Symplectic Topology
Example Sheet 7

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Exercise 7.1. Let (V, ω) be a symplectic vector space and consider the space

$$\mathcal{J}(V,\omega) := \left\{ J \in \mathrm{End}(V) \;\middle|\; \begin{array}{l} J^2 = -\mathbb{1}, \\ \omega(\cdot, J\cdot) + \omega(J\cdot, \cdot) = 0, \\ \omega(v, Jv) > 0 \; \forall \, v \in V \setminus \{0\} \end{array} \right\}$$

of ω-compatible linear complex structure on V.

(i) Prove that the formula

$$\left\langle \widehat{J}_1, \widehat{J}_2 \right\rangle := \tfrac{1}{2}\mathrm{trace}(\widehat{J}_1 \widehat{J}_2) \tag{1}$$

for $\widehat{J}_i \in T_J\mathcal{J}(V,\omega)$ defines a Riemannian metric on $\mathcal{J}(V,\omega)$ and the formula

$$\Omega_J(\widehat{J}_1, \widehat{J}_2) := \tfrac{1}{2}\mathrm{trace}(\widehat{J}_1 J \widehat{J}_2) \tag{2}$$

defines a nondegenerate 2-form that is compatible with the Riemannian matric (1) and the almost complex structure $\widehat{J} \mapsto -J\widehat{J}$.

(ii) Prove that the formulas (1) and (2) define a Kähler structure on $\mathcal{J}(V,\omega)$.
Hint: Show that the Levi-Civita connection of the metric (1) is given by

$$\nabla_t \widehat{J} = \partial_t \widehat{J} - \tfrac{1}{2} J\big((\partial_t J)\widehat{J} + \widehat{J}(\partial_t J)\big) \tag{3}$$

for every smooth curve $\mathbb{R} \to \mathcal{J}(M,\omega) : t \mapsto J(t)$ and every smooth vector field $\widehat{J}(t) \in T_{J(t)}\mathcal{J}(V,\omega)$ along this curve. Show that it preserves the almost complex structure $\widehat{J} \mapsto -J\widehat{J}$.

Exercise 7.2. Prove that the diffeomorphism $\mathcal{S}_n \to \mathcal{J}(\mathbb{R}^{2n}, \omega_0) : Z \mapsto J(Z)$ in Exercise 2.11 is a Kähler isometry, where $\mathcal{S}_n$ denotes Siegel upper half space with its standard complex structure and the Riemannian metric on $\mathcal{S}_n$ is determined by the norm

$$|\widehat{Z}|_Z := \sqrt{\mathrm{trace}\left(\left(\widehat{X}Y^{-1}\right)^2 + \left(\widehat{Y}Y^{-1}\right)^2\right)}.$$

Exercise 7.3. (i) Prove that the Riemann curvature tensor of the Riemannian metric (1) on $\mathcal{J}(V, \omega)$ is given by

$$R(\widehat{J}_1, \widehat{J}_2)\widehat{J}_3 = -\frac{1}{2}\big[[\widehat{J}_1, \widehat{J}_2], \widehat{J}_3\big]. \tag{4}$$

Deduce that $\mathcal{J}(V, \omega)$ has nonpositive sectional curvature.

(ii) Prove that the geodesics on $\mathcal{J}(M, \omega)$ are given by $t \mapsto J\exp(-J\widehat{J}t)$ for $\widehat{J} \in T_J\mathcal{J}(V, \omega)$ and $t \in \mathbb{R}$. Deduce that $\mathcal{J}(V, \omega)$ is geodesically complete.

(iii) Prove that the exponential map

$$T_J\mathcal{J}(V, \omega) \to \mathcal{J}(V, \omega) : \widehat{J} \mapsto J\exp(-J\widehat{J}) \tag{5}$$

is a diffeomorphism for every $J \in \mathcal{J}(V, \omega)$.

