Quantum products for mapping tori and the Atiyah-Floer conjecture

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1 Introduction

Let Σ be a compact oriented Riemann surface with area form ω and $f_0, f_1 \ldots, f_n$ be orientation and area preserving diffeomorphisms of Σ such that

$$f_n \circ \cdots \circ f_0 = \mathrm{id}.$$

Suppose that $P \to \Sigma$ is a principal SO(3)-bundle with nonzero second Stiefel-Whitney class and choose lifts

$$\begin{array}{ccc} P & \xrightarrow{\tilde{f}_j} & P \\ \downarrow & & \downarrow \\ \Sigma & \xrightarrow{f_j} & \Sigma \end{array}$$

to bundle automorphisms that also satisfy $\tilde{f}_n \circ \cdots \circ \tilde{f}_0 = \mathrm{id}$. These lifts induce symplectomorphisms $\phi_{\tilde{f}_j} : M_\Sigma \to M_\Sigma$ on the moduli space of flat connections on P. They also determine SO(3)-bundles $Q_{\tilde{f}_j} \to Y_{f_j}$ over mapping tori. There are natural product structures

$$\mathrm{HF}(Y_{f_0}, Q_{\tilde{f}_0}) \otimes \mathrm{HF}(Y_{f_1}, Q_{\tilde{f}_1}) \otimes \cdots \otimes \mathrm{HF}(Y_{f_n}, Q_{\tilde{f}_n}) \to \mathbb{Z} \tag{1}$$

in instanton Floer homology and

$$\mathrm{HF}^{\mathrm{symp}}(M_\Sigma, \phi_{\tilde{f}_0}) \otimes \mathrm{HF}^{\mathrm{symp}}(M_\Sigma, \phi_{\tilde{f}_1}) \otimes \cdots \otimes \mathrm{HF}^{\mathrm{symp}}(M_\Sigma, \phi_{\tilde{f}_n}) \to \mathbb{Z} \tag{2}$$

in symplectic Floer homology. Both were discovered by Donaldson. They are well defined up to an overall sign. The homomorphism (1) can be interpreted as a relative Donaldson invariant on a 4-manifold with boundary. In other words, this product is obtained by counting anti-self-dual connections over a 4-dimensional cobordism with $n+1$ cylindrical ends corresponding to Y_{f_j}. The second homomorphism is obtained by counting pseudo-holomorphic sections of a symplectic fibre bundle over the punctured sphere with fibre M_Σ and holonomies $\phi_{\tilde f_j}$ around the $n+1$ punctures. The next theorem is the main result of this paper. It can be viewed as an extension of the Atiyah-Floer conjecture to product structures (cf. [1, 17]).

Theorem 1.1. *The natural isomorphism* $\mathrm{HF}(Y_f, Q_{\tilde f}) \to \mathrm{HF}^{\mathrm{symp}}(M_\Sigma, \phi_{\tilde f})$ *constructed in [6] intertwines the two product structures (1) and (2) (up to a sign).*

Corollary 1.2. *The quantum cohomology ring of* M_Σ *is isomorphic to the instanton Floer homology ring of* $S^1 \times \Sigma$.

Proof. Theorem 1.1 with $\tilde f_0 = \tilde f_1 = \tilde f_2 = \mathrm{id}$ and Theorem 5.1 in [16]. □

Remark 1.1. **(i)** The techniques of [4, 5, 6] suggest a general relation between holomorphic curves in M_Σ and anti-self-dual instantons on the 4-manifold $S \times \Sigma$ for any compact Riemann surface S. The technique of proof is an adiabatic limit argument where the metric on the fibre Σ converges to zero. In this limit the anti-self-dual instantons on the product $S \times \Sigma$ degenerate to holomorphic curves $S \to M_\Sigma$. The proof of Theorem 1.1 is the analogue of this argument for Riemann surfaces S with $n+1$ cylindrical ends. Instead of holomorphic maps $S \to M_\Sigma$ one obtains holomorphic sections of a fibre bundle $W \to S$ with fibres M_Σ and holonomies $\phi_{\tilde f_j}$ at the $n+1$ cylindrical ends. The comparison theorem for Floer homologies in [6] corresponds to the case of two cylindrical ends.

(ii) It is interesting to relate the results of this paper to the recent work of Donaldson about symplectic Lefschetz fibration (cf. [2, 3]). He proved that every symplectic manifold, after blowing up a suitable number of points, admits the structure of a topological Lefschetz fibration

$$\begin{array}{ccc} \Sigma & \hookrightarrow & X \\ & & \downarrow \\ & & S^2. \end{array}$$

The projection $\pi : X \to S^2$ has finitely many nondegenerate singular points (with distinct singular values) near which π is holomorphic. Moreover, each regular fibre of π is a symplectic submanifold of X. Thus the holonomy around each singular value is a positive Dehn twist f_j, and the composition of these Dehn twists is the identity. Cutting out neighbourhoods of the singular fibres one obtains a fibration $X \to S$ over the punctured sphere. Now the neighbourhoods of the singular fibres determine natural Floer homology classes

$$a(\tilde f_j) \in \mathrm{HF}(Y_{f_j}, Q_{\tilde f_j})$$

and their product under (1) is the degree-0 Donaldson invariant of X.

(iii) In his recent thesis [11] Handfield discusses the relation between anti-self-dual instantons on $S \times \Sigma$ and holomorphic curves $S \to M_\Sigma$ for a closed Riemann surface S. His work is based on the ideas developed in [4, 5, 6].

(iv) Corollary 1.2 was established independently by Munoz [14, 15] who computed both ring structures separately, without using the natural isomorphism.

2 The punctured sphere

Fix an integer $n \geq 2$ and let

$$S = \mathbb{C}P^1 \setminus \{z_0, \ldots, z_n\}$$

be the Riemann sphere with punctures at $n+1$ distinct points $z_0, \ldots, z_n$. We assume throughout that the points $z_0, \ldots, z_n$ are ordered and in general position, i.e.

$$0 = |z_0| < |z_1| < \cdots < |z_{n-1}| < |z_n| = \infty.$$

It is convenient to use polar coordinates $z = e^{2\pi w}$. Let us fix lifts $w_j = s_j + it_j \in \mathbb{C}$ of the points z_j. Then

$$-\infty = s_0 < s_1 < \cdots < s_{n-1} < s_n = \infty.$$

and S can be identified with the quotient $S = U/\sim$ of the open set

$$U = \{s + it \in \mathbb{C} \,|\, \text{ if } s = s_j \text{ then } t_j < t < t_j + 1\}$$

by the equivalence relation $(s,t) \sim (s, t+1)$. It is convenient to choose a metric on S in which the punctures become cylindrical ends. Hence let

$$\lambda : \mathbb{C} \setminus \{w_j + ik \,|\, j, k \in \mathbb{Z},\, 0 \leq j \leq n\} \to \mathbb{R}$$

be a smooth positive function such that $\lambda(s, t+1) = \lambda(s,t)$ and, for some $T > 1$ and every $w = s + it \in U$,

$$\lambda(w) = \begin{cases} 1/2\pi|w - w_j|, & \text{if } |w - w_j| \leq e^{-2\pi T} \text{ for some } j, \\ 1, & \text{if } |s| \geq T. \end{cases}$$

Consider the volume form

$$\mathrm{dvol}_S = \lambda^2 ds \wedge dt$$

on S and note that the function $\zeta \mapsto w_j + e^{2\pi\zeta}$ is an isometry from the half cylinder $Z = (-\infty, -T) \times \mathbb{R}/\mathbb{Z}$ with the standard metric to the punctured ball $\{w \in \mathbb{C} \,|\, 0 < |w - w_j| < e^{-2\pi T}\}$ with the metric $\lambda^2(ds^2 + dt^2)$.

3 Fibre bundles

Throughout we identify $S^1 = \mathbb{R}/\mathbb{Z}$. Let Σ be a compact oriented Riemann surface with volume form ω. The mapping torus of a symplectomorphism $f \in \mathrm{Diff}(\Sigma, \omega)$ is a fibre bundle

$$Y_f \to S^1$$

with holonomy f. It is defined by $Y_f := \mathbb{R} \times \Sigma/\sim$, where

$$(t+1, z) \sim (t, f(z)).$$

Let $f_0, f_1, \ldots, f_n \in \mathrm{Diff}(\Sigma, \omega)$ such that

$$f_n \circ f_{n-1} \circ \cdots \circ f_0 = \mathrm{id}.$$

Then there is a natural fibre bundle

$$\begin{array}{ccc} \Sigma & \hookrightarrow & X \\ & & \downarrow \\ & & S \end{array}$$

over the punctured sphere with holonomy f_j around the jth puncture. Thus the restriction of the bundle to a circle around w_j is diffeomorphic to Y_{f_j}. To construct the bundle explicitly it is convenient to introduce the maps

$$g_j = f_j \circ f_{j-1} \circ \cdots \circ f_0$$

for $j = 0, 1, \ldots, n-1$. Then $X := U \times \Sigma / \sim$ under the equivalence relation

$$s_j < s < s_{j+1} \qquad \Longrightarrow \qquad (s, t+1, z) \sim (s, t, g_j(z)).$$

For $j = 0, \ldots, n$ there is a natural embedding $\iota_j : (-\infty, -T) \times Y_{f_j} \hookrightarrow X$, defined by $\iota_0(s, t, z) = [s, t, z]$ and $\iota_n(s, t, z) = [-s, -t, z]$ for $j = 0, n$ and by

$$\iota_j(s, t, z) = \begin{cases} [w_j + e^{2\pi(s+it)}, {f_j}^k(z)], & \text{if } k - 1/4 < t < k + 3/4, \\ [w_j + i + e^{2\pi(s+it)}, {g_j}^{-1} \circ {f_j}^k(z)], & \text{if } k - 3/4 < t < k + 1/4, \end{cases}$$

for $j = 1, \ldots, n-1$. These maps are well defined and satisfy

$$\iota_j(s, t+1, z) = \iota_j(s, t, f_j(z))$$

for $j = 0, \ldots, n$.

Let $\mathcal{J}(\Sigma)$ denote the space of complex structures on Σ that are compatible with the orientation given by ω. A **vertical complex structure** on X is an almost complex structure on the vertical tangent bundle. Explicitly, such a complex structure can be represented by a smooth map $U \to \mathcal{J}(\Sigma) : (s, t) \mapsto J(s, t)$ that satisfies

$$s_j < s < s_{j+1} \qquad \Longrightarrow \qquad J(s, t+1) = {g_j}^* J(s, t).$$

The pullback $J_j = {\iota_j}^* J$ is a vertical complex structure on $(-\infty, -T) \times Y_{f_j}$. It is given by

$$J_j(s, t) = \begin{cases} ({f_j}^k)^* J(w_j + e^{2\pi(s+it)}), & \text{if } k - 1/4 < t < k + 3/4, \\ ({f_j}^k)^* {g_j}_* J(w_j + i + e^{2\pi(s+it)}), & \text{if } k - 3/4 < t < k + 1/4, \end{cases}$$

for $j = 1, \ldots, n-1$ and it satisfies $J_j(s, t+1) = {f_j}^* J_j(s, t)$. We assume throughout that $J_j(s, t)$ is independent of the s-variable for $-s$ sufficiently large. Let $*_{s,t}$ denote the Hodge $*$-operator on Σ corresponding to the metric $\omega(\cdot, J(s, t)\cdot)$ and $*_{j;s,t}$ the Hodge $*$-operator of the metric $\omega(\cdot, J_j(s, t)\cdot)$. Note that $*_{s,t}\alpha = -\alpha \circ J(s, t)$ for $\alpha \in \Omega^1(\Sigma, \mathfrak{g}_P)$.

4 Connections and gauge transformations

Let $P \to \Sigma$ be a principal bundle with structure group $\mathrm{G} = \mathrm{SO}(3)$ and nonzero second Stiefel-Whitney class. Denote by $\mathfrak{g} = \mathrm{Lie}(\mathrm{G}) = \mathfrak{so}(3)$ the Lie algebra of G, by $\mathcal{A}(P) \subset \Omega^1(P, \mathfrak{g})$ the space of connection 1-forms on P, and by $\mathcal{G}(P)$ the identity component in the group of gauge transformations (thought of as equivariant maps from P to G). A lift $\tilde{f} : P \to P$ of $f \in \mathrm{Diff}(\Sigma, \omega)$ determines a principal SO(3)-bundle

$$\begin{array}{ccc} P & \hookrightarrow & Q_{\tilde{f}} \\ & & \downarrow \\ & & Y_f \end{array}$$

given by the mapping torus $Q_{\tilde{f}} := \mathbb{R} \times Q/\sim$, where

$$(t+1, p) \sim (t, \tilde{f}(p)).$$

Choose $n+1$ such lifts $\tilde{f}_0, \ldots, \tilde{f}_n$ of the maps $f_0, \ldots, f_n$ such that

$$\tilde{f}_n \circ \cdots \circ \tilde{f}_0 = \mathrm{id}$$

and define $\tilde{g}_j = \tilde{f}_j \circ \cdots \circ \tilde{f}_0$. Then there is a principal SO(3)-bundle

$$\begin{array}{ccc} P & \hookrightarrow & Q \\ & & \downarrow \\ & & X \end{array}$$

defined as the quotient $Q := U \times P/\sim$ under the equivalence relation

$$s_j < s < s_{j+1} \qquad \Longrightarrow \qquad (s, t+1, p) \sim (s, t, \tilde{g}_j(p)).$$

The embedding ι_j lifts to a bundle map $\tilde{\iota}_j : (-\infty, -T) \times Q_{\tilde{f}_j} \to Q$, given by

$$\tilde{\iota}_j(s,t,p) = \begin{cases} [w_j + e^{2\pi(s+it)}, {\tilde{f}_j}^{\,k}(p)], & \text{if } k - 1/4 < t < k + 3/4, \\ [w_j + i + e^{2\pi(s+it)}, {\tilde{g}_j}^{\,-1} \circ {\tilde{f}_j}^{\,k}(p)], & \text{if } k - 3/4 < t < k + 1/4, \end{cases}$$

for $j = 1, \ldots, n-1$. A connection on Q is a 1-form

$$\Xi = A + \Phi\, ds + \Psi\, dt$$

on $U \times P$ such that the functions $A : U \to \mathcal{A}(P)$ and $\Phi, \Psi : U \to C^\infty(\Sigma, \mathfrak{g}_P)$ satisfy the periodicity conditions

$$\begin{gathered} A(s, t+1) = {\tilde{g}_j}^{\,*} A(s,t), \\ \Phi(s, t+1) = \Phi(s,t) \circ \tilde{g}_j, \qquad \Psi(s, t+1) = \Psi(s,t) \circ \tilde{g}_j, \end{gathered} \tag{3}$$

for $s_j < s < s_{j+1}$. For $j = 1, \ldots, n-1$ the pullback connection

$$\Xi_j = {\tilde{\iota}_j}^{\,*} \Xi = A_j + \Phi_j\, ds + \Psi_j\, dt$$

on $(-\infty, -T) \times Q_{\tilde{f}_j}$ is given by

$$A_j(s,t) = \begin{cases} (\tilde{f}_j{}^k)^* A(w_j + e^{2\pi(s+it)}), & \text{if } k - 1/4 < t < k + 3/4, \\ (\tilde{g}_j{}^{-1} \circ \tilde{f}_j{}^k)^* A(w_j + i + e^{2\pi(s+it)}), & \text{if } k - 3/4 < t < k + 1/4, \end{cases}$$

$$\Phi_j(s,t) = 2\pi e^{2\pi s}\left(\cos(2\pi t)\Phi(w_j + e^{2\pi(s+it)}) + \sin(2\pi t)\Psi(w_j + e^{2\pi(s+it)})\right),$$

$$\Psi_j(s,t) = 2\pi e^{2\pi s}\left(\cos(2\pi t)\Psi(w_j + e^{2\pi(s+it)}) - \sin(2\pi t)\Phi(w_j + e^{2\pi(s+it)})\right)$$

for $-1/4 < t < 3/4$. For general t these functions are determined by the periodicity conditions $\Phi_j(s,t+1) = \Phi_j(s,t) \circ \tilde{f}_j$ and $\Psi_j(s,t+1) = \Psi_j(s,t) \circ \tilde{f}_j$. Let $\mathcal{A}(Q)$ denote the set of all connections Ξ that satisfy (3).

A gauge transformation of Q is a smooth function $u : U \to \mathcal{G}(P)$ that satisfies

$$s_j < s < s_{j+1} \qquad \Longrightarrow \qquad u(s,t+1) = u(s,t) \circ \tilde{g}_j.$$

The pullback gauge transformation $u_j = u \circ \tilde{\iota}_j : (-\infty, -T) \times Q_{\tilde{f}_j} \to \mathrm{G}$ is given by

$$u_j(s,t) = \begin{cases} u(w_j + e^{2\pi(s+it)}) \circ \tilde{f}_j{}^k, & \text{if } k - 1/4 < t < k + 3/4, \\ u(w_j + i + e^{2\pi(s+it)}) \circ \tilde{g}_j{}^{-1} \circ \tilde{f}_j{}^k, & \text{if } k - 3/4 < t < k + 1/4, \end{cases}$$

and satisfies $u_j(s,t+1) = u_j(s,t) \circ \tilde{f}_j$. Let $\mathcal{G}(Q)$ denote the group of gauge transformations such that $u_j(s,t) = \mathbb{1}$ for $-s$ sufficiently large. This group acts on $\mathcal{A}(Q)$ by

$$A \mapsto u^{-1}du + u^{-1}Au,$$
$$\Phi \mapsto u^{-1}\partial_s u + u^{-1}\Phi u, \qquad \Psi \mapsto u^{-1}\partial_t u + u^{-1}\Psi u.$$

The action of u_j on Ξ_j is given by the same formulae.

5 Anti-self-dual instantons

The curvature of $\Xi = (A, \Phi, \Psi) \in \mathcal{A}(Q)$ is the 2-form

$$\begin{aligned} F_\Xi &= F_A + (\partial_s\Psi - \partial_t\Phi + [\Phi,\Psi])\, ds \wedge dt \\ &\quad - (\partial_s A - d_A\Phi) \wedge ds - (\partial_t A - d_A\Psi) \wedge dt. \end{aligned}$$

Since

$$\begin{aligned} *F_\Xi &= \lambda^{-2} * (\partial_s\Psi - \partial_t\Phi + [\Phi,\Psi]) + \lambda^2(*F_A)\, ds \wedge dt \\ &\quad - (*_{s,t}(\partial_t A - d_A\Psi)) \wedge ds + (*_{s,t}(\partial_s A - d_A\Phi)) \wedge dt, \end{aligned}$$

the connection Ξ is anti-self-dual if and only if

$$\begin{aligned} (\partial_s A - \mathrm{d}_A\Phi) + *_{s,t}(\partial_t A - \mathrm{d}_A\Psi) &= 0, \\ \partial_s\Psi - \partial_t\Phi + [\Phi,\Psi] + \lambda^2 * F_A &= 0. \end{aligned} \tag{4}$$

Note that the Hodge $*$-operator on 1-forms depends on the complex structure $J(s,t)$ but is invariant under rescaling, while the Hodge $*$-operator on 2-forms depends only on the volume form. Note also that $\Xi_j = \tilde{\iota}_j{}^*\Xi$ satisfies the same equations with $\lambda = 1$ and $*_{s,t}$ replaced by $*_{j;s,t}$.

The Yang-Mills action of Ξ is given by

$$\mathcal{YM}(\Xi) = \int_0^1 \int_{-\infty}^{\infty} \left(\|\partial_s A - \mathrm{d}_A \Phi\|^2_{L^2(\Sigma,J)} + \lambda^2 \|F_A\|^2_{L^2(\Sigma)} \right) \mathrm{d}s\mathrm{d}t. \tag{5}$$

If this action is finite then the connection has limits at the cylindrical ends. To be more precise, we shall assume that the connection is in temporal gauge on the cylindrical ends. This means that $\Phi_j(s,t) = 0$ for $-s$ sufficiently large. If this holds, and the flat connections on $Q_{\tilde{f}_j}$ are nondegenerate for all j, then there exist flat connections

$$a_j = A_j(t) + \Psi_j(t)\,\mathrm{d}t \in \mathcal{A}^{\mathrm{flat}}(Q_{\tilde{f}_j})$$

such that

$$\lim_{s\to-\infty} (A_j(s,t) + \Psi_j(s,t)\,\mathrm{d}t) = a_j. \tag{6}$$

The convergence is exponential and in the C^∞-topology. Moreover,

$$\lim_{s\to-\infty} \partial_s A_j(s,t) = 0, \qquad \lim_{s\to-\infty} \partial_s \Psi_j(s,t) = 0.$$

Warning: When referring to (6) below we shall always mean convergence in the C^1-norm together with uniform convergence of $\partial_s A_j$ and $\partial_s \Psi_j$ to zero and with $\partial_s \Phi_j = 0$ for $-s$ sufficiently large.

In general, the flat connections on the mapping tori $Q_{\tilde{f}_j}$ may occur in families and, to obtain smooth moduli spaces, we must choose suitable perturbations. The quantum product structure (1) can then be obtained by counting the solutions of the perturbed instanton equations with prescribed limit connections.

6 Perturbations

Throughout this section let g denote the genus of Σ. For any loop $\gamma : \mathbb{R}/\mathbb{Z} \to P$ denote by $\rho_\gamma : \mathcal{A}(P) \to \mathrm{SU}(2)$ the holonomy along γ. Thus $\rho_\gamma(A) = u(1)$ where $u : [0,1] \to \mathrm{SU}(2)$ is the solution of the ordinary differential equation

$$\dot{u} + A(\dot{\gamma})u = 0, \qquad u(0) = \mathbb{1}.$$

The differential of ρ_γ at A has the form

$$\rho_\gamma(A)^{-1}\mathrm{d}\rho_\gamma(A)\alpha = -\int_0^1 u(\theta)^{-1}\alpha(\dot{\gamma}(\theta))u(\theta)\,\mathrm{d}\theta$$

for $\alpha \in \Omega^1(\Sigma, \mathfrak{g}_P)$. Now fix a path of basepoints $[-1,1] \to P : \tau \mapsto p_\tau$ and $2g$ maps $[-1,1] \times \mathbb{R}/\mathbb{Z} \to P : (\tau,\theta) \mapsto \gamma_j(\tau,\theta)$ such that the projections $\pi \circ \gamma_j$ are orientation preserving embeddings, generate the fundamental group of Σ, and satisfy $\gamma_j(\tau,0)=p_\tau$. For $\tau\in[-1,1]$ let

$$\rho_\tau:\mathcal{A}(P)\to\mathrm{SU}(2)^{2g}$$

denote the holonomy along the loops $\theta\mapsto\gamma_j(\tau,\theta)$, $j=1,\ldots,2g$. Choose a smooth cutoff function $\beta:(-1,1)\to\mathbb{R}$ with compact support and mean value 1. Then every smooth function $h:\mathrm{SU}(2)^{2g}\to\mathbb{R}$, that is invariant under conjugacy, determines a Hamiltonian function

$$\mathcal{A}(P)\to\mathbb{R}:A\mapsto H(A)=\int_{-1}^{1}\beta(\tau)h(\rho_\tau(A))\,\mathrm{d}\tau.$$

The partial derivative of h with respect to the jth coordinate u_j can be represented by a function $\eta_j:\mathrm{SU}(2)^{2g}\to\mathfrak{su}(2)$ such that

$$\frac{\partial h}{\partial u_j}(u)u_j\xi=\langle\eta_j(u),\xi\rangle.$$

for $u=(u_1,\ldots,u_{2g})\in\mathrm{SU}(2)^{2g}$ and $\xi\in\mathfrak{g}$. Consider the vector field

$$\mathcal{A}(P)\to\Omega^1(\mathfrak{g}_P):A\mapsto v(A)=\sum_{j=1}^{2g}v_j(A),$$

where $v_j(A)\in\Omega^1(\Sigma,\mathfrak{g}_P)$ is supported in $\pi^{-1}(\mathrm{im}\,\pi\circ\gamma_j)$ and satisfies

$$\gamma_j{}^*v_j(A)=\beta(\tau)u_j(\tau,\theta)\eta_j(\rho_\tau(A))u_j(\tau,\theta)^{-1}\mathrm{d}\tau.$$

Here $\theta\mapsto u_j(\tau,\theta)$ is the holonomy of A along the loop $\theta\mapsto\gamma_j(\tau,\theta)$. Direct computation shows that

$$\mathrm{d}H(A)\alpha=\int_\Sigma\langle v(A)\wedge\alpha\rangle$$

for $A\in\mathcal{A}(P)$ and $\alpha\in\Omega^1(\Sigma,\mathfrak{g}_P)$. This means that v is the Hamiltonian vector field of H. Any such Hamiltonian function is invariant under $\mathcal{G}(P)$ and hence

$$H(u^*A)=H(A),\qquad v(u^*A)=u^{-1}v(A)u,\qquad \mathrm{d}_Av(A)=0 \tag{7}$$

for $A\in\mathcal{A}(P)$ and $u\in\mathcal{G}(P)$. The vector fields v that arise from the holonomy are smooth with respect to the $W^{k,p}$-norm for all k and p.

