\beta \circ \phi_{\beta\alpha} \quad \forall \alpha, \beta$$

and vice versa.
Each Riemannian metric g on an oriented manifold M^m determines a volume form $d\mathrm{Vol}_g \in \Omega^m(M)$.

Exterior Algebra: V n-dimensional real vector space
$V^* = \mathcal{L}(V, \mathbb{R})$ dual space
$\Lambda^k V^* = \{$alternating multilinear k-forms on V$\}$
Let $\omega \in \Lambda^k V^*$. Then $\omega : \underbrace{V \times V \times \cdots \times V}_{\text{k times}} \to \mathbb{R}$
$\omega(v_{\delta(1)}, \ldots, v_{\delta(k)}) = \mathrm{sign}(\delta) \cdot \omega(v_1, \ldots, v_k)$ for every permutation $\delta \in \rho_k$.
$\dim \Lambda^k V^* = \binom{n}{k} \quad k = 0, \ldots, n.$

Exercise: Suppose V is oriented and is equipped with an inner product

$$V \times V \to \mathbb{R} \quad (v,w) \mapsto \langle v,w\rangle.$$

Show that there is a unique n-form $\omega \in \Lambda^k V^*$ satisfying

$$\omega(e_1, \ldots, e_n) = 1$$

for every positive oriented ONB $e_1, \ldots, e_n$.

Exterior Product:

$$\Lambda^k V^* \times \Lambda^l V^* \to \Lambda^{k+l} V^* \quad (\omega, \tau) \mapsto \omega \wedge \tau$$

$$\omega \wedge \tau(v_1, \ldots, v_{k+l} = \sum_{\delta\in\rho_{k,l}} \mathrm{sign}(\delta)\omega(v_{\delta(1)}, \ldots, v_{\delta(k)})\tau(v_{\delta(k+1)}, \ldots, v_{\delta(k+l)})$$

$$\rho_{k,l} = \{\delta \in S_{k+l} \mid \delta(1) < \delta(2) < \ldots < \delta(k), \delta(k+1) < \ldots < \delta(k+l)\}$$

Pullback:

$$\Phi : W \to V \text{ linear} \qquad \Phi^* : \lambda^k V^* \to \Lambda^k W^*$$
$$(\Phi^*\omega)(\omega_1, \dots, \omega_k) := \omega(\Phi\omega_1, \dots, \Phi\omega_k)$$

Rules:

- bilinear: $\omega_1(\tau_1 + \tau_2) = \omega_1\tau_1 + \omega_1\tau_2$
- associative: $\omega_1(\tau_1\rho) = (\omega_1\tau_1)\rho$ and $\Phi^*(\omega_1\tau) = \Phi^*\omega_1\Phi^*\tau$.

Example: $V = \mathbb{R}^n \ni \xi = (\xi^1, \dots, \xi^n)$. $I = (i_1, \dots, i_k)$, where $i_1 < i_2 < \dots < i_k$. Define $dx^I \in \Lambda^k(\mathbb{R}^n)^*$ by

$$dx^I(\xi_1, \dots, \xi_k) = \det \begin{pmatrix} \xi_1^{i_1} & \dots & \xi_k^{i_1} \\ \vdots & & \vdots \\ \xi_1^{i_k} & \dots & \xi_k^{i_k} \end{pmatrix}$$

Check:

(1) $dx^I = dx^{i_1} \wedge \dots \wedge dx^{i_k}$

(2) dx^I form a basis of $\Lambda^k(\mathbb{R}^i)^*$.

(3) If $\Phi \in \mathbb{R}^{n\times n}$ then

$$\Phi^* dx^I = \sum_J \det(\Phi^I_J) dx^J$$
$$\Phi^I_J = \begin{pmatrix} \Phi^{i_1}_{j_1} & \dots & \Phi^{i_k}_{j_k} \\ \vdots & & \vdots \\ \Phi^{i_k}_{j_1} & \dots & \Phi^{i_k}_{j_k} \end{pmatrix},$$
$$\xi = \Phi_n \qquad \xi^i = \sum_j \Phi^i_j \eta^j$$

M^n manifold. $T^*M \to M$, where $T^*M = \cup_{p\in M}\{p\} \times T^*M$

$$\Lambda^k T^*M \to M \text{ vector bundle of alternating k-forms}$$
$$\Lambda^k T^*M \ni (p, \omega) \quad \omega : \underbrace{T_p^*M \times \dots \times T_p^*M}_{\text{k times multilinear alternation}} \quad \to \mathbb{R}$$
$$\Omega^k(M) = \{\text{sections of the vector bundle } \Lambda^k T^*M\}$$
$$= \{\text{differential k-forms on } M\} \ni \omega$$

ω is a family of alternating k-forms $\omega_p \in \Lambda^k T_p^*M$, one for each $p \in M$ depending smoothly on p.

Pullback:

$$f : N \to M \qquad \omega \in \Omega^k(M)$$

Define $f^*\omega \in \Omega^k(N)$ by

$$(f^*\omega)_q(\omega_1, \dots, \omega_k) := \omega_{f(q)}(df(q)\omega_1, \dots, df(q)\omega_k)$$

local coordinates:

$$\sum_I \omega_{\alpha,I}(x) dx^I = \omega_\alpha := (\phi_\alpha^{-1})^*\omega \in \Omega^k(\phi_\alpha(U_\alpha))$$

$$\omega_\alpha = \phi^*_{\beta\alpha}\omega_\beta$$
$$f : N \to M \text{ then}$$

$$\begin{aligned}(f^*\omega)_\beta &= f^*_{\alpha\beta}\omega_\alpha \\ &= \sum_J \det\left(\frac{\partial f^I_{\alpha\beta}}{\partial y^J}(y)\right)\omega_{\alpha,I}(f_{\alpha\beta}(y)) dy^J\end{aligned}$$

Integration: M^n oriented, compact. $\omega \in \Omega^n(M)$.

$$\omega_\alpha = \omega_\alpha(x) dx^1 \wedge \ldots \wedge dx^n \in \Omega^n(\phi_\alpha(U_\alpha))$$

$$\omega_\alpha(x) = \omega_\eta(\phi_{\beta\alpha}(x)) \det(d\phi_{\beta\alpha}(x))$$

$$\int_{\phi_\alpha(U_\alpha \cap U_\beta)} \omega_\alpha = \int_{\phi_\beta(U_\alpha \cap U_\beta)\omega_\beta}$$