Exercise 7.4. Let (M, ω) be a compact symplectic manifold and denote by $\mathcal{J}(M, \omega)$ the space of ω-compatible almost complex structures on M. Fix an element $J_0 \in \mathcal{J}(M, \omega)$ and define the distance function

$$d : \mathcal{J}(M, \omega) \times \mathcal{J}(M, \omega)$$

by

$$d(J_1, J_2) := \sum_{k=0}^{\infty} 2^{-k} \frac{\|\widehat{J}_1 - \widehat{J}_2\|_{C^k}}{1 + \|\widehat{J}_1 - \widehat{J}_2\|_{C^k}}, \qquad J_i =: J_0 \exp(-J_0\widehat{J}_i).$$

(Define what is meant by the C^k-norm of a section of a vector bundle.) Prove that $(\mathcal{J}(M, \omega), d)$ is a complete metric space. Prove that a sequence $J_\nu \in \mathcal{J}(M, \omega)$ converges to $J \in \mathcal{J}(M, \omega)$ with respect to the metric d if and only if it converges to J with respect to the C^∞ topology. Deduce that every countable intersection of open and dense (both with respect to the C^∞ topology) subsets of $\mathcal{J}(M, \omega)$ is dense.

Exercise 7.5. Let X, Y, Z be Banach spaces and let $D : X \to Y$ and $\Phi : Z \to Y$ be bounded linear operators such that D has a closed image and the operator

$$D \oplus \Phi : X \times Z \to Y, \qquad (D \oplus \Phi)(x, z) := Dx + \Phi z$$

is surjective. Consider the bounded linear operator $\Pi : W \to Z$ defined by

$$W := \ker(D \oplus \Phi) = \{(x, z) \in X \times Z \,|\, Dx + \Phi z = 0\}, \qquad \Pi(x, z) := z.$$

Prove that Π has a closed image and that the linear operators

$$\ker D \to \ker \Pi : x \mapsto (x, 0), \qquad \frac{Z}{\operatorname{im} \Pi} \to \frac{Y}{\operatorname{im} D} : [z] \mapsto [\Phi z]$$

are (well defined) Banach space isomorphisms.

Exercise 7.6. Let (M, J) be an almost complex manifold, (Σ, j) be a Riemann surface, and $u : \Sigma \to M$ be a J-holomorphic curve. Denote by

$$D_u : \Omega^0(\Sigma, u^*TM) \to \Omega^{0,1}_J(\Sigma, u^*TM)$$

the linearized operator defined by

$$D_u \widehat{u} := \tfrac{1}{2} \left(\nabla \widehat{u} + J(u) \nabla \widehat{u} \circ j \right) - \frac{1}{2} J(u) \left(\nabla_{\widehat{u}} J(u) \right) \partial_J(u)$$

for $\widehat{u} \in \Omega^0(\Sigma, u^*TM)$. (See Exercises 6.6 and 6.8.) Prove that

$$D_u(du \cdot \xi) = du \cdot \bar{\partial}_j \xi$$

for $\xi \in \mathrm{Vect}(\Sigma) = \Omega^0(\Sigma, T\Sigma)$. Here $\bar{\partial}_j : \Omega^0(\Sigma, T\Sigma) \to \Omega^{0,1}_j(\Sigma, T\Sigma)$ is the Cauchy–Riemann on the tangent bundle of Σ, determined by the complex structrure j.

Exercise 7.7. Let (Σ, j) be closed connected Riemann surfaces of positive genus and define $M := \Sigma \times \mathbb{C}\mathrm{P}^1$ with the product complex structure J.

(i) Let $u : \mathbb{C}\mathrm{P}^1 \to M$ be a J-holomorphic curve of the form

$$u(z) := (p, \phi(z))$$

for $z \in \mathbb{C}\mathrm{P}^1 \cong \mathbb{C} \cup \{\infty\}$, where $p \in \Sigma$ and $\phi : \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$ is a rational map of degree d. Compute the Fredholm index of D_u and show that D_u is surjective.

(ii) Let $u : \Sigma \to M$ be the J-holomorphic curve $u(z) := (z, q)$ for $z \in \Sigma$, where $q \in \mathbb{C}\mathrm{P}^1$. Compute the Fredholm index of D_u and show that D_u is not surjective.