Now let $U\times\mathcal{A}(P)\to\mathbb{R}:(s,t,A)\mapsto H_{s,t}(A)$ be a smooth family of such Hamiltonian functions and $U\times\mathcal{A}(P)\to\Omega^1(\Sigma,\mathfrak{g}_P):(s,t,A)\mapsto v_{s,t}(A)$ be the corresponding family of Hamiltonian vector fields. Suppose that

$$H_{s,t}(A)=H_{s,t+1}(\tilde{g}_j{}^*A),\qquad \tilde{g}_j{}^*v_{s,t}(A)=v_{s,t+1}(\tilde{g}_j{}^*A)$$

for $s_j<s<s_{j+1}$. Let $K_{s,t}$ be another such family of Hamiltonian functions with corresponding vector fields $w_{s,t}$ and consider the perturbed anti-self-duality equations

$$\begin{array}{rcl}(\partial_sA-\mathrm{d}_A\Phi-v_{s,t}(A))+*_{s,t}(\partial_tA-\mathrm{d}_A\Psi-w_{s,t}(A)) &=& 0,\\ \partial_s\Psi-\partial_t\Phi+[\Phi,\Psi]+\lambda^2*F_A &=& 0.\end{array} \tag{8}$$

The perturbed Yang-Mills action of a solution of (8) is given by

$$\mathcal{YM}(\Xi; H, K) = \int_0^1 \int_{-\infty}^{\infty} \Big(\|\partial_s A - \mathrm{d}_A \Phi - v_{s,t}(A)\|^2_{L^2(\Sigma,J)} + \lambda^2 \|F_A\|^2_{L^2(\Sigma)} \Big).$$

If this action is finite then the limits (6) will exist in the case of a suitably chosen perturbation. To be more precise, consider the pullback perturbations under $\tilde{\iota}_j$. They are given by

$$v_{j;s,t} = 2\pi e^{2\pi s}(\cos(2\pi t)v + \sin(2\pi t)w),$$

$$w_{j;s,t} = 2\pi e^{2\pi s}(\cos(2\pi t)w - \sin(2\pi t)v),$$

for $j = 1, \dots, n-1$ and $-1/4 < t < 3/4$, where v and w are evaluated at the point $w_j + e^{2\pi(s+it)}$. These functions extend to $(-\infty, -T) \times \mathbb{R}$ via the periodicity conditions

$$\tilde{f_j}^* v_{j;s,t}(A) = v_{j;s,t+1}(\tilde{f_j}^* A), \qquad \tilde{f_j}^* w_{j;s,t}(A) = w_{j;s,t+1}(\tilde{f_j}^* A).$$

We shall assume throughout that $v_{j;s,t} = 0$ and $w_{j;s,t}$ is independent of s for $-s$ sufficiently large. Call $a_j = A_j(t) + \Psi_j(t)\,\mathrm{d}t$ a **perturbed flat connection** for the jth cylindrical end if

$$A_j(t+1) = \tilde{f_j}^* A(t), \qquad \Psi_j(t+1) = \Psi_j(t) \circ \tilde{f}_j. \tag{9}$$

and

$$F_{A_j(t)} = 0, \qquad \dot{A}_j(t) - \mathrm{d}_{A_j(t)}\Psi_j(t) - w_{j;-\infty,t}(A_j(t)) = 0. \tag{10}$$

Condition (9) asserts that a_j is a connection on $Q_{\tilde{f}_j}$. Denote the space of solutions of (9) and (10) by

$$\mathcal{A}^{\mathrm{flat}}(Q_{\tilde{f}_j}, K_j) = \left\{ a_j \in \mathcal{A}(Q_{\tilde{f}_j}) \,|\, (10) \right\}.$$

The gauge equivalence classes of such connections correspond naturally to the fixed points of a Hamiltonian deformation of the symplectomorphism $\phi_{\tilde{f}_j} : M_\Sigma \to M_\Sigma$ (defined below) and the perturbation K_j can be chosen such that these fixed points are all nondegenerate (cf. [4, 5]). Under this assumption the quotient space $\mathcal{A}^{\mathrm{flat}}(Q_{\tilde{f}_j}, K_j)/\mathcal{G}(Q_{\tilde{f}_j})$ is a finite set and, for every solution Ξ of (8) with finite perturbed Yang-Mills energy, the limits (6) exists. Given $n+1$ perturbed flat connections

$$a_j \in \mathcal{A}^{\mathrm{flat}}(Q_{\tilde{f}_j}, K_j), \qquad j = 0, \dots, n,$$

we denote the moduli space of solutions of (8) with these limits by

$$\mathcal{M}_1(a_0, \dots, a_n; H, K) = \frac{\{\Xi \in \mathcal{A}(Q) \,|\, (8),\, (6)\}}{\mathcal{G}(Q)}.$$

7 Pseudoholomorphic sections

Consider the moduli space

$$M_\Sigma = \mathcal{A}^{\mathrm{flat}}(P)/\mathcal{G}(P).$$

This space is a smooth compact manifold of dimension $6g(\Sigma) - 6$, where $g(\Sigma)$ denotes the genus of Σ. It carries a natural symplectic form, and every complex structure $J \in \mathcal{J}(\Sigma)$ determines a complex structure on M_Σ via the Hodge $*$-operator. Each lift $\tilde{f} : P \to P$ of a symplectomorphism of Σ determines a symplectomorphism

$$\phi_{\tilde{f}} : M_\Sigma \to M_\Sigma$$

given by $\phi_{\tilde{f}}([A]) = [\tilde{f}^* A]$. As above, choose $n+1$ such lifts $\tilde{f}_0, \ldots, \tilde{f}_n$ such that

$$\tilde{f}_n \circ \cdots \circ \tilde{f}_0 = \mathrm{id}$$

and define $\tilde{g}_j = \tilde{f}_j \circ \cdots \circ \tilde{f}_0$. Then there is a fibre bundle

$$\begin{array}{ccc} M_\Sigma & \hookrightarrow & M_X \\ & & \downarrow \\ & & S \end{array}$$

over the punctured sphere with holonomy $\phi_{\tilde{f}_j}$ around the jth puncture. It is defined as the quotient $M_X = U \times M_\Sigma / \sim$ under the equivalence relation

$$s_j < s < s_{j+1} \qquad \Longrightarrow \qquad (s, t, [A]) \sim (s, t+1, [\tilde{g}_j{}^* A]).$$

The complex structure $J : U \to \mathcal{J}(\Sigma)$ determines a vertical complex structure on M_X. The (perturbed) holomorphic sections of M_X with respect to this complex structure can be expressed as connections $\Xi \in \mathcal{A}(Q)$ that satisfy the equations

$$\begin{array}{rcl} (\partial_s A - d_A \Phi - v_{s,t}(A)) + *_{s,t}(\partial_t A - d_A \Psi - w_{s,t}(A)) & = & 0, \\ F_A & = & 0. \end{array} \tag{11}$$

As in [4], the additional terms Φ and Ψ are uniquely determined by the requirement that the 1-forms $\partial_s A - d_A \Phi$ and $\partial_t A - d_A \Psi$ in $\Omega^1(\Sigma, \mathfrak{g}_P)$ are harmonic with respect to A. The quantum product structure (2) in symplectic Floer homology is obtained by counting the solutions $\Xi \in \mathcal{A}(Q)$ of (11) that are in temporal gauge near the cylindrical ends and satisfy (6). Given $n+1$ connections $a_j \in \mathcal{A}^{\mathrm{flat}}(Q_{\tilde{f}_j}, K_j)$, denote by

$$\mathcal{M}_0(a_0, \ldots, a_n; H, K) = \frac{\{\Xi \in \mathcal{A}(Q) \,|\, (11),\, (6)\}}{\mathcal{G}(Q)}$$

the moduli space of gauge equivalence classes of pseudoholomorphic sections of M_X with prescribed limiting data at the cylindrical ends. The task at hand is to compare the solutions of (11) with those of (8).

A perturbation (H,K) is called **regular** if the perturbed flat connections corresponding to the $n+1$ cylindrical ends are all nondegenerate (see for example [4]) and if the linearized operator $\mathcal{D}_0(\Xi)$ (defined in Section 9 below) is surjective for all a_j and every solution Ξ of (11) and (6). The set of regular perturbations will be denoted by $\mathcal{HK}_{\mathrm{reg}}$. It was proved in [20] and [10] that $\mathcal{HK}_{\mathrm{reg}}$ is of the second category in the sense of Baire in the space of all smooth perturbations.

8 Adiabatic limits

The strategy for the proof of Theorem 1.1 is, precisely as in [6], to show that the solutions of (4) degenerate to pseudoholomorphic sections in the limit where the metric on the fibre converges to zero. More precisely, if we multiply the metric on Σ by a factor ε^2 then the anti-self-duality equations for this metric take the form

$$\begin{array}{rcl}(\partial_s A - d_A\Phi - v_{s,t}(A)) + *_{s,t}(\partial_t A - d_A\Psi - w_{s,t}(A)) & = & 0,\\ \partial_s\Psi - \partial_t\Phi + [\Phi,\Psi] + \varepsilon^{-2}\lambda^2 * F_A & = & 0.\end{array} \tag{12}$$

The moduli space of solutions of these equations with given limit connections a_j will be denoted by

$$\mathcal{M}_\varepsilon(a_0,\ldots,a_n;H,K) = \frac{\{\Xi\in\mathcal{A}(Q)\,|\,(12),\,(6)\}}{\mathcal{G}(Q)}.$$

As before we consider only solutions that are in temporal gauge near the cylindrical ends. The Yang-Mills action for the ε-dependent metric is given by

$$\mathcal{YM}_\varepsilon(\Xi) = \int_0^1\int_{-\infty}^{\infty}\Big(\|\partial_s A - d_A\Phi - v_{s,t}(A)\|^2_{L^2(\Sigma,J)} + \frac{\lambda^2}{\varepsilon^2}\|F_A\|^2_{L^2(\Sigma)}\Big). \tag{13}$$

It depends on the a_j but not on ε.

Theorem 8.1. *Suppose that the perturbed flat connections $a_j\in\mathcal{A}^{\mathrm{flat}}(Q_{\tilde f_j},K_j)$ are nondenerate for $j=0,\ldots,n$. Then the moduli spaces $\mathcal{M}_0(a_0,\ldots,a_n;H,K)$ and $\mathcal{M}_\varepsilon(a_0,\ldots,a_n;H,K)$ have the same virtual dimensions.*

Theorem 8.2. *Let $(H,K)\in\mathcal{HK}_{\mathrm{reg}}$ and suppose that the moduli space $\mathcal{M}_0(a_0,\ldots,a_n;H,K)$ has virtual dimension zero. Then, for $\varepsilon>0$ sufficiently small, there is a natural orientation preserving bijection*

$$\mathcal{T}_\varepsilon:\mathcal{M}_0(a_0,\ldots,a_n;H,K)\to\mathcal{M}_\varepsilon(a_0,\ldots,a_n;H,K).$$

Theorem 1.1 is an immediate corollary of these results. Theorem 8.1 will be proved in the next section. Theorem 8.2 then follows easily from the techniques developed in [6] and we shall summarize the main points in Section 10.

9 Proof of the index formula

The proof of Theorem 8.1 rests on estimates for the differential operators that arise from the linearized equations. To examine these operators in detail we introduce some further notation. Denote by $\mathcal{X}$ the space of triples $\xi = (\alpha, \phi, \psi)$, where $\alpha : U \to \Omega^1(\Sigma, \mathfrak{g}_P)$ and $\phi, \psi : U \to \Omega^0(U, \mathfrak{g}_P)$ are smooth functions that have compact support in $S = U/\sim$ and that, for $s_j < s < s_{j+1}$, satisfy the periodicity conditions

$$\begin{gathered}\alpha(s,t+1) = \tilde{g}_j{}^*\alpha(s,t),\\ \phi(s,t+1) = \phi(s,t)\circ\tilde{g}_j, \qquad \psi(s,t+1) = \psi(s,t)\circ\tilde{g}_j.\end{gathered} \tag{14}$$

Such a triple can be identified with a 1-form $\xi = \alpha + \phi\,\mathrm{d}s + \psi\,\mathrm{d}t \in \Omega^1(X, \mathfrak{g}_Q)$ with values in the Lie algebra bundle $\mathfrak{g}_Q$ associated to Q. The self-duality operator

$$\Omega^1(X,\mathfrak{g}_Q) \to \Omega^{2,+}(X,\mathfrak{g}_Q) \oplus \Omega^0(X,\mathfrak{g}_Q) : \xi \mapsto \mathcal{D}\xi = (\mathrm{d}_\Xi^+\xi, \mathrm{d}_\Xi^*\xi)$$

has the form

$$\mathrm{d}_\Xi^+\xi = -\tilde{\alpha}\wedge\mathrm{d}s + *_{s,t}\tilde{\alpha}\wedge\mathrm{d}t + \tilde{\psi}\,\mathrm{d}s\wedge\,\mathrm{d}t + \lambda^{-2}\tilde{\psi}\omega, \qquad \mathrm{d}_\Xi^*\xi = -\lambda^{-2}\tilde{\phi},$$

where $\tilde{\xi} = (\tilde{\alpha}, \tilde{\phi}, \tilde{\psi})$ is given by

$$\begin{aligned}\tilde{\alpha} &= \nabla_s\alpha - \mathrm{d}_A\phi - \mathrm{d}v_{s,t}(A)\alpha + *_{s,t}(\nabla_t\alpha - \mathrm{d}_A\psi - \mathrm{d}w_{s,t}(A)\alpha),\\ \tilde{\phi} &= \nabla_s\phi + \nabla_t\psi + \lambda^2 * \mathrm{d}_A *_{s,t}\alpha,\\ \tilde{\psi} &= \nabla_s\psi - \nabla_t\phi + \lambda^2 * \mathrm{d}_A\alpha.\end{aligned}$$

The L^p and $\mathrm{W}^{1,p}$-norms of the 1-form $\xi = \alpha + \phi\,\mathrm{d}s + \psi\,\mathrm{d}t$ are given by

$$\|\xi\|_{\mathrm{L}^p}^p = \int_0^1\int_{-\infty}^{\infty}\Big(\lambda^2\,\|\alpha\|_{L^p(\Sigma,J)}^p + \lambda^{2-p}\,\|\phi\|_{L^p(\Sigma)}^p + \lambda^{2-p}\,\|\psi\|_{L^p(\Sigma)}^p\Big)$$

$$\begin{aligned}\|\xi\|_{\mathrm{W}^{1,p}}^p \;=\; & \int_0^1\int_{-\infty}^{\infty}\Big(\lambda^2\,\|\alpha\|_{W^{1,p}(\Sigma,J)}^p\\ & + \lambda^{2-p}\,\|\nabla_s\alpha\|_{L^p(\Sigma,J)}^p + \lambda^{2-p}\,\|\nabla_t\alpha\|_{L^p(\Sigma,J)}^p\\ & + \lambda^{2-p}\,\|\phi\|_{W^{1,p}(\Sigma,J)}^p + \lambda^{2-p}\,\|\psi\|_{W^{1,p}(\Sigma,J)}^p\\ & + \lambda^{2-2p}\,\|\nabla_s\phi\|_{L^p(\Sigma)}^p + \lambda^{2-2p}\,\|\nabla_s\psi\|_{L^p(\Sigma)}^p\\ & + \lambda^{2-2p}\,\|\nabla_t\phi\|_{L^p(\Sigma)}^p + \lambda^{2-2p}\,\|\nabla_t\psi\|_{L^p(\Sigma)}^p\Big),\end{aligned}$$

where $\nabla_s = \partial_s + \Phi$, $\nabla_t = \partial_t + \Psi$, and

$$\|\phi\|_{W^{1,p}(\Sigma,J)}^p = \|\phi\|_{L^p(\Sigma)}^p + \|d_A\phi\|_{L^p(\Sigma,J)}^p,$$

$$\|\alpha\|_{W^{1,p}(\Sigma,J)}^p = \|\alpha\|_{L^p(\Sigma,J)}^p + \|d_A\alpha\|_{L^p(\Sigma)}^p + \|d_A(\alpha\circ J)\|_{L^p(\Sigma)}^p.$$

Thus the $\mathrm{W}^{1,p}$-norm depends on a connection $\Xi = (A, \Phi, \Psi)$ which should be chosen to be in temporal gauge near the cylindrical ends. We assume further that the limits (6) exist. The L^p-norm of the triple $\tilde{\xi} \cong \mathcal{D}\xi$ is given by

$$\|\tilde{\xi}\|_{\tilde{\mathrm{L}}^p}^p = \int_0^1 \int_{-\infty}^{\infty} \Big(\lambda^{2-p} \|\tilde{\alpha}\|_{L^p(\Sigma,J)}^p + \lambda^{2-2p} \|\tilde{\phi}\|_{L^p(\Sigma)}^p + \lambda^{2-2p} \|\tilde{\psi}\|_{L^p(\Sigma)}^p \Big).$$

For $1 < p < \infty$ we denote by L^p, $\mathrm{W}^{1,p}$, and $\widetilde{\mathrm{L}}^p$ the completions of $\mathcal{X}$ with respect to these norms. They are independent of the choice of the connection Ξ. Note that $\tilde{\xi} \in \widetilde{\mathrm{L}}^p$ if and only if $\lambda^{-1}\tilde{\xi} \in \mathrm{L}^p$.

Linearizing the equations (12) we obtain the ε-dependent self-duality operator

$$\mathcal{D}_\varepsilon = \mathcal{D}_\varepsilon(\Xi) : \mathrm{W}^{1,p} \to \widetilde{\mathrm{L}}^p,$$

given by

$$\mathcal{D}_\varepsilon \xi = \tilde{\xi} = (\tilde{\alpha}, \tilde{\phi}, \tilde{\psi}),$$

$$\begin{aligned} \tilde{\alpha} &= \nabla_s \alpha - \mathrm{d}_A \phi - \mathrm{d}v_{s,t}(A)\alpha + *_{s,t}(\nabla_t \alpha - \mathrm{d}_A \psi - \mathrm{d}w_{s,t}(A)\alpha), \\ \tilde{\phi} &= \nabla_s \phi + \nabla_t \psi + (\lambda/\varepsilon)^2 * \mathrm{d}_A *_{s,t} \alpha, \\ \tilde{\psi} &= \nabla_s \psi - \nabla_t \phi + (\lambda/\varepsilon)^2 * \mathrm{d}_A \alpha. \end{aligned} \tag{15}$$

Next we discuss the Cauchy-Riemann operator along the section $S \to \mathcal{M} : (s,t) \mapsto [A(s,t)]$. The pullback vertical tangent bundle under this section is the bundle $\mathrm{H}_A \to S$ whose fibre over $s + it$ is the space

$$\mathrm{H}^1_{A(s,t)} = \ker \mathrm{d}_{A(s,t)} \cap \ker \mathrm{d}_{A(s,t)} *_{s,t}$$

of harmonic 1-forms with respect to $A(s,t)$ and $*_{s,t}$. Let

$$\pi_A = \pi_{A(s,t)} : L^2(\Sigma, T^*\Sigma \otimes \mathfrak{g}_P) \to \mathrm{H}^1_{A(s,t)}$$

denote the L^2 orthogonal projection onto the harmonic part. We introduce the subspaces

$$\mathrm{L}^p(\mathrm{H}_A) \subset \mathrm{L}^p, \qquad \mathrm{W}^{1,p}(\mathrm{H}_A) \subset \mathrm{W}^{1,p}, \qquad \widetilde{\mathrm{L}}^p(\mathrm{H}_A) \subset \widetilde{\mathrm{L}}^p,$$

of those triples (α, ϕ, ψ) that satisfy $\phi = \psi = 0$ and $\pi_A(\alpha) = \alpha$. The operator

$$\mathcal{D}_0 = \mathcal{D}_0(\Xi) : \mathrm{W}^{1,p}(\mathrm{H}_A) \to \widetilde{\mathrm{L}}^p(\mathrm{H}_A)$$

is given by

$$\mathcal{D}_0 \alpha_0 = \pi_A \bigg(\nabla_s \alpha_0 - \mathrm{d}v_{s,t}(A)\alpha_0 + *_{s,t} \nabla_t \alpha_0 - *_{s,t} \mathrm{d}w_{s,t}(A)\alpha_0 \bigg).$$

Note that $\tilde{\alpha}_0 = \mathcal{D}_0 \alpha_0$ can be interpreted as a $(0,1)$-form $\tilde{\alpha}_0 \wedge \mathrm{d}s - *_{s,t}\tilde{\alpha}_0 \wedge \mathrm{d}t$ on S with values in the bundle $H_A \to S$. The L^p-norm of this 1-form is precisely the norm of the space $\widetilde{\mathrm{L}}^p$.

Both operators $\mathcal{D}_\varepsilon(\Xi)$ and $\mathcal{D}_0(\Xi)$ are Fredholm whenever the limit connections $a_0, \dots, a_n$ of Ξ are all nondegenerate. To examine the relation between these operators it will be convenient to introduce the ε-dependent norms

$$\|\xi\|_{0,p,\varepsilon}^p = \int_0^1 \int_{-\infty}^{\infty} \Big(\lambda^2 \|\alpha\|_{L^p(\Sigma,J)}^p + \varepsilon^p \lambda^{2-p} \|\phi\|_{L^p(\Sigma)}^p + \varepsilon^p \lambda^{2-p} \|\psi\|_{L^p(\Sigma)}^p\Big),$$

$$\begin{aligned}\|\xi\|_{1,p,\varepsilon}^p \quad = \quad & \int_0^1 \int_{-\infty}^{\infty} \Big(\lambda^2 \|\alpha\|_{W^{1,p}(\Sigma,J)}^p \\ & + \varepsilon^p \lambda^{2-p} \|\nabla_s \alpha\|_{L^p(\Sigma,J)}^p + \varepsilon^p \lambda^{2-p} \|\nabla_t \alpha\|_{L^p(\Sigma,J)}^p \\ & + \varepsilon^p \lambda^{2-p} \|\phi\|_{W^{1,p}(\Sigma,J)}^p + \varepsilon^p \lambda^{2-p} \|\psi\|_{W^{1,p}(\Sigma,J)}^p \\ & + \varepsilon^{2p} \lambda^{2-2p} \|\nabla_s \phi\|_{L^p(\Sigma)}^p + \varepsilon^{2p} \lambda^{2-2p} \|\nabla_s \psi\|_{L^p(\Sigma)}^p \\ & + \varepsilon^{2p} \lambda^{2-2p} \|\nabla_t \phi\|_{L^p(\Sigma)}^p + \varepsilon^{2p} \lambda^{2-2p} \|\nabla_t \psi\|_{L^p(\Sigma)}^p\Big),\end{aligned}$$

$$\|\tilde\xi\|_{\widetilde{0},p,\varepsilon}^p = \int_0^1 \int_{-\infty}^{\infty} \Big(\lambda^{2-p} \|\tilde\alpha\|_{L^p(\Sigma,J)}^p + \varepsilon^p \lambda^{2-2p} \|\tilde\phi\|_{L^p(\Sigma)}^p + \varepsilon^p \lambda^{2-2p} \|\tilde\psi\|_{L^p(\Sigma)}^p\Big).$$

Proposition 9.1. *Fix a connection $\Xi = (A, \Phi, \Psi) \in \mathcal{A}(Q)$ that is in temporal gauge near the cylindrical ends and satisfies $F_{A(s,t)} = 0$ for all s and t. Suppose that the limits (6) exist. Then, for every $p \geq 2$, there exist constants $\varepsilon_0 > 0$ and $c \geq 1$ such that, for $0 < \varepsilon < \varepsilon_0$ and $\xi \in \mathcal{X}$,*

$$\|\xi\|_{1,p,\varepsilon} \leq c \Big(\varepsilon \|\mathcal{D}_\varepsilon \xi\|_{\widetilde{0},p,\varepsilon} + \|\pi_A(\xi)\|_{\mathrm{L}^p}\Big), \tag{16}$$

$$\|\xi - \pi_A(\xi)\|_{1,p,\varepsilon} \leq c\varepsilon \Big(\|\mathcal{D}_\varepsilon \xi\|_{\widetilde{0},p,\varepsilon} + \|\pi_A(\xi)\|_{\mathrm{L}^p}\Big), \tag{17}$$

$$\|\pi_A(\mathcal{D}_\varepsilon \xi) - \mathcal{D}_0 \pi_A(\xi)\|_{\widetilde{\mathrm{L}}^p} \leq c \|\xi - \pi_A(\xi)\|_{0,p,\varepsilon}. \tag{18}$$

Proof. Appendix D. (For $p = 2$ see [6, Lemmata 4.2 and 4.3].) □

Proof of Theorem 8.1. Choose an L^2-orthonormal basis $\alpha_1, \dots, \alpha_k \in \mathrm{W}^{1,2}(\mathrm{H}_A)$ of the kernel of $\mathcal{D}_0$ and an $\widetilde{\mathrm{L}}^2$-orthonormal basis $\beta_1, \dots, \beta_\ell \in \widetilde{\mathrm{L}}^2(\mathrm{H}_A)$ of the cokernel of $\mathcal{D}_0$. Define the operator

$$\widehat{\mathcal{D}}_\varepsilon : \mathrm{W}^{1,2} \oplus \mathbb{R}^\ell \to \widetilde{\mathrm{L}}^2 \oplus \mathbb{R}^k$$

by

$$\widehat{\mathcal{D}}_\varepsilon \hat\xi := \left(\mathcal{D}_\varepsilon \xi + \sum_{j=1}^{\ell} \lambda_j \beta_j, \langle \alpha_1, \pi_A(\xi)\rangle, \dots, \langle \alpha_k, \pi_A(\xi)\rangle\right)$$

for $\hat{\xi} = (\xi, \lambda_1, \ldots, \lambda_\ell) \in \mathrm{W}^{1,2} \oplus \mathbb{R}^\ell$. We prove that $\widehat{\mathcal{D}}_\varepsilon$ is injective. To see this, note first that there is a constant $c_0 \geq 1$ such that

$$\|\pi_A(\xi)\|_{\mathrm{L}^2} + \sum_{j=1} |\lambda_j| \leq c_0 \left\| \mathcal{D}_0 \pi_A(\xi) + \sum_{j=1}^{\ell} \lambda_j \beta_j \right\|_{\widetilde{\mathrm{L}}^2}$$

for all $\hat{\xi} = (\xi, \lambda_1, \ldots, \lambda_\ell) \in \mathrm{W}^{1,2} \oplus \mathbb{R}^\ell$. Hence it follows from (18) and (17) with $p = 2$ that

$$\begin{aligned}
&\|\pi_A(\xi)\|_{\mathrm{L}^2} + \sum_{j=1} |\lambda_j| \\
&\leq\ c_0 \left\| \pi_A(\mathcal{D}_\varepsilon \xi) + \sum_{j=1}^{\ell} \lambda_j \beta_j \right\|_{\widetilde{\mathrm{L}}^2} + c_0 \left\| \pi_A(\mathcal{D}_\varepsilon \xi) - \mathcal{D}_0 \pi_A(\xi) \right\|_{\widetilde{\mathrm{L}}^2} \\
&\leq\ c_0 \left\| \mathcal{D}_\varepsilon \xi + \sum_{j=1}^{\ell} \lambda_j \beta_j \right\|_{\widetilde{0},2,\varepsilon} + c_0 c \left\| \xi - \pi_A(\xi) \right\|_{0,2,\varepsilon} \\
&\leq\ c_0 \left\| \mathcal{D}_\varepsilon \xi + \sum_{j=1}^{\ell} \lambda_j \beta_j \right\|_{\widetilde{0},2,\varepsilon} + c_0 c^2 \varepsilon \left(\|\mathcal{D}_\varepsilon \xi\|_{\widetilde{0},2,\varepsilon} + \|\pi_A(\xi)\|_{\mathrm{L}^2} \right) \\
&\leq\ c_0(1 + c^2\varepsilon) \left\| \mathcal{D}_\varepsilon \xi + \sum_{j=1}^{\ell} \lambda_j \beta_j \right\|_{\widetilde{0},2,\varepsilon} + c_0 c^2 \varepsilon \left(\|\pi_A(\xi)\|_{\mathrm{L}^2} + \sum_{j=1}^{\ell} |\lambda_j| \right).
\end{aligned}$$