<u>Orientation:</u>

$$\det(d\phi_{\beta\alpha}(x)) > 0 \quad \forall x \in \phi_\alpha(U_\alpha \cap U_\beta) \quad \forall \alpha, \beta$$

("change of variables").
Choose a partition of unity $\rho_\alpha : M \to [0,1]$, $\text{supp}(\rho_\alpha) \subset U_\alpha$ and $\sum_\alpha \rho_\alpha \equiv 1$. Define

$$\int_M \omega := \sum_\alpha \int_{\phi_\alpha(U_\alpha)} (\rho_\alpha \circ \phi_\alpha^{-1}) \omega_\alpha$$

independent of ϕ_α and ρ_α. Changing the orientation of M changes the sign of $\int_M \omega$.
M^n oriented Riemannian manifold. g Riemannian metric. (This gives us the volume form). $d\text{Vol}_g \in \Omega^n(M)$ unique n-form on each tangent space T_pM associated to the given inner product and orientation as in Exercise 1.

In local coordinates: $\phi : U_\alpha \to \mathbb{R}^n$ positively oriented coordinate chart.
<u>metric:</u> $g_\alpha = (g_{\alpha,ij}) : \phi_\alpha(U_\alpha) \to \mathbb{R}^{n \times n}$ positive definite

Exercise 2: In local coordinated the volume form of g is given by

$$(d\text{Vol}_g)_\alpha = \sqrt{\det(g_\alpha(x))} dx^1 \wedge \ldots \wedge dx^n$$

well-defined.

$$g_\alpha(x) = d\phi_{\beta\alpha}(x)^T g_\beta(\phi_{\beta\alpha}(x)) d\phi_{\beta\alpha}(x)$$

$$\Rightarrow \det g_\alpha(x) = \det(g_\beta(\phi_{\beta\alpha}(x)) \underbrace{(\det d\phi_{\beta\alpha}(x))^2}_{>0}$$

$$\Rightarrow \sqrt{\det g_\alpha(x)} = \sqrt{\det g_\beta(\phi_{\beta\alpha}(x))} \det d\phi_{\beta\alpha}(x)$$

$$\Rightarrow (d\text{Vol}_g)_\alpha = \phi_{\beta\alpha}^*(d\text{Vol}_g)_\beta$$

Observations:

1. $d\text{Vol}_g$ defines a Borel measure on M. We can integrate function $f : M \to \mathbb{R}$
2. <u>Inner product on $\Omega^0(M) = C^\infty(M, \mathbb{R})$</u>

$$\langle f, h \rangle_{L^2} := \int_M fh \cdot d\text{Vol}_g$$

3. <u>Inner product on $\Omega^1(M)$</u>
   Pointwise inner product on $T_p^*M \ni \omega, \tau$

$$\langle \omega, \tau \rangle_g := \sum_{i=1}^n \omega(e_1)\tau(e_i) \quad e_1, \ldots, e_n \text{ ONB wrt } g$$

   in local coordinates

$$\omega = \sum_i \omega_i dx^i \quad \tau = \sum_i \tau_i dx^i$$

$$\langle \omega, \tau \rangle = \sum_{i,j} \omega_i g_\alpha^{ij}(x) \tau_j$$

$$x = \phi_\alpha(p) \qquad (g_\alpha^{ij}(x)) := (g_{\alpha,ij}(x))^{-1}$$

So $\sum_j g_\alpha^{ij} g_{\alpha,jk} = \delta_k^i$. For $\omega, \tau \in \Omega^1(M)$ define

$$\langle \omega, \tau \rangle_{L^2} := \int_M \langle \omega, \tau \rangle_g d\mathrm{Vol}_g,$$

where $\langle \omega, \tau \rangle_g : M \to \mathbb{R}$.

4. The derivative of a smooth function $f : M \to \mathbb{R}$ assigns to each $p \in M$ the linear functional

$$df(p) : T_pM \to \mathbb{R}$$

This gives an operator

$$d : \Omega^0(M) \to \Omega^1(M)$$

in local coordinates

$$(df)_\alpha = \sum_{i=1}^n \frac{\partial f_\alpha}{\partial x^i}(x) dx^i$$

Definition: Let $d^* : \Omega^1(M) \to \Omega^0(M)$ be the *formal adjoint operator* of $d : \Omega^0(M) \to \Omega^1(M)$ with respect to the L^2-inner products in 2. and 3.
The decomposition

$$d^*d : \Omega^0(M) \to \Omega^0(M)$$

is called the *Laplace-Beltrami operator* associated to g.

Lemma 1: In local coordinates, if $\omega_\alpha = \sum_{i=1}^n \omega_{\alpha,i}(x) dx^i$ then

$$(d^*\omega)_\alpha = -\frac{1}{\sqrt{g_\alpha}} \sum_{i,j} \frac{\partial}{\partial x_i} \left(\sqrt{\det g_\alpha} g_\alpha^{ij} \frac{\partial f_\alpha}{\partial x^j} \right)$$

Proof: Exercise (integration by parts). □

Corollary: In local coordinates

$$(d^*df)_\alpha = -\frac{1}{\sqrt{\det g_\alpha}} \sum_{i,j} \frac{\partial}{\partial x^i} \left(\sqrt{\det g_\alpha} g_\alpha^{ij} \frac{\partial f_\alpha}{\partial x^j} \right)$$

2nd order elliptic operator

Sobolev spaces:

$$f \in W^{k,p}(M) :\overset{\text{Def}}{\Leftrightarrow} f \circ \phi_\alpha^{-1} \in W^{k,p}_{loc}(\phi_\alpha(U_\alpha)) \quad \forall \alpha$$

(independent of choice of atlas $(\phi_\alpha)_\alpha$). $W^{k,p}$ norm: M compact. Partition of unity $\rho_\alpha : M \to [0,1]$, $\mathrm{supp}(\rho_\alpha) \subset U_\alpha$.

$$\|f\|_{W^{k,p}(M)} := \|f\|_{k,p} := \sum_\alpha \|(\rho_\alpha f) \circ \phi_\alpha^{-1}\|_{W^{k,p}(\phi_\alpha(U_\alpha))}$$

Lemma 2: M compact, connected, oriented, no boundary.

