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FLOWS ON VECTOR BUNDLES AND HYPERBOLIC SETS

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ABSTRACT. This note deals with C. Conley's topological approach to hyperbolic invariant sets for continuous flows. It is based on the notions of isolated invariant sets and Morse decompositions and it leads to the concept of weak hyperbolicity.

1. Introduction. It is our aim to give an exposition of a small part of C. Conley's lectures on dynamical systems, which he gave in 1984 in Madison. It deals with a topological approach to hyperbolic invariant sets of flows.

In order to describe the contents we recall at first that a linear flow $(e, t) \mapsto e \cdot t$ on a vector bundle $\pi\colon E \to S$ over a compact metric space S is called hyperbolic, if there is a splitting of E into a direct sum $E^s \oplus E^u = E$ of two invariant subbundles such that for two positive numbers K and ε the following exponential estimates hold:

$$\|e \cdot t\| \leqslant K\exp(-\varepsilon t)\|e\| \quad \text{if } e \in E^s \text{ and } t \geqslant 0,$$
$$\|e \cdot (-t)\| \leqslant K\exp(-\varepsilon t)\|e\| \quad \text{if } e \in E^u \text{ and } t \geqslant 0.$$

Due to the linearity of the flow the zero section $Z \subset E$ of the bundle E, which is homeomorphic to S, is an invariant set. Moreover, due to the above estimates every bounded orbit of the flow on E is contained in Z, such that Z is an isolated invariant set in the sense of Conley [4].

It is now tempting to start with the latter property as the crucial concept replacing the hyperbolicity assumption with the topological assumption requiring only the zero section Z to be our isolated invariant set of the flow. In order to describe this approach we at first do not assume Z to be isolated and define the stable and unstable invariant sets of Z as follows:

$$E^s = \{e \in E \mid \varnothing \neq \omega(e) \subset Z\}, \qquad E^u = \{e \in E \mid \varnothing \neq \omega^*(e) \subset Z\}$$

where $\omega(e)$ and $\omega^*(e)$ denote the positive and the negative limit set of a point $e \in E$. In view of the linearity of the flow one shows easily that E^s and E^u intersect every fiber in a linear subspace. However, E^s and E^u are not necessarily subbundles of E.

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In order to apply the theory of flows on compact spaces it is useful to study the induced flow on the projective bundle PE of E, using again the linearity of the flow. It turns out (Theorem 2.7) that Z is an isolated invariant set if and only if in the projective bundle PE the invariant set PE^u is an attractor and PE^s is its complementary repeller. The exponential estimates for E^s and E^u are easily established if Z is isolated.

If one assumes, in addition that the induced flow on Z is chain transitive, then E^s and E^u are not just invariant sets but actually invariant subbundles of E with $E = E^s \oplus E^u$, so that in this case the flow is indeed hyperbolic. This result, due to Selgrade [**11**], will also be derived in the first part (Theorem 2.13).

In the second part we consider the flow of a C^1-vector field on a compact Riemannian manifold M. Linearizing this flow one can associate with it a flow on the tangent bundle TM of M and another one on the cotangent bundle T^*M of M. Let now $S \subset M$ be a compact invariant set and assume the vector field to have no singular points on S, then it defines a one-dimensional invariant subbundle of T_SM which we shall denote by E^0.

The invariant set S is then called weakly hyperbolic, if the zero section in the quotient bundle T_SM/E^0 is an isolated invariant set. Using the results of the first part it will be proved that S is weakly hyperbolic precisely if the projective bundle P_SM admits a three set Morse decomposition such that the attractor corresponds to the unstable set E^u, the repeller to the stable set E^s and the third set to the distinguished bundle E^0 (Theorem 3.3). The difference to the hyperbolicity in the classical sense is that the stable and unstable sets are not necessarily subbundles of T_SM.

It also turns out that $S \subset M$ is hyperbolic precisely if the annihilator of E^0 in T_S^*M defines a hyperbolic vector bundle over S.

Finally, a general perturbation result for attractor-repeller pairs for flows will be used in a natural way to conclude that the above defined weak hyperbolicity is a stable property, i.e. that nearby invariant sets S' for nearby flows are also weakly hyperbolic (Theorem 3.9). For Anosov flows on compact manifolds this is, of course, well known [**1**, **2**, **9**] and follows also readily from our considerations.

2. Flows on vector bundles. On the vector bundle $\pi\colon E \to S$ over a compact metric space S with finite dimensional fiber $V \cong \pi^{-1}(p)$ we consider a linear flow. This is a continuous map from $E \times \mathbf{R}$ into E, denoted by $(e,t) \mapsto e \cdot t$, which satisfies $e \cdot 0 = e$, $e \cdot (t+s) = (e \cdot t) \cdot s$ for all $e \in E$ and for all real numbers $t, s \in \mathbf{R}$. Moreover, the flow is linear which requires for $e, e' \in E$ with $\pi(e) = \pi(e') \in S$ that

$$(2.1)\quad \pi(e \cdot t) = \pi(e' \cdot t), \quad e \cdot t + e' \cdot t = (e + e') \cdot t, \quad \lambda(e \cdot t) = (\lambda e) \cdot t$$

for all $t \in \mathbf{R}$ and $\lambda \in \mathbf{R}$. Therefore the flow on E induces a flow on the zero section $Z \subset E$ which is homeomorphic to S. Moreover, it induces also a flow on the projective bundle PE of E. For our notation we refer to the appendix.

LEMMA 2.1. (i) *There is a unique flow on S such that*

$$(2.2)\qquad \pi(e \cdot t) = \pi(e) \cdot t$$

for every $e \in E$ *and every* $t \in \mathbf{R}$. *This flow is given by*

$$p \cdot t = \pi(\sigma(p) \cdot t), \qquad p \in S,\ t \in \mathbf{R}. \tag{2.3}$$

(ii) *There exists a unique flow on PE such that*

$$P(e \cdot t) = Pe \cdot t \tag{2.4}$$

for every $e \in E \setminus Z$ *and every* $t \in \mathbf{R}$.

Proof. If a flow on S satisfies (2.2) then (2.3) follows from the fact that $\pi \circ \sigma = 1_S$. Conversely, let $p \cdot t \in S$ be defined by (2.3) for $p \in S$, $t \in \mathbf{R}$. Then $p \cdot 0 = \pi(\sigma(p)) = p$ and

$$\sigma(p \cdot t) = \sigma(\pi(\sigma(p) \cdot t)) = \sigma(p) \cdot t$$

since $\sigma(p) \cdot t \in Z$. This implies $p \cdot (t+s) = (p \cdot t) \cdot s$ and therefore (2.3) defines a flow on M. Finally, since $\pi(e) = \pi(\sigma(\pi(e)))$, it follows from (2.1) and (2.3) that

$$\pi(e \cdot t) = \pi(\sigma(\pi(e)) \cdot t) = \pi(e) \cdot t$$

for $e \in E$ and $t \in \mathbf{R}$. This proves statement (i). Statement (ii) is an immediate consequence of (2.1). □

Lemma 2.2. *There exist constants* $K \geqslant 1$, $\omega > 0$ *such that the following inequality holds for* $e \in E$, $t \in \mathbf{R}$

$$\|e \cdot t\| \leqslant K \exp(\omega|t|)\|e\|.$$

Proof. It follows from the continuity of the flow together with the compactness of S that

$$K(t) = \sup\{\|e \cdot s\| \,|\, e \in E,\ \|e\| = 1,\ -t \leqslant s \leqslant t\} < \infty$$

for every $t \geqslant 0$. Furthermore the function $K(t)$ is nondecreasing in t and satisfies $K(0) = 1$ as well as $K(t+s) \leqslant K(t)K(s)$ for $t, s \geqslant 0$. Defining $K = K(1)$ and $\omega = \log K(1)$ we obtain for $n \leqslant t \leqslant n+1$,

$$K(t) \leqslant K(n+1) \leqslant K(1)^{n+1} = K\exp(\omega n) \leqslant K\exp(\omega t).$$

This proves the statement of the lemma. □

Lemma 2.3. (i) $\varnothing \neq \omega(e) \subset Z$ *if and only if* $\lim_{t \to \infty}\|e \cdot t\| = 0$.
(ii) $\varnothing \neq \omega^*(e) \subset Z$ *if and only if* $\lim_{t \to -\infty}\|e \cdot t\| = 0$.

Proof. If $\varnothing \neq \omega(e) \subset Z$ and $\varepsilon > 0$ then there exists a $T > 0$ such that $\|e \cdot t\| \leqslant \varepsilon$ for all $t \geqslant T$ since $N_\varepsilon = \{e \in E \,|\, \|e\| \leqslant \varepsilon\}$ is a neighborhood of Z. Conversely, suppose that $\lim_{t \to \infty}\|e \cdot t\| = 0$. Then it follows from the continuity of the norm function that $\|e'\| = 0$ for every $e' \in \omega(e)$ and hence $\omega(e) \subset Z$. This proves statement (i). Statement (ii) is proved similarly. □

In the remainder of this section we consider flows on E for which the zero section Z is an *isolated invariant set*. This means that there exists a compact (isolating) neighborhood N of Z in E such that $e \cdot \mathbf{R} \subset N$ implies $e \in Z$.

Lemma 2.4. *If Z is an isolated invariant set in E then*
(i) $\sup\{\|e \cdot t\| \,|\, t \in \mathbf{R}\} < \infty \Leftrightarrow e \in Z$,
(ii) $\sup\{\|e \cdot t\| \,|\, t \geqslant 0\} < \infty \Leftrightarrow \varnothing \neq \omega(e) \subset Z$,
(iii) $\sup\{\|e \cdot t\| \,|\, t \leqslant 0\} < \infty \Leftrightarrow \varnothing \neq \omega^*(e) \subset Z$.

PROOF. Let N be an isolated neighborhood for Z and choose $\varepsilon > 0$ such that $\|e\| \leqslant \varepsilon$ implies $e \in N$. Now suppose that $\|e \cdot t\| \leqslant K$ for all $t \in \mathbf{R}$. Then $ce \cdot \mathbf{R} \subset N$ for $c = \varepsilon/K > 0$. This implies $ce \in Z$ and hence $e \in Z$. Thus we have proved statement (i).

In order to establish statement (ii) let us first assume that $\|e \cdot t\| \leqslant K$ for all $t \geqslant 0$. Then $ce \cdot [0, \infty) \subset N$ for $c = \varepsilon/K$ and therefore $\omega(ce) \subset Z$. From this we conclude that $\|ce \cdot t\| = c\|e \cdot t\|$ converges to zero as t tends to infinity (Lemma 2.3) and hence $\omega(e) \subset Z$. Conversely, it follows from Lemma 2.3 that the forward orbit of e is bounded if $\varnothing \neq \omega(e) \subset Z$. This proves statement (ii).

Statement (iii) follows by reversing the time. $\square$

Let us now introduce the stable and unstable sets of Z by

$$E^s = \{e \in E \mid \varnothing \neq \omega(e) \subset Z\}, \quad E^u = \{e \in E \mid \varnothing \neq \omega^*(e) \subset Z\}. \tag{2.5}$$

It follows from Lemma 2.3 that these sets intersect each fiber E_p, $p \in S$, in a linear subspace, even if Z is not an isolated invariant set. The following example shows that these subspaces need not span the whole space, even if Z is an isolated invariant set. See Figure 1.

$$\dot{x} = x(1-x), \quad \dot{y} = (x - 1/2)y, \qquad 0 \leqslant x \leqslant 1. \tag{2.6}$$

In the "dual" flow the zero section Z is not an isolated invariant set, the sets E^u and E^s are not closed and their intersection is bigger than Z. See Figure 2.

$$\dot{x} = x(1-x), \quad \dot{z} = (1/2 - x)z, \qquad 0 \leqslant x \leqslant 1. \tag{2.7}$$

LEMMA 2.5. *If Z is an isolated invariant set in E, then the sets E^s and E^u are closed and $E^s \cap E^u = Z$. Furthermore, there exist constants $K > 0$, $\varepsilon > 0$ such that*

$$\|e \cdot t\| \leqslant K \exp(-\varepsilon t)\|e\| \quad \forall e \in E^s \; \forall t \geqslant 0, \tag{2.8}$$

$$\|e \cdot t\| \leqslant K \exp(\varepsilon t)\|e\| \quad \forall e \in E^u \; \forall t \leqslant 0. \tag{2.9}$$

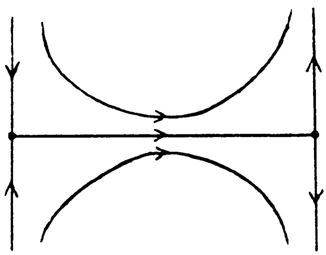

FIGURE 1

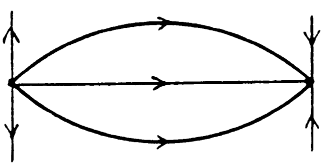

FIGURE 2

PROOF. First note that $E^u \cap E^s$ consists of those $e \in E$ whose orbits are bounded and hence it follows from Lemma 2.4 that $E^u \cap E^s = Z$.

