SPIN GEOMETRY AND SEIBERG-WITTEN INVARIANTS

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PREFACE

Over the last year remarkable new developments have no less than revolutionized the subject of 4-manifold topology. When Seiberg and Witten discovered their monopole equations in October 1994 it was soon realized by Kronheimer, Mrowka, Taubes, and others that these new invariants led to remarkably simpler proofs of many of Donaldson's theorems and gave rise to new interconnections between Riemannian geometry, 4-manifolds, and symplectic topology. For example, manifolds with nontrivial invariants do not admit metrics of positive scalar curvature, Kronheimer and Mrowka finally settled the Thom conjecture, and Taubes proved that symplectic 4-manifolds have nontrivial invariants, thus settling a longstanding conjecture related to the existence of symplectic structures. One of the deepest and most striking new results in this circle of ideas is Taubes' theorem about the relation between the Seiberg-Witten and the Gromov invariants in the symplectic case. This can be interpreted as an existence theorem for J-holomorphic curves and it gave rise to a number of new theorems about symplectic 4-manifolds which extend known results from Kähler geometry. There were also new theorems about Kähler surfaces such as minimal Kähler surfaces which admit a metric of positive scalar curvature are rational or ruled, and for minimal surfaces of general type the canonical class is, up to sign, a differentiable invariant. Witten conjectured that the new invariants should, in the case of 4-manifolds of simple type, be equivalent to the Donaldson invariants. A geometric approach for proving this conjecture was developed by Pidstrigach and Tyurin and was first announced by Pidstrigach in his lectures at the Newton Institute in December 1994.

The purpose of this book is to give a comprehensive and largely self-contained introduction to the Seiberg-Witten invariants, including the necessary background material from geometry and analysis and many of the applications to 4-manifold topology and symplectic and Kähler geometry. A notable exception is that the book says nothing about the physics and quantum field theory background from which these new ideas originated. Although this is a subject of great importance which will undoubtedly lead to many more fruitful interactions with geometry and other branches of mathematics, the author lacks the expertise required for an exposition of these ideas. Two other omissions are that the Pidstrigach-Tyurin approach to the proof of Witten's conjecture will not be discussed and the proof of Taubes' theorem about the Seiberg-Witten and the Gromov invariants will only be briefly sketched. Moreover, the book does not contain an exposition of classical material from 4-manifold topology. An excellent reference for this is the first chapter of [21], for example.

The book has four parts. The first part is devoted to background material from gauge theory (Chapter 1), Riemannian geometry (Chapter 2) and complex geometry (Chapter 3). In particular, the latter chapter contains an extensive discussion of Hermitian connections on the tangent bundle of symplectic manifolds and their relation with Cauchy-Riemann operators, and of the Dolbeault cohomology of Kähler manifolds and the Hirzebruch-Riemann-Roch theorem (without proof).

The reader who is primarily interested in the Seiberg-Witten invariants is advised to begin with Part II and refer back to the earlier chapters as necessary. Chapter 4 gives an exposition of foundational material about spin geometry and Clifford algebras. This chapter lays the foundation for the classification of spin and spin^c structures on vector bundles in Chapter 5. Chapter 6 is devoted to Dirac operators. In particular, the relation of the Dirac operator to the Cauchy-Riemann operator is examined in the symplectic case, the Weitzenböck formula is proved, and applications to manifolds with positive scalar curvature are discussed.

The heart of the book is Part III which begins the introduction to the Seiberg-Witten invariants. Chapter 7 discusses the fundamental properties of the solutions of the Seiberg-Witten monopole equations and shows how they can be used to construct 4-manifold invariants. In the case $b^+(X) \geq 2$ these have the form of a map

$$\mathrm{SW} : \mathcal{S}^c(X) \to \mathbb{Z}$$

which assigns to every spin^c structure Γ on TX an integer $\mathrm{SW}(X, \Gamma)$.* Denote by $c = c_1(L_\Gamma) \in H^2(X, \mathbb{Z})$ the characteristic class of the spin^c structure Γ. This is an integral lift of $\mathrm{w}_2(TX) \in H^2(X, \mathbb{Z}_2)$. c is called a **basic class** if $\mathrm{SW}(X, \Gamma) \neq 0$ for some spin^c structure Γ with $c_1(L_\Gamma) = c$. The Seiberg-Witten invariants satisfy the following axioms.

(Naturality) If X and Y are compact oriented smooth 4-manifolds with $b^+ \geq 2$, $f : X \to Y$ is an orientation preserving diffeomorphism, and $\Gamma \in \mathcal{S}^c(Y)$ then†

$$\mathrm{SW}(X, f^*\Gamma) = \mathrm{SW}(Y, \Gamma).$$

(Dimension) Every basic class of X satisfies

$$c \cdot c \geq 2\chi(X) + 3\sigma(X).$$

*The map is only defined up to a sign which depends on a choice of orientation of $H^1(X) \oplus H^{2,+}(X)$. Moreover, the invariant is only defined under the assumption that $b^+ - b_1$ is odd and we shall use the convention $\mathrm{SW}(X, \Gamma) = 0$ when this condition is not satisfied.

†Here the orientation of $H^1(X) \oplus H^{2,+}(X)$ is understood to be induced by f.

(Symmetry) The invariants of Γ and its dual structure $\bar{\Gamma}$ are related by[‡]

$$\mathrm{SW}(X, \bar{\Gamma}) = (-1)^{\frac{\chi(X)+\sigma(X)}{4}} \mathrm{SW}(X, \Gamma).$$

(Finiteness) Every compact oriented smooth 4-manifold with $b^+ \geq 2$ has only finitely many basic classes.

(Scalar curvature) If X has a metric of positive scalar curvature then all the Seiberg-Witten invariants of X are zero.

(Connected sum) If X_1 and X_2 are compact oriented smooth 4-manifolds with $b^+ > 0$, then the Seiberg-Witten invariants of $X_1 \# X_2$ are all zero.

(Blowup) If X and N are compact oriented smooth 4-manifolds with $b^+(X) \geq 2$, $b_1(N) = b^+(N) = 0$, and $\Gamma_N \in \mathcal{S}^c(N)$, $\Gamma \in \mathcal{S}^c(X)$ are spin^c structures whose characteristic classes $c = c_1(L_\Gamma) \in H^2(X, \mathbb{Z})$ and $e = c_1(L_{\Gamma_N}) \in H^2(N, \mathbb{Z})$ satisfy

$$c \cdot c - 2\chi(X) - 3\sigma(X) + e \cdot e + b_2(N) \geq 0$$

then

$$\mathrm{SW}(X \# N, \Gamma \# \Gamma_N) = \mathrm{SW}(X, \Gamma).$$

In particular, the basic classes of $X \# N$ have the form $c' = c + e$ where $c \in H^2(X, \mathbb{Z})$ is a basic class of X and $e \in H^2(N, \mathbb{Z})$ is a characteristic vector.

(Genus) Let X be a compact oriented smooth 4-manifold with $b^+ \geq 2$ and $\Sigma \subset X$ be a compact connected oriented embedded 2-manifold which represents a nontorsion homology class $[\Sigma] \in H_2(X, \mathbb{Z})$. Suppose that $\Sigma \cdot \Sigma \geq 0$. Then the genus of Σ satisfies

$$2g(\Sigma) - 2 \geq \Sigma \cdot \Sigma + |c \cdot \Sigma|$$

for every basic class c of X.

(Symplectic) Let (X, ω) be a compact symplectic 4-manifold with $b^+ \geq 1$ and $\Gamma_{\mathrm{can}} \in \mathcal{S}^c(X)$ be the canonical spin^c structure of an almost complex structure on TX which is compatible with ω. Then[*]

$$\mathrm{SW}(X, \Gamma_{\mathrm{can}}) = 1.$$

The proofs of the naturality, dimension, symmetry, finiteness and scalar curvature axioms are given in Chapter 7 while the remaining axioms are

[‡]Note that $\chi + \sigma = 2(1 + b^+ - b_1)$ is divisible by 4 if and only if $b^+ - b_1$ is odd.

[*]On a symplectic manifold the space $H^1(X, \mathbb{R}) \oplus H^{2,+}(X, \mathbb{R})$ has a natural orientation which is explained in Remark 13.35.

deeper theorems whose proofs are deferred to the later chapters. The genus axiom is due to Kronheimer and Mrowka and the normalization axiom in the stated form is due to Taubes. The existence of an invariant with all these properties has some immediate nontrivial consequences. For example, the genus and normalization axioms together imply the Thom conjecture for the case $b^+ \geq 2$ and for holomorphic curves with nonnegative self-intersection. The scalar curvature and normalization axioms imply that symplectic 4-manifolds with $b^+ \geq 2$ do not admit metrics of positive scalar curvature. This can be used to prove nonexistence results for symplectic structures on certain 4-manifolds (Taubes) and to give a new proof of Donaldson's theorem that Kähler surfaces of general type cannot be diffeomorphic to connected sums of several copies of the projective plane (with both orientations) even though, by Freedman's theorem, every simply connected nonspin Kähler surface is homeomorphic to such a connected sum.

Chapter 8 deals with some of the more technical aspects of the theory such as the proof of the compactness, regularity, and transversality theorems as well as a removable singularity theorem for the solutions of the Seiberg-Witten equations on Euclidean space. In a first section it also contains an explicit discussion of the Seiberg-Witten equations on flat $\mathbb{R}^4$ and this might be a good starting point for the reader to get a feel for the equations.

Chapter 9 contains several applications, new and old, of the Seiberg-Witten invariants in general 4-manifold topology, including a proof of Donaldson's classical theorem about the diagonalizability of definite intersection forms, an account of Furuta's proof of the 10/8-*conjecture*, and a proof of the general wall-crossing formula of Li-Liu and Ohta-Ono in the case $b^+ = 1$. In this case there are two invariants $\mathrm{SW}^{\pm}(X, \Gamma)$ depending on the choice of the metric and the wall-crossing formula is an expression for the difference of these invariants.

The subject of Chapter 10 are the connected sum and blowup axioms. The proof of the vanishing theorem for connected sums is based on the limiting behavious of solutions of the Seiberg-Witten equations for a sequence of metrics which *pinch the neck*. This result only uses compactness theorems and the removable singularity theorem (both proved in Chapter 8). The proof of the blowup formula is considerably harder and requires gluing techniques for solutions of the Seiberg-Witten equations on 4-manifolds with cylindrical ends. Geometrically this corresponds to *stretching the neck* rather than *pinching* it. The analysis in the proof is also needed for the construction of Seiberg-Witten Floer homology (which will not be carried out in this book).

Part IV introduces applications of the Seiberg-Witten invariants in Kähler geometry (Chapter 11), gives a proof of the Thom conjecture and other vanishing theorems (Chapter 12), and discusses applications to symplectic 4-manifolds (Chapter 13). In Chapter 11 it is proved that Kähler surfaces have nontrivial Seiberg-Witten invariants and that for minimal surfaces plus and minus the canonical class are the only basic classes. As a result the canonical class is, up to sign, a diffeomorphism invariant. Another new theorem is that Kähler surfaces are irreducible (at least in the simply connected case) and that the only minimal Kähler surfaces with positive scalar curvature are blowups of rational and ruled surfaces. Another interesting observation is that the moduli space of Seiberg-Witten monopoles can in the Kähler case be naturally identified with the set of divisors (in the class e where $c_1(L_\Gamma) = 2e - c_1(K)$). The chapter also contains a computation by Mrowka of the Seiberg-Witten invariants for elliptic surfaces.

Chapter 12 gives a proof of the Thom conjecture which asserts that embedded complex curves in Kähler surfaces minimize the genus among all embedded surfaces representing the same homology class. This conjecture has now been confirmed for all Kähler surfaces under the assumption of nonnegative self-intersection number.

Chapter 13 deals with applications to symplectic 4-manifolds. It begins with a brief introduction to the existence question for symplectic structures and then discusses the basic theorems of Taubes about the nontriviality of the invariants. Some of the immediate consequences include, for example, the result that all almost complex structures on the 4-torus which are compatible with some symplectic form must have Chern classes zero, and that the manifold $\mathbb{C}P^2 \# \mathbb{C}P^2 \# \mathbb{C}P^2$, for example, does not admit a symplectic structure. Much more interesting consequences can be derived from Taubes' existence theorem for J-holomorphic curves which can then be combined with the work of Gromov and McDuff. Some of the corollaries are that minimal symplectic 4-manifolds with positive scalar curvature or $K \cdot [\omega] < 0$ or $K \cdot K < 0$ are rational or ruled (Ohta-Ono, Li-Liu), that symplectic structures on rational and ruled surfaces are unique up to diffeomorphism and deformation (Taubes, Li-Liu, Lalonde-McDuff), that smooth blowup is equivalent to symplectic blowup (Taubes), and that simply connected symplectic 4-manifolds are irreducible (Kotschick). Many of these results are symplectic versions of known theorems in Kähler geometry. A notable exception is the result that for minimal Kähler surfaces of general type plus and minus the canonical class are the only basic classes. There is no symplectic analogue of this theorem and Kähler and symplectic geometry appear to diverge here. The chapter closes with a brief sketch of the proof of Taubes' theorem about the Seiberg-Witten and the Gromov invariants.

The book includes an appendix about various topics in analysis which form essential background material for the construction of moduli spaces in geometry. Appendix A is devoted to linear Fredholm theory and determinant line bundles and Appendix B deals with the implicit function theorem, the Sard-Smale theorem, and applications to transversality problems. Sobolev spaces and elliptic operators are discussed in Appendix C.

Appendix D gives a proof of an existence and uniqueness theorem for solutions of the Kazdan-Warner equation. Appendix E contains a proof of a unique continuation theorem for first order operators, based on the Agmon-Nirenberg technique, and with applications to Dirac operators. Finally, Appendix F discusses line bundles and divisors and includes a brief introduction to several complex variables.

I include here some remarks about the relation between 4-manifolds, Yang-Mills equations, and Seiberg-Witten invariants which arose out of a conversation with Simon Donaldson. His original work on 4-manifold invariants from Yang-Mills moduli spaces can be viewed in two ways. *Either* the topology of these moduli spaces would be described in terms of known invariants of 4-manifolds *or* the Yang-Mills equations would give rise to new 4-manifold invariants. Of course, we know now that the outcome of his work was the second alternative. However, both possibilities would have been interesting. The new Seiberg-Witten invariants can in some sense be viewed as an intermediate answer to the above dichotomy. They give information about 4-manifolds and, if Witten's conjecture is true, then Donaldson's invariants can be expressed in terms of these. From this point of view the 4-manifold invariants of Seiberg-Witten would then give us information about the Yang-Mills moduli spaces. That Donaldson's invariants were discovered earlier is an accident of history and it might as well have been the other way round, as one could imagine from the work of Gromov and Lawson in [48] on manifolds of positive scalar curvature which also involves the Dirac operator in a crucial way. To put it in different terms *"mathematical discoveries are not necessarily made in the logical order"*. (Quotation from Rebecca Earle.)

This book had its starting point in two lectures given by Peter Kronheimer in Oxford in the beginning of November 1994 about the Seiberg-Witten invariants which at the time had just been discovered for a few weeks. In the following spring I gave a lecture course about this subject at Warwick and proceeded with writing this manuscript alongside. I am indebted to many people who made helpful comments and suggestions at various stages and others whose lectures on new developments were a source of inspiration and influenced the contents of this book. I would like to thank them all. In particular, I would like to thank Miguel Abreu, Stefan Bauer, Simon Donaldson, Peter Kronheimer, John Jones, Dusa McDuff, Mario Micallef, Tom Mrowka, John Rawnsley, Miles Reid, and Joel Robbin for numerous discussions about various aspects of the theory which greatly aided my understanding.

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Part I

FOUNDATIONS

1

CONNECTIONS AND CURVATURE

The purpose of this chapter is to give an exposition of background material about vector bundles, connections, characteristic classes, and an application in K-theory. The first section is devoted to connections and curvature from the principal bundle and vector bundle point of view. Section 1.4 gives a brief introduction to the Chern classes via Chern-Weil theory. Section 1.7 gives an application which expresses the integral of a characteristic class over the projectivized *kernel manifold* of a regular family of Fredholm operators in terms of the topological index. This result will play a crucial role in the proof of the wall-crossing formula for the Seiberg-Witten invariants.

1.1 Fiber bundles

A smooth map $\pi : E \to X$ between smooth manifolds is called a **locally trivial fibration** if there exists an open cover $\{U_\alpha\}_\alpha$ of X, a smooth manifold F, and a collection of diffeomorphisms $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ such that the following diagrams commute

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[rr, "\varphi_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times F \arrow[dl, "\mathrm{pr}"] \\ & U_\alpha & \end{tikzcd} .$$

The maps φ_α are called **local trivializations**. Let $E_x = \pi^{-1}(x)$ denote the fiber over x and $\varphi_\alpha(x) : E_x \to F$ the restriction of φ_α to E_x followed by the projection onto F. The **transition maps** $u_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{Diff}(F)$ are defined by

$$u_{\beta\alpha}(x) = \varphi_\beta(x) \circ \varphi_\alpha(x)^{-1}.$$

Thus

$$\varphi_\beta \circ {\varphi_\alpha}^{-1}(x, v) = (x, u_{\beta\alpha}(x)v)$$

for $x \in U_\alpha \cap U_\beta$ and $v \in F$. The transition maps satisfy the cocycle condition

$$u_{\gamma\beta} u_{\beta\alpha} = u_{\gamma\alpha}, \qquad u_{\alpha\alpha} = 1. \tag{1.1}$$

The bundle E can be recovered from the $u_{\beta\alpha}$ as the set of equivalence classes $[\alpha, x, v]$ with $x \in U_\alpha$ and $v \in F$ under the equivalence relation $[\alpha, x, v] \equiv [\beta, x, u_{\beta\alpha}(x)v]$. A section $s : X \to E$ is in local coordinates represented by functions $s_\alpha : U_\alpha \to F$ defined by $s_\alpha(x) = \varphi_\alpha(x)s(x)$. The s_α satisfy $s_\beta(x) = u_{\beta\alpha}(x)s_\alpha(x)$ for $x \in U_\alpha \cap U_\beta$. Conversely, any such collection $\{s_\alpha\}_\alpha$ determines a section s. The space of sections will be denoted by $C^\infty(X, E)$ or sometimes, in the vector bundle case, by $\Omega^0(X, E)$.

Exercise 1.1 Let M be a compact smooth manifold with boundary and $f : M \to [0, 1]$ be a smooth surjection without critical points such that

$$\partial M = f^{-1}(0) \cup f^{-1}(1).$$

Prove that there is a diffeomorphism

$$\varphi : f^{-1}(0) \times [0, 1] \to M.$$

Hint: Choose a Riemannian metric on M and, for every $x \in M$ let $H_x \subset T_xM$ be the orthogonal complement of the vertical subspace $V_x = \ker df(x)$. For $x_0 \in f^{-1}(0)$ there is a unique path $x : [0, 1] \to M$ such that $x(0) = x_0$, $f(x(t)) = t$, and $\dot{x}(t) \in H_{x(t)}$ for every t. Prove that $x(t)$ is the solution of the differential equation

$$\dot{x} = \frac{\nabla f(x)}{|\nabla f(x)|^2}, \qquad x(0) = x_0.$$

Use the solutions to define φ. □

Exercise 1.2 Suppose that E and X are compact connected manifolds and $\pi : E \to X$ is a surjective submersion. Prove that E admits the structure of a locally trivial fibration. **Hint:** Choose a splitting $TX = V \oplus H$ where $V_\eta = \ker d\pi(\eta)$ for $\eta \in E$. □

Let $\mathrm{G} \subset \mathrm{Diff}(F)$ be a Lie group acting on F via

$$\mathrm{G} \times F \to F : (g, v) \mapsto gv.$$

This is to be understood as a left action, i.e. $g(hv) = (gh)v$ for $g, h \in \mathrm{G}$ and $v \in F$. The bundle E is said to have **structure group** G if there exists a system $\{U_\alpha, \varphi_\alpha\}_\alpha$ of local trivializations such that all the transition maps take values in G. In this case E is called a **G-bundle** and $\{U_\alpha, \varphi_\alpha\}_\alpha$ a **G-atlas**. An **automorphism** or **gauge transformation** of a G-bundle $E \to X$ is a smooth map $u : E \to E$ such that $\pi \circ u = \pi$ and the maps $u_\alpha : U_\alpha \to \mathrm{Diff}(F)$ defined by $u_\alpha(x) = \varphi_\alpha(x) \circ u \circ \varphi_\alpha(x)^{-1}$ for $x \in U_\alpha$ take values in G. The maps $u_\alpha : U_\alpha \to \mathrm{G}$ satisfy

$$u_\beta u_{\beta\alpha} = u_{\beta\alpha} u_\alpha$$

and, conversely, any such collection $\{u_\alpha\}_\alpha$ determines an automorphism $u : E \to E$. The group of such automorphisms is called the **gauge group** and will be denoted by $\mathcal{G}(E)$.

There are a number of interesting structure groups related to additional structures on the bundle E. For example, if $F = \mathbb{R}^n$ and $\mathrm{G} = \mathrm{GL}(n)$ then E is a vector bundle (of rank n). If in addition the bundle is oriented, then the structure group reduces to $\mathrm{G} = \mathrm{SL}(n, \mathbb{R})$ and the frame bundle (see below) consists of oriented bases. If an oriented vector bundle is equipped with a Riemannian metric then the structure group reduces to $\mathrm{SO}(n)$ and the frame bundle consists of oriented orthonormal frames. If E is a real vector bundle of rank $2n$ and is equipped with a complex structure $J \in \mathrm{Aut}(E)$ with $J^2 = -\mathbb{1}$ then the structure group reduces to $\mathrm{GL}(n, \mathbb{C})$ and the frame bundle consists of complex bases. If in addition E is equipped with a Hermitian structure then the structure group reduces to $\mathrm{U}(n)$ and the frame bundle consists of unitary bases.

Frame bundles

Note that the group G acts on the *model fiber* F but not on the *actual fibers* E_x of the bundle E. An action of G on E_x depends on the choice of the identification of F with E_x, i.e. on a choice of *frame*. More precisely, a **G-frame** at x is a diffeomorphism $e : F \to E_x$ such that $\varphi_\alpha(x) \circ e \in \mathrm{G}$ for any α with $x \in U_\alpha$. They form the **G-frame bundle**

$$\mathcal{F}(E) = \{(x, e) \,|\, x \in X,\ e : F \to E_x \text{ is a G-frame}\}$$

with right G-action $(x, e) \mapsto (x, e \circ g)$ for $g \in \mathrm{G} \subset \mathrm{Diff}(F)$. More generally, a **principal G-bundle** is a locally trivial fiber bundle

$$\pi : P \to X$$

with fiber $F = \mathrm{G}$ which is equipped with a smooth right G-action

$$P \times \mathrm{G} \to P : (p, g) \mapsto pg$$

which preserves the fibers (i.e. $\pi(pg) = \pi(p)$) and an atlas of *equivariant* local trivializations $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathrm{G}$. In this case the transition maps have the form $\varphi_\beta \circ {\varphi_\alpha}^{-1}(x, g) = (x, u_{\beta\alpha}(x)g)$ for $x \in U_\alpha \cap U_\beta$ and $g \in \mathrm{G}$. The bundle P can be recovered from the transition maps as the set of equivalence classes $[\alpha, x, g]$ with $x \in U_\alpha$ and $g \in \mathrm{G}$ under the equivalence relation $[\alpha, x, g] \equiv [\beta, x, u_{\beta\alpha}(x)g]$. Denote by

$$TP \times \mathrm{G} \to TP : (v, g) \mapsto vg$$

the action of G on the tangent bundle and by

$$P \times \mathfrak{g} \to TP : (p, \xi) \mapsto p\xi = \left.\frac{d}{dt}\right|_{t=0} p \exp(t\xi)$$

the **infinitesimal action** of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(\mathrm{G})$.

Remark 1.3 If the $u_{\beta\alpha}$ are the transition maps of a fiber bundle $E \to X$ with structure group $\mathrm{G} \subset \mathrm{Diff}(F)$ and $P = \mathcal{F}(E)$ is the principal G-frame bundle then the two descriptions of P (as pairs (x, e) and as equivalence classes $[\alpha, x, g]$) are related by

$$p = (x, e) = [\alpha, x, g], \qquad g = \varphi_\alpha(x) \circ e$$

for $x \in U_\alpha$ and $g \in \mathrm{G}$. □

Exercise 1.4 Let P be a compact smooth manifold and G be a compact Lie group acting smoothly and freely on P. Prove that the quotient $X = P/\mathrm{G}$ admits the structure of a smooth manifold and that the natural projection $\pi : P \to X$ is a principal G-bundle. **Hint:** Prove the existence of local slices. □

Remark 1.5. (Associated bundles) Given any principal G-bundle $\pi : P \to X$ and a representation

$$\rho : \mathrm{G} \to \mathrm{Diff}(F)$$

there is a locally trivial G-bundle

$$E = P \times_\rho F.$$

This bundle is defined as the set of equivalence classes of pairs $[p, v]$ in $P \times F$ under the equivalence relation

$$[p, v] \equiv [pg, \rho(g)^{-1}v]$$

for $g \in \mathrm{G}$. The reader may check that if $\rho : \mathrm{G} \to \mathrm{Diff}(F)$ is injective then the frame bundle $\mathcal{F}(P \times_\rho F)$ is isomorphic to P and, conversely, the associated bundle $\mathcal{F}(E) \times_{\mathrm{G}} F$ is isomorphic to E. The reader may also check that the sections of $P \times_\rho F$ can be identified with smooth maps $s : P \to F$ which satisfy

$$s(pg) = \rho(g)^{-1}s(p)$$

for $p \in P$ and $g \in \mathrm{G}$. An important special case is the adjoint representation of G on its Lie algebra $\mathfrak{g} = \mathrm{Lie}(\mathrm{G})$. The corresponding associated bundle is denoted by

$$\mathfrak{g}_P = P \times_{\mathrm{ad}} \mathfrak{g}.$$

The sections of this bundle are maps $\xi : P \to \mathfrak{g}$ which satisfy

$$\xi(pg) = g^{-1}\xi(p)g.$$

They form a Lie algebra denoted by $\Omega^0(X, \mathfrak{g}_P) = \Omega^0_{\mathrm{ad}}(P, \mathfrak{g})$. □

Remark 1.6. (Gauge transformations) A **gauge transformation** of a principal bundle $\pi : P \to X$ is an equivariant diffeomorphism $\varphi : P \to P$ such that

$$\pi \circ \varphi = \pi.$$

Any such gauge transformation has the form

$$\varphi(p) = pu(p)$$

where $u : P \to \mathrm{G}$ satisfies

$$u(pg) = g^{-1}u(p)g.$$

These maps u form the **gauge group** $\mathcal{G}(P)$. Its Lie algebra is

$$\mathrm{Lie}(\mathcal{G}(P)) = \Omega^0_{\mathrm{ad}}(P, \mathfrak{g}).$$

In the case of the G-frame bundle $P = \mathcal{F}(E)$ there is a one-to-one correspondence between automorphisms of E and gauge transformations of P. Namely, if $u : E \to E$ is an automorphism then the corresponding gauge transformation of $\mathcal{F}(E)$ is given by

$$\mathcal{F}(E) \to \mathrm{G} : (x, e) \mapsto e^{-1} \circ u|_{E_x} \circ e. \qquad \Box$$

Remark 1.7 For any vector bundle $E \to X$ denote by $\Omega^k(X, E)$ the space of differential k-forms τ on X with values in E. In local trivializations such a form can be represented by a collection of vector valued forms

$$\tau_\alpha \in \Omega^k(U_\alpha, \mathbb{R}^n)$$

which satisfy $\tau_\beta = u_{\beta\alpha}\tau_\alpha$. If

$$E = P \times_\rho V$$

for some principal G-bundle, some vector space V, and some representation $\rho : \mathrm{G} \to \mathrm{Aut}(V)$, then the k-forms with values in E can be identified with forms $\tau \in \Omega^k(P, V)$ which satisfy

$$\tau_{pg}(v_1 g, \dots, v_k g) = \rho(g)^{-1}\tau_p(v_1, \dots, v_k)$$

for $g \in \mathrm{G}$ and

$$\tau_p(p\xi, v_2, \dots, v_k) = 0$$

for $\xi \in \mathfrak{g}$ and $v_j \in T_pP$. This means that τ is **equivariant** and **horizontal**. The space of such forms will be denoted by

$$\Omega^k_\rho(P, V) \cong \Omega^k(X, P \times_\rho V). \qquad \Box$$

1.2 Connections

Let $E \to X$ be a smooth vector bundle. A **connection** on E is a linear operator $\nabla : C^\infty(X, E) \to \Omega^1(X, E)$ such that

$$\nabla(fs) = f\nabla s + df \otimes s$$

for $f : X \to \mathbb{R}$ and $s \in C^\infty(X, E)$. The difference of any two connections on E is an operator $a = \nabla^2 - \nabla^1 : C^\infty(X, E) \to \Omega^1(X, E)$ which is linear over the functions, i.e. $a(fs) = fa(s)$ for $f : X \to \mathbb{R}$ and $s \in C^\infty(X, E)$. Any such operator is given by multiplication with an endomorphism valued 1-form which we also denote by $a \in \Omega^1(X, \mathrm{End}(E))$. Thus the space of connections on E is an affine space with associated vector space $\Omega^1(X, \mathrm{End}(E))$.