With $c_0 c^2 \varepsilon \leq 1/2$ we obtain

$$\|\pi_A(\xi)\|_{\mathrm{L}^2} + \sum_{j=1}^{\ell} |\lambda_j| \leq 3c_0 \left\| \widehat{\mathcal{D}}_\varepsilon \hat{\xi} \right\|_{\widetilde{0},2,\varepsilon}.$$

Combining this inequality with (16) gives

$$\begin{aligned}
\|\hat{\xi}\|_{1,2,\varepsilon} &= \sqrt{\|\xi\|_{1,2,\varepsilon}^2 + \sum_{j=1}^{\ell} {\lambda_j}^2} \\
&\leq \|\xi\|_{1,2,\varepsilon} + \sum_{j=1}^{\ell} |\lambda_j| \\
&\leq c \left(\varepsilon \|\mathcal{D}_\varepsilon \xi\|_{\widetilde{0},2,\varepsilon} + \|\pi_A(\xi)\|_{\mathrm{L}^2} \right) + \sum_{j=1}^{\ell} |\lambda_j| \\
&\leq c\varepsilon \left\| \mathcal{D}_\varepsilon \xi + \sum_{j=1}^{\ell} \lambda_j \beta_j \right\|_{\widetilde{0},2,\varepsilon} + c' \left(\|\pi_A(\xi)\|_{\mathrm{L}^2} + \sum_{j=1}^{\ell} |\lambda_j| \right) \\
&\leq c'' \|\widehat{\mathcal{D}}_\varepsilon \hat{\xi}\|_{\widetilde{0},2,\varepsilon}.
\end{aligned}$$

This shows that $\widehat{\mathcal{D}}_\varepsilon$ is injective for $\varepsilon > 0$ sufficiently small. To prove surjectivity we examine the adjoint operator

$$\mathcal{D}_\varepsilon{}' : \mathrm{W}^{1,2} \to \widetilde{\mathrm{L}}^2.$$

It is defined by the identity

$$\left\langle \lambda^{-1}\mathcal{D}_\varepsilon{}'\xi', \xi\right\rangle_{0,2,\varepsilon} = \left\langle \xi', \lambda^{-1}\mathcal{D}_\varepsilon\xi\right\rangle_{0,2,\varepsilon}$$

for $\xi, \xi' \in \mathcal{X}$. Direct computation shows that $\mathcal{D}_\varepsilon{}'\xi' = \tilde{\xi}'$ is given by

$$\begin{aligned}
\tilde{\alpha}' &= -\widetilde{\nabla}_s\alpha' - \mathrm{d}_A\phi' - \mathrm{d}v_{s,t}(A)\alpha' + *_{s,t}(\widetilde{\nabla}_t\alpha' - \mathrm{d}_A\psi - \mathrm{d}w_{s,t}(A)\alpha'),\\
\tilde{\phi}' &= -\widetilde{\nabla}_s\phi' + \widetilde{\nabla}_t\psi' + (\lambda/\varepsilon)^2 * \mathrm{d}_A *_{s,t} \alpha',\\
\tilde{\psi}' &= -\widetilde{\nabla}_s\psi' - \widetilde{\nabla}_t\phi' + (\lambda/\varepsilon)^2 * \mathrm{d}_A\alpha',
\end{aligned}$$

where

$$\widetilde{\nabla}_s\alpha' = \nabla_s\alpha' + \lambda^{-1}(\partial_s\lambda)\alpha' + \alpha' \circ (J\partial_s J), \qquad \widetilde{\nabla}_t\alpha' = \nabla_s\alpha' + \lambda^{-1}(\partial_t\lambda)\alpha',$$

and

$$\widetilde{\nabla}_s\phi' = \nabla_s\phi' - \lambda^{-1}(\partial_s\lambda)\phi', \qquad \widetilde{\nabla}_t\phi' = \nabla_t\phi' - \lambda^{-1}(\partial_t\lambda)\phi'.$$

Similarly, the operator

$$\mathcal{D}_0{}' : \mathrm{W}^{1,2}(\mathrm{H}_A) \to \widetilde{\mathrm{L}}^2(\mathrm{H}_A)$$

is given by

$$\mathcal{D}_0{}'\alpha_0{}' = \pi_A\left(\widetilde{\nabla}_s\alpha_0{}' - \mathrm{d}v_{s,t}(A)\alpha_0{}' + *_{s,t}\widetilde{\nabla}_t\alpha_0{}' - *_{s,t}\mathrm{d}w_{s,t}(A)\alpha_0{}'\right).$$

Note that there exists a constant $c_1 > 0$ such that

$$|\partial_s J| + |\lambda^{-1}\partial_s\lambda| + |\lambda^{-1}\partial_t\lambda| \le c_1\lambda.$$

Hence it follows from Proposition 9.1 that the adjoint operator $\mathcal{D}_\varepsilon{}'$ satisfies the same estimates as $\mathcal{D}_\varepsilon$. Namely, for every $p \ge 2$, there exist constants $\varepsilon_0 > 0$ and $c \ge 1$ such that, for $0 < \varepsilon < \varepsilon_0$ and $\xi' \in \mathcal{X}$,

$$\|\xi'\|_{1,p,\varepsilon} \le c\left(\varepsilon\left\|\mathcal{D}_\varepsilon{}'\xi'\right\|_{\tilde{0},p,\varepsilon} + \|\pi_A(\xi')\|_{\mathrm{L}^p}\right),$$

$$\|\xi' - \pi_A(\xi')\|_{1,p,\varepsilon} \le c\varepsilon\left(\left\|\mathcal{D}_\varepsilon{}'\xi'\right\|_{\tilde{0},p,\varepsilon} + \|\pi_A(\xi')\|_{\mathrm{L}^p}\right),$$

$$\left\|\pi_A(\mathcal{D}_\varepsilon{}'\xi') - \mathcal{D}_0{}'\pi_A(\xi')\right\|_{\widetilde{\mathrm{L}}^p} \le c\,\|\xi' - \pi_A(\xi')\|_{0,p,\varepsilon}.$$

Now it follows by the same arguments as above that the operator $\widehat{\mathcal{D}}_\varepsilon{}'$ is injective. This shows that $\widehat{\mathcal{D}}_\varepsilon$ is surjective. Hence

$$\operatorname{index}\mathcal{D}_\varepsilon = k - \ell = \operatorname{index}\mathcal{D}_0.$$

This proves the theorem. $\square$

The Fredholm index will be denoted by

$$\mu(a_0, \ldots, a_n; H, K) = \operatorname{index} \mathcal{D}_\varepsilon = \operatorname{index} \mathcal{D}_0.$$

This index depends only on the limit connections a_j but not on the connection Ξ used to define it. It is the virtual dimension of the moduli spaces $\mathcal{M}_0(a_0, \ldots, a_n; H, K)$ and $\mathcal{M}_\varepsilon(a_0, \ldots, a_n; H, K)$. As pointed out above, the space $\mathcal{HK}_{\mathrm{reg}}$ of regular perturbations consists of those pairs (H, K) such that the operator $\mathcal{D}_0(\Xi)$ is surjective for all solutions Ξ of (11) and (6). This set is a countable intersection of open and dense sets in the space of all smooth perturbations (cf. [20, 10]). If $(H, K) \in \mathcal{HK}_{\mathrm{reg}}$ then $\mathcal{M}_0(a_0, \ldots, a_n; H, K)$ is a smooth orientable manifold of dimension

$$\dim \mathcal{M}_0(a_0, \ldots, a_n; H, K) = \mu(a_0, \ldots, a_n; H, K).$$

If, moreover, $\mu(a_0, \ldots, a_n; H, K) = 0$, then there exists a constant $\varepsilon_0 > 0$ such that $\mathcal{M}_\varepsilon(a_0, \ldots, a_n; H, K)$ is a zero-dimensional manifold for $0 < \varepsilon \leq \varepsilon_0$.

Remark 9.1. The same argument as in the proof of Theorem 8.1 shows that if $\mathcal{D}_0$ is injective then so is $\mathcal{D}_\varepsilon$ for $\varepsilon > 0$ sufficiently small, and similarly for surjectivity. The argument also provides a uniform estimate for the left, respectively right, inverse of $\mathcal{D}_\varepsilon$ (cf. [6]).

10 Proof of the main result

To restate Theorem 8.2 more precisely we shall need some notation. Fix a reference connection

$$\hat{\Xi} = \hat{A} + \hat{\Phi}\, \mathrm{d}s + \hat{\Psi}\, \mathrm{d}t \in \mathcal{A}(Q)$$

such that the pullback connections on the cylindrical ends $(-\infty, -T) \times Q_{\tilde{f}_j}$ satisfy (6) and

$$\partial_s \hat{A}_j = 0, \qquad \hat{\Phi}_j = 0, \qquad \partial_s \hat{\Psi}_j = 0$$

for $-s$ sufficiently large. Then denote

$$\mathcal{A}^{1,p}(a_0, \ldots, a_n) = \{\hat{\Xi} + \xi \,|\, \xi \in \mathrm{W}^{1,p}\}.$$

Likewise, denote by $\mathcal{G}^{2,p}$ the completion of the group $\mathcal{G}(Q)$ (of gauge transformations that are equal to $\mathbb{1}$ near the cylindrical ends) with respect to the $\mathrm{W}^{1,p}$-norm on $u^*\hat{\Xi}$. Next define

$$\mathcal{A}_0^{1,p}(a_0, \ldots, a_n; H, K) = \left\{\Xi \in \mathcal{A}^{1,p}(a_0, \ldots, a_n) \,|\, (11)\right\},$$

and

$$\mathcal{A}_\varepsilon^{1,p}(a_0, \ldots, a_n; H, K) = \left\{\Xi \in \mathcal{A}^{1,p}(a_0, \ldots, a_n) \,|\, (12)\right\}.$$

Thus, for $\varepsilon \geq 0$,

$$\mathcal{M}_\varepsilon(a_0, \ldots, a_n; H, K) \cong \frac{\mathcal{A}_\varepsilon^{1,p}(a_0, \ldots, a_n; H, K)}{\mathcal{G}^{2,p}}.$$

Theorem 10.1. *Assume* $(H,K) \in \mathcal{HK}_{\mathrm{reg}}$ *and* $\mu(a_0,\dots,a_n;H,K)=0$. *Then there exist constants* $\varepsilon_0 > 0$ *and* $c > 0$ *such that, for* $0 < \varepsilon < \varepsilon_0$, *there exists a map*

$$\mathcal{T}_\varepsilon : \mathcal{A}_0^{1,p}(a_0,\dots,a_n;H,K) \to \mathcal{A}_\varepsilon^{1,p}(a_0,\dots,a_n;H,K)$$

that satisfies the following conditions.

(i) *If* $\Xi_0 \in \mathcal{A}_0^{1,p}(a_0,\dots,a_n;H,K)$ *then* $\Xi_\varepsilon = \mathcal{T}_\varepsilon(\Xi_0)$ *satisfies*[1]

$$\mathrm{d}_{\Xi_0}^{*_\varepsilon}(\Xi_\varepsilon - \Xi_0) = 0, \tag{19}$$

$$\|\Xi_\varepsilon - \Xi_0\|_{1,p,\varepsilon} \le c\varepsilon^2. \tag{20}$$

Here the $1,p,\varepsilon$*-norm is the one determined by* Ξ_0.

(ii) $\mathcal{T}_\varepsilon$ *is equivariant under the action of* $\mathcal{G}^{2,p}$, *i.e.*

$$\mathcal{T}_\varepsilon(u^*\Xi_0) = u^*\mathcal{T}_\varepsilon(\Xi_0)$$

for every $\Xi_0 \in \mathcal{A}_0^{1,p}(a_0,\dots,a_n;H,K)$ *and every* $u \in \mathcal{G}^{2,p}$. *The induced map of the moduli spaces will also be denoted by* $\mathcal{T}_\varepsilon$.

(iii) $\mathcal{T}_\varepsilon$ *is injective.*

(iv) $\mathcal{T}_\varepsilon$ *is surjective.*

The first assertion of Theorem 10.1 is a refined version of the implicit function theorem and follows from a Newton type iteration argument. The details are word by word the same as in the proof of Theorem 5.1 in [6] and will be omitted. The next two assertions follow from standard arguments in gauge theory, e.g. the gauge equivariance follows from the uniqueness part of the implicit function theorem (Proposition 10.2 below), and injectivity follows from the standard observation that the $\mathrm{W}^{1,p}$-norms of two gauge equivalent connections control the $\mathrm{W}^{2,p}$-norm of the gauge transformation by which they are related (cf. Proposition 5.7 in [6] for the present context). The hardest part of the proof of Theorem 10.1 is surjectivity. It relies on the following four propositions.

Proposition 10.2. *Assume* $(H,K) \in \mathcal{HK}_{\mathrm{reg}}$ *and* $\mu(a_0,\dots,a_n;H,K) = 0$. *Then there exist constants* $\delta > 0$ *and* $\varepsilon_0 > 0$ *such that the following holds. If* $\Xi_0 \in \mathcal{A}_0^{1,p}(a_0,\dots,a_n;H,K)$ *and* $\Xi \in \mathcal{A}_\varepsilon^{1,p}(a_0,\dots,a_n;H,K)$ *for some* $\varepsilon \in (0,\varepsilon_0)$ *and (19) is satisfied and*

$$\|\Xi - \Xi_0\|_{1,p,\varepsilon} \le \delta\varepsilon^{1/2+2/p}$$

then $\Xi = \mathcal{T}_\varepsilon(\Xi_0)$.

[1] Here $\mathrm{d}_\Xi^{*_\varepsilon} : \Omega^1(X,\mathfrak{g}_Q) \to \Omega^0(X,\mathfrak{g}_Q)$ is the adjoint of the covariant differential $\mathrm{d}_\Xi : \Omega^0(X,\mathfrak{g}_Q) \to \Omega^1(X,\mathfrak{g}_Q)$ with respect to the ε-dependent L^2-norm on 1-forms. In explicit terms

$$\mathrm{d}_\Xi^{*_\varepsilon}\xi = -*\mathrm{d}_A *_{s,t} \alpha - (\varepsilon/\lambda)^2\nabla_s\phi - (\varepsilon/\lambda)^2\nabla_t\psi.$$

Proposition 10.3. *Assume $(H,K) \in \mathcal{HK}_{\rm reg}$ and $\mu(a_0,\dots,a_n;H,K) = 0$. Then there exist constants $\varepsilon_0 > 0$ and $\delta > 0$ such that the following holds. If $\Xi_0 \in \mathcal{A}_0^{1,p}(a_0,\dots,a_n;H,K)$ and $\Xi \in \mathcal{A}_\varepsilon^{1,p}(a_0,\dots,a_n;H,K)$ for some $\varepsilon \in (0,\varepsilon_0)$ and*

$$\|\Xi - \Xi_0\|_{1,p,\varepsilon} \le \delta\varepsilon^{1/2+2/p}, \tag{21}$$

then there exists a $u \in \mathcal{G}^{2,p}$ such that $u^*\Xi = \mathcal{T}_\varepsilon(\Xi_0)$.

As pointed out above, Proposition 10.2 is the uniqueness part of the implicit function theorem (compare with [6, Theorem 5.2]). Proposition 10.3 is an easy consequence of Proposition 10.2. One can use the gauge freedom to achieve condition (19) and has to show that this can be done without destroying the estimate (21). (See [6, Section 6] for details.)

Proposition 10.4. *Assume $(H,K) \in \mathcal{HK}_{\rm reg}$ and $\mu(a_0,\dots,a_n;H,K) = 0$. Then, for every constant $c_0 > 0$, there exists a constant $\varepsilon_0 > 0$ such that the following holds. If $\Xi \in \mathcal{A}_\varepsilon^{1,p}(a_0,\dots,a_n;H,K)$ for some $\varepsilon \in (0,\varepsilon_0)$ and*

$$\sup_{s,t}\left(\varepsilon^{-2}\|F_A\|_{L^\infty(\Sigma)} + \lambda^{-1}\|\partial_t A - d_A\Psi\|_{L^\infty(\Sigma,J)}\right) \le c_0 \tag{22}$$

then there exists a connection $\Xi_0 \in \mathcal{A}_0^{1,p}(a_0,\dots,a_n;H,K)$ such that $\Xi = \mathcal{T}_\varepsilon(\Xi_0)$.

The proof of this result relies on Proposition 10.3. The key idea is to project the first component of $\Xi = (A,\Phi,\Psi)$ onto the moduli space M_Σ of flat connections and then prove estimates for the resulting section A' of the bundle $\mathcal{M}$. One can show that, firstly, A' is approximately holomorphic in the sense that the complex anti-linear part of its differential is bounded by a constant times ε^2. Secondly, one can control the $\mathrm{W}^{1,p}$ norm of this section. Hence it can be approximated by a pseudo-holomorphic section A'' (Theorem 2.5 in [6]). A further modification of A'' (by a gauge transformation, pointwise for all s and t) then gives rise to a connection $\Xi_0 \in \mathcal{A}_0^{1,p}(a_0,\dots,a_n;H,K)$ such that

$$d_{A(s,t)} *_{s,t} (A(s,t) - A_0(s,t)) = 0$$

for all s and t. This connection satisfies an estimate

$$\|\Xi - \Xi_0\|_{1,2,\varepsilon} \le c\varepsilon^{1+2/p},$$

where c is independent of Ξ. With $c\varepsilon^{1/2} \le \delta$ it follows from Proposition 10.3 that Ξ is gauge equivalent to $\mathcal{T}_\varepsilon(\Xi_0)$. The details are carried out in [6, Sections 7 and 8] and will not be reproduced here.

Proposition 10.5. *Assume $(H,K) \in \mathcal{HK}_{\rm reg}$ and $\mu(a_0,\dots,a_n;H,K) = 0$. Then there exist constants $c_0 > 0$ and $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0,\varepsilon_0)$ and every $\Xi \in \mathcal{A}_\varepsilon^{1,p}(a_0,\dots,a_n;H,K)$,*

$$\sup_{s,t}\left(\varepsilon^{-2}\|F_A\|_{L^\infty(\Sigma)} + \lambda^{-1}\|\partial_t A - d_A\Psi\|_{L^\infty(\Sigma,J)}\right) \le c_0.$$

To prove this one argues by contradiction. If the result were false, then bubbling would have to occur, and this would give rise to nonempty moduli spaces of negative virtual dimension, contradicting the assumption $(H, K) \in \mathcal{HK}_{\mathrm{reg}}$. For more details see [6, Section 9].

Proof of Theorem 10.1 (iv). Choose c_0 and ε_0 as in Proposition 10.5. Making ε_0 smaller, if necessary, we may assume that Proposition 10.4 holds with these constants. Now let $\Xi \in \mathcal{A}_\varepsilon^{1,p}(a_0, \dots, a_n; H, K)$ and suppose that $0 < \varepsilon < \varepsilon_0$. Then, by Proposition 10.5, Ξ satisfies (22). Hence, by Proposition 10.4, Ξ lies in the image of $\mathcal{T}_\varepsilon$. Hence $\mathcal{T}_\varepsilon$ is surjective for $0 < \varepsilon < \varepsilon_0$. □

Proof of Theorem 8.2. By Theorem 10.1, there is a natural bijection

$$\mathcal{T}_\varepsilon : \mathcal{M}_0(a_0, \dots, a_n; H, K) \to \mathcal{M}_\varepsilon(a_0, \dots, a_n; H, K).$$

It remains to show that this bijection preserves orientations. To see this consider the space

$$\mathcal{A}_\Sigma^{1,p}(a_0, \dots, a_n) = \left\{ \Xi \in \mathcal{A}^{1,p}(a_0, \dots, a_n; H, K) \,|\, F_{A(s,t)} = 0 \,\forall\, s,\, t \right\}.$$

Since $\mathcal{A}^{\mathrm{flat}}(P)$ is connected and simply connected, this space is nonempty. For every $\Xi \in \mathcal{A}_\Sigma^{1,p}(a_0, \dots, a_n)$, there are two Fredholm operators $\mathcal{D}_0(\Xi)$ and $\mathcal{D}_\varepsilon(\Xi)$ for $\varepsilon > 0$. Their determinants give rise to two line bundles

$$\mathcal{L}_0 \to \mathcal{A}_\Sigma^{1,p}(a_0, \dots, a_n), \qquad \mathcal{L}_\varepsilon \to \mathcal{A}_\Sigma^{1,p}(a_0, \dots, a_n).$$

The argument in the proof of Theorem 8.1 establishes, for every Ξ, a natural identification of $\det(\mathcal{D}_0(\Xi))$ with $\det(\mathcal{D}_\varepsilon(\Xi))$ for sufficiently small $\varepsilon > 0$. Hence the line bundles $\mathcal{L}_0$ and $\mathcal{L}_\varepsilon$ are isomorphic along any given loop in $\mathcal{A}_\Sigma^{1,p}(a_0, \dots, a_n)$ for ε sufficiently small. Since $\mathcal{L}_\varepsilon$ extends to the affine space $\mathcal{A}^{1,p}(a_0, \dots, a_n)$, it follows that both bundles are orientable and there is a natural bijection from the (two element) set $\mathrm{Or}_0(a_0, \dots, a_n)$ of orientations of $\mathcal{L}_0$ to the set $\mathrm{Or}_\varepsilon(a_0, \dots, a_n)$ of orientations of $\mathcal{L}_\varepsilon$. Denote this bijection by

$$\tau_\varepsilon(a_0, \dots, a_n) : \mathrm{Or}_0(a_0, \dots, a_n) \to \mathrm{Or}_\varepsilon(a_0, \dots, a_n). \tag{23}$$

Now suppose that we are given a system of orientations

$$\sigma_0(a_0, \dots, a_n) \in \mathrm{Or}_0(a_0, \dots, a_n), \tag{24}$$

one for each $(n+1)$-tuple of perturbed flat connections $a_j \in \mathcal{A}^{\mathrm{flat}}(Q_{\tilde{f}_j}, K_j)$. Suppose further that we are given $n+1$ systems of coherent orientations for the $n+1$ (symplectic) Floer homologies:

$$\sigma_{\tilde{f}_j}(a_j, b_j) \in \mathrm{Or}_{\tilde{f}_j}(a_j, b_j)$$

(cf. Floer–Hofer [9]). The orientations (24) are called *compatible with the coherent orientations* if all the Floer gluing maps are orientation preserving. Given the coherent orientations of the Floer chain complexes, there exists a compatible collection of orientations (24) and such a collection is uniquely determined up to an overall sign. The same holds with the subscript 0 replaced by $\varepsilon > 0$.

Now one can show as in [6, Proposition 10.3] that the maps (23) commute with the Floer gluing maps. Hence the maps τ_ε map any compatible collection of orientations (for the symplectic Floer theory) to a compatible collection

$$\sigma_\varepsilon(a_0, \ldots, a_n) \in \mathrm{Or}_\varepsilon(a_0, \ldots, a_n) \tag{25}$$

for the corresponding instanton theory.