$$\Rightarrow d^*d : W^{k+2,p}(M) \to W^{k,p}(M)$$

is a *Fredholm operator* of index zero

$$\ker d^*d = \{ f \equiv \text{ const } \}$$

$$\mathrm{im}\, d^*d = \{ f \in W^{k,p} \mid \int_M f d\mathrm{Vol}_g = 0 \}$$

Proof:

1. Let $u \in W^{k+2,p}(M)$. Then

$$\begin{aligned}
\|u\|_{k+2,p} &= \sum_\alpha \|\rho_\alpha u\|_{W^{k+2,p}(U_\alpha)} \\
&= \sum_\alpha \|(\rho_\alpha u)\circ\phi_\alpha^{-1}\|_{W^{k+2,p}(\phi_\alpha(U_\alpha))} \\
&\overset{\text{Chapter III}}{\leq} c\sum_\alpha \|(d^*d(\rho_\alpha u))_\alpha\|_{W^{k,p}(\phi_\alpha(U_\alpha))} \\
&\leq c\sum_\alpha \|\rho_\alpha(d^*du)\|_{W^{k,p}(U_\alpha)} \\
&\quad +c\sum_\alpha \|d^*d(\rho_\alpha u)-\rho_\alpha(d^*du)\|_{W^{k,p}(U_\alpha)} \\
&\leq c'\|d^*du\|_{k,p}+c'\|u\|_{k+1,p}
\end{aligned}$$

$W^{k+2,p}(M) \hookrightarrow W^{k+1,p}(M)$ compact operator (Rellich)

$\Rightarrow d^*d$ has a finite dimensional kernel and a closed image

2. $d^*du = 0 \overset{\text{Chapter IV}}{\Rightarrow} u$ smooth.

$$0 = \int_M ud^*du d\mathrm{Vol}_g = \int_M \langle du, du\rangle d\mathrm{Vol}_g = \|du\|_{L^2}^2$$

$$\Rightarrow du \equiv 0$$

3. <u>$k = 0$:</u>
$\mathcal{X} := \{f \in L^p(M) \mid \int_M f d\mathrm{Vol}_g = 0\}$. Then

$$d^*du \in \mathcal{X} \quad \forall u \in W^{2,p}(\Omega)$$

$$\int_M 1\cdot d^*du d\mathrm{Vol}_g = \int_M \langle \underbrace{d1}_{=0}, du\rangle d\mathrm{Vol}_g = 0$$

So $\operatorname{im} d^*d \subset \mathcal{X}$. $\operatorname{im} d^*d$ is closed by 1.
To show: is d^*d dense in $\mathcal{X}$?
Let $h \in L^q(M) = L^p(M)^*$ such that

$$\int_M hd^*du d\mathrm{Vol}_g = 0 \quad \forall h$$

$$\overset{III.\text{ Thm }8}{\Rightarrow} h \in W^{2,q}(M), d^*dh$$

$$\overset{2.}{\Rightarrow} h \equiv \text{ const.}$$

$$h \perp \mathcal{X}$$

$$\overset{\text{Hahn-Banach}}{\Rightarrow} \operatorname{im} d^*d \text{ dense in } \mathcal{X}$$

<u>$k \geq 1$:</u> Let $f \in W^{k,p}(M)$ with $\int_M f d\mathrm{Vol}_g = 0$

$$\overset{k=0}{\Rightarrow} u \in W^{2,p}(M) \quad d^*du = f \in W^{k,p}$$

$$\overset{\text{III.8}}{\Rightarrow} u \in W^{k+2,p}$$

□

Gauss curvature $\Sigma^2 \subset \mathbb{R}^3$
Gauss map: $\nu : \Sigma \to S^2 := \{v \in \mathbb{R} \mid Å\ vÅ\ = 1\}$
$d\nu(x) : T_x\Sigma \to T_{\nu(x)}S^2 = \nu(x)^\perp$
The two tangent spaces $T_x\Sigma$ and $T_{\nu(x)}S^2$ are equal.

$$K(x) := \det(d\nu(x)), \text{ where } d\nu(x) : \nu(x)^\perp \to \nu(x)^\perp$$

Gauss curvature $K : \Sigma \to \mathbb{R}$.

1. $K > 0$
2. $K = 0$
3. $K < 0$

Gauss-Bonnet:

$$\int_\Sigma K_y d\mathrm{Vol}_g = 2\pi\xi(\Sigma) = 2\pi(2 - 2\text{genus})$$

Levi-Civita Connection: tangent vectors in every point
$\gamma : \mathbb{R} \to M$
$\Phi_\gamma(t_1, t_0) : T_{\underbrace{\gamma(t_0)}_{v_0}} M \to T_{\underbrace{\gamma(t_1)}_{v_1}} M$
"Parallel transport of tangent vectors $v(t) \in T_{\gamma(t)}M$ along a curve".
Solve a differential equation. Define a *covariant derivative*

$$T_{\gamma(t)}M \ni \nabla_t v \sim \text{"}\frac{\partial}{\partial t}v(t)\text{"}$$

$$\text{Linearity: } \nabla_t (v + w) = \nabla_t v + \nabla_t w$$

$$\text{Leibnitz rule: } \nabla_t (\lambda v) = (\partial_t\lambda)v + \lambda \nabla_t v, \lambda : \mathbb{R} \to \mathbb{R}$$

In local coordinates:

$$\phi_\alpha : U_\alpha \to \mathbb{R}^n \quad x(t) := \phi_\alpha(\gamma(t))$$

$$v(t)_\alpha = \xi(t) = d\phi_\alpha(\gamma(t))v(t)$$

$$\text{i.e. } v(t) = [\alpha, \xi(t)]$$

Ansatz:

$$(\nabla\xi)^k(t) = (\nabla_t v(t))^k_\alpha = \xi'^k(t) + \Gamma^k_{ij}(x'^i(t))\xi^j(x)$$

Axioms:

1. If $\Phi_\gamma(t_1, t_0) : T_{\gamma(t_0)}M \to T_{\gamma(t_1)}M$ is defined by

$$\Phi_\gamma(t_1, t_0)v_0 = v_1,$$

where $v_1 := v(t_1)$ and $v(t) \in T_{\gamma(t)}M$ is the unique vectorfield along γ such that

$$\nabla_t v \equiv 0,$$

then $\Phi_\gamma(t_1, t_0)$ preserves the Riemannian. Equivalent:

(1)
$$\frac{\partial}{\partial t}\langle v, w\rangle = \langle \nabla_t v, w\rangle + \langle v, \nabla_t w\rangle$$

2. If $\gamma : \mathbb{R}^2 \to M \quad (s, t) \in \mathbb{R}^2$, then

(2)
$$\nabla_s \partial_t \gamma = \nabla_t \partial_s \gamma$$

Definition: $\bigtriangledown$ is called *Riemannian* if (1) holds and *torsion free* if it satisfies (2).

