Symplectic Topology
Example Sheet 4

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Exercise 4.1. Let (M,ω) be a symplectic manifold and let $U \subset \mathbb{C}$ be an open set. Identify $\mathbb{C}$ with $\mathbb{R}^2$ and let $s + \mathbf{i}t$, respectively (s,t), be the coordinate on U. Let $\{J_{s,t}\}_{(s,t)\in\Omega}$ be a smooth family of ω-compatible almost complex structures on M, let $F, G : U \times M \to \mathbb{R}$ be smooth functions, and define $F_{s,t}, G_{s,t} : M \to \mathbb{R}$ by

$$F_{s,t} := F(s,t,\cdot), \qquad G_{s,t} := G(s,t,\cdot)$$

for $(s,t) \in U$. Consider the partial differential equation

$$\partial_s u + X_{F_{s,t}}(u) + J_{s,t}(u)\big(\partial_s u + X_{G_{s,t}}(u)\big) = 0 \tag{1}$$

for a smooth function $u : U \to M$.

(i) Define

$$\widetilde{M} := U \times M,$$

choose a function $c : U \to \mathbb{R}$, and let $\widetilde{\omega} \in \Omega^2(\widetilde{M})$ be the 2-form

$$\widetilde{\omega} := \omega - d^{\widetilde{M}}(F\,ds + G\,dt) + c\,ds \wedge dt. \tag{2}$$

Here we identify the 2-forms $\omega \in \Omega^2(M)$ and $c\,ds \wedge dt \in \Omega^2(U)$ with their pullbacks to $\widetilde{M}$. Prove that $\widetilde{\omega}$ is a symplectic form on $\widetilde{M}$ whenever

$$c(s,t) > \partial_s G_{s,t}(p) - \partial_t F_{s,t}(p) + \{F_{s,t}, G_{s,t}\}(p)$$

for all $(s,t) \in U$ and all $p \in M$. **Hint:** Compute $\widetilde{\omega}^{n+1}$.

(ii) Prove that the almost complex structure $\widetilde{J} \in \mathcal{J}(\widetilde{M})$, defined by

$$\widetilde{J} := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ JX_F - X_G & JX_G + X_F & J \end{pmatrix}, \tag{3}$$

is compatible with $\widetilde{\omega}$. Here we abbreviate

$$J := J_{s,t}(p), \qquad X_F := X_{F_{s,t}}(p), \qquad X_G := X_{G_{s,t}}(p)$$

for $\widetilde{p} = (s,t,p) \in U \times M = \widetilde{M}$. Note that $T_{\widetilde{p}}\widetilde{M} = \mathbb{R}^2 \times T_pM$ and that, with these abbreviations, the right hand side of (3) defines an automorphism of $T_{\widetilde{p}}\widetilde{M}$ in block form.

(iii) Prove that $u : U \to M$ is a solution of (1) if and only if the function $\widetilde{u} : U \to \widetilde{M}$, defined by $\widetilde{u}(s,t) := (s,t,u(s,t))$ is a $\widetilde{J}$-holomorphic curve.

(iv) Examine the energy identity for $\widetilde{u}$.

Exercise 4.2. Let $U \subset \mathbb{R}^d$ be an open set and $x_0 \in U$. Prove that the following are equivalent for every smooth function $w : U \to \mathbb{R}$ and every integer $k \geq d$.

(i)

$$\lim_{r \to 0} \frac{1}{r^k} \int_{B_r(x_0)} |w| = 0.$$

(ii) $\partial^\alpha w(x_0) = 0$ for every $\alpha = (\alpha_1, \dots, \alpha_d)$ with $|\alpha| = \alpha_1 + \cdots + \alpha_d \leq k - d$.

A smooth function $w : U \to \mathbb{R}$ is said to **vanish to infinite order at** x_0 if it satisfies these equivalent conditions for every integer $k \geq d$. An L^1-function $w : U \to \mathbb{R}$ is said to **vanish to infinite order at** x_0 if it satisfies (i) for every integer $k \geq d$.

Exercise 4.3. Denote $B_\varepsilon := \{z \in \mathbb{C} \,|\, |z| < \varepsilon\}$. Let $J : B_\varepsilon \to \mathrm{GL}_{\mathbb{R}}(\mathbb{C}^n)$ be a continuous map such that $J(z)^2 = -\mathbb{1}$ for every $z \in B_\varepsilon$. Prove that there exists a constant $0 < \delta \leq \varepsilon$ and a continuous function $\Psi : B_\delta \to \mathrm{GL}_{\mathbb{R}}(\mathbb{C}^n)$ such that

$$\Psi(z)^{-1} J(z) \Psi(z) = \mathbf{i}\mathbb{1}$$

for every $z \in B_\delta$. If J is continuously differentiable, show that Ψ can be chosen continuously differentiable. If J is smooth prove that Ψ can be chosen smooth. If J is of class $W^{1,p}$, $p > 2$, show that Ψ can be chosen of class $W^{1,p}$.