Let us fix two such compatible collections of orientations, which are related by τ_ε. Then the sign $\nu_0(\Xi_0)$ of a solution $\Xi_0 \in \mathcal{A}_0^{1,p}(a_0, \ldots, a_n; H, K)$ is determined by comparing the natural orientation of $\det(\mathcal{D}_0(\Xi_0)) \cong \mathbb{R}$ with the orientation given by (24). Similarly for $\Xi_\varepsilon \in \mathcal{A}_\varepsilon^{1,p}(a_0, \ldots, a_n; H, K)$. But since $\sigma_\varepsilon(a_0, \ldots, a_n)$ is the image of $\sigma_0(a_0, \ldots, a_n)$ under $\tau_\varepsilon(a_0, \ldots, a_n)$, it follows that the two orientations of $\det(\mathcal{D}_0(\Xi_0))$ agree if and only if the two orientations of $\det(\mathcal{D}_\varepsilon(\Xi_0))$ agree. Since $\Xi_\varepsilon = \mathcal{T}_\varepsilon(\Xi_0)$ is close to Ξ_0 it follows that

$$\nu_\varepsilon(\mathcal{T}_\varepsilon(\Xi_0)) = \nu_0(\Xi_0)$$

for $\varepsilon > 0$ sufficiently small. Hence the map $\mathcal{T}_\varepsilon : \mathcal{M}_0(a_0, \ldots, a_n; H, K) \to \mathcal{M}_\varepsilon(a_0, \ldots, a_n; H, K)$ is orientation preserving. This completes the proof of Theorem 8.2 □

Proof of Theorem 1.1. The Floer homology groups $\mathrm{HF}^{\mathrm{symp}}(\phi_{\tilde{f}})$ and $\mathrm{HF}(Q_{\tilde{f}})$ are both derived from the chain complex

$$\mathrm{CF}(\tilde{f}, K) = \bigoplus_{[a] \in \mathcal{A}^{\mathrm{flat}}(Q_{\tilde{f}}, K)/\mathcal{G}(Q_{\tilde{f}})} \mathbb{Z}a$$

for a regular perturbation K. Given $n+1$ automorphisms $\tilde{f}_0, \ldots, \tilde{f}_n$ such that $\tilde{f}_n \circ \cdots \circ \tilde{f}_0 = \mathrm{id}$ and a regular perturbation $(H, K) \in \mathcal{HK}_{\mathrm{reg}}$, there are two homomorphisms

$$\psi_0, \psi_\varepsilon : \mathrm{CF}(\tilde{f}_0, K_0) \otimes \mathrm{CF}(\tilde{f}_1, K_1) \otimes \cdots \otimes \mathrm{CF}(\tilde{f}_n, K_n) \to \mathbb{Z}$$

defined by

$$\psi_0(a_0, \ldots, a_n) = \sum_{[\Xi_0] \in \mathcal{M}_0(a_0, \ldots, a_n; H, K)} \nu_0(\Xi_0),$$

$$\psi_\varepsilon(a_0, \ldots, a_n) = \sum_{[\Xi_\varepsilon] \in \mathcal{M}_\varepsilon(a_0, \ldots, a_n; H, K)} \nu_\varepsilon(\Xi_\varepsilon),$$

whenever $\mu(a_0, \ldots, a_n; H, K) = 0$ and $\psi_0(a_0, \ldots, a_n) = \psi_\varepsilon(a_0, \ldots, a_n) = 0$ whenever $\mu(a_0, \ldots, a_n; H, K) \neq 0$. Here the definition of the signs $\nu_0(\Xi_0)$ and $\nu_\varepsilon(\Xi_\varepsilon)$ requires the choice of two compatible collections of orientations (24) and (25). Changing an overall sign, if necessary, we may assume that these systems of orientations are related by the map (23). Under this assumption Theorem 8.2 asserts that $\nu_\varepsilon(\mathcal{T}_\varepsilon(\Xi_0)) = \nu_0(\Xi_0)$ for $\varepsilon > 0$ sufficiently small. Hence $\psi_0 = \psi_\varepsilon$ for $\varepsilon > 0$ sufficiently small. This proves that the product structures (1) and (2) agree up to a sign. □

A Cauchy-Riemann operators in Hilbert space

Let V and H be separable Hilbert spaces and $V \hookrightarrow H$ be a compact linear inclusion with a dense image. Throughout we identify H with its dual space H^* via the Riesz representation theorem. Then the adjoint of the inclusion $V \hookrightarrow H$ is an inclusion $H \hookrightarrow V^*$ which is again compact and has a dense image. Thus there are two inclusions

$$V \subset H \subset V^*$$

and we shall think of V as a subset of H and of H as a subset of V^*. We assume that

$$\|x\|_H \leq \|x\|_V$$

for every $x \in V$. Let $\mathcal{L}(H)$ denote the space of bounded linear operators on H and $\mathcal{L}(V,H)$ the space of bounded linear operators from $V \to H$. Denote by $\mathcal{P}(H)$ the set of self-adjoint Hilbert space isomorphisms $Q : H \to H$ such that $QV = V$ and

$$0 < \inf_{0 \neq x \in H} \frac{\langle x, Qx \rangle_H}{\|x\|_H^2} \leq \sup_{0 \neq x \in H} \frac{\langle x, Qx \rangle_H}{\|x\|_H^2} < \infty.$$

Every operator $Q \in \mathcal{P}(H)$ determines an inner product

$$\langle x, y \rangle_Q := \langle x, Qy \rangle_H$$

such that the corresponding norm is equivalent to the standard norm on H. An operator $D \in \mathcal{L}(V,H)$ is called **Q-symmetric** if, for all $x, y \in V$,

$$\langle Dx, Qy \rangle_H = \langle x, QDy \rangle_H \,.$$

Lemma A.1. *Suppose $D \in \mathcal{L}(V,H)$ is Q-symmetric and let $\pi_D : H \to H$ denote the Q-orthogonal projection onto the kernel of D. Then the following are equivalent.*

(i) *There exists a constant $c_0 > 0$ such that, for every $x \in V$,*

$$\|x\|_V \leq c\left(\|Dx\|_H + \|x\|_H\right). \tag{26}$$

Moreover, for every $x \in H$ we have

$$x \in V \qquad \Longleftrightarrow \qquad \sup_{0 \neq y \in V} \frac{|\langle x, QDy \rangle_H|}{\|y\|_H} < \infty. \tag{27}$$

(ii) *There exists a constant $c_0 > 0$ such that, for every $x \in V$,*

$$\|x\|_V \leq c_0\left(\|Dx\|_H + \|\pi_D(x)\|_H\right). \tag{28}$$

Moreover, if $x \in H$ satisfies $\langle x, QDy \rangle = 0$ for every $y \in V$, then $x \in V$.

(iii) *D is a Fredholm operator of index zero.*

Proof. We prove that (i) implies (ii). Suppose, by contradiction, that (28) does not hold. Then there exists a sequence $x_n \in H$ such that

$$\|x_n\|_V = 1, \qquad \|Dx_n\|_H + \|\pi_D(x_n)\|_H \le 1/n.$$

Since the inclusion $V \hookrightarrow H$ is compact there exists a subsequence x_{n_k} that converges in H to a vector x. By (26), x_{n_k} is a Cauchy sequence in V. Since V is complete, we have $x \in V$ and

$$\lim_{k\to\infty} \|x_{n_k} - x\|_V = 0.$$

Hence

$$\|x\|_V = 1, \qquad Dx = 0, \qquad \pi_D(x) = 0.$$

Such a vector cannot exist and this contradiction proves (28). The second assertion in (ii) is an obvious consequence of (27).

We prove that (ii) implies (iii). By assumption, the composition of $\pi_D : H \to H$ with the inclusion $V \hookrightarrow H$ is a compact operator. Hence it follows from standard arguments in functional analysis that every operator that satisfies (28) has a finite dimensional kernel and a closed range (e.g. [19, Appendix A]). Now the second assertion in (ii) shows that the Q-orthogonal complement of the image of D agrees with the kernel of D. Hence the cokernel of D is finite dimensional and has the same dimension as the kernel of D. This proves (iii).

We prove that (iii) implies (i). If D is Fredholm then the operator $V \to H \oplus \ker D : x \mapsto (Dx, \pi_D(x))$ is injective and has a closed range. Hence (28) follows from the open mapping theorem. Hence D satisfies (26). To prove (27) suppose first that D is bijective. Suppose that $x \in H$ satisfies

$$\sup_{0\ne y\in V} \frac{|\langle x, QDy\rangle_H|}{\|y\|_H} < \infty$$

and choose $w \in H$ such that

$$\langle x, QDy\rangle_H = \langle w, Qy\rangle_H$$

for $y \in V$. Denote

$$\xi := D^{-1}w \in V.$$

Since D is Q-symmetric, we have

$$\langle x - \xi, QDy\rangle_H = \langle x, QDy\rangle_H - \langle D\xi, Qy\rangle_H = \langle x, QDy\rangle_H - \langle w, Qy\rangle_H = 0$$

for every $y \in V$. Since D is surjective, it follows that $x = \xi \in V$. This proves (i) under the assumption that D is bijective. In general, the kernel of D is contained in the Q-orthogonal complement of the image of D. Since D has index zero both must be equal. Hence the identity on H is an isomorphism from the kernel of D to a complement of the image. This implies that $D + \varepsilon\mathbb{1} : V \to H$ is bijective for $\varepsilon > 0$ sufficiently small. By what we have just proved, the operator $D + \varepsilon\mathbb{1}$ satisfies (27) and hence, so does D. This proves the lemma. $\square$

Let $\mathcal{S}(V,H;Q)$ denote the set of Q-symmetric operators $D : V \to H$ that satisfy the equivalent conditions of Lemma A.1. This set is open (with respect to the operator norm) in the Banach space of Q-symmetric operators in $\mathcal{L}(V,H)$. Moreover, the inequality (28) is stable under small perturbations. Namely, if $D \in \mathcal{S}(V,H;Q)$ satisfies (28) then every operator $D' \in \mathcal{S}(V,H;Q)$ that is sufficiently close to D in the operator norm satisfies (28) with c_0 replaced by $2c_0$. (The proof is an exercise.) This shows that for every compact subset $\mathcal{K} \subset \mathcal{S}(V,H;Q)$ there exists a constant $c_0 > 0$ such that (28) holds for every $D \in \mathcal{K}$.

If $Q \in \mathcal{P}(H)$ then the restriction of Q to V will still be denoted by Q. Its adjoint is an operator on V^* and is an extension of the original operator under the inclusion $H \hookrightarrow V^*$. This extension will also be denoted by Q. If $D \in \mathcal{S}(V,H;Q)$, then the dual operator D^* can be thought of as an operator from $H \to V^*$. Since D is Q-symmetric, the operator $Q^{-1}D^*Q : H \to V^*$ is an extension of $D : V \to H$. Let us denote this extension again by

$$D = Q^{-1}D^*Q : H \to V^*.$$

Thus D can be thought of both as an operator from $V \to H$ and as an operator from $H \to V^*$. The self-adjoint property (27) can then be expressed in the form that $x \in V$ if and only if $x \in H$ and $Dx \in H$.

Now let $Q, J : \mathbb{R}^2 \to \mathcal{L}(H)$ and $D : \mathbb{R}^2 \to \mathcal{L}(V,H)$ be operator valued functions that are continuously differentiable with respect to the strong operator topology (and hence are continuous in the norm topology). In the remainder of this section we assume that these functions satisfy $Q(s,t) \in \mathcal{P}(H)$, $D(s,t) \in \mathcal{S}(V,H;Q(s,t))$, and

$$J^*Q + QJ = 0, \qquad J^2 = -\mathbb{1}, \qquad DJ + JD = 0 \tag{29}$$

for all $(s,t) \in \mathbb{R}^2$. We also assume that there exist constants $c_0, c_1, c_2 \geq 1$ and $\delta > 0$ such that the operator $D = D(s,t)$ satisfies (28) for every $(s,t) \in \mathbb{R}^2$ and

$$\delta \|x\|_H^2 \leq \langle x, Qx \rangle_H \leq \delta^{-1} \|x\|_H^2 , \tag{30}$$

$$\|Dx\|_H + \|(\partial_s D)x\|_H + \|(\partial_t D)x\|_H \leq c_1 \|x\|_V , \tag{31}$$

$$\|(\partial_s J)x\|_H + \|(\partial_t J)x\|_H + \|(\partial_s Q)x\|_H + \|(\partial_t Q)x\|_H \leq c_2 \|x\|_H . \tag{32}$$

for every $x \in V$ and every $(s,t) \in \mathbb{R}^2$.

Proposition A.2. *For every $p \geq 2$ there exist constants $c \geq 1$ and $\varepsilon_0 > 0$ (depending only on δ, c_0, c_1, c_2, and p) such that the following holds. If $\xi \in L^p(\mathbb{R}^2,V) \cap W^{1,p}(\mathbb{R}^2,H)$ and $\tilde{\xi} \in L^p(\mathbb{R}^2,H)$ satisfy*

$$\partial_s\xi(s,t) + J(\varepsilon s,\varepsilon t)\partial_t\xi(s,t) + D(\varepsilon s,\varepsilon t)\xi(s,t) = \tilde{\xi}(s,t) \tag{33}$$

for some $\varepsilon \in (0,\varepsilon_0)$ then

$$\int_{\mathbb{R}^2} \|\xi\|_H^p \leq c \int_{\mathbb{R}^2} \left(\|\tilde{\xi}\|_H^p + \|\pi_D(\xi)\|_H^p \right). \tag{34}$$

Proof. Suppose, without loss of generality, that $\xi:\mathbb{R}^2\to V$ is twice continuously differentiable and has compact support. Then the function $\tilde{\xi}:\mathbb{R}^2\to H$, given by (33), is continuously differentiable and has compact support. It satisfies the equation

$$\partial_s\partial_s\xi+\partial_t\partial_t\xi-D^2\xi=f, \tag{35}$$

where $f:\mathbb{R}^2\to V^*$ is given by

$$f=\partial_s\tilde{\xi}-J\partial_t\tilde{\xi}-D\tilde{\xi}+\varepsilon(\partial_tJ)\partial_s\xi-\varepsilon(\partial_sJ)\partial_t\xi+\varepsilon(\partial_t(JD)-\partial_sD)\xi. \tag{36}$$

Here J and D are evaluated at the point $(\varepsilon s,\varepsilon t)\in\mathbb{R}^2$. Take the inner product with $\|\xi\|_Q^{p-2}Q\xi$ and compute

$$\begin{aligned}
\int_{\mathbb{R}^2}\|\xi\|_Q^{p-2}\langle Q\xi,f\rangle_{V,V^*} &= \int_{\mathbb{R}^2}\|\xi\|_Q^{p-2}\left\langle Q\xi,D^2\xi-\partial_s\partial_s\xi-\partial_t\partial_t\xi\right\rangle_{V,V^*}\\
&= \int_{\mathbb{R}^2}\|\xi\|_Q^{p-2}\left(\|D\xi\|_Q^2+\|\partial_s\xi\|_Q^2+\|\partial_t\xi\|_Q^2\right)\\
&\quad+\varepsilon\int_{\mathbb{R}^2}\|\xi\|_Q^{p-2}\left(\langle(\partial_sQ)\xi,\partial_s\xi\rangle_H+\langle(\partial_tQ)\xi,\partial_t\xi\rangle_H\right)\\
&\quad+\int_{\mathbb{R}^2}\left(\partial_s\|\xi\|_Q^{p-2}\right)\langle\xi,\partial_s\xi\rangle_Q\\
&\quad+\int_{\mathbb{R}^2}\left(\partial_t\|\xi\|_Q^{p-2}\right)\langle\xi,\partial_t\xi\rangle_Q\\
&= \int_{\mathbb{R}^2}\|\xi\|_Q^{p-2}\left(\|D\xi\|_Q^2+\|\partial_s\xi\|_Q^2+\|\partial_t\xi\|_Q^2\right)\\
&\quad+\varepsilon\int_{\mathbb{R}^2}\|\xi\|_Q^{p-2}\left(\langle(\partial_sQ)\xi,\partial_s\xi\rangle_H+\langle(\partial_tQ)\xi,\partial_t\xi\rangle_H\right)\\
&\quad+(p-2)\int_{\mathbb{R}^2}\|\xi\|_Q^{p-4}\left(\langle\xi,\partial_s\xi\rangle_Q^2+\langle\xi,\partial_t\xi\rangle_Q^2\right)\\
&\quad+\frac{\varepsilon(p-2)}{2}\int_{\mathbb{R}^2}\|\xi\|_Q^{p-4}\langle(\partial_sQ)\xi,\xi\rangle_H\langle\xi,\partial_s\xi\rangle_Q\\
&\quad+\frac{\varepsilon(p-2)}{2}\int_{\mathbb{R}^2}\|\xi\|_Q^{p-4}\langle(\partial_tQ)\xi,\xi\rangle_H\langle\xi,\partial_t\xi\rangle_Q.
\end{aligned}$$

Let $c_3:=(p-1)c_2/\delta^2$. Then, by (30) and (32),

$$\begin{aligned}
&\int_{\mathbb{R}^2}\|\xi\|_Q^{p-2}\left(\|D\xi\|_Q^2+\|\partial_s\xi\|_Q^2+\|\partial_t\xi\|_Q^2\right)\\
&\qquad\le\int_{\mathbb{R}^2}\|\xi\|_Q^{p-2}\langle Q\xi,f\rangle_{V,V^*}+\varepsilon c_3\int_{\mathbb{R}^2}\|\xi\|_Q^{p-1}\left(\|\partial_s\xi\|_Q+\|\partial_t\xi\|_Q\right).
\end{aligned}$$

Since $\|x\|_H\le\|x\|_V$ for every $x\in V$ it follows from (28) and (30) that

$$\|\xi\|_Q\le\delta^{-1}\|\xi\|_V\le c_0\delta^{-2}\left(\|D\xi\|_Q+\|\pi_D(\xi)\|_Q\right). \tag{37}$$

Let $c_4 := c_0c_3/\delta^2$. Then the last two inequalities show that

$$\begin{aligned}
&\int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \Big(\|D\xi\|_Q^2 + \|\partial_s\xi\|_Q^2 + \|\partial_t\xi\|_Q^2 \Big) \\
&\quad \le \int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \langle Q\xi, f\rangle_{V,V^*} \\
&\qquad + \varepsilon c_4 \int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \Big(\|\partial_s\xi\|_Q + \|\partial_t\xi\|_Q \Big) \Big(\|D\xi\|_Q + \|\pi_D(\xi)\|_Q \Big) \\
&\quad \le \int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \langle Q\xi, f\rangle_{V,V^*} + \varepsilon^2 {c_4}^2 \int_{\mathbb{R}^2} \|\pi_D(\xi)\|_Q^2 \\
&\qquad + \varepsilon c_4 \int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \Big(\|D\xi\|_Q^2 + \|\partial_s\xi\|_Q^2 + \|\partial_t\xi\|_Q^2 + \|\pi_D(\xi)\|_Q^2 \Big).
\end{aligned}$$

With $\varepsilon c_4 \le 1/2$ this gives

$$\begin{aligned}
&\int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \Big(\|D\xi\|_Q^2 + \|\partial_s\xi\|_Q^2 + \|\partial_t\xi\|_Q^2 \Big) \\
&\quad \le 2 \int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \Big(\langle Q\xi, f\rangle_{V,V^*} + \varepsilon^2 {c_4}^2 \|\pi_D(\xi)\|_Q^2 \Big).
\end{aligned} \tag{38}$$

Now recall from (36) that $f = g + h$, where $g : \mathbb{R}^2 \to V^*$ is given by

$$g = \partial_s\tilde{\xi} - J\partial_t\tilde{\xi} - D\tilde{\xi}$$

and $h : \mathbb{R}^2 \to H$ is given by

$$h = \varepsilon\Big((\partial_t J)\partial_s\xi - (\partial_s J)\partial_t\xi + (\partial_t J)D\xi + J(\partial_t D)\xi - (\partial_s D)\xi \Big).$$

Let $c_5 := (c_2 + c_0c_1)/\delta^2$. Then, by (30), (31), (32), and (37),

$$\begin{aligned}
\|h\|_Q &\le \varepsilon c_2\delta^{-2} \Big(\|D\xi\|_Q + \|\partial_s\xi\|_Q + \|\partial_t\xi\|_Q \Big) + \varepsilon c_1\delta^{-1} \|\xi\|_V \\
&\le \varepsilon c_5 \Big(\|D\xi\|_Q + \|\partial_s\xi\|_Q + \|\partial_t\xi\|_Q + \|\pi_D(\xi)\|_Q \Big).
\end{aligned}$$

Hence, by (37),

$$\begin{aligned}
&\int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \langle Q\xi, h\rangle_H \\
&\quad \le \int_{\mathbb{R}^2} \|\xi\|_Q^{p-1} \|h\|_Q \\
&\quad \le \varepsilon c_5 \int_{\mathbb{R}^2} \|\xi\|_Q^{p-1} \Big(\|D\xi\|_Q + \|\partial_s\xi\|_Q + \|\partial_t\xi\|_Q + \|\pi_D(\xi)\|_Q \Big) \\
&\quad \le \varepsilon c_0 c_5 \delta^{-2} \int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \Big(\|D\xi\|_Q + \|\pi_D(\xi)\|_Q \Big) \cdot \\
&\qquad\qquad \cdot \Big(\|D\xi\|_Q + \|\partial_s\xi\|_Q + \|\partial_t\xi\|_Q + \|\pi_D(\xi)\|_Q \Big) \\
&\quad \le 3\varepsilon c_0 c_5 \delta^{-2} \int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \Big(\|D\xi\|_Q^2 + \|\partial_s\xi\|_Q^2 + \|\partial_t\xi\|_Q^2 + \|\pi_D(\xi)\|_Q^2 \Big).
\end{aligned}$$

Suppose that $3\varepsilon c_0 c_5 \delta^{-2} \le 1/4$ and let $c_6 := c_4 + 3c_0c_5/\delta^2$. Then, inserting the last inequality into (38), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \left(\|D\xi\|_Q^2 + \|\partial_s\xi\|_Q^2 + \|\partial_t\xi\|_Q^2 \right) \\
&\qquad \le 4 \int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \left(\langle Q\xi, g\rangle_{V,V^*} + \varepsilon c_6 \|\pi_D(\xi)\|_Q^2 \right). \qquad (39)
\end{aligned}$$

Now

$$\begin{aligned}
\int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \langle Q\xi, g\rangle_{V,V^*} &= \int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \langle Q\xi, -D\tilde{\xi} + \partial_s\tilde{\xi} - J\partial_t\tilde{\xi}\rangle_{V,V^*} \\
&= \int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \left(-\langle D\xi, \tilde{\xi}\rangle_Q - \langle \partial_s\xi, \tilde{\xi}\rangle_Q + \langle \partial_t\xi, J\tilde{\xi}\rangle_Q \right) \\
&\quad - \varepsilon \int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \langle (\partial_s Q)\xi, \tilde{\xi}\rangle_H \\
&\quad + \varepsilon \int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \left(\langle (\partial_t Q)\xi, J\tilde{\xi}\rangle_H + \langle \xi, (\partial_t J)\tilde{\xi}\rangle_Q \right) \\
&\quad - \int_{\mathbb{R}^2} \left(\partial_s \|\xi\|_Q^{p-2} \right) \langle \xi, \tilde{\xi}\rangle_Q \\
&\quad + \int_{\mathbb{R}^2} \left(\partial_t \|\xi\|_Q^{p-2} \right) \langle \xi, J\tilde{\xi}\rangle_Q \\
&= \int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \left(-\langle D\xi, \tilde{\xi}\rangle_Q - \langle \partial_s\xi, \tilde{\xi}\rangle_Q + \langle \partial_t\xi, J\tilde{\xi}\rangle_Q \right) \\
&\quad - \varepsilon \int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \langle (\partial_s Q)\xi, \tilde{\xi}\rangle_H \\
&\quad + \varepsilon \int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \left(\langle (\partial_t Q)\xi, J\tilde{\xi}\rangle_H + \langle \xi, (\partial_t J)\tilde{\xi}\rangle_Q \right) \\
&\quad - (p-2) \int_{\mathbb{R}^2} \|\xi\|_Q^{p-4} \langle \xi, \partial_s\xi\rangle_Q \langle \xi, \tilde{\xi}\rangle_Q \\
&\quad - \frac{\varepsilon(p-2)}{2} \int_{\mathbb{R}^2} \|\xi\|_Q^{p-4} \langle (\partial_s Q)\xi, \xi\rangle_H \langle \xi, \tilde{\xi}\rangle_Q \\
&\quad + (p-2) \int_{\mathbb{R}^2} \|\xi\|_Q^{p-4} \langle \xi, \partial_t\xi\rangle_Q \langle \xi, J\tilde{\xi}\rangle_Q \\
&\quad + \frac{\varepsilon(p-2)}{2} \int_{\mathbb{R}^2} \|\xi\|_Q^{p-4} \langle (\partial_t Q)\xi, \xi\rangle_H \langle \xi, J\tilde{\xi}\rangle_Q.
\end{aligned}$$

Let $c_7 := (p+1)c_2/\delta^2$. Then, by (30) and (32),

$$\begin{aligned}
&\int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \langle Q\xi, g\rangle_{V,V^*} \\
&\qquad \le (p-1) \int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \|\tilde{\xi}\|_Q \left(\|D\xi\|_Q + \|\partial_s\xi\|_Q + \|\partial_t\xi\|_Q \right) \\
&\qquad\quad + \varepsilon c_7 \int_{\mathbb{R}^2} \|\xi\|_Q^{p-1} \|\tilde{\xi}\|_Q.
\end{aligned}$$

Inserting this inequality into (39) and using (37) we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \left(\|D\xi\|_Q^2 + \|\partial_s\xi\|_Q^2 + \|\partial_t\xi\|_Q^2 \right) \\
&\quad \le 4(p-1) \int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \|\tilde{\xi}\|_Q \left(\|D\xi\|_Q + \|\partial_s\xi\|_Q + \|\partial_t\xi\|_Q \right) \\
&\qquad + 4\varepsilon c_7 \int_{\mathbb{R}^2} \|\xi\|_Q^{p-1} \|\tilde{\xi}\|_Q + 4\varepsilon c_6 \int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \|\pi_D(\xi)\|_Q^2 \\
&\quad \le 4(p-1) \int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \|\tilde{\xi}\|_Q \left(\|D\xi\|_Q + \|\partial_s\xi\|_Q + \|\partial_t\xi\|_Q \right) \\
&\qquad + 4\varepsilon c_0 c_7 \delta^{-2} \int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \|\tilde{\xi}\|_Q \left(\|D\xi\|_Q + \|\pi_D(\xi)\|_Q \right) \\
&\qquad + 4\varepsilon c_6 \int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \|\pi_D(\xi)\|_Q^2 .
\end{aligned}$$

Suppose that $\varepsilon c_0 c_7/\delta^2 \le 1$ and let $c_8 := 4c_0c_7/\delta^2$. Then

$$\begin{aligned}
&\int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \left(\|D\xi\|_Q^2 + \|\partial_s\xi\|_Q^2 + \|\partial_t\xi\|_Q^2 \right) \\
&\quad \le 4p \int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \|\tilde{\xi}\|_Q \left(\|D\xi\|_Q + \|\partial_s\xi\|_Q + \|\partial_t\xi\|_Q \right) \\
&\qquad + \frac{c_8}{2} \int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \|\tilde{\xi}\|_Q^2 + \frac{8\varepsilon c_6 + \varepsilon^2 c_8}{2} \int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \|\pi_D(\xi)\|_Q^2 \\
&\quad \le \frac{1}{2} \int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \left(\|D\xi\|_Q^2 + \|\partial_s\xi\|_Q^2 + \|\partial_t\xi\|_Q^2 \right) \\
&\qquad + \frac{c_8 + 48p^2}{2} \int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \|\tilde{\xi}\|_Q^2 + \frac{8\varepsilon c_6 + \varepsilon^2 c_8}{2} \int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \|\pi_D(\xi)\|_Q^2
\end{aligned}$$

Let $c_9 := \max\{c_8 + 48p^2, 8c_6 + 4\}$. Since $\varepsilon c_8 \le 4$, we obtain

$$\int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \|D\xi\|_Q^2 \le c_9 \int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \left(\|\tilde{\xi}\|_Q^2 + \varepsilon \|\pi_D(\xi)\|_Q^2 \right). \tag{40}$$

Let $c_{10} := 2(c_0/\delta^2)^2(c_9+1)$. Then, by (37) and (40),

$$\begin{aligned}
\int_{\mathbb{R}^2} \|\xi\|_Q^p &\le 2(c_0\delta^{-2})^2 \int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \left(\|D\xi\|_Q^2 + \|\pi_D(\xi)\|_Q^2 \right) \\
&\le c_{10} \int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \left(\|\tilde{\xi}\|_Q^2 + \|\pi_D(\xi)\|_Q^2 \right).
\end{aligned}$$

By Hölder's inequality, $\int_{\mathbb{R}^2} \|\xi\|_Q^{p-2} \|\tilde{\xi}\|_Q^2 \le (\int_{\mathbb{R}^2} \|\xi\|_Q^p)^{1-2/p} (\int_{\mathbb{R}^2} \|\tilde{\xi}\|_Q^p)^{2/p}$. Hence

$$\left(\int_{\mathbb{R}^2} \|\xi\|_Q^p \right)^{2/p} \le c_{10} \left(\left(\int_{\mathbb{R}^2} \|\tilde{\xi}\|_Q^p \right)^{2/p} + \left(\int_{\mathbb{R}^2} \|\pi_D(\xi)\|_Q^p \right)^{2/p} \right).$$

This proves the proposition. □

Proposition A.3. *Assume Q, J, and D are independent of s and t and satisfy $Q \in \mathcal{P}(H)$, $D \in \mathcal{S}(V, H; Q)$, and (29). Suppose that $\xi \in C^2(\mathbb{R}^2, H) \cap C^1(\mathbb{R}^2, V)$ is a compactly supported function such that*

$$\partial_s \xi + J \partial_t \xi + D\xi = 0.$$

Then $\xi = 0$.