Definition: Every Riemannian manifold (M, g) admits a unique torsion free Riemannian connection $\bigtriangledown$ (on TM), called the *Levi-Civita connection.*
Formula:

(*)
$$\Gamma_{ij}^k(x) = \sum_l g^{kl}(x)\Gamma_{lij}(x),$$

where
$$\Gamma_{lij}(x) := \frac{1}{2}\left(\frac{\partial g_{li}}{\partial x^j}(x) + \frac{\partial g_{lj}}{\partial x^i}(x) - \frac{\partial g_{ij}}{\partial x^l}(x)\right)$$

Character symbols.

Remark:
$$(2) \Leftrightarrow \Gamma_{ij}^k = \Gamma_{ji}^k$$
$$(1) \Leftrightarrow \frac{\partial g_{ij}}{\partial x^l} = \sum_{\nu=1}^{n}(\Gamma_{li}^{\nu} g_{\nu_j} + g_{i\nu}\Gamma_{lj}^{\nu}) \Leftrightarrow (*)$$

Do covariant derivatives commute?

$$\gamma : \mathbb{R}^2 \to M \quad X : \mathbb{R}^2 \to TM$$
$$X(s,t) \in T_{\gamma(s,t)}M$$
$$\bigtriangledown_s \bigtriangledown_t X \stackrel{?}{=} \bigtriangledown_t \bigtriangledown_s X \qquad \text{NO!}$$

Riemann curvature tensor:

$$p \in M \quad R_p : T_pM \times T_pM \to \mathcal{L}(T_pM, T_pM)$$
$$R_\gamma(\partial_s\gamma, \partial_t\gamma)X = \bigtriangledown_s \bigtriangledown_t X - \bigtriangledown_t \bigtriangledown_s X \quad \forall\gamma, X.$$

Remark: In the cas $n = 2$ the Gauss curvature is

$$K_{(p)} = \frac{\langle R(u,v)v, u\rangle}{\text{Å } u\text{Å }^2\text{Å } v\text{Å }^2 - \langle u,v\rangle^2}$$

for any basis $u, v \in T_pM$.
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1^{st} *Bianchi identity*

$$R(u,v)w + R(v,w)u + R(w,u)v = 0$$
$$\Leftrightarrow$$
$$R_{kij}^l + R_{ijk}^l + R_{jki}^l = 0$$

Exercise: 4.$\Rightarrow$ 3.

Sectional curvature:

$$K(p,E) := \frac{\langle R(u,v)v, u\rangle}{\text{Å } u\text{Å }^2\text{Å } v\text{Å }^2 - \langle u,v\rangle^2}$$
$$E \subset T_pM \quad u, v \text{ Basis of } E \quad \dim E = 2$$

Scalar curvature:

$$s(p) := \sum_{i,j} \langle R(e_i, e_j)e_j, e_i \rangle,$$

where $e_1, \dots, e_n$ form an ONB of T_pM. Exercise:

a) independent of e_i

b) in local coordinates $s = \sum_{i,j,k} g^{ij} R^k_{ikj}$

c) $\dim M = 2$, then $\delta = 2K$. (Rmk)

Lemma 3: Let (M, g) be a Riemannian n-mfld with scalar curvature $s : M \to \mathbb{R}$ and Laplace-Beltrami operator

$$\Delta_g = d * d : \Omega(M) \to \Omega^0(M).$$

Let $u : M \to (0, \infty)$ be smooth and define $\tilde{g} := u^2 g$
$\Rightarrow$ the scalar curvature of $\tilde{g}$ is the function

$$\tilde{s} = u^{-2}s + 2(n-1)u^{-3}\Delta_g u - (n-1)(n-4)u^{-4} \mathring{A}\ du \mathring{A}\ ^2_g$$

Remark:

$$\begin{aligned} a^k_{ij} &= \tilde{\Gamma}^k_{ij} - \Gamma^k_{ij} \\ &= u^{-1}(\delta^k_i \partial_j u + \delta^k_j \partial_i u - \sum_l g_{ij} g^{kl} \partial_l u) \\ \Rightarrow \tilde{R}^l_{kij} - R^l_{kij} &= \partial_i a^l_{jk} - \partial_j a^l_{ik} + \sum_\nu (a^l_{i\nu} a^\nu_{jk} - a^l_{j\nu} a^\nu_{ik}) \\ &= \Delta_g u + \sum_\nu \mathring{A}\ du \mathring{A}\ ^2 \end{aligned}$$

multiply by $\tilde{g}^{kj} = u^{-2} g^{kj}$, $l = i$, sum over (all) i, j, k.

Simplify equation:

$n > 2$: Choose $\nu := u^{(n-2)/2}$ and $\tilde{g} = v^{4/(n-2)} g$
Exercise: (*Yamalie equation*)

$$\frac{4(n-1)}{n-2} \Delta_g v + vs = v^{(n+2)/(n-2)} \tilde{s}$$

$n = 2$ $u := e^f$, $\tilde{g} = e^{2f} g$, (plug this in)
$\tilde{s} = u^{-2}s + 2u^{-3}\Delta_g u + 2u^{-4} \mathring{A}\ du \mathring{A}\ ^2_g$
$\Delta_g e^f = e^f \Delta_g f - e^f \mathring{A}\ df \mathring{A}\ ^2$
finally we get $\mathring{A}\ de^f \mathring{A}\ ^2_g = e^{2f} \mathring{A}\ df \mathring{A}\ ^2_g$

$$\Rightarrow \tilde{s} = e^{-2f}s + 2e^{-f}\Delta_g f - 2e^{-2f} \mathring{A}\ df \mathring{A}\ ^2 + 2e^{-2f} \mathring{A}\ df \mathring{A}\ ^2$$

$$\tilde{s} = e^{-2f}(s + \Delta_g f) \text{ with Beltrami operator}$$

$\tilde{K} = \frac{1}{2}\tilde{s} =$ Gauss curvature of $e^{2f} g$

$$\Rightarrow \tilde{K} = e^{-2f}(\Delta_g f + K_g$$

Question: Can we solve this equation $\Delta_g + K_g - e^{-2f}\tilde{K} = 0$? YES!