In a vector bundle atlas $\{U_\alpha, \varphi_\alpha\}_\alpha$ a connection can be represented by a collection of matrix valued 1-forms $A_\alpha \in \Omega^1(U_\alpha, \mathbb{R}^{n\times n})$ via

$$(\nabla s)_\alpha = ds_\alpha + A_\alpha s_\alpha.$$

The A_α are called the **connection potentials** of ∇. They satisfy the condition

$$A_\alpha = {u_{\beta\alpha}}^* A_\beta = {u_{\beta\alpha}}^{-1} du_{\beta\alpha} + {u_{\beta\alpha}}^{-1} A_\beta u_{\beta\alpha}. \tag{1.2}$$

Conversely, any such collection $A = \{A_\alpha\}_\alpha$ determines a connection ∇. Sometimes it is convenient to write $\nabla = \nabla_A$.

Suppose that the bundle E has structure group $\mathrm{G} \subset \mathrm{GL}(n, \mathbb{R})$ and that $\{U_\alpha, \varphi_\alpha\}_\alpha$ is a vector bundle atlas with transition maps $u_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{G}$. A connection ∇ is called a **G-connection** if all its connection potentials take values in the Lie algebra $\mathfrak{g} = \mathrm{Lie}(\mathrm{G}) \subset \mathbb{R}^{n\times n}$, i.e.

$$A_\alpha \in \Omega^1(U_\alpha, \mathfrak{g}).$$

The important structure groups for this book are $\mathrm{SO}(n)$, $\mathrm{U}(n)$, $\mathrm{Spin}^c(2n)$. The space of connections on E will be denoted by $\mathcal{A}(E)$. If E has structure group G then $\mathcal{A}(E)$ is understood to be the space of G-connections unless otherwise mentioned.

Exercise 1.8. (Existence of a connection) Prove that on any vector bundle $E \to X$ with any structure group G there exists a G-connection. **Hint:** Choose a partition of unity $\{\rho_\alpha\}_\alpha$, subordinate to the cover $\{U_\alpha\}_\alpha$ and define

$$A_\alpha = \sum_\gamma \rho_\gamma ({u_{\gamma\alpha}}^{-1} du_{\gamma\alpha}).$$

Prove that these 1-forms satisfy (1.2). □

Exercise 1.9. (Action of the gauge group) The group $\mathcal{G}(E)$ of automorphisms of E acts on the space $\mathcal{A}(E)$ via

$$\nabla \mapsto u^*\nabla = u^{-1} \circ \nabla \circ u.$$

If $\nabla = \nabla_A$ with connection potentials $A_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})$ and u is represented by $u_\alpha : U_\alpha \to \mathrm{G}$ prove that $u^{-1} \circ \nabla_A \circ u = \nabla_{u^*A}$ is represented by

$$u_\alpha{}^* A_\alpha = u_\alpha{}^{-1} du_\alpha + u_\alpha{}^{-1} A_\alpha u_\alpha. \qquad \Box$$

Exercise 1.10. (Pullback connection) Let $E \to X$ be a vector bundle with G-connection ∇ and $f : Y \to X$ be a smooth map. Prove that there is a natural connection $f^*\nabla$ on the pullback bundle

$$f^*E = \{(x, \eta) \in Y \times E \,|\, \pi(\eta) = f(y)\}$$

such that $(f^*\nabla)(f^*s) = f^*(\nabla s)$ for $s \in C^\infty(X, E)$. An important case is that of a curve $\gamma : \mathbb{R} \to X$. A section of γ^*E is a smooth map $\eta : \mathbb{R} \to E$ with $\eta(t) \in E_{\gamma(t)}$ and we abbreviate $\nabla\eta(t) = (\gamma^*\nabla)_{\partial/\partial t}\eta(t)$. In the case $\eta(t) = s(\gamma(t))$ for a section $s : X \to E$ we have

$$\nabla(s \circ \gamma)(t) = \nabla_{\dot\gamma(t)} s(\gamma(t)).$$

Hint: Given a vector bundle atlas $\{U_\alpha, \varphi_\alpha\}_\alpha$ for E with transition maps $u_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{G}$ construct a vector bundle atlas $\{V_\alpha, \psi_\alpha\}_\alpha$ for f^*E with $V_\alpha = f^{-1}(U_\alpha)$ and transition maps $v_{\beta\alpha} = u_{\beta\alpha} \circ f$. If $A_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})$ are the connection potentials for ∇ show that $f^*A_\alpha \in \Omega^1(f^{-1}(U_\alpha), \mathfrak{g})$ are connection potentials for $f^*\nabla$. $\Box$

Exercise 1.11. (Riemannian connection) Let $E \to X$ be a real Riemannian vector bundle of rank n. A connection $\nabla : C^\infty(X, E) \to \Omega^1(X, E)$ is called **Riemannian** if

$$\partial_v \langle s_1, s_2 \rangle = \langle \nabla_v s_1, s_2 \rangle + \langle s_1, \nabla_v s_2 \rangle \tag{1.3}$$

for two sections $s_1, s_2 \in C^\infty(X, E)$ and a vector field $v \in \mathrm{Vect}(X)$. Prove that ∇ is a Riemannian connection if and only if it is an $\mathrm{O}(n)$-connection in the above sense. $\Box$

Exercise 1.12. (Hermitian connection) Let $E \to X$ be an complex vector bundle of rank n with a Hermitian form $\langle \cdot, \cdot \rangle$. Our convention is that the Hermitian form be complex anti-linear in the first argument and complex linear in the second. A connection $\nabla : C^\infty(X, E) \to \Omega^1(X, E)$ is called **Hermitian** if it satisfies 1.3. In (non-unitary) local trivializations a Hermitian structure on E is given by a collection of matrix functions $H_\alpha : U_\alpha \to \mathbb{C}^{n\times n}$ such that $H_\alpha(x)^* = H_\alpha(x)$ is positive definite for $x \in U_\alpha$ and

$$H_\alpha(x) = u_{\beta\alpha}(x)^* H_\beta(x) u_{\beta\alpha}(x).$$

for $x \in U_\alpha \cap U_\beta$. Prove that a connection with connection potentials $A_\alpha \in \Omega^1(U_\alpha, \mathbb{C}^{r\times r})$ is Hermitian if and only if

$$dH_\alpha = {A_\alpha}^* H_\alpha + H_\alpha A_\alpha$$

for every α. Note that in the case of local unitary frames with $H_\alpha(x) = \mathbb{1}$ this means that A_α takes values in the Lie algebra $\mathfrak{u}(n) = \mathrm{Lie}(\mathrm{U}(n))$ of skew-Hermitian matrices and hence is a $\mathrm{U}(n)$-connection as above. □

Parallel transport

Let $E \to X$ be a vector bundle with structure group $\mathrm{G} \subset \mathrm{GL}(n, \mathbb{R})$ and connection ∇. Given a path $\gamma : \mathbb{R} \to X$ there are linear isomorphisms

$$\Phi^\nabla(\gamma; t_1, t_0) = \Phi(\gamma; t_1, t_0) : E_{\gamma(t_0)} \to E_{\gamma(t_1)}$$

defined by $\Phi(\gamma; t_1, t_0)\eta_0 = \eta(t_1)$ where $\eta : \mathbb{R} \to E$ is the unique parallel section of $\gamma^* E$ with $\eta(t_0) = \eta_0$. This means that $\eta(t) \in E_{\gamma(t)}$ for all t and $\nabla\eta = 0$ (see Exercise 1.10). Note that the function $t \mapsto \Phi(\gamma; t, t_0)\eta_0$ is smooth for every smooth path γ and every $\eta_0 \in E_{\gamma(t_0)}$. The maps $\Phi(\gamma; t_1, t_0)$ satisfy

$$\Phi(\gamma; t_2, t_1) \circ \Phi(\gamma; t_1, t_0) = \Phi(\gamma; t_2, t_0), \qquad \Phi(\gamma; t_0, t_0) = \mathrm{id}.$$

Moreover, they are independent of the parametrization of γ in the sense that for every diffeomorphism $\beta : \mathbb{R} \to \mathbb{R}$

$$\Phi(\gamma \circ \beta; t_1, t_0) = \Phi(\gamma; \beta(t_1), \beta(t_0)).$$

A collection of isomorphisms $\Phi(\gamma; t_1, t_0)$ with these properties is called a **parallel transport structure**. There is a one-to-one correspondence between such parallel transport structures and connections.

Exercise 1.13 Let $E \to X$ be a Riemannian vector bundle with connection ∇. Prove that ∇ is a Riemannian connection (as in Exercise 1.11) if and only if the parallel transport maps are orthogonal, i.e.

$$\Phi(\gamma; t_1, t_0)^* = \Phi(\gamma; t_0, t_1) = \Phi(\gamma; t_1, t_0)^{-1}.$$

Prove a similar assertion for Hermitian connections. More generally, suppose that E has structure group $\mathrm{G} \subset \mathrm{GL}(n, \mathbb{R})$ and show that ∇ is a G-connection if and only if $e_1^{-1} \circ \Phi^\nabla(\gamma; t_1, t_0) \circ e_0 \in \mathrm{G}$ for any two G-frames $e_0 : \mathbb{R}^n \to E_{\gamma(t_0)}$ and $e_1 : \mathbb{R}^n \to E_{\gamma(t_1)}$. □

Exercise 1.14 If ∇ is a connection and $u \in \mathcal{G}(E)$ is an automorphism show that

$$\Phi^{u^*\nabla}(\gamma; t_1, t_0) = u(\gamma(t_1))^{-1}\Phi^{\nabla}(\gamma; t_1, t_0)u(\gamma(t_0)).$$

for $\gamma : \mathbb{R} \to X$ and $t_0, t_1 \in \mathbb{R}$. □

Connections on principal bundles

Let $E \to X$ be a vector bundle with structure group $\mathrm{G} \subset \mathrm{GL}(n, \mathbb{R})$ and $P = \mathcal{F}(E)$ be the corresponding principal G-frame bundle. A G-connection ∇ on E determines a splitting of the tangent bundle TP into horizontal and vertical subbundles, namely

$$TP = V \oplus H$$

where $V_p = \{p\xi \,|\, \xi \in \mathfrak{g}\} = \ker\, d\pi(p)$ and

$$H_{(x,e)} = \left\{ \left.\frac{d}{dt}\right|_{t=0} (\gamma(t), \Phi^{\nabla}(\gamma; t, 0)e) \,\middle|\, \gamma : \mathbb{R} \to X,\ \gamma(0) = x \right\}.$$

The horizontal subbundle is equivariant under the right action of G in the sense that $H_{pg} = H_p g$ for $p \in P$ and $g \in \mathrm{G}$. Of course, this also holds for the vertical bundle. Conversely, any horizontal distribution $H \subset TP$ with this property determines a G-connection on E.

Exercise 1.15 Prove that $T_pP = H_p \oplus V_p$, where $p = (x, e)$ and V_p and H_p are defined as above. **Hint:** Recall the identification of P with the set of equivalence classes of triples $[\alpha, x, g]$ with $x \in U_\alpha$ and $g \in \mathrm{G}$ under the equivalence relation $[\alpha, x, g] \equiv [\beta, x, u_{\beta\alpha}(x)g]$. Deduce that the tangent space T_pP with $p = [\alpha, x, g]$ is the set of equivalence classes $[\alpha, v, g\xi]$ with $v \in T_xX$ and $\xi \in \mathfrak{g}$ under the equivalence relation

$$[\alpha, v, g\xi] \equiv [\beta, v, u_{\beta\alpha}(x)g\xi + (du_{\beta\alpha}(x)v)\xi].$$

Show that with this identification

$$V_p = \{[\alpha, 0, g\xi] \,|\, \xi \in \mathfrak{g}\}, \qquad H_p = \{[\alpha, v, -A_\alpha(v)g] \,|\, v \in T_xX\}$$

where the $A_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})$ are the connection potentials of ∇. □

A **connection** on a principal G-bundle $P \to X$ is an equivariant horizontal distribution $H \subset TP$. Any such distribution can be uniquely represented as the kernel of a 1-form $A \in \Omega^1(P, \mathfrak{g})$ which satisfies

$$A_{pg}(vg) = g^{-1}A_p(v)g, \qquad A_p(p\xi) = \xi$$

for $v \in T_pP$, $g \in \mathrm{G}$, and $\xi \in \mathfrak{g}$. The second condition guaranties that $H_p = \ker A_p$ is a complement of V_p and the first condition guarantees that H is equivariant. A 1-form with these properties is called a **connection-1-form** and the set of such 1-forms is denoted by $\mathcal{A}(P)$. Note that the difference of two connections $a = A_1 - A_2$ is an equivariant and horizontal 1-form on P and hence, by Remark 1.7, can be identified with a 1-form on X with values in $\mathfrak{g}_P = P \times_{\mathrm{ad}} \mathfrak{g}$. Conversely, if $A \in \mathcal{A}(P)$ and $a \in \Omega^1(X, \mathfrak{g}_P) = \Omega^1_{\mathrm{ad}}(P, \mathfrak{g})$ then $A + a \in \mathcal{A}(P)$. Thus $\mathcal{A}(P)$ is an affine space with associated vector space $\Omega^1(X, \mathfrak{g}_P)$. The group $\mathcal{G}(P)$ of gauge transformations acts on $\mathcal{A}(P)$ via

$$u^*A = u^{-1}du + u^{-1}Au$$

for $A \in \mathcal{A}(P)$ and $u \in \mathcal{G}(P)$ (see Remark 1.6). Note that this action is contravariant.

Exercise 1.16 Show that connections on a principal bundle can in local trivializations be described by Lie algebra valued 1-forms $A_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})$ which satisfy (1.2). Deduce that $\mathcal{A}(P)$ is nonempty (Exercise 1.8). □

Exercise 1.17 Recall from Remark 1.6 that an automorphism $\varphi : P \to P$ has the form $\varphi(p) = pu(p)$ where $u \in \mathcal{G}(P)$. Show that the pullback of a 1-form $A \in \Omega^1(P, \mathfrak{g})$ under φ is given by $\varphi^*A = u^*A = u^{-1}du + u^{-1}Au$. Show that if A is a connection 1-form then so is u^*A. (Compare with Exercise 1.9.) □

Exercise 1.18 Let $E \to X$ be a vector bundle with structure group $\mathrm{G} \subset \mathrm{GL}(n, \mathbb{R})$. Prove that there is a one-to-one correspondence between G-connections on E and G-connections on $P = \mathcal{F}(E)$. □

Exercise 1.19 Let $P \to X$ be a principal G-bundle and $\rho : \mathrm{G} \to \mathrm{Aut}(V)$ be a representation. Define $\dot\rho : \mathfrak{g} \to \mathrm{End}(V)$ by

$$\dot\rho(\xi) = \left.\frac{d}{dt}\right|_{t=0} \rho(\exp(t\xi))$$

for $\xi \in \mathfrak{g}$. Prove that there is a one-to-one correspondence between G-connections on the associated bundle $E = P \times_\rho V$ and 1-forms $B \in \Omega^1(P, \mathrm{End}(V))$ which satisfy

$$B_{pg}(vg) = \rho(g)^{-1}B_p(v)\rho(g), \qquad B_p(p\xi) = \dot\rho(\xi) \tag{1.4}$$

for $v \in T_pP$, $g \in \mathrm{G}$, and $\xi \in \mathfrak{g}$. **Hint:** A 1-form $B \in \Omega^1(P, \mathrm{End}(V))$ which satisfies (1.4) induces a collection of covariant derivative operators

$$d_B : \Omega^k(X, E) = \Omega^k_\rho(P, V) \to \Omega^{k+1}(X, E) = \Omega^{k+1}_\rho(P, V)$$

defined by $d_B\tau = d\tau + B \wedge \tau$. Show that if τ is equivariant and horizontal then so is $d_B\tau$. If $A \in \mathcal{A}(P)$ then $B = \dot\rho(A)$ satisfies (1.4) and we shall also use the notation $d_A\tau = d\tau + \dot\rho(A) \wedge \tau$. □

1.3 Curvature

Let $E \to X$ be a vector bundle with connection ∇. The curvature of ∇ is the endomorphism valued 2-form $F^\nabla \in \Omega^2(X, \mathrm{End}(E))$ defined by

$$F^\nabla(v,w)s = \nabla_v \nabla_w s - \nabla_w \nabla_v s + \nabla_{[v,w]} s.$$

for $v, w \in \mathrm{Vect}(X)$ and $s \in C^\infty(X, E)$. (See the footnote on page 35 for the sign conventions in the definition of the Lie bracket.) The reader may check that F^∇ is well defined. If $u \in \mathcal{G}(E)$ is an automorphism then the curvature of the connection

$$u^*\nabla = u^{-1} \circ \nabla \circ u$$

is given by

$$F^{u^*\nabla} = u^{-1} F^\nabla u.$$

Exercise 1.20 A connection $\nabla : C^\infty(X, E) \to \Omega^1(X, E)$ extends to an operator $d^\nabla : \Omega^k(X, E) \to \Omega^{k+1}(X, E)$ defined by

$$d^\nabla(\tau \otimes s) = (d\tau) \otimes s + (-1)^{\deg(\tau)} \tau \wedge \nabla s$$

for $\tau \in \Omega^k(X)$ and $s \in C^\infty(X, E)$. Prove that this operator is well defined. Prove that for every 1-form $\alpha \in \Omega^1(X, E)$ and any two vector fields $v, w \in \mathrm{Vect}(X)$

$$d^\nabla \alpha(v,w) = \nabla_v(\alpha(w)) - \nabla_w(\alpha(v)) + \alpha([v,w]).$$

(See the footnote on page 35.) Deduce that the curvature satisfies

$$d^\nabla d^\nabla \tau = F^\nabla \wedge \tau. \qquad \square$$

Exercise 1.21 Prove the **Bianchi identity**

$$d^\nabla F^\nabla = 0.$$

Hint: Use the formula

$$\begin{aligned} d^\nabla \omega(u,v,w) &= \nabla_u(\omega(v,w)) + \nabla_v(\omega(w,u)) + \nabla_w(\omega(u,v)) \\ &\quad + \omega([u,v],w) + \omega([v,w],u) + \omega([w,u],v) \end{aligned}$$

for $\omega \in \Omega^2(X, E)$. (See the footnote on page 35.) $\square$

Exercise 1.22 Prove that the curvature is in local trivializations given by

$$F_\alpha = dA_\alpha + A_\alpha \wedge A_\alpha \in \Omega^2(U_\alpha, \mathbb{R}^{n\times n})$$

or, more explicitly, by

$$F_\alpha(u,v) = dA_\alpha(u,v) + [A_\alpha(u), A_\alpha(v)]$$

for $u, v \in T_xX$. Show that

$$F_\alpha = {u_{\beta\alpha}}^{-1} F_\beta u_{\beta\alpha}.$$

If $\alpha : U_\alpha \to \mathbb{R}^m$ is a chart on X show that

$$\alpha_* F_\alpha = \sum_{i<j} F_{ij} dx^i \wedge dx^j, \qquad F_{ij} = \frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} + [A_i, A_j],$$

where $\alpha_* A_\alpha = \sum_i A_i dx^i$. □

Exercise 1.23 Given a connection ∇ and a 1-form $a \in \Omega^1(X, \mathrm{End}(E))$ prove that

$$F^{\nabla + a} = F^\nabla + d^\nabla a + a \wedge a,$$

where $d^\nabla a \in \Omega^2(X, \mathrm{End}(E))$ is defined by

$$(d^\nabla a)s = d^\nabla(as) - a \wedge d^\nabla s.$$ □

Flat connections

Assume now that the bundle E has structure group $\mathrm{G} \subset \mathrm{GL}(n, \mathbb{R})$. Denote by $\mathrm{End}^{\mathfrak{g}}(E)$ the bundle of those endomorphisms $A : E_x \to E_x$ which satisfy $e^{-1} \circ A \circ e \in \mathfrak{g}$ for some (and hence every) G-frame $e : \mathbb{R}^n \to E_x$. It is an easy consequence of Exercise 1.22 that if ∇ is a G-connection then $F^\nabla \in \Omega^2(X, \mathrm{End}^{\mathfrak{g}}(E))$. For example, if E is a Riemannian vector bundle with $\mathrm{G} = \mathrm{O}(n)$ or $\mathrm{G} = \mathrm{SO}(n)$ then $\mathrm{End}^{\mathfrak{g}}(E)$ is the bundle of skew-symmetric endomorphisms and the curvature of a Riemannian connection ∇ thus satisfies

$$F^\nabla(u,v)^* + F^\nabla(u,v) = 0$$

for $u, v \in T_xX$. Similarly for Hermitian connections in the complex case. The next lemma shows that the curvature is the obstruction to integrability of the horizontal subbundle $H \subset TP$ of the frame bundle $P = T\mathcal{F}(E)$ determined by ∇. Given a vector field $v \in \mathrm{Vect}(X)$ denote by $v^\sharp \in \mathrm{Vect}(P)$ the horizontal lift.

Proposition 1.24 *Let $E \to X$ be a vector bundle with structure group* G *and* G*-frame bundle $P = \mathcal{F}(E)$. Let ∇ be a* G*-connection on E with corresponding horizontal distribution $H \subset TP$ and connection* 1*-form $A \in \mathcal{A}(P)$. Then*

$$F^\nabla(u(x), v(x)) = eA_p([u^\sharp, v^\sharp](p))e^{-1}$$

for $p = (x, e) \in \mathcal{F}(E)$ and $u, v \in \mathrm{Vect}(X)$.

Proof: In the notation of Exercise 1.15 the horizontal lift of v is given by

$$v^\sharp(p) = [\alpha, v(x), -A_\alpha(v(x))g]$$

for $p = [\alpha, x, g] \equiv (x, e)$ with $g = \varphi_\alpha(x) \circ e$. (See also Remark 1.3.) The Lie bracket of two such horizontal lifts is given by

$$[u^\sharp, v^\sharp](p) = [u, v]^\sharp(p) + [\alpha, 0, F_\alpha(u(x), v(x))g].$$

Since $A_p([u, v]^\sharp(p)) = 0$ it follows that

$$A_p([u^\sharp, v^\sharp](p)) = g^{-1}F_\alpha(u(x), v(x))g = e^{-1}F^\nabla(u(x), v(x))e$$

as claimed. □

This shows that the horizontal distribution $H \subset TP$ is integrable if and only if the curvature F^∇ vanishes. In this case ∇ is called a **flat connection**. Now the integral curves of the horizontal distribution have the form $t \mapsto (\gamma(t), \Phi^\nabla(\gamma; t, 0)e)$ where $\gamma : [0,1] \to X$ and it follows from integrability that the endpoint of this curve depends only on the homotopy class of γ. Thus every flat connection ∇ gives rise to a representation

$$\rho^\nabla : \pi_1(X, x_0) \to \mathrm{G}$$

defined by

$$\rho^\nabla(\gamma) = {e_0}^{-1}\Phi^\nabla(\gamma; 1, 0)e_0$$

for every loop $\gamma : [0,1] \to X$ with $\gamma(0) = \gamma(1) = x_0$. Here $e_0 : \mathbb{R}^n \to E_{x_0}$ is a fixed G-frame. This representation is called the **holonomy** of ∇. Obviously, only the conjugacy class of ρ^∇ is determined by ∇ and the representative depends on the choice of frame. Moreover, using Exercise 1.14, one can show that two flat G-connections ∇ and ∇' are gauge equivalent if and only if $\rho^{\nabla'}$ and ρ^∇ are conjugate. Furthermore, for every homomorphism $\rho : \pi_1(X, x_0) \to \mathrm{G}$ there exists a flat G-connection (on some G-bundle $E \to X$) with holonomy $\rho^\nabla = \rho$. This shows that there is a natural bijection

$$\frac{\{\text{flat G-connections}\}}{\text{gauge equivalence}} \cong \frac{\mathrm{Hom}(\pi_1(X), \mathrm{G})}{\text{conjugacy}}.$$

The details of the proof will not be carried out. In the case $\mathrm{G} = S^1$ two homomorphisms are conjugate if and only if they are equal and hence the space of gauge equivalence classes of flat S^1-connections can be identified with $\mathrm{Hom}(\pi_1(X), S^1)$.

Remark 1.25 If $H_1(X;\mathbb{Z})$ has no torsion, then

$$\mathrm{Hom}(\pi_1(X), S^1) \cong \frac{H^1(X; i\mathbb{R})}{H^1(X; 2\pi i\mathbb{Z})} =: T.$$

In general, $\mathrm{Hom}(\pi_1(X), S^1)$ is a principal space which carries a free action of T. Each component of $\mathrm{Hom}(\pi_1(X), S^1)$ is diffeomorphic to T and corresponds to the isomorphism class of a line bundle whose first Chern class descends to zero in $H^2(X;\mathbb{R})$. (See Section 1.4.) For example, if $\pi_1(X) = \mathbb{Z}_2$ then there are precisely two homomorphisms $\pi_1(X) \to S^1$ corresponding to the two isomorphism classes of line bundles $E \to X$ with $c_1(E) = 0 \in H^2(X;\mathbb{R})$. □

Curvature on principal bundles

The above discussion suggests that the curvature of a connection 1-form $A \in \Omega^1(P, \mathfrak{g})$ on a principal bundle $P \to X$ should be the 2-form $F_A \in \Omega^2_{\mathrm{ad}}(P, \mathfrak{g})$ defined by

$$F_A(u, v) = \begin{cases} dA_p(u, v), & \text{if } u, v \in H_p, \\ \quad 0, & \text{if } u \in V_p \text{ or } v \in V_p. \end{cases}$$

Since $dA(u, v) = A([u, v])$ for horizontal vector fields this agrees with the formula in Proposition 1.24. The reader may check that this 2-form is indeed equivariant and horizontal and can be expressed in the form

$$F_A = dA + \frac{1}{2}[A \wedge A].$$

This curvature 2-form satisfies $F_{u^*A} = u^{-1}F_A u$ for $u \in \mathcal{G}(P)$ and the **Bianchi identity**

$$d_A F_A = 0,$$

where $d_A : \Omega^2(X, \mathfrak{g}_P) \to \Omega^3(X, \mathfrak{g}_P)$.

Exercise 1.26 Prove that the infinitesimal action of the gauge group is given by the covariant derivative

$$d_A\xi = \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi)^* A$$

for $A \in \mathcal{A}(P)$ and $\xi \in \Omega^0(X, \mathfrak{g}_P)$. (See Exercise 1.19.) □

Exercise 1.27 Consider any associated bundle $E = P \times_\rho V$. Prove that

$$d_A d_A \tau = \dot\rho(F_A) \wedge \tau$$

for every $\tau \in \Omega^k(X, E)$. Prove that $d_{u^*A}\tau = \rho(u)^{-1} d_A(\rho(u)\tau)$ for $u \in \mathcal{G}(P)$, $A \in \mathcal{A}(P)$, and $\tau \in \Omega^k(X, E)$. □

1.4 Chern classes

From an axiomatic point of view the Chern classes can be defined as a functor c which assigns to every every complex vector bundle $E \to X$ of rank k over a finite dimensional compact manifold X the **total Chern class**

$$c(E) = 1 + c_1(E) + \cdots + c_k(E).$$

Here $c_j(E) \in H^2(X;\mathbb{Z})$ is an integral cohomology class on X, called the **j-th Chern class**, and $1 \in H^0(X;\mathbb{Z})$ is the Poincaré dual of the fundamental class $[X] \in H_{2n}(X;\mathbb{Z})$. It is the generator whenever X is connected.

Theorem 1.28 *There is a unique functor c, called the* **Chern class***, which assigns to every complex vector bundle E over a compact manifold X an integral cohomology class $c(E) \in H^{\mathrm{ev}}(X;\mathbb{Z})$ and satisfies the following axioms.*

(Naturality) *Isomorphic vector bundles have the same Chern classes.*

(Functoriality) *For every smooth map $f : Y \to X$ and every complex vector bundle $E \to X$, $c(f^*E) = f^*c(E)$.*

(Direct sum) *If E_1 and E_2 are complex vector bundles over X then*

$$c(E_1 \oplus E_2) = c(E_1)c(E_2).$$

(Zero) *If E is the trivial bundle then $c(E) = 1$.*

(Normalization) *The first Chern class of the canonical bundle $H \to \mathbb{C}P^n$ with fiber $H_\ell = \ell^* = \mathrm{Hom}(\ell, \mathbb{C})$ over a point $\ell \in \mathbb{C}P^n$ is the canonical generator**

$$c_1(H) = h = \mathrm{PD}([\mathbb{C}P^{n-1}]) \in H^2(\mathbb{C}P^n;\mathbb{Z})$$

and $c_k(H) = 0$ for $k > 1$.

Proof: We only sketch the main idea. The proof is based on the following three observations.

(i) For every complex vector bundle $E \to X$ there exists a bundle $F \to X$ such that the direct sum $E \oplus F \cong X \times \mathbb{C}^n$ is isomorphic to the trivial bundle. Equivalently, there exists an embedding of E into the trivial bundle $X \times \mathbb{C}^n$. This embedding can be thought of as a smooth map $f : X \to \mathrm{Gr}(k, n)$ to the complex Grassmannian such that E is isomorphic to the pullback under f of the tautological bundle $E(k, n) \to \mathrm{Gr}(k, n)$, whose fiber over a subspace $\Lambda \subset \mathbb{C}^n$ is the subspace itself.

(ii) Two pullback bundles $E_0 = {f_0}^*E(k, n)$ and $E_1 = {f_1}^*E(k, n)$ over X are isomorphic if and only if the functions $f_0 : X \to \mathrm{Gr}(k, n)$ and $f_1 : X \to \mathrm{Gr}(k, n)$ are homotopic (for n sufficiently large).

*Think of $\ell \in \mathbb{C}P^n$ as a one-dimensional complex linear subspace of $\mathbb{C}^{n+1}$.