Proof. The function ξ satisfies equation (35) with $f = 0$. Denote by $\Delta := \partial_s{}^2 + \partial_t{}^2$ the positive definite Laplacian. Then

$$\begin{aligned}
\Delta \|\xi\|_Q^2 &= 2\|\partial_s\xi\|_Q^2 + 2\|\partial_t\xi\|_Q^2 + 2\langle \xi, \partial_s\partial_s\xi + \partial_t\partial_t\xi\rangle_Q \\
&= 2\|\partial_s\xi\|_Q^2 + 2\|\partial_t\xi\|_Q^2 + 2\langle \xi, D^2\xi\rangle_Q \\
&= 2\|\partial_s\xi\|_Q^2 + 2\|\partial_t\xi\|_Q^2 + 2\|D\xi\|_Q^2 \\
&\geq 0
\end{aligned}$$

Since the function $\mathbb{R}^2 \to \mathbb{R} : (s,t) \mapsto \|\xi(s,t)\|_Q^2$ has compact support the integral of $\Delta\|\xi\|_Q^2$ over $\mathbb{R}^2$ is zero. Hence ξ is constant, and hence $\xi \equiv 0$. □

B Estimates on a Riemann surface

In this section we collect some standard estimates for connections over Riemann surfaces. We include the proofs for the sake of completeness. Let Σ be a compact oriented Riemann surface with volume form ω. Denote by $\mathcal{J}(\Sigma)$ the space of complex structures on Σ compatible with the orientation. Let $P \to \Sigma$ be a principal SO(3)-bundle with nonzero second Stiefel-Whitney class. Let $\mathcal{A}(P) \subset \Omega^1(\Sigma, \mathfrak{g}_P)$ denote the space of connection 1-forms on P, $\mathcal{A}^{\mathrm{flat}}(P) \subset \mathcal{A}(P)$ the submanifold of flat connections, and $\mathcal{G}(P) \subset \mathrm{Map}(P, \mathrm{G})$ the identity component of the gauge group.

Lemma B.1. *Fix a complex structure $J_0 \in \mathcal{J}(\Sigma)$ and a connection $A_0 \in \mathcal{A}(P)$. Then, for every $C > 0$ and every $p > 1$, there exists a constant $c = c(C, p, J_0, A_0) \geq 1$ such that, if $J \in \mathcal{J}(\Sigma)$ and $A \in \mathcal{A}(P)$ satisfy*

$$\|J\|_{C^1(\Sigma, J_0)} + \|A - A_0\|_{L^\infty(\Sigma, J_0)} \leq C \tag{41}$$

then, for every $\alpha \in \Omega^1(\Sigma, \mathfrak{g}_P)$,

$$c^{-1}\|\alpha\|_{L^p(\Sigma, J_0)}^p \leq \|\alpha\|_{L^p(\Sigma, J)}^p \leq c\|\alpha\|_{L^p(\Sigma, J_0)}^p, \tag{42}$$

$$\|\nabla_{A_0}\alpha\|_{L^p(\Sigma, J_0)}^p \leq c\left(\|\mathrm{d}_A\alpha\|_{L^p(\Sigma)}^p + \|\mathrm{d}_A(\alpha \circ J)\|_{L^p(\Sigma)}^p + \|\alpha\|_{L^p(\Sigma, J)}^p\right). \tag{43}$$

Here ∇_{A_0} denotes the covariant derivative with respect to the connection on $T^*\Sigma \otimes \mathfrak{g}_P$ determined by A_0 and the metric $\omega(\cdot, J_0 \cdot)$.

Proof. For a fixed complex structure $J = J_1$ the estimate (42) is obvious and it holds with a uniform constant c in some C^0-neighbourhood of J_1. By the Arzéla-Ascoli theorem, we can cover the set of all $J \in \mathcal{J}(\Sigma)$ with $\|J\|_{C^1(\Sigma,J_0)} \leq C$ by finitely many such neighbourhoods, and this proves (42).

We prove (43). For $A = A_0$ and a fixed complex structure $J = J_1$ this follows from the Calderon-Zygmund inequality. Now, for every $J_1 \in \mathcal{J}(\Sigma)$ there exists a constant $c_1 > 0$ such that, for every $\alpha \in \Omega^1(\Sigma, \mathfrak{g}_P)$,

$$\begin{aligned} \|\mathrm{d}_{A_0}(\alpha \circ J) - \mathrm{d}_{A_0}(\alpha \circ J_1)\|_{L^p(\Sigma)} &\leq c_1 \|J - J_1\|_{C^1(\Sigma,J_0)} \|\alpha\|_{L^p(\Sigma,J_0)} \\ &\quad + c_1 \|J - J_1\|_{L^\infty(\Sigma,J_0)} \|\nabla_{A_0}\alpha\|_{L^p(\Sigma,J_0)} . \end{aligned}$$

Hence there exist constants $\delta = \delta(J_1) > 0$ and $c_2 = c_2(J_1) > 0$ such that, if $\|J - J_1\|_{L^\infty(\Sigma,J_0)} < \delta$ then

$$\begin{aligned} \|\nabla_{A_0}\alpha\|_{L^p(\Sigma,J_0)} &\leq c_2 \left(\|\mathrm{d}_{A_0}\alpha\|_{L^p(\Sigma)} + \|\mathrm{d}_{A_0}(\alpha \circ J)\|_{L^p(\Sigma)}\right) \\ &\quad + c_2 \left(1 + \|J - J_1\|_{C^1(\Sigma,J_0)}\right) \|\alpha\|_{L^p(\Sigma,J_0)} . \end{aligned}$$

By the Arzéla-Ascoli theorem, we can cover the set of all $J \in \mathcal{J}(\Sigma)$ with $\|J\|_{C^1(\Sigma,J_0)} \leq C$ by finitely many such δ-neighbourhoods. This proves (43) for $A = A_0$. Since $\mathrm{d}_A\alpha - \mathrm{d}_{A_0}\alpha = [(A - A_0) \wedge \alpha]$ the estimate (43) holds, with a larger constant c, for every pair (A, J) that satisfies (41). This proves the lemma. □

Lemma B.2. *Fix a complex structure $J_0 \in \mathcal{J}(\Sigma)$ and a connection $A_0 \in \mathcal{A}(P)$. Then, for every $\delta > 0$, $C > 0$, and $p \geq 2$, there exists a constant $c = c(\delta, C, p, J_0, A_0) \geq 1$ such that, if $J \in \mathcal{J}(\Sigma)$ and $A \in \mathcal{A}(P)$ satisfy (41) then, for every $\phi \in \Omega^0(\Sigma, \mathfrak{g}_P)$ and every $\alpha \in \Omega^1(\Sigma, \mathfrak{g}_P)$,*

$$\|\phi\|_{L^p(\Sigma)}^p \leq \delta \|\mathrm{d}_A\phi\|_{L^p(\Sigma,J)}^p + c \|\phi\|_{L^2(\Sigma)}^p , \tag{44}$$

$$\|\alpha\|_{L^p(\Sigma,J)}^p \leq \delta \left(\|\mathrm{d}_A\alpha\|_{L^p(\Sigma)}^p + \|\mathrm{d}_A(\alpha \circ J)\|_{L^p(\Sigma)}^p\right) + c \|\alpha\|_{L^2(\Sigma,J)}^p . \tag{45}$$

Proof. For $p = 2$ there is nothing to prove. For $p > 2$ this is a standard estimate in partial differential equations. For the sake of completeness we give a proof. We show first that, if $u : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable, then

$$|u(x)| \leq \frac{\varepsilon^{1-2/p}}{(2\pi)^{1/p}} \left(\frac{p-1}{p-2}\right)^{1-1/p} \|\nabla u\|_{L^p(B_\varepsilon)} + \frac{2}{\pi\varepsilon^2} \|u\|_{L^1(B_\varepsilon)} , \tag{46}$$

where

$$B_\varepsilon = \left\{y \in \mathbb{R}^2 \,|\, |y - x| < \varepsilon\right\} .$$

To see this observe that, for $|\xi| = 1$ and $0 < t \le \varepsilon$,

$$
\begin{aligned}
|u(x)| &\le |u(x+t\xi)| + \int_0^\varepsilon |\nabla u(x+s\xi)|\, \mathrm{d}s \\
&\le |u(x+t\xi)| + \left(\int_0^\varepsilon s^{-1/(p-1)}\, \mathrm{d}s\right)^{1-1/p} \left(\int_0^\varepsilon |\nabla u(x+s\xi)|^p\, s\, \mathrm{d}s\right)^{1/p} \\
&= |u(x+t\xi)| + \varepsilon^{1-2/p} \left(\frac{p-1}{p-2}\right)^{1-1/p} \left(\int_0^\varepsilon |\nabla u(x+s\xi)|^p\, s\, \mathrm{d}s\right)^{1/p}.
\end{aligned}
$$

Here the second estimate follows from Hölder's inequality. Now integrate over $\xi \in S^1$ and use Hölder's inequality again to obtain

$$
|u(x)| \le \frac{1}{2\pi}\int_0^{2\pi} |u(x+te^{i\theta})|\, \mathrm{d}\theta + \frac{\varepsilon^{1-2/p}}{(2\pi)^{1/p}} \left(\frac{p-1}{p-2}\right)^{1-1/p} \|\nabla u\|_{L^p(B_\varepsilon)}.
$$

Now integrate over the interval $\varepsilon/2 \le t \le \varepsilon$ to obtain (46). It follows from (46) that there exists a constant $c_0 > 0$ such that

$$
\|\phi\|_{L^p(\Sigma)} \le \varepsilon^{1-2/p} \|\mathrm{d}_{A_0}\phi\|_{L^p(\Sigma,J_0)} + c_0\varepsilon^{-1} \|\phi\|_{L^2(\Sigma)}, \tag{47}
$$

$$
\|\alpha\|_{L^p(\Sigma,J_0)} \le \varepsilon^{1-2/p} \|\nabla_{A_0}\alpha\|_{L^p(\Sigma,J_0)} + c_0\varepsilon^{-1} \|\alpha\|_{L^2(\Sigma)} \tag{48}
$$

for $\phi \in \Omega^0(\Sigma, \mathfrak{g}_P)$, $\alpha \in \Omega^1(\Sigma, \mathfrak{g}_P)$, and $0 < \varepsilon \le 1$. Now (45) follows from (48) and (43) if $\varepsilon > 0$ is chosen sufficiently small. To prove (44) note that, by Lemma B.1,

$$
\|\mathrm{d}_{A_0}\phi\|_{L^p(\Sigma,J_0)} \le c\,\|\mathrm{d}_{A_0}\phi\|_{L^p(\Sigma,J)} \le c\,\|\mathrm{d}_{A}\phi\|_{L^p(\Sigma,J)} + c'\,\|\phi\|_{L^p(\Sigma)}
$$

Combining this with (47) gives (44) provided that $\varepsilon > 0$ is chosen sufficiently small. This proves the lemma. $\square$

Lemma B.3. *Fix a complex structure $J_0 \in \mathcal{J}(\Sigma)$. Then, for every $C > 0$ and every $p \ge 2$, there exists a constant $c = c(C, p, J_0) \ge 1$ such that, if $J \in \mathcal{J}(\Sigma)$ satisfies*

$$
\|J\|_{C^1(\Sigma,J_0)} \le C, \tag{49}
$$

and $A \in \mathcal{A}^{\mathrm{flat}}(P)$ then, for every $\phi \in \Omega^0(\Sigma, \mathfrak{g}_P)$ and every $\alpha \in \Omega^1(\Sigma, \mathfrak{g}_P)$,

$$
\|\phi\|^p_{L^p(\Sigma)} \le c\,\|\mathrm{d}_A\phi\|^p_{L^p(\Sigma,J)}, \tag{50}
$$

$$
\|\alpha\|^p_{L^p(\Sigma,J)} \le c\Big(\|\mathrm{d}_A\alpha\|^p_{L^p(\Sigma)} + \|\mathrm{d}_A(\alpha\circ J)\|^p_{L^p(\Sigma)} + \|\pi_A(\alpha)\|^p_{L^p(\Sigma,J)}\Big). \tag{51}
$$

Proof. We prove (50). If $A_1 \in \mathcal{A}^{\mathrm{flat}}(P)$ then $d_{A_1} : \Omega^0(\Sigma, \mathfrak{g}_P) \to \Omega^1(\Sigma, \mathfrak{g}_P)$ is injective. Hence there exists a constant $c = c(A_1) > 0$ such that (50) holds for $A = A_1$ and $J = J_0$. Hence

$$\|\phi\|_{L^p(\Sigma)} \leq c \left(\|\mathrm{d}_A \phi\|_{L^p(\Sigma, J_0)} + \|A - A_1\|_{L^\infty(\Sigma, J_0)} \|\phi\|_{L^p(\Sigma)} \right)$$

This shows that (50) holds, with $J = J_0$ and a uniform constant c, in some C^0-neighbourhood of A_1. Cover the set

$$\mathcal{A}^{\mathrm{flat}}_C(P) = \left\{ \mathcal{A}^{\mathrm{flat}}(P) \mid \|A - A_0\|_{C^2(\Sigma, J_0)} \leq C \right\}$$

by finitely many such neighbourhoods to obtain (50) with $J = J_0$ for some constant $c > 0$, every $A \in \mathcal{A}^{\mathrm{flat}}_C(P)$, and every $\phi \in \Omega^0(\Sigma, \mathfrak{g}_p)$. Now use (42) to obtain (50) for any $A \in \mathcal{A}^{\mathrm{flat}}_C(P)$ and any $J \in \mathcal{J}(\Sigma)$ that satisfies (49). If C is sufficiently large then every flat connection is gauge equivalent to one in $\mathcal{A}^{\mathrm{flat}}_C(P)$ and this proves (50).

Next we prove (51) for $p = 2$. Write

$$\alpha = \pi_A(\alpha) + \mathrm{d}_A \zeta - (\mathrm{d}_A \eta) \circ J$$

for $\eta, \zeta \in \Omega^0(\Sigma, \mathfrak{g}_P)$. Since the three terms on the right are pairwise L^2-orthogonal, with respect to the metric determined by J, we have

$$\|\alpha\|^2_{L^2(\Sigma,J)} = \|\mathrm{d}_A \zeta\|^2_{L^2(\Sigma,J)} + \|\mathrm{d}_A \eta\|^2_{L^2(\Sigma,J)} + \|\pi_A(\alpha)\|^2_{L^2(\Sigma,J)}.$$

Since A is flat, we have

$$\mathrm{d}_A{}^{*_J} \mathrm{d}_A \eta = - * \mathrm{d}_A \alpha.$$

Taking the L^2-inner product with η and using the Cauchy-Schwarz inequality, we obtain

$$\|\mathrm{d}_A \eta\|^2_{L^2(\Sigma,J)} \leq \|\eta\|_{L^2(\Sigma)} \|\mathrm{d}_A \alpha\|_{L^2(\Sigma)} \leq c \|\mathrm{d}_A \eta\|_{L^2(\Sigma,J)} \|\mathrm{d}_A \alpha\|_{L^2(\Sigma)}.$$

The last inequality follows from (50) with $p = 2$. Hence

$$\|\mathrm{d}_A \eta\|_{L^2(\Sigma,J)} \leq c \|\mathrm{d}_A \alpha\|_{L^2(\Sigma)}, \qquad \|\mathrm{d}_A \zeta\|_{L^2(\Sigma,J)} \leq c \|\mathrm{d}_A(\alpha \circ J)\|_{L^2(\Sigma)}.$$

This proves (51) for $p = 2$. If $A \in \mathcal{A}^{\mathrm{flat}}_C(P)$ then (51) for general p follows from the case $p = 2$ and (45). Since there exists a constant $C > 0$ such that every flat connection is gauge equivalent to one in $\mathcal{A}^{\mathrm{flat}}_C(P)$, this proves the lemma. □

Lemma B.4. *Fix a complex structure $J_0 \in \mathcal{J}(\Sigma)$. Then, for every $C > 0$ and every $p \geq 2$, there exists a constant $c = c(C, p, J_0) \geq 1$ such that, if $J \in \mathcal{J}(\Sigma)$ satisfies (49) and $A \in \mathcal{A}^{\mathrm{flat}}(P)$ then, for every $\phi \in \Omega^0(\Sigma, \mathfrak{g}_P)$ and every $\alpha \in \Omega^1(\Sigma, \mathfrak{g}_P)$,*

$$\|\mathrm{d}_A \phi\|^p_{L^p(\Sigma,J)} \leq c \left(\|\alpha\|^p_{L^p(\Sigma,J)} + \|\mathrm{d}_A((\mathrm{d}_A \phi) \circ J + \alpha)\|^p_{L^p(\Sigma)} \right). \tag{52}$$

Proof. By (42) in Lemma B.1, it suffices to prove the inequality

$$\|\mathrm{d}_A\phi\|_{L^p(\Sigma,J_0)} \le c\left(\|\alpha\|_{L^p(\Sigma,J_0)} + \|\mathrm{d}_A((\mathrm{d}_A\phi)\circ J+\alpha)\|_{L^p(\Sigma)}\right) \tag{53}$$

instead of (52). Let

$$\psi := *\mathrm{d}_A((\mathrm{d}_A\phi)\circ J+\alpha) \in \Omega^0(\Sigma,\mathfrak{g}_P).$$

Then

$$\mathrm{d}_A{}^{*_J}\mathrm{d}_A\phi = \psi - *\mathrm{d}_A\alpha.$$

For $p=2$ the estimate follows by taking the inner product with ϕ and using (50). The general case can be reduced to $p=2$ via the Calderon-Zygmund inequality and (44). Hence, for every pair (J,A), there exists a constant $c=c(J,A)\ge 1$ such that (53) holds for every $\phi\in\Omega^0(\Sigma,\mathfrak{g}_P)$ and every $\alpha\in\Omega^1(\Sigma,\mathfrak{g}_P)$. We prove that the constant can be chosen independent of A. To see this note that

$$\begin{aligned}
&\mathrm{d}_A((\mathrm{d}_A\phi)\circ J) - \mathrm{d}_{A_1}((\mathrm{d}_{A_1}\phi)\circ J)\\
&\quad = [(A-A_1)\wedge((\mathrm{d}_A\phi)\circ J)] + [(A-A_1)\circ J\wedge[(A-A_1),\phi]]\\
&\qquad + [\mathrm{d}_{A_1}((A-A_1)\circ J),\phi] - [(A-A_1)\circ J\wedge \mathrm{d}_A\phi].
\end{aligned}$$

Let $c_1 = c(J,A_1)$ and use (53) with $A=A_1$ to obtain

$$\begin{aligned}
&\|\mathrm{d}_A\phi\|_{L^p(\Sigma,J_0)}\\
&\le \|A-A_1\|_{L^\infty(\Sigma,J_0)}\|\phi\|_{L^p(\Sigma)}\\
&\quad + c_1\left(\|\alpha\|_{L^p(\Sigma,J_0)} + \|\mathrm{d}_{A_1}((\mathrm{d}_{A_1}\phi)\circ J+\alpha)\|_{L^p(\Sigma)}\right)\\
&\le c_1\left(1+\|A-A_1\|_{L^\infty(\Sigma,J_0)}\right)\|\alpha\|_{L^p(\Sigma,J_0)} + c_1\|\mathrm{d}_A((\mathrm{d}_A\phi)\circ J+\alpha)\|_{L^p(\Sigma)}\\
&\quad + c_2\left(\|A-A_1\|^2_{L^\infty(\Sigma,J_0)} + \|A-A_1\|_{C^1(\Sigma,J_0)}\right)\|\mathrm{d}_A\phi\|_{L^p(\Sigma,J_0)}.
\end{aligned}$$

Here we have used the inequality $\|\phi\|_{L^p(\Sigma)} \le c\|\mathrm{d}_A\phi\|_{L^p(\Sigma,J_0)}$. It follows that every flat connection A_1 has a C^1-neighbourhood in which (53) holds with $c = 2c_1(J,A_1)$. By the Arzéla-Ascoli theorem, cover the set $\mathcal{A}_C^{\mathrm{flat}}(P)$ by finitely many such neighbourhoods. Since (53) is gauge invariant, and every flat connection is gauge equivalent to one in $\mathcal{A}_C^{\mathrm{flat}}(P)$, there exists, for every $J\in\mathcal{J}(\Sigma)$, a constant $c_3=c_3(J)>0$ such that (53) holds with $c=c_3(J)$ for every flat connection A. Now apply (53) with $J=J_1$ to the pair $(\phi,\alpha+(\mathrm{d}_A\phi)\circ(J-J_1))$ to obtain

$$\begin{aligned}
\|\mathrm{d}_A\phi\|_{L^p(\Sigma,J_0)} \quad\le\quad & c_3(J_1)\left(\|\alpha\|_{L^p(\Sigma,J_0)} + \|\mathrm{d}_A((\mathrm{d}_A\phi)\circ J+\alpha)\|_{L^p(\Sigma)}\right)\\
& + c_3(J_1)\|J-J_1\|_{L^\infty(\Sigma,J_0)}\|\mathrm{d}_A\phi\|_{L^p(\Sigma,J_0)}.
\end{aligned}$$

Hence (53) holds with $c=2c_3(J_1)$ whenever $c_3(J_1)\|J-J_1\|_{L^\infty(\Sigma,J_0)}\le 1/2$. By the Arzéla-Ascoli theorem, cover the set of all $J\in\mathcal{J}(\Sigma)$ that satisfy (49) by finitely many such neighbourhoods. This proves the lemma. □

C An L^p estimate

Throughout we fix a principal G-bundle $P \to \Sigma$ over a compact oriented Riemann surface Σ, a smooth reference connection $A_0 \in \mathcal{A}(P)$, a volume form dvol_Σ compatible with the orientation, and a complex structure $J_0 \in \mathcal{J}(\Sigma)$ compatible with the orientation. We introduce the spaces

$$\mathbf{X} := \Omega^1(\Sigma, \mathfrak{g}_P) \oplus \Omega^0(\Sigma, \mathfrak{g}_P) \oplus \Omega^0(\Sigma, \mathfrak{g}_P), \qquad \mathcal{X} := C_0^\infty\left(\mathbb{R}^2, \mathbf{X}\right).$$

Thus the elements of $\mathcal{X}$ are triples $\xi = (\alpha, \phi, \psi)$, where $\phi, \psi : \mathbb{R}^2 \to \Omega^0(\Sigma, \mathfrak{g}_P)$ and $\alpha : \mathbb{R}^2 \to \Omega^1(\Sigma, \mathfrak{g}_P)$ are smooth functions with compact support. For a real number $\lambda > 0$, a complex structure $J \in \mathcal{J}(\Sigma)$, and a connection $A \in \mathcal{A}(P)$ we introduce the operators

$$\mathbf{D} = \mathbf{D}_{\lambda,J,A} : \mathbf{X} \to \mathbf{X}, \qquad \mathbf{J} = \mathbf{J}_J : \mathbf{X} \to \mathbf{X}, \qquad \mathbf{Q} = \mathbf{Q}_{\lambda,J} : \mathbf{X} \to \mathbf{X}$$

by

$$\mathbf{D} = \begin{pmatrix} 0 & -\mathrm{d}_A & -*_J d_A \\ \lambda^2 * \mathrm{d}_A *_J & 0 & 0 \\ \lambda^2 * \mathrm{d}_A & 0 & 0 \end{pmatrix},$$

$$\mathbf{J} = \begin{pmatrix} *_J & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$\mathbf{Q} = \begin{pmatrix} -*_0 *_J & 0 & 0 \\ 0 & \lambda^{-2} & 0 \\ 0 & 0 & \lambda^{-2} \end{pmatrix},$$

where $*_J$ denotes the Hodge $*$-operator on $\Omega^1(\Sigma, \mathfrak{g}_P)$ induced by J and we abbreviate $*_0 := *_{J_0}$. These operators satisfy the conditions

$$\mathbf{JD} + \mathbf{DJ} = 0, \qquad \mathbf{QJ} + \mathbf{J}^*\mathbf{Q} = 0, \qquad \mathbf{QD} - \mathbf{D}^*\mathbf{Q} = 0,$$

where $\mathbf{J}^*$ and $\mathbf{D}^*$ denote the formal adjoint operators with respect to the inner product on $\mathbf{X}$ determined by ω and J_0. If A is flat then the kernel of the operator $\mathbf{D} = \mathbf{D}_{\lambda,A,J}$ consists of all triples $\xi = (\alpha, \phi, \psi) \in \mathbf{X}$ such that $\phi = \psi = 0$ and $\alpha = \pi_A(\alpha)$, i.e. α is harmonic with respect to A and $*_J$.