Secondly we establish the inequality (2.8) with $\varepsilon = 0$. If (2.8) would not hold with $\varepsilon = 0$, then there would exist a sequence $e_k \in E^s$ such that $\|e_k\| = 1$ and $\|e_k \cdot t_k\|$ tends to infinity where $t_k \geqslant 0$ is chosen such that $\|e_k \cdot t_k\| \geqslant \|e_k \cdot t\|$ for all $t \geqslant 0$. Now replace $e_k \cdot t_k$ by a subsequence such that $\|e_k \cdot t_k\|^{-1} e_k \cdot t_k$ converges to $e^* \in E$, $\|e^*\| = 1$. If the sequence t_k is bounded then a subsequence converges to some $t^* \geqslant 0$ and we obtain $\|e^* \cdot (-t^*)\| = \lim_{k \to \infty} \|e_k\| / \|e_k \cdot t_k\| = 0$. This implies $e^* \cdot (-t^*) \in Z$ and hence $e^* \in Z$ contradicting $\|e^*\| = 1$. But if the sequence t_k is unbounded, then we obtain $\|e^* \cdot t\| \leqslant 1$ for all $t \in \mathbf{R}$ which again implies $e^* \in Z$ (Lemma 2.4). We conclude that (2.8) holds with $\varepsilon = 0$. In connection with Lemma 2.4 this implies that E^s is closed.

Now we claim that for every $\alpha > 0$ there exists a $T > 0$ such that $\|e \cdot t\| \leqslant \alpha \|e\|$ for all $e \in E^s$ and all $t \geqslant T$. Otherwise, there would exist a sequence $e_k \in E^s$ such that $\|e_k \cdot t_k\| \geqslant \alpha \|e_k\|$ for some $t_k \geqslant k$. But this would imply that any limit point $e^* \in E^s$ of $\|e_k \cdot t_k\|^{-1} e_k \cdot t_k$ satisfies $\|e^*\| = 1$ and

$$\|e^* \cdot t\| = \lim_{k \to \infty} \frac{\|e_k \cdot (t_k + t)\|}{\|e_k \cdot t_k\|} \leqslant \lim_{k \to \infty} \frac{\|e_k \cdot (t_k + t)\|}{\alpha \cdot \|e_k\|} \leqslant \frac{K}{\alpha}$$

for all $t \in \mathbf{R}$. This contradiction proves the above claim.

Let us now choose $\alpha < 1$ and $T > 0$ such that $\|e \cdot t\| \leqslant \alpha \|e\|$ for all $e \in E^s$ and all $t \geqslant T$ and define $\varepsilon = -(\log \alpha)/T > 0$. Then $\alpha = \exp(-\varepsilon T)$ and hence the following inequality holds for $kT \leqslant t \leqslant (k+1)T$ and $e \in E^s$:

$$\|e \cdot t\| \leqslant K \|e \cdot kT\| \leqslant K \alpha^k \|e\| \leqslant K \alpha^{-1} \exp(-\varepsilon t) \|e\|.$$

This proves inequality (2.8).

The assertions on E^u are proved in the same way. □

In the following we will discuss the properties of the induced flow on the projective bundle PE (Lemma 2.1(ii)).

For this purpose let us first recall that a compact invariant set A in a compact metric flow M is said to be an attractor if it admits a neighborhood U such that $\omega(U) = A$. In this case $A^* = \{x \in M \mid \omega(x) \cap A = \varnothing\}$ is its complementary repeller. This means that there exists a neighborhood U^* of A^* with $\omega^*(U^*) = A^*$ and that $\omega^*(x) \subset A^*$, $\omega(x) \subset A$ for all $x \in M \setminus (A \cup A^*)$ [**4**, **10**].

LEMMA 2.6. *Let $(x, t) \mapsto x \cdot t$ be a flow on the compact metric space M. Then a pair A, A^* of disjoint compact invariant sets in an attractor-repeller pair in X if and only if*

(2.10) $x \in M \setminus A^* \Rightarrow x \cdot [0, \infty) \cap U \neq \varnothing$ *for every neighborhood U of A and*

(2.11) $x \in M \setminus A \Rightarrow x \cdot (-\infty, 0] \cap U^* \neq \varnothing$ *for every neighborhood U^* of A^*.*

PROOF. The necessity of the condition is obvious. Conversely, suppose that (2.10) holds for every neighborhood U of A and (2.11) for every neighborhood U^* of A^*. Let W be a compact neighborhood of A with $W \cap A^* = \varnothing$. Then (2.11) shows that $x \cdot (-\infty, 0] \not\subset W$ for all $x \in W \setminus A$. By a lemma due to Conley [**4**, II, 5.1, D; **10**, Lemma 3.1], this implies that A is an attractor. Moreover, it follows from (2.10) that

$\omega(x) \cap A \neq \varnothing$ for all $x \in M \setminus A^*$. Hence $A^* = \{x \in M | \omega(x) \cap A = \varnothing\}$ is the complementary repeller of A in M. $\square$

Now we are in the position to prove the following characterization of flows in E for which Z is an isolated invariant set.

THEOREM 2.7. *The zero section Z is an isolated invariant set in E if and only if there exist closed subsets E^s and E^u of E which intersect each fiber in a linear subspace and satisfy*

(i) *PE^u is an attractor in PE and PE^s is its complementary repeller,*

(ii) *$\varnothing \neq \omega(e) \subset Z$ for every $e \in E^s$ and $\varnothing \neq \omega^*(e) \subset Z$ for every $e \in E^u$.*

If these conditions are satisfied then E^s and E^u are given by (2.5).

PROOF. Let us first assume that Z is an isolated invariant set in E and let E^s, E^u be given by (2.5). Then $E_1^u = \{e \in E^u | \|e\| = 1\}$ is a compact set with $E_1^u \cap E^s = \varnothing$. See Figure 3.

We claim that for every neighborhood W of E_1^u there exists an $\varepsilon > 0$ such that for every $e \in E \setminus E^s$ with $\|e\| \leqslant \varepsilon$ we have

$$t(e) = \sup\{t > 0 | \|e \cdot s\| < 1, 0 \leqslant s \leqslant t\} < \infty$$

and $e \cdot t(e) \in W$. First note that $t(e)$ has to be finite by Lemma 2.4(ii). Now suppose that there exists a sequence $e_k \in E \setminus E^s$ such that $\|e_k\|$ tends to zero and $e_k \cdot t(e_k) \notin W$. Then the sequence $t(e_k)$ tends to infinity. Otherwise there would exist a subsequence, still denoted by e_k, such that e_k converges to $e^* \in Z$ and $t(e_k)$ converges to t^*, leading to the contradiction $\|e^* \cdot t^*\| = 1$. Now let $e^* \in E$ be a limit point of $e_k \cdot t(e_k)$. Then $e^* \notin \operatorname{int} W$. But on the other hand $\|e^*\| = 1$ and $\|e^* \cdot t\| \leqslant 1$ for all $t \leqslant 0$ which implies $e^* \in E_1^u$. This contradiction proves the claim.

The above claim shows that $Pe \cdot [0, \infty) \cap PW \neq \varnothing$ for every neighborhood PW of PE^u and every $Pe \in PE \setminus PE^s$. By duality, we obtain $Pe \cdot (-\infty, 0] \cap PW^* \neq \varnothing$ for every neighborhood PW^* of PE^s and every $Pe \in PE \setminus PE^u$. Hence it follows from Lemma 2.6 that PE^u is an attractor in PE and PE^s is its complementary repeller.

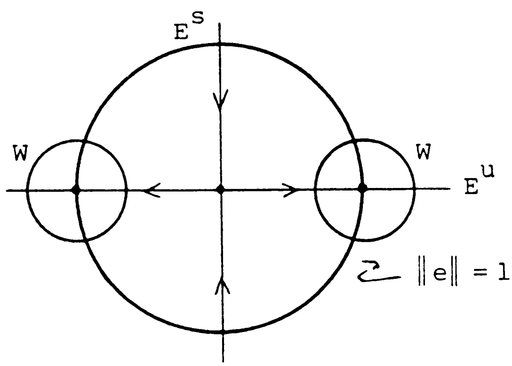

FIGURE 3

Conversely, suppose that there exist closed subsets E^s and E^u of E which intersect each fiber in a linear subspace and satisfy the conditions (i) and (ii). Then $E^s \cap E^u = Z$ since $PE^s \cap PE^u = \varnothing$. We prove in four steps that Z is an isolated invariant set in E.

Step 1. $\inf\{\|e \cdot t\| \mid t \geqslant 0\} > 0\ \forall e \in E^u \setminus Z$.

Suppose that $\inf\{\|e \cdot t\| \mid t \geqslant 0\} = 0$ for some $e \in E^u \setminus Z$ and choose a sequence t_k tending to infinity such that $\|e \cdot t_k\|$ tends to zero and $\|e \cdot t\| \geqslant \|e \cdot t_k\|$ for $0 \leqslant t \leqslant t_k$. For example, t_k can be chosen to be the largest time at which the function $\|e \cdot t\|$ achieves its minimum on the interval $0 \leqslant t \leqslant k$. Now choose a subsequence, still denoted by t_k, such that $\|e \cdot t_k\|^{-1} e \cdot t_k$ converges to $e^* \in E^u$. Then $\|e^*\| = 1$ and $\|e^* \cdot t\| = \lim_{k \to \infty} \|e \cdot t_k\|^{-1} \|e \cdot (t_k + t)\| \geqslant 1$ for every $t \leqslant 0$. But this implies $\omega^*(e^*) \cap Z = \varnothing$, contradicting condition (ii).

Step 2. $\sup\{\|e \cdot t\| \mid t \geqslant 0\} = \infty\ \forall e \in E^u \setminus Z$.

Suppose that there exists an $e \in E^u \setminus Z$ such that $\varepsilon \leqslant \|e \cdot t\| \leqslant \varepsilon^{-1}$ for all $t \geqslant 0$ and some $\varepsilon > 0$. Then $\varnothing \neq \omega(e) \subset E^u$ and $\varepsilon \leqslant \|e^* \cdot t\| \leqslant \varepsilon^{-1}$ for every $e^* \in \omega(e)$ and every $t \in \mathbf{R}$. Again, this contradicts condition (ii).

Step 3. E^s and E^u are given by (2.5).

Let $e \in E \setminus E^s$ and suppose that $\varnothing \neq \omega(e) \subset Z$. Choose a sequence t_k tending to infinity such that $\|e \cdot t\| \leqslant \|e \cdot t_k\|$ for all $t \geqslant t_k$ and such that $\|e \cdot t_k\|^{-1} e \cdot t_k$ converges to $e^* \in E$, $\|e^*\| = 1$. Then $Pe^* = \lim_{k \to \infty} Pe \cdot t_k \in PE^u$ and hence $e^* \in E^u$. Furthermore, it follows from the choice of t_k that $\|e^* \cdot t\| \leqslant 1$ for all $t \geqslant 0$. This contradicts Step 2.

Step 4. Z is an isolated invariant set.

