Symplectic Topology
Example Sheet 9

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Exercise 9.1. Denote the standard basis of $\mathbb{R}^{2n}$ by $e_1, \ldots, e_{2n}$. Let $\lambda > 0$ and let $A \in \mathbb{R}^{2n \times 2n}$ be a matrix that satisfies

$$Ae_1 = \lambda e_1, \qquad Ae_2 = \lambda e_2.$$

Prove that the transposed matrix A^T maps the closed unit ball $B^{2n}(1)$ into $B^2(\lambda) \times \mathbb{R}^{2n-2}$.

Exercise 9.2. Let $f : (0, \infty) \to (0\infty)$ be a smooth function and define $\omega_f \in \Omega^2(\mathbb{R}^{2n} \setminus \{0\})$ by

$$\omega_f := F^*\omega_0, \qquad F(z) := f(|z|)\frac{z}{|z|}.$$

Prove that ω_f is compatible with the standard complex structure J_0. **Hint:** Use complex notation and show that ω_f is a $(1,1)$-form.

In the next exercise we denote the coordinates on $\mathbb{C}^n$ by $z = (z_1, \ldots, z_n)$ and abbreviate

$$\begin{aligned} dz \wedge d\overline{z} &:= \sum_{j=1}^n dz_j \wedge d\overline{z}_j, \\ z \cdot d\overline{z} &:= \sum_{j=1}^n z_j d\overline{z}_j, \qquad (1) \\ \overline{z} \cdot dz &:= \sum_{j=1}^n \overline{z}_j dz_j. \end{aligned}$$

Exercise 9.3. Define the 1-forms $\alpha_0 \in \Omega^1(\mathbb{C}^n)$ and $\alpha_{\mathrm{FS}} \in \Omega^1(\mathbb{C}^n \setminus \{0\})$ by

$$\begin{aligned} \alpha_0 &:= \frac{\mathbf{i}}{4}\big(z \cdot d\overline{z} - \overline{z} \cdot dz\big), \\ \alpha_{\mathrm{FS}} &:= \frac{\mathbf{i}}{4|z|^2}\big(z \cdot d\overline{z} - \overline{z} \cdot dz\big). \end{aligned} \tag{2}$$

Prove that

$$\begin{aligned} \omega_0 &:= d\alpha_0 = \frac{\mathbf{i}}{2} dz \wedge d\overline{z} = \frac{\mathbf{i}}{2}\partial\bar{\partial}|z|^2, \\ \rho_{\mathrm{FS}} &:= d\alpha_{\mathrm{FS}} = \frac{\mathbf{i}}{2}\left(\frac{dz \wedge d\overline{z}}{|z|^2} - \frac{\overline{z} \cdot dz \wedge z \cdot d\overline{z}}{|z|^4}\right) = \frac{\mathbf{i}}{2}\partial\bar{\partial}\log\left(|z|^2\right). \end{aligned} \tag{3}$$

Thus ρ_{FS} is the pullback of the Fubini–Study form ω_{FS} under the projection $\mathrm{pr} : \mathbb{C}^n \setminus \{0\} \to \mathbb{C}\mathrm{P}^{n-1}$. Define $F_\lambda : \mathbb{C}^n \setminus \{0\} \to \mathbb{C}^n \setminus B^{2n}(\lambda)$ by

$$F_\lambda(z) := \sqrt{\lambda^2 + |z|^2}\,\frac{z}{|z|} = \sqrt{1 + \frac{\lambda^2}{|z|^2}}\,z$$

and prove that

$$F_\lambda^*\alpha_0 = \alpha_0 + \lambda^2\alpha_{\mathrm{FS}}, \qquad F_\lambda^*\omega_0 = \omega_0 + \lambda^2\rho_{\mathrm{FS}}.$$

Exercise 9.4. Let $u : \mathbb{C} \to \mathbb{C}^n$ be a holomorphic function of the form

$$u(z) = z^m v(z)$$

where $v(0) \neq 0$. Prove that

$$\lim_{\delta \to 0} \int_{|z|=\delta} u^*\alpha_{\mathrm{FS}} = m\pi. \tag{4}$$

Hint: Consider first the case $v(z) \equiv a$ for some nonzero vector $a \in \mathbb{C}^n$.

Exercise 9.5. Prove that the set

$$\widetilde{\mathbb{C}}^n := \big\{([w_1 : \cdots : w_n], (z_1, \ldots, z)) \in \mathbb{C}\mathrm{P}^{n-1} \times \mathbb{C}^n \,|\, z_j w_k = z_k w_j \ \forall\, j, k\big\}$$

is a complex submanifold of $\mathbb{C}\mathrm{P}^{n-1} \times \mathbb{C}^n$ and that

$$Z := \mathbb{C}\mathrm{P}^{n-1} \times \{0\}$$

is a complex submanifold of $\widetilde{\mathbb{C}}^n$. Prove that the pullback of $\omega_0 + \lambda^2\rho_{\mathrm{FS}}$ under the projection $\pi : \widetilde{\mathbb{C}}^n \setminus Z \to \mathbb{C}^n \setminus \{0\}$ extends to a Kähler form on $\widetilde{\mathbb{C}}^n$.

Exercise 9.6. Let $J \in \mathcal{J}(\mathbb{C}\mathrm{P}^2, \omega_{\mathrm{FS}})$ be any almost complex structure on $\mathbb{C}\mathrm{P}^2$ that is compatible with the Fubini–Study form ω_{FS}. Let $A := [\mathbb{C}\mathrm{P}^1]$ be the positive generator of $H_2(\mathbb{C}\mathrm{P}^2; \mathbb{Z})$, i.e. the homology class of the line. Consider the evaluation map

$$\mathrm{ev}_2 : \mathcal{M}_2(A; J) := \frac{\mathcal{M}(A; J) \times S^2 \times S^2}{\mathrm{PSL}(2, \mathbb{C})} \to \mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^2.$$

Prove that an element $(p_1, p_2) \in \mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^2$ is a regular value of ev_2 if and only if $p_1 \neq p_2$. Deduce that any two distinct points in $\mathbb{C}\mathrm{P}^2$ are contained in the image of a *unique* (up to reparametrization) J-holomorphic sphere representing the homology class A.