Lemma C.1. *Fix a real number $p \geq 2$, a complex structure $J_0 \in \mathcal{J}(\Sigma)$, and a reference connection $A_0 \in \mathcal{A}(P)$. Then for every constant $C > 0$ there exists a constant $c_1 = c_1(C, p, J_0, A_0) > 0$ such that the following holds. If $\lambda > 0$, $J \in \mathcal{J}(\Sigma)$, and $A \in \mathcal{A}^{\mathrm{flat}}(P)$ satisfy*

$$\lambda + 1/\lambda + \|J\|_{C^1(\Sigma,J_0)} + \|A - A_0\|_{L^\infty(\Sigma,J_0)} \leq C \tag{54}$$

then, for every $\xi = (\alpha, \phi, \psi) \in \mathbf{X}$, we have

$$\|\xi\|^p_{W^{1,p}(\Sigma,J_0)} \leq c_1 \left(\|\mathbf{D}_{\lambda,J,A}\xi\|^p_{L^p(\Sigma,J)} + \|\pi_A(\alpha)\|^p_{L^p(\Sigma,J)} \right). \tag{55}$$

Proof. Let $c = c(C, p, J_0, A_0)$ be the maximum of the constants in Lemmata B.1, B.3, and B.4. Then

$$\begin{aligned}
\|\xi\|^p_{W^{1,p}(\Sigma,J_0)} &= \|\alpha\|^p_{L^p(\Sigma,J_0)} + \|\phi\|^p_{L^p(\Sigma)} + \|\psi\|^p_{L^p(\Sigma)} \\
&\quad + \|\nabla_{A_0}\alpha\|^p_{L^p(\Sigma,J_0)} + \|d_{A_0}\phi\|^p_{L^p(\Sigma,J_0)} + \|d_{A_0}\psi\|^p_{L^p(\Sigma,J_0)} \\
&\le c\,\|\alpha\|^p_{L^p(\Sigma,J)} + (1+(2C)^p)\left(\|\phi\|^p_{L^p(\Sigma)} + \|\psi\|^p_{L^p(\Sigma)}\right) \\
&\quad + \|\nabla_{A_0}\alpha\|^p_{L^p(\Sigma,J_0)} + 2^p\left(\|d_A\phi\|^p_{L^p(\Sigma,J_0)} + \|d_A\psi\|^p_{L^p(\Sigma,J_0)}\right).
\end{aligned}$$

The last inequality follows from Lemma B.1, the triangle inequality, and the fact that $\|A - A_0\|_{L^\infty(\Sigma,J_0)} \le C$. By Lemma B.1, we obtain

$$\begin{aligned}
\|\xi\|^p_{W^{1,p}(\Sigma,J_0)} &\le 2c\,\|\alpha\|^p_{L^p(\Sigma,J)} + (1+(2C)^p)\left(\|\phi\|^p_{L^p(\Sigma)} + \|\psi\|^p_{L^p(\Sigma)}\right) \\
&\quad + c\left(\|d_A\alpha\|^p_{L^p(\Sigma)} + \|d_A(\alpha\circ J)\|^p_{L^p(\Sigma)}\right) \\
&\quad + 2^p c\left(\|d_A\phi\|^p_{L^p(\Sigma,J)} + \|d_A\psi\|^p_{L^p(\Sigma,J)}\right).
\end{aligned}$$

Now it follows from Lemma B.3 that

$$\begin{aligned}
\|\xi\|^p_{W^{1,p}(\Sigma,J_0)} &\le (2c^2+c)\left(\|d_A\alpha\|^p_{L^p(\Sigma)} + \|d_A(\alpha\circ J)\|^p_{L^p(\Sigma)}\right) \\
&\quad + 2c^2\,\|\pi_A(\alpha)\|^p_{L^p(\Sigma,J)} \\
&\quad + c(2^p+1+(2C)^p)\left(\|d_A\phi\|^p_{L^p(\Sigma,J)} + \|d_A\psi\|^p_{L^p(\Sigma,J)}\right).
\end{aligned}$$

By Lemma B.4, with $\alpha := d_A\psi - *d_A\phi$, we have

$$\|d_A\phi\|^p_{L^p(\Sigma,J)} \le c\,\|d_A\phi + *d_A\psi\|^p_{L^p(\Sigma,J)}.$$

Similarly

$$\|d_A\psi\|^p_{L^p(\Sigma,J)} \le c\,\|d_A\phi + *d_A\psi\|^p_{L^p(\Sigma,J)}.$$

Hence

$$\begin{aligned}
\|\xi\|^p_{W^{1,p}(\Sigma,J_0)} &\le (2c^2+c)\left(\|d_A\alpha\|^p_{L^p(\Sigma)} + \|d_A(\alpha\circ J)\|^p_{L^p(\Sigma)}\right) \\
&\quad + 2c^2\,\|\pi_A(\alpha)\|^p_{L^p(\Sigma,J)} \\
&\quad + 2c^2(2^p+1+(2C)^p)\,\|d_A\phi + *d_A\psi\|^p_{L^p(\Sigma,J)} \\
&\le c_1\left(\left\|\lambda^2 d_A\alpha\right\|^p_{L^p(\Sigma)} + \left\|\lambda^2 d_A(\alpha\circ J)\right\|^p_{L^p(\Sigma)}\right) \\
&\quad + c_1\left(\|d_A\phi + *d_A\psi\|^p_{L^p(\Sigma,J)} + \|\pi_A(\alpha)\|^p_{L^p(\Sigma,J)}\right) \\
&= c_1\left(\|\mathbf{D}_{\lambda,J,A}\xi\|^p_{L^p(\Sigma,J)} + \|\pi_A(\alpha)\|^p_{L^p(\Sigma,J)}\right)
\end{aligned}$$

This proves the lemma. □

Now let

$$\lambda : \mathbb{R}^2 \to (0,\infty), \qquad J : \mathbb{R}^2 \to \mathcal{J}(\Sigma), \qquad A : \mathbb{R}^2 \to \mathcal{A}(P)$$

be smooth functions. Given such functions we denote by $\mathbf{D}(s,t)$, $\mathbf{J}(s,t)$, and $\mathbf{Q}(s,t)$ the operators on $\mathbf{X}$ determined as above by $\lambda(s,t)$, $J(s,t)$, and $A(s,t)$ for $(s,t) \in \mathbb{R}$. Consider the operator[2]

$$\mathcal{D}^\varepsilon := \mathcal{D}^\varepsilon_{\lambda,J,A} := \partial_s + \mathbf{J}(\varepsilon s, \varepsilon t)\partial_t + \mathbf{D}(\varepsilon s, \varepsilon t) : \mathcal{X} \to \mathcal{X}.$$

Thus two triples $\xi = (\alpha, \phi, \psi) \in \mathcal{X}$ and $\tilde{\xi} = (\tilde{\alpha}, \tilde{\phi}, \tilde{\psi}) \in \mathcal{X}$ satisfy $\mathcal{D}^\varepsilon \xi = \tilde{\xi}$ if and only if

$$\begin{array}{rcl} \tilde{\alpha} &=& \partial_s \alpha - \mathrm{d}_{A(\varepsilon s,\varepsilon t)}\phi + *_{\varepsilon s,\varepsilon t}(\partial_t \alpha - \mathrm{d}_{A(\varepsilon s,\varepsilon t)}\psi), \\ \tilde{\phi} &=& \partial_s \phi + \partial_t \psi + \lambda(\varepsilon s,\varepsilon t)^2 * \mathrm{d}_{A(\varepsilon s,\varepsilon t)} *_{\varepsilon s,\varepsilon t} \alpha, \\ \tilde{\psi} &=& \partial_s \psi - \partial_t \phi + \lambda(\varepsilon s,\varepsilon t)^2 * \mathrm{d}_{A(\varepsilon s,\varepsilon t)}\alpha. \end{array} \tag{56}$$

Here $*_{s,t} := *_{J(s,t)} : \Omega^1(\Sigma, \mathfrak{g}_P) \to \Omega^1(\Sigma, \mathfrak{g}_P)$. In the following we assume that $A(s,t)$ is flat for all $(s,t) \in \mathbb{R}^2$. We shall also assume that

$$\sup_{s,t} \left(|\lambda(s,t)| + |1/\lambda(s,t)| + |\partial_s \lambda(s,t)| + |\partial_t \lambda(s,t)|\right) < \infty, \tag{57}$$

$$\sup_{s,t} \Big(\|J(s,t)\|_{C^2(\Sigma)} + \|\partial_s J(s,t)\|_{C^1(\Sigma)} + \|\partial_t J(s,t)\|_{C^1(\Sigma)} \Big) < \infty, \tag{58}$$

$$\sup_{s,t} \Big(\|A(s,t) - A_0\|_{C^1(\Sigma)} + \|\partial_s A(s,t)\|_{L^\infty(\Sigma)} + \|\partial_s A(s,t)\|_{L^\infty(\Sigma)} \Big) < \infty. \tag{59}$$

Here all norms are understood with respect to the metric induced by J_0.

Proposition C.2. *Fix a real number $p \geq 2$. Let $\lambda : \mathbb{R}^2 \to (0,\infty)$, $J : \mathbb{R}^2 \to \mathcal{J}(\Sigma)$, and $A : \mathbb{R}^2 \to \mathcal{A}^{\mathrm{flat}}(P)$ be continuously differentiable functions that satisfy (57), (58), and (59). Then there exist positive constants ε_0 and c_2 such that, for every $\xi \in \mathcal{X}$ and every $\varepsilon \in (0, \varepsilon_0)$, we have*

$$\begin{aligned} &\int_{\mathbb{R}^2} \Big(\|\partial_s \xi\|^p_{L^p(\Sigma,J)} + \|\partial_t \xi\|^p_{L^p(\Sigma,J)} + \|\mathbf{D}\xi\|^p_{L^p(\Sigma,J)} \Big) \\ &\qquad \leq c_2 \int_{\mathbb{R}^2} \Big(\|\mathcal{D}^\varepsilon \xi\|^p_{L^p(\Sigma,J)} + \varepsilon^p \|\pi_A(\alpha)\|^p_{L^p(\Sigma,J)} \Big). \end{aligned} \tag{60}$$

Here we abbreviate $\mathbf{D} = \mathbf{D}(\varepsilon s, \varepsilon t)$, denote by

$$\pi_{A(\varepsilon s,\varepsilon t)} : \Omega^1(\Sigma, \mathfrak{g}_P) \to \mathrm{H}^1_{A(\varepsilon s,\varepsilon t)}$$

the L^2-orthogonal projection with respect to the metric induced by $J(\varepsilon s, \varepsilon t)$, and denote by $\|\cdot\|_{L^p(\Sigma,J)}$ the L^p-norm with respect to the same metric.

[2]**Warning:** The operator $\mathcal{D}^\varepsilon$ in this section should not be confused with the operator $\mathcal{D}_\varepsilon$ introduced in Section 9. In the case $\lambda \equiv 1$ the two operators are related by rescaling on $\mathbb{R}^2$ with a factor ε (see Step 1 in the proof of Lemma D.1).

Remark C.1. The constant c_2 in (60) does not depend on the support of ξ. This is consistant with the observation that, in the Calderon-Zygmund inequality, the L^p-norms of the first derivatives of a compactly supported function $u : \mathbb{C} \to \mathbb{C}$ can be estimated by the L^p-norm of $\bar{\partial} u$ with a constant that does not depend on the support of u. This is also consistent with the presence of the factor ε on the right hand side of (60). Taking the limit $\varepsilon \to 0$ we obtain the estimate

$$\|\partial_s\xi\|_{L^p} + \|\partial_t\xi\|_{L^p} + \|\mathbf{D}\xi\|_{L^p} \le c\,\|\partial_s\xi + \mathbf{J}\partial_t\xi + \mathbf{D}\xi\|_{L^p}$$

whenever λ, J, and A are independent of s and t. In contrast, the inequality

$$\|\xi\|_{L^p} \le c_r\,\|\partial_s\xi + \mathbf{J}\partial_t\xi + \mathbf{D}\xi\|_{L^p}$$

only holds with a constant that depends on the size of the support of ξ, i.e. it holds for every $\xi \in \mathcal{X}$ with support in $B_r \times \Sigma$. Hence, if we replace the L^p norms of $\partial_s\xi$ and $\partial_t\xi$ on the left hand side of (60) by the L^p-norm of ξ itself, then we must remove the factor ε on the right.

The strategy of the proof of Proposition C.2 is as follows. We first show how to apply Proposition A.2 to the present case to obtain the inequality

$$\int_{\mathbb{R}^2} \|\xi\|^p_{L^2(\Sigma)} \le c\int_{\mathbb{R}^2}\left(\|\mathcal{D}^\varepsilon\xi\|^p_{L^2(\Sigma)} + \|\pi_A(\alpha)\|^p_{L^2(\Sigma)}\right).$$

(see Lemma C.3). Here we use the L^p norm for functions on $\mathbb{R}^2$ with values in $L^2(\Sigma)$. This norm is weaker than the L^p-norm on $\mathbb{R}^2 \times \Sigma$ and stronger than the L^2-norm on $\mathbb{R}^2 \times \Sigma$. We shall use this inequality in the case $\pi_A(\alpha) = 0$ in order to delete lower order terms on the right hand sides of the estimates.

The second step is to establish the estimate (60) with $\varepsilon = 0$ in the case where λ, J, and A are independent of s and t (Lemma C.4). The proof in this case is based on the Calderon-Zygmund inequality. This gives rise to an additional lower order term on the right hand side of the estimate. This lower order term can be cancelled, by the first step of the proof, when $\pi_A(\alpha) = 0$, and it is not present in the case $\alpha = \pi_A(\alpha)$. The result can therefore be proved by using the Hodge decomposition on Σ.

The third step of the proof (Lemma C.5) is to establish the estimate (60) in the case where $\xi = (\alpha, \phi, \psi) \in \mathcal{X}$ satisfies

$$\pi_{A(\varepsilon s,\varepsilon t)}(\alpha(s,t)) = 0.$$

Here we must assume that $\varepsilon > 0$ is sufficiently small. The idea of the proof is to first establish the estimate for every ξ with support in a ball of radius one. Here we use the fact that the variations of λ, J, and A are small on such a ball, whenever ε is sufficiently small. Next one can use cutoff functions to obtain an estimate of the form

$$\int_{\mathbb{R}^2}\left(\|\partial_s\xi\|^p_{L^p(\Sigma)} + \|\partial_t\xi\|^p_{L^p(\Sigma)}\right) \le c\int_{\mathbb{R}^2}\left(\|\mathcal{D}^\varepsilon\xi\|^p_{L^p(\Sigma)} + \|\xi\|^p_{L^p(\Sigma)}\right).$$

To remove the additional term $\|\xi\|_{L^p}$ on the right we need an inequality of the form

$$\|\xi\|_{L^p(\Sigma)} \le \delta \|\mathbf{D}\xi\|_{L^p(\Sigma)} + c_\delta \|\xi\|_{L^2(\Sigma)}$$

for all $(s,t) \in \mathbb{R}^2$, where $\delta > 0$ can be chosen arbitrarily small. With this established, one can use the first step with $\pi_A(\alpha) = 0$ to complete the third step.

The fourth step of the proof (Lemma C.6) is to establish the estimate

$$\int_{\mathbb{R}^2} \left(\|\partial_s\xi\|^p_{L^p(\Sigma)} + \|\partial_t\xi\|^p_{L^p(\Sigma)} \right) \le c \int_{\mathbb{R}^2} \left(\|\mathcal{D}^\varepsilon\xi\|^p_{L^p(\Sigma)} + \varepsilon^p \|\xi\|^p_{L^p(\Sigma)} \right)$$

for all $\xi \in \mathcal{X}$. Here the idea is again to use the Hodge decomposition to express $\xi = (\alpha, \phi, \psi)$ as a sum of the harmonic part ξ_0 and the nonharmonic part ξ_1:

$$\xi = \xi_0 + \xi_1, \qquad \xi_0 := (\pi_A(\alpha), 0, 0), \qquad \xi_1 := (\alpha - \pi_A(\alpha), \phi, \psi).$$

The harmonic part $\alpha_0 := \pi_A(\alpha)$ satisfies the identity

$$\Delta\alpha_0 = \partial_s(\tilde{\alpha}_0 - \varepsilon\alpha_0 \circ \partial_t J) + \partial_t(\tilde{\alpha}_0 \circ J + \varepsilon\alpha_0 \circ \partial_s J),$$

where $\Delta = \partial_s\partial_s + \partial_t\partial_t$ is the Laplace operator on $\mathbb{R}^2$ and

$$\tilde{\alpha}_0 := \partial_s\alpha_0 - (\partial_t\alpha_0) \circ J.$$

Hence the estimate for ξ_0 follows from the Calderon-Zygmund inequality for the Laplace operator, and for ξ_1 it has already been established in the third step. The estimate for $\xi = \xi_0 + \xi_1$ then follows from the fact that L^p-norm of the harmonic part of $\mathcal{D}^\varepsilon\xi_1$ is controlled by ε times the L^p-norm of ξ_1, and likewise, the L^p-norm of the nonharmonic part of $\mathcal{D}^\varepsilon\xi_0$ is controlled by ε times the L^p-norm of ξ_0.

The final step in the proof of Proposition C.2 is to replace the term $\|\xi\|_{L^p}$ on the right by the term $\|\pi_A(\alpha)\|_{L^p}$.

Lemma C.3. *Fix a real number $p \ge 2$. Suppose λ, J, and A satisfy (57), (58), and (59). Then there exist positive constants ε_0 and c_3 such that, for every $\xi \in \mathcal{X}$ and every $\varepsilon \in (0, \varepsilon_0)$, we have*

$$\int_{\mathbb{R}^2} \|\xi\|^p_{L^2(\Sigma,J)} \le c_3 \int_{\mathbb{R}^2} \left(\|\mathcal{D}^\varepsilon\xi\|^p_{L^2(\Sigma,J)} + \|\pi_A(\alpha)\|^p_{L^2(\Sigma,J)} \right).$$

Proof. Consider the Hilbert spaces

$$H = L^2(\Sigma, T^*\Sigma \otimes \mathfrak{g}_P) \oplus L^2(\Sigma, \otimes\mathfrak{g}_P) \oplus L^2(\Sigma, \otimes\mathfrak{g}_P),$$
$$V = W^{1,2}(\Sigma, T^*\Sigma \otimes \mathfrak{g}_P) \oplus \mathrm{W}^{1,2}(\Sigma, \otimes\mathfrak{g}_P) \oplus \mathrm{W}^{1,2}(\Sigma, \otimes\mathfrak{g}_P).$$

These are two completions of $\mathbf{X}$ and we consider the inner product on H that is determined by the reference metric $\langle\cdot,\cdot\rangle = \mathrm{dvol}_\Sigma(\cdot, J_0\cdot)$. Then the operators $\mathbf{Q}(s,t), \mathbf{J}(s,t) \in \mathcal{L}(H)$ and $\mathbf{D}(s,t) \in \mathcal{L}(V,H)$ satisfy the requirements of Proposition A.2. Explicitly, the conditions (57), (58), and (59) guarantee that $\mathbf{Q}$, $\mathbf{J}$, and $\mathbf{D}$ satisfy (28), (30), (31), and (32). Hence the assertion follows from Proposition A.2. $\square$

Lemma C.4. *Fix a real number $p \geq 2$, a complex structure $J_0 \in \mathcal{J}(\Sigma)$, and a reference connection $A_0 \in \mathcal{A}(P)$. Then for every constant $C > 0$ there exists a constant $c_4 = c_4(C, p, J_0, A_0) > 0$ such that the following holds. If $\lambda > 0$, $J \in \mathcal{J}(\Sigma)$, and $A \in \mathcal{A}^{\mathrm{flat}}(P)$ satisfy (54) then, for every $\xi \in \mathcal{X}$, we have*

$$\int_{\mathbb{R}^2} \left(\|\partial_s \xi\|^p_{L^p(\Sigma,J)} + \|\partial_t \xi\|^p_{L^p(\Sigma,J)} \right) \leq c_4 \int_{\mathbb{R}^2} \|\partial_s \xi + \mathbf{J}\partial_t \xi + \mathbf{D}\xi\|^p_{L^p(\Sigma,J)}\,. \quad (61)$$

Proof. The proof consists of seven steps. Whenever we refer to continuous dependence on J and A, it is to be understood with respect to the C^1-topology on $\mathcal{J}(\Sigma)$ and the C^0-topology on $\mathcal{A}(P)$.

Step 1 *The lemma holds whenever $\phi(s,t) = \psi(s,t) = 0$ and $\alpha_0(s,t) = \alpha(s,t)$ is harmonic with respect to A for all s and t.*

In this case

$$\partial_s \xi + \mathbf{J}\partial_t \xi + \mathbf{D}\xi = \partial_s \xi + \mathbf{J}\partial_t \xi = (\partial_s \alpha_0 + *\partial_t \alpha_0, 0, 0),$$

The Calderon-Zygmund inequality asserts that there is a constant $c_{CZ}(p) > 0$ such that, for every smooth function

$$\alpha_0 : \mathbb{R}^2 \to \ker d_A \cap \ker d_A^*$$

with compact support, we have

$$\int_{\mathbb{R}^2} \left(\|\partial_s \alpha_0\|^p_{L^2(\Sigma,J)} + \|\partial_t \alpha_0\|^p_{L^2(\Sigma,J)} \right) \leq c_{CZ}(p) \int_{\mathbb{R}^2} \|\partial_s \alpha_0 + *\partial_t \alpha_0\|^p_{L^2(\Sigma,J)}\,.$$

The constant $c_{CZ}(p)$ depends only on $p > 1$. Now, by Lemma B.2, there exists a constant $c = c(C, p, J_0, A_0)$ such that, for every pair $(J, A) \in \mathcal{J}(\Sigma) \times \mathcal{A}(P)$ that satisfies (54) and every $\alpha_0 \in \Omega^1(\Sigma, \mathfrak{g}_P)$ that satisfies

$$d_A \alpha_0 = d_A(\alpha_0 \circ J) = 0,$$

we have

$$c^{-1} \|\alpha_0\|^p_{L^2(\Sigma,J)} \leq \|\alpha_0\|^p_{L^p(\Sigma,J)} \leq c \|\alpha_0\|^p_{L^2(\Sigma,J)}\,.$$

Combining these two inequalities we obtain

$$\int_{\mathbb{R}^2} \left(\|\partial_s \alpha_0\|^p_{L^p(\Sigma,J)} + \|\partial_t \alpha_0\|^p_{L^p(\Sigma,J)} \right) \leq c^2 c_{CZ}(p) \int_{\mathbb{R}^2} \|\partial_s \alpha_0 + *\partial_t \alpha_0\|^p_{L^p(\Sigma,J)}\,.$$

This proves Step 1.

Step 2 *There exists a constant $c = c(C, p, J_0, A_0) > 0$, such that, for all λ, J, and A that satisfy (54) and all $\xi \in \mathcal{X}$, we have*

$$\|\xi\|^p_{W^{1,p}(\mathbb{R}^2 \times \Sigma)} \leq c \left(\|\partial_s \xi + \mathbf{J}\partial_t \xi + \mathbf{D}\xi\|^p_{L^p(\mathbb{R}^2 \times \Sigma)} + \|\xi\|^p_{L^p(\mathbb{R}^2 \times \Sigma)} \right).$$

The Calderon-Zygmund inequality implies a standard L^p-estimate for the self-duality operator. Namely, there is a constant $c' = c'(\lambda, J, A)$, depending continuously on λ, J, and A, such that, for every $\xi \in \mathcal{X}$ and every integer vector $k \in \mathbb{Z}^2$, we have

$$\|\xi\|^p_{W^{1,p}(B_1(k)\times\Sigma)} \le c' \left(\|\partial_s\xi + \mathbf{J}\partial_t\xi + \mathbf{D}\xi\|^p_{L^p(B_2(k)\times\Sigma)} + \|\xi\|^p_{L^p(B_2(k)\times\Sigma)} \right).$$

Now take the sum over all $k \in \mathbb{Z}^2$ to obtain the required inequality with $c = 16c'$.

Step 3 *There exists a constant $c = c(C, p, J_0, A_0) > 0$, such that, for all λ, J, and A that satisfy (54) and all $\xi \in \mathcal{X}$, we have*

$$\|\xi\|^p_{W^{1,p}(\mathbb{R}^2\times\Sigma)} \le c \left(\|\partial_s\xi + \mathbf{J}\partial_t\xi + \mathbf{D}\xi\|^p_{L^p(\mathbb{R}^2\times\Sigma)} + \int_{\mathbb{R}^2} \|\xi\|^p_{L^2(\Sigma)} \right).$$

Step 3 follows from Step 2 and Lemma B.2.

Step 4 *There exists a constant $c = c(C, p, J_0, A_0) > 0$, such that, for all λ, J, and A that satisfy (54) and all $\xi \in \mathcal{X}$, that satisfy*

$$\alpha(s,t) \in \operatorname{im} d_A \oplus \operatorname{im} d_A^*$$

for $(s,t) \in \mathbb{R}^2$, we have

$$\int_{\mathbb{R}^2} \|\xi\|^p_{L^2(\Sigma,J)} \le c \int_{\mathbb{R}^2} \|\partial_s\xi + \mathbf{J}\partial_t\xi + \mathbf{D}\xi\|^p_{L^2(\Sigma,J)}.$$

Since A is flat, there exists a constant $c_0 = c_0(J, A)$, depending continuously on J and A, such that

$$\|\alpha\|^2_{W^{1,2}(\Sigma,J)} \le c_0 \left(\|d_A\alpha\|^2_{L^2(\Sigma)} + \|d_A(\alpha\circ J)\|^2_{L^2(\Sigma)} \right)$$

for every $\alpha \in \operatorname{im} d_A \oplus \operatorname{im} d_A^*$ and

$$\|\phi\|^2_{W^{1,2}(\Sigma)} + \|\psi\|^2_{W^{1,2}(\Sigma)} \le c_0 \|d_A\phi + *d_A\psi\|^2_{L^2(\Sigma,J)}$$

for all $\phi, \psi \in \Omega^0(\Sigma, \mathfrak{g}_P)$ (see Lemmata B.3 and B.4). Hence Step 4 follows from Lemma C.3.

Step 5 *There exists a constant $c = c(C, p, J_0, A_0) > 0$, such that, for all λ, J, and A that satisfy (54) and all $\xi \in \mathcal{X}$, that satisfy*

$$\alpha(s,t) \in \operatorname{im} d_A \oplus \operatorname{im} d_A^*$$

for $(s,t) \in \mathbb{R}^2$, we have

$$\|\xi\|^p_{W^{1,p}(\mathbb{R}^2\times\Sigma)} \le c \|\partial_s\xi + \mathbf{J}\partial_t\xi + \mathbf{D}\xi\|^p_{L^p(\mathbb{R}^2\times\Sigma)}$$

Step 5 follows from Steps 3 and 4 and the fact that, for $\tilde{\xi} \in \mathcal{X}$ and $p \geq 2$, we have $\|\tilde{\xi}\|_{L^2(\Sigma)} \leq \mathrm{Vol}(\Sigma)^{(p-2)/2p} \|\tilde{\xi}\|_{L^p(\Sigma)}$.