$$\int_\Sigma K_g d\mathrm{Vol}_g = 2\pi\chi(\Sigma)$$

Σ compact oriented 2-mfld without boundary
g Riemannian metric
$d\mathrm{Vol}_g \in \Omega^2(\Sigma)$ volume form
$K_g : \Sigma \to \mathbb{R}$ Gauss curvature
$d^*d = \Delta_g : \Omega^0(\Sigma) \to \Omega^0(\Sigma)$ Laplace-Beltrami operator

Remark 1: K satisfies the Gauss-Bonnet formula

$$\int_\Sigma K_g d\mathrm{Vol}_g = 2\pi(2 - 2\mathrm{genus}(\Sigma))$$

Remark 2: For $u : \Sigma \to \mathbb{R}$

$$K_{e^{2u}g} = e^{-2u}(K_g + \Delta_g u) \quad \text{(Lemma 4).}$$

Note:

$$d\mathrm{Vol}_{e^{2u}g} = e^{2u} d\mathrm{Vol}_g, \text{ so}$$

$$K_{e_g^{2u}} d\mathrm{Vol}_{e^{2u}g} = K_g d\mathrm{Vol}_g + (\Delta_g u) d\mathrm{Vol}_g,$$

where $\int(\Delta_g u)d\mathrm{Vol}_g = 0$.

Theorem 1: (Σ, g) compact oriented Riemannian 2-mfld without boundary $\Rightarrow \exists u \in C^\infty(\Sigma)$ such that

$$K_{e^{2u}g} \equiv \text{const} = \begin{cases} 1 & \text{genus } = 0 \\ 0 & \text{genus } = 1 \\ -1 & \text{genus } > 1 \end{cases}$$

u is unique in the case genus ¿1.

Remark 3: <u>genus = 0:</u>

$$\Rightarrow \int_\Sigma K_g d\mathrm{Vol}_g = 0$$

$$\overset{\text{Lemma 2}}{\Rightarrow} \exists u \in W^{2,p}(\Sigma) \text{ such that}$$

$$\Delta_g u = -K_g \quad u \text{ smooth}$$

$$\overset{\text{Rmk 2}}{\Rightarrow} K_{e^{2u}g} \equiv 0.$$

u is unique up to an additive constant. Why is u smooth? u smooth by Chapter III, Thm 8.

Definition: An *almost complex structure* on Σ is an automorphism $J : T\Sigma \to T\Sigma$ such that $J^2 = -1$. For each $p \in \Sigma$ we have an automorphism $J(p) : T_p\Sigma \to T_p\Sigma$, depending smoothly on p, and

$$J(p)J(p)v = -v \quad \forall v \in T_p\Sigma$$

A) Every almost complex structure on a 2-mfld Σ is integrable, ie. we can cover Σ by coordinate charts $\phi : u \to \mathbb{C}$ such that the derivative of this coordinate charts

$$d\phi(p)J(p)v = id\phi(p)v$$

(this is a hard theorem (without proof)

B) Every Riemannian metric g on an *oriented* 2-mfld Σ determines a unique almost complex structure J_g such that $\forall p \in \Sigma, \forall v, w \in T_p\Sigma$

$$g(v, w) := \langle v, w\rangle = d\mathrm{Vol}_g(v, J_g(P)w)$$

$$\tilde{g} \text{ another Riemannian metric with } J_{\tilde{g}} = J_g$$

$$\Leftrightarrow \exists u \in C^\infty(M) \text{ such that } \tilde{g} = e^{2u}g$$

Example: $\Sigma = S^2$, $g = g_0$ standard metric, $S^2 = \{x \in \mathbb{R}^3 \mid \text{Å } x\text{Å } = 1\}$
What is J_0?

$$J_0(x)\xi = X \times \xi = \begin{pmatrix} x_2\xi_3 - x_3\xi_2 \\ x_3\xi_1 - x_1\xi_3 \\ x_1\xi_2 - x_2\xi_1 \end{pmatrix}$$

$\Rightarrow$ stereographic projection

$$\psi(x) = \frac{x_1 + ix_2}{1 - x_3} \quad \psi : S^2 \setminus \{(0,0,1)\} \to \mathbb{C}, S^2 \to \mathbb{C} \cup \{\infty\}$$

$$\psi^* i = J_0 \quad g_0 = \psi^* \left(\frac{dx^2 + dy^2}{(1 + x^2 + y^2)^2} \right)$$

Remark 5: For every compact oriented Riemannian 2-mfld (Σ, g) of genus = 0
$\exists$ diffeomorphism $\phi : \Sigma \to S^2$ such that $J_g = \phi^* J_0$.
Define

$$\tilde{g} := \phi^* g_0 \Rightarrow J_{\tilde{g}} = J_{\phi^* g_0} = \phi^* J_{g_0} = \phi^* J_0 = J_g$$

$$\overset{\text{Rmk 4}}{\Rightarrow} \exists u : \Sigma \to \mathbb{R} \text{ such that } \tilde{g} = e^{2u} g$$

$$\Rightarrow K_{e^{2u} g} = K_{\tilde{g}} = K_{\phi^* g_0} = \phi^* K_{g_0} = K_{g_0} \circ \phi \equiv 1$$

Remark 5: (Non uniqueness in Thm 1 for genus = 0):

$$SL(2, \mathbb{C}) = \{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{C}^{2 \times 2} \mid ad - bc = 1\}$$

$$PSL(2, \mathbb{C} = SL(2, \mathbb{C} \setminus \pm 1 \overset{\approx}{\to} \text{Diff}(S^2, J_0)$$

$$SO(3) \cong SU(2) \setminus \pm 1 \overset{\approx}{\to} \text{Diff}(S^2, g_0)$$

$$SU(2) = \{A \in \mathbb{C}^{2 \times 2} \mid A^* A = 1\}$$

$$\phi_A^* J_0 = J_0 \Rightarrow \phi_A^* g_0 = e^{2u_A} g_0$$

$$K_{e^{2u} g_0} = K_{\phi_A^* g_0} = K_{g_0} \circ \phi_A \equiv 1$$

$$u_A \equiv 0 \Leftrightarrow \phi_A^* g_0 = g_0 \Leftrightarrow A \in SU(2)$$

Remark 7:

