The Exponential Vandermonde Matrix

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Fix real numbers $a_0 < a_1 < a_2 < \cdots < a_n$. Given positive numbers $x_0, x_1, \ldots, x_n$ define the $(n+1) \times (n+1)$ matrix

$$W(x) = \begin{bmatrix} x_0^{a_0} & -x_0^{a_1} & -x_0^{a_2} & \cdots & -x_0^{a_n} \\ x_1^{a_0} & x_1^{a_1} & -x_1^{a_2} & \cdots & -x_1^{a_n} \\ x_2^{a_0} & x_2^{a_1} & x_2^{a_2} & \cdots & -x_2^{a_n} \\ & & & \ddots & \\ x_n^{a_0} & x_n^{a_1} & x_n^{a_2} & \cdots & x_n^{a_n}; \end{bmatrix};$$

the entries above the diagonal are negative, those on or below the diagonal are positive.

Theorem. *The signed* **exponential Vandermonde determinant**

$$w(x_0, x_1, x_2, \ldots, x_n) = \det\big(W(x)\big)$$

is positive for $0 < x_0 \le x_1 \le x_2 \le \cdots \le x_n$.

Proof: If we divide each row of the matrix $W(x)$ by its leading entry we get another matrix of the same form with a_i replaced by $a_i - a_0$. Hence we assume w.l.o.g. that $a_0 = 0$. We prove the following stronger statement by induction on n: *The function*

$$w_m(x) := \frac{\partial^m w(x_0, x_1, \ldots, x_n)}{\partial x_n \partial x_{n-1} \cdots \partial x_{n-m+1}}$$

is positive for $m = 0, 1, 2, \ldots, n$ *and* $0 < x_0 \le x_1 \le x_2 \le \cdots \le x_n$.

Since the determinant of a matrix is linear in each row and the ith row of the matrix $W(x)$ depends only on x_i we have that

$$w_m(x) = \det\bigl(W_m(x)\bigr)$$

where the matrix $W_m(x)$ results from $W(x)$ by replacing the ith row by its derivative with respect to x_i for $i = n{-}m{+}1, \ldots, n$. Note that $W(x) = W_0(x)$ and

(1) for $i = 0, 1, \ldots, n - m$ the ith row of $W_m(x)$ is the same as the ith row of $W(x)$ and begins with 1, and

(2) for $i = n - m + 1, \ldots, n$ the ith row of $W_m(x)$ begins with 0.

Lemma. If $0 < k \leq n - m$ and $x_{k-1} = x_k$ then

$$w_m(x) = 2x_k^{a_k} w_m(x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n). \qquad (\#)$$

The term on the right in (#) is the determinant of the $n \times n$ matrix which results by deleting the kth row and column from $W_m(x)$. Prove the lemma as follows: Subtract the $(k-1)$st row of $W_m(x)$ from the kth row. The result has the same determinant, its kth row vanishes off the diagonal, and its (k, k) entry is $2x_k^{a_k}$. The formula (#) now follows by expansion by minors on the kth row.

We now prove that $w_m(x) > 0$ by backwards induction on m. First consider the case $m = n$. The off diagonal entries in the 0th column of $W_n(x)$ vanish and for $i, j > 0$ the (i, j) entry is $\pm a_j x_i^{a_j - 1}$. Hence

$$w_n(x) = a_1 a_2 \cdots a_n w_0(x_1, x_2, \ldots, x_n)$$

by expansion by minors in the top row and then factoring out a_j from the jth column. The term $w_0(x_1, x_2, \ldots, x_n)$ on the right is of the same type as $w(x)$ but it is the determinant of an $n \times n$ matrix and the exponents are $a_i - 1$. Hence by the induction hypothesis (on n) $w_n(x)$ is positive. Now assume by the induction hypothesis (on m) that $w_{m+1} = \partial w_m / \partial x_{n-m}$ is positive. By the lemma and the induction hypothesis (on n) w_m is positive when $x_{n-m} = x_{n-m-1}$. Hence w_m is positive by integration with respect to x_{n-m}.

Remark. The unsigned exponential Vandermonde determinant is the same but without the minus signs above the diagonal. A slight simplification of our argument shows that it is positive for $0 < x_0 < x_1 < x_2 < \cdots < x_n$: the lemma is not needed since the analogue of w_m is zero when $x_{n-m} = x_{n-m-1}$.