Step 6 *There exists a constant $c = c(C, p, J_0, A_0) > 0$, such that, for every pair $(J, A) \in \mathcal{J}(\Sigma) \times \mathcal{A}^{\mathrm{flat}}(P)$ that satisfies (54) and every $\alpha \in \Omega^1(\Sigma, \mathfrak{g}_P)$, we have*

$$\|\pi_A(\alpha)\|_{L^p(\Sigma,J)} \leq c \|\alpha\|_{L^p(\Sigma,J)}.$$

Write

$$\alpha =: \pi_A(\alpha) + d_A\eta + *d_A\zeta.$$

Then

$$d_A^* d_A \eta = d_A^* \alpha, \qquad d_A^* d_A \zeta = - * d_A \alpha.$$

Hence, by Lemma B.4, there is an inequality

$$\|d_A\eta\|_{L^p(\Sigma,J)} + \|d_A\zeta\|_{L^p(\Sigma,J)} \leq c' \|\alpha\|_{L^p(\Sigma,J)},$$

where the constant $c' = c'(J)$ is independent of A and depends continuously on J. This implies Step 6 with $c = 2c' + 1$.

Step 7 *We prove the lemma.*

Write $\xi = \xi_0 + \xi_1$, where

$$\xi_0 = (\alpha_0, 0, 0), \qquad \xi_1 = (\alpha_1, \phi, \psi),$$

and

$$\alpha_0(s,t) \in \ker d_A \cap \ker d_A^*, \qquad \alpha_1(s,t) \in \mathrm{im}\, d_A \oplus \mathrm{im}\, d_A^*.$$

Thus $\alpha_0(s,t) = \pi_A(\alpha(s,t))$ is the harmonic part of $\alpha(s,t)$. Let $c = c(C, p, J_0, A_0)$ be the maximum of the constants in Steps 1 and 5. Then

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left(\|\partial_s\xi\|^p_{L^p(\Sigma,J)} + \|\partial_t\xi\|^p_{L^p(\Sigma,J)} \right) \\
&\leq 2^p \int_{\mathbb{R}^2} \left(\|\partial_s\xi_0\|^p_{L^p(\Sigma,J)} + \|\partial_t\xi_0\|^p_{L^p(\Sigma,J)} \right) \\
&\quad + 2^p \int_{\mathbb{R}^2} \left(\|\partial_s\xi_1\|^p_{L^p(\Sigma,J)} + \|\partial_t\xi_1\|^p_{L^p(\Sigma,J)} \right) \\
&\leq 2^p c \int_{\mathbb{R}^2} \|\partial_s\xi_0 + \mathbf{J}\partial_t\xi_0\|^p_{L^p(\Sigma,J)} \\
&\quad + 2^p c \int_{\mathbb{R}^2} \|\partial_s\xi_1 + \mathbf{J}\partial_t\xi_1 + \mathbf{D}\xi\|^p_{L^p(\Sigma,J)} \\
&\leq 2^p c c'' \int_{\mathbb{R}^2} \|\partial_s\xi + \mathbf{J}\partial_t\xi + \mathbf{D}\xi\|^p_{L^p(\Sigma,J)}.
\end{aligned}$$

The penultimate inequality follows from Steps 1 and 5. The last inequality follows from Step 6. This proves the lemma with a constant that depends continuously on λ, J, and A. Since the inequality is gauge invariant and the moduli space of flat connections on P is compact, the constant can be chosen independent of A. $\square$

Lemma C.5. *Fix a real number $p \geq 2$. Suppose that λ, J, and A satisfy (57), (58), and (59). Suppose further that $A(s,t)$ is flat for all s and t. Then there exist constants $\varepsilon_0 > 0$ and $c_5 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$ and every $\xi = (\alpha, \phi, \psi) \in \mathcal{X}$ that satisfies*

$$\pi_{A(\varepsilon s, \varepsilon t)}(\alpha(s,t)) = 0$$

for $(s,t) \in \mathbb{R}^2$, we have

$$\int_{\mathbb{R}^2} \left(\|\partial_s \xi\|^p_{L^p(\Sigma,J)} + \|\partial_t \xi\|^p_{L^p(\Sigma,J)} \right) \leq c_5 \int_{\mathbb{R}^2} \|\mathcal{D}^\varepsilon \xi\|^p_{L^p(\Sigma,J)}. \tag{62}$$

Proof. Throughout the proof we use the notation $z = (s,t) \in \mathbb{R}^2$.

Step 1 *There exists a constant $c > 0$ such that*

$$\|\xi(z)\|^p_{W^{1,p}(\Sigma,J_0)} \leq c_1 \, \|\mathbf{D}(\varepsilon z)\xi(z)\|^p_{L^p(\Sigma,J(\varepsilon z))}$$

for all $z \in \mathbb{R}^2$, $\varepsilon > 0$, and $\xi \in \mathcal{X}$ such that $\pi_{A(\varepsilon z)}(\alpha(z)) = 0$.

Denote

$$C := \sup_{z \in \mathbb{R}^2} \Big(\lambda(z) + 1/\lambda(z) + \|J(z)\|_{C^1(\Sigma,J_0)} + \|A(z) - A_0\|_{L^\infty(\Sigma,J_0)} \Big).$$

By (57), (58), and (59) we have $C < \infty$. Hence Step 1 follows from Lemma C.1 with $c = c_1(C, p, J_0, A_0)$.

Step 2 *There exists a constant $c_2 > 0$ such that*

$$\begin{aligned} &\|(\mathcal{D}^\varepsilon \xi)(z) - \partial_s \xi(z) - \mathbf{J}(\varepsilon z_1)\partial_t \xi(z) - \mathbf{D}(\varepsilon z_1)\xi(z)\|^p_{L^p(\Sigma,J(\varepsilon z))} \\ &\leq c_2 |z - z_1|^p \varepsilon^p \|\xi(z)\|^p_{W^{1,p}(\Sigma,J_0)}. \end{aligned}$$

for all $\xi \in \mathcal{X}$, $z, z_1 \in \mathbb{R}^2$, and $\varepsilon > 0$.

Abbreviate $\lambda_1 := \lambda(\varepsilon z_1)$, $J_1 := J(\varepsilon z_1)$, $A_1 := A(\varepsilon z_1)$, $\mathbf{D}_1 := \mathbf{D}_{\lambda_1, J_1, A_1}$, $\mathbf{J}_1 := \mathbf{J}_{J_1}$, and

$$\tilde{\xi} := (\tilde{\alpha}, \tilde{\phi}, \tilde{\psi}) := \mathcal{D}^\varepsilon \xi, \qquad \tilde{\xi}_1 := (\tilde{\alpha}_1, \tilde{\phi}_1, \tilde{\psi}_1) := \partial_s \xi + \mathbf{J}_1 \partial_t \xi + \mathbf{D}_1 \xi.$$

Then

$$\begin{aligned} \tilde{\alpha} - \tilde{\alpha}_1 &= [A_1 - A, \phi] + [(A - A_1) \circ J_1, \psi] + (\partial_t \alpha - \mathrm{d}_A \psi) \circ (J_1 - J), \\ \tilde{\phi} - \tilde{\phi}_1 &= ({\lambda_1}^2 - \lambda^2) * \mathrm{d}_{A_1}(\alpha \circ J_1) + \lambda^2 * [(A_1 - A) \wedge (\alpha \circ J)], \\ &\quad + \lambda^2 * \mathrm{d}_{A_1}(\alpha \circ (J_1 - J)) \\ \tilde{\psi} - \tilde{\psi}_1 &= ({\lambda_1}^2 - \lambda^2) * \mathrm{d}_{A_1} \alpha + \lambda^2 * [(A_1 - A) \wedge \alpha]. \end{aligned} \tag{63}$$

Since $A = A(\varepsilon z)$, $J = J(\varepsilon z)$, and $\lambda = \lambda(\varepsilon z)$, the assertion follows from (57), (58), and (59).

Step 3 *There exist positive constants ε_0 and c_3 such that*

$$\int_{\mathbb{R}^2} \left(\|\partial_s \xi\|^p_{L^p(\Sigma,J)} + \|\partial_t \xi\|^p_{L^p(\Sigma,J)} \right) \le c_3 \int_{\mathbb{R}^2} \|\mathcal{D}^\varepsilon \xi\|^p_{L^p(\Sigma,J)} .$$

for every $\varepsilon > 0$ and every $\xi \in \mathcal{X}$ with support in a ball of radius $\varepsilon_0/\varepsilon$ that satisfies $\pi_{A(\varepsilon z)}(\alpha(z)) = 0$ for every $z \in \mathbb{R}^2$.

By Step 1, we have

$$\begin{aligned} \|\xi(z)\|^p_{W^{1,p}(\Sigma,J_0)} &\le c_1 \|\mathbf{D}(\varepsilon z)\xi(z)\|^p_{L^p(\Sigma,J(\varepsilon z))} \\ &\le 3^p c_1 \left(\|\partial_s \xi(z)\|^p_{L^p(\Sigma,J(\varepsilon z))} + \|\partial_t \xi(z)\|^p_{L^p(\Sigma,J(\varepsilon z))} \right) \\ &\quad + 3^p c_1 \|(\mathcal{D}^\varepsilon \xi)(z)\|^p_{L^p(\Sigma,J(\varepsilon z))} \end{aligned}$$

for all $\xi \in \mathcal{X}$, $z \in \mathbb{R}^2$, and $\varepsilon > 0$. Using Lemma B.1, Lemma C.4, and Step 2, we obtain the following estimate for every $\xi \in \mathcal{X}$ with support in a ball of radius r. We denote by $z_1 \in \mathbb{R}^2$ the center of the ball and abbreviate $J_1 := J(\varepsilon z_1)$, $A_1 := A(\varepsilon z_1)$, $J = J(\varepsilon z)$, $A = A(\varepsilon z)$. Then

$$\begin{aligned} &\int_{\mathbb{R}^2} \left(\|\partial_s \xi\|^p_{L^p(\Sigma,J)} + \|\partial_t \xi\|^p_{L^p(\Sigma,J)} \right) \\ &\le c \int_{\mathbb{R}^2} \left(\|\partial_s \xi\|^p_{L^p(\Sigma,J_1)} + \|\partial_t \xi\|^p_{L^p(\Sigma,J_1)} \right) \\ &\le c c_4 \int_{\mathbb{R}^2} \|\partial_s \xi + \mathbf{J}_1 \partial_t \xi + \mathbf{D}_1 \xi\|^p_{L^p(\Sigma,J_1)} \\ &\le c^2 c_4 \int_{\mathbb{R}^2} \|\partial_s \xi + \mathbf{J}_1 \partial_t \xi + \mathbf{D}_1 \xi\|^p_{L^p(\Sigma,J)} \\ &\le 2^p c^2 c_4 \int_{\mathbb{R}^2} \left(\|\mathcal{D}^\varepsilon \xi\|^p_{L^p(\Sigma,J)} + \|\mathcal{D}^\varepsilon \xi - \partial_s \xi - \mathbf{J}_1 \partial_t \xi - \mathbf{D}_1 \xi\|^p_{L^p(\Sigma,J)} \right) \\ &\le 2^p c^2 c_4 \int_{\mathbb{R}^2} \left(\|\mathcal{D}^\varepsilon \xi\|^p_{L^p(\Sigma,J)} + c_2 (r\varepsilon)^p \|\xi\|^p_{W^{1,p}(\Sigma,J_0)} \right) \\ &\le 2^p c^2 c_4 (1 + 3^p c_1 c_2 (r\varepsilon)^p) \int_{\mathbb{R}^2} \|\mathcal{D}^\varepsilon \xi\|^p_{L^p(\Sigma,J)} \\ &\quad + (6r\varepsilon)^p c^2 c_1 c_2 c_4 \int_{\mathbb{R}^2} \left(\|\partial_s \xi\|^p_{L^p(\Sigma,J)} + \|\partial_t \xi\|^p_{L^p(\Sigma,J)} \right). \end{aligned}$$

This proves Step 3 with $(6\varepsilon_0)^p c^2 c_1 c_2 c_4 = 1/2$ and $c_3 = 2^{p+1} c^2 c_4 + 1$.

Step 4 *We prove the lemma.*

Let $\rho : \mathbb{R}^2 \to [0, 1]$ be a smooth cutoff function, supported in the open ball of radius one centered at zero, such that

$$\sum_{k \in \mathbb{Z}^2} \rho(k + z) = 1$$

for every $z \in \mathbb{R}^2$. For $i = (i_0, i_1) \in \mathbb{Z}^2$ with $i_0, i_1 \in \{0, 1\}$ denote

$$\xi_i := \rho_i \xi, \qquad \rho_i(z) := \sum_{k \in \mathbb{Z}^2} \rho(i + 2k + z).$$

Then, by Step 3, we have

$$\int_{\mathbb{R}^2} \left(\|\partial_s \xi_i\|^p_{L^p(\Sigma,J)} + \|\partial_t \xi_i\|^p_{L^p(\Sigma,J)} \right) \le c_3 \int_{\mathbb{R}^2} \|\mathcal{D}^\varepsilon \xi_i\|^p_{L^p(\Sigma,J)} .$$

whenever $\varepsilon \le \varepsilon_0$. Take the sum of these four functions to obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left(\|\partial_s \xi\|^p_{L^p(\Sigma,J)} + \|\partial_t \xi\|^p_{L^p(\Sigma,J)} \right) \\
&\le 4^p c_3 \sum_i \int_{\mathbb{R}^2} \|\mathcal{D}^\varepsilon \xi_i\|^p_{L^p(\Sigma,J)} \\
&= 4^p c_3 \sum_i \int_{\mathbb{R}^2} \|\rho_i \mathcal{D}^\varepsilon \xi + (\partial_s \rho_i)\xi + (\partial_t \rho_i)\mathbf{J}\xi\|^p_{L^p(\Sigma,J)} \\
&\le 12^p c_3 \sum_i \int_{\mathbb{R}^2} \left(\|\rho_i \mathcal{D}^\varepsilon \xi\|^p_{L^p(\Sigma,J)} + \left(|\partial_s \rho_i|^p + |\partial_t \rho_i|^p\right) \|\xi\|^p_{L^p(\Sigma,J)} \right) \\
&\le c' \int_{\mathbb{R}^2} \left(\|\mathcal{D}^\varepsilon \xi\|^p_{L^p(\Sigma,J)} + \|\xi\|^p_{L^p(\Sigma,J)} \right),
\end{aligned}$$

where $c' := 4(12\|\rho\|_{C^1})^p c_3$. Now it follows from Lemma B.2 and Step 1 that, for every $\delta > 0$ there exists a constant $c_\delta > 0$ such that, for every $z \in \mathbb{R}^2$ and every $\varepsilon > 0$, we have

$$\|\xi(z)\|^p_{L^p(\Sigma,J(\varepsilon z))} \le \delta \|\mathbf{D}(\varepsilon z)\xi(z)\|^p_{L^p(\Sigma,J(\varepsilon z))} + c_\delta \|\xi(z)\|^p_{L^2(\Sigma,J(\varepsilon z))} .$$

Hence

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left(\|\partial_s \xi\|^p_{L^p(\Sigma,J)} + \|\partial_t \xi\|^p_{L^p(\Sigma,J)} \right) \\
&\le c' \int_{\mathbb{R}^2} \left(\|\mathcal{D}^\varepsilon \xi\|^p_{L^p(\Sigma,J)} + \delta \|\mathbf{D}\xi\|^p_{L^p(\Sigma,J)} + c_\delta \|\xi\|^p_{L^2(\Sigma,J)} \right) \\
&\le c'(1 + 3^p \delta) \int_{\mathbb{R}^2} \|\mathcal{D}^\varepsilon \xi\|^p_{L^p(\Sigma,J)} + c_\delta c' \int_{\mathbb{R}^2} \|\xi\|^p_{L^2(\Sigma,J)} \\
&\quad + 3^p c' \delta \int_{\mathbb{R}^2} \left(\|\partial_s \xi\|^p_{L^p(\Sigma,J)} + \|\partial_t \xi\|^p_{L^p(\Sigma,J)} \right).
\end{aligned}$$

If $3^p c' \delta \le 1/2$ we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left(\|\partial_s \xi\|^p_{L^p(\Sigma,J)} + \|\partial_t \xi\|^p_{L^p(\Sigma,J)} \right) \\
&\le (2c' + 1) \int_{\mathbb{R}^2} \|\mathcal{D}^\varepsilon \xi\|^p_{L^p(\Sigma,J)} + 2c_\delta c' \int_{\mathbb{R}^2} \|\xi\|^p_{L^2(\Sigma,J)} \\
&\le c'' \int_{\mathbb{R}^2} \|\mathcal{D}^\varepsilon \xi\|^p_{L^p(\Sigma,J)} .
\end{aligned}$$

The last inequality follows from Lemma C.3. □

Lemma C.6. *Fix a real number $p \geq 2$. Suppose that λ, J, and A satisfy (57), (58), and (59). Suppose further that $A(s,t)$ is flat for all s and t. Then there exist constants $\varepsilon_0 > 0$ and $c_4 > 0$ such, for every $\xi \in \mathcal{X}$ and every $\varepsilon \in (0, \varepsilon_0)$, we have*

$$\int_{\mathbb{R}^2} \left(\|\partial_s \xi\|^p_{L^p(\Sigma,J)} + \|\partial_t \xi\|^p_{L^p(\Sigma,J)} \right) \leq c_6 \int_{\mathbb{R}^2} \left(\|\mathcal{D}^\varepsilon \xi\|^p_{L^p(\Sigma,J)} + \varepsilon^p \|\xi\|^p_{L^p(\Sigma,J)} \right). \tag{64}$$

Proof. Given $\xi = (\alpha, \phi, \psi) \in \mathcal{X}$ and $\varepsilon > 0$ we write

$$\xi = \xi_0 + \xi_1, \qquad \xi_0 = (\alpha_0, 0, 0), \qquad \xi_1 = (\alpha_1, \phi, \psi),$$

where $\alpha_0(z) := \pi_{A(\varepsilon z)}(\alpha(z))$ for $z \in \mathbb{R}^2$. Then, by Lemma C.5,

$$\int_{\mathbb{R}^2} \left(\|\partial_s \xi_1\|^p_{L^p(\Sigma,J)} + \|\partial_t \xi_1\|^p_{L^p(\Sigma,J)} \right) \leq c_5 \int_{\mathbb{R}^2} \|\mathcal{D}^\varepsilon \xi_1\|^p_{L^p(\Sigma,J)}. \tag{65}$$

Moreover, $\mathbf{D}(\varepsilon z)\xi_0(z) = 0$ and hence $\mathcal{D}^\varepsilon \xi_0 = (\tilde{\alpha}_0, 0, 0)$, where

$$\begin{aligned} \tilde{\alpha}_0(z) &:= \partial_s \alpha_0(z) - (\partial_t \alpha_0(z)) \circ J(\varepsilon z), \\ \tilde{\alpha}_0(z) \circ J(\varepsilon z) &= \partial_t \alpha_0(z) + (\partial_s \alpha_0(z)) \circ J(\varepsilon z). \end{aligned}$$

Denote by $\Delta := {\partial_s}^2 + {\partial_t}^2$ the Laplace operator on $\mathbb{R}^2$. Then

$$\Delta \alpha_0 = \partial_s \beta + \partial_t \gamma$$

where the functions $\beta, \gamma : \mathbb{R}^2 \to \Omega^1(\Sigma, \mathfrak{g}_P)$ are defined by

$$\begin{aligned} \beta(z) &:= \tilde{\alpha}_0(z) - \varepsilon \alpha_0(z) \circ \partial_t J(\varepsilon z), \\ \gamma(z) &:= \tilde{\alpha}_0(z) \circ J(\varepsilon z) + \varepsilon \alpha_0(z) \circ \partial_s J(\varepsilon z). \end{aligned}$$

Hence it follows from the Calderon-Zygmund inequality for functions with values in a Hilbert space that

$$\begin{aligned} \int_{\mathbb{R}^2} \left(\|\partial_s \alpha_0\|^p_{L^2(\Sigma)} + \|\partial_t \alpha_0\|^p_{L^2(\Sigma)} \right) &\leq c_{CZ} \int_{\mathbb{R}^2} \left(\|\beta\|^p_{L^2(\Sigma)} + \|\gamma\|^p_{L^2(\Sigma)} \right) \\ &\leq c' \int_{\mathbb{R}^2} \left(\|\tilde{\alpha}_0\|^p_{L^2(\Sigma)} + \varepsilon^p \|\alpha_0\|^p_{L^2(\Sigma)} \right). \end{aligned}$$

Now consider the identities

$$\begin{aligned} d_A(\partial_s \alpha_0) &= -\varepsilon [\partial_s A \wedge \alpha_0], \\ d_A((\partial_s \alpha_0) \circ J) &= -\varepsilon [\partial_s A \wedge (\alpha_0 \circ J)] - \varepsilon d_A(\alpha_0 \circ \partial_s J). \end{aligned}$$

Similar identities hold for $\partial_t \alpha_0$. Hence, by Lemmata B.1 and B.2, we have

$$\begin{aligned} &\int_{\mathbb{R}^2} \left(\|\partial_s \alpha_0\|^p_{L^p(\Sigma,J)} + \|\partial_t \alpha_0\|^p_{L^p(\Sigma,J)} \right) \\ &\leq c'' \int_{\mathbb{R}^2} \left(\|\partial_s \alpha_0\|^p_{L^2(\Sigma,J)} + \|\partial_t \alpha_0\|^p_{L^2(\Sigma,J)} + \varepsilon^p \|\alpha_0\|^p_{L^p(\Sigma,J)} \right). \end{aligned}$$

Combining this with the previous inequality gives

$$\begin{aligned}
&\int_{\mathbb{R}^2} \Big(\|\partial_s \alpha_0\|^p_{L^p(\Sigma,J)} + \|\partial_t \alpha_0\|^p_{L^p(\Sigma,J)} \Big) \\
&\le c''' \int_{\mathbb{R}^2} \Big(\|\tilde{\alpha}_0\|^p_{L^p(\Sigma,J)} + \varepsilon^p \|\alpha_0\|^p_{L^p(\Sigma,J)} \Big) .
\end{aligned}$$

This inequality can be written in the form

$$\begin{aligned}
&\int_{\mathbb{R}^2} \Big(\|\partial_s \xi_0\|^p_{L^p(\Sigma,J)} + \|\partial_t \xi_0\|^p_{L^p(\Sigma,J)} \Big) \\
&\le c''' \int_{\mathbb{R}^2} \Big(\|\mathcal{D}^\varepsilon \xi_0\|^p_{L^p(\Sigma,J)} + \varepsilon^p \|\xi_0\|^p_{L^p(\Sigma,J)} \Big) .
\end{aligned} \tag{66}$$

Now it follows from the definitions that

$$\begin{aligned}
d_A \tilde{\alpha}_0 &= \varepsilon \big(-[\partial_s A \wedge \alpha_0] + [\partial_t A \wedge (\alpha_0 \circ J)] + d_A(\alpha_0 \circ \partial_t J) \big) \\
d_A(\tilde{\alpha}_0 \circ J) &= \varepsilon \big(-[\partial_t A \wedge \alpha_0] - [\partial_s A \wedge (\alpha_0 \circ J)] - d_A(\alpha_0 \circ \partial_s J) \big).
\end{aligned}$$

Hence, by Lemmata B.3 and B.4, there exists a constant $C_0 > 0$ such that

$$\|\tilde{\alpha}_0 - \pi_A(\tilde{\alpha}_0)\|^p_{L^p(\Sigma,J)} \le C_0 \varepsilon^p \|\alpha_0\|^p_{L^p(\Sigma,J)} \tag{67}$$

for every $z \in \mathbb{R}^2$. Hence the nonharmonic part of $\tilde{\alpha}_0$ is bounded by ε times α_0. Likewise, the harmonic part of $\tilde{\alpha}_1$, the first component of $\tilde{\xi}_1 := \mathcal{D}^\varepsilon \xi_1$, is bounded by ε times α_1. To see this write

$$\alpha_1 =: d_A \eta - (d_A \zeta) \circ J,$$

where $\eta, \zeta : \mathbb{R}^2 \to \Omega^0(\Sigma, \mathfrak{g}_P)$ and note that

$$\pi_A(\tilde{\alpha}_1) = \pi_A(\partial_s \alpha_1 + (\partial_t \alpha_1) \circ J).$$

Since $A = A(\varepsilon z)$ and $J = J(\varepsilon z)$, we obtain

$$\pi_A(\partial_s \alpha_1) = \varepsilon\, \pi_A\big([\partial_s A, \eta] - [(\partial_s A) \circ J, \zeta] - (d_A \zeta) \circ \partial_s J\big)$$

and similarly for $\partial_t \alpha_1$. Hence, by Lemmata B.3 and B.4, there exists a constant $C_1 > 0$ such that

$$\|\pi_A(\tilde{\alpha}_1)\|^p_{L^p(\Sigma,J)} \le C_1 \varepsilon^p \|\alpha_1\|^p_{L^p(\Sigma,J)} \tag{68}$$

for every $z \in \mathbb{R}^2$. Denote $\tilde{\alpha} := \tilde{\alpha}_0 + \tilde{\alpha}_1$. Then

$$\tilde{\alpha}_0 = \pi_A(\tilde{\alpha}) - \pi_A(\tilde{\alpha}_1) + (\tilde{\alpha}_0 - \pi_A(\tilde{\alpha}_0))$$

and

$$\tilde{\alpha}_1 = (\tilde{\alpha} - \pi_A(\tilde{\alpha})) - (\tilde{\alpha}_0 - \pi_A(\tilde{\alpha}_0)) + \pi_A(\tilde{\alpha}_1).$$

Hence, by (67) and (68), we have

$$
\begin{aligned}
&\|\tilde{\alpha}_0\|^p_{L^p(\Sigma,J)} + \|\tilde{\alpha}_1\|^p_{L^p(\Sigma,J)} \\
&\le 3^p \Big(\|\pi_A(\tilde{\alpha})\|^p_{L^p(\Sigma,J)} + \|\pi_A(\tilde{\alpha}_1)\|^p_{L^p(\Sigma,J)} + \|\tilde{\alpha}_0 - \pi_A(\tilde{\alpha}_0)\|^p_{L^p(\Sigma,J)} \Big) \\
&\quad + 3^p \Big(\|\tilde{\alpha} - \pi_A(\tilde{\alpha})\|^p_{L^p(\Sigma,J)} + \|\tilde{\alpha}_0 - \pi_A(\tilde{\alpha}_0)\|^p_{L^p(\Sigma,J)} + \|\pi_A(\tilde{\alpha}_1)\|^p_{L^p(\Sigma,J)} \Big) \\
&\le 3^p \Big(\|\pi_A(\tilde{\alpha})\|^p_{L^p(\Sigma,J)} + \|\tilde{\alpha} - \pi_A(\tilde{\alpha})\|^p_{L^p(\Sigma,J)} \Big) \\
&\quad + 2(3\varepsilon)^p \Big(C_0 \|\alpha_0\|^p_{L^p(\Sigma,J)} + C_1 \|\alpha_1\|^p_{L^p(\Sigma,J)} \Big) \\
&\le c \Big(\|\tilde{\alpha}\|^p_{L^p(\Sigma,J)} + \varepsilon^p \|\alpha\|^p_{L^p(\Sigma,J)} \Big).
\end{aligned}
$$