If $\|e \cdot t\| \leqslant 1$ for all $t \in \mathbf{R}$ and $e \notin Z$, then it follows from Step 2 that $e \notin E^u$. Since $\omega^*(e) \neq \varnothing$ it follows from Step 3 that $\omega^*(e) \not\subset Z$. Choose $e^* \in \omega^*(e) \setminus Z$. Then $Pe^* \in \omega^*(Pe) \subset PE^s$ and hence $e^* \in E^s$. Furthermore $\|e^* \cdot t\| \leqslant 1$ for all $t \in \mathbf{R}$, contradicting the dual result of Step 2. This proves Step 4 and the statement of the theorem. □

LEMMA 2.8. *Suppose that Z is an isolated invariant set in E and let E^s and E^u be defined by* (2.5). *Then there exists an* $\varepsilon > 0$ *such that*

$$\lim_{t \to \infty} \exp(-\varepsilon t) \|e \cdot t\| = \infty \quad \forall e \in E \setminus E^s, \tag{2.12}$$

$$\lim_{t \to -\infty} \exp(\varepsilon t) \|e \cdot t\| = \infty \quad \forall e \in E \setminus E^u. \tag{2.13}$$

PROOF. We show first that $\|e \cdot t\|$ tends to infinity as t goes to infinity for $e \in E \setminus E^s$. Otherwise there would exist a sequence t_k (tending to infinity) such that $\|e \cdot t_k\|$ is bounded and $\|e \cdot \tau_k\|$ tends to infinity where $\tau_k \in [t_k, t_{k+1}]$ is chosen such that

$$\|e \cdot \tau_k\| \geqslant \|e \cdot t\|, \qquad t_k \leqslant t \leqslant t_{k+1}.$$

Now let us choose a subsequence k_n such that $\|e \cdot \tau_{k_n}\|^{-1} e \cdot \tau_{k_n}$ converges to $e^* \in E$, $\|e^*\| = 1$. If the sequence $\tau_{k_n} - t_{k_n}$ does not go to infinity then it has a limit point $t^* \geqslant 0$ leading to the contradiction

$$\|e^* \cdot (-t^*)\| = \lim_{n \to \infty} \|e \cdot t_{k_n}\| / \|e \cdot \tau_{k_n}\| = 0$$

for a suitably chosen subsequence. In the same manner one can show that the sequence $t_{k_n+1} - \tau_{k_n}$ goes to infinity. But this implies $\|e^* \cdot t\| \leqslant 1$ for all $t \in \mathbf{R}$, and it follows from Lemma 2.4 that $e^* \in Z$, once again contradicting $\|e^*\| = 1$. Thus we have established statement (2.12) for $\varepsilon = 0$.

Now it follows from Lemma 2.5 that E^s and E^u satisfy conditions (i) and (ii) of Theorem 2.7 with respect to the perturbed flow $(e, t) \mapsto \exp(-\varepsilon t)e \cdot t$ if $\varepsilon > 0$ is sufficiently small. This proves (2.12) for some $\varepsilon > 0$.

Statement (2.13) follows by duality. $\square$

The statement of Theorem 2.7 is illustrated by Figure 4, a diagram of a flow on a vector bundle over the one point space (antipodal points are to be identified).

The next result shows that this situation is in a sense typical for attractors in PE. It has first been established by Selgrade [**11**]. We present a simplified proof.

PROPOSITION 2.9. *Let A be an attractor in PE and let $e, e' \in E \setminus Z$ be given such that $\pi(e) = \pi(e')$ and $Pe \in A$, $Pe' \notin A$. Then*

$$\lim_{t \to -\infty} \|e \cdot t\| / \|e' \cdot t\| = 0. \tag{2.14}$$

Furthermore A intersects each fiber in a projective linear subspace.

PROOF. Let us introduce the two-dimensional subspace $L = \{ce + c'e' \mid c, c' \in \mathbf{R}\}$ in E and suppose that $Pe \in A$ is a boundary point of $A \cap PL$ relative to PL. Moreover choose $\varepsilon < d(A, A^*)$ and note that, by Lemma A3, there exists a $\delta > 0$ such that the implication

$$\langle e_0, e_1 \rangle^2 / \|e_0\|^2 \|e_1\|^2 \geqslant 1 - \delta \Rightarrow d(Pe_0, Pe_1) \leqslant \varepsilon$$

holds for all $e_0, e_1 \in E \setminus Z$. Now suppose that (2.14) would not hold. Then there would exist a sequence t_k tending to $-\infty$ and a constant $K > 0$ such that

$$\|e' \cdot t_k\| / \|e \cdot t_k\| \leqslant K, \qquad k \in \mathbf{N}.$$

For $c \in \mathbf{R}$ with $|c|$ sufficiently small this implies that

$$\begin{aligned} &\frac{\langle ce' \cdot t_k + e \cdot t_k, e \cdot t_k \rangle^2}{\|ce' \cdot t_k + e \cdot t_k\|^2 \|e \cdot t_k\|^2} \\ &\quad = \frac{c^2 \langle e' \cdot t_k, e \cdot t_k \rangle^2 + 2c \|e \cdot t_k\|^2 \langle e' \cdot t_k, e \cdot t_k \rangle + \|e \cdot t_k\|^4}{c^2 \|e' \cdot t_k\|^2 \|e \cdot t_k\|^2 + 2c \|e \cdot t_k\|^2 \langle e' \cdot t_k, e \cdot t_k \rangle + \|e \cdot t_k\|^4} \geqslant 1 - \delta \end{aligned}$$

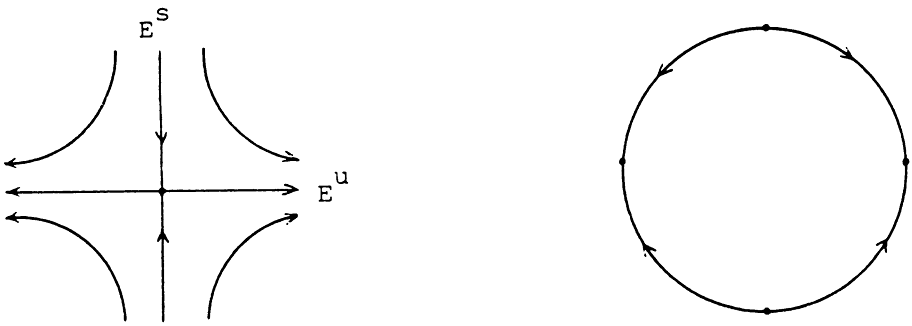

FIGURE 4

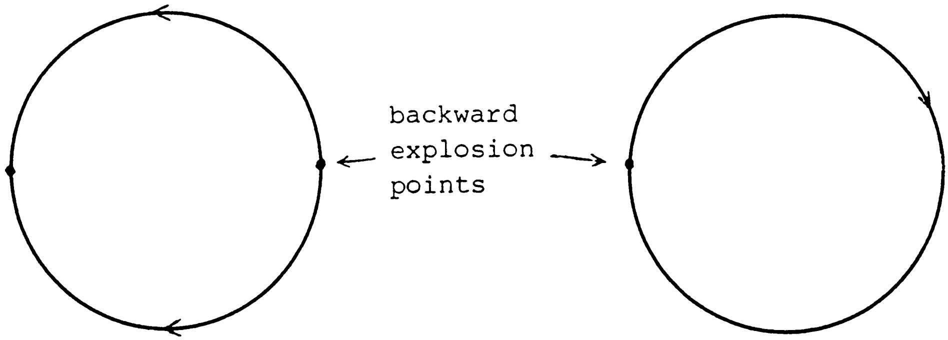

FIGURE 5

for all $k \in \mathbf{N}$. But this would imply that $d(P(ce' + e) \cdot t_k, A) \leqslant \varepsilon$ for all $k \in \mathbf{N}$, hence $\omega^*(P(ce' + e)) \not\subset A^*$ and therefore $P(ce' + e) \in A$ for $|c|$ sufficiently small. This would contradict the fact that Pe is a boundary point of $A \cap PL$ in PL and thus we have established (2.14) in this case.

It remains to show that $A \cap PL$ consists of a single point. For this purpose note that any point in $PL \setminus \{Pe\}$ is given by $P(e' + ce)$ for some $c \in \mathbf{R}$. It follows from (2.14) that

$$\lim_{t \to -\infty} \frac{\langle e' \cdot t + ce \cdot t, e' \cdot t \rangle^2}{\|e' \cdot t + ce \cdot t\|^2 \|e' \cdot t\|^2}$$

$$= \lim_{t \to -\infty} \frac{\|e' \cdot t\|^4 + 2c\|e' \cdot t\|^2 \langle e \cdot t, e' \cdot t \rangle + c^2 \langle e \cdot t, e' \cdot t \rangle^2}{\|e' \cdot t\|^4 + 2c\|e' \cdot t\|^2 \langle e \cdot t, e' \cdot t \rangle + c^2 \|e \cdot t\|^2 \|e' \cdot t\|^2} = 1.$$

By Lemma A3, this implies

$$\lim_{t \to -\infty} d\big(P(e' + ce) \cdot t, Pe' \cdot t\big) = 0$$

and hence $P(e' + ce) \notin A$. Therefore $A \cap PL$ consists of a single point.

In particular, we have shown that, for any two-dimensional subspace L in E, $A \cap PL$ is either empty or consists of a single point or equals PL. This implies that A intersects each fiber in a projective linear subspace. □

The proof of the previous proposition shows that, if A is an attractor in PE and L is a two-dimensional subspace in E with $A \cap PL = \{Pe\}$, then Pe is a *backward explosion point* in PL. This means that the diameter of $PW \cdot t$ tends to zero as t goes to $-\infty$ for any closed subset $PW \subset PL \setminus \{Pe\}$. Two typical examples are illustrated in Figure 5. The concept of a backward (forward) explosion point has been introduced by Charles Conley. It has also been discussed by Selgrade [**11**].

Assume that M is a compact metric flow. An (ε, T)-chain from $x \in M$ to $y \in M$ consists of a sequence $x_0, \ldots, x_k$ in M and a sequence $t_0, \ldots, t_{k-1}$ in $\mathbf{R}$ such that $x_0 = x$, $x_k = y$, $t_j \geqslant T$ and $d(x_j \cdot t_j, x_{j+1}) \leqslant \varepsilon$ for $j = 0, \ldots, k-1$. Let $X \subset M$ be any subset. Then $\Omega(X)$ denotes the set of all points $y \in M$ such that for every $\varepsilon > 0$ and every $T > 0$ there exists an (ε, T)-chain from some point in X to y. Likewise, $\Omega^*(X)$ denotes the set of all points $y \in M$ such that for every $\varepsilon > 0$ and every $T > 0$ there exists an (ε, T)-chain from y to some point in X. In [**4**, II, 6.1, C] C.

Conley has shown that

$$\begin{aligned}\Omega(X) &= \bigcap\{A \subset M \mid A \text{ is an attractor, } \omega(X) \subset A\},\\ \Omega^*(X) &= \bigcap\{A^* \subset M \mid A^* \text{ is a repeller, } \omega^*(X) \subset A^*\}.\end{aligned} \tag{2.15}$$

The flow on M is said to be *chain recurrent* if $x \in \Omega(x)$ for all $x \in M$ or equivalently $M = A \cup A^*$ for every attractor-repeller pair A, A^* in M. The flow on M is said to be *chain transitive* if $y \in \Omega(x)$ for all $x, y \in M$ or equivalently $A = M$ and $A = \varnothing$ are the only attractors in M. Note that the flow on M is chain transitive if and only if it is chain recurrent and M is connected.

With these preparations we are in the position to state a very useful lemma which is due to Selgrade [**11**]. For the sake of completeness we include the proof.

LEMMA 2.10. *If $L_p \subset E_p$ is a linear subspace and $q \in \Omega(p)$, then*

$$L_q = \{e \in E_q \mid e \notin Z \Rightarrow Pe \in \Omega(PL_p)\}$$

is a linear subspace of E_q and $\dim L_q \geqslant \dim L_p$.

PROOF. Since $\Omega(PL_p)$ is the intersection of attractors, it intersects each fiber in a projective linear subspace (Proposition 2.9). Therefore, L_q is a linear subspace of E_q.

Now let us define the set $L_q(\varepsilon, T)$ to be the closure of all points $e \in E_q$ such that there exists an (ε, T)-chain from some point in L_p to $e \cdot (-T)$. Then for every $e \in L_q(\varepsilon, T) \setminus Z$ there exists an (ε, T)-chain from some point in PL_p to Pe. This implies $\bigcap_{n \in \mathbf{N}} L_q(1/n, n) \subset L_q$. Now the following construction shows that $L_q(\varepsilon, T)$ contains a linear subspace of dimension at least that of L_p. First note that, by (2.15), $\Omega(p)$ is an invariant set and hence $q \cdot (-T) \in \Omega(p)$. Secondly, choose $\delta > 0$ such that $d_\alpha(e, e') \leqslant \delta$ implies $d(e, e') \leqslant \varepsilon$ whenever $e, e' \in U_\alpha$. Then there exists a (δ, T)-chain $p_0, \ldots, p_k$, $t_0, \ldots, t_{k-1}$ from p to $q \cdot (-T)$ such that $p_{j-1} \cdot t_{j-1}$ and p_j lie in the same set U_{α_j} for $j = 1, \ldots, k$. Given any $e \in L_p$ define the sequence $e_0, \ldots, e_k$ in E such that $e_0 = e$, $\pi(e_j) = p_j$ and that $\varphi_{\alpha_j}(e_j)$ coincides with $\varphi_{\alpha_j}(e_{j-1} \cdot t_{j-1})$ in the V-component for $j = 1, \ldots, k$. Then it follows from the choice of δ that this sequence defines an (ε, T)-chain from $e \in L_p$ to $e_k \in \pi^{-1}(q \cdot (-T))$, and therefore $e' = e_k \cdot T \in L_q(\varepsilon, T)$. Furthermore, the points $e' \in \pi^{-1}(q)$ obtained this way form a linear subspace of the same dimension as L_p.

