Symplectic Topology
Example Sheet 2

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For a real vector space V denote the space of symplectic bilinear forms, linear symplectic structures, respectively inner products by

$$\begin{aligned}\mathcal{S}(V) &:= \{\omega : V \times V \to \mathbb{R} \,|\, \omega \text{ is skew symmetric and nondegenerate}\}, \\ \mathcal{J}(V) &:= \left\{J \in \mathrm{GL}(V) \,|\, J^2 = -\mathbb{1}\right\}, \\ \mathcal{M}(V) &:= \{g : V \times V \to \mathbb{R} \,|\, g \text{ is an inner product}\}.\end{aligned}$$

A linear complex structure $J \in \mathcal{J}(V)$ is called **compatible with the symplectc form** $\omega \in \mathcal{S}(V)$ if

$$\omega(J\cdot, J\cdot) = \omega, \qquad \omega(v, Jv) > 0 \text{ for all } v \in V \setminus \{0\}.$$

It is called **compatible with the inner product** $g \in \mathcal{M}(V)$ if

$$g(J\cdot, J\cdot) = g.$$

A symplectic form $\omega \in \mathcal{S}(V)$ is called **compatible with the inner product** $g \in \mathcal{M}(V)$ if there exists a linear complex structure $J \in \mathcal{J}(V)$ such that $g = \omega(\cdot, J\cdot)$. For $g \in \mathcal{M}(V)$ and $\omega \in \mathcal{S}(V)$ define

$$\begin{aligned}\mathcal{J}(V,\omega) &:= \{J \in \mathcal{J}(V) \,|\, J \text{ is compatible with } \omega\}, \\ \mathcal{S}(V,g) &:= \{\omega \in \mathcal{S}(V) \,|\, \omega \text{ is compatible with } g\}, \\ \mathcal{J}(V,g) &:= \{J \in \mathcal{J}(V) \,|\, J \text{ is compatible with } g\}, \\ \mathcal{SJ}(V) &:= \{(\omega, J) \in \mathcal{S}(V) \times \mathcal{J}(V) \,|\, J \text{ is compatible with } \omega\}.\end{aligned}$$

The first of these manifolds is contractible, the second and third are diffeomorphic to each other, and the last three are homotopy equivalent to each other and to $\mathcal{J}(V)$ and $\mathcal{S}(V)$.

Exercise 2.1. Let V be a $2n$-dimensional real vector space and let $J \in \mathcal{J}(V)$ and $\omega \in \mathcal{S}(V)$. Prove that J is compatible with ω if and only if the formula

$$g_J := \omega(\cdot, J\cdot)$$

defines an inner product on V. Prove that, if J is compatible with ω, then J is compatible with g_J. Prove that the following are equivalent.

(a) $J \in \mathcal{J}(V, \omega)$

(b) There exist $v_1, \ldots, v_n \in V$ such that the vectors $v_1, Jv_1, \ldots, v_n, Jv_n$ form a basis of V and $\omega(v_i, Jv_j) = \delta_{ij}$ and $\omega(v_i, v_j) = \omega(Jv_i, Jv_j) = 0$ for all i, j.

(c) There is a vector space isomorphism $\Psi : \mathbb{R}^{2n} \to V$ such that $\Psi^*\omega = \omega_0$ and $\Psi^*J = J_0$.

Exercise 2.2. Prove that $\mathcal{J}(V, \omega) \neq \emptyset$ for every $\omega \in \mathcal{S}(V)$.

Exercise 2.3. Fix an inner product $g \in \mathcal{M}(V)$. Prove that the projection $\pi_{\mathcal{S}} : \mathcal{SJ}(V) \to \mathcal{S}(V)$ is a homotopy equivalence with homotopy inverse

$$\mathcal{S}(V) \to \mathcal{SJ}(V) : \omega \mapsto (\omega, J_{g,\omega}).$$

Here $J_{g,\omega} := Q^{-1}A \in \mathcal{J}(V, \omega) \cap \mathcal{J}(V, g)$ is defined by $g(A\cdot, \cdot) = \omega$ and $Q := \sqrt{A^*A}$ is the unique g-self-adjoint, g-positive-definite automorphism of V whose square is $A^*A = -A^2$.

Exercise 2.4. Fix an inner product $g \in \mathcal{M}(V)$. Prove that the projection $\pi_{\mathcal{J}} : \mathcal{SJ}(V) \to \mathcal{J}(V)$ is a homotopy equivalence with homotopy inverse

$$\mathcal{J}(V) \to \mathcal{SJ}(V) : J \mapsto (\omega_{g,J}, J), \qquad \omega_{g,J} := \frac{g(J, \cdot, \cdot) - g(\cdot, J\cdot)}{2}.$$

Exercise 2.5. Prove that the map

$$\mathrm{GL}(2n, \mathbb{R}) \to \mathcal{J}(\mathbb{R}^{2n}) : \Psi \mapsto \Psi J \Psi^{-1}$$

descends to a diffeomorphism $\mathrm{GL}(2n, \mathbb{R})/\mathrm{GL}(n, \mathbb{C}) \to \mathcal{J}(\mathbb{R}^{2n})$. (Here we identify $\mathrm{GL}(n, \mathbb{C})$ with the subgroup of all nonsingular real $2n \times 2n$-matrices that commute with J_0.)

Exercise 2.6. Prove that the map

$$\mathrm{GL}(2n, \mathbb{R}) \to \mathcal{S}(\mathbb{R}^{2n}) : \Psi \mapsto (\Psi^{-1})^*\omega_0$$

descends to a diffeomorphism $\mathrm{GL}(2n, \mathbb{R})/\mathrm{Sp}(2n) \to \mathcal{S}(\mathbb{R}^{2n})$.