The last inequality uses Lemma B.4 and the constant c is independent of $z \in \mathbb{R}^2$. It follows that

$$
\|\mathcal{D}^\varepsilon \xi_0\|^p_{L^p(\Sigma,J)} + \|\mathcal{D}^\varepsilon \xi_1\|^p_{L^p(\Sigma,J)} \le c \Big(\|\mathcal{D}^\varepsilon \xi\|^p_{L^p(\Sigma,J)} + \varepsilon^p \|\xi\|^p_{L^p(\Sigma,J)} \Big)
$$

for every $z \in \mathbb{R}^2$. Hence the assertion follows from (65) and (66). □

Proof of Proposition C.2. Let c_6 be the constant of Lemma C.6, denote

$$
C := \sup_{z \in \mathbb{R}^2} \Big(\lambda(z) + 1/\lambda(z) + \|J(z)\|_{C^1(\Sigma,J_0)} + \|A(z) - A_0\|_{L^\infty(\Sigma,J_0)} \Big),
$$

and let $c_1 = c_1(C, p, J_0, A_0)$ be the constant of Lemma C.1. Then, by (64), we have

$$
\begin{aligned}
\int_{\mathbb{R}^2} \|\mathbf{D}\xi\|^p_{L^p(\Sigma,J)} &= \int_{\mathbb{R}^2} \|\mathcal{D}^\varepsilon \xi - \partial_s \xi - \partial_t \xi\|^p_{L^p(\Sigma,J)} \\
&\le 3^p \int_{\mathbb{R}^2} \Big(\|\mathcal{D}^\varepsilon \xi\|^p_{L^p(\Sigma,J)} + \|\partial_s \xi\|^p_{L^p(\Sigma,J)} + \|\partial_t \xi\|^p_{L^p(\Sigma,J)} \Big) \\
&\le 3^p (c_6 + 1) \int_{\mathbb{R}^2} \Big(\|\mathcal{D}^\varepsilon \xi\|^p_{L^p(\Sigma,J)} + \varepsilon^p \|\xi\|^p_{L^p(\Sigma,J)} \Big).
\end{aligned}
$$

Moreover, by Lemmata B.1 and C.1, we have

$$
\|\xi\|^p_{L^p(\Sigma,J)} \le c \|\xi\|^p_{L^p(\Sigma,J_0)} \le c c_1 \Big(\|\mathbf{D}\xi\|^p_{L^p(\Sigma,J)} + \|\pi_A(\alpha)\|^p_{L^p(\Sigma,J)} \Big)
$$

for every $(s,t) \in \mathbb{R}^2$. Now let $c_7 := c_6 + 3^p(c_6 + 1)$. Then, combining the last two inequalities with (64), we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \Big(\|\partial_s \xi\|^p_{L^p(\Sigma,J)} + \|\partial_t \xi\|^p_{L^p(\Sigma,J)} + \|\mathbf{D}\xi\|^p_{L^p(\Sigma,J)} \Big) \\
&\le c_7 \int_{\mathbb{R}^2} \Big(\|\mathcal{D}^\varepsilon \xi\|^p_{L^p(\Sigma,J)} + \varepsilon^p \|\xi\|^p_{L^p(\Sigma,J)} \Big) \\
&\le c_7 \int_{\mathbb{R}^2} \Big(\|\mathcal{D}^\varepsilon \xi\|^p_{L^p(\Sigma,J)} + \varepsilon^p c c_1 \|\mathbf{D}\xi\|^p_{L^p(\Sigma,J)} + \varepsilon^p c c_1 \|\pi_A(\alpha)\|^p_{L^p(\Sigma,J)} \Big).
\end{aligned}
$$

If $\varepsilon^p cc_1c_7 \le 1/2$ then

$$\begin{aligned}&\int_{\mathbb{R}^2}\Big(\|\partial_s\xi\|^p_{L^p(\Sigma,J)}+\|\partial_t\xi\|^p_{L^p(\Sigma,J)}+\|\mathbf{D}\xi\|^p_{L^p(\Sigma,J)}\Big)\\&\le 2c_7\int_{\mathbb{R}^2}\Big(\|\mathcal{D}^\varepsilon\xi\|^p_{L^p(\Sigma,J)}+\varepsilon^p cc_1\,\|\pi_A(\alpha)\|^p_{L^p(\Sigma,J)}\Big).\end{aligned}$$

This proves the proposition. □

D Proof of Proposition 9.1

Let us return to the notation of Sections 3, 4, and 5. Fix a perturbation H, K as in Section 6 and $n+1$ perturbed flat connections $a_j \in \mathcal{A}^{\mathrm{flat}}(Q_{\tilde{f}_j}, K_j)$. Let

$$\Xi = (A, \Phi, \Psi) \in \mathcal{A}(Q)$$

be a connection that is in temporal gauge near the cylindrical ends and satisfies (6) and $F_{A(s,t)} = 0$ for all s and t. Let $\mathcal{D}_\varepsilon = \mathcal{D}_\varepsilon(\Xi) : \mathrm{W}^{1,p} \to \widetilde{\mathrm{L}}^p$ be the operator introduced in Section 9. Recall from (15) that $\tilde{\xi} = \mathcal{D}_\varepsilon\xi$ if and only if

$$\begin{aligned}\tilde{\alpha} &= \nabla_s\alpha - d_A\phi - \mathrm{d}v_{s,t}(A)\alpha + *_{s,t}(\nabla_t\alpha - d_A\psi - dw_{s,t}(A)\alpha),\\ \tilde{\phi} &= \nabla_s\phi + \nabla_t\psi + (\lambda/\varepsilon)^2 * d_A *_{s,t}\alpha,\\ \tilde{\psi} &= \nabla_s\psi - \nabla_t\phi + (\lambda/\varepsilon)^2 * d_A\alpha.\end{aligned}$$

Lemma D.1. *For every Ξ as above and every $p \ge 2$ there exist constants $c > 0$ and $\varepsilon_0 > 0$ such that the following holds. If $\xi = (\alpha, \phi, \psi) \in \mathrm{W}^{1,p}$ and $\varepsilon \in (0, \varepsilon_0)$ then*

$$\begin{aligned}&\int_0^1\int_{-\infty}^{\infty}\lambda^2\Big(\|\mathrm{d}_A\alpha\|^p_{L^p(\Sigma)}+\|\mathrm{d}_A(\alpha\circ J)\|^p_{L^p(\Sigma)}\Big)\\&+\int_0^1\int_{-\infty}^{\infty}\varepsilon^p\lambda^{2-p}\Big(\|\nabla_s\alpha\|^p_{L^p(\Sigma,J)}+\|\nabla_t\alpha\|^p_{L^p(\Sigma,J)}\Big)\\&+\int_0^1\int_{-\infty}^{\infty}\varepsilon^p\lambda^{2-p}\Big(\|\mathrm{d}_A\phi\|^p_{L^p(\Sigma,J)}+\|\mathrm{d}_A\psi\|^p_{L^p(\Sigma,J)}\Big)\\&+\int_0^1\int_{-\infty}^{\infty}\varepsilon^{2p}\lambda^{2-2p}\Big(\|\nabla_s\phi\|^p_{L^p(\Sigma)}+\|\nabla_t\phi\|^p_{L^p(\Sigma)}\Big)\\&+\int_0^1\int_{-\infty}^{\infty}\varepsilon^{2p}\lambda^{2-2p}\Big(\|\nabla_s\psi\|^p_{L^p(\Sigma)}+\|\nabla_t\psi\|^p_{L^p(\Sigma)}\Big)\\&\le c\int_0^1\int_{-\infty}^{\infty}\Big(\varepsilon^p\lambda^2\,\|\pi_A(\alpha)\|^p_{L^p(\Sigma,J)}\\&\quad+\ \varepsilon^p\lambda^{2-p}\,\|\tilde{\alpha}\|^p_{L^p(\Sigma,J)}+\varepsilon^{2p}\lambda^{2-2p}\|\tilde{\phi}\|^p_{L^p(\Sigma)}+\varepsilon^{2p}\lambda^{2-2p}\|\tilde{\psi}\|^p_{L^p(\Sigma)}\Big).\end{aligned}\tag{69}$$

where $\tilde{\xi} = (\tilde{\alpha}, \tilde{\phi}, \tilde{\psi}) \in \widetilde{\mathrm{L}}^p$ is given by $\tilde{\xi} = \mathcal{D}_\varepsilon\xi$.

Proof. The proof consists of six steps. The first two steps are proved by direct calculation.

Step 1 *Assume $v = w = 0$. Define*

$$\Xi' = (A', \Phi', \Psi'), \qquad \xi' = (\alpha', \phi', \psi'), \qquad \tilde{\xi}' = (\tilde{\alpha}', \tilde{\phi}', \tilde{\psi}'),$$

by

$$A'(s,t) = A(\varepsilon s, \varepsilon t), \quad \Phi'(s,t) = \varepsilon \Phi(\varepsilon s, \varepsilon t), \quad \Psi'(s,t) = \varepsilon \Psi(\varepsilon s, \varepsilon t),$$

$$\alpha'(s,t) = \alpha(\varepsilon s, \varepsilon t), \quad \phi'(s,t) = \varepsilon \phi(\varepsilon s, \varepsilon t), \quad \psi'(s,t) = \varepsilon \psi(\varepsilon s, \varepsilon t),$$

$$\tilde{\alpha}'(s,t) = \varepsilon \alpha(\varepsilon s, \varepsilon t), \quad \tilde{\phi}'(s,t) = \varepsilon^2 \phi(\varepsilon s, \varepsilon t), \quad \tilde{\psi}'(s,t) = \varepsilon^2 \psi(\varepsilon s, \varepsilon t),$$

Denote

$$\lambda'(s,t) = \lambda(\varepsilon s, \varepsilon t), \qquad \nabla_s{}' = \partial_s + \Phi', \qquad \nabla_t{}' = \partial_t + \Psi'.$$

Then $\tilde{\xi} = \mathcal{D}_\varepsilon \xi$ if and only if

$$\begin{array}{rcl} \tilde{\alpha}' & = & \nabla_s{}'\alpha' - \mathrm{d}_{A'}\phi' + *_{\varepsilon s, \varepsilon t}(\nabla_t{}'\alpha' - \mathrm{d}_{A'}\psi'), \\ \tilde{\phi}' & = & \nabla_s{}'\phi' + \nabla_t{}'\psi' + (\lambda')^2 * \mathrm{d}_{A'} *_{\varepsilon s, \varepsilon t} \alpha', \\ \tilde{\psi}' & = & \nabla_s{}'\psi' - \nabla_t{}'\phi' + (\lambda')^2 * \mathrm{d}_{A'}\alpha'. \end{array} \tag{70}$$

Step 2 *Let Ξ', ξ', and $\tilde{\xi}'$ be as in Step 1, and denote $J'(s,t) = J(\varepsilon s, \varepsilon t)$. Then (69) is equivalent to*

$$\begin{aligned} &\int_0^{1/\varepsilon} \int_{-\infty}^{\infty} \lambda'^2 \left(\|\mathrm{d}_{A'}\alpha'\|^p_{L^p(\Sigma)} + \|\mathrm{d}_{A'}(\alpha' \circ J')\|^p_{L^p(\Sigma)} \right) \\ &+ \int_0^{1/\varepsilon} \int_{-\infty}^{\infty} \lambda'^{2-p} \left(\left\|\nabla_s{}'\alpha'\right\|^p_{L^p(\Sigma, J')} + \left\|\nabla_t{}'\alpha'\right\|^p_{L^p(\Sigma, J')} \right) \\ &+ \int_0^{1/\varepsilon} \int_{-\infty}^{\infty} \lambda'^{2-p} \left(\|\mathrm{d}_{A'}\phi'\|^p_{L^p(\Sigma, J')} + \|\mathrm{d}_{A'}\psi'\|^p_{L^p(\Sigma, J')} \right) \\ &+ \int_0^{1/\varepsilon} \int_{-\infty}^{\infty} \lambda'^{2-2p} \left(\left\|\nabla_s{}'\phi'\right\|^p_{L^p(\Sigma)} + \left\|\nabla_t{}'\phi'\right\|^p_{L^p(\Sigma)} \right) \\ &+ \int_0^{1/\varepsilon} \int_{-\infty}^{\infty} \lambda'^{2-2p} \left(\left\|\nabla_s{}'\psi'\right\|^p_{L^p(\Sigma)} + \left\|\nabla_t{}'\psi'\right\|^p_{L^p(\Sigma)} \right) \\ &\le c \int_0^{1/\varepsilon} \int_{-\infty}^{\infty} \Big(\varepsilon^p \lambda'^2 \, \|\pi_{A'}(\alpha')\|^p_{L^p(\Sigma, J')} \\ &\quad + \lambda'^{2-p} \, \|\tilde{\alpha}'\|^p_{L^p(\Sigma, J')} + \lambda'^{2-2p} \|\tilde{\phi}'\|^p_{L^p(\Sigma)} + \lambda'^{2-2p} \|\tilde{\psi}'\|^p_{L^p(\Sigma)} \Big). \end{aligned} \tag{71}$$

Step 3 *Assume $v = w = 0$ and $\Phi = \Psi = 0$. Fix an integer $j \in \{0, \dots, n\}$ and denote by*

$$Q_j = \iota_j((-\infty, -T) \times Q_{\tilde{f}_j}) \subset Q$$

the jth cylindrical end. Then there exist constants $c > 0$ and $\varepsilon_0 > 0$ such that every $\xi \in \mathrm{W}^{1,p}$ that vanishes oustside Q_j satisfies (69) for $0 < \varepsilon < \varepsilon_0$.

Suppose, without loss of generality, that $j = 0$. Hence assume that $\xi(s,t) = 0$ for $s \geq -T$ and recall that $\lambda(s,t) = 1$ for $s \leq -T$. Choose a cutoff function $\beta : \mathbb{R} \to [0,1]$ such that

$$\beta(t) = \begin{cases} 1, & \text{for } |t| \leq \delta, \\ 0, & \text{for } |t| \geq 1/2 - \delta, \end{cases} \qquad \beta(-t) = \beta(t),$$

and

$$0 \leq t \leq 1/2 \qquad \Longrightarrow \qquad \beta(t) + \beta(t - 1/2) = 1.$$

Consider the functions

$$\xi_0(s,t) = \beta(t)\xi(s,t), \qquad \xi_1(s,t) = \beta(t - 1/2)\xi(s,t).$$

Both functions have compact support. By Step 1, the rescaled function ξ_0' satisfies

$$\begin{aligned} \beta(\varepsilon t)\tilde{\alpha}' + \varepsilon\dot{\beta}(\varepsilon t) *_{\varepsilon s, \varepsilon t} \alpha' &= \nabla_s{}'\alpha_0' - \mathrm{d}_{A'}\phi_0' + *_{\varepsilon s,\varepsilon t}(\nabla_t{}'\alpha_0' - \mathrm{d}_{A'}\psi_0'), \\ \beta(\varepsilon t)\tilde{\phi}' + \varepsilon\dot{\beta}(\varepsilon t)\psi' &= \nabla_s{}'\phi_0' + \nabla_t{}'\psi_0' + *\mathrm{d}_{A'} *_{\varepsilon s,\varepsilon t} \alpha_0', \\ \beta(\varepsilon t)\tilde{\psi}' - \varepsilon\dot{\beta}(\varepsilon t)\phi' &= \nabla_s{}'\psi_0' - \nabla_t{}'\phi_0' + *\mathrm{d}_{A'}\alpha_0'. \end{aligned}$$

and similarly for ξ_1'. Now apply Proposition C.2 to ξ_0' and ξ_1' to obtain that ξ' satisfies (71) for $\varepsilon > 0$ sufficiently small. This proves Step 3 for $j = 0$. For general j apply the same argument to the pullback 1-form $\xi_j = \tilde{\iota}_j{}^*\xi$ on Q_j. This shows that ξ_j satisfies (71) with $\lambda = 1$. Step 3 then follows by expressing this estimate in terms of ξ.

Step 4 *Assume $v = w = 0$ and $\Phi = \Psi = 0$. Let $B = \{s + it \in \mathbb{C} \,|\, s^2 + t^2 \leq 1\}$ and $\iota : B \hookrightarrow S$ be a holomorphic embedding. Then there exist constants $c > 0$ and $\varepsilon_0 > 0$ such that every $\xi \in \mathrm{W}^{1,p}$ with support in $\iota(B)$ satisfies (69) for $0 < \varepsilon < \varepsilon_0$.*

Trivialize the bundle X over B to obtain an embedding

$$\iota : B \times \Sigma \to X.$$

Then trivialize the bundle Q over the image of ι to obtain an embedding

$$\tilde{\iota} : B \times P \to Q.$$

By Step 1, the rescaled pullback 1-form $(\tilde{\iota}^*\xi)'$ satisfies (70). Hence, by Proposition C.2, it satisfies (71). Hence, by Step 2, it satisfies (69).

Step 5 *Assume $v = w = 0$ and $\Phi = \Psi = 0$. Then there exist constants $c > 0$ and $\varepsilon_0 > 0$ such that every $\xi \in \mathrm{W}^{1,p}$ with compact support in X satisfies (69) for $0 < \varepsilon < \varepsilon_0$.*

Cover S by finitely many open sets U_ν each of which is either a cylindrical end or is contained in the holomorphic image of a ball B. Choose a partition of unity $\rho_\nu : S \to [0,1]$ subordinate to the cover, and apply Steps 3 and 4 to the functions $(\rho_\nu \circ \pi)\xi$. Then the error terms in the estimate (69) arising from the cutoff functions $\rho_\nu = \rho_\nu(s,t)$ are arbitrarily small as ε tends to zero. They can be dominated by positive terms on the left hand side of the inequality, and hence ξ satisfies (69).

Step 6 *We prove the lemma.*

By Lemma B.3, there exists a constant $c_0 > 0$ such that, for every $(s,t) \in \mathbb{R}^2$, every $\phi \in \Omega^0(\Sigma, \mathfrak{g}_P)$, and every $\alpha \in \Omega^1(\Sigma, \mathfrak{g}_P)$,

$$\|\phi\|_{W^{1,p}(\Sigma)} \le c_0 \|\mathrm{d}_A \phi\|_{L^p(\Sigma,J)},$$

$$\|\alpha\|_{L^p(\Sigma,J)} \le c_0 \Big(\|\mathrm{d}_A \alpha\|_{L^p(\Sigma)} + \|\mathrm{d}_A(\alpha \circ J)\|_{L^p(\Sigma)} + \|\pi_A(\alpha)\|_{L^p(\Sigma,J)} \Big).$$

Moreover, the limit condition (6) implies that there exists a constant $c_1 > 0$ such that, for every $(s,t) \in \mathbb{R}^2$ and every $\alpha \in \Omega^1(\Sigma, \mathfrak{g}_P)$,

$$\|\Phi\|_{L^\infty(\Sigma)} + \|\Psi\|_{L^\infty(\Sigma)} \le c_1 \lambda,$$

$$\|\mathrm{d}v_{s,t}(A)\alpha\|_{L^p(\Sigma,J)} + \|\mathrm{d}w_{s,t}(A)\alpha\|_{L^p(\Sigma,J)} \le c_1 \lambda \|\alpha\|_{L^p(\Sigma,J)}. \tag{72}$$

Hence the error terms in (69) arising from nonzero terms v, w, Φ, or Ψ can be controlled by larger positive terms on the left hand side provided that $\varepsilon > 0$ sufficiently small. This proves the lemma. □

Proof of Proposition 9.1. The inequality (16) follows directly from Lemma D.1 and Lemma B.3. To prove (17) write

$$\alpha = \pi_A(\alpha) + \mathrm{d}_A \zeta - (\mathrm{d}_A \eta) \circ J$$

for $\eta, \zeta \in \Omega^0(\Sigma, \mathfrak{g}_P)$ and abbreviate

$$B_s = \partial_s A - \mathrm{d}_A \Phi, \qquad B_t = \partial_t A - \mathrm{d}_A \Psi, \qquad B = B_s - B_t \circ J.$$

By (6), there exists a constant $c_1 > 0$ such that, for every $(s,t) \in \mathbb{R}^2$,

$$\|B_s\|_{L^\infty(\Sigma,J)} + \|B_t\|_{L^\infty(\Sigma,J)} \le c_1 \lambda. \tag{73}$$

Moreover,

$$\mathrm{d}_A *_{s,t} \mathrm{d}_A \eta = \mathrm{d}_A(\alpha - \pi_A(\alpha)), \qquad \mathrm{d}_A *_{s,t} \mathrm{d}_A \zeta = \mathrm{d}_A *_{s,t} (\alpha - \pi_A(\alpha)).$$

Hence, by Lemma B.4, there exists a constant $c_2 > 0$ such that, for all $(s,t) \in \mathbb{R}^2$,

$$\|\mathrm{d}_A\eta\|_{L^p(\Sigma,J)} + \|\mathrm{d}_A\zeta\|_{L^p(\Sigma,J)} \leq c_2 \|(\alpha - \pi_A(\alpha))\|_{L^p(\Sigma,J)}. \tag{74}$$

Moreover,

$$\begin{aligned}\nabla_s(\alpha - \pi_A(\alpha)) &= \nabla_s(\mathrm{d}_A\zeta - (\mathrm{d}_A\eta)\circ J) \\ &= \mathrm{d}_A\nabla_s\zeta - (\mathrm{d}_A\nabla_s\eta)\circ J \\ &\quad + [B_s,\zeta] - [B_s\circ J,\eta] - (\mathrm{d}_A\eta)\circ\partial_s J,\end{aligned} \tag{75}$$

and similarly for $\nabla_t(\alpha - \pi_A(\alpha))$. Hence

$$\begin{aligned}\mathrm{d}_A *_{s,t} \mathrm{d}_A\nabla_s\eta &= \mathrm{d}_A\nabla_s\alpha + [B_s\wedge\pi_A(\alpha)] \\ &\quad - \mathrm{d}_A\left([B_s,\zeta] + [*_{s,t}B_s,\eta] - (\mathrm{d}_A\eta)\circ\partial_s J\right),\end{aligned}$$

$$\begin{aligned}\mathrm{d}_A *_{s,t} \mathrm{d}_A\nabla_s\zeta &= \mathrm{d}_A *_{s,t} \nabla_s\alpha + [B_s\wedge *_{s,t}\pi_A(\alpha)] \\ &\quad - \mathrm{d}_A\left([*_{s,t}B_s,\zeta] - [B_s,\eta] - (\mathrm{d}_A\eta)\circ J\partial_s J - \pi_A(\alpha)\circ\partial_s J\right).\end{aligned}$$

Hence, by Lemma B.4, Lemma B.3, (73), and (74), there exists a constant $c_3 > 0$ such that, for all $(s,t) \in \mathbb{R}^2$,

$$\|\mathrm{d}_A\nabla_s\eta\|_{L^p(\Sigma,J)} + \|\mathrm{d}_A\nabla_s\zeta\|_{L^p(\Sigma,J)} \leq c_3\left(\|\nabla_s\alpha\|_{L^p(\Sigma,J)} + \lambda\|\alpha\|_{L^p(\Sigma,J)}\right).$$

By (74) and (75), this implies

$$\|\nabla_s(\alpha - \pi_A(\alpha))\|_{L^p(\Sigma,J)} \leq c_4\left(\|\nabla_s\alpha\|_{L^p(\Sigma,J)} + \lambda\|\alpha\|_{L^p(\Sigma,J)}\right).$$

A similar inequality holds for $\nabla_t(\alpha - \pi_A(\alpha))$. Hence (17) follows from Lemma D.1 and Lemma B.3.

To prove (18) note that

$$\pi_A(\mathcal{D}_\varepsilon\xi) - \mathcal{D}_0\pi_A(\xi) = \pi_A(\theta_0 - \theta_1), \tag{76}$$

where

$$\theta_0 = \nabla_s(\alpha - \pi_A(\alpha)) + *_{s,t}\nabla_t(\alpha - \pi_A(\alpha)),$$
$$\theta_1 = \mathrm{d}v_{s,t}(A)(\alpha - \pi_A(\alpha)) + *_{s,t}\mathrm{d}w_{s,t}(\alpha - \pi_A(\alpha)).$$

By (75),

$$\begin{aligned}\pi_A(\theta_0) &= \pi_A(\nabla_s(\alpha - \pi_A(\alpha)) - (\nabla_t(\alpha - \pi_A(\alpha)))\circ J) \\ &= \pi_A\left([B,\zeta] - [B\circ J,\eta] - (\mathrm{d}_A\eta)\circ(\partial_s J + J\partial_t J)\right).\end{aligned}$$

Hence, by (72), (73), (74), (76), and Lemma B.3,

$$
\begin{aligned}
&\|\pi_A(\mathcal{D}_\varepsilon\xi) - \mathcal{D}_0\pi_A(\xi)\|_{\widetilde{\mathrm{L}}^p}^p \\
&\le \|\pi_A(\theta_0)\|_{\widetilde{\mathrm{L}}^p}^p + \|\pi_A(\theta_1)\|_{\widetilde{\mathrm{L}}^p}^p \\
&= \int_0^1\int_{-\infty}^{\infty} \lambda^{2-p}\left(\|\pi_A(\theta_0)\|_{L^p(\Sigma,J)}^p + \|\pi_A(\theta_1)\|_{L^p(\Sigma,J)}^p\right) \\
&\le c_5\int_0^1\int_{-\infty}^{\infty} \lambda^{2}\left(\|\mathrm{d}_A\zeta\|_{L^p(\Sigma,J)}^p + \|\mathrm{d}_A\eta\|_{L^p(\Sigma,J)}^p + \|\alpha-\pi_A(\alpha)\|_{L^p(\Sigma,J)}^p\right) \\
&\le c_6\int_0^1\int_{-\infty}^{\infty} \lambda^{2}\,\|\alpha-\pi_A(\alpha)\|_{L^p(\Sigma,J)}^p \\
&\le c_6\,\|\xi-\pi_A(\xi)\|_{0,p,\varepsilon}^p\,.
\end{aligned}
$$

This proves the proposition. □

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