We conclude that the set $\mathfrak{L}_q(\varepsilon, T)$ of m-dimensional linear subspaces of E_q contained in $L_q(\varepsilon, T)$ is nonempty for $m = \dim L_p$. Therefore the intersection of the decreasing sequence $\mathfrak{L}_q(1/n, n)$ of nonempty compact sets is nonempty. This proves the statement of the lemma. $\square$

Note that the statement of Lemma 2.10 remains valid if $\Omega(p)$ and $\Omega(PL_p)$ are replaced by $\omega(p)$ and $\omega(PL_p)$, respectively. In that case the proof becomes much simpler.

The main difference between the previous results of this section (Lemma 2.5, Theorem 2.7 and Lemma 2.8) and those by Selgrade [**11**] is that we do not assume the flow on S to be chain transitive. Using Lemma 2.10 we shall recover some of Selgrade's results.

COROLLARY 2.11. *Let A be an attractor in PE. Then*

$$A_p = \{ e \in E_p | e \notin Z \Rightarrow Pe \in A \}$$

is a linear subspace of E_p for every $p \in S$ and $\dim A_q \geqslant \dim A_p$, $q \in \Omega(p)$.

PROOF. It follows from Proposition 2.9 that A_p is a linear subspace of E_p. Furthermore, since A is an attractor containing $\omega(PA_p)$, it follows from (2.15) that $\Omega(PA_p) \subset A$ and hence $\{e \in E_q | e \notin Z \Rightarrow Pe \in \Omega(PA_p)\} \subset A_q$. Therefore we obtain from Lemma 2.10 that $\dim A_q \geqslant \dim A_p$. □

COROLLARY 2.12. *If Z is an isolated invariant set in E and E^u is a subbundle then so is E^s and*

$$E = E^s \oplus E^u. \tag{2.16}$$

PROOF. Suppose that the dimension of $E_p^s = E^s \cap E_p$ is less than $\dim E - \dim E^u$ for some $p \in S$ and define

$$L_p = \{ e \in E \,|\, \pi(e) = p,\ e \perp E_p^s \}.$$

Then $\dim L_p > \dim E^u$. Furthermore, $PL_p \cap PE^s = \varnothing$ and therefore $\omega(PL_p) \subset PE^u$ (Theorem 2.7). By (2.15) this implies that $\Omega(PL_p) \subset PE^u$ and hence, by Lemma 2.10, $\dim E^u \geqslant \dim L_p$ which is a contradiction. Therefore $\dim E_p^s = \dim E - \dim E^u$ for all $p \in S$ and thus the statement of the corollary follows from Lemma A2. □

THEOREM 2.13. *We consider a continuous linear flow on the vector bundle π: $E \to S$ over a compact metric space S. Then the zero-section, $Z \subset E$, is an invariant set and the stable and unstable invariant sets of Z are defined by*

$$E^s = \{ e \in E \,|\, \varnothing \neq \omega(e) \subset Z \}, \quad E^u = \{ e \in E \,|\, \varnothing \neq \omega^*(e) \subset Z \}.$$

If the flow on S is chain transitive, then $Z \subset E$ is an isolated invariant set if and only if

(i) *E^s and E^u are subbundles of E with $E = E^s \oplus E^u$ and,*

(ii) *there are positive constants K and ε such that*

$$\|e \cdot t\| \leqslant K \exp(-\varepsilon t) \|e\| \quad \text{if } e \in E^s \text{ and } t \geqslant 0,$$
$$\|e \cdot (-t)\| \leqslant K \exp(-\varepsilon t) \|e\| \quad \text{if } e \in E^u \text{ and } t \geqslant 0.$$

PROOF. If Z is an isolated invariant set in E then E^u is closed (Lemma 2.5) and PE^u is an attractor in PE (Theorem 2.7). Since the flow in S is chain transitive this implies that the dimension of $E_p^u = E^u \cap E_p$ is independent of $p \in S$ (Corollary 2.11). Therefore E^u is a subbundle of E (Lemma A2) and so is E^s and (2.16) holds (Corollary 2.12). Conversely, it is a trivial consequence of (2.8), (2.9) and (2.16) that every bounded orbit in E lies in Z. □

As a special case of Corollary 2.13 consider the almost periodic differential equation

$$\dot{x} = A(\theta)x, \qquad \dot{\theta}_j = \omega_j, \qquad j = 1, \ldots, m, \tag{2.17}$$

on $T^m \times \mathbf{R}^n$ where $x \in \mathbf{R}^n$ and $\theta \in \mathbf{R}^m$ represents the component in $T^m = \mathbf{R}^m/\mathbf{Z}^m$. We assume that $A(\theta + e_j) = A(\theta)$ for $j = 1, \ldots, m$ where $e_j \in \mathbf{R}^m$ denotes the jth unit vector. Then the flow on the base T^m is chain transitive and hence (2.17) defines a hyperbolic flow if and only if every solution $(x(t), \theta(t))$, $t \in \mathbf{R}$, of (2.17) with a bounded x-component satisfies $x(t) \equiv 0$. This corresponds to a finitely generated frequency module for the almost periodic matrix function, $F(t) = A(\omega_1 t, \ldots, \omega_m t)$. If $F(t)$ is an arbitrary almost periodic matrix then the differential equation $\dot{x}(t) = F(t)x(t)$ can also be formulated in the framework of this section but the base space becomes more complicated. The interested reader is referred to Johnson and Moser [**8**].

As a side remark we point out that the spectrum of a flow on E may be defined by

$$\sigma(E) = \{\lambda \in \mathbf{R} \mid Z \text{ is not an isolated invariant set} \\ \text{for the flow } (e,t) \mapsto \exp(-\lambda t) e \cdot t\}.$$

The spectrum has been discussed in some detail by Sacker and Sell [**13**] and by Selgrade [**11**]. It depends on E as well as on the flow but there should not arise any confusion since in this section we consider only one flow on E.

It follows from Lemma 2.2 that $\sigma(E)$ is bounded and from Lemma 2.8 that $\sigma(E)$ is closed. If S consists of a single point then $\sigma(E)$ corresponds to the real parts of the eigenvalues of the induced linear flow on V. Of course, Z is an isolated invariant set if and only if $0 \notin \sigma(E)$. Furthermore, the spectrum of the invariant subsets E^u and E^s can be defined analogously and we obtain $\sigma(E) = \sigma(E^s) \cup \sigma(E^u)$, $\sigma(E^s) \subset (-\infty, 0)$ and $\sigma(E^u) \subset (0, \infty)$. In general, the spectrum of an attractor in PE need not be disjoint from the spectrum of its complementary repeller. An example can be constructed as follows with a chain recurrent flow in the base space.

EXAMPLE 2.14. Consider the differential equation

$$\dot{x} = \sin^2 2\pi x, \quad \dot{y} = (\cos 2\pi x) y, \quad \dot{z} = (2 + \cos 2\pi x) z \tag{2.18}$$

as a flow on the vector bundle $E = S^1 \times \mathbf{R}^2$, $S^1 = \mathbf{R}/\mathbf{Z}$, where $x \in \mathbf{R}$ represents the S^1-component. See Figure 6. Define the subbundles $E^y = S^1 \times \mathbf{R} \times 0$ and $E^z = S^1 \times 0 \times \mathbf{R}$. Then the projectivized equation

$$\dot{x} = \sin^2 2\pi x, \quad \dot{\eta} = -2\zeta^2\eta, \quad \dot{\zeta} = 2\eta^2\zeta, \quad \eta^2 + \zeta^2 = 1, \tag{2.19}$$

shows that PE^z is an attractor in PE and PE^y is its complementary repeller. Furthermore, it follows from (2.18) that $\sigma(E^y) = [-1, 1]$ and $\sigma(E^z) = [1, 3]$.

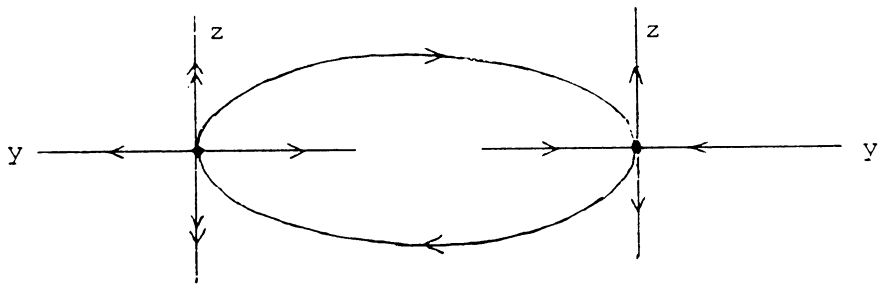

FIGURE 6

At the end of this section we indicate how similar ideas can be applied to Hamiltonian systems on $\mathbf{R}^{2n}$. Let the function $H\colon \mathbf{R}^{2n} \to \mathbf{R}$ be twice differentiable and denote its arguments by $z = (x, y) \in \mathbf{R}^{2n}$ where $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^n$. Consider the differential equation

$$\dot{z} = JH_z(z), \tag{2.20}$$

$$\dot{\zeta} = JH_{zz}(z)\zeta \tag{2.21}$$

on $\mathbf{R}^{4n}$ where $\zeta = (\xi, \eta) \in \mathbf{R}^{2n}$ is to be understood as a tangent vector and $J \in \mathbf{R}^{2n \times 2n}$ denotes the symplectic matrix

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

A Lagrangian plane $L \subset \mathbf{R}^{2n}$ is an n-dimensional subspace with the property $\langle \zeta_0, J\zeta_1 \rangle = 0$ for all $\zeta_0, \zeta_1 \in L$. Note that any subspace L with this property is at most of dimension n since $J\zeta$ is orthogonal to L for every $\zeta \in L$. Furthermore, the flow defined by (2.20), (2.21) maps Lagrangian planes into itself since the expression $\langle \zeta_0(t), J\zeta_1(t) \rangle$ is constant for any two solutions $\zeta_0(t)$, $\zeta_1(t)$ of (2.21) over the same solution $z(t)$ of (2.20). Now let $\mathfrak{L} \subset \mathrm{Gr}_n(\mathbf{R}^{2n})$ denote the space of Lagrangian planes. Then the differential equation (2.20), (2.21) induces a flow on the bundle $\mathbf{R}^{2n} \times \mathfrak{L}$ and one can study how isolated invariant sets, attractors, repellers, explosion points in this flow can be characterized. Of course, the same question can be posed in the context of a Hamiltonian system on an arbitrary symplectic manifold. In particular, one might consider the case that the function H is periodic with period 1 in all variables and study the differential equation (2.20) on the torus $T^{2n} = \mathbf{R}^{2n}/\mathbf{Z}^{2n}$.

3. Hyperbolic invariant sets. Let M be an n-dimensional, smooth, compact manifold without boundary and let $X\colon M \to TM$ be a smooth vector field. Whenever necessary, we will assume that M is equipped with a Riemannian metric.

Let $\pi\colon TM \to M$ and $\pi^*\colon T^*M \to M$ be the canonical projection maps and let $\langle v^*, v \rangle$ stand for the duality pairing between $v^* \in T_p^*M$ and $v \in T_pM$. For any subset $N \subset M$ we denote by T_NM and T_N^*M, respectively, the (co)tangent bundle of M restricted to N. The corresponding projectivized bundles will be denoted by P_NM and P_N^*M. In particular, $PM = P_MM$ can be obtained from the sphere bundle

$$\Sigma M = \{ v \in TM \,|\, \langle v, v \rangle = 1 \}$$

by identifying v with $-v$. Note that in this case the inner product $\langle v_0, v_1 \rangle$ for $v_0, v_1 \in T_pM$ is induced by the Riemannian metric.

We will always denote the local coordinates of $v \in T_pM$ and $v^* \in T_p^*M$ by $(x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n$ and $(x, \xi^*) \in \mathbf{R}^n \times \mathbf{R}^{n*}$, respectively, where $x \in \mathbf{R}^n$ denotes the local coordinates of $p = \pi(v) = \pi^*(v^*) \in M$. A tangent vector $w \in T_{v^*}T^*M$ will in local coordinates be represented by $\{x, \xi^*, y, \eta^*\} \in \mathbf{R}^n \times \mathbf{R}^{n*} \times \mathbf{R}^n \times \mathbf{R}^{n*}$. Finally, let $(x, f(x))$ denote the local coordinates of $X(p) \in T_pM$.