FACT: Any two connected, simply connected, complete Riemannian 2-mflds (Σ_0, g_0) and (Σ_1, g_1) with the same constant Guass curvature $K_0 \equiv \text{const} \equiv K_1$ are isometric, ie. $\exists$ a diffeo $\phi : \Sigma_0 \to \Sigma_1$ such that $\phi^* g_1 = g_0$.

$$\begin{array}{ll} K = 0 & \Sigma = \mathbb{R}^2 \\ K = 1 & \Sigma = S^2 \\ K = -1 & \Sigma = \mathbb{D} = \{z \in \mathbb{C} \mid \text{Å } z\text{Å } < 1\} \quad z = x + iy \end{array}$$

$$g = \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}$$

Theorem 2: (Kozdan-Warner): $\mathbf{p > \frac{n}{2}}$
(M^n, g) compact oriented Riemannian n-mfld
$\Delta_g = d^* d$ Laplace-Beltrami operator
$d\mathrm{Vol}_g \in \Omega^n(M)$ volume form
Let $h, f \in L^p(M)$ such that

$$h \geq 0 \qquad \int_M h d\mathrm{Vol}_g > 0 \qquad \int_M f d\mathrm{Vol}_g > 0$$

$\Rightarrow \exists!$solution $u \in W^{2,p}(M)$ of the Kazdan-Warner equation

(KW) $$\Delta_g u + e^u h = f$$

If f, h are smooth, so is u.

Thm 2 $\Rightarrow$ Thm 1: (genus¿1) $M = \Sigma$. By Lemma 4

$$-1 = K_{e^u g} = e^{-u}(K_g + \frac{1}{2}\Delta_g u)$$

$$\Leftrightarrow \Delta_g u + 2e^u = -2K_g \quad | \, K_g < 0$$

(The sign is very important $\rightarrow h \geq 0$!)

Remark 8: Example $h = f = 1 \Rightarrow u \equiv 0$ is a solution of (KW). Why is $u \equiv 0$ unique?
Let u be any solution of

$$\Delta_g u + e^u = 1.$$

Choose $x_0 \in M$ such that

$$u(x_0) = \sup_M u$$

Claim: $u(x_0) \leq 0$.
Suppose $u(x_0) > 0 \Rightarrow (\Delta_g u)(x_0) = 1 - e^{u(x_0)} < 0$.

$$\Delta_g u = -\frac{1}{\sqrt{\det g}} \sum_{i,j} \partial_i(\sqrt{\det g} g^{ij} \partial_j u)$$

$\Rightarrow u$ is subharmonic near x_0
$\Rightarrow u \equiv \mathrm{const} \overset{!}{=} 0 \Rightarrow$ Contradiction! (the same for infimum)

Exercise: If $u(x_1) = \inf_M u$ then $u(x_1) \geq 0 \Rightarrow u \equiv 0$.

Remark 9: It is enough to prove Thm 2 for $f \equiv \mathrm{const}$. Denote

$$A = \frac{1}{\mathrm{Vol}(M)} \int_M f d\mathrm{Vol}_g \quad \mathrm{Vol}(M) := \int_M d\mathrm{Vol}_g$$

$$\Rightarrow \int_M (f - A) d\mathrm{Vol}_g = 0$$

$$\overset{\text{Lemma 2}}{\Rightarrow} \exists! u_0 : M \rightarrow \mathbb{R} \text{ smooth}$$

$$\Delta_g u_0 : f - A \qquad \int_M u_0 d\mathrm{Vol}_g = 0$$

Suppose

$$\Delta u + e^u h = f$$

and define $v := u - u_0$

$$\begin{aligned} \Rightarrow \Delta_g v &= \Delta_g u - \Delta_g u_0 \\ &= f - e^u h - f + A \\ &= A - e^v e^{u_0} h \end{aligned}$$

so

$$(KW) \Leftrightarrow \Delta_g v + e^v(e^{u_0}h) = A$$

$$\Delta u + e^u h = A$$

(*)

$h \in L^p(M) \quad h \geq 0,\ \Delta = d^*d \quad \int_M h > 0.\ p > \frac{n}{2}.$
To show: $\exists! u \in W^{2,p}(M)$ satisfying (*).

Lemma 5: $\exists$ continuous function

$$\phi : (0,\infty)^2 \to (0,\infty) \quad \forall h \in L^p(M)$$

with $h \geq 0$ and $\forall u \in W^{2,p}(M)$ satisfying (*) we have

$$\|u\|_{L^\infty(M)} \leq \phi\left(\underbrace{\frac{1}{\text{Vol}(M)}\int_M h}_{B}\ \underbrace{\|u\|_{L^p}}_{C}\right)$$

Proof of Thm 2:

Step 1: Result holds for $h_0 \equiv A$ with $u_0 \equiv 0$ (Rmk 8). Fix a function $h \in L^p(M)$ with $h \geq 0$ and $\int_M h > 0$. For $t \in [0,1]$ define

$$h_t(x) := (1-t)A + th(x)$$

For $t \in [0,1]$ define

$$\mathcal{F}_t : W^{2,p}(M) \to L^p(M)$$

by

$$\mathcal{F}_t(u) := \Delta u + e^u h_t.$$

Because $p > \frac{n}{2}$ we have a Sobolev embedding $W^{2,p}(M) \hookrightarrow C^0(M)$.

Step 2 $\mathcal{F}_t : W^{2,p} \to L^p$ is a smooth map between Banach spaces and

$$d\mathcal{F}_t(u) : W^{2,p}(M) \to L^p(M)$$

is bijective for every $t \in [0,1]$ and every $u \in W^{2,p}(M)$.

