Homogeneous quasimorphisms on the symplectic linear group

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Let G be a group. A **quasimorphism** on G is a map $\rho : \mathrm{G} \to \mathbb{R}$ satisfying

$$|\rho(gh) - \rho(g) - \rho(h)| \le C$$

for all $g, h \in \mathrm{G}$ and a suitable constant C. It is called **homogeneous** if $\rho(g^k) = k\rho(g)$ for every $g \in \mathrm{G}$ and every integer $k \ge 0$. Let

$$\mathrm{Sp}(2n) := \left\{ \Psi \in \mathbb{R}^{2n\times 2n} \,|\, \Psi J_0 \Psi^T = J_0 \right\}, \qquad J_0 := \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix},$$

denote the group of symplectic matrices and $\widetilde{\mathrm{Sp}}(2n)$ its universal cover. Think of an element of $\widetilde{\mathrm{Sp}}(2n)$ as a homotopy class $[\Psi]$ (with fixed endpoints) of a smooth path $\Psi : [0,1] \to \mathrm{Sp}(2n)$ satisfying $\Psi(0) = \mathbb{1}$.

Theorem 1. *There is a unique homogeneous quasimorphism μ on $\widetilde{\mathrm{Sp}}(2n)$ that descends to the determinant homomorphism on $\mathrm{U}(n)$ in the sense that*

$$\det(X + iY) = \exp\left(2\pi i \mu([\Psi])\right), \qquad \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} := \Psi(1),$$

for every $[\Psi] \in \widetilde{\mathrm{Sp}}(2n)$ with $\Psi(1) \in \mathrm{Sp}(2n) \cap \mathrm{O}(2n) \cong \mathrm{U}(n)$.

The quasimomorphism of Theorem 1 plays a central role in [3] and this motivated the present note. Two explicit constructions of the quasimorphism can be found in [1] and [5]. The construction in [1] uses the unitary part in a polar decomposition and homogenization. The construction in [5] uses the eigenvalue decomposition of a symplectic matrix (but does not mention the term *quasimorphism*).

Lemma 1 *If $\rho : \mathrm{G} \to \mathbb{R}$ is a homogeneous quasimorphism then ρ is invariant under conjugation and $\rho(g^{-1}) = -\rho(g)$ for every $g \in \mathrm{G}$.*

Proof of Lemma 1. Let C be the constant in the definition of quasimorphism. By homgeneity, we have $\rho(1) = 0$. Hence $\left|\rho(g^k) + \rho(g^{-k})\right| \le C$ for every $g \in \mathrm{G}$ and every integer $k \ge 0$. By homogeneity, we obtain $|\rho(g) + \rho(g^{-1})| \le C/k$ for every k and so $\rho(g^{-1}) = -\rho(g)$. Hence

$$\left|\rho(ghg^{-1}) - \rho(h)\right| = \left|\rho(ghg^{-1}) - \rho(g) - \rho(h) - \rho(g^{-1})\right| \le 2C.$$

Using homogeneity again we obtain $\rho(ghg^{-1}) = \rho(h)$ for all $g, h \in \mathrm{G}$. □

Proof of Theorem 1. Let $\mathcal{P} \subset \mathrm{Sp}(2n)$ denote the set of symmetric positive definite symplectic matrices. This space is contractible and hence there is a natural injection $\iota : \mathcal{P} \to \widetilde{\mathrm{Sp}}(2n)$. Explicitly, the map ι assigns to a matrix $P \in \mathcal{P}$ the unique homotopy class of paths $\Phi : [0, 1] \to \mathcal{P}$ with endpoints $\Phi(0) = \mathbb{1}$ and $\Phi(1) = P$.