On M, TM and T^*M, respectively, we consider the differential equations

$$\dot{x} = f(x), \tag{3.1}$$

$$\dot{x} = f(x), \qquad \dot{\xi} = df(x)\xi, \tag{3.2}$$

$$\dot{x} = f(x), \qquad \dot{\xi}^* = -\xi^* df(x), \tag{3.3}$$

and denote the corresponding flows by $p \cdot t$, $v \cdot t$ and $v^* \cdot t$ for $p \in M$, $v \in TM$, $v^* \in T^*M$ and $t \in \mathbf{R}$. The fundamental duality relation between (3.2) and (3.3) can be expressed by the identity

$$\langle v^* \cdot t, v \cdot t \rangle = \langle v^*, v \rangle \tag{3.4}$$

for all $v \in TM$, $v^* \in T^*M$ with $\pi(v) = \pi^*(v^*)$ and all $t \in \mathbf{R}$.

Equation (3.3) can be understood as a Hamiltonian system in the following way. There exists a unique 1-form λ: $TT^*M \to \mathbf{R}$ such that $\alpha^*\lambda = \alpha$ for every 1-form α: $M \to T^*M$. The 2-form $\omega = d\lambda$ defines the standard symplectic structure on T^*M. Given any Hamiltonian function h: $T^*M \to \mathbf{R}$ a Hamiltonian vector field X_h: $T^*M \to TT^*M$ can be defined by

$$\omega\big(w, X_h(v^*)\big) = Th(v^*)w, \qquad w \in T_{v^*}T^*M. \tag{3.5}$$

The corresponding flow on T^*M has the property that the energy h remains constant along its orbits.

Note that, in local coordinates, λ and ω are given by $\lambda(w) = \xi^*y$ and $\omega(w_0, w_1) = \eta_0^* y_1 - \eta_1^* y_0$. Therefore the right-hand side of (3.3) defines a Hamiltonian vector field on T^*M. In this case the Hamiltonian function h: $T^*M \to \mathbf{R}$ is given by

$$h(v^*) = \langle v^*, X(p) \rangle, \qquad v^* \in T_p^*M, \tag{3.6}$$

or in local coordinates $h(v^*) = \xi^* f(x)$.

We will now introduce the topological concept of a weakly hyperbolic invariant set.

DEFINITION 3.1. Let $S \subset M$ be a compact invariant set (not necessarily isolated) and suppose that $X(p) \neq 0$ for all $p \in S$. Let $E^0 \subset T_S M$ and $E_0^* \subset T_S^* M$ be defined by

$$E^0 = \{ cX(p) \,|\, p \in S, c \in \mathbf{R} \}, \quad E_0^* = \big\{ v^* \in T_S^*M \,|\, \langle v^*, X\big(\pi^*(v^*)\big) \rangle = 0 \big\} \tag{3.7}$$

and note that both subbundles are invariant under the respective flow. Then S is said to be *weakly hyperbolic for equation* (3.2) if the zero section in the quotient bundle $T_S M / E^0$ is an isolated invariant set. S is said to be *weakly hyperbolic for equation* (3.3) if the zero section in the subbundle E_0^* is an isolated invariant set.

Note that the projection of a vector $v \in T_p M$ on the orthogonal complement of $X(p)$ is given by $v - \|X(p)\|^{-2} \langle v, X(p) \rangle X(p)$ and the norm of this vector is the square root or $\|v\|^2 - \|X(p)\|^{-2} \langle v, X(p) \rangle^2$. Therefore a compact invariant set $S \subset M$ is weakly hyperbolic for equation (3.2) if and only if

$$E^0 = \Big\{ v \in T_S M \,|\, \exists K > 0 \; \forall t \in \mathbf{R} \; \|v \cdot t\|^2 - \|X(p \cdot t)\|^{-2} \langle v \cdot t, X(p \cdot t) \rangle^2 \leqslant K, \; p = \pi(v) \Big\}. \tag{3.8}$$

This implies that every bounded orbit in TS lies in E^0. But the following example shows that the latter condition is not enough to guarantee weak hyperbolicity.

EXAMPLE 3.2. Consider the differential equation

$$\dot{x} = \sin 2\pi y, \qquad \dot{\xi} = 2\pi(\cos 2\pi y)\eta$$
$$\dot{y} = 0, \qquad \dot{\eta} = 0$$

for $[x, y] \in T^2 = \mathbf{R}^2/\mathbf{Z}^2 = M$ and $(\xi, \eta) \in \mathbf{R}^2$ and define $S = \{[x, y] \mid y = 1/8\}$. Then $\sin 2\pi y = \cos 2\pi y = 1/\sqrt{2}$ for $[x, y] \in S$ and therefore the bounded orbits in $T_S M = S \times \mathbf{R}^2$ lie in $E^0 = S \times \mathbf{R} \times \{0\}$. But the induced flow on TS/E^0 is constant so that the zero section is not an isolated invariant set.

Recall from [**4**] that a *Morse decomposition* of a compact flow M is an ordered collection $\Lambda_1, \ldots, \Lambda_m$ of isolated compact invariant sets such that for every $x \in M \setminus \bigcup \Lambda_j$ there exist indices $i < j$ with $\omega(x) \subset \Lambda_i$, $\omega^*(x) \subset \Lambda_j$. Equivalently, the sets

$$A_j = \left\{ x \in M \mid \omega^*(x) \subset \Lambda_1 \cup \cdots \cup \Lambda_j \right\}, \qquad j = 0, \ldots, m,$$

define a filtration of attractors in M with $A_0 = \varnothing$, $A_m = M$ and $A_{j-1} \subset A_j$ for $j = 1, \ldots, m$. Note that Λ_j is the complementary repeller of A_{j-1} in A_j for $j = 1, \ldots, m$.

THEOREM 3.3. *Let $S \subset M$ be a compact invariant set such that $X(p) \neq 0$ for all $p \in S$ and let $E^0 \subset T_S M$ be defined by* (3.7). *Then S is weakly hyperbolic for equation* (3.2) *if and only if there exist closed subsets E^s and E^u of $T_S M$ which intersect each fiber in a linear subspace and satisfy*

(i) *the ordered triple PE^u, PE^0, PE^s is a Morse decomposition of $P_S M$,*

(ii) $\lim_{t \to \infty} \|v \cdot t\| = 0\ \forall v \in E^s$, $\lim_{t \to -\infty} \|v \cdot t\| = 0\ \forall v \in E^u$.

If these conditions are satisfied then

$$E^s = \left\{ v \in T_S M \mid \lim_{t \to \infty} \|v \cdot t\| = 0 \right\}, \tag{3.9}$$

$$E^u = \left\{ v \in T_S M \mid \lim_{t \to -\infty} \|v \cdot t\| = 0 \right\} \tag{3.10}$$

and there exist constants $K > 0$, $\varepsilon > 0$ such that

$$\|v \cdot t\| \leqslant K \exp(-\varepsilon t) \|v\| \quad \forall v \in E^s,\ t \geqslant 0, \tag{3.11}$$

$$\|v \cdot t\| \leqslant K \exp(\varepsilon t) \|v\| \quad \forall v \in E^u,\ t \leqslant 0. \tag{3.12}$$

PROOF. Let us first assume that E^s and E^u are closed subsets of $T_S M$ which intersect each fiber in a linear subspace and satisfy the conditions (i) and (ii). Moreover, define the subsets

$$\overline{E}^s = \left\{ v \in T_S M \mid \|v\| \neq 0 \Rightarrow \omega(Pv) \cap PE^u = \varnothing \right\},$$
$$\overline{E}^u = \left\{ v \in T_S M \mid \|v\| \neq 0 \Rightarrow \omega^*(Pv) \cap PE^s = \varnothing \right\},$$

of $T_S M$. Then $P\overline{E}^s$ is the complementary repeller of PE^u in $P_S M$ and $P\overline{E}^u$ is the complementary attractor of PE^s. Hence it follows from Proposition 2.9 that $\overline{E}^s$ and $\overline{E}^u$ intersect each fiber in a linear subspace. Furthermore, we obtain from condition (i) that $\overline{E}^s \cap \overline{E}^u = E^0$ and that $P(\overline{E}^u/E^0)$ is an attractor in the projectivized quotient bundle $P(T_S M/E^0)$ and that $P(\overline{E}^s/E^0)$ is its complementary repeller.

Now we will show that

$$\bar{E}^s = E^s \oplus E^0, \qquad \bar{E}^u = E^u \oplus E^0. \tag{3.13}$$

Assume first that $v \in E^s \oplus E^0$ and $\|v\| \neq 0$. Then $Pv \cdot \mathbf{R} \subset P(E^s \oplus E^0)$ and hence $\omega(Pv) \cap PE^u = \varnothing$. This implies $v \in \bar{E}^s$ and thus $E^s \oplus E^0 \subset \bar{E}^s$. Now let $L \subset T_pM$ be the orthogonal complement of $E^s \cap T_pM$ in $\bar{E}^s \cap T_pM$. Then $PL \cap PE^s = \varnothing$ and hence $\omega(PL) \subset PE^0$. Now the remark following Lemma 2.10 shows that $\dim L \leqslant 1$. This proves $\bar{E}^s = E^s \oplus E^0$ and the second assertion in (3.13) follows by duality.

Equation (3.13) and condition (ii) imply that

$$\lim_{t \to \infty} \left[\|v \cdot t\|^2 - \|X(p \cdot t)\|^{-2} \langle v \cdot t, X(p \cdot t)\rangle^2 \right] = 0$$

for every $v \in \bar{E}^s \cap T_pM$ and an analogous statement holds for $v \in \bar{E}^u$. Hence the subsets $\bar{E}^s/E^0$ and $\bar{E}^u/E^0$ of the quotient bundle T_SM/E^0 satisfy all the requirements of Theorem 2.7. Therefore the zero section in T_SM/E^0 is an isolated invariant set and

$$\bar{E}^s = \left\{ v \in T_SM \,\middle|\, \lim_{t \to \infty} \left[\|v \cdot t\|^2 - \|X(\pi(v) \cdot t)\|^{-2} \langle v \cdot t, X(\pi(v) \cdot t)\rangle^2 \right] = 0 \right\}, \tag{3.14}$$

$$\bar{E}^u = \left\{ v \in T_SM \,\middle|\, \lim_{t \to -\infty} \left[\|v \cdot t\|^2 - \|X(\pi(v) \cdot t)\|^{-2} \langle v \cdot t, X(\pi(v) \cdot t)\rangle^2 \right] = 0 \right\}. \tag{3.15}$$

Furthermore, it follows from Lemma 2.5 that there exist constants $K_0 > 0$, $\varepsilon > 0$ such that

$$\begin{aligned} &\|v \cdot t\|^2 - \|X(p \cdot t)\|^{-2} \langle v \cdot t, X(p \cdot t)\rangle^2 \\ &\qquad \leqslant K_0^2 \exp(-2\varepsilon t) \left[\|v\|^2 - \|X(p)\|^{-2} \langle v, X(p)\rangle^2 \right], \\ &v \in \bar{E}^s \cap T_pM,\ t \geqslant 0, \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} &\|v \cdot t\|^2 - \|X(p \cdot t)\|^{-2} \langle v \cdot t, X(p \cdot t)\rangle^2 \\ &\qquad \leqslant K_0^2 \exp(2\varepsilon t) \left[\|v\|^2 - \|X(p)\|^{-2} \langle v, X(p)\rangle^2 \right], \\ &v \in \bar{E}^u \cap T_pM,\ t \leqslant 0. \end{aligned} \tag{3.17}$$

We will use these inequalities for proving (3.11) and (3.12). For this purpose note that

$$\alpha = \sup\left\{ \|X(p)\|^{-2} \langle v, X(p)\rangle^2 \,|\, v \in E^s,\ \|v\| = 1,\ p = \pi(v) \right\} < 1$$

and hence

$$(1 - \alpha)\|v\|^2 \leqslant \|v\|^2 - \|X(p)\|^{-2} \langle v, X(p)\rangle^2, \qquad p = \pi(v),$$

for all $v \in E^s$. Therefore (3.11) follows from (3.16) with $K = K_0/\sqrt{1 - \alpha}$. In the same way (3.12) follows from (3.17).