Proof of Step 2:

$$W^{2,p}(M) \hookrightarrow C^0(M) \to C^0(M) \to L^p(M),$$

where $C^0(M) \ni u \mapsto e^u \in C^0(M)$, $C^0(M) \ni v \mapsto vh_t \in L^p(M)$. Then $W^{2,p}(M) \ni u \mapsto e^u h_t \in L^p(M)$ smoothly.

$$\begin{aligned} d\mathcal{F}_t(u)\widehat{u} &:= \frac{d}{ds}|_{s=0}\ \mathcal{F}_t(u+s\widehat{u}) \\ &= \frac{d}{ds}|_{s=0}\ (\Delta(u+s\widehat{u}) + e^{u+s\widehat{u}}h_t) \\ &= \Delta\widehat{u} + (e^u h_t)\widehat{u} \end{aligned}$$

bounded linear operator from $W^{2,p}(M) \to L^p(M)$.
Exercise: prove that this operator is bounded!
The linear operator

$$W^{2,p}(M) \hookrightarrow C^0(M) \to L^p(M)$$

$$\widehat{u} \overset{cpct}{\mapsto} \widehat{u} \overset{bded}{\mapsto} e^u h_t \widehat{u}$$

by Rellich $\Rightarrow L - \Delta$ is compact
$\overset{\text{Lemma 2}}{\Rightarrow}$ L Fredholm with index $= 0$. Why is $\ker L = 0$?
Let $\widehat{u} \in W^{2,p}(M)$ with $L\widehat{u} = 0$

$$\begin{aligned}
\Rightarrow 0 &= \int_M \widehat{u} L \widehat{u} \\
&= \int_M \widehat{u}(d^* d\widehat{u} + e^u h_t \widehat{u}) \\
&= \int_M \langle d\widehat{u}, d\widehat{u}\rangle + \int_M e^u h_t \mathring{A}\ \widehat{u} \mathring{A}\ ^2 \\
&= \int_M \mathring{A}\ d\widehat{u} \mathring{A}\ ^2 + \int_M e^u h_t \mathring{A}\ \widehat{u} \mathring{A}\ \geq 0
\end{aligned}$$

$$\Rightarrow \int_M \underbrace{\mathring{A}\ d\widehat{u} \mathring{A}\ ^2}_{\geq 0} = 0 \qquad \int_M \underbrace{e^u h_t \mathring{A}\ \widehat{u} \mathring{A}\ ^2}_{\geq 0} = 0$$

$$\Rightarrow d\widehat{u} \equiv 0 \text{ a.e.} \quad e^u h_t \mathring{A}\ \widehat{u} \mathring{A}\ ^2 \equiv 0 \text{ a.e.}$$

$$\Rightarrow \widehat{u} \equiv 0$$

$\widehat{u} \mathring{A}\ _E = 0$, where $E := \{x \mid h_t(x) > 0\}$, not measure zero! $\Rightarrow \widehat{u} \equiv 0$.
The set

$$\mathcal{M} := \{(t,u) \mid u \in W^{2,p}(M), \Delta u + e^u h_t = A\}$$

is a 1-dimensional submanifold of $[0,1] \times W^{2,p}(M)$ and the projection $\pi : \mathcal{M} \to [0,1]$ with $\pi(t,u) = t$ is a *submersion* (the set $\mathcal{M}$ is locally near each point $(t_0, u_0) \in \mathcal{M}$ the graph of a smooth function $[0,1] \cap (t_0 - \varepsilon, t_0 + \varepsilon) \to W^{2,p}(M) : t \mapsto u_t$).

Step 3: need to copy page PD196

Step 4: $\mathcal{L}$ is compact.

Proof of Step 4:

$$c_0 := \sup_{0 \leq t \leq 1} \phi\left(\frac{1}{\text{Vol}(M)} \int_M h_t, \|h_t\|_{L^p}\right)$$

$$\Rightarrow \|u\|_{L^\infty} \leq c_0 \quad \forall (t,u) \in \mathcal{M}$$

$$\Rightarrow \|u\|_{W^{2,p}(M)} \leq c_1 (\|\Delta u\|_p + \|u\|_p)$$

$$\leq c_1 \|h\|_p e^{c_0} + c_0 c_1 \text{Vol}(M)^{1/p} \leq c_2$$

$$\Rightarrow e^u \in W^{2,p}(M)$$

Let $(t_\nu, u_\nu) \in \mathcal{M}$.

$$\Rightarrow t_\nu \in [0,1] \quad \|u_\nu\|_{2,p} \leq c \quad \forall \nu$$

$\overset{\text{Rellich}}{\Rightarrow}$ because the inclusion $W^{2,p}(M) \hookrightarrow C^0(M)$ is compact $\exists$ subsequence ν_i such that t_{ν_i} converges to $t \in [0,1]$. u_{ν_p} converges in $C^0(M)$ to u.

$$\begin{aligned}
\|u_{\nu_i} - u_{\nu_j}\|_{2,p} &\leq c_1(\|\Delta u_{\nu_i} - \Delta u_{\nu_j}\|_p + \|u_{\nu_i} - u_{\nu_j}\|_p) \\
&\leq c_1 \|e^{u_{\nu_i}} h_{t_{\nu_i}} - e^{u_{\nu_j}} h_{t_{\nu_j}}\|_p \\
&\leq \|\exp(u_{nu_i} - u_{\nu_j})\|_\infty \|h_{t_{\nu_i}}\|_p + \|\exp(u_{\nu_j})\|_\infty \|h_{t_{\nu_i}} - h_{t_{\nu_j}}\|_p \\
&\to 0
\end{aligned}$$

$\Rightarrow u_{\nu_i}$ is a Cauchy sequence in $W^{2,p}(M)$
$\Rightarrow u \in W^{2,p}(M)$ and $\lim_{i \to \infty} \|u\nu_i - u\|_{2,p} = 0$.
$\Rightarrow (t,u) \in \mathcal{M}$.

Step 5: Let $(t, u) \in \mathcal{M}$

$$\Rightarrow \exists \varepsilon > 0 \forall t' \in [0,1] \forall u', u'' \in W^{2,p}(M)$$

If $(t', u') \in \mathcal{M}$ and $(t', u'') \in \mathcal{M}$ and

$$\mathring{A}\ t' - t_0 \mathring{A}\ < \varepsilon \quad \|u' - u_0\|_{2,p} < \varepsilon \quad \|u'' - u_0\|_{2,p} < \varepsilon$$

then $u' = u''$.

Proof of Step 5: Inverse function theorem (Exercise (Analysis II)).

Step 6: $\exists! u \in W^{2,p}(M)$ of (*) (that's the claim).