Let $\mu : \widetilde{\mathrm{Sp}}(2n) \to \mathbb{R}$ be a homogeneous quasimorphism that descends to the determinant homomorphism on $\mathrm{U}(n)$. It suffices to prove that the restriction of μ to $\iota(\mathcal{P})$ is bounded. (If μ' is another quasimorphism satisfying the requirements of Theorem 1 and μ, μ' are bounded on $\iota(\mathcal{P})$ then, by polar decomposition and the determinant assumption, their difference is bounded and so, by homogeneity, they are equal.) We prove that μ vanishes on $\iota(\mathcal{P})$. For every unitary matrix $Q \in \mathrm{U}(n) \subset \mathrm{Sp}(2n)$ and every $P \in \mathcal{P}$ we have

$$\mu(\iota(QPQ^T)) = \mu(\iota(P)). \tag{1}$$

To see this, choose two paths $\Phi : [0, 1] \to \mathcal{P}$ and $\Psi : [0, 1] \to \mathrm{U}(n)$ such that $\Phi(0) = \Psi(0) = 1$ and $\Phi(1) = P$, $\Psi(1) = Q$. Then $\mu([\Phi]) = \mu([\Psi\Phi\Psi^{-1}])$, by Lemma 1, and so (1) follows from the fact that $\Psi^{-1} = \Psi^T$. Now let $P \in \mathcal{P}$. Since P is a symmetric symplectic matrix we have $PJ_0P = J_0$ and hence

$$\mu(\iota(P)) = \mu(\iota(J_0P^{-1}J_0^{-1})) = \mu(\iota(P^{-1})) = \mu(\iota(P)^{-1}) = -\mu(\iota(P)).$$

Here the second equation follows from (1) and the last from Lemma 1. This shows that $\mu(\iota(P)) = 0$ for every $P \in \mathcal{P}$. □

Remark 1. Lemma 1 is well known to the experts [2]. We included a proof to give a self-contained exposition.

Remark 2. Related results, obtained with different methods, are contained in [1] and [4]. Our main theorem can in fact be deduced from these results.

Remark 3. The determinant homomorphism $\det : \mathrm{U}(n) \to S^1$ is uniquely determined by the condition that it induces an isomorphism on fundamental groups. Hence it follows from Theorem 1 that the homogeneous quasimorphism $\mu : \widetilde{\mathrm{Sp}}(2n) \to \mathbb{R}$ is uniquely determined by the condition that it restricts to an isomorphism of the fundamental group of $\mathrm{Sp}(2n)$ to the integers.

Remark 4. The referee pointed out to us the following generalization.

Let G *be a uniformly perfect group and* $Z \to \widetilde{\mathrm{G}} \to \mathrm{G}$ *be a central extension. If* ρ *is a homogeneous quasimorphism on* $\widetilde{\mathrm{G}}$ *that vanishes on* Z *then* $\rho \equiv 0$.

To see this we first observe that, since ρ vanishes on Z, we have

$$\rho(zg) = \lim_{k\to\infty} k^{-1}\rho(z^k g^k) = \lim_{k\to\infty} k^{-1}\rho(g^k) = \rho(g)$$

for all $z \in Z$ and $g \in \widetilde{\mathrm{G}}$. Hence ρ descends to G. Now let $C > 0$ be the constant in the definition of quasimorphism. Then, by Lemma 1, we have $|\rho(ghg^{-1}h^{-1})| = |\rho(ghg^{-1}h^{-1}) - \rho(g) - \rho(hg^{-1}h^{-1})| \le C$ for all $g, h \in \mathrm{G}$. Since every element of G can be expressed as a product of at most N commutators we have $|\rho(g)| \le (2N-1)C$ for all $g \in \mathrm{G}$. Thus the quasimorphism is bounded and hence vanishes identically.

Theorem 1 follows from this generalization because $\mathrm{Sp}(2n)$ is uniformly perfect and $\widetilde{\mathrm{Sp}}(2n)$ is a central extension of $\mathrm{Sp}(2n)$. However, the geometric properties of the Maslov quasimorphism $\mu : \widetilde{\mathrm{Sp}}(2n) \to \mathbb{R}$ derived in the proof of Theorem 1 do not follow from the above algebraic argument.

References

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