In order to establish (3.9), suppose that $\lim_{t\to\infty}\|v\cdot t\| = 0$. Then it follows from (3.14) that $v \in \bar{E}^s$. Now we obtain from (3.13) that $v = v_s + cX(p)$ for some $v_s \in E^s$ and some $c \in \mathbf{R}$. But this implies $\lim_{t\to\infty}|c|\,\|X(p\cdot t)\| = 0$ and hence $c = 0$. We conclude that $v = v_s \in E^s$ and therefore E^s is given by (3.9). Equation (3.10) can be established analogously.

It remains to prove that the conditions (i) and (ii) are necessary for weak hyperbolicity. For this purpose let us assume that the zero section in T_SM/E^0 is an isolated invariant set and let E^s, E^u and $\bar{E}^s$, $\bar{E}^u$ be defined by (3.9), (3.10) and (3.14), (3.15), respectively. Then it follows from (3.8) that $\bar{E}^s \cap \bar{E}^u = E^0$. Furthermore, it follows from Lemma 2.5 and Lemma 2.8, applied to the quotient bundle T_SM/E^0, that there exist constants $K_0 > 0$, $\varepsilon > 0$ such that (3.16) and (3.17) are satisfied as well as

$$\lim_{t\to\infty} \exp(-2\varepsilon t)\left[\|v\cdot t\|^2 - \|X(p\cdot t)\|^{-2}\langle v\cdot t, X(p\cdot t)\rangle^2\right] = \infty, \qquad v \in T_SM\setminus \bar{E}^s,\ p = \pi(v), \tag{3.18}$$

and

$$\lim_{t\to-\infty} \exp(2\varepsilon t)\left[\|v\cdot t\|^2 - \|X(p\cdot t)\|^{-2}\langle v\cdot t, X(p\cdot t)\rangle^2\right] = \infty, \qquad v \in T_SM\setminus \bar{E}^u,\ p = \pi(v). \tag{3.19}$$

We will now prove that the zero section in T_SM is an isolated invariant set with respect to the perturbed flow which maps $v \in T_SM$ and $t \in \mathbf{R}$ into $\exp(\mu t)v\cdot t$, provided that $0 < |\mu| < \varepsilon$. In fact, if $\exp(\mu t)\|v\cdot t\| \leqslant 1$ and $-\varepsilon < \mu < \varepsilon$ then it follows from (3.18) and (3.19) that $v \in \bar{E}^s \cap \bar{E}^u = E^0$. If in addition $\mu \neq 0$, then this implies $\|v\| = 0$.

Defining

$$E^s_\mu = \left\{v \in T_SM \mid \lim_{t\to\infty} \exp(\mu t)\|v\cdot t\| = 0\right\},$$

$$E^u_\mu = \left\{v \in T_SM \mid \lim_{t\to-\infty} \exp(\mu t)\|v\cdot t\| = 0\right\},$$

we obtain from Theorem 2.7 that PE^u_μ is an attractor in P_SM and PE^s_μ is its complementary repeller, provided that $0 < |\mu| < \varepsilon$.

Our next aim is to establish that $E^u_\mu = \bar{E}^u$, $E^s_\mu = E^s$, $0 < \mu < \varepsilon$. First of all it follows directly from the definitions along with (3.18) and (3.19) that $E^0 \subset E^u_\mu \subset \bar{E}^u$ and $E^s_\mu \subset E^s \subset \bar{E}^s$ for $0 < \mu < \varepsilon$. Now let $v \in T_SM\setminus E^u_\mu$ be given. Then $\|v\cdot t\|^{-1}v\cdot t$ converges to E^s_μ as t goes to $-\infty$. From this we conclude that $P(E^s_\mu \oplus E^0/E^0)$ is a repeller in the projectivized quotient bundle $P(T_SM/E^0)$ whose complementary attractor is contained in $P(E^u_\mu/E^0)$. Taking into account that $P(\bar{E}^s/E^0)$ is also a repeller in $P(T_SM/E^0)$ containing $P(E^s_\mu \oplus E^0/E^0)$ and whose complementary attractor $P(\bar{E}^u/E^0)$ contains $P(E^u_\mu/E^0)$ we obtain $\bar{E}^u = E^u_\mu$ and $\bar{E}^s = E^s_\mu \oplus E^0 = E^s \oplus E^0$. Since $E^s_\mu \subset E^s$ we conclude that $E^s_\mu = E^s$. This proves the desired equations.

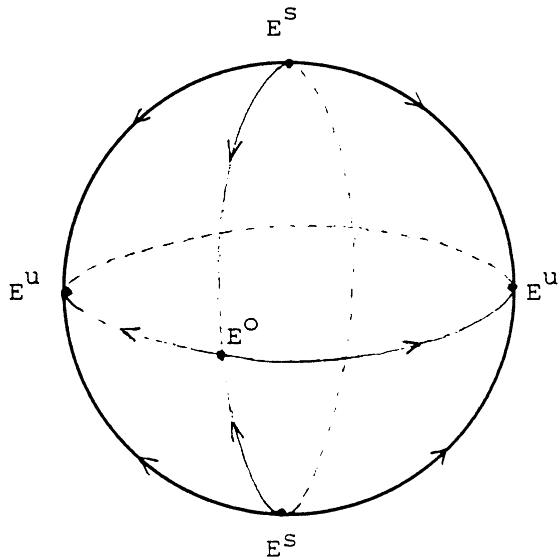

FIGURE 7

Hence $P\bar{E}^u$ is an attractor in P_SM and PE^s is its complementary repeller. With the same methods one can show that PE^u is an attractor in P_SM with complementary repeller $P\bar{E}^s$. Since $P\bar{E}^u \cap P\bar{E}^s = PE^0$ we conclude that the ordered triple PE^u, PE^0, PE^s is a Morse decomposition of P_SM. This finishes the proof of Theorem 3.3. □

In fact, we have proven a little more than what is stated in Theorem 3.3, namely

COROLLARY 3.4. *Let $S \subset M$ be a weakly hyperbolic invariant set for equation* (3.2) *and let the subsets E^0, E^s, E^u of T_SM be defined by* (3.7), (3.9), (3.10). *Then $\bar{E}^s = E^s \oplus E^0$ is given by* (3.14) *and $\bar{E}^u = E^u \oplus E^0$ by* (3.15). *Furthermore, $P(E^s \oplus E^0)$ is the complementary repeller of PE^u and $P(E^u \oplus E^0)$ is the complementary attractor of PE^s in P_SM.*

PROOF. Proof of Theorem 3.3. □

The Morse decomposition PE^u, PE^0, PE^s of P_SM is illustrated in Figure 7, a diagram of the induced flow on the sphere bundle ΣM. Since the imagination of the authors is unfortunately restricted to three dimensions, the reader will have to content himself with the diagram of a single fiber.

Theorem 3.3 shows that the only difference between weak hyperbolicity and the classical concept of a hyperbolic invariant set (see below) is the bundle property of the stable and unstable manifolds E^s and E^u of the zero section in T_SM.

LEMMA 3.5. *Let $S \subset M$ be a compact invariant set such that $X(p) \neq 0$ for all $p \in S$ and let $E^0 \subset T_SM$ and $E_0^* \subset T_S^*M$ be defined by* (3.7). *Then the following statements are equivalent.*

(i) *TS decomposes into three invariant subbundles E^s, E^0 and E^u such that the inequalities* (3.11) *and* (3.12) *are satisfied for some constants $K > 0$ and $\varepsilon > 0$.*

(ii) *E_0^* decomposes into two invariant subbundles E_0^{*s} and E_0^{*u} such that the following inequalities hold for some constants $K > 0$ and $\varepsilon > 0$*

$$\|v^* \cdot t\| \leqslant K \exp(-\varepsilon t)\|v^*\| \quad \forall v^* \in E_0^{*s}\ \forall t \geqslant 0, \tag{3.20}$$

$$\|v^* \cdot t\| \leqslant K \exp(\varepsilon t)\|v^*\| \quad \forall v^* \in E_0^{*u}\ \forall t \leqslant 0. \tag{3.21}$$

If these conditions are satisfied then S is said to be a hyperbolic invariant set.

PROOF. Let us first assume that statement (i) is satisfied. Then E_0^* decomposes into the subbundles

$$E_0^{*s} = \left\{ v^* \in E_0^* | v^* \perp E^s \cap T_pM,\ p = \pi^*(v^*) \right\},$$

$$E_0^{*u} = \left\{ v^* \in E_0^* | v^* \perp E^u \cap T_pM,\ p = \pi^*(v^*) \right\},$$

and it remains to establish (3.20) and (3.21). For this purpose let us first show that there exists a constant $\delta > 0$ such that

$$\delta\|v^*\| \leqslant \sup\left\{ \langle v^*, v\rangle | v \in E^s,\ \|v\| = 1,\ \pi(v) = \pi^*(v^*) \right\}$$

for all $v^* \in E_0^{*u}$. Otherwise, there would exist a sequence $v_k^* \in E_0^{*u}$ such that $\|v_k^*\| = 1$ and $\sup\{\langle v_k^*, v\rangle | v \in E^s,\ \|v\| = 1,\ \pi(v) = \pi^*(v_k^*)\}$ tends to zero. Any limit point $v^* \in E_0^{*u}$ of v_k^* would then annihilate $(E^s + E^0 + E^u) \cap T_pM = T_pM$, $p = \pi^*(v^*)$, and therefore be in the zero section of T_S^*M, contradicting $\|v^*\| = 1$.

Now let $v^* \in E_0^{*u}$ and $t \leqslant 0$ be given and choose $v \in E^s$ such that $\pi(v) = \pi^*(v^* \cdot t)$, $\|v\| = 1$ and $\delta\|v^* \cdot t\| \leqslant \langle v^* \cdot t, v\rangle$. Then it follows from (3.4) and (3.11) that

$$\|v^* \cdot t\| \leqslant \delta^{-1}\langle v^* \cdot t, v\rangle = \delta^{-1}\langle v^*, v \cdot (-t)\rangle \leqslant K\delta^{-1}\exp(\varepsilon t)\|v^*\|.$$

This proves (3.21) and (3.20) can be established analogously.

Conversely, suppose that statement (ii) is satisfied and define the invariant subbundles

$$\bar{E}^s = \left\{ v \in T_SM | v \perp E_0^{*s} \cap T_p^*M,\ p = \pi(v) \right\},$$

$$\bar{E}^u = \left\{ v \in T_SM | v \perp E_0^{*u} \cap T_p^*M,\ p = \pi(v) \right\}$$

of T_SM. Then $\bar{E}^s \cap \bar{E}^u = E^0$ and $\bar{E}^s + \bar{E}^u = T_pM$. Furthermore, one can show as above that there exists a constant $\delta > 0$ such that the following inequality holds for every $v \in \bar{E}^u$ and $p = \pi(v)$

$$\delta^2\left[\|v\|^2 - \|X(p)\|^{-2}\langle v, X(p)\rangle^2\right] \leqslant \sup\left\{ \langle v^*, v\rangle^2 | v^* \in E_0^{*s},\ \pi(v^*) = p,\ \|v^*\| = 1 \right\}.$$

Now let $v \in \bar{E}^u$ and $t < 0$ be given and choose $v^* \in E_0^{*s}$ such that $\pi^*(v^*) = \pi(v \cdot t) = p \cdot t$, $\|v^*\| = 1$ and

$$\delta^2\left[\|v \cdot t\|^2 - \|X(p \cdot t)\|^{-2}\langle v \cdot t, X(p \cdot t)\rangle^2\right] \leqslant \langle v^*, v \cdot t\rangle.$$

Then it follows from (3.4) and (3.20) that

$$\begin{aligned}\|v\cdot t\|^2-\|X(p\cdot t)\|^{-2}\langle v\cdot t, X(p\cdot t)\rangle^2 &\leqslant \delta^{-2}\langle v^*\cdot(-t), v\rangle^2\\ &\leqslant \delta^{-2}\|v^*\cdot(-t)\|^2\Big[\|v\|^2-\|X(p)\|^{-2}\langle v, X(p)\rangle\Big]^2\\ &\leqslant K^2\delta^{-2}\exp(2\varepsilon t)\Big[\|v\|^2-\|X(p)\|^{-2}\langle v, X(p)\rangle^2\Big].\end{aligned}$$

This proves (3.17) with $K_0 = K\delta^{-1}$. The inequality (3.16) can be established analogously.