(iii) There are cohomology classes

$$c(k,n) = \sum_{i=1}^{k} c_i(k,n)$$

with $c_i(k,n) \in H^{2i}(\mathrm{Gr}(k,n);\mathbb{Z})$ which satisfy

$$c_0(k,n) = 1, \qquad c_1(1,n) = h \in H^2(\mathbb{C}P^{n-1};\mathbb{Z}),$$

and the relation

$$\pi_1{}^*c(k_1,n_1) \cup \pi_2{}^*c(k_2,n_2) = \iota^* c(k_1+k_2, n_1+n_2) \tag{1.5}$$

where

$$\pi_j : \mathrm{Gr}(k_1,n_1) \times \mathrm{Gr}(k_2,n_2) \to \mathrm{Gr}(k_j,n_j)$$

is the obvious projection and

$$\iota : \mathrm{Gr}(k_1,n_1) \times \mathrm{Gr}(k_2,n_2) \to \mathrm{Gr}(k_1+k_2,n_1+n_2)$$

is the obvious inclusion. The class $c_i(k,n)$ will serve as the ith Chern class of the tautological bundle. It is defined as $(-1)^i$ times the Poincaré dual of the Schubert cycle $\xi_i \subset \mathrm{Gr}(k,n)$. Given any flag

$$V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n$$

in $\mathbb{C}^n$, this Schubert cycle can be defined as the set of all k-dimensional subspaces $\Lambda \subset \mathbb{C}^n$ which satisfy the following, for $k-n \le j \le k$,

$$\dim\left(\Lambda \cap V_{n-k+j}\right) = \begin{cases} 0, & \text{if } j < 0, \\ j+1, & \text{if } 0 \le j \le i-1, \\ j, & \text{if } i \le j \le k. \end{cases}$$

The Chern class of a complex vector bundle $E \to X$ of rank n can now be defined as follows. Choose a sufficiently large integer n and a smooth map $f : X \to \mathrm{Gr}(k,n)$ such that E is isomorphic to $f^*E(k,n)$ and define

$$c(E) = f^*c(k,n).$$

That such a map f exists follows from (i), and that the cohomology class $f^*c(k,n)$ is independent of the choice of f follows from (ii). Using (ii) one checks easily that these classes satisfy *"Naturality"*, *"Functoriality"*, and *"Zero"* axioms. The *"Normalization"* axiom follows from the definition of the Schubert cycles in (iii) above, and the *"Direct sum"* axiom follows from (1.5). That the axioms uniquely determine the Chern classes follows from the fact that the obvious map $\pi : F(n) \to \mathrm{Gr}(k,n)$ from the flag manifold to the Grassmannian induces an injective map in cohomology and that the pullback $\pi^* E(k,n)$ of the tautological bundle is a direct sum of line bundles. For more details see [45] or [93]. □

Exercise 1.29 Use the axioms to prove that the first Chern class of a complex line bundle $L \to X$ is Poincaré dual to the zero set of a generic section. **Hint:** Let $s_0 : X \to L$ and $s_1 : X \to L$ be two transverse sections. Prove that the zero sets of s_0 and s_1 represent the same homology class. Find a transverse section of the canonical bundle $s : \mathbb{C}P^n \to H$ whose zero set is equal to $\mathbb{C}P^{n-1}$. Given a line bundle $L \to X$ find a smooth map $f : X \to \mathbb{C}P^n$ such that the pullback bundle f^*H is isomorphic to L and f is transverse to $\mathbb{C}P^{n-1}$. Consider the pullback section f^*s. □

Remark 1.30 The axioms imply, in particular, that $c_0(E) = 1$ for every bundle E. To see this choose a bundle F with $E \oplus F = \mathbb{C}^N$ and note that $c_0(E)c_0(F) = 1$. The axioms also imply that $c_j(E) = 0$ whenever $j > \mathrm{rank}\, E$. For line bundles this follows from the functoriality and normalization axioms, for the tautological bundle over $\mathrm{G}(k,n)$ from the fact that the pullback $\pi^* E(k,n)$ over the flag manifold $F(n)$ is a direct sum of line bundles, and for general bundles from the functoriality axiom and (i) above. □

Consider the natural homomorphism $H^k(X;\mathbb{Z}) \to H^k_{\mathrm{DR}}(X)$. Its image is the set of deRham cohomology classes whose integral over every smooth cycle is an integer. Its kernel is the **torsion subgroup** of all cohomology classes $a \in H^k(X;\mathbb{Z})$ such that $ma = 0$ for some integer m. Chern-Weil theory gives a construction of the image of the Chern classes in $H^*_{\mathrm{DR}}(X)$ which we shall still denote by $c_k(E)$. The Chern classes as integral classes are only determined by this construction if the cohomology of X is torsion free.

1.5 Chern-Weil theory

The goal of this section is to explain the construction of the Chern classes via Chern Weil theory. We follow the discussion in Milnor-Stasheff [93], Appendix C. Let $E \to X$ be a vector bundle with structure group $\mathrm{G} \subset \mathrm{GL}(n,\mathbb{R})$ and

$$p : \mathfrak{g} \to \mathbb{R}$$

be a homogeneous polynomial of degree k on the Lie algebra $\mathfrak{g} = \mathrm{Lie}(\mathrm{G})$. Assume that p is invariant under the adjoint action of G, i.e.

$$p(\xi) = p(g^{-1}\xi g)$$

for $\xi \in \mathfrak{g}$ and $g \in \mathrm{G}$. The strategy is to define a differential form $p(F^\nabla) \in \Omega^{2k}(X)$ for every connection ∇ on E, then show that this form is closed, and deduce that the resulting cohomology class

$$[p(F^\nabla)] \in H^{2k}_{\mathrm{DR}}(X)$$

is independent of the choice of ∇.

Example 1.31. (Chern classes) Consider the polynomials $c_k : \mathfrak{u}(n) \to \mathbb{R}$ defined as the k-th symmetric function in the eigenvalues of $i\xi/2\pi$. Thus

$$c_k(\xi) = \sum_{j_1 < \cdots < j_k} x_{j_1} \cdots x_{j_k}$$

where $x_1, \ldots, x_n$ are the eigenvalues of $i\xi/2\pi$. Note that

$$\det\left(\lambda \mathbb{1} + \frac{i\xi}{2\pi}\right) = \sum_{k=0}^{n} \lambda^{n-k} c_k(\xi).$$

For example $c_0(\xi) = 1$, $c_1(\xi) = \sum_i x_i$, and $c_2(\xi) = \sum_{i<j} x_i x_j$. It is also interesting to consider the **Chern character**

$$\mathrm{ch}(\xi) = \sum_{i=1}^{n} e^{x_i} = \mathrm{trace}\left(\exp\left(\frac{i\xi}{2\pi}\right)\right) = \sum_{k=0}^{\infty} \frac{1}{k!} \mathrm{trace}\left(\left(\frac{i\xi}{2\pi}\right)^k\right).$$

It is easy to see that

$$\mathrm{ch}(\xi \oplus \xi') = \mathrm{ch}(\xi) + \mathrm{ch}(\xi'), \qquad \mathrm{ch}(\xi \otimes \xi') = \mathrm{ch}(\xi)\mathrm{ch}(\xi')$$

for $\xi \in \mathfrak{u}(n)$ and $\xi' \in \mathfrak{u}(n')$. □

Let $E_1, \ldots, E_N$ be a basis of $\mathfrak{g} \subset \mathbb{R}^{n\times n}$. Then any polynomial on $\mathfrak{g}$ can be expressed in the form

$$p(\xi) = \sum_{|\nu|=k} a_\nu \xi^\nu, \qquad \xi = \sum_i \xi^i E_i$$

where $\nu = (\nu_1, \ldots, \nu_N)$ is a multi-index. Now recall that in a local trivialization $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ the curvature form $F^\nabla \in \Omega^2(X, \mathrm{End}(E))$ is given by a 2-form $F_\alpha \in \Omega^2(U_\alpha, \mathfrak{g})$ which in the basis $E_1, \ldots, E_N$ can be written in the form

$$F_\alpha = \sum_i \omega^i E_i$$

where $\omega^i \in \Omega^2(U_\alpha)$. The restriction of the $2k$-form $p(F^\nabla) \in \Omega^2(X)$ to U_α is defined by

$$p(F^\nabla)|_{U_\alpha} = p(F_\alpha) = \sum_{|\nu|=k} a_\nu \omega^\nu$$

where $\omega^\nu = (\omega^1)^{\nu_1} \wedge \ldots \wedge (\omega^N)^{\nu_N}$ is defined in terms of the exterior product. Since p is invariant under the adjoint action this definition is independent of α. The crucial step is the following **fundamental lemma**.

Lemma 1.32 *The form $p(F^\nabla)$ is closed.*

Proof: Let the a_{ij}^k be defined by

$$[E_i, E_j] = \sum_k a_{ij}^k E_k.$$

Then it follows from the invariance of p that

$$\sum_{j,k} \partial_k P(\xi) \xi^j a_{ij}^k = 0$$

for all i. To see this consider the curve $\xi(t) = \exp(-tE_i)\xi \exp(tE_i)$ and differentiate the function $t \mapsto P(\xi(t))$ at $t = 0$. Then with $F_\alpha = \omega = \sum_i \omega^i E_i \in \Omega^2(U_\alpha, \mathfrak{g})$ it follows that

$$\sum_{j,k} \partial_k p(\omega) \wedge \omega^j a_{ij}^k = 0.$$

Moreover, with $A_\alpha = \sum_i a^i E_i \in \Omega^1(U_\alpha, \mathfrak{g})$ the Bianchi identitiy $d^\nabla F^\nabla = 0$ takes the form

$$d\omega^k + \sum_{i,j} a^i \wedge \omega^j a_{ij}^k = 0$$

and this implies

$$d(p(\omega)) = \sum_k \partial_k p(\omega) \wedge d\omega^k = -\sum_{i,j,k} \partial_k p(\omega) \wedge a^i \wedge \omega^j a_{ij}^k = 0$$

as claimed. □

Corollary 1.33 *The cohomology class of $p(F^\nabla)$ is independent of ∇.*

Proof: Consider the convex combination $\nabla_t = t\nabla_0 + (1-t)\nabla_1$ of two connections on E. On the bundle $\widetilde{E} = E \times \mathbb{R} \to \widetilde{X} = X \times \mathbb{R}$ there is a unique connection $\widetilde{\nabla}$ such that the pullback connection under the obvious inclusion $\iota_t : X \to X \times \mathbb{R}$ is given by ${\iota_t}^* \widetilde{\nabla} = \nabla_t$. For any invariant polynomial p on $\mathfrak{g}$ write

$$p(F^{\widetilde{\nabla}}) = \widetilde{\omega} = \omega(t) + \beta(t)\, dt \in \Omega^{2k}(X \times \mathbb{R}),$$

where $\omega(t) = \iota_t{}^*\widetilde{\omega} = p(F^{\nabla_t}) \in \Omega^{2k}(X)$ and $\beta(t) \in \Omega^{2k-1}(X)$. That $\widetilde{\omega}$ is closed is equivalent to $d\omega(t) = 0$ and $\dot{\omega}(t) = d\beta(t)$ for every t. Thus the form

$$p(F^{\nabla_1}) - p(F^{\nabla_0}) = d\int_0^1 \beta(t)\,dt$$

is exact. □

The (deRham version of the) Chern classes $c_k(E) \in H^{2k}_{\mathrm{DR}}(X)$ of a complex vector bundle over X are defined as the cohomology classes of $c_k(F^\nabla)$ where ∇ is a Hermitian connection on E and the $c_k : \mathfrak{u}(n) \to \mathbb{R}$ are defined as in Example 1.31. That these classes satisfy the naturality, functoriality, direct sum, and zero axioms is obvious from the definitions.

Proof of the normalization axiom: Think of $\mathbb{C}P^n$ as the space of complex lines $\ell \subset \mathbb{C}^{n+1}$ and consider the canonical bundle $H \to \mathbb{C}P^n$ whose fiber over $\ell \in \mathbb{C}P^n$ is the dual line $H_\ell = \ell^* = \mathrm{Hom}(\ell, \mathbb{C})$. This bundle can be identified with the quotient

$$H = S^{2n+1} \times_{S^1} \mathbb{C}$$

under the equivalence relation $[z, w] \equiv [\lambda z, \lambda w]$ for $z = (z_0, z_1, \ldots, z_n) \in S^{2n+1} \subset \mathbb{C}^{n+1}$ and $w \in \mathbb{C}$. Thus a connection on L is an imaginary valued 1-form $\alpha \in \Omega^1(S^{2n+1}, i\mathbb{R})$ such that

$$\alpha_{\lambda z}(\lambda\zeta) = \alpha_z(\zeta), \qquad \alpha_z(iz) = -i,$$

for $z \in S^{2n+1}$ and $\zeta \in T_z S^{2n+1}$. In more abstract terms, think of S^{2n+1} as the total space of a principal S^1-bundle. Then H is the associated bundle corresponding to the representation $S^1 \to \mathrm{Aut}(\mathbb{C}) : \lambda \mapsto \bar{\lambda}$. Hence the minus sign. An example of such a 1-form is given by $\alpha_z(\zeta) = -i|z|^{-2}\mathrm{Im}\,\langle z, \zeta\rangle$ or, equivalently,

$$\alpha = \frac{1}{2|z|^2}\sum_{j=0}^n (z_j d\bar{z}_j - \bar{z}_j dz_j).$$

α can also be expressed in the form

$$\alpha = \frac{1}{2}(\bar{\partial} f - \partial f), \qquad d\alpha = \partial\bar{\partial} f, \qquad f(z) = \log|z|^2.$$

Hence the Chern form

$$\frac{i}{2\pi} d\alpha = \frac{1}{2\pi i}\bar{\partial}\partial f$$

of the connection α agrees with the Kähler form ω of the Fubini-Study metric. (See Example 3.49 in Chapter 3.) It is an easy exercise to show that the integral of this form over $\mathbb{C}P^1$ is 1 and hence $c_1(H) = h$ is the generator of $H^2(\mathbb{C}P^n; \mathbb{Z})$. (See Exercise 1.36 below.) □

The normalization and functoriality axioms can now be used to define the Chern classes as integral classes. For further details the reader is referred to the excellent book by Milnor and Stasheff [93]. The next proposition asserts that the Euler class of a complex vector bundle agrees with the top Chern class. Given a section $s : X \to E$ note that the linear map $\nabla s(x) : T_xX \to E_x$ is at a zero of s independent of the connection ∇. If X is oriented and has real dimension $2n$, where E has rank n, then a zero x of s is called **nondegenerate** if $\nabla s(x) : T_xX \to E_x$ is an isomorphism and in this case the index $\nu(s,x) = \pm 1$ is determined by whether or not this isomorphism is orientation preserving. (Note that the fiber E_x carries a natural orientation as a complex vector space.)

Proposition 1.34 *Let $E \to X$ be a complex rank-n bundle over a compact oriented $2n$-manifold. If $s : X \to E$ is a section with only nondegenerate zeros then*

$$\langle c_n(E), [X] \rangle = \sum_{s(x)=0} \nu(s,x)$$

where the sum runs over all zeros of s.

Proof: We only sketch the main idea in the case $n = 1$. In this case $E = L$ is a complex Hermitian line bundle over a Riemann surface $X = \Sigma$. Suppose without loss of generality that Σ is connected, choose a splitting

$$\Sigma = \Sigma_1 \cup_C \Sigma_2,$$

and orient $C = \partial\Sigma_1 = -\partial\Sigma_2$ as the boundary of Σ_1. Then choose nonzero sections $s_i : \Sigma_i \to L$ with $|s_i(x)| = 1$ for $x \in \Sigma_i$. Define $\gamma : C \to S^1$ by

$$s_2(x) = \gamma(x) s_1(x), \qquad x \in C.$$

Now fix a Hermitian connection ∇ and define $\alpha_i \in \Omega^1(\Sigma_i, \sqrt{-1}\mathbb{R})$ by $\nabla s_i = \alpha_i s_i$. Then $F^\nabla|_{\Sigma_i} = d\alpha_i$ and $\alpha_2|_C = \alpha_1|_C + \gamma^{-1} d\gamma$. Hence it follows from Stokes' theorem that

$$\int_\Sigma F^\nabla = \int_{\Sigma_1} d\alpha_1 + \int_{\Sigma_2} d\alpha_2 = \int_C \alpha_1 - \alpha_2 = -\int_C \gamma^{-1} d\gamma = -2\pi\sqrt{-1}\deg(\gamma).$$

This shows that

$$\deg(\gamma) = \frac{\sqrt{-1}}{2\pi} \int_\Sigma F^\nabla = \langle c_1(L), [\Sigma] \rangle. \tag{1.6}$$

The proposition now follows by applying this formula to the splitting where Σ_1 is a union of small discs centered at the zeros of s and Σ_2 is the closure of the complement. The details of this are left to the reader. □

Corollary 1.35 *The Euler characteristic of an almost complex $2n$-manifold X is given by*

$$\langle c_n(TX), [X]\rangle = \chi(X).$$

In particular, if Σ is a compact oriented Riemann surface of genus g, then

$$\langle c_1(T\Sigma), [\Sigma]\rangle = 2 - 2g.$$

Proof: Proposition 1.34 and the Poincaré-Hopf theorem. □

1.6 Examples and exercises

Exercise 1.36. (Line bundles) Let $L \to X$ be a complex line bundle over a smooth manifold X. Suppose that L is equipped with a Hermitian structure and denote by $\pi : P \to X$ the unit circle bundle in L. Then the action of S^1 on P is generated by a vector field $v : P \to TP$ (i.e. the flow of v is given by $X \times \mathbb{R} \to X : (x, t) \mapsto x \cdot e^{2\pi it}$). A connection 1-form $A \in \Omega^1(P, i\mathbb{R})$ satisfies

$$\iota(v)dA = 0, \qquad \iota(v)A = 2\pi i.$$

Show that $\mathcal{L}_v A = \iota(v)dA + d\iota(v)A = 0$ and deduce that the curvature form dA descends to X. Show that the (deRham) first Chern class of L is given by

$$c_1(L) = [\omega], \qquad \frac{i}{2\pi} dA = \pi^*\omega.$$

Compare this with the proof of the normalization axiom. □

Remark 1.37 **(i)** Let $L \to \Sigma$ be a complex line bundle over a compact oriented Riemann surface. Then $\langle c_1(L), [\Sigma]\rangle$ is the selfintersection number of the zero section in L.
(ii) For a line bundle $L \to X$ over a general compact manifold the first Chern class can be defined as the Poincaré dual of the zero set of a generic section $s : X \to L$.
(iii) If (X, J) is an almost complex 4-manifold and $C \subset X$ is a pseudo-holomorphic submanifold of real dimension 2 then the genus of C is given by the **adjunction formula**

$$2g(C) - 2 = C \cdot C + c_1(K) \cdot C. \tag{1.7}$$

Here $C \cdot C$ denotes the self-intersection number and $c_1(K) = -c_1(TX, J) \in H^2(X; \mathbb{Z})$ denotes the canonical class. To see this consider the splitting $T_C X = TC \oplus \nu_C$ into tangent and normal bundle. Both are complex line bundles over C and, by (i), $\langle c_1(\nu_C), [C]\rangle = C \cdot C$. □

Example 1.38 Isomorphism classes of **line bundles over the torus** $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$ can be described as equivalence classes of **cocycles**

$$\mathbb{Z}^m \to \mathrm{Map}(\mathbb{R}^m, S^1) : k \mapsto \varphi_k$$

which satisfy $\varphi_{k+\ell}(x) = \varphi_\ell(x+k)\varphi_k(x)$ for $x \in \mathbb{R}^m$ and $k, \ell \in \mathbb{Z}^m$. Two such cocycles φ and ψ are **equivalent** if there exists a function $g : \mathbb{R}^m \to S^1$ such that $\psi_k(x) = g(x+k)^{-1}\varphi_k(x)g(x)$. For any cocycle φ the corresponding line bundle $L = L(\varphi) \to \mathbb{T}^m$ can be explicitly described as the quotient $L = \mathbb{R}^m \times \mathbb{C}/\mathbb{Z}^m$ under the action $k \cdot (x, z) = (x+k, \varphi_k(x)z)$. Thus a section of L is a smooth map $s : \mathbb{R}^m \to \mathbb{C}$ which satisfies $s(x+k) = \varphi_k(x)s(x)$ for $x \in \mathbb{R}^m$ and $k \in \mathbb{Z}^m$. A connection on L has the form

$$\nabla_A s = ds + As, \qquad A = \sum_{\nu=1}^{n} A_\nu(x)dx_\nu,$$

where the functions $A_\nu : \mathbb{R}^m \to i\mathbb{R}$ satisfy

$$A_\nu(x+k) - A_\nu(x) = -\varphi_k(x)^{-1}\frac{\partial \varphi_k}{\partial x_\nu}(x).$$

This can be used to compute the curvature and hence the first Chern class of the bundle. For example, for any integer matrix $B \in \mathbb{Z}^{m\times m}$ consider the cocycle

$$\varphi_k(x) = \exp(2\pi i k^T Bx). \tag{1.8}$$

A corresponding connection is given by $A = -2\pi i \sum_{\nu,\mu=1}^m x_\nu B_{\nu\mu} dx_\mu$ with curvature form $F_A = dA = -2\pi i \sum_{\nu<\mu} (B_{\nu\mu} - B_{\mu\nu})\, dx_\nu \wedge dx_\mu$. Hence the bundle $L(\varphi)$ has first Chern class

$$c_1(L(\varphi)) = \left[\sum_{\nu<\mu}^{m} C_{\nu\mu} dx_\nu \wedge dx_\mu\right], \qquad C = B - B^T.$$

This bundle admits a trivialization whenever B is symmetric and it admits a square root whenever B is skew-symmetric. (Prove this.) The reader may check that another cocycle with first Chern class C is given by

$$\varphi_k(x) = \varepsilon(k)\exp(\pi i k^T Cx).$$

where the numbers $\varepsilon(k) = \pm 1$ are chosen such that

$$\varepsilon(k+\ell) = \varepsilon(k)\varepsilon(\ell)\exp(\pi i k^T C\ell).$$

If $C = B - B^T$ then the numbers $\varepsilon(k) = \exp(\pi i k^T Bk)$ satisfy this condition. That every cocycle is equivalent to one of the form (1.8) is a consequence of Exercise 1.39 below. But it can also be proved directly by choosing two Yang-Mills connections A and A' (with constant curvature form) for two cocycles φ and φ' with the same Chern class. Then $F_{A'} = F_A$ and hence $A' = A + d\xi$ for some function $\xi : \mathbb{R}^m \to i\mathbb{R}$. Now the function $g = \exp(\xi) : \mathbb{R}^m \to S^1$ transforms φ into φ'. □

Exercise 1.39 Prove that for every compact manifold X the map

$$\frac{\{\text{complex line bundles } L \to X\}}{\text{isomorphisms}} \longrightarrow H^2(X;\mathbb{Z}) : L \mapsto c_1(L)$$

is a bijection. (In fact, it is a group isomorphism with respect to the tensor product of line bundles.) **Hint:** Triangulate X, denote by $X_k \subset X$ the k-skeleton, and by $\mathcal{S}_k$ the set of k-simplices in the triangulation (thought of as submanifolds with corners). Fix a unitary section $s : X_1 \to L$ over the 1-skeleton. For every 2-simplex $\Delta \in \mathcal{S}_2$ choose a unitary section $s_\Delta : \Delta \to L$ and define the map $\varphi_s : \mathcal{S}_2 \to \mathbb{Z}$ by

$$\varphi_s(\Delta) = \deg(\gamma_\Delta)$$

where the loop $\gamma_\Delta : \partial\Delta \to S^1$ is defined by $s_\Delta(x) = \gamma_\Delta(x)s(x)$ for $x \in \partial\Delta$. Show that φ_s is a simplicial cocycle, that its cohomology class $[\psi_s]$ is independent of s, and that this class agrees with the first Chern class:

$$c_1(L) = [\varphi_s] \in H^2(X;\mathbb{Z}).$$

Prove that if φ_s is a coboundary then L admits a trivialization over the 2-skeleton and hence over all of X. □

Exercise 1.40 Prove that, up to isomorphism, there are precisely two complex vector bundles over $\mathbb{R}P^2$ of any given rank. Relate this to the fact that $H^2(\mathbb{R}P^2;\mathbb{Z}) = \mathbb{Z}_2$. **Hint:** Show that the set of paths $\Psi : [0,1] \to \mathrm{U}(n)$ with $\Psi(0)\Psi(1) = \mathbb{1}$ has two components. □

Example 1.41 The previous exercise shows that up to isomorphism there is a unique nontrivial complex line bundle $L \to \mathbb{R}P^2$. Explicitly, such a bundle is given by

$$L = S^2 \times_{\mathbb{Z}_2} \mathbb{C}$$

where $\mathbb{Z}_2$ acts in the obvious way on both factors. Thus L is the set of equivalence classes of pairs $[x,z]$ in $S^2 \times \mathbb{C}$ under the equivalence relation $[x,z] \equiv [-x,-z]$. A section of this bundle is a smooth map $s : S^2 \to \mathbb{C}$ which satisfies $s(-x) = -s(x)$. By the Borsuk-Ulam theorem every such map must have a zero. This shows that the bundle is nontrivial. □

Exercise 1.42 Let $E \to X$ be a Hermitian vector bundle over a smooth oriented 4-manifold. Prove that

$$\frac{1}{4\pi^2}\int_X \mathrm{trace}^c(F_A \wedge F_A) = \langle 2c_2(E) - c_1(E)^2, [X]\rangle$$

for every Hermitian connection A on E. □

Exercise 1.43 Prove that two complex vector bundles over a smooth compact 4-manifold X are isomorphic if and only if they have the same rank and the same Chern classes. **Hint:** Assume X is connected. Triangulate X and use the same notation as in Exercise 1.39. Choose an isomorphism $\Psi_{10}(x) : E_{0x} \to E_{1x}$ over the 1-skeleton. For every 2-simplex $\Delta \in \mathcal{S}_2$ choose trivializations $\Phi_0^\Delta(x) : \mathbb{C}^n \to E_{0x}$ and $\Phi_1^\Delta(x) : \mathbb{C}^n \to E_{1x}$ over Δ. Then define $\gamma_\Delta : \partial\Delta \to \mathrm{U}(n)$ by

$$\gamma_\Delta(x) = \Phi_1^\Delta(x)^{-1}\Psi_{10}(x)\Phi_0^\Delta(x).$$

Consider the map $\rho : \mathcal{S}_2 \to \mathbb{Z}$ defined by

$$\rho(\Delta) = \deg(\det \circ \gamma_\Delta).$$

If $c_1(E_0) = c_1(E_1)$ show that ρ is a simplicial cocycle and use this to construct and isomorphism $E_0 \to E_1$ over the 2-skeleton. Using $\pi_2(\mathrm{U}(n)) = 0$ extend this isomorphism over the 3-skeleton and in fact over the complement of a single 4-simplex Δ. Finally use the same argument as above to construct a map $\partial\Delta \cong S^3 \to \mathrm{U}(n)$ and show that this map is contractible if and only if $c_2(E_0) = c_2(E_1)$. □

Exercise 1.44 The space

$$\mathcal{J}(\mathbb{R}^4) = \left\{ J \in \mathbb{R}^{4\times 4} \,|\, J^2 = -\mathbb{1} \right\}$$

of complex structures on $\mathbb{R}^4$ is homotopy equivalent to S^2 (see e.g. [85]). Hence the space of homotopy classes of almost complex structures on the trivial bundle $E = S^4 \times \mathbb{R}^4 \to S^4$ can be identified with $\pi_4(S^2) = \mathbb{Z}_2$. Thus there are precisely two homotopy classes of almost complex structures on E. Let these be represented by $J_0, J_1 : S^4 \to \mathcal{J}(\mathbb{R}^4)$. (In fact J_0 can be chosen constant.) Use the previous exercise to show that (E, J_0) and (E, J_1) are isomorphic as complex vector bundles, i.e. there exists a smooth map $\Phi : S^4 \to \mathrm{GL}(4, \mathbb{R})$ such that

$$J_1(x) = \Phi(x)^{-1} J_0 \Phi(x)$$

for every $x \in S^4$. □

The **Hirzebruch signature theorem** expresses the signature of a complex $2k$-dimensional manifold (that is, the signature of the intersection form on the middle dimensional homology) in terms of the Chern classes of TX. In the case of a complex surface this formula is

$$\operatorname{sign}(Q_X) = \frac{1}{3}\langle c_1(TX)^2 - 2c_2(TX), [X]\rangle. \tag{1.9}$$

The following lemma relates the first Chern class to the intersection form Q_X. It is an important tool in deciding whether or not the intersection form Q_X is even.

Lemma 1.45 *Assume that X is a complex surface. Then*

$$\langle c_1(TX), \alpha\rangle = Q_X(\alpha, \alpha) (\bmod\ 2)$$

for every $\alpha \in H_2(X;\mathbb{Z})$.