Exercise 2.7. Prove that the map

$$\mathrm{GL}(2n,\mathbb{R}) \to \mathcal{SJ}(\mathbb{R}^{2n}) : \Psi \mapsto \big((\Psi^{-1})^*\omega_0, \Psi J_0 \Psi^{-1}\big)$$

descends to a diffeomorphism $\mathrm{GL}(2n,\mathbb{R})/\mathrm{U}(n) \to \mathcal{SJ}(\mathbb{R}^{2n})$.

Exercise 2.8. Let g_0 denote the standard inner product on $\mathbb{R}^{2n}$. Prove that the spaces $\mathcal{J}(\mathbb{R}^{2n}, g_0)$ and $\mathcal{S}(\mathbb{R}^{2n}, g_0)$ are both diffeomorphic to the homogeneous space $\mathrm{O}(2n)/\mathrm{U}(n)$. Construct explicit diffeomorphisms.

Exercise 2.9. Let $J \in \mathbb{R}^{2n\times 2n}$. Prove that $J \in \mathcal{J}(\mathbb{R}^{2n}, \omega_0)$ if and only if the matrix $P := -J_0 J$ is symplectic, symmetric, and positive definite. Deduce that the space $\mathcal{J}(\mathbb{R}^{2n}, \omega_0)$ is contractible.

Exercise 2.10. Denote **Siegel upper half space** by

$$\mathcal{S}_n := \left\{ Z = X + \mathbf{i}Y \in \mathbb{C}^{n\times n} \,|\, X = X^T,\, Y = Y^T > 0 \right\}.$$

Prove that a matrix

$$\Psi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \qquad A, B, C, D \in \mathbb{R}^{n\times n},$$

is symplectic if and only if

$$A^T D - C^T B = \mathbb{1}, \qquad A^T C = C^T A, \qquad B^T D = D^T B.$$

Prove that the formula

$$\Psi_* Z := (AZ + B)(CZ + D)^{-1}$$

defines a transitive group action of $\mathrm{Sp}(2n)$ on Siegel upper half space. Prove that $\mathrm{Sp}(2n)\cap\mathrm{SO}(2n) \cong \mathrm{U}(n)$ is the stabilizer subgroup of the matrix $\mathbf{i}\mathbb{1} \in \mathcal{S}_n$.

Exercise 2.11. Prove that the formula

$$\mathrm{Sp}(2n) \times \mathcal{J}(\mathbb{R}^{2n}, \omega_0) \to \mathcal{J}(\mathbb{R}^{2n}, \omega_0) : (\Psi, J) \mapsto \Psi J \Psi^{-1}$$

defines a transitive group action of the linear symplectic group $\mathrm{Sp}(2n)$ on the space of ω_0-compatible linear complex structures on $\mathbb{R}^{2n}$, and that the stabilizer subgroup of J_0 is $\mathrm{Sp}(2n) \cap \mathrm{SO}(2n) \cong \mathrm{U}(n)$. Deduce that there is a unique $\mathrm{Sp}(2n)$-equivariant diffeomorphism $\mathcal{S}_n \to \mathcal{J}(\mathbb{R}^{2n}, \omega_0) : Z \mapsto J(Z)$, such that $J(\mathbf{i}\mathbb{1}) = J_0$. Prove that an explicit formula for this diffeomorphism is given by

$$J(X + \mathbf{i}Y) = \begin{pmatrix} XY^{-1} & -Y - XY^{-1}X \\ Y^{-1} & -Y^{-1}X \end{pmatrix}. \tag{1}$$

Deduce that $\mathcal{J}(\mathbb{R}^{2n}, \omega_0)$ is contractible.

Exercise 2.12. Let $\omega \in \mathcal{S}(V)$. A linear complex structure $J \in \mathcal{J}(V)$ is called **ω-tame** if

$$\omega(v, Jv) > 0 \qquad \text{for all } v \in V \setminus \{0\}.$$

This exercise shows that the space $\mathcal{J}_\tau(V, \omega)$ of all ω-tame linear complex structures on V is contractible. Let $J \in \mathbb{R}^{2n \times 2n}$. Prove that the following assertions are equivalent.

(a) $J \in \mathcal{J}_\tau(\mathbb{R}^{2n}, \omega_0)$.

(b) The matrix $Z := -J_0 J$ satisfies $Z^{-1} = J_0^{-1} Z J_0$ and $\langle v, Zv \rangle > 0$ for every nonzero vector $v \in \mathbb{R}^{2n}$.

(c) The matrix $W := (\mathbb{1} - Z)(\mathbb{1} + Z)^{-1}$ satisfies $\|W\| < 1$ and $J_0 W + W J_0 = 0$.

The set of matrices $W \in \mathbb{R}^{2n \times 2n}$ satisfying (c) is convex, hence $\mathcal{J}_\tau(\mathbb{R}^{2n}, \omega_0)$ is contractible, and hence so is the space $\mathcal{J}_\tau(V, \omega)$ of ω-tame linear complex structures for every symplectic vector space (V, ω).

Exercise 2.13. Assume $\dim(V) = 2$ and let $\omega \in \mathcal{S}(V)$ and $J \in \mathcal{J}(V)$. Prove that ω and J are compatible if and only if they induce the same orientation on V.

Exercise 2.14. Assume $\dim(V) = 4$ and let $g \in \mathcal{M}(V)$. Prove that $\mathcal{J}(V, g)$ is diffeomorphic to two disjoint copies of the 2-sphere.

Exercise 2.15. Assume $\dim(V) = 6$ and let $g \in \mathcal{M}(V)$. Prove that $\mathcal{J}(V, g)$ is diffeomorphic to two disjoint copies of a 2-sphere bundle over a 4-sphere.