Now let $v \in T_SM$ be given, define $p = \pi(v)$ and suppose that

$$\sup\Big\{\|v\cdot t\|^2-\|X(p\cdot t)\|^{-2}\langle v\cdot t, X(p\cdot t)\rangle^2 \big| t\in\mathbf{R}\Big\} < \infty.$$

Furthermore, note that $v = v_s + v_u$ for some $v_s \in \bar{E}^s$ and $v_u \in \bar{E}^u$. Then it follows from (3.16) that

$$\sup\Big\{\|v_u\cdot t\|^2-\|X(p\cdot t)\|^{-2}\langle v_u\cdot t, X(p\cdot t)\rangle^2 \big| t\geqslant 0\Big\} < \infty$$

and hence we obtain from (3.17) that $v_u \in E^0$. This implies $v \in \bar{E}^s$ and it follows again from (3.16) (negative time) that $v \in E^0$.

We conclude that the zero section in the quotient bundle T_SM/E^0 is an isolated invariant set. Analogous arguments show that $\bar{E}^s$ is given by (3.14) and $\bar{E}^u$ by (3.15). Now let E^s and E^u be defined by (3.9) and (3.10), respectively. Then it follows from Theorem 3.3 that the inequalities (3.11) and (3.12) are satisfied. Furthermore, Corollary 3.4 shows that $\bar{E}^s = E^0 \oplus E^s$ and $\bar{E}^u = E^0 \oplus E^u$. Therefore E^s and E^u are subbundles of T_SM (Lemma A2) and satisfy $T_SM = E^s \oplus E^0 \oplus E^u$. This proves Lemma 3.5. □

The proof of Lemma 3.5 shows that every hyperbolic invariant set $S \subset M$ is weakly hyperbolic with respect to both equations (3.2) and (3.3). In this context it would be interesting to know whether a compact, connected invariant set $S \subset M$ is hyperbolic if it is weakly hyperbolic with respect to both equations (3.2) and (3.3). The next result shows that all three notions of hyperbolicity are equivalent if the flow on S is chain transitive.

COROLLARY 3.6. *Let $S \subset M$ be a chain transitive, compact invariant set such that $X(p) \neq 0$ for all $p \in S$. Then the following statements are equivalent.*

(i) *S is hyperbolic.*

(ii) *S is weakly hyperbolic for equation* (3.2).

(iii) *S is weakly hyperbolic for equation* (3.3).

PROOF. It follows from Lemma 3.5 that (i) implies (ii) and (iii). Furthermore, it follows from Theorem 2.13 that (iii) implies (i). Finally, it follows from Theorem 3.3, Corollary 2.11 and Corollary 2.12 that (ii) implies (i). □

Our next aim is to prove a topological perturbation theorem for hyperbolic invariant sets. For this purpose we consider the parametrized differential equations

(3.22) $\qquad \dot{x} = f(x,\lambda),$

(3.23) $\qquad \dot{x} = f(x,\lambda), \qquad \dot{\xi} = d_xf(x,\lambda)\xi,$

(3.24) $\qquad \dot{x} = f(x,\lambda), \qquad \dot{\xi}^* = -\xi^* d_xf(x,\lambda),$

on M, TM and T^*M, respectively, corresponding to a continuous family X_λ: $M \to TM$, $\lambda \in \Lambda$, of smooth vector fields on M. The corresponding flows will be denoted by $(p, \lambda) \cdot t \in M$, $(v, \lambda) \cdot t \in TM$ and $(v^*, \lambda) \cdot t \in T^*M$ for $p \in M$, $v \in TM$, $v^* \in T^*M$, $t \in \mathbf{R}$ and $\lambda \in \Lambda$. We assume that the parameter space Λ is a compact metric space.

THEOREM 3.7. *Suppose that $S_0 \subset M$ is a weakly hyperbolic invariant set for the equation* (3.24) *at $\lambda = \lambda_0$. Then there exist neighborhoods U of S_0 in M and W of λ_0 in Λ such that every compact set $S \subset U$ which is invariant under* (3.22) *with $\lambda \in W$ is again weakly hyperbolic for equation* (3.24).

PROOF. Suppose that the statement of the theorem were false, then there would exist sequences $v_k^* \in T^*M$ and $\lambda_k \in \Lambda$ such that λ_k converges to λ_0, $\langle v_k^*, X(p_k)\rangle = 0$ and

$$
\begin{aligned}
&d\big((p_k, \lambda_k) \cdot t, S_0\big) \leqslant 1/k \quad \forall t \in \mathbf{R}\, \forall k \in \mathbf{N},\\
&\sup\big\{ \big\|\big(v_k^*, \lambda_k\big) \cdot t\big\| \big| t \in \mathbf{R}\big\} = 1 \quad \forall k \in \mathbf{N}
\end{aligned}
\tag{3.25}
$$

where $p_k = \pi^*(v_k^*) \in M$. Now choose any $\hat{v}_k^* \in \operatorname{cl}\{(v_k^*, \lambda_k) \cdot \mathbf{R}\}$ such that $\|\hat{v}_k^*\| = 1$. Then the inequalities (3.25) are still satisfied with v_k^* replaced by $\hat{v}_k^*$ and $p_k = \pi^*(\hat{v}_k^*)$. Moreover, $\langle \hat{v}_k^*, X(p_k)\rangle = 0$. Therefore any limit point $v^* \in T^*M$ of $\hat{v}_k^*$ satisfies $\|(v^*, \lambda_0) \cdot t\| \leqslant \|v^*\| = 1$, $p = \pi^*(v^*) \in S_0$ and $\langle v^*, X(p)\rangle = 0$. This contradicts the assumption that S_0 is a weakly hyperbolic invariant set for equation (3.24) at $\lambda = \lambda_0$. $\square$

An analogous perturbation theorem for invariant sets which are weakly hyperbolic with respect to equation (3.23) can be proved in exactly the same manner as Theorem 3.7. We will however give a completely different proof using a perturbation result for attractor-repeller pairs which is due to Conley [**4**]. This proof shows that the stable and unstable sets E^s and E^u of the perturbed invariant set lie close to those of the unperturbed invariant set. We will first formulate Conley's perturbation result for attractor-repeller pairs.

LEMMA 3.8 [**4**, II.5.3.C]. *Let S be a compact invariant set in a compact flow M, let A, A^* be an attractor-repeller pair in S and let U and U^* be disjoint compact neighborhoods of A and A^*, respectively, in M. Then there exists a neighborhood W of S in M such that, if $\bar{S}$ is any compact invariant set in W then $\bar{A} = \omega(\bar{S} \cap U) \subset U$ is an attractor in $\bar{S}$ and $\bar{A}^* = \omega^*(\bar{S} \cap U^*) \subset U^*$ is its complementary repeller. See Figure* 8.

We point out that the statement of Lemma 3.8 in [**4**] is formulated without the requirement $\bar{A} \subset U$. However, there exists a $T > 0$ such that $(U \cap S) \cdot T \subset \operatorname{int} U$ and therefore $(U \cap \bar{S}) \cdot T \subset \operatorname{int} U$ if W is chosen small enough. By [**4**, II.5.1.C], this shows that $\bar{A} = \omega(U \cap \bar{S}) \subset U$. The inclusion $\bar{A}^* \subset U^*$ follows from similar arguments.

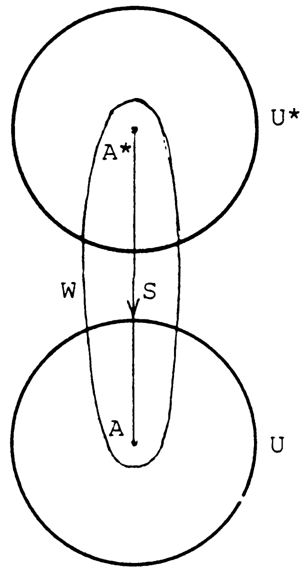

FIGURE 8

For any compact set $S \subset M$ which is invariant under (3.22) with $\lambda \in \Lambda$ we define the subsets

$$E^0(S, \lambda) = \{cX_\lambda(p) \mid p \in S,\ c \in \mathbf{R}\}, \tag{3.26}$$

$$E^s(S, \lambda) = \left\{v \in T_S M \mid \lim_{t \to \infty} \|(v, \lambda) \cdot t\| = 0\right\}, \tag{3.27}$$

$$E^u(S, \lambda) = \left\{v \in T_S M \mid \lim_{t \to -\infty} \|(v, \lambda) \cdot t\| = 0\right\} \tag{3.28}$$

of $T_S M$. Moreover, a subset $U \subset TM$ is said to be a *cone* if $v \in U$ implies $cv \in U$ for all $c \in \mathbf{R}$.

THEOREM 3.9. *Suppose that $S_0 \subset M$ is a weakly hyperbolic invariant set for equation* (3.23) *at $\lambda = \lambda_0$. Moreover, let U^0, U^s and U^u be closed cones in TM such that PU^0, PU^s and PU^u are disjoint compact neighborhoods of $PE^0(S_0, \lambda_0)$, $PE^s(S_0, \lambda_0)$ and $PE^u(S_0, \lambda_0)$, respectively, in PM. Then there exist neighborhoods U of S_0 in M and W of λ_0 in Λ such that every compact set $S \subset U$ which is invariant under* (3.22) *with $\lambda \in W$ is weakly hyperbolic for equation* (3.23) *and satisfies $E^0(S, \lambda) \subset U^0$, $E^s(S, \lambda) \subset U^s$ and $E^u(S, \lambda) \subset U^u$.*

PROOF. Let us first choose closed cones $\overline{U}^s$ and $\overline{U}^u$ in TM such that $P\overline{U}^s$ is a neighborhood of $P(E^s(S_0, \lambda_0) \oplus E^0(S_0, \lambda_0))$, $P\overline{U}^u$ is a neighborhood of $P(E^u(S_0, \lambda_0) \oplus E^0(S_0, \lambda_0))$ and $P\overline{U}^s \cap PU^u = \varnothing$, $P\overline{U}^u \cap PU^s = \varnothing$, $P\overline{U}^s \cap P\overline{U}^u \subset PU^0$. Moreover, it follows from Theorem 3.3 that there exist constants $\alpha < 1$, $T > 0$ such that $\|(v, \lambda_0) \cdot T\| < \alpha\|v\|$ for all $v \in E^s(S_0, \lambda_0)$ and $\|(v, \lambda_0) \cdot (-T)\| < \alpha\|v\|$ for all $v \in E^u(S_0, \lambda_0)$. Hence we can assume without loss of generality that

$$\|(v, \lambda_0) \cdot T\| < \alpha\|v\| \quad \forall v \in U^s, \qquad \|(v, \lambda_0) \cdot (-T)\| < \alpha\|v\| \quad \forall v \in U^u.$$

Finally, we assume that U^0 is chosen small enough such that every linear subspace of a fiber T_pM contained in U^0 is at most one dimensional.

Now we can apply Lemma 3.8 to the compact invariant set $P_{S_0}M \times \lambda_0$ in the flow $PM \times \Lambda$. We conclude that there exist neighborhoods U of S_0 in M and W of Λ_0 in Λ such that, if $P_SM \times \lambda$ is any compact invariant set in $P_UM \times W$, then $\omega(P_SM \cap P\bar{U}^u \times \lambda) = \bar{A} \subset P\bar{U}^u \times \lambda$ is an attractor in $P_SM \times \lambda$ with complementary repeller

$$\omega^*(P_SM \cap PU^s \times \lambda) = A^* \subset PU^s \times \lambda$$

and likewise $\omega(P_SM \cap PU^u \times \lambda) = A \subset PU^u \times \lambda$ is an attractor in $P_SM \times \lambda$ with complementary repeller $\omega^*(P_SM \cap P\bar{U}^s \times \lambda) = \bar{A}^* \subset P\bar{U}^s \times \lambda$. Furthermore, we assume that U is chosen small enough such that $0 \neq X(p) \in \bar{U}^u \cap \bar{U}^s$ for all $p \in U$ and W is chosen small enough such that

$$\|(v, \lambda) \cdot T\| \leqslant \alpha\|v\| \quad \forall v \in U^s\ \forall \lambda \in W, \tag{3.29}$$

$$\|(v, \lambda) \cdot (-T)\| \leqslant \alpha\|v\| \quad \forall v \in U^u\ \forall \lambda \in W. \tag{3.30}$$

Then $\bar{A} \cap \bar{A}^*$ intersects each fiber in a projective linear subspace (Proposition 2.9) and contains $PE^0(S, \lambda) \times \lambda$. Since $\bar{A} \cap \bar{A}^* \subset PU^0 \times \lambda$ we obtain $\bar{A} \cap \bar{A}^* = PE^0(S, \lambda) \times \lambda$. Hence it follows from (3.29) and (3.30) that the subsets

$$E^s = \{v \in T_SM \,|\, v \neq 0 \Rightarrow (Pv, \lambda) \in A\} \subset U^s,$$

$$E^u = \{v \in T_SM \,|\, v \neq 0 \Rightarrow (Pv, \lambda) \in A^*\} \subset U^u.$$

Satisfy all the requirements of Theorem 3.3. Therefore it follows from Theorem 3.3 that S is weakly hyperbolic for equation (3.23) and that $E^s(S, \lambda) = E^s \subset U^s$ as well as $E^u(S, \lambda) = E^u \subset U^u$. □

The refined perturbation Theorem 3.9 will be used to derive a perturbation result for hyperbolic invariant sets.