Proof: We prove this lemma first under the assumption that the homology class α can be represented by an oriented Riemann surface Σ which is embedded into X as a complex submanifold. Then the normal bundle ν_Σ is a complex line bundle over Σ and, by the above remark, the number $\langle c_1(\nu_\Sigma), [\Sigma]\rangle$ is the self-intersection number of Σ in X. Hence

$$\begin{aligned}
\langle c_1(TX), [\Sigma]\rangle &= \langle c_1(T_\Sigma X), [\Sigma]\rangle \\
&= \langle c_1(T\Sigma), [\Sigma]\rangle + \langle c_1(\nu_\Sigma), [\Sigma]\rangle \\
&= 2 - 2g + \Sigma \cdot \Sigma \\
&\equiv Q_X([\Sigma], [\Sigma]) (\bmod\ 2).
\end{aligned}$$

In general, every integral homology class α can be represented by an oriented embedded surface $\Sigma \subset X$ which may or may not be a complex submanifold.* If Σ is not complex then the same argument works modulo 2 if c_1 is replaced by the second Stiefel-Whitney class $\mathrm{w}_2(TX) \in H^2(X;\mathbb{Z}_2)$. This class agrees with the mod 2 reduction of c_1. □

* Every 2-dimensional integral homology class α in a compact manifold X can be represented by an oriented embedded submanifold $\Sigma \subset X$. To see this note first that α can be represented by a finite cycle, because X can be triangulated. Every such cycle can be thought of as a continuous map defined on a compact 2-dimensional simplicial complex without boundary. Every such complex can be given the structure of a smooth 2-dimensional compact manifold without boundary (which in the case of integer coefficients is orientable). Hence α is represented by a continuous map $f : \Sigma \to X$ defined on a smooth compact 2-manifold Σ. Approximate f by a smooth map and use a general position argument to make f an immersion with finitely many transverse self-intersections. Use a local surgery argument to remove the self-intersections.

If X is a smooth 4-manifold there is an alternative proof which uses the correspondence between complex line bundles and $H^2(X;\mathbb{Z})$ (see Exercise 1.39). Given a homology class $\alpha \in H_2(X;\mathbb{Z})$ choose a complex line bundle $L \to X$ with first Chern class $c_1(L) = \mathrm{PD}(\alpha)$. Choose a smooth section $s : X \to L$ which is transverse to the zero section. Then the submanifold $\Sigma = s^{-1}(0)$ represents the class α. Use surgery along curves to obtain a connected submanifold.

1.7 A crossing index for Fredholm families

K-theory

Let M be a finite dimensional compact manifold. A **K-theory class** on M is an equivalence class of pairs (E, F) of complex vector bundles over M under the equivalence relation

$$(E_1, F_1) \equiv (E_2, F_2) \qquad \Longleftrightarrow \qquad E_1 \oplus F_2 \oplus \mathbb{C}^N \cong E_2 \oplus F_1 \oplus \mathbb{C}^N$$

for some integer N. The additional summand $\mathbb{C}^N$ is needed to obtain an equivalence relation. Denote by $E \ominus F$ the equivalence class of a pair (E, F). The set of equivalence classes is denoted by $K(M)$. Note that the number $\operatorname{rank} E - \operatorname{rank} F$ only depends on the equivalence class $E \ominus F$. Thus there is an **augmentation homomorphism**

$$\varepsilon : K(M) \to \mathbb{Z}, \qquad \varepsilon(E \ominus F) = \operatorname{rank} E - \operatorname{rank} F$$

and its kernel is denoted by $\widetilde{K}(M)$. It consists of all equivalence classes $E \ominus F$ with $\operatorname{rank} E = \operatorname{rank} F$. For every bundle E the pair (E, E) is equivalent to $(0, 0)$ and thus $E \ominus E \equiv 0 \in \widetilde{K}(M)$. Note also that, with the convention $E = E \ominus \{0\}$, one has $E \equiv F \oplus G$ if and only if $E \ominus F \equiv G$. This justifies the notation $E \ominus F$.

The correspondence $E \mapsto E \ominus \mathbb{C}^{\operatorname{rank} E}$ induces a map from isomorphism classes of vector bundles to $\widetilde{K}(M)$. This map is surjective but not injective. The kernel consists of all isomorphism classes of vector bundles $E \to M$ such that $E \oplus \mathbb{C}^N \cong \mathbb{C}^{\operatorname{rank} E + N}$ is the trivial bundle for some integer N. That the map is onto follows from the fact that for every bundle F there exists a bundle F' such that $F \oplus F'$ is isomorphic to the trivial bundle and hence $E \ominus F \equiv (E \oplus F') \ominus \mathbb{C}^N$ with $N = \operatorname{rank}(E \oplus F')$ whenever $\operatorname{rank} E = \operatorname{rank} F$. Thus $\widetilde{K}(M)$ can be defined as the set of equivalence classes of vector bundles over X under the equivalence relation

$$E \equiv F \qquad \Longleftrightarrow \qquad E \oplus \mathbb{C}^{\operatorname{rank} F + N} = F \oplus \mathbb{C}^{\operatorname{rank} E + N} \tag{1.10}$$

for some integer N. This is called *stable equivalence* and $\widetilde{K}(M)$ is called the **reduced K-theory** of M.

The obvious operations of Whitney sum and tensor product carry over to K-theory. Recall that the Chern character is a homomorphism from isomorphism classes of vector bundles over M (with Whitney sum and tensor product) to the rational cohomology of M (with sum and cup-product):

$$\operatorname{ch}(E \oplus F) = \operatorname{ch}(E) + \operatorname{ch}(F), \qquad \operatorname{ch}(E \otimes F) = \operatorname{ch}(E)\operatorname{ch}(F)$$

In particular this shows that the difference of the Chern characters of two bundles E and F depends only on the equivalence class $E \ominus F \in K(M)$. Hence it is natural to define

$$\mathrm{ch}(E \ominus F) = \mathrm{ch}(E) - \mathrm{ch}(F).$$

The total Chern class is multiplicative in the sense that

$$c(E \oplus F) = c(E)c(F), \qquad c(E \ominus F) = c(E)c(F)^{-1}.$$

To prove the second formula note, firstly, that the Chern classes of E depend only on the equivalence class of E under (1.10), secondly, that the Chern classes of the trivial bundle are all zero and, thirdly, that $c(F)c(F') = 1$ whenever $F \oplus F'$ is the trivial bundle.

A crossing index

Recall from Remark 1.37 that the n-th Chern class of a complex rank-n bundle over a $2n$-dimensional manifold agrees with the Euler class. Thus it can be thought of as the number of zeros of a generic section, counted with appropriate signs. This observation can be generalized as follows. Let E and F be two complex vector bundles over a smooth compact oriented manifold of real dimension $\dim M = 2n$ and consider a section $D \in C^\infty(M, \mathrm{Hom}(E, F))$ of the bundle of complex linear maps $E \to F$:

$$\begin{tikzcd} E \arrow[rr, "D"] \arrow[dr] & & F \arrow[dl] \\ & M & \end{tikzcd}.$$

If

$$\mathrm{rank}\, E - \mathrm{rank}\, F + n - 1 = 0 \tag{1.11}$$

then a generic section D will be injective at all but finitely many points. A point $x \in M$ is called a **crossing** if $\ker\, D(x) \neq \{0\}$. A crossing is called **regular** if $\dim^c \ker\, D(x) = 1$ and

$$\mathrm{im}\, D(x) \oplus \{(\nabla_v D)(x)\zeta \,|\, v \in T_x M\} = F_x$$

for some (and hence every) nonzero vector $\zeta \in \ker\, D(x)$. For every regular crossing $x \in M$ define the **crossing index at** x by $\nu(x, D) = +1$ or $\nu(x, D) = -1$ according to the orientations in the direct sum.

Proposition 1.46 *Assume (1.11) and let $D \in C^\infty(M, \mathrm{Hom}(E, F))$ be a section with only regular crossings. Then the crossing index of D is given by*

$$\nu(D) = \sum_{\ker\, D(x) \neq \{0\}} \nu(x, D) = \int_M c_n(F \ominus E).$$

Proof: Assume first that $E = \mathbb{C}$. Then condition (1.11) says that $\mathrm{rank}\, F = n$ and a section of $\mathrm{Hom}(E, F)$ is a section s of F. A crossing x is simply a zero of s, it is regular if s intersects the zero section of F transversally at x, and the crossing index is the intersection number. Hence the assertion of the proposition in this case reduces to the fact that the n-th Chern class agrees with the Euler class.

Secondly, suppose that $E = \mathbb{C}^{N+1}$ for some integer N. Then (1.11) reads $\operatorname{rank} F = N + n$. A general position argument shows that under this condition N generic sections $s_1, \dots, s_N$ of F are everywhere linearly independent. Denote by $F_0 \subset F$ the rank-N subbundle spanned by such generic sections. This subbundle is obviously isomorphic to the trivial bundle and hence the n-th Chern class of the quotient agrees with that of F. Now choose a further section $s : M \to F$ such that the induced section $\bar{s}$ of the rank-n bundle F/F_0 intersects the zero section transversally. Then the crossing index of $\bar{s} : M \to F/F_0$ agrees with the crossing index of

$$D = s_1 \oplus \cdots \oplus s_n \oplus s \in C^\infty(M, \operatorname{Hom}(\mathbb{C}^{N+1}, F))$$

and hence

$$\nu(D) = \nu(\bar{s}) = \int_M c_n(F/F_0) = \int_M c_n(F).$$

This proves the proposition in the case where E is the trivial bundle. The general case can easily be reduced to this. Choose a bundle $E' \to M$ such that $E' \oplus E = \mathbb{C}^{N+1}$ is the trivial bundle and consider the section $D' = D \oplus \mathrm{id}$ of $\operatorname{Hom}(E \oplus E', F \oplus E')$. The formula $c(F \oplus E') = c(F)c(E') = c(F)c(E)^{-1} = c(F \ominus E)$ shows that

$$\nu(D) = \nu(D') = \int_M c_n(F \oplus E') = \int_M c_n(F \ominus E).$$

This proves the proposition. □

It is also interesting to consider the case where (1.11) does not hold but instead

$$d = \operatorname{rank} E - \operatorname{rank} F + n - 1 > 0. \tag{1.12}$$

Denote by SE the unit sphere bundle in E and by $PE = SE/S^1$ the corresponding projective bundle. There is a natural line bundle

$$L = PE \times_{S^1} \mathbb{C} \to PE$$

which restricts to the canonical bundle over projective space in each fiber. A section $D \in C^\infty(M, \operatorname{Hom}(E, F))$ is called **transverse** if the restriction of D to SE is transverse to the zero section in F or, equivalently, D itself is transverse to the zero section in F. The reader may check that in the case (1.11) this condition is equivalent to D having only regular crossings. For transverse sections the space

$$\mathcal{M}(D) = \frac{\{(x,\zeta)\,|\,x \in M,\, \zeta \in E_x,\, |\zeta| = 1,\, D(x)\zeta = 0\}}{S^1} \subset PE$$

is a smooth submanifold of real dimension dim $\mathcal{M}(D) = 2d$ where d is given by (1.12).

Proposition 1.47 *Assume (1.12) and suppose* $D \in C^\infty(M, \mathrm{Hom}(E,F))$ *is transverse. Then*

$$\nu(D) = \int_{\mathcal{M}(D)} c_1(L)^d = \int_M c_n(F \ominus E).$$

Proof: This result reduces to Proposition 1.46 as follows. An interesting class of sections of the canonical bundle $H \to \mathbb{C}P^k$ (with fiber $H_\ell = \ell^* = \mathrm{Hom}(\ell, \mathbb{C})$ for $\ell \in \mathbb{C}P^k$) is given by choosing a linear functional $\varphi : \mathbb{C}^{k+1} \to \mathbb{C}$ and defining $s(\ell) = \varphi|_\ell$. Similarly, a section of the bundle $L \to PE$ can be obtained from any fiberwise linear map $\varphi : E \to \mathbb{C}$. Given $D \in C^\infty(M, \mathrm{Hom}(E,F))$ choose d such maps

$$\varphi_1, \ldots, \varphi_d \in C^\infty(M, \mathrm{Hom}(E, \mathbb{C}))$$

such that the section

$$D' = D \oplus \varphi_1 \oplus \cdots \oplus \varphi_d \in C^\infty(M, \mathrm{Hom}(E, F \oplus \mathbb{C}^d))$$

has only regular crossings. This is possible by a general position argument. Now each φ_ν determines a section $s_\nu : PE \to L$ as above. Moreover, D' has only regular crossings if and only if the common zero set of the sections

$$s = s_1 \oplus \cdots \oplus s_d : PE \to L^d = L \oplus \cdots \oplus L$$

is transverse to $\mathcal{M}(D)$. The reader may check that in this case the crossing index of $D' = D \oplus \varphi$ agrees with the intersection number of the zero set of $s : PE \to L^d$ with $\mathcal{M}(D)$. Hence, by Propositioon 1.46,

$$\int_{\mathcal{M}(D)} c_1(L)^d = s^{-1}(0) \cdot \mathcal{M}(D) = \nu(D \oplus \varphi) = \int_M c_n(F \ominus E).$$

This proves the proposition. □

The infinite dimensional case

The previous proposition generalizes easily to the infinite dimensional setting with complex Banach space bundles $E, F \to M$ over a finite dimensional compact oriented manifold M and a section of Fredholm operators

$D(x) : E_x \to F_x$ for $x \in M$. However, in this case the Chern classes of $E \ominus F$ are undefined and should be replaced by the topological index

$$\mathcal{IND}(D) = \ker D \ominus \operatorname{coker} D \in K(M).$$

This is a well defined element in the K-theory of M and hence has Chern classes. (See Section A.1.) As in the finite dimensional case there is a canonical line bundle $L \to PE$ over the infinite dimensional projective bundle.

Proposition 1.48 *Let $D : E \to F$ be a bundle of complex linear Fredholm operators over a compact oriented $2n$-manifold M. Suppose that the restriction $D|_{SE} : SE \to F$ is transverse to the zero section in F and*

$$d = \operatorname{index}^c D_x + n - 1 \geq 0 \tag{1.13}$$

for $x \in M$. Then

$$\nu(D) = \int_{\mathcal{M}(D)} c_1(L)^d = \int_M c_n(\mathcal{IND}(D^*)).$$

If $d = 0$ then the left hand side is to be understood as the oriented number of points $x \in M$ where $D(x)$ is not injective:

$$\nu(D) = \sum_{\ker D(x) \neq \{0\}} \nu(x, D) = \int_M c_n(\mathcal{IND}(D^*)).$$

The sign $\nu(x, D)$ is determined, as before, by whether or not the isomorphism $T_xM \to \operatorname{coker} D(x) : v \mapsto \nabla_v D(x)\zeta$ is orientation preserving for $0 \neq \zeta \in \ker D(x)$.

Proof: Choose a subbundle $E_1 \subset E$ of finite codimension such that $D|_{E_1}$ is injective. Then $F_1 = DE_1 \subset F$ is also a subbundle of finite codimension and there exist complements $E_0 \subset E$ and $F_0 \subset F$ such that

$$E = E_0 \oplus E_1, \qquad F = F_0 \oplus F_1.$$

Write $D = D_{00} + D_{01} + D_{11}$ where $D_{ij} : E_i \to F_j$. Suppose, by making a small perturbation if necessary, that $D_{00}|_{SE_0} : SE_0 \to F_0$ is transverse to the zero section in F_0. Then the map

$$D_0 = D_{00} + D_{11}$$

is transverse to the zero section in F. Now choose a family of sections $K_t : E \to F$ such that $K_t(x) : E_x \to F_x$ is a compact operator for every $x \in M$ and

$$K_0 = 0, \qquad K_1 = D_{01} = D - D_0.$$

A generic such family gives rise to a cobordism from $\mathcal{M}(D_0)$ to $\mathcal{M}(D)$ in PE and hence the two integrals of $c_1(L)^d$ agree. Moreover, D_0 can be replaced by the finite dimensional bundle homomorphism D_{00} since D_1 is bijective. Hence

$$\int_{\mathcal{M}(D)} c_1(L)^d = \int_{\mathcal{M}(D_{00})} c_1(L)^d = \int_M c_n(F_0 \ominus E_0) = \int_M c_n(\mathcal{IND}(D^*)).$$

The last identity follows from the fact that D, D_0, and D_{00} have the same topological index with $\mathcal{IND}(D_{00}) = E_0 \ominus F_0$ (see Section A.1). This proves the proposition. □

Proposition 1.49 *Let $D : E \to F$ be a bundle of complex linear Fredholm operators over a compact oriented m-manifold M. Suppose that the restriction $D|_{SE} : SE \to F$ is transverse to the zero section in F and*

$$\mathrm{index}^c\, D_x = k + 1 \geq 1. \tag{1.14}$$

for $x \in M$. Then $\mathcal{M}(D)$ is a manifold of dimension $2k + m$ and

$$\int_{\mathcal{M}(D)} c_1(L)^k \wedge \pi^*\omega = \int_M \omega$$

for every $\omega \in \Omega^m(M)$, where $\pi : PE \to M$ denotes the obvious projection.

Proof: If $k = 0$ then $\dim \mathcal{M}(D) = \dim M = m$ and the projection $\pi : \mathcal{M}(D) \to M$ has degree 1. To see this just note that the operator D_x is onto for a generic point $x \in M$. Any such point is a regular value of the projection $\pi : \mathcal{M}(D) \to M$ with a single preimage. This proves the assertion (up to a sign) for $k = 0$. The verification of the sign is left to the reader. The general case reduces to the case $k = 0$ as in the proof of Proposition 1.47. Namely, choose k maps $\varphi_1, \ldots, \varphi_k \in C^\infty(M, \mathrm{Hom}(E, \mathbb{C}))$ such that the section

$$D' = D \oplus \varphi_1 \oplus \cdots \oplus \varphi_k \in C^\infty(M, \mathrm{Hom}(E, F \oplus \mathbb{C}^k)),$$

when restricted to SE is transverse to the zero section in $F \oplus \mathbb{C}^k$. Consider the sections $s_\nu : PE \to L$ determined by φ_ν and denote by $s^{-1}(0) \subset PE$ their common zero set. This is a codimension-$2k$ submanifold transverse to $\mathcal{M}(D)$ and $\mathcal{M}(D') = s^{-1}(0) \cap \mathcal{M}(D)$. Since $s^{-1}(0)$ is the Poincaré dual of the class $c_1(\mathcal{L})^k \in H^{2k}(PE; \mathbb{Z})$ it follows that

$$\int_{\mathcal{M}(D)} c_1(L)^k \wedge \pi^*\omega = \int_{\mathcal{M}(D')} \pi^*\omega = \int_M \omega.$$

This proves the proposition. □

2

RIEMANNIAN GEOMETRY

This chapter is devoted to foundational material about Riemannian manifolds. The first section collects some basic facts about the Levi-Civita connection and the curvature tensor. Section 2.2 is devoted to a discussion of the scalar curvature. It contains a brief discussion of Einstein metrics and proofs of the uniformization theorem for Riemann surfaces as well as the Gromov-Lawson theorem that the posititive scalar curvature condition is preserved under connected sums. The last two sections on the covariant divergence and differential forms contain material which will play a crucial role in the discussion of spin representations and Dirac operators.

2.1 The Levi-Civita connection

Let X be an n-dimensional Riemannian manifold. The **Levi-Civita connection** is the unique torsion free Riemannian connection ∇ on the tangent bundle TX. The Riemannian condition asserts that for any three vector fields $u, v, w \in \mathrm{Vect}(X)$ we have

$$\partial_u \langle v, w\rangle = \langle \nabla_u v, w\rangle + \langle v, \nabla_u w\rangle$$

and the torsion free condition asserts that the Lie bracket of two vector fields v and w is given by*

$$[v, w] = \nabla_w v - \nabla_v w.$$

*The Lie bracket of two vector fields $v, w : X \to TX$ is defined by

$$[v, w] = -\mathcal{L}_v w = -\left.\frac{d}{dt}\right|_{t=0} {\varphi_t}^* w,$$

where $\varphi_t \in \mathrm{Diff}(X)$ denotes the flow of v. In local coordinates,

$$[v, w]_\alpha = dv_\alpha \cdot w_\alpha - dw_\alpha \cdot v_\alpha.$$

With this convention the operator $\mathrm{Vect}(X) \to \mathrm{Der}(C^\infty(X)) : v \mapsto \mathcal{L}_v$ defined by $\mathcal{L}_v f = df \circ v = \left.\frac{d}{dt}\right|_{t=0} f \circ \varphi_t$ is a Lie algebra anti-homomorphism:

$$\mathcal{L}_{[v,w]} = \mathcal{L}_w \mathcal{L}_v - \mathcal{L}_v \mathcal{L}_w = -[\mathcal{L}_v, \mathcal{L}_w].$$

This corresponds to the fact that it is the differential at the identity of the Lie group anti-homomorphism $\mathrm{Diff}(X) \to \mathrm{Aut}(C^\infty(X))$ which assigns to each diffeomnorphism $\varphi \in \mathrm{Diff}(X)$ the linear operator $f \mapsto f \circ \varphi$ on $C^\infty(X)$.

Exercise 2.1. (Christoffel symbols) Prove that the tangent bundle of a Riemannian manifold admits a unique torsion free Riemannian connection. **Hint:** In local coordinates a connection can be expressed in the form

$$\nabla_\xi \eta = \sum_k \left(\sum_i \frac{\partial \eta^k}{\partial x^i} \xi^i + \sum_{i,j} \Gamma^k_{ij} \xi^i \eta^j \right) \frac{\partial}{\partial x^k}$$

where $\xi = \sum_i \xi^i \partial/\partial x^i$ and $\eta = \sum_j \eta^j \partial/\partial x^j$ are vector fields, g_{ij} denotes the Riemannian metric, and the Γ^k_{ij} are suitable real valued functions. The torsion free condition takes the form $\Gamma^k_{ij} = \Gamma^k_{ji}$ and the connection is Riemannian if and only if

$$\frac{\partial g_{ij}}{\partial x^\ell} = \sum_{\nu=1}^n \left(\Gamma^\nu_{\ell i} g_{\nu j} + g_{i\nu} \Gamma^\nu_{\ell j} \right).$$

Prove that these two conditions are satisfied if and only if the Γ^k_{ij} are the **Christoffel symbols**, given by

$$\Gamma^k_{ij} = \sum_\nu g^{k\nu} \Gamma_{\nu ij}, \qquad \Gamma_{kij} = \frac{1}{2} \left(\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{kj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right).$$

Here the g^{ij} are the entries of the inverse matrix, i.e. $\sum_\nu g_{i\nu} g^{\nu j} = \delta_i^j$. □

The **curvature tensor** is a skew-symmetric bilinear form $R_x : T_xX \times T_xX \to \mathrm{End}(T_xX)$ defined by

$$R(u,v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w + \nabla_{[u,v]} w$$

for $u, v, w \in \mathrm{Vect}(X)$. The Riemannian condition on the Levi-Civita connection implies $R(u,v) \in \mathfrak{so}(TX)$ for all u and v.

Lemma 2.2. (Bianchi's first identity) *For $u, v, w, z \in \mathrm{Vect}(X)$*

$$R(u,v)w + R(v,w)u + R(w,u)v = 0 \tag{2.1}$$

and hence

$$\langle R(u,v)w, z \rangle = \langle R(w,z)u, v \rangle. \tag{2.2}$$

Proof: The first identity follows from the Jacobi identity for the Lie bracket of vector fields and the second identity follows from the first. □

The covariant derivative of the tensor R can be defined by

$$(\nabla_u R)(v,w) = \nabla_u (R(v,w)) - R(\nabla_u v, w) - R(v, \nabla_u w).$$

It satisfies the following.

Lemma 2.3. (Bianchi's second identity) *For $u, v, w \in \mathrm{Vect}(X)$*

$$(\nabla_u R)(v, w) + (\nabla_v R)(w, u) + (\nabla_w R)(u, v) = 0. \tag{2.3}$$

Proof: The Bianchi identity for any connection ∇ on any vector bundle $E \to X$ with curvature form $R^\nabla \in \Omega^2(X, \mathrm{End}(E))$ asserts that $d^\nabla R^\nabla = 0$ (see Chapter 1). In the case of the Levi-Civita connection the left hand side of (2.3) is $d^\nabla R^\nabla(u, v, w)$ and hence must be zero. □

Exercise 2.4 Prove that the curvature tensor can, in local coordinates, be expressed in the form

$$\langle R(u,v)w, z\rangle = \sum_{i,j,k,\ell} R_{\ell kij} u^i v^j w^k z^\ell, \qquad R_{\ell kij} = \sum_\nu g_{\ell\nu} R^\nu_{kij},$$

where

$$R^\ell_{kij} = \frac{\partial \Gamma^\ell_{jk}}{\partial x^i} - \frac{\partial \Gamma^\ell_{ik}}{\partial x^j} + \sum_\nu \left(\Gamma^\ell_{i\nu}\Gamma^\nu_{jk} - \Gamma^\ell_{j\nu}\Gamma^\nu_{ik} \right).$$

Prove that the formula (2.2) reads

$$R_{ijk\ell} = R_{k\ell ij},$$

that Bianchi's first identity (2.1) takes the form

$$R^\ell_{kij} + R^\ell_{jki} + R^\ell_{ijk} = 0,$$

and Bianchi's second identity (2.3) takes the form

$$\begin{aligned} \partial_m R^i_{jk\ell} + \partial_k R^i_{j\ell m} + \partial_\ell R^i_{jmk} &= \sum_\nu \left(R^i_{\nu k\ell}\Gamma^\nu_{mj} + R^i_{\nu\ell m}\Gamma^\nu_{kj} + R^i_{\nu mk}\Gamma^\nu_{\ell j} \right) \\ &\quad - \sum_\nu \left(\Gamma^i_{m\nu} R^\nu_{jk\ell} + \Gamma^i_{k\nu} R^\nu_{j\ell m} + \Gamma^i_{\ell\nu} R^\nu_{jmk} \right) \end{aligned}$$

for i, j, k, ℓ, m. □

Orthonormal frames

It is sometimes useful to describe the Levi-Civita connection and the curvature tensor in terms of a local orthonormal frame $e_1, \ldots, e_n$ of TX. Thus we define

$$\Gamma^k_{ij} = \langle \nabla_{e_i} e_j, e_k\rangle, \qquad R^\ell_{kij} = \langle R(e_i, e_j)e_k, e_\ell\rangle.$$

Note that with this definition the numbers Γ^k_{ij} are not symmetric in i and j unless the Lie bracket $[e_i, e_j]$ vanishes. In fact, we have

$$[e_i, e_j] = \sum_k (\Gamma^k_{ji} - \Gamma^k_{ij}) e_k, \qquad \Gamma^k_{ij} + \Gamma^j_{ik} = 0.$$

The first equation expresses the fact that the Levi-Civita connection is torsion free and the second that it is Riemannian. The curvature is now given by the formula

$$R^{\ell}_{kij} = \partial_i \Gamma^{\ell}_{jk} - \partial_j \Gamma^{\ell}_{ik} + \sum_{\nu} \left(\Gamma^{\ell}_{i\nu} \Gamma^{\nu}_{jk} - \Gamma^{\ell}_{j\nu} \Gamma^{\nu}_{ik} \right) - \sum_{\nu} \left(\Gamma^{\nu}_{ij} - \Gamma^{\nu}_{ji} \right) \Gamma^{\ell}_{\nu k},$$

where ∂_j denotes the derivative in the direction e_j. Also note that in an orthonormal frame we have

$$R_{ijk\ell} = R_{k\ell ij} = R^{k}_{\ell ij} = R^{\ell k}_{ij}$$

and that this expression is anti-symmetric under interchanging i and j, respectively k and ℓ. In an orthonormal frame the first Bianchi identity takes the form

$$R_{\ell ijk} + R_{\ell jki} + R_{\ell kij} = 0 \tag{2.4}$$

and, if $\nabla_{e_i} e_j = 0$ for all i, j at some point $x \in X$, then at this point the second Bianchi identity reads

$$\frac{\partial R_{ijk\ell}}{\partial x^m} + \frac{\partial R_{ij\ell m}}{\partial x^k} + \frac{\partial R_{ijmk}}{\partial x^\ell} = 0. \tag{2.5}$$

The proofs are left to the reader.

Exercise 2.5 Prove that on any Riemannian manifold X near every point $x \in X$ there exists a local orthonormal frame $e_1, \ldots, e_n$ such that the covariant derivatives $\nabla_{e_j} e_k$ all vanish at the point x. □

2.2 Scalar curvature

The **Ricci tensor** is a symmetric bilinear form $S_x : T_xX \times T_xX \to \mathbb{R}$ defined by

$$S(u, v) = \sum_{i=1}^{n} \langle R(e_i, u)v, e_i \rangle \tag{2.6}$$

for any orthonormal frame $e_1, \ldots, e_n$ of TX. The reader may check, using Lemma 2.2, that this expression is independent of the orthonormal frame used to define it and that the resulting bilinear form is symmetric. In an orthonormal frame the Ricci tensor can be represented by a symmetric matrix. The **scalar curvature** of X is defined as the trace of this matrix and is denoted by s. Thus

$$s = \sum_{j=1}^{n} S(e_j, e_j)$$

for any orthonormal frame $e_1, \ldots, e_n$. In particular, a manifold with positive definite Ricci tensor has positive scalar curvature.

If the same orthonormal frame is used to define the Ricci tensor and the scalar curvature we obtain the expression

$$s = \sum_{i,j} \langle R(e_i, e_j)e_j, e_i \rangle = \sum_{i,j} R_{ijij}. \tag{2.7}$$

This formula holds only in orthonormal frames. It relates the scalar curvature to the **sectional curvature** of a plane $E \subset T_x X$. The latter is defined by

$$K_x(E) = \langle R(u,v)v, u \rangle$$

for any orthonormal frame u, v of E and its value is independent of the choice of this frame. Hence the scalar curvature is twice the sum of the sectional curvatures over all planes which are spanned by pairs of vectors in a given orthonormal frame.