Proof of Step 6: Define

$$T := \{t \in [0,1] \mid \#\{u \in W^{2,p}(M) \mid (t,u) \in \mathcal{M}\} = 1\}$$

To show: $1 \in T$.

1. $T \neq \varnothing$ (because $0 \in T$)

2. T is an open subset of $[0,1]$

3. T is closed $\Rightarrow T = [0,1]$.

T is open: Let $t_o \in T$, $u_0 \in W^{2,p}$, $(t_0, u_0) \in \mathcal{M}$
$\overset{\text{Step 3}}{\Rightarrow} \exists$ smooth map

$$[0,1] \cap (t_0 - \varepsilon, t_0 + \varepsilon) \to W^{2,p}(M), t \mapsto u_t$$

such that

$$(t, u_t) \in \mathcal{M} \forall t \text{ and } u_{t_0} = u_0$$

$$\Rightarrow \#\{u \in W^{2,p} \mid (t,u) \in \mathcal{M}\} \geq 1 \quad \forall t \in [0,1] \cap (t_0 - \varepsilon, t_0 + \varepsilon)$$

Claim: $\# = 1$ for t sufficiently close to t_0.
Suppose otherwise. Then $\exists t_\nu \to t_0$, $u_\nu \in W^{2,p}$ such that $(t_\nu, u_\nu) \in \mathcal{M}$, $u_\nu \neq u_{t_\nu}$.
$\overset{\text{Step 4}}{\Rightarrow}$ w.l.o.g assume

$$u_\nu \to \tilde{u}_0 \in W^{2,p}$$

$$\Rightarrow \tilde{u}_0 = u_0$$

$$\Rightarrow u_\nu \overset{W^{2,p}}{\to} u_0$$

$$u_\nu \neq u_{t_\nu}, \quad t_\nu \to t_0$$

contradicts Step 5!

T is closed: Let $t_\nu \in T$ converge to t^*. Then $\exists! u_\nu \in W^{2,p}(M)$ such that $(t_\nu, u_\nu) \in \mathcal{M}$.
To show: $t^* \in T$.
w.l.o.g assume that $u_\nu \overset{W^{2,p}}{\to} u^*$.

$$\Rightarrow (t^*, u^*) \in \mathcal{M}$$

$$\Delta u^* + e^{u^*} h_{t^*} = \lim_{\nu \to \infty} (\Delta u_\nu + e^{u_\nu} h_{t_\nu}) = A$$

Claim: u^* is the only solution of $\Delta u + e^u h_{t^*} = A$.
Suppose $\exists$ another solution $\widehat{u}$ of $\Delta\widehat{u} + e^{\widehat{u}} h_{t^*} = A$, where $\widehat{u} \neq u^*$.
$\Rightarrow$ by the implicit function theorem $\exists$ solutions $\widehat{u}_\nu$ of

$$\Delta\widehat{u}_\nu + e^{\widehat{u}_\nu} h_{t_\nu} = A$$

for ν large such that $\widehat{u}_\nu \to \widehat{u}$
contradicts $t_\nu \in T$ $(\widehat{u}_\nu \neq u_\nu)$. □

Proof of Lemma 5:

Step 1: $\exists c_0 > 0 \forall u \in W^{2,p}(M)$

$$\int_M u = 0 \Rightarrow \|u\|_{L^\infty} \leq c_0 \|\Delta u\|_{L^p}$$

Proof: By Lemma 2

$$\Delta : \{u \in W^{2,p} \mid \int_M u = 0\} \to L^p(M)$$

is injective and has a closed image

$$\Rightarrow \|u\|_{L^\infty} \leq c \|\Delta u\|_{L^p}$$

$\Rightarrow$ Step 1 with $c_0 = c$ (???).

(*)
$$\Delta u + e^u h = A$$

$$B := \frac{1}{\mathrm{Vol}(M)} \int_M h d\mathrm{Vol}_g$$

$$C := \|h\|_{L^p}$$

Step 2: $u, h \in C^\infty(M), (*)$

$$\Rightarrow u(x) \leq 4c_0 \frac{Ac}{B} + \log\left(\frac{A}{B}\right)$$

Proof:

$$\int_M (h - B) = 0, \text{ B is the mean value.}$$

$$\Rightarrow \exists! v_0 \in C^\infty(M)$$

$$\Delta v_0 = B - h \quad \int_M v_0 = 0$$

$$\begin{aligned} \Rightarrow \|v_0\|_{L^\infty} &\leq c_0 \|B - h\|_{L^p} \\ &\leq c_0(\|h\|_p + \mathrm{Vol}(M)^{1/p} B) \\ &\leq c_0(\|h\|_p + \|h\|_p) \\ &\leq 2c_0 C \end{aligned}$$

Claim 1:

$$u(x) \leq \log\left(\frac{A}{B}\right) + \frac{A}{B}(v_0(x) + 2c_0 C)$$

(this implies Step 2). Define

$$w_\varepsilon(x) := \log\left(\frac{A+\varepsilon}{B}\right) + \left(\frac{A+\varepsilon}{B}\right) \underbrace{(v_0 + 2c_0 C)}_{\geq 0} - u(x)$$

Claim 2: $w_\varepsilon(x) \geq 0 \quad \forall x \forall \varepsilon$ (this implies Claim 1).
Choose $x_\varepsilon \in M$ such that

$$w_\varepsilon(x_\varepsilon) = \inf_M w_\varepsilon$$

$$\Rightarrow (d^* dw_\varepsilon)(x_\varepsilon) \leq 0$$

$$\begin{aligned} \Rightarrow 0 &\geq \Delta w_\varepsilon \\ &= \frac{A+\varepsilon}{B}\Delta v_0(x_\varepsilon) - \Delta u(x_\varepsilon) \\ &= \frac{A+\varepsilon}{B}(B - h(x_\varepsilon)) + e^{u(x_\varepsilon)}h(x_\varepsilon) - A \\ &= \varepsilon + h(x_\varepsilon)\left(e^{u(x_\varepsilon)} - \frac{A+\varepsilon}{B}\right) \end{aligned}$$

$$\Rightarrow h(x_\varepsilon) > 0 \quad e^{u(x_\varepsilon)} < \frac{A+\varepsilon}{B}$$

$$u(x_\varepsilon) < \log\left(\frac{A+\varepsilon}{B}\right)$$

$$\Rightarrow w_\varepsilon(x_\varepsilon) \geq 0$$

□

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