THEOREM 3.10. *Suppose that $S_0 \subset M$ is a hyperbolic invariant set for equation* (3.22) *at $\lambda = \lambda_0$. Then there exist neighborhoods U of S_0 in M and W of λ_0 in Λ such that every compact set $S \subset U$ which is invariant under* (3.22) *with $\lambda \in W$ is hyperbolic.*

PROOF. Define $m = \dim E^s(S_0, \lambda_0)$ and $k = \dim E^u(S_0, \lambda_0)$ and note that $m + k + 1 = n$. Now choose closed cones U^0, U^s, U^u in TM as in Theorem 3.9. These cones can be chosen small enough such that $\dim L \leqslant m$ for every linear subspace L of a fiber T_pM with $L \subset U^s$ and analogously $\dim L \leqslant k$ if $L \subset U^u$ and $\dim L \leqslant 1$ if $L \subset U^0$. Now choose neighborhoods U of S_0 in M and W of λ_0 in Λ such that the statement of Theorem 3.9 is satisfied and let $S \subset U$ be a compact set which is invariant under (3.22) with $\lambda \in W$. Then S is weakly hyperbolic with respect to equation (3.23) (Theorem 3.9) and it remains to show that $E^s(S, \lambda)$ and $E^u(S, \lambda)$ are subbundles of T_SM (Theorem 3.3). First note that $E^s(S, \lambda) \subset U^s$ and $E^u(S, \lambda) \subset U^u$ and therefore $\dim(E^s(S, \lambda) \cap T_pM) \leqslant m$ and $\dim(E^u(S, \lambda) \cap T_pM) \leqslant k$ for every $p \in S$. Now suppose that $\dim(E^s(S, \lambda) \cap T_pM) < m$ for some $p \in S$ and let $L \subset T_pM$ be the orthogonal complement of $(E^s(S, \lambda) \oplus E^0(S, \lambda)) \cap T_pM$. Then it follows from Corollary 3.4 that $\omega(PL) \subset PE^u(S, \lambda)$ and therefore $\Omega(PL) \subset PE^u(S, \lambda)$. Hence it follows from Lemma 2.10 that

$$\dim(E^u(S, \lambda) \cap T_qM) \geqslant \dim L > n - m - 1 = k$$

for $q \in \Omega(p) \subset S$. This contradiction shows that $\dim(E^s(S, \lambda) \cap T_pM) = m$ for all $p \in S$ and therefore $E^s(S, \lambda)$ is a subbundle of T_SM (Lemma 1.4). In the same way one can prove that $E^u(S, \lambda)$ is a subbundle of T_SM and therefore S is a hyperbolic invariant set. $\square$

A special case of Theorem 3.10 is that M itself is a hyperbolic invariant set for the vector field $X\colon M \to TM$. In this situation the vector field X is said to be *Anosov*. In this situation Theorem 3.10 states that the set of Anosov vector fields is open in the set of all vector fields with respect to the C^0-topology which is, of course, well-known. Note that our proof of this result is, however, based on topological methods.

Finally, we point out that related results have been discussed by Fenichel [**5**] and Floer [**6**, **7**], as well as Churchill, Franke and Selgrade [**12**].

Appendix.

Vector bundles. Let V be a finite dimensional Hilbert space over the reals $\mathbf{R}$. Let S be a compact metric space and $\pi\colon E \to S$ be a vector bundle over S with fiber V. This means that there exists a finite open cover $\{U_\alpha | \alpha \in A\}$ of M and homeomorphisms $\varphi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times V$ such that the diagrams

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \overset{\varphi_\alpha}{\to} & U_\alpha \times V \\ & \pi \searrow \qquad \swarrow & \\ & U_\alpha & \end{array}$$

commute and the maps $L_{\beta\alpha}(p)\colon V \to V$ defined by $\varphi_\beta\varphi_\alpha^{-1}(p, v) = (p, L_{\beta\alpha}(p)v)$ are linear for $p \in U_\alpha \cap U_\beta$. Without loss of generality we can assume that φ_α extends to a homeomorphism of $\mathrm{cl}(\pi^{-1}(U_\alpha))$ onto $\mathrm{cl}(U_\alpha) \times V$ for all α. At some places we use the notation $E_p = \pi^{-1}(p)$ for $p \in S$.

The zero section of E is defined by

$$Z = \{e \in E | \varphi_\alpha(e) = (\pi(e), 0) \text{ whenever } \pi(e) \in U_\alpha\}$$

and is the homeomorphic image of S under the map $\sigma\colon S \to E$ defined by

$$\sigma(p) = \varphi_\alpha^{-1}(p, 0), \qquad p \in U_\alpha.$$

Note that the fiber E_p can be given a vector space structure by defining

$$ce = \varphi^{-1}(p, cv), \qquad e + e' = \varphi^{-1}(p, v + v')$$

for $e = \varphi_\alpha^{-1}(p, v) \in E_p$ and $e' = \varphi_\alpha^{-1}(p, v') \in E$.

A continuous, positive definite bilinear form on E is given by

$$\langle e, e' \rangle = \sum_\alpha d(\pi(e), S \setminus U_\alpha) \langle e, e' \rangle_\alpha, \qquad \pi(e) = \pi(e'),$$

where $\langle e, e' \rangle_\alpha = \langle v, v' \rangle$ for $e = \varphi_\alpha^{-1}(p, v)$, $e' = \varphi_\alpha^{-1}(p, v')$ and $\langle e, e' \rangle_\alpha = 0$ for $\pi(e) = \pi(e') \notin U_\alpha$. Since φ_α extends to a homeomorphism on $\mathrm{cl}(\pi^{-1}(U_\alpha))$ there exists a $\delta > 0$ such that

$$\delta \|e\|_\alpha \leqslant \|e\| = \sqrt{\langle e, e \rangle} \leqslant \delta^{-1} \|e\|_\alpha, \qquad \pi(e) \in U_\alpha.$$

In particular, this implies that the compact sets $\{e \in E | \|e\| \leqslant \varepsilon\}$ define a neighborhood basis for the zero section $Z \subset E$.

LEMMA A1. *E is metrizable.*

PROOF. Define

$$d_\alpha(e,e') = \max\left\{ d\big(\pi(e),\pi(e')\big),\ \|v - v'\| \right\}$$

for $e = \varphi_\alpha^{-1}(p,v)$, $e' = \varphi_\alpha^{-1}(p',v')$ with $p, p' \in U_\alpha$. For any sequence $e_0,\dots,e_m \in E$ we define

$$\rho(e_0,\dots,e_m) = \max\left\{ \sum_{i=1}^{m} d_{\alpha_j}(e_{j-1},e_j) \,\middle|\, \pi(e_{j-1}),\pi(e_j) \in U_{\alpha_j},\ \alpha_j \in A \right\},$$

where the maximum over the empty set is by definition $+\infty$. Then the distance function

$$d(e,e') = \inf\left\{ \rho(e_0,\dots,e_m) \,|\, m \in \mathbf{N},\ e_j \in E,\ e_0 = e,\ e_m = e' \right\}$$

defines a metric on E which is compatible with the original topology. $\square$

A *subbundle* of E is a closed subset $F \subset E$ which intersects each fiber in a linear subspace and—with the induced topology—is again a vector bundle.

LEMMA A2. *Let $F \subset E$ be a closed subset of E which intersects each fiber in a linear subspace F_p, $p \in S$. Then F is a subbundle if and only if* $\dim F_p = \dim F_q$ *for all* $p, q \in S$.

PROOF. Suppose that $m = \dim F_p$ is independent of $p \in S$. For $p \in U_\alpha$ define $\Pi_\alpha(p)\colon V \to V$ to be the orthogonal projection of V onto the subspace

$$W_p = \left\{ v \in V \,|\, (p,v) \in \varphi_\alpha(F_p) \right\}.$$

In order to establish the continuity of Π_α as a map from U_α into $\mathscr{L}(V)$ it is enough to show that every sequence $p_k \in U_\alpha$ converging to $p \in U_\alpha$ has a subsequence (still denoted by p_k) such that $\Pi_\alpha(p_k)$ converges to $\Pi_\alpha(p)$. For this purpose let $v_j(p_k)$, $j = 1,\dots,n$, be an orthonormal basis of V such that $v_1(p_k),\dots,v_m(p_k)$ is an orthonormal basis of W_{p_k} and choose a subsequence in such a way that $v_j(p_k)$ converge to v_j for $j = 1,\dots,n$. Then $v_1,\dots,v_n$ form an orthonormal basis of V. Furthermore, it follows from the closedness of F that $v_1,\dots,v_m$ form an orthonormal basis of W_p. Therefore the following inequality holds for $v = \sum c_j v_j \in V$

$$\begin{aligned}\|\Pi_\alpha(p_k)v - \Pi_\alpha(p)v\| &\leqslant \sum_{j=1}^{n} |c_j|\,\|\Pi_\alpha(p_k)\|\,\|v_j - v_j(p_k)\| \\ &\quad + \sum_{j=1}^{m} |c_j|\,\|v_j(p_k) - v_j\|.\end{aligned}$$

This proves the continuity of Π_α.

Now let $p_0 \in U_\alpha$ be given and choose a neighborhood U_0 of p_0 in U_α such that the restriction of $\Pi_\alpha(p)$ to $W = W_{p_0}$ remains injective for $p \in U_0$. Then the map $\psi_0^{-1}\colon U_0 \times W \to F$ given by

$$\psi_0^{-1}(p,w) = \varphi_\alpha^{-1}(p,\Pi_\alpha(p)w), \qquad p \in U_\alpha,\ w \in W,$$

defines the desired bundle structure on F in a neighborhood of p_0. $\square$

Let us now introduce the projectivized line bundle $PE = E \setminus Z/\sim$ where $e \sim e'$ if and only if $\pi(e) = \pi(e')$ and there exists a constant $c \in \mathbf{R} \setminus \{0\}$ such that $e' = ce$. The canonical projection map will be denoted by $P\colon E \setminus Z \to PE$. For $L \subset E$ we define $PL = \{Pe \mid e \in L \setminus Z\}$. In particular $PL = \varnothing$ if $L \subset Z$. Of course, there exists a projection $P\pi\colon PE \to S$ such that the following diagram commutes:

$$\begin{tikzcd} E \setminus Z \arrow[rr, "P"] \arrow[dr, "\pi"'] & & PE \arrow[dl, "P\pi"] \\ & S & \end{tikzcd}$$

This map is a fibration whose fibers are isomorphic to the projective space PV. More precisely, there exist (unique) homeomorphisms $P\varphi_\alpha\colon P\pi^{-1}(U_\alpha) \to U_\alpha \times PV$ such that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \setminus Z \arrow[rr, "\varphi_\alpha"] \arrow[d, "P"'] & & U_\alpha \times V \setminus \{0\} \arrow[d, "1 \times P"] \\ P\pi^{-1}(U_\alpha) \arrow[rr, "P\varphi_\alpha"] \arrow[dr, "P\pi"'] & & U_\alpha \times PV \arrow[dl] \\ & U_\alpha & \end{tikzcd}$$

Note that PE is a compact metric space. A metric can be defined by

$$d(Pe, Pe') = \min\left\{ d\left(\frac{e}{\|e\|}, \frac{e'}{\|e'\|}\right), d\left(\frac{e}{\|e\|}, -\frac{e'}{\|e'\|}\right)\right\}$$

for $e, e' \in E \setminus Z$.

LEMMA A3. *There exists a constant $\delta > 0$ such that*

$$\delta d(Pe, Pe') \leqslant 1 - \frac{\langle e, e'\rangle^2}{\|e\|^2\|e'\|^2} \leqslant \delta^{-1} d(Pe, Pe')$$

for all $e, e' \in E \setminus Z$ with $\pi(e) = \pi(e')$.

This result follows from the strict convexity of the finite dimensional Hilbert space V along with the compactness of S.

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