Exercise 2.6 Prove that the Ricci curvature tensor is in local coordinates given by

$$S(\xi, \eta) = \sum_{i,j} R_{ij} \xi^i \eta^j, \qquad R_{ij} = \sum_{k=1}^{n} R_{ikj}^k = \sum_{k,\ell} g^{k\ell} R_{ki\ell j}.$$

Hence prove that the scalar curvature is

$$s = \sum_{i,j} g^{ij} R_{ij} = \sum_{i,j,k} g^{ij} R_{ikj}^k = -\sum_{i,j} R_{ij}^{ij}. \qquad \square$$

Lemma 2.7 *Let X be a Riemannian manifold of dimension $n = \dim X \geq 3$ and assume that there exists a function $\lambda : X \to \mathbb{R}$ such that the Ricci tensor satisfies*

$$S(u,v) = \lambda \langle u, v \rangle \tag{2.8}$$

for all $x \in X$ and all $u, v \in T_x X$. Then λ is constant and, moreover, the scalar curvature is given by

$$s = n\lambda.$$

Proof: Choose a local orthonormal frame $e_1, \ldots, e_n$ near a point $x \in X$ such that all the covariant derivatives $\nabla_{e_j} e_k = 0$ at x (see Exercise 2.5). Denote by $R_{ijk\ell}$ the curvature tensor in this frame. Then the Ricci tensor is given by

$$R_{ij} = S(e_i, e_j) = \sum_k R_{kikj}.$$

Since $\langle e_i, e_j \rangle = 1$ the condition $S = \lambda g$ takes the form

$$R_{ij} = \lambda \delta_{ij}$$

and, in particular, the scalar curvature is

$$s = \sum_j R_{jj} = n\lambda.$$

By (2.5), the second Bianchi identity at the point x reduces to

$$\partial_m R_{ijk\ell} + \partial_k R_{ij\ell m} + \partial_\ell R_{ijmk} = 0$$

for all i, j, k, ℓ, m. Choose $k = i$ and $\ell = j$, and rename m into k to obtain

$$\partial_k R_{ijij} + \partial_i R_{ijjk} + \partial_j R_{ijki} = 0.$$

Now take the sum over all i and j and use the formulae $R_{ijk\ell} = R_{k\ell ij} = -R_{jik\ell}$ to obtain

$$\begin{aligned} 0 &= \partial_k \sum_{i,j} R_{ijij} + \sum_{i,j} \partial_i R_{ijjk} + \sum_{i,j} \partial_j R_{ijki} \\ &= \partial_k \sum_j R_{jj} - \sum_{i,j} \partial_i R_{jijk} - \sum_{i,j} \partial_j R_{ijik} \\ &= n\partial_k\lambda - \sum_i \partial_i R_{ik} - \sum_i \partial_j R_{jk} \\ &= (n-2)\partial_k\lambda \end{aligned}$$

for all k. Hence λ must be constant. □

A Riemannian metric on X is called an **Einstein metric** if its Ricci-tensor is a constant multiple of the metric g, i.e. if (2.8) holds for some constant $\lambda \in \mathbb{R}$. The question of the existence of an Einstein metric on a given manifold X is an important problem in Riemannian geometry. A deep theorem by Yau guarantees the existence of such metrics on a large class of Kähler manifolds. We shall discuss his result in Chapter 3.

Remark 2.8. (Second fundamental form) Assume that the manifold X is embedded into $\mathbb{R}^N$ and inherits the metric from the ambient space. Consider the map $\pi : X \to \mathbb{R}^{N\times N}$ which assigns to each point $x \in X$ the orthogonal projection $\pi(x) : \mathbb{R}^N \to T_xX$. Its differential at x in the direction $v \in T_xX$ is a matrix $d\pi(x)v \in \mathbb{R}^{N\times N}$ which sends T_xX to $T_xX^\perp$. Thus we have a map

$$h_x : T_xX \to \mathrm{Hom}(T_xX, T_xX^\perp)$$

defined by

$$h_x(v)w = (d\pi(x)v)w$$

for $v, w \in T_xX$. This map is called the **second fundamental form**. It satisfies $h_x(v)w = h_x(w)v$. The **Gauss equation** asserts that the curvature of X at $x \in X$ is given by

$$R_x(u,v) = h_x(u)^* h_x(v) - h_x(v)^* h_x(u). \tag{2.9}$$

If $f : T_xX \to T_xX^\perp$ is a function whose graph agrees with X (with the point x shifted to the origin) then it is easy to see that $h_x(v)w$ is given by the Hessian of f at 0.

Example 2.9 Consider the n-sphere $S^n \subset \mathbb{R}^{n+1}$ with its standard metric. Prove that

$$R_x(u,v) = uv^* - vu^* \in \mathrm{End}(T_xX)$$

for $u, v \in T_xX$. Deduce that the curvature tensor, when regarded as a linear map $\Lambda^2 \to \Lambda^2$, is the identity. Hence prove that the scalar curvature of the n-sphere with its standard metric is $s = n(n-1)$. □

Exercise 2.10 In holomorphic coordinates $x + iy$ the standard metric on $S^2 = \mathbb{C} \cup \{\infty\}$ is

$$g = 4\frac{dx^2 + dy^2}{(1 + x^2 + y^2)^2}.$$

Use this to prove that the 2-sphere has constant scalar curvature 2. □

Exercise 2.11 Consider the manifold $X = S^2 \times S^2$ with the metric $g_X = \lambda_1 g \times \lambda_2 g$ where g denotes the standard metric on S^2 and $\lambda_i > 0$. Prove that g_X is an Einstein metric if and only if $\lambda_1 = \lambda_2$. Prove that the manifold $X = S^2 \times \Sigma$ does not admit an Einstein metric whenever Σ is a Riemann surface with genus $g \geq 1$. □

Exercise 2.12 Let X and Y be Riemannian manifolds. Prove that the scalar curvature of the product metric on $X \times Y$ is given by

$$s_{X\times Y}(x,y) = s_X(x) + s_Y(y)$$

for $x \in X$ and $y \in Y$. In particular $X \times Y$ admits a metric with positive scalar curvature whenever either X or Y does. □

Exercise 2.13 Let $B^n = \{x \in \mathbb{R}^n \,|\, |x| \leq 1\}$ dnote the unit ball in $\mathbb{R}^n$. Consider the diffeomorphism $f : \mathbb{R}^n - \{0\} \to \mathbb{R} \times S^{n-1}$ given by

$$f(x) = \left(\log|x|, \frac{x}{|x|}\right).$$

Prove that the pullback of the product metric g on $\mathbb{R} \times S^{n-1}$ under f is given by

$$f^*g(\xi,\eta) = g(df(x)\xi, df(x)\eta)) = \frac{1}{|x|^2}\langle \xi, \eta\rangle$$

for $x \in B^n$ and $\xi, \eta \in \mathbb{R}^n = T_xB^n$. □

Exercise 2.14 Let $f : X \to Y$ be a diffeomorphism of smooth n-manifolds. Let g be a Riemannian metric on Y with scalar curvature s_g and denote by f^*g the pullback metric on X. Prove that the scalar curvature of this metric is given by

$$s_{f^*g} = s_g \circ f.$$

Hint: Use the naturality of the curvature on general vector bundles. □

Exercise 2.15 Prove that the positive definite Laplace-Beltrami operator $\Delta_g = d^*d$ of the metric g is in local coordinates given by

$$\Delta_g u = -\frac{1}{\sqrt{\det(g)}} \sum_{i,j} \frac{\partial}{\partial x^i} \left(g^{ij} \sqrt{\det(g)} \frac{\partial u}{\partial x^j} \right) \tag{2.10}$$

or, equivalently,

$$\Delta_g u = -\sum_{i,j} g^{ij} \frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i,j,k} \frac{\partial u}{\partial x^k} \Gamma_{ij}^k g^{ij}. \tag{2.11}$$

for $u : \mathbb{R}^n \to \mathbb{R}$. □

The following proposition gives an explicit formula for the change in the scalar curvature under a conformal change in the metric.

Lemma 2.16 *Let X be an n-manifold with Riemannian metric g and scalar curvature $s = s_g : X \to \mathbb{R}$. Consider the metric*

$$\tilde{g} = u^2 g$$

for some positive function $u : X \to \mathbb{R}$. The corresponding scalar curvature $\tilde{s} = s_{u^2 g} : X \to \mathbb{R}$ is given by

$$\tilde{s} - u^{-2} s = 2(n-1) u^{-3} \Delta_g u - (n-1)(n-4) u^{-4} |du|_g^2. \tag{2.12}$$

Proof: The proof is by a rather lengthy, but not very enlightening calculation which we leave to the reader as an exercise with hints. The first hint is to use the formula (2.11) of Exercise 2.15 for the Laplace-Beltrami operator. Secondly, it is convenient to use geodesics to construct a local coordinate chart such that $g_{ij}(0) = \delta_{ij}$ and the Christoffel symbols Γ_{ij}^k all vanish at $x = 0$. Note, however, that the derivatives of the Γ_{ij}^k and also the individual derivatives of the g_{ij} will in general not vanish at $x = 0$. The curvature tensor at $x = 0$ is given by

$$R_{kij}^\ell = \frac{\partial \Gamma_{jk}^\ell}{\partial x^i} - \frac{\partial \Gamma_{ik}^\ell}{\partial x^j}.$$

A simple calculation shows that the Christoffel symbols $\widetilde{\Gamma}_{ij}^k$ of the metric $\tilde{g} = u^2 g$ are given by

$$\widetilde{\Gamma}_{ij}^k - \Gamma_{ij}^k = a_{ij}^k = u^{-1}\left(\frac{\partial u}{\partial x^i}\delta_j^k + \frac{\partial u}{\partial x^j}\delta_i^k - \sum_\nu g^{k\nu}\frac{\partial u}{\partial x_\nu}g_{ij}\right).$$

It is important here to distinguish between δ_{ij} and g_{ij} because the a_{ij}^k are still to be differentiated. In fact, the lemma follows by inserting this expression in the formula

$$\widetilde{R}_{kij}^\ell - R_{kij}^\ell = \frac{\partial a_{jk}^\ell}{\partial x^i} - \frac{\partial a_{ik}^\ell}{\partial x^j} + \sum_\nu a_{i\nu}^\ell a_{jk}^\nu - \sum_\nu a_{j\nu}^\ell a_{ik}^\nu,$$

setting $\ell = i$, multiplying by g^{jk}, and finally summing over all i, j, and k. These calculations are left to the reader. □

Exercise 2.17 With $n \neq 2$ and $v = u^{n/2-1}$ prove that (2.12) is equivalent to the **Yamabe equation**

$$\frac{4(n-1)}{n-2}\Delta_g v + sv = \tilde{s}v^{\frac{n+2}{n-2}}. \tag{2.13}$$

Here $\tilde{s}$ is the scalar curvature of the metric

$$\tilde{g} = v^{\frac{4}{n-2}}g$$

and thus a positive solution of (2.13) with a given constant $\tilde{s} = \lambda$ gives rise to a metric with constant scalar curvature. **Hint:** Use the identities

$$|du^m|_g^2 = m^2u^{2m-2}|du|_g^2$$

and

$$\Delta_g u^m = mu^{m-1}\Delta_g u + m(m-1)u^{m-2}|du|_g^2. \qquad \square$$

Theorem 2.18. (Gromov-Lawson) *Let $n \geq 3$ and assume that X and Y are Riemannian n-manifolds with positive scalar curvature. Then the connected sum $X\#Y$ admits a metric of positive scalar curvature.*

Proof: The proof was explained to me by Mario Micallef. It relies on the formula of Lemma 2.16. Fix a point $x_0 \in X$ and identify a neighbourhood of x_0 in X with a neighbourhood of 0 in $\mathbb{R}^n$. Thus we are given a metric g on $\mathbb{R}^n$ with positive scalar curvature. We may assume without loss of generality that $g_{ij}(0) = \delta_{ij}$ and $\Gamma_{ij}^k(0) = 0$. The main idea is to multiply this metric by a function $u^2 : \mathbb{R}^n \to (0, \infty)$ which at 0 has a singularity of the form

$$u(x) \sim |x|^{-1}.$$

In view of Exercise 2.13 this implies that the rescaled metric near zero approximates the metric on the cylinder $\mathbb{R} \times S^{n-1}$. The main point is to show that this can be done without destroying the positivity of the scalar curvature. To see this recall from Lemma 2.16 that the scalar curvature $\tilde{s}$ of the rescaled metric $\tilde{g} = u^2 g$ is given by

$$\tilde{s} = u^{-2}s + 2(n-1)u^{-3}\Delta_g u - (n-1)(n-4)u^{-4}|\nabla u|_g^2.$$

Now consider the ordinary Laplacian

$$\Delta = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

for the function $u_0(x) = 1/|x|$. The reader may check that for this choice of u the scalar curvature $\tilde{s}$ is positive near zero. (Note that $\Delta u_0 = 0$ in the case $n = 3$.) More generally, if we define

$$u(x) = f(|x|)$$

for some function $f : \mathbb{R} \to \mathbb{R}$ we find

$$\Delta u = -f''(r) - (n-1)\frac{f'(r)}{r}, \qquad |\nabla u|^2 = f'(r)^2$$

where $r = |x|$. Thus we must find a function f such that

$$f(r) = \begin{cases} \delta/r, & \text{for } r \text{ near } 0, \\ 1, & \text{for } r \text{ near } 1, \end{cases}$$

and

$$s_0 - 2(n-1)\frac{f''}{f} - 2(n-1)^2\frac{f'}{rf} - (n-1)(n-4)\left(\frac{f'}{f}\right)^2 - C\frac{r|f'|}{f} > 0$$

for all r where $s_0 > 0$ is the infimum of the scalar curvature of the metric g. Here we have worked with the ordinary Laplacian but this is only a small perturbation of the Laplace-Beltrami operator near $x = 0$. The last term in the inequality accounts for the error terms. Following Micallef and Wang [88] we introduce the function $\alpha : \mathbb{R} \to \mathbb{R}$ by

$$\frac{f'}{f} = -\frac{\alpha}{r}, \qquad \frac{f''}{f} = -\frac{\alpha'}{r} + \frac{\alpha + \alpha^2}{r^2}.$$

Then the above inequality becomes

$$s_0 + 2(n-1)\frac{\alpha'}{r} + (n-1)(n-2)\frac{\alpha(2-\alpha)}{r^2} - C\alpha > 0. \tag{2.14}$$

It is now fairly easy to find a function $\alpha : \mathbb{R} \to \mathbb{R}$ which satisfies (2.14) and

$$\alpha(r) = \begin{cases} 1, & \text{for } r \text{ near } 0, \\ 0, & \text{for } r \text{ near } 1. \end{cases}$$

The term of the form α/r^2 can be used to compensate for the negative term $-C\alpha$ for small r. We can then choose $\alpha : [0, r_0] \to [0,1]$ with $r_0 > 0$ sufficiently small such that $\alpha(r_0) = 0$, $\alpha(0) = 1$, and

$$s_0 + 2(n-1)\frac{\alpha'}{r} + (n-1)(n-2)\frac{\alpha(1+\varepsilon-\alpha)}{r^2} > 0.$$

To find this function it is useful to consider the curve

$$\gamma(t) = \alpha(r_0 e^{-t}).$$

Then the differential inequality translates into

$$\dot{\gamma} < (1+\varepsilon-\gamma)\gamma + c{r_0}^2 e^{-2t}.$$

A solution of the differential equation

$$\dot{\gamma} = (1+\varepsilon-\gamma)\gamma$$

is given by the explicit formula

$$\gamma(t) = \frac{(1+\varepsilon)Ae^{(1+\varepsilon)t}}{1+Ae^{(1+\varepsilon)t}}, \qquad 0 < A \ll 1.$$

Perturbing this function slightly near $t = 0$ and $t = \infty$ gives a solution of the required differential inequality with $\gamma(0) = 0$ and $\gamma(t) = 1$ for $t \geq T$. Thus we have constructed a metric on $X - \{x_0\}$ which has positive scalar curvature and which near x_0 converges to a product metric on the cylinder $\mathbb{R} \times S^{n-1}$. We can now perturb this metric to make it equal to the metric on the cylinder in a neighbourhood of x_0. Choosing a similar metric on $Y - \{y_0\}$ we can construct a metric of positive scalar curvature on the connected sum $X \# Y$. This proves the theorem. □

In Chapter 3 we shall see that $\mathbb{C}P^2$ has positive sectional curvature and hence positive scalar curvature.

Corollary 2.19 *For any two positive integers ℓ and m the 4-manifold*

$$X = \ell\mathbb{C}P^2 \# m\overline{\mathbb{C}P}^2$$

admits a metric of positive scalar curvature.

Proof: Theorem 2.18 and Example 3.49 below. □

Theorem 2.20. (Uniformization) *Every compact oriented Riemann surface Σ admits a metric with constant scalar curvature. There is such a metric in every conformal class and it is unique up to a constant factor.*

Proof: Let g be a Riemannian metric on Σ with volume form ω_g. For any function $u : \Sigma \to \mathbb{R}$ consider the rescaled metric $\tilde{g} = e^{2u}g$. By Lemma 2.16 the scalar curvature $s_{e^{2u}g} : \Sigma \to \mathbb{R}$ of the rescaled metric is given by

$$s_{e^{2u}g} = e^{-2u}s_g + 2e^{-3u}\Delta_g e^u + 2e^{-4u}|de^u|_g^2$$

where $\Delta_g = d^*d$ denotes the positive definite Laplace-Beltrami operator of the metric g. Since $\Delta e^u = e^u\Delta u - e^u|du|^2$ and $|de^u|^2 = e^{2u}|du|^2$ it follows that $s_{e^{2u}g} = e^{-2u}(s_g + 2\Delta_g u)$. Hence the Gauss curvature $K = s/2$ of the rescaled metric is given by

$$K_{e^{2u}g} = e^{-2u}(\Delta_g u + K_g). \tag{2.15}$$

By the Gauss-Bonnet theorem, the integral of K_g is given by*

$$\frac{1}{2\pi}\int_\Sigma K_g\omega_g = \chi(\Sigma).$$

In the case of the torus this integral is zero. Hence the linear equation $\Delta_g u = -K_g$ has a solution u and it follows that $K_{e^{2u}g} = 0$. Uniqueness in this case is obvious. That the 2-sphere with the standard complex structure admits a metric of constant scalar curvature was shown in Examples 2.9 and 3.48. The uniqueness proof in this case will be omitted. For surfaces of higher genus write (2.15) in the form

$$\Delta_g u - e^{2u}K_{e^{2u}g} = -K_g.$$

With $K_{e^{2u}g} = -1$ this equation becomes

$$\Delta_g u + e^{2u} = -K_g.$$

Since $\int_\Sigma(-K_g)\omega_g = -2\pi\chi(\Sigma) > 0$ this is a special case of the Kazdan-Warner equation (D.1) in Appendix D and hence, by Theorem D.1, it has a unique solution u. This proves the theorem. □

*This also follows from the fact that the Ricci form $\rho_{\omega_g} = K_g\omega_g$ represents 2π times the first Chern class of the tangent bundle. (See Lemma 3.44 below.)

2.3 Divergence

The covariant derivative of a vector field $v : X \to TX$ determines an endomorphism $TX \to TX : w \mapsto \nabla_w v$ which we denote by ∇v. Its trace is called the **covariant divergence** of v and is denoted by

$$\mathrm{div}(v) = \mathrm{trace}(\nabla v) = \sum_i \langle \nabla_{e_i} v, f_i \rangle$$

where $e_1, \dots, e_n$ is a basis of TX and $f_1, \dots, f_n$ is the dual basis, i.e.

$$\langle e_i, f_j \rangle = \delta_{ij}.$$

Note that the divergence of v is independent of the choice of the basis used to define it. The divergence has the following formal properties.

Lemma 2.21 *For any function $f : X \to \mathbb{R}$ and any vector field $v : X \to TX$ we have*

$$\mathrm{div}(fv) = f\mathrm{div}(v) + df(v)$$

and

$$\int_X \mathrm{div}(v)\,\mathrm{dvol} = 0. \tag{2.16}$$

Proof: The first identity follows by direct calculation:

$$\begin{aligned}
\mathrm{div}(fv) &= \sum_i \langle f_i, \nabla_{e_i}(fv) \rangle \\
&= \sum_i \langle f_i, f\nabla_{e_i} v + df(e_i)v \rangle \\
&= f \cdot \sum_i \langle f_i, \nabla_{e_i} v \rangle + df\left(\sum_i \langle f_i, v \rangle e_i \right) \\
&= f \cdot \mathrm{div}(v) + df(v).
\end{aligned}$$

To prove the second identity note that the volume form is in local coordinates given by

$$\mathrm{dvol} = \sqrt{\det g} dx^1 \wedge \ldots \wedge dx^n.$$

The divergence of a vector field $\xi = \sum_i \xi^i \partial/\partial x^i$ is given by

$$\begin{aligned}
\mathrm{div}_g(\xi) &= \sum_i \left(\frac{\partial \xi^i}{\partial x^i} + \frac{1}{2} \sum_{\nu,\mu} g^{\nu\mu} \frac{\partial g_{\nu\mu}}{\partial x^i} \xi^i \right) \\
&= \sum_i \left(\frac{\partial \xi^i}{\partial x^i} + \frac{1}{2} \mathrm{trace}\left(g^{-1} \frac{\partial g}{\partial x^i} \right) \xi^i \right)
\end{aligned}$$

$$= \sum_i \left(\frac{\partial \xi^i}{\partial x^i} + \xi^i \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \sqrt{\det g} \right)$$

Hence

$$\operatorname{div}_g(\xi) \cdot \sqrt{\det g} = \sum_i \frac{\partial}{\partial x_i} \left(\xi^i \sqrt{\det g} \right)$$

and this proves the lemma. □

Lemma 2.22 *Let $D : C^\infty(X, E) \to \Omega^1(X, E)$ be any Riemannian connection on any Riemannian vector bundle $E \to X$. Then the L^2-adjoint of the first order operator $D_v : C^\infty(X, E) \to C^\infty(X, E)$ is given by the formula*

$$D_v^* s + D_v s = -\operatorname{div}(v) s$$

for $s \in C^\infty(X, E)$.

Proof: For two sections $s, s' \in C^\infty(X, E)$ consider the function

$$f = \langle s, s' \rangle.$$

Since D is a Riemannian connection we have

$$\langle D_v s, s' \rangle + \langle s, D_v s' \rangle = df(v) = \operatorname{div}(fv) - f \operatorname{div}(v)$$

for any vector field $v : X \to TX$. By Lemma 2.21, the integral of $\operatorname{div}(fv)$ over X vanishes, and hence

$$\int_X (\langle D_v s, s' \rangle + \langle s, D_v s' \rangle) \,\mathrm{dvol} = -\int_X \operatorname{div}(v) \cdot \langle s, s' \rangle \,\mathrm{dvol}.$$

This proves the lemma. □

Lemma 2.23 *If $e_1, \dots, e_n$ is a local frame of TX and $f_1, \dots, f_n$ is the dual frame then*

$$\sum_i \left(\nabla_{e_i} f_i + \operatorname{div}(e_i) f_i \right) = 0.$$

Proof: The pointwise inner product with e_j gives

$$\sum_i \langle \nabla_{e_i} f_i + \operatorname{div}(e_i) f_i, e_j \rangle = \operatorname{div}(e_j) - \sum_i \langle f_i, \nabla_{e_i} e_j \rangle = 0$$

for every j. □

Exercise 2.24 Prove that the covariant divergence of a vector field v satisfies the identity

$$\mathcal{L}_v \mathrm{dvol} = \operatorname{div}(v) \mathrm{dvol}$$

(and hence depends only on the volume form). □

Remark 2.25 Consider a local orthonormal frame $e_1, \ldots, e_n$ of TX with $\Gamma_{ij}^k = \langle \nabla_{e_i} e_j, e_k \rangle$ as in Section 2.1. Then the covariant divergence of the vector field e_j is given by

$$\mathrm{div}(e_j) = -\sum_i \Gamma_{ii}^j. \qquad \square$$

2.4 Differential forms

Any connection on TX induces a connection on the bundle $\Lambda^k T^*X$ whose sections are differential k-forms on X. This induced connection is uniquely characterized by the formula

$$\nabla_v(\iota(w)\alpha) = \iota(\nabla_v w)\alpha + \iota(w)\nabla_v \alpha. \tag{2.17}$$

For $\alpha \in \Omega^1(X)$ the left hand side is the ordinary derivative of the function $\alpha(w)$ and so the equation determines the covariant derivative of 1-forms. Now it can be used inductively to define the covariant derivative of k-forms for $k = 2, 3, \ldots$. It is easy to see that

$$\nabla_v(\alpha \wedge \beta) = (\nabla_v \alpha) \wedge \beta + \alpha \wedge \nabla_v \beta. \tag{2.18}$$

Recall that the pointwise inner product of k-forms $\alpha, \beta \in \Omega^k(X)$ at $x \in X$ is given by

$$\langle \alpha, \beta \rangle = \sum_{i_1 < \cdots < i_k} \alpha(e_{i_1}, \ldots, e_{i_k})\beta(e_{i_1}, \ldots, e_{i_k})$$

where $e_1, \ldots, e_n$ is any orthonormal basis of T_xX. Exercise 2.28 below shows that this expression is independent of the choice of the orthonormal basis. Recall also that the Riemannian metric induces an isomorphism

$$TX \to T^*X : v \mapsto v^*$$

where $v^* : T_xX \to \mathbb{R}$ denotes the linear functional $w \mapsto \langle v, w \rangle$ for $v \in T_xX$.

Lemma 2.26 *For $\tau \in \Omega^k(X)$, $\alpha \in \Omega^{k-1}(X)$, and a vector field $v : X \to TX$ we have*

$$\langle \iota(v)\tau, \alpha \rangle = \langle \tau, v^* \wedge \alpha \rangle.$$

Proof: The formula is equivalent to

$$\langle \tau, v_1^* \wedge \ldots \wedge v_k^* \rangle = \tau(v_1, \ldots, v_k)$$

for vector fields $v_i : X \to TX$. If the vector fields v_i are orthonormal then this is obvious from the above definition of the norm (and its independence of the choice of orthonormal frame). In general it follows by writing the v_i as linear combinations of a fixed orthonormal basis $e_1, \ldots, e_n$ and using multi-linearity on both sides of the equation. □

Lemma 2.27 *Let $e_1, \ldots, e_n$ be a local frame of TX and $f_1, \ldots, f_n$ be the dual frame so that $\langle e_i, f_j \rangle = \delta_{ij}$. Then the differential and codifferential of a k-form $\alpha \in \Omega^k(X)$ are given by*

$$d\alpha = \sum_i f_i^* \wedge \nabla_{e_i} \alpha, \qquad d^*\alpha = -\sum_i \iota(f_i) \nabla_{e_i} \alpha.$$

Proof: We prove that the right hand side in the equation for $d\alpha$ is independent of the choice of the basis. To see this take any other bases

$$e_k' = \sum_i a_k^i e_i, \qquad f_\ell' = \sum_j b_\ell^j f_j.$$

The condition that the f_ℓ' form the dual basis of e_k' can be expressed in the form

$$\sum_k a_k^i b_k^j = \delta^{ij}, \qquad \sum_i a_k^i b_\ell^i = \delta_{k\ell}.$$

These two equations are equivalent. Now

$$\sum_i f_i'^* \wedge \nabla_{e_i'} \alpha = \sum_{i,k,\ell} a_i^k b_i^\ell f_\ell^* \wedge \nabla_{e_k} \alpha = \sum_{i,k,\ell} f_k^* \wedge \nabla_{e_k} \alpha.$$

With $e_k = \partial/\partial x^k$, $f_k = dx^k$ we obtain the formula for $d\alpha$. The formula for $d^*\alpha$ follows from the identity (with L^2 inner products)

$$\begin{aligned}
\langle d^*\alpha, \beta \rangle &= \langle \alpha, d\beta \rangle \\
&= \sum_i \langle \alpha, f_i^* \wedge \nabla_{e_i} \beta \rangle \\
&= \sum_i \langle \iota(f_i)\alpha, \nabla_{e_i} \beta \rangle \\
&= -\sum_i \langle \nabla_{e_i} \iota(f_i)\alpha, \beta \rangle - \sum_i \operatorname{div}(e_i) \langle \iota(f_i)\alpha, \beta \rangle \\
&= -\sum_i \langle \iota(f_i) \nabla_{e_i} \alpha, \beta \rangle - \sum_i \langle \iota(\nabla_{e_i} f_i + \operatorname{div}(e_i) f_i)\alpha, \beta \rangle \\
&= -\sum_i \langle \iota(f_i) \nabla_{e_i} \alpha, \beta \rangle.
\end{aligned}$$

The second equation follows from our formula for $d\beta$ in terms of the local frame, the third from Lemma 2.26, the fourth from Lemma 2.22, the fifth from (2.17), and the last from Lemma 2.23. □

Exercise 2.28 Let $\alpha = \sum_I \alpha_I dx^I$ be the local coordinate representation of a k-form on X where the sum runs over all multi-indices $I = \{i_1, \dots, i_k\}$ with $i_1 < i_2 < \cdots < i_k$. Prove that the pointwise norm of α is given by

$$|\alpha|^2 = \sum_{I,J} \alpha_I g^{IJ} \alpha_J, \qquad g^{IJ} = \det\left((g^{i_\nu j_\mu})_{\nu,\mu=1}^k\right). \qquad \Box$$

Exercise 2.29 Prove that in local coordinates the covariant derivative of a 1-form $\alpha = \sum_k \alpha_k dx^k$ in the direction of a vector field $\xi = \sum \xi^i \partial/\partial x^i$ is given by

$$\nabla_\xi \alpha = \sum_k \sum_i \xi^i \left(\frac{\partial \alpha_k}{\partial x^i} - \sum_{i,j} \Gamma_{ik}^j \alpha_j \right) dx^k$$

For 2-forms $\tau = \sum_{i<j} \tau_{ij} dx^i \wedge dx^j$ the formula is

$$\nabla_\xi \tau = \sum_{k<\ell} \sum_i \xi^i \left(\frac{\partial \tau_{k\ell}}{\partial x^i} - \sum_j (\Gamma_{ik}^j \tau_{j\ell} - \Gamma_{i\ell}^j \tau_{jk}) \right) dx^k \wedge dx^\ell. \qquad \Box$$

Exercise 2.30 Let $e_1, \dots, e_n$ be a local orthonormal frame of TX. Prove that the positive definite Laplace-Beltrami operator on $C^\infty(X)$ is given by

$$\Delta_g f = d^* df = -\sum_{i=1}^n \Big(\partial_i \partial_i f + \operatorname{div}(e_i) \partial_i f \Big)$$

where $\partial_i f = df(e_i)$ denotes the derivative of the function $f : X \to \mathbb{R}$ in the direction e_i. $\Box$

Exercise 2.31. (Weitzenböck formula) Prove that in a local orthonormal frame $e_1, \dots, e_n$ the **Hodge Laplacian** on 1-forms is given by

$$d^* d\alpha + dd^* \alpha = \nabla^* \nabla \alpha + \sum_{i,j} \alpha(e_i) S(e_i, e_j) {e_j}^*$$

where $\nabla^* \nabla = \sum_i {\nabla_i}^* \nabla_i$ is the Bochner Laplacian and $S : S^2TX \to \mathbb{R}$ denotes the Ricci tensor. (**Hint:** Use the formulae of Lemma 2.27 with $e_i = f_i$. Assume without loss of generality that $\nabla_i e_j = 0$ at a given point $x_0 \in X$ for all i and j.) Deduce that

$$\langle d\beta, d\alpha \rangle_{L^2} + \langle d^*\beta, d^*\alpha \rangle_{L^2} = \langle \nabla\beta, \nabla\alpha \rangle_{L^2} + \int_X S(\alpha^*, \alpha^*) \mathrm{dvol}$$

for $\alpha, \beta \in \Omega^1(X)$. With $\alpha = \beta$ this formula shows that $H^1(X; \mathbb{R}) = 0$ for every manifold with positive Ricci tensor. $\Box$

For any endomorphism $\Phi \in C^\infty(X, \mathrm{End}(TX))$ consider the operator $\Omega^*(X) \to \Omega^*(X) : \tau \mapsto \iota(\Phi)\tau$ defined by

$$\iota(\Phi)\tau(v_1, \ldots, v_k) = \sum_{j=1}^k \tau(v_1, \ldots, v_{j-1}, \Phi v_j, v_{j+1}, \ldots, v_k) \tag{2.19}$$

for $\tau \in \Omega^k(X)$. Thus $\iota(\Phi)\alpha = \Phi^*\alpha$ for $\alpha \in \Omega^1(X)$ and for forms of higher degree one finds

$$\iota(\Phi)(\sigma \wedge \tau) = \iota(\Phi)\sigma \wedge \tau + \sigma \wedge \iota(\Phi)\tau.$$

Here we use the convention $\iota(\Phi)\sigma \wedge \tau = (\iota(\Phi)\sigma) \wedge \tau$. This formula can also be used as a definition of $\iota(\Phi)$. For any connection ∇ on the tangent bundle the covariant derivative of Φ is defined by the Leibnitz rule

$$\nabla_u(\Phi v) = (\nabla_u \Phi)v + \Phi \nabla_u v.$$

With this convention we have

$$\nabla_u(\iota(\Phi)\tau) = \iota(\nabla_u \Phi)\tau + \iota(\Phi)\nabla_u \tau.$$

The proof is left to the reader.

Exercise 2.32 If $\Phi\Psi = \Psi\Phi$ prove that

$$\iota(\Phi)\iota(\Psi)\tau = \iota(\Psi)\iota(\Phi)\tau$$

for $\tau \in \Omega^*(X)$. Moreover, prove that

$$\iota(v)\iota(\Phi)\tau - \iota(\Phi)\iota(v)\tau = \iota(\Phi v)\tau$$

for $v \in \mathrm{Vect}(X)$, $\Phi \in C^\infty(X, \mathrm{End}(TX))$, and $\tau \in \Omega^*(X)$. □

Exercise 2.33 Denote by ∇ the Levi-Civita connection on TX and by $R \in \Omega^2(X, \mathrm{End}(TX))$ the curvature tensor. Prove that

$$\nabla_u \nabla_v \tau - \nabla_u \nabla_v \tau + \nabla_{[u,v]}\tau = -\iota(R(u,v))\tau$$

for $\tau \in \Omega^*(X)$ and $u, v \in \mathrm{Vect}(X)$. □

Exercise 2.34. (Weitzenböck formula) Prove that in a local orthonormal frame $e_1, \ldots, e_n$ the Laplace-Beltrami operator on k-forms is given by

$$d^*d\tau + dd^*\tau = \nabla^*\nabla\tau + \sum_{i,j} {e_i}^* \wedge \iota(e_j)\iota(R(e_i, e_j))\tau$$

where $\nabla^*\nabla = \sum_i {\nabla_i}^*\nabla_i$ is the Bochner Laplacian and R denotes the Riemann curvature tensor (Compare with Exercise 2.31). □

Exercise 2.35 Assume that $\Phi \in \mathrm{End}(TX)$ satisfies

$$\nabla\Phi = 0, \qquad \Phi^*\Phi = 1\!\!1$$

and

$$\Phi R(u,v) = R(u,v)\Phi, \qquad R(u,v) = R(\Phi u, \Phi v).$$

Prove that

$$\Delta\iota(\Phi)\tau = \iota(\Phi)\Delta\tau$$

where $\Delta = d^*d + dd^*$ denotes the Laplace-Beltrami operator. In Chapter 3 we shall see that the complex structure on a Kähler manifold satisfies the conditions we have imposed on Φ. □

Exercise 2.36 Prove that for any 1-form $\alpha \in \Omega^1(X)$ on a manifold with boundary we have

$$\int_X \Big(|\nabla\alpha|^2 - \langle\alpha, \nabla^*\nabla\alpha\rangle\Big)\mathrm{dvol}_X = \int_{\partial X} \langle\alpha, \nabla_\nu\alpha\rangle\,\mathrm{dvol}_{\partial X}$$

where $\nabla_\nu\alpha$ denotes the covariant derivative in the direction of the outward unit normal on ∂X. **Hint:** Use Stokes' formula

$$\int_X \mathrm{div}(v)\mathrm{dvol}_X = \int_{\partial X} \langle\nu, v\rangle\mathrm{dvol}_{\partial X}$$

for $v \in \mathrm{Vect}(X)$. □

Duality

Recall from Lemma 2.26 that for every vector field $v \in \mathrm{Vect}(X)$ the operator $\Omega^k(X) \to \Omega^{k+1}(X) : \tau \mapsto v^* \wedge \tau$ is the adjoint operator of $\iota(v) : \Omega^{k+1}(X) \to \Omega^k(X)$. If we denote by $\alpha = v^* \in \Omega^1(X)$ the differential form which is dual to v and by $v = \alpha^*$ the vector field dual to α then we see that the adjoint operator of $\tau \mapsto \alpha\wedge\tau$ is given by $\tau \mapsto \iota(\alpha^*)\tau$. Since no confusion can arise we shall delete the $*$ in this formula and use the notation $\iota(\alpha)\tau$ instead. This extends naturally to forms of higher degree as follows. Define

$$\iota(\alpha)\tau = \sum_I \alpha_I\iota(e_I)\tau, \qquad \iota(e_I)\tau = \iota(e_{i_1})\cdots\iota(e_{i_k})\tau, \tag{2.20}$$

where the summation is over all multi-indices $I = \{i_1, \ldots, i_k\}$ with $i_1 < \ldots < i_k$, the vector fields $e_1, \ldots, e_n$ form a local orthonormal frame of TX and the k-form α is given by

$$\alpha = \sum_I \alpha_I {e_I}^*, \qquad {e_I}^* = {e_{i_1}}^* \wedge \ldots \wedge {e_{i_k}}^*.$$

The reason for this definition lies in the following lemma. In particular, the lemma proves that the right hand side of (2.20) is independent of the choice of the orthonormal frame.

Lemma 2.37 *For $\alpha \in \Omega^k(X)$, $\tau \in \Omega^\ell(X)$ and $\sigma \in \Omega^{\ell-k}(X)$ we have*

$$\langle \tau, \alpha \wedge \sigma \rangle = (-1)^{\frac{k(k-1)}{2}} \langle \iota(\alpha)\tau, \sigma \rangle.$$

Proof: Consider the form $\alpha = {e_I}^*$ and apply Lemma 2.26 repeatedly. The factor $(-1)^{\frac{k(k-1)}{2}}$ arises from reversing the order in the exterior product ${e_{i_1}}^* \wedge \ldots \wedge {e_{i_k}}^*$. □

3

COMPLEX GEOMETRY

This chapter is concerned with various topics revolving around almost complex, symplectic, and Kähler manifolds. Section 3.1 lays the foundations with a discussion of the canonical splitting of the complex exterior algebra of a symplectic vector space. Section 3.2 examines the corresponding splitting of the space of complex valued differential forms on an almost complex manifold. Contrary to the Kähler case, the ordinary differential d does not split into the sum $\partial + \bar{\partial}$. There are additional terms to consider which arise from the Nijenhuis tensor. Section 3.3 deals with compatible almost complex structures J on a symplectic manifold (X, ω). Here the splitting into (p, q)-forms is not invariant under the Levi-Civita connection, but there is a canonical Hermitian connection on TX which does respect the splitting. The section contains a proof of a Weitzenböck type formula and some other useful identities. Section 3.4 is devoted to the Dolbeault cohomology of a Kähler manifold and Section 3.5 to holomorphic line bundles and their relation with Hermitian Yang-Mills connections. Section 3.6 discusses the Hirzebruch-Riemann-Roch theorem. A proof is given in the case of the trivial bundle over a symplectic 4-manifold. Section 3.7 gives a brief discussion of Kähler-Einstein metrics (without proofs). Section 3.9 contains Milnor's calculation of the characteristic classes and Betti numbers of hypersurfaces in projective space.

3.1 Complex exterior algebra

Hermitian vector spaces

Let V be a $2n$-dimensional real vector space, $\omega : V \times V \to \mathbb{R}$ be a skew-symmetric bilinear form, and $J \in \mathrm{End}(V)$ be a complex structure which is **compatible with** ω. This means that the bilinear form

$$g(v, w) = \omega(v, Jw)$$

defines an inner product on V and hence

$$\langle v, w \rangle = g(v, w) + i\omega(v, w)$$

is a Hermitian form. It is complex anti-linear in the first argument and complex linear in the second. The triple (V, J, ω) is called a **Hermitian vector space**. The volume form of the inner product g is given by

$$\mathrm{dvol}_g = \frac{\omega^n}{n!}.$$

The standard example is $\mathbb{R}^{2n}$ with the Euclidean metric. In the coordinates $x_1, y_1, \ldots, x_n, y_n$ the standard symplectic and complex structures are given by

$$\omega_0 = \sum_{j=1}^n dx_j \wedge dy_j, \qquad J_0 = \begin{pmatrix} 0 & -1 & \cdots & 0 & 0 \\ 1 & 0 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 0 & -1 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

If we identify $\mathbb{R}^{2n} = \mathbb{C}^n$ via $z_j = x_j + iy_j$ then the matrix J_0 represents multiplication by $i = \sqrt{-1}$ and the symplectic form ω_0 can be written as

$$\omega_0 = \frac{1}{2i} \sum_{j=1}^n d\bar{z}_j \wedge dz_j$$

where $dz_j = dx_j + idy_j$ and $d\bar{z}_j = dx_j - idy_j$. The resulting Hermitian form on $\mathbb{C}^n$ is given by

$$\langle z, \zeta \rangle = \sum_{\nu=1}^n \bar{z}_\nu \zeta_\nu.$$

Any Hermitian vector space (V, J, ω) admits a **unitary basis**, i.e. an orthonormal basis of the form $v_1, Jv_1, \ldots, v_n, Jv_n$. Any such basis induces a vector space isomorphism $\Psi : \mathbb{R}^{2n} \to V$ which identifies ω, J, and g with the standard structures on $\mathbb{R}^{2n}$.

Example 3.1 Consider the Euclidean space $\mathbb{R}^2$ with coordinates (x, y) and its standard orientation. An inner product on $\mathbb{R}^2$ is given by a symmetric matrix

$$g = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

with $E > 0$ and $EG - F^2 > 0$. The corresponding area form

$$\omega = \sqrt{EG - F^2}\, dx \wedge dy$$

is always compatible with g. The corresponding complex structure is given by the matrix

$$J = \frac{1}{\sqrt{EG - F^2}} \begin{pmatrix} -F & -G \\ E & F \end{pmatrix}.$$

Thus every inner product on an oriented 2-dimensional real vector space determines a unique complex structure and Hermitian form on this vector space such that the real part of the Hermitian form is the given inner product. □

Endomorphisms

Let (V, J, ω) be a Hermitian vector space of complex dimension n. Denote by $\mathrm{End}(V, J)$ the space of complex linear endomorphisms and by $\mathrm{End}_0(V, J)$ the space of complex linear traceless endomorphisms. It is standard convention to define the metric on $\mathrm{End}(V, J)$ as half the complex trace, i.e.

$$\langle T_1, T_2 \rangle = \frac{1}{2}\mathrm{trace}^c({T_1}' T_2).$$

for $T_1, T_2 \in \mathrm{End}(V, J)$. Here $T' = T^* \in \mathrm{End}(V, J)$ denotes the adjoint operator defined by $\langle T'w, v\rangle = \langle w, Tv\rangle$. The corresponding norm on $\mathrm{End}(V, J)$ is given by

$$|T| = \sqrt{\frac{1}{2}\mathrm{trace}(T'T)}.$$

For $v, w \in V$ let $vw' \in \mathrm{End}(V, J)$ be given by

$$vw'\theta = v\langle w, \theta\rangle$$

for $\theta \in V$. The traceless part of vw' is given by

$$(vw')_0 = vw' - \frac{1}{n}\langle w, v\rangle \mathbb{1}.$$

Lemma 3.2 *Let (V, J, ω) be a Hermitian vector space. Then*

$$|(vw')_0|^2 = \frac{1}{2}|v|^2 |w|^2 - \frac{1}{2n}|\langle v, w\rangle|^2$$

for $v, w \in V$, and

$$\langle T, (vw')_0\rangle = \frac{1}{2}\langle Tw, v\rangle$$

for $v, w \in V$ and $T \in \mathrm{End}_0(V, J)$.

Proof: To prove the first assertion compute

$$(wv')_0(vw')_0 = wv'vw' - \frac{1}{n}ww'vv' - \frac{1}{n}vv'ww' + \frac{1}{n^2}|\langle v, w\rangle|^2 \mathbb{1}$$

and take half the trace. To prove the second assertion note that T has zero trace and so $\langle T, \mathbb{1}\rangle = 0$. Hence

$$\langle T, (vw')_0\rangle = \frac{1}{2}\mathrm{trace}(T'vw') = \frac{1}{2}\mathrm{trace}(w'T'v) = \frac{1}{2}\langle Tw, v\rangle.$$

This proves the lemma. □

Dual space

We denote by

$$V^* = \mathrm{Hom}(V, \mathbb{R})$$

the dual space of V as a real vector space and by

$$V \to V^* : v \mapsto v^* = g(v, \cdot)$$

the natural isomorphism induced by the inner product g. The space V^* carries a natural complex structure J^*. It is important to note that, by the compatibility condition, we have

$$J^* v^* = -(Jv)^*$$

and hence the isomorphism $v \mapsto v^*$ is complex anti-linear. At some places we shall not mention the complex structure explicitly and denote by $\bar{V}$ the real vector space V with the reversed complex structure $-J$. Thus we denote by $\mathrm{Hom}(V, \mathbb{C})$ the space of complex linear and by $\mathrm{Hom}(\bar{V}, \mathbb{C})$ the space of complex anti-linear functionals $V \to \mathbb{C}$.

The space $V^* \otimes \mathbb{C}$ is the space of complex valued real linear functionals on V and decomposes as

$$V^* \otimes \mathbb{C} = \mathrm{Hom}(V, \mathbb{C}) \oplus \mathrm{Hom}(\bar{V}, \mathbb{C})$$

into the subspaces of complex linear and complex anti-linear functionals. There are natural isomorphisms

$$\bar{V} \to \mathrm{Hom}(V, \mathbb{C}) : v \mapsto v', \qquad V \to \mathrm{Hom}(\bar{V}, \mathbb{C}) : v \mapsto v''$$

given by

$$v' = v^* + i(Jv)^* = \langle v, \cdot \rangle, \qquad v'' = v^* - i(Jv)^* = \langle \cdot, v \rangle. \tag{3.1}$$

Note that

$$(Jv)' = -iv', \qquad (Jv)'' = iv''.$$

The Hermitian structure on $\Lambda^* V^* \otimes \mathbb{C}$ is induced by the standard inner product on $\Lambda^* V^*$ and the complex structure. Thus we define

$$\langle \sigma, \tau \rangle = \sum_{i_1 < \cdots < i_k} \overline{\sigma(e_{i_1}, \ldots, e_{i_k})} \tau(e_{i_1}, \ldots, e_{i_k})$$

for $\sigma, \tau \in \Lambda^k V^* \otimes \mathbb{C}$ and an orthonormal basis $e_1, \ldots, e_{2n}$ of V. (See Section 2.4 and, in particular, Exercise 2.28.)

Remark 3.3 In our standard model $\mathbb{R}^{2n} \cong \mathbb{C}^n$ the isomorphisms $\bar{\mathbb{C}}^n \to \Lambda^{1,0}\mathbb{C}^{n*} : v \mapsto v'$ and $\mathbb{C}^n \to \Lambda^{0,1}\mathbb{C}^{n*} : v \mapsto v''$ are given by

$$v' = \sum_{\nu=1}^{n} \bar{v}_\nu dz_\nu, \qquad v'' = \sum_{\nu=1}^{n} v_\nu d\bar{z}_\nu$$

for $v = (v_1, \dots, v_n) \in \mathbb{C}^n$. Moreover the forms

$$2^{-k/2} dz_I \wedge d\bar{z}_J = 2^{-k/2} dz_{i_1} \wedge \dots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q}$$

with $p + q = k$ form a unitary basis of $\Lambda^k V^* \otimes \mathbb{C}$. □

Exterior algebra

The exterior algebra $\Lambda^* V^* \otimes \mathbb{C}$ of complex valued real multi-linear forms on V decomposes as a direct sum

$$\Lambda^* V^* \otimes \mathbb{C} = \bigoplus_{p,q} \Lambda^{p,q} V^*. \tag{3.2}$$

The space $\Lambda^{p,q} V^*$ is generated by elements of the form $\sigma \wedge \tau$ where $\sigma \in \Lambda^{p,0} V^*$ is complex linear in each argument and $\tau \in \Lambda^{0,q} V^*$ is complex anti-linear in each argument. Given $\tau \in \Lambda^k V^* \otimes \mathbb{C}$ denote by $\tau^{p,q} \in \Lambda^{p,q} V^*$ the projection of τ onto $\Lambda^{p,q} V^*$. In particular we have $v^{*1,0} = \frac{1}{2} v'$ and $v^{*0,1} = \frac{1}{2} v''$.

Lemma 3.4 *For $\tau \in \Lambda^{0,k} V^*$, $\sigma \in \Lambda^{0,k-1} V^*$, and $v \in V$ we have*

$$\langle \tau, v'' \wedge \sigma \rangle = 2 \langle \iota(v)\tau, \sigma \rangle.$$

Proof: The fact that τ is complex anti-linear in all variables can be expressed in the form $\iota(Jv)\tau = -i\iota(v)\tau$. Hence for $\tau \in \Lambda^{0,k} V^*$ and $\sigma \in \Lambda^{0,k-1} V^*$ we obtain

$$\begin{aligned} \langle \tau, v'' \wedge \sigma \rangle &= \langle \tau, v^* \wedge \sigma \rangle - \langle \tau, i(Jv)^* \wedge \sigma \rangle \\ &= \langle \tau, v^* \wedge \sigma \rangle - i \langle \tau, (Jv)^* \wedge \sigma \rangle \\ &= \langle \iota(v)\tau, \sigma \rangle - i \langle \iota(Jv)\tau, \sigma \rangle \\ &= \langle \iota(v)\tau, \sigma \rangle + i \langle i\iota(v)\tau, \sigma \rangle \\ &= 2 \langle \iota(v)\tau, \sigma \rangle. \end{aligned}$$

Here we have used the formula $\langle \tau, v^* \wedge \sigma \rangle = \langle \iota(v)\tau, \sigma \rangle$ of Lemma 2.26 with the real inner product replaced by the Hermitian form. The reader may check that this is permitted. (For real valued forms the imaginary part of the Hermitian inner product is zero.) □

As in the case of the real exterior algebra we generalize the formula of Lemma 3.4 to forms of higher degree. Choose an orthonormal basis of TX of the form $e_1, Je_1, \ldots, e_n, Je_n$. Then every form $\alpha \in \Lambda^{0,k}$ which is complex anti-linear in all variables can be expressed as

$$\alpha = \sum_I \alpha_I {e_I}'', \qquad {e_I}'' = {e_{i_1}}'' \wedge \ldots \wedge {e_{i_k}}'',$$

where the sum runs over all multi-indices $I = \{i_1, \ldots, i_k\}$ with $i_1 < \cdots < i_k$. Define the linear map

$$\iota(\bar\alpha) : \Lambda^{0,\ell} V^* \to \Lambda^{0,\ell-k} V^*$$

by

$$\iota(\bar\alpha)\tau = \sum_I \bar\alpha_I \iota(e_I)\tau \tag{3.3}$$

for $\tau \in \Lambda^{0,\ell} V^*$ where $\iota(e_I)\tau = \iota(e_{i_1}) \cdots \iota(e_{i_k})\tau$. The reader may check that the right hand side of this equation is independent of the choice of the unitary basis $e_1, \ldots, e_n$ used to define it. In fact, the formula (3.3) continues to hold for any complex basis. Note that the map $(\alpha, \tau) \mapsto \iota(\bar\alpha)\tau$ is complex linear in τ and complex anti-linear in α.

Lemma 3.5 *For $\alpha \in \Lambda^{0,k} V^*$, $\tau \in \Lambda^{0,\ell} V^*$ and $\sigma \in \Lambda^{0,\ell-k} V^*$ we have*

$$\langle \tau, \alpha \wedge \sigma \rangle = 2^k (-1)^{\frac{k(k-1)}{2}} \langle \iota(\bar\alpha)\tau, \sigma \rangle.$$

Proof: Consider the form $\alpha = {e_I}^*$ and apply Lemma 3.4 repeatedly. □

Remark 3.6 The components of a 1-form σ are given by

$$\sigma^{1,0}(v) = \frac{1}{2} \left(\sigma(v) - i\sigma(Jv) \right),$$

$$\sigma^{0,1}(v) = \frac{1}{2} \left(\sigma(v) + i\sigma(Jv) \right).$$

If $V = \mathbb{C}^n$ and $\sigma \in V^* \otimes \mathbb{C}$ is given by

$$\sigma = \sum_{j=1}^n A_j dx_j + \sum_{j=1}^n B_j dy_j$$

then $\sigma^{1,0}, \sigma^{0,1} \in V^* \otimes \mathbb{C}$ are given by

$$\sigma^{1,0} = \sum_{j=1}^n \frac{1}{2}(A_j - iB_j) dz_j, \qquad \sigma^{0,1} = \sum_{j=1}^n \frac{1}{2}(A_j + iB_j) d\bar z_j. \qquad \square$$

Remark 3.7 The components of a 2-form τ are given by

$$\begin{aligned}
\tau^{2,0}(v,w) &= \frac{1}{4}\left(\tau(v,w) - \tau(Jv,Jw) - i\tau(Jv,w) - i\tau(v,Jw)\right), \\
\tau^{1,1}(v,w) &= \frac{1}{2}\left(\tau(v,w) + \tau(Jv,Jw)\right), \\
\tau^{0,2}(v,w) &= \frac{1}{4}\left(\tau(v,w) - \tau(Jv,Jw) + i\tau(Jv,w) + i\tau(v,Jw)\right).
\end{aligned}$$

If $V = \mathbb{C}^n$ and $F \in \Lambda^2 V^* \otimes \mathbb{C}$ is given by

$$F = \sum_{j<k} A_{jk} dx_j \wedge dx_k + \sum_{j,k} B_{jk} dx_j \wedge dy_k + \sum_{j<k} C_{jk} dy_j \wedge dy_k$$

with $A_{jk} = -A_{kj}$ and $C_{jk} = -C_{kj}$ for $j \geq k$ then

$$F^{2,0} = \frac{1}{4}\sum_{j<k}\Big(A_{jk} - C_{jk} - i(B_{jk} - B_{kj})\Big) dz_j \wedge dz_k,$$

$$F^{1,1} = \frac{1}{4}\sum_{j,k}\Big(A_{jk} + C_{jk} - i(B_{jk} + B_{kj})\Big) d\bar{z}_j \wedge dz_k,$$

$$F^{0,2} = \frac{1}{4}\sum_{j<k}\Big(A_{jk} - C_{jk} + i(B_{jk} - B_{kj})\Big) d\bar{z}_j \wedge d\bar{z}_k.$$

In particular, $F^{0,2} = 0$ if and only if $A_{jk} = C_{jk}$ and $B_{jk} = B_{kj}$. □

3.2 Cauchy-Riemann operators and the Nijenhuis tensor

Complex valued differential forms

Let X be a smooth manifold of dimension $2n$. An **almost complex structure** on X is an automorphism $J : TX \to TX$ of the tangent bundle which satisfies $J^2 = -\mathbb{1}$. Any such almost complex structure gives rise to a splitting of the space $\Omega^k(X, \mathbb{C})$ of complex valued k-forms as a direct sum

$$\Omega^k(X, \mathbb{C}) = \bigoplus_{p+q=k} \Omega^{p,q}(X), \tag{3.4}$$

where $\Omega^{p,q}(X)$ denotes the space of (p,q)-forms on X, i.e. of sections of the bundle $\Lambda^{p,q}T^*X$. There are natural operators $\bar{\partial} : \Omega^{p,q} \to \Omega^{p,q+1}$ and $\partial : \Omega^{p,q} \to \Omega^{p+1,q}$ defined by the operator d followed by the projection onto the relevant subspace in the decomposition (3.4). Thus

$$\partial\tau = (d\tau)^{p+1,q}, \qquad \bar{\partial}\tau = (d\tau)^{p,q+1}$$

for $\tau \in \Omega^{p,q}(X)$.

Exercise 3.8 Prove that

$$\bar\partial(\sigma \wedge \tau) = (\bar\partial\sigma) \wedge \tau + (-1)^{\deg(\sigma)} \sigma \wedge \bar\partial\tau$$

for $\sigma, \tau \in \Omega^*(X)$. □

The Nijenhuis tensor

An almost complex structure J is called **integrable** if it arises from complex coordinate charts on X with holomorphic transition maps. In this case it follows from the local coordinate representation of the operators $\bar\partial$ and ∂ that $d = \partial + \bar\partial$. In the nonintegrable case the difference between these operators gives rise to the **Nijenhuis tensor**

$$N_J : TX \otimes TX \to TX$$

which is defined by

$$N_J(v, w) = [v, w] + J[Jv, w] + J[v, Jw] - [Jv, Jw] \tag{3.5}$$

for $v, w \in \mathrm{Vect}(X)$.

Exercise 3.9 Prove that (3.5) is a tensor. This means that the value of $N_J(v, w)$ at $x \in X$ depens only on $v(x)$ and $w(x)$ but not on the derivatives. **Hint:** Show that N_J is bilinear over the functions, i.e.

$$N_J(fv, w) = f N_J(v, w)$$

for $v, w \in \mathrm{Vect}(X)$ and $f \in C^\infty(X)$. □

Exercise 3.10 Prove that N_J is skew-symmetric and complex anti-linear in both variables, i.e.

$$N(v, w) + N(w, v) = 0, \qquad N(Jv, w) = N(v, Jw) = -JN(v, w).$$

for $v, w \in \mathrm{Vect}(X)$. □

Exercise 3.11 Prove that

$$\alpha \in \Omega^{1,0}(X) \qquad \Longrightarrow \qquad (d\alpha)^{0,2} = \frac{1}{4} \alpha \circ N_J$$

and similarly for $\alpha \in \Omega^{0,1}(X)$. **Hint:** Use the formula

$$d\alpha(v, w) = \mathcal{L}_v(\alpha(w)) - \mathcal{L}_w(\alpha(v)) + \alpha([v, w]).$$

and Remark 3.7. □

Exercise 3.12 Define the operator $\iota(N_J) : \Omega^k(X, \mathbb{C}) \to \Omega^{k+1}(X, \mathbb{C})$ by $\iota(N_J)\alpha = \alpha \circ N_J$ for $\alpha \in \Omega^1$ and by

$$\iota(N_J)(\sigma \wedge \tau) = (\iota(N_J)\sigma) \wedge \tau + (-1)^{\deg(\sigma)} \sigma \wedge \iota(N_J)\tau$$

in general. Prove that

$$d\tau - \partial\tau - \bar{\partial}\tau = \frac{1}{4}\iota(N_J)\tau \in \Omega^{p+2,q-1}(X) \oplus \Omega^{p-1,q+2}(X)$$

for $\tau \in \Omega^{p,q}(X)$. **Hint:** Use Exercises 3.8 and 3.11. □

Exercise 3.13 Prove that

$$\bar{\partial}\bar{\partial}f = -\frac{1}{4}(\partial f) \circ N_J, \qquad \partial\partial f = -\frac{1}{4}(\bar{\partial} f) \circ N_J.$$

Hint: Use Exercise 3.11 with $\alpha = \partial f$ and $\alpha = \bar{\partial} f$. □

The Nijenhuis tensor can be viewed as an obstruction to integrability. It follows easily from Exercise 3.11 or by direct calculation that $N_J = 0$ whenever J is integrable. The Newlander-Nirenberg theorem asserts that the converse is true as well. Its proof goes beyond the scope of this book.

Theorem 3.14. (Newlander-Nirenberg) *An almost complex structure is integrable if and only if the Nijenhuis tensor vanishes.*

In the integrable case Exercise 3.12 asserts that the d operator splits as $d = \partial + \bar{\partial}$. This gives rise to the Dolbeault cohomology groups. These will be discussed in Section 3.4.

Cauchy-Riemann operators on vector bundles

The decomposition (13.8) extends to complex vector bundles $E \to X$ over almost complex manifolds (X, J):

$$\Omega^k(X, E) = \bigoplus_{p+q=k} \Omega^{p,q}(X, E).$$

Here $\Omega^{p,q}(X, E) = C^\infty(X, \Lambda^{p,q}T^*X \otimes E)$ denotes the space of (p, q)-forms on X with values in E. A **Cauchy-Riemann operator** on E is an operator $D'' : C^\infty(X, E) \to \Omega^{0,1}(X, E)$ such that

$$D''(fs) = \bar{\partial} f \otimes s + fD''s$$

for $s \in C^\infty(X, E)$ and $f \in C^\infty(X, \mathbb{C})$. Note that any such operator extends naturally to an operator $D'' : \Omega^{p,q}(X, E) \to \Omega^{p,q+1}(X, E)$ via

$$D''(\tau \otimes s) = \bar{\partial}\tau \otimes s + (-1)^{\deg(\tau)}\tau \wedge D''s$$

for $s \in C^\infty(X, E)$ and $\tau \in \Omega^{p,q}(X, E)$.

Relation with Hermitian connections

Choose a Hermitian structure on E and let $P \to X$ denote the principal unitary frame bundle associated to E. Then any Hermitian connection $B \in \mathcal{A}(P)$ determines a covariant derivative operator

$$d_B : C^\infty(X, E) \to \Omega^1(X, E)$$

and its complex linear and complex anti-linear parts

$$\partial_B : C^\infty(X, E) \to \Omega^{1,0}(X, E), \qquad \bar{\partial}_B : C^\infty(X, E) \to \Omega^{0,1}(X, E)$$

are given by

$$\partial_B s = \frac{1}{2}(d_B s + i d_B s \circ J), \qquad \bar{\partial}_B s = \frac{1}{2}(d_B s - i d_B s \circ J)$$

for $s \in C^\infty(X, E)$. Note that $\bar{\partial}_B$ is a Cauchy-Riemann operator. The induced Cauchy-Riemann operator $\bar{\partial}_B : \Omega^{p,q}(X, E) \to \Omega^{p,q+1}(X, E)$ is also given by the covariant derivative followed by the projection onto the $(p, q+1)$-part. Similarly for $\partial_B : \Omega^{p,q} \to \Omega^{p+1,q}$.

Proposition 3.15 *For every Cauchy-Riemann operator D'' on E there exists a unique Hermitian connection $B \in \mathcal{A}(P)$ such that $D'' = \bar{\partial}_B$.*

Proof: A general Cauchy-Riemann operator can in local coordinates be represented in the form

$$(D''s)^\alpha = \bar{\partial} s^\alpha + C^\alpha s^\alpha$$

where $C^\alpha \in \Omega^{0,1}(\alpha(U_\alpha), \mathbb{C}^{m\times m})$ with

$$(\beta \circ \alpha^{-1})C^\beta = \Phi_{\beta\alpha} C^\alpha \Phi_{\beta\alpha}^{-1} - (\bar{\partial}\Phi_{\beta\alpha})\Phi_{\beta\alpha}^{-1}.$$

The 1-forms $B^\alpha = C^\alpha - (C^\alpha)^*$ take values in the skew-Hermitian matrices and satisfy (1.2). Hence they are the connection potential of a Hermitian connection $B \in \mathcal{A}(E)$. If $J^\alpha : \alpha(U_\alpha) \to \mathbb{R}^{2n\times 2n}$ represents the almost complex structure on X then the associated $\bar{\partial}$-operator is given by

$$(\bar{\partial}_B s)^\alpha = \bar{\partial} s^\alpha + \frac{1}{2}(B^\alpha + iB^\alpha \circ J^\alpha)s^\alpha = \bar{\partial} s^\alpha + C^\alpha s^\alpha = (D''s)^\alpha.$$

This prove existence. To prove uniqueness assume that $b \in \Omega^1(X, \mathfrak{u}(E))$ is the difference of two Hermitian connections which induce the same Cauchy-Riemann operator. Then $b^{0,1} = 0$. But since b is skew-Hermitian and $b^{0,1}(v) = (b(v) + ib(Jv))/2$ for $v \in TX$ it follows that $b = 0$ (every complex matrix decomposes uniquely into a Hermitian and a skew-Hermitian matrix). This proves the proposition. □

Proposition 3.16 *For every Hermitian connection $B \in \mathcal{A}(E)$ the associated Cauchy-Riemann operator satisfies*

$$\bar{\partial}_B \bar{\partial}_B s = F_B^{0,2} s - \frac{1}{4}(\partial_B s) \circ N_J$$

for $s \in C^\infty(X, E)$.

Proof: By Exercise 3.11, $(d_B \eta)^{0,2} = \eta \circ N_J/4$ for $\eta \in \Omega^{1,0}(X)$. Hence

$$\bar{\partial}_B \bar{\partial}_B s = (d_B \bar{\partial}_B s)^{0,2} = (d_B d_B s)^{0,2} - (d_B \partial_B s)^{0,2} = F_B^{0,2} s - \frac{1}{4}(\partial_B s) \circ N_J$$

as claimed. □

If (X, J) is a complex manifold then $N_J = 0$ and so we have

$$\bar{\partial}_B \circ \bar{\partial}_B = F_B^{0,2}.$$

It is a deep theorem in complex geometry that the $(0, 2)$-part of the curvature vanishes precisely when locally near every point the bundle E has a basis of holomorphic sections. (These are sections in the kernel of $\bar{\partial}_B$.) This implies that there is a system of coordinate charts and local trivializations of E with holomorphic transition matrices, i.e. E is a **holomorphic vector bundle**. Conversely, whenever E is a holomorphic vector bundle with a Hermitian structure there is a canonical Cauchy-Riemann operator on E given by the ordinary $\bar{\partial}$-operator in any holomorphic trivialization. This operator obviously satisfies $\bar{\partial} \circ \bar{\partial} = 0$. In view of Proposition 3.15 this Cauchy-Riemann operator corresponds to a Hermitian connection B which is called the **Chern connection** of E and satisfies $F_B^{0,2} = 0$. These findings are summarized in the following classical theorem of complex geometry which we shall not prove here.

Theorem 3.17. (Newlander-Nirenberg) *Let (X, J) be a complex manifold and $E \to X$ be a Hermitian vector bundle with a connection B. Then the Cauchy-Riemann operator $\bar{\partial}_B : \Omega^0(X, E) \to \Omega^{0,1}(X, E)$ determines a holomorphic structure on E if and only if $F_B^{0,2} = 0$.*

3.3 Almost complex structures on symplectic manifolds

Let (X, ω) be a **symplectic manifold**. This means that $\omega \in \Omega^2(X)$ is a closed nondegenerate 2-form. An almost complex structure J on X is called **compatible with** ω if the expression

$$g(v, w) = \omega(v, Jw) \tag{3.6}$$

defines a Riemannian metric on X. Such almost complex structures always exist and they form a contractible space denoted by $\mathcal{J}(X, \omega)$ (cf. [85]). Fix an almost complex structure $J \in \mathcal{J}(X, \omega)$ and denote the corresponding Hermitian structure on TX by

$$\langle v, w \rangle = g(v, w) + i\omega(v, w).$$

This structure is complex anti-linear in the first argument and complex linear in the second. The goal of this section is to show how to construct a natural Hermitian connection on the tangent bundle TX. This connection will preserve the canonical splitting of the space of differential forms. The induced connection on the bundle of forms of type (p, q) can in fact be defined as the Levi-Civita connection followed by the L^2-orthogonal projection onto $\Omega^{p,q}$. We begin with a brief discussion of the Nijenhuis tensor in the symplectic case.

The Nijenhuis tensor on symplectic manifolds

The next lemma summarizes some basic facts about the almost complex structure in symplectic manifolds. The first two assertions also hold when ω is not closed. The third assertion shows that J is integrable if and only if $\nabla J = 0$ where ∇ denotes the Levi-Civita connection of the metric induced by J. The last assertion will play a crucial role in relating the Dirac operator on a symplectic manifold to the Cauchy-Riemann operator.

Lemma 3.18 *Let (X, ω) be a symplectic manifold and $J \in \mathcal{J}(X, \omega)$. denote by ∇ the Levi-Civita connection of the metric (3.6) and by $N = N_J$ the Nijenhuis tensor of J. Then the following holds for all $u, v, w \in \mathrm{Vect}(X)$.*

(i) $(\nabla_v J)J + J(\nabla_v J) = 0$ *and* $g((\nabla_u J)v, w) + g(v, (\nabla_u J)w) = 0$.

(ii) $g((\nabla_u J)v, w) + g((\nabla_v J)w, u) + g((\nabla_w J)u, v) = 0$.

(iii) $g(u, N(v, w)) = 2g(J(\nabla_u J)v, w)$.

(iv) $J(\nabla_{Ju} J) = \nabla_u J$.

(v) $\langle u, N(v, w) \rangle + \langle v, N(w, u) \rangle + \langle w, N(u, v) \rangle = 0$.

Proof: To prove the first assertion differentiate the identities

$$g(v, Jw) + g(Jv, w) = 0, \qquad J^2 = -\mathbb{1}.$$

Statement (ii) is based on the formula

$$\begin{aligned} d\omega(u, v, w) &= \nabla_u(\omega(v, w)) + \nabla_v(\omega(w, u)) + \nabla_w(\omega(u, v)) \\ &\quad + \omega([u, v], w) + \omega([v, w], u) + \omega([w, u], v). \end{aligned}$$

Choose vector fields such that all six covariant derivatives $\nabla_u v$ etc vanish at a given point $x \in X$. Then the Lie brackets all vanish at x and with $\omega(v, w) = g(Jv, w)$ the first three terms on the right give the expression in (ii). The left hand side vanishes because ω is closed. The formula for the

Nijenhuis tensor in (iii) follows by direct calculation using $[u, v] = \nabla_v u - \nabla_u v$ and (ii). Now we have

$$\begin{aligned} 2g(J(\nabla_{Ju}J)v, w) &= g(Ju, N(v, w)) \\ &= -g(u, JN(v, w)) \\ &= g(u, N(v, Jw)) \\ &= 2g(J(\nabla_u J)v, Jw) \\ &= 2g((\nabla_u J)v, w). \end{aligned}$$

The first and fourth equalities follow from (iii) and the third follows from Exercise 3.10. This proves (iv). To prove (v) note that

$$\begin{aligned} &g(u, N(v, w)) + g(v, N(w, u)) + g(w, N(u, v)) \\ &\quad = 2g(J(\nabla_u J)v, w) + 2g(J(\nabla_v J)w, u) + 2g(J(\nabla_w J)u, v) \\ &\quad = -2g((\nabla_{Ju}J)v, w) - 2g((\nabla_v J)w, Ju) - 2g((\nabla_w J)Ju, v) \\ &\quad = 0. \end{aligned}$$

The first equation follows from (iii), the second from (iv) and (i), and the last from (ii). This shows that the real part of the left hand side in (v) is zero. But, by Exercise 3.10, the left hand side is a $(0, 3)$-form on X and hence the imaginary part vanishes as well. This proves the lemma. □

The previous lemma shows that the Nijenhuis tensor on a symplectic manifold can be interpreted as a complex anti-linear map

$$\mathrm{Vect}(X) \to \Omega^{0,2}(X) : u \mapsto \Theta_u$$

which assigns to every vector field $u \in \mathrm{Vect}(X)$ the 2-form

$$\Theta_u(v, w) = \langle u, N(v, w)\rangle = 2\langle w, J(\nabla_u J)v\rangle. \tag{3.7}$$

Here we have used the Hermitian form rather than the real inner product. The second equality requires the assertions (iv) and (v) of Lemma 3.18. It follows from (i) in Lemma 3.18 that the form Θ_u is complex anti-linear in both variables.

Exercise 3.19 Prove that in a local unitary frame $e_1, \ldots, e_n$

$$\Theta_u = \frac{1}{2}\sum_{i,j}\langle u, N(e_i, e_j)\rangle {e_i}'' \wedge {e_j}'', \qquad \sum_i {e_i}'' \wedge \Theta_{e_i} = 0.$$

The last equation is equivalent to assertion (vi) in Lemma 3.18. □

The Hermitian connection

The Levi-Civita connection will not in general preserve the spaces $\Omega^{p,q}(X)$. There is a canonical connection which has this property. It is given by

$$\widetilde{\nabla}_v w = \nabla_v w - \frac{1}{2} J(\nabla_v J) w \tag{3.8}$$

for $v, w \in \mathrm{Vect}(X)$. The induced connection on Ω^k is given by

$$\widetilde{\nabla}_v \tau = \nabla_v \tau + \frac{1}{2} \iota(J\nabla_v J)\tau$$

for $\tau \in \Omega^k(X)$, where $\iota(J\nabla_v J)\tau$ is defined by (2.19). For $\tau \in \Omega^{p,q}(X)$ this formula can be interpreted as the Levi-Civita connection followed by projection onto $\Omega^{p,q}$.

Lemma 3.20 **(i)** *For* $u, v, w \in \mathrm{Vect}(X)$

$$\widetilde{\nabla}_v(Jw) = J\widetilde{\nabla}_v w, \qquad \mathcal{L}_u(g(v,w)) = g(\widetilde{\nabla}_u v, w) + g(v, \widetilde{\nabla}_u w).$$

(ii) *For* $\alpha, \beta \in \Omega^*(X)$ *and* $u, v \in \mathrm{Vect}(X)$

$$\widetilde{\nabla}_u(\alpha \wedge \beta) = (\widetilde{\nabla}_u \alpha) \wedge \beta + \alpha \wedge (\widetilde{\nabla}_u \beta)$$

$$\widetilde{\nabla}_u(\iota(v)\alpha) = \iota(\widetilde{\nabla}_u v)\alpha + \iota(v)\widetilde{\nabla}_u \alpha.$$

(iii) *If* $\tau \in \Omega^{p,q}(X)$ *then* $\widetilde{\nabla}_v \tau \in \Omega^{p,q}(X)$ *for all* $v \in \mathrm{Vect}(X)$.

Proof: Statement (i) is proved by direct calculation. The formula

$$\widetilde{\nabla}_v(Jw) - J\widetilde{\nabla}_v w = (\nabla_v J)w - \frac{1}{2} J(\nabla_v J)Jw + \frac{1}{2} J^2(\nabla_v J)w = 0,$$

shows that $\widetilde{\nabla} J = 0$ and the second identity follows from the fact that $J(\nabla_v J)$ is a skew symmetric endomorphism of TX. Statement (ii) follows easily by induction. To prove (iii) let $\alpha \in \Omega^{0,1}$. Then, since $\widetilde{\nabla} J = 0$,

$$\begin{aligned}
(\widetilde{\nabla}_v \alpha)(Jw) &= \mathcal{L}_v(\alpha(Jw)) - \alpha(\widetilde{\nabla}_v(Jw)) \\
&= -i\mathcal{L}_v(\alpha(w)) - \alpha(J\widetilde{\nabla}_v w) \\
&= -i\mathcal{L}_v(\alpha(w)) + i\alpha(\widetilde{\nabla}_v J) \\
&= -i(\widetilde{\nabla}_v \alpha)(w)
\end{aligned}$$

and hence $\widetilde{\nabla}_v \alpha \in \Omega^{0,1}(X)$. For $\alpha \in \Omega^{1,0}(X)$ the argument is similar. For general p and q take exterior products and use (ii) to prove that if $\tau \in \Omega^{p,q}$ then $\widetilde{\nabla}_v \tau \in \Omega^{p,q}$. This proves the lemma. $\square$

The curvature of the Hermitian connection

We examine the curvature tensor $\widetilde{R} \in \Omega^2(X, \mathrm{End}(TX))$ of $\widetilde{\nabla}$. As in the general case this 2-form is defined by

$$\widetilde{R}(u,v) = \widetilde{\nabla}_u \widetilde{\nabla}_v w - \widetilde{\nabla}_v \widetilde{\nabla}_u w + \widetilde{\nabla}_{[u,v]} w$$

for $u, v, w \in \mathrm{Vect}(TX)$. It takes values in the space of skew-Hermitian endomorphisms on TX and hence has a complex trace. As before let R denote the curvature tensor of the Levi-Civita connection ∇.

Lemma 3.21 *The curvature tensor $\widetilde{R}$ of $\widetilde{\nabla}$ is given by*

$$\widetilde{R}(u,v) = \frac{1}{2}R(u,v) - \frac{1}{2}JR(u,v)J - \frac{1}{4}[\nabla_u J, \nabla_v J].$$

For all $u, v \in T_xX$ this is a complex linear skew Hermitian endomorphism of T_xX. Its complex trace is given by

$$\rho_{\omega,J} = i\,\mathrm{trace}^c\left(\widetilde{R}(u,v)\right) = \frac{1}{2}\mathrm{trace}(JR(u,v)) - \frac{1}{8}\mathrm{trace}(J[\nabla_u J, \nabla_v J]).$$

The second term on the right is a form of type $(1,1)$. The form $\rho_{\omega,J}$ is closed and $c_1(TX, J) = (2\pi)^{-1}[\rho_{\omega,J}]$.

Proof: The formula for $\widetilde{R}$ follows by a direct computation which is left to the reader. The formula for the trace follows from the general assertion that for every complex linear tranformation $T : V \to V$ of a complex vector space (V, J) the complex trace is related to the real trace by

$$\mathrm{trace}^c(T) = \frac{1}{2}\mathrm{trace}(T) - \frac{i}{2}\mathrm{trace}(JT).$$

That the form $(u,v) \mapsto \mathrm{trace}(J[\nabla_u J, \nabla_v J])$ is of type $(1,1)$ can, by Remark 3.7, be expressed in the form $[\nabla_{Ju} J, \nabla_{Jv} J] = [\nabla_u J, \nabla_v J]$. This follows from the identity $\nabla_{Ju} J = -J\nabla_u J$ in Lemma 3.18 (v). The assertion about the Chern class is a general fact about Hermitian connections. □

Remark 3.22 Assume that $E \to X$ is a complex line bundle with connection B and denote by $\widetilde{\nabla}_B$ the connection on $\Lambda^{0,*}T^*X \otimes E$ induced by the Hermitian $\widetilde{\nabla}$ and B as in Exercise 3.24. Recall from By Lemma 3.21, the curvature of the connection induced by $\widetilde{\nabla}$ on the anti-canonical bundle $K^* = \Lambda^{0,n}T^*X$ is the scalar 2-form

$$\widetilde{F} = \mathrm{trace}^c(\widetilde{R}) = -\frac{i}{2}\mathrm{trace}(JR) + \frac{i}{8}\mathrm{trace}(J[\nabla_\cdot J, \nabla_\cdot J]). \tag{3.9}$$

In the terminology of Section 6.3 below this is twice the curvature $F_{A_{\mathrm{can}}}$ where A_{can} is the virtual connection on the bundle ${L_{\Gamma_{\mathrm{can}}}}^{1/2} = K^{-1/2}$. Thus $\widetilde{F} = 2F_{A_{\mathrm{can}}}$. (See Lemma 6.16.) □

Relation with the Cauchy-Riemann operator

Proposition 3.23 *Let $e_1, Je_1, \dots, e_n, Je_n$ be a local orthonormal frame of TX. Then for $\tau \in \Omega^{0,k}(X)$ we have*

$$\bar\partial\tau = \frac{1}{2}\sum_{j=1}^n e_j{}'' \wedge \left(\widetilde\nabla_{e_j}\tau + i\widetilde\nabla_{Je_j}\tau\right),$$

$$\bar\partial^*\tau = -\sum_{j=1}^n \left(\iota(e_j)\widetilde\nabla_{e_j}\tau + \iota(Je_j)\widetilde\nabla_{Je_j}\tau\right).$$

Proof: By Lemma 2.27,

$$\begin{aligned}
\bar\partial\tau &= (d\tau)^{0,k+1} \\
&= \left(\sum_j e_j{}^* \wedge \nabla_{e_j}\tau + \sum_j (Je_j)^* \wedge \nabla_{Je_j}\tau\right)^{0,k+1} \\
&= \frac{1}{2}\left(\sum_j e_j{}'' \wedge \widetilde\nabla_{e_j}\tau + \sum_j (Je_j)'' \wedge \widetilde\nabla_{Je_j}\tau\right).
\end{aligned}$$

This proves the first assertion. It continues to hold when ω is not closed. In the proof of the second assertion we shall use the identity

$$(\iota(v)\nabla_u\tau)^{0,k-1} = \iota(v)\widetilde\nabla_u\tau - \frac{1}{2}\iota(J(\nabla_u J)v)\tau$$

for $\tau \in \Omega^{0,k}(X)$ and $u, v \in \mathrm{Vect}(X)$ and the formula for $d^*\tau$ in Lemma 2.27:

$$\begin{aligned}
\bar\partial^*\tau &= (d^*\tau)^{0,k-1} \\
&= -\sum_j \left(\iota(e_j)\nabla_{e_j}\tau + \iota(Je_j)\nabla_{Je_j}\tau\right)^{0,k-1} \\
&= -\sum_j \left(\iota(e_j)\widetilde\nabla_{e_j}\tau + \iota(Je_j)\widetilde\nabla_{Je_j}\tau\right) \\
&\quad + \frac{1}{2}\sum_j \left(\iota(J(\nabla_{e_j}J)e_j)\tau + \iota(J(\nabla_{Je_j}J)Je_j)\tau\right) \\
&= -\sum_j \left(\iota(e_j)\widetilde\nabla_{e_j}\tau + \iota(Je_j)\widetilde\nabla_{Je_j}\tau\right) \\
&\quad + \frac{1}{2}\sum_j \left(\iota(J(\nabla_{e_j}J)e_j + (\nabla_{Je_j}J)e_j)\tau\right).
\end{aligned}$$

By Lemma 3.18 (v), the last term vanishes. Here we have used the fact that ω is closed. $\square$

Exercise 3.24 Prove that the formulae of Proposition 3.23 continue to hold for forms $\tau \in \Omega^{0,k}(X, E)$ with values in a line bundle E with connection ∇_B provided that on the left we consider the Cauchy-Riemann operator $\bar\partial_B$ induced by B and on the right we replace $\widetilde\nabla$ by the connection $\widetilde\nabla_B$ on $\Lambda^{0,*}T^*X \otimes E$ defined by

$$\widetilde\nabla_B(\tau \otimes s) = (\widetilde\nabla\tau) \otimes s + \tau \otimes \nabla_B s$$

for $\tau \in \Omega^{0,k}(X)$ and $s \in C^\infty(X, E)$. □

The Weitzenböck formula

Given $\tau \in \Omega^2(X, \mathbb{C})$ define the function $\tau_\omega : X \to \mathbb{C}$ by

$$\tau \wedge \omega^{n-1} = \tau_\omega \omega^n. \tag{3.10}$$

It is easy to see that in a local unitary frame $e_1, Je_1, \ldots, e_n, Je_n$ with $J \in \mathcal{J}(X, \omega)$ the function τ_ω can be expressed in the form

$$\tau_\omega = \frac{1}{n}\sum_{j=1}^n \tau(e_j, Je_j).$$

Proposition 3.25 *Let (X, ω) be a symplectic manifold of dimension $2n$ and $J \in \mathcal{J}(X, \omega)$. Suppose that $E \to X$ is a Hermitian line bundle and $B \in \mathcal{A}(E)$ is a Hermitian connection. Then*

$$\bar\partial_B^*\bar\partial_B\varphi_0 = \frac{1}{2}d_B{}^*d_B\varphi_0 - \frac{n}{2}i(F_B)_\omega\varphi_0.$$

for $\varphi_0 \in \Omega^{0,0}(X, E)$. Moreover, if (X, ω) is a symplectic 4-manifold, then

$$\bar\partial_B\bar\partial_B^*\varphi_2 = \frac{1}{2}\widetilde\nabla_B{}^*\widetilde\nabla_B\varphi_2 + i(F_B + \widetilde F)_\omega\varphi_2.$$

for $\varphi_2 \in \Omega^{0,2}(X, E)$, where $\widetilde F$ is given by (3.9).

Proof: Denote the connection on E by $\nabla_{B,v}s = d_Bs(v)$ for $v \in TX$. Then the formulae of Proposition 3.23 have the form

$$\bar\partial_B\varphi_0 = \frac{1}{2}\sum_{j=1}^n e_j{}'' \wedge \left(\nabla_{B,e_j}\varphi_0 + \sqrt{-1}\nabla_{B,Je_j}\varphi_0\right),$$

$$\bar\partial_B^*\alpha = \sum_{j=1}^n\left(\nabla_{B,e_j}{}^*(\iota(e_j)\alpha) + \nabla_{B,Je_j}{}^*(\iota(Je_j)\alpha)\right)$$

for $\varphi_0 \in C^\infty(X, E)$ and $\alpha \in \Omega^{0,1}(X, E)$. With $\alpha = \bar\partial_B\varphi_0$ one finds

$$\begin{aligned}
\bar{\partial}_B^* \bar{\partial}_B \varphi_0 &= \frac{1}{2} \sum_{i,j=1}^{n} {\nabla_{B,e_i}}^* \Big(\iota(e_i){e_j}'' \wedge \Big(\nabla_{B,e_j}\varphi_0 + \sqrt{-1}\nabla_{B,Je_j}\varphi_0 \Big)\Big) \\
&\quad + \frac{1}{2} \sum_{i,j=1}^{n} {\nabla_{B,Je_i}}^* \Big(\iota(Je_i){e_j}'' \wedge \Big(\nabla_{B,e_j}\varphi_0 + \sqrt{-1}\nabla_{B,Je_j}\varphi_0 \Big)\Big) \\
&= \frac{1}{2} \sum_{j=1}^{n} {\nabla_{B,e_j}}^* \Big(\nabla_{B,e_j}\varphi_0 + \sqrt{-1}\nabla_{B,Je_j}\varphi_0 \Big) \\
&\quad - \frac{\sqrt{-1}}{2} \sum_{j=1}^{n} {\nabla_{B,Je_j}}^* \Big(\nabla_{B,e_j}\varphi_0 + \sqrt{-1}\nabla_{B,Je_j}\varphi_0 \Big) \\
&= \frac{1}{2} \sum_{j=1}^{n} \Big({\nabla_{B,e_j}}^*\nabla_{B,e_j}\varphi_0 + {\nabla_{B,Je_j}}^*\nabla_{B,Je_j}\varphi_0 \Big) \\
&\quad + \frac{\sqrt{-1}}{2} \sum_{j=1}^{n} \Big({\nabla_{B,e_j}}^*\nabla_{B,Je_j}\varphi_0 - {\nabla_{B,Je_j}}^*\nabla_{B,e_j}\varphi_0 \Big) \\
&= \frac{1}{2} {d_B}^* d_B \varphi_0 \\
&\quad - \frac{\sqrt{-1}}{2} \sum_{j=1}^{n} \Big(\nabla_{B,e_j}\nabla_{B,Je_j}\varphi_0 - \nabla_{B,Je_j}\nabla_{B,e_j}\varphi_0 + \nabla_{B,[e_j,Je_j]}\varphi_0 \Big) \\
&= \frac{1}{2} {d_B}^* d_B \varphi_0 - \frac{\sqrt{-1}}{2} \sum_{j=1}^{n} F_B(e_j, Je_j)\varphi_0 \\
&= \frac{1}{2} {d_B}^* d_B \varphi_0 - \frac{n\sqrt{-1}}{2} (F_B)_\omega \varphi_0.
\end{aligned}$$

This computation relies on the formulae ${\nabla_{B,v}}^* = -\nabla_{B,v} - \mathrm{div}(v)$, on the fact that F_B is the curvature of the connection ∇_B on E, and on the identity

$$\sum_{j=1}^{n} [e_j, Je_j] = \sum_{j=1}^{n} \Big(\mathrm{div}(e_j)Je_j - \mathrm{div}(Je_j)e_j \Big)$$

for an orthonormal frame $e_1, Je_1, \dots, e_n, Je_n$. (See Lemma 2.23.) This proves the first equation. The proof of the second equation is similar and is left as an exercise. One uses the formula

$$\bar{\partial}_B^* \varphi_2 = -\sum_{j=1}^{n} \Big(\iota(e_j)\widetilde{\nabla}_{B,e_j}\varphi_2 + \iota(Je_j)\widetilde{\nabla}_{B,Je_j}\varphi_2 \Big).$$

of Proposition 3.23 and the dual identity

$$\bar{\partial}_B\alpha = -\frac{1}{2}\sum_{j=1}^{n}\left(\widetilde{\nabla}_{B,e_j}{}^{*}(e_j{}'' \wedge \alpha) + \widetilde{\nabla}_{B,Je_j}{}^{*}((Je_j)'' \wedge \alpha)\right)$$

for $\alpha = \bar{\partial}_B^*\varphi_2$. The proof uses the fact that $\Omega^{0,3}(X,E) = \{0\}$ for symplectic 4-manifolds. The curvature term has the required form because φ_2 is a section of the bundle $\Lambda^{0,2}T^*X \otimes E$ with curvature

$$F^{\widetilde{\nabla}_B} = F_B + \widetilde{F}.$$

This proves the proposition. $\square$

Corollary 3.26 *Let (X,ω) be a symplectic manifold and $J \in \mathcal{J}(X,\omega)$. Then, for every function $f : X \to \mathbb{C}$,*

$$\bar{\partial}^*\bar{\partial}f = \frac{1}{2}d^*df.$$

Proof: Proposition 3.25 with $E = X \times \mathbb{C}$ and $d_B = d$. $\square$

Duality

Here are some useful equations relating the Hodge-$*$-operator to a compatible almost complex structure on a symplectic manifold (X,ω). The Hodge-$*$-operator $* : \Omega^k(X,\mathbb{C}) \to \Omega^{2n-k}(X,\mathbb{C})$ is defined by

$$\bar{\alpha} \wedge *\beta = \langle \alpha, \beta \rangle \mathrm{dvol}$$

for $\alpha, \beta \in \Omega^k(X,\mathbb{C})$. This operator maps $\Omega^{p,q}(X) \to \Omega^{n-q,n-p}(X)$ and, since the manifold X is even dimensional, it satisfies

$$d^*\tau = - * d * \tau$$

Lemma 3.27 *Let (X,ω) be a symplectic manifold and $J \in \mathcal{J}(X,\omega)$. Denote by $*$ the Hodge-$*$-operator of the corresponding metric (3.6). Then, for every 1-form $\alpha \in \Omega^1(X)$,*

$$*(\alpha \circ J) = \alpha \wedge \frac{\omega^{n-1}}{(n-1)!}, \qquad d^*(\alpha \circ J) = -n(d\alpha)_\omega.$$

Proof: The first identity is symplectic linear algebra and can be proved by an explicit calculation in a symplectic vector space with a unitary basis. The second follows from the first:

$$d^*(\alpha \circ J) = - * d * (\alpha \circ J) = - * (d\alpha) \wedge \frac{\omega^{n-1}}{(n-1)!} = -n(d\alpha)_\omega.$$

This proves the lemma. $\square$

The previous identities were noted by LeHong Van in [69]. She proved that a compatible pair (ω, J) satisfies the second identity in Lemma 3.27 if and only if ω is closed.

Corollary 3.28 *Let (X, ω) be a $2n$-dimensional symplectic manifold and $J \in \mathcal{J}(X, \omega)$. Then, for every $\alpha \in \Omega^1(X, \mathbb{C})$*

$$\bar{\partial}^* \alpha^{0,1} = \frac{1}{2}(d^*\alpha - in(d\alpha)_\omega), \qquad \partial^* \alpha^{1,0} = \frac{1}{2}(d^*\alpha + in(d\alpha)_\omega).$$

Proof: Lemma 3.27, $\alpha^{0,1} = (\alpha + i(\alpha \circ J)/2$, and $\bar{\partial}^*\alpha^{0,1} = d^*\alpha^{0,1}$. □

Corollary 3.29 *Let (X, ω) be a $2n$-dimensional symplectic manifold and $J \in \mathcal{J}(X, \omega)$. Then, for $f \in C^\infty(X)$,*

$$(\partial\bar{\partial} f)_\omega = \frac{i}{2n} d^* df.$$

Proof: If $\alpha \in \Omega^{0,1}(X)$ then $d^*\alpha = \bar{\partial}^*\alpha = \bar{\partial}^*\alpha^{0,1}$ and, by Corollary 3.28,

$$in(d\alpha)_\omega = -\bar{\partial}^*\alpha.$$

With $\alpha = \bar{\partial} f$, this gives $in(\partial\bar{\partial} f)_\omega = -\bar{\partial}^*\bar{\partial} f = -d^*df/2$. The last equation follows from Corollary 3.26. □

Exercise 3.30 Prove that $\partial^* = - * \bar{\partial} *$ and $\bar{\partial}^* = - * \partial *$. □

Exercise 3.31 Prove that, for every $\tau \in \Omega^{0,q}(X)$,

$$*\left(\tau \wedge \frac{\omega^k}{k!}\right) = (-1)^{\frac{q(q-1)}{2}} i^q \tau \wedge \frac{\omega^{n-q-k}}{(n-q-k)!},$$

and, for every $\sigma \in \Omega^{p,0}(X)$,

$$*\left(\sigma \wedge \frac{\omega^k}{k!}\right) = (-1)^{\frac{p(p+1)}{2}} i^p \sigma \wedge \frac{\omega^{n-p-k}}{(n-p-k)!}.$$

Hint: Prove this with standard coordinates $z_j = x_j + iy_j$ on $\mathbb{R}^{2n} = \mathbb{C}^n$ and use the formula

$$\left|\tau \wedge \frac{\omega^k}{k!}\right|^2 = \binom{n-q}{k} |\tau|^2$$

for $\tau \in \Omega^{0,q}(X, \mathbb{C})$ and $0 \le k \le n-q$. This last formula can be proved easily for the standard basis of $\Lambda^{0,q}\mathbb{C}^n$ and in general follows from the fact that the map $\Lambda^{0,q} \to \Lambda^{k,k+q} : \tau \mapsto \tau \wedge \omega^k$ is conformal for $0 \le k \le n-q$ (i.e. the image of the standard basis is orthogonal). □

Lemma 3.32 *Let (X,ω) be a symplectic manifold of dimension $2n$ and $J \in \mathcal{J}(X,\omega)$. Let $\tau \in \Omega^{0,q}(X)$ and $\sigma \in \Omega^{p,0}(X)$. Then*

$$\partial^* \left(\tau \wedge \frac{(i\omega)^k}{k!} \right) = (\bar\partial\tau) \wedge \frac{(i\omega)^{k-1}}{(k-1)!}, \tag{3.11}$$

$$\bar\partial^* \left(\sigma \wedge \frac{(i\omega)^k}{k!} \right) = -(\partial\sigma) \wedge \frac{(i\omega)^{k-1}}{(k-1)!} \tag{3.12}$$

for every $k \geq 1$ and

$$(\partial\tau) \wedge \frac{(i\omega)^{n-q}}{(n-q)!} = (\bar\partial^*\tau) \wedge \frac{(i\omega)^{n-q+1}}{(n-q+1)!}, \tag{3.13}$$

$$(\bar\partial\sigma) \wedge \frac{(i\omega)^{n-p}}{(n-p)!} = -(\partial^*\sigma) \wedge \frac{(i\omega)^{n-p+1}}{(n-p+1)!}. \tag{3.14}$$

Proof: By Exercises 3.30 and 3.31 we have

$$\begin{aligned}
\partial^* \left(\tau \wedge \frac{(i\omega)^k}{k!} \right) &= - * \bar\partial * \left(\tau \wedge \frac{(i\omega)^k}{k!} \right) \\
&= -(-1)^{\frac{q(q-1)}{2}} i^{q+k} * \left((\bar\partial\tau) \wedge \frac{\omega^{n-q-k}}{(n-q-k)!} \right) \\
&= -(-1)^{\frac{q(q-1)}{2}} i^{q+k} (-1)^{\frac{q(q+1)}{2}} i^{q+1} (\bar\partial\tau) \wedge \frac{\omega^{k-1}}{(k-1)!} \\
&= -(-1)^q i^{2q+k+1} (\bar\partial\tau) \wedge \frac{\omega^{k-1}}{(k-1)!} \\
&= (\bar\partial\tau) \wedge \frac{(i\omega)^{k-1}}{(k-1)!}.
\end{aligned}$$

This proves (3.11). (3.12) follows by complex conjugation. To prove (3.13) note first that the map

$$\Lambda^{0,q} \to \Lambda^{n-q,n} : \tau \mapsto \tau \wedge \frac{\omega^{n-q}}{(n-q)!}$$

is a bijection. (The hint in Exercise 3.31 shows that this map is injective and both spaces have the same dimension.) Hence it follows from Exercise 3.31 with $\alpha = \tau \wedge \omega^{n-q}/(n-q)!$ that

$$(*\alpha) \wedge \frac{\omega^{n-q}}{(n-q)!} = (-1)^{\frac{q(q-1)}{2}} i^q \alpha, \qquad \alpha \in \Lambda^{n-q,n}.$$

Hence

$$
\begin{aligned}
(\bar\partial^*\tau)\wedge\frac{(i\omega)^{n-q+1}}{(n-q+1)!}
&= -(*\partial*\tau)\wedge\frac{(i\omega)^{n-q+1}}{(n-q+1)!} \\
&= -(-1)^{\frac{q(q-1)}{2}}i^{n+1}\left\{*\left((\partial\tau)\wedge\frac{\omega^{n-q}}{(n-q)!}\right)\right\}\wedge\frac{\omega^{n-q+1}}{(n-q+1)!} \\
&= -(-1)^{\frac{q(q-1)}{2}}i^{n+1}(-1)^{\frac{(q-1)(q-2)}{2}}i^{q-1}(\partial\tau)\wedge\frac{\omega^{n-q}}{(n-q)!} \\
&= -(-1)^{q-1}i^{2q}(\partial\tau)\wedge\frac{(i\omega)^{n-q}}{(n-q)!} \\
&= (\partial\tau)\wedge\frac{(i\omega)^{n-q}}{(n-q)!}.
\end{aligned}
$$

This proves (3.13) and (3.14) follows again by conjugation. □

3.4 Dolbeault cohomology

Let (X, J, ω) be a **Kähler manifold** of real dimension $2n$. The Kähler condition means that ω is a closed nondegenerate 2-form and J is an integrable complex structure which is compatible with ω. By Lemma 3.18 The Kähler condition is equivalent to $\nabla J = 0$, where ∇ denotes the Levi-Civita connection of the **Kähler metric** $g(v, w) = \omega(v, Jw)$. Let $R \in \Omega^2(X, \mathrm{End}(TX))$ denote the curvature tensor and $S : S^2TX \to \mathbb{R}$ the Ricci tensor of the Kähler metric.

Lemma 3.33 *Let (X, J, ω) be a Kähler manifold. Then*

$$R(u,v)J = JR(u,v), \qquad R(Ju,Jv) = R(u,v)$$

and

$$S(u,v) = S(Ju,Jv) = \frac{1}{2}\mathrm{trace}(JR(u,Jv))$$

for all $x \in X$ and all $u, v \in T_xX$.

Proof: Exercise. **Hint:** Use $\nabla J = 0$ and the formulae of Section 2.1. □

By Exercise 3.12, the d operator on differential forms splits as

$$d = \partial + \bar\partial.$$

Hence the equation $d \circ d = 0$ decomposes as

$$\partial\partial = 0, \qquad \partial\bar\partial + \bar\partial\partial = 0, \qquad \bar\partial\bar\partial = 0.$$

This gives rise to the Dolbeault double complex

$$
\begin{tikzcd}
\Omega^{p,q-2} \arrow[dr, "\bar\partial"] & & \Omega^{p+1,q-1} \arrow[dr, "\bar\partial"] & & \Omega^{p+2,q} \\
& \Omega^{p,q-1} \arrow[ur, "\partial"] \arrow[dr, "\bar\partial"] & & \Omega^{p+1,q} \arrow[ur, "\partial"] \arrow[dr, "\bar\partial"] & \\
\Omega^{p-1,q-1} \arrow[ur, "\partial"] \arrow[dr, "\bar\partial"] & & \Omega^{p,q} \arrow[ur, "\partial"] \arrow[dr, "\bar\partial"] & & \Omega^{p+1,q+1} \\
& \Omega^{p-1,q} \arrow[ur, "\partial"] \arrow[dr, "\bar\partial"] & & \Omega^{p,q+1} \arrow[ur, "\partial"] \arrow[dr, "\bar\partial"] & \\
\Omega^{p-2,q} \arrow[ur, "\partial"] & & \Omega^{p-1,q+1} \arrow[ur, "\partial"] & & \Omega^{p,q+2}
\end{tikzcd}
$$

with corresponding **Dolbeault cohomology** groups

$$H^{p,q}(X) = \frac{\Omega^{p,q}(X) \cap \ker d}{\Omega^{p,q}(X) \cap \operatorname{im} d}.$$

There is an obvious embedding $H^{p,q}(X) \hookrightarrow H^{p+q}(X;\mathbb{C})$ and hence the Dolbeault cohomology groups $H^{p,q}(X)$ are finite dimensional. They can be identified with the spaces of harmonic forms of type (p,q)

$$H^{p,q}(X) \cong \Omega^{p,q}(X) \cap \ker d \cap \ker d^*.$$

These groups can also be defined on general complex manifolds. But in the Kähler case their direct sum agrees with the ordinary de Rham cohomology.

Theorem 3.34 *On a Kähler manifold (X, J, ω) there is a natural isomorphism*

$$H^k(X;\mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X).$$

Proof: By Lemma 3.33, the complex structure J satisfies the conditions of Exercise 2.35. Hence

$$\Delta\iota(J)\tau = \iota(J)\Delta\tau \tag{3.15}$$

for every $\tau \in \Omega^*(X,\mathbb{C})$, where $\Delta = d^*d + dd^*$ denotes the Laplace-Beltrami operator and and $\iota(J)\tau$ is defined by (2.19). Now the space $\Omega^{p,q}(X)$ can be uniquely characterized as the subspace of those forms $\tau \in \Omega^{p+q}(X,\mathbb{C})$ for which

$$\iota(J)\tau = i(p-q)\tau.$$

In other words, the orthogonal projection $\Pi_{p,q} : \Omega^{p+q}(X,\mathbb{C}) \to \Omega^{p,q}(X)$ is the eigenspace projection of the operator $\iota(J) : \Omega^{p+q}(X,\mathbb{C}) \to \Omega^{p+q}(X,\mathbb{C})$ onto the eigenspace with eigenvalue $i(p-q)$. Hence it can be expressed in the form of a contour integral

$$\Pi_{p,q} = \int_{C_{p,q}} (\lambda\mathbb{1} - \iota(J))^{-1}\, d\lambda$$

where $C_{p,q}$ is a simple closed curve encircling the eigenvalue $i(p-q)$ (say, of radius 1/2). The formula (3.15) shows that this projection operator commutes with the Laplace-Beltrami operator $\Delta = d^*d + dd^*$. Hence the kernel of Δ, that is, the space of harmonic $(p+q)$-forms, is invariant under the projection $\Pi_{p,q}$. It follows that every harmonic k-form $\tau \in H^k(X;\mathbb{C})$ decomposes as a sum

$$\tau = \sum_{p+q=k} \Pi_{p,q}\tau$$

of harmonic forms of type (p,q). This proves the theorem. □

Remark 3.35 Consider the $\bar\partial$-Laplacian and the ∂-Laplacian

$$\Delta_{\bar\partial} = \bar\partial^*\bar\partial + \bar\partial\bar\partial^*, \qquad \Delta_\partial = \partial^*\partial + \partial\partial^*.$$

Comparing these operators with Δ we find that

$$\Delta_\partial + \Delta_{\bar\partial} = \Delta = d^*d + dd^*$$

is the standard Laplace-Beltrami operator. To see this just recall that $d = \partial + \bar\partial$ and $d^* = \partial^* + \bar\partial^*$ and note that, by (3.15), the spaces $\Omega^{p,q}$ are invariant under Δ. Hence

$$\partial^*\bar\partial + \bar\partial\partial^* = 0, \qquad \bar\partial^*\partial + \partial\bar\partial^* = 0.$$

The first operator is the component of Δ which goes from $\Omega^{p,q} \to \Omega^{p-1,q+1}$ and the second goes to $\Omega^{p+1,q-1}$. Moreover, the two operators $\Delta_{\bar\partial}$ and Δ_∂ agree and hence

$$\Delta_{\bar\partial} = \Delta_\partial = \frac{1}{2}\Delta.$$

The proof is left as an exercise. It follows that on a Kähler manifold the Dolbeault cohomology group $H^{p,q}(X) = H^{p+q}(X) \cap \Omega^{p,q}(X)$ is naturally isomorphic to the quotients $\ker\bar\partial/\operatorname{im}\bar\partial$ and $\ker\partial/\operatorname{im}\partial$ in $\Omega^{p,q}(X)$. □

Recall that the Hodge-$*$-operator maps $\Omega^{p,q}(X) \to \Omega^{n-q,n-p}(X)$ and preserves the space of harmonic forms. Hence we have the following.

Corollary 3.36 *On a compact Kähler manifold of real dimension $2n$ the Hodge-$*$-operator induces a natural isomorphism*

$$H^{p,q}(X) \to H^{n-q,n-p}(X).$$

Moreover, complex conjugation gives an isomorphism $H^{p,q}(X) \to H^{q,p}(X)$ and thus the odd cohomology groups $H^{2k+1}(X;\mathbb{C})$ are even dimensional.

If X has real dimension 4 then a 2-form $\eta \in \Omega^2(X,\mathbb{C})$ is called **self-dual** (respectively **anti-self-dual**) if $\eta = *\eta$ (respectively $\eta = -*\eta$). Denote by $\Omega^{2,\pm}(X) = \Omega^{2,\pm}(X, g)$ the space of self-dual, respectively anti-self-dual, 2-forms. For $\eta \in \Omega^2(X)$ denote by $\eta^\pm = \frac{1}{2}(\eta \pm *\eta)$ its self-dual, respectively anti-self-dual, part. Since the Hodge-$*$-operator preserves the space $H^2(X)$ of (real valued) harmonic forms this space splits as a direct sum

$$H^2(X) = H^{2,+}(X) \oplus H^{2,-}(X)$$

into the spaces of self-dual and anti-self-dual harmonic 2-forms. It is important to note that this splitting is orthogonal with respect to the standard inner product. Define $b^\pm = b^\pm(X) = \dim H^{2,\pm}(X)$.

If (X, J, ω) is a Kähler surface then the 2-form ω is always self-dual and hence b^+ is at least 1. Moreover, the bundle $\Lambda^{2,+}T^*X \otimes \mathbb{C}$ of complex valued self-dual 2-forms decomposes as

$$\Lambda^{2,+}T^*X \otimes \mathbb{C} = \Lambda^{2,0}T^*X \oplus \mathbb{C}\omega \oplus \Lambda^{0,2}T^*X.$$

It is easy to see, by examining the real and imaginary parts of $dz_1 \wedge dz_2$, that every form of type $(2, 0)$ and of type $(0, 2)$ is self-dual. Since the space of self-dual 2-forms is 3-dimensional this proves the above identity. It follows that the bundle $\Lambda^{2,-}T^*X \otimes \mathbb{C}$ consists entirely of forms of type $(1, 1)$ and

$$\Lambda^{1,1}T^*X \otimes \mathbb{C} = \mathbb{C}\omega \oplus \Lambda^{2,-}T^*X \otimes \mathbb{C}.$$

These observations have some important consequences for the topology of Kähler surfaces which we shall discuss next.

Proposition 3.37 *Let (X, J, ω) be a Kähler surface and $E \to X$ be a holomorphic line bundle with a nonzero section $s : X \to E$. Then either E is the trivial bundle or*

$$[\omega] \cdot c_1(E) > 0.$$

Proof: The zero set of s is an analytic hypersurface $V \subset X$ and any such hypersurface decomposes as a union $V = V_1 \cup \cdots \cup V_\ell$ of irreducible ones. (See Theorem F.18 in Appendix F.) Each V_i has a well-defined multiplicity $m_i > 0$ as a zero set of s (see the discussion on page 573). In other words the pair (E, s) is represented by the effective divisor $D = \sum_i m_i V_i$ and it follows easily (from standard arguments in differential topology) that the first Chern class of E is given by

$$c_1(E) = \sum_i m_i \mathrm{PD}([V_i]).$$

By Proposition F.19, each V_i is the image of some nonconstant holomorphic map $u_i : \Sigma_i \to X$ defined on a compact connected Riemann surface Σ_i. For each i the number $\mathrm{PD}([V_i]) \cdot [\omega]$ is given by the integral of ω over V_i (or the integral of ${u_i}^*\omega$ over Σ_i) and this integral is always positive. Hence

$$c_1(E) \cdot [\omega] = \sum_i m_i \int_{\Sigma_i} {u_i}^*\omega \geq 0.$$

Equality can only occur if s has no zeros and in this case E admits a holomorphic trivialization. This proves the proposition. □

Proposition 3.38 *If (X, J, ω) is a compact Kähler surface with $b^+ > 1$ then $[\omega] \cdot c_1(TX) \leq 0$. Moreover, b^+ is always odd and*

$$\dim^c H^{2,0}(X) = \frac{b^+ - 1}{2} = p_g.$$

This number is called the **geometric genus** *of X.*

Proof: The cohomology group $H^{1,1}(X)$ can be identified with the space of harmonic 2-forms η of type $(1,1)$ Any such form can be written as $\eta = f\omega + \eta^-$ where $\eta^+ = f\omega$ and η^- are both harmonic. This implies $df \wedge \omega = d(f\omega) - fd\omega = 0$. Now it is a simple exercise in 4-dimensional linear algebra to prove that $*\alpha = -(\alpha \circ J) \wedge \omega$ for every 1-form α. This shows that the map $\Omega^1 \to \Omega^3 : \alpha \mapsto \alpha \wedge \omega$ is a bijection and hence we obtain $df = 0$. Thus f is a constant function and we conclude that

$$H^{1,1}(X) = \mathbb{C}\omega \oplus H^{2,-}(X; \mathbb{C}).$$

The Hodge decomposition $H^2(X;\mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$ of Theorem 3.34 now shows that

$$H^{2,+}(X;\mathbb{C}) = H^{2,0}(X) \oplus \mathbb{C}\omega \oplus H^{0,2}(X).$$

Hence $b^+ > 1$ if and only if $H^{2,0}(X) \neq 0$. Now $H^{2,0}(X)$ can be identified with the space of all forms $\tau \in \Omega^{2,0}(X)$ which satisfy $\bar{\partial}\tau = 0$. To see this just note that, by Remark 3.35, $2\bar{\partial}^*\bar{\partial}\tau = d^*d\tau + dd^*\tau$. Hence $H^{2,0}(X)$ is the space of holomorphic sections of the bundle $\Lambda^{2,0}T^*X$ with Chern class $-c_1(TX)$. Hence it follows from Proposition 3.37 that $[\omega] \cdot c_1(TX) \leq 0$ as claimed. □

Theorem 3.39. (Hodge index theorem) *Let E and F be holomorphic line bundles over a compact Kähler surface X and denote $a = c_1(E)$ and $b = c_1(F)$. Assume $a \cdot a \geq 0$. Then*

$$(a \cdot a)(b \cdot b) \leq |a \cdot b|^2$$

and if equality holds then a and b are linearly dependent.

Proof: Every holomorphic line bundle has a connection whose curvature form is of type $(1,1)$. Hence $a, b \in H^{1,1}(X;\mathbb{R})$. The proof of Proposition 3.38 shows that the space $H^{1,1}(X;\mathbb{R})$ of real valued harmonic forms splits as a direct sum

$$H^{1,1}(X;\mathbb{R}) = \mathbb{R}\omega \oplus H^{2,-}(X;\mathbb{R}).$$

Hence the quadratic form $(x,y) \mapsto x \cdot y$ on $H^{1,1}(X;\mathbb{R})$ is nondegenerate and has a 1-dimensional positive subspace. This implies that every 2-dimensional subspace $W \subset H^{1,1}(X;\mathbb{R})$ must contain a vector x with $x \cdot x < 0$. Assume that the vectors a and b are linearly independent and consider the subspace W spanned by a and b. The quadratic form $W \to \mathbb{R} : x \mapsto x \cdot x = x^2$ is represented by the matrix

$$A = \begin{pmatrix} a^2 & ab \\ ab & b^2 \end{pmatrix}$$

Since W contains a vector with negative self-intersection number this matrix must have a negative eigenvalue. Since $a^2 \geq 0$ it follows that $\det(A) < 0$ and this proves the required inequality. □

3.5 Holomorphic line bundles

Let (X, J, ω) be a Kähler manifold of real dimension $2n$. The goal of this section is to show that the space of holomorphic structures on a line bundle $E \to X$ with first Chern class in $H^{1,1}(X)$ can be identified with the torus $H^1(X; i\mathbb{R})/H^1(X; 2\pi i\mathbb{Z})$. Consider the space of **Hermitian Yang-Mills** connections

$$\mathcal{A}^\omega(E) = \left\{ B \in \mathcal{A}(E) \,|\, F_B^{0,2} = 0,\; (F_B)_\omega = \mu \right\}.$$

Here μ is a constant. The formula

$$c_1(E) \cdot [\omega]^{n-1} = \int_X \frac{iF_B}{2\pi} \wedge \omega^{n-1} = \frac{i\mu}{2\pi} n!\mathrm{Vol}(X)$$

shows that the natural choice for this constant is

$$\mu = -\frac{2\pi i c_1(E) \cdot [\omega]^{n-1}}{n!\mathrm{Vol}(X)} \tag{3.16}$$

The group $\mathcal{G}(E) = \mathrm{Map}(X, S^1)$ of unitary gauge transformations acts naturally on $\mathcal{A}^\omega(E)$. The next theorem asserts that the quotient space can be naturally identified with the space of holomorphic structures on E. It also asserts that this space is nonempty whenever $c_1(E)$ is of type $(1,1)$ and in this case can be identified with the torus $H^1(X; i\mathbb{R})/H^1(X; 2\pi i\mathbb{Z})$. This latter identification is not natural. It requires the choice of a base point. To be more explicit let us denote by $\mathcal{CR}(E)$ the set of Cauchy-Riemann operators $\bar\partial : C^\infty(X,E) \to \Omega^{0,1}(X,E)$ which satisfy $\bar\partial \circ \bar\partial = 0$. The complexified gauge group $\mathcal{G}(E)^c = \mathrm{Map}(X,\mathbb{C}^*)$ acts on this space by $\bar\partial \mapsto u^{-1}\circ\bar\partial\circ u$ for $u \in \mathcal{G}(E)^c$. The quotient space $\mathcal{CR}(E)/\mathcal{G}(E)^c$ is the set of holomorphic structures on E.

Theorem 3.40 *Let (X,J,ω) ba a Kähler manifold and $E \to X$ be a complex line bundle. Then the map $\mathcal{A}^\omega(E) \to \mathcal{CR}(E) : B \mapsto \bar\partial_B$ induces a bijection of quotient spaces*

$$\frac{\mathcal{A}^\omega(E)}{\mathcal{G}(E)} \to \frac{\mathcal{CR}(E)}{\mathcal{G}(E)^c}.$$

If the first Chern class $c_1(E) \in H^2(X;\mathbb{Z})$ projects to a class in $H^{1,1}(X;\mathbb{C})$, then $\mathcal{A}^\omega(E) \neq \emptyset$ and there exists a bijection

$$\frac{\mathcal{A}^\omega(E)}{\mathcal{G}(E)} \cong \frac{H^1(X;i\mathbb{R})}{H^1(X;2\pi i\mathbb{Z})}.$$

Lemma 3.41 *Let (X,ω) be a Kähler manifold of real dimension $2n$ and $\alpha \in \Omega^1(X,\mathbb{R})$. Then*

$$(d\alpha)^{0,2} = 0 \qquad \Longrightarrow \qquad \|d\alpha\|_{L^2} = n\,\|(d\alpha)_\omega\|_{L^2}.$$

Proof: Suppose that $(d\alpha)^{0,2} = 0$. Since the Dolbeault Laplacian is equal to half the Hodge Laplacian (see Remark 3.35), we have

$$dd^*\alpha^{0,1} + d^*d\alpha^{0,1} = 2\bar\partial\bar\partial^*\alpha^{0,1} = 2\bar\partial d^*\alpha^{0,1}.$$

Since $2\alpha^{0,1} = \alpha + i\alpha\circ J$ we obtain

$$\begin{aligned}\Delta\alpha^{0,1} &= \bar\partial d^*\alpha + i\bar\partial d^*(\alpha\circ J)\\ &= \frac{1}{2}\Big(dd^*\alpha + i(dd^*\alpha)\circ J + idd^*(\alpha\circ J) - (dd^*(\alpha\circ J))\circ J\Big).\end{aligned}$$

The real part on the left is $2^{-1}(d^*d\alpha + dd^*\alpha)$. Comparing this with the real part on the right gives

$$d^*d\alpha = -(dd^*(\alpha\circ J))\circ J.$$

Take the L^2-inner product with α to obtain

$$\|d\alpha\|^2 = \|d^*(\alpha\circ J)\|^2 = n^2\,\|(d\alpha)_\omega\|^2.$$

Here we have used Lemma 3.27. □

Proof of Theorem 3.40: For a Hermitian connection $B \in \mathcal{A}(E)$ and a complex gauge transformation $u \in \mathrm{Map}(X, \mathbb{C}^*) = \mathcal{G}(E)^c$ define

$$u^*B = B + u^{-1}\bar{\partial}u - \bar{u}^{-1}\partial\bar{u}.$$

Note that this agrees with the usual action whenever $u \in \mathrm{Map}(X, S^1) = \mathcal{G}(E)$ and that

$$\bar{\partial}_{u^*B} = u^{-1} \circ \bar{\partial}_B \circ u$$

for $u \in \mathcal{G}(E)^c$.

We prove that, for every $B \in \mathcal{A}^{0,2}(E)$, there exists a $u \in \mathcal{G}^c(E)$ such that $u^*B \in \mathcal{A}^\omega(E)$. It suffices to consider gauge transformations of the form $u = e^\theta$, where $\theta : X \to \mathbb{R}$. In this case

$$u^*B = B + \bar{\partial}\theta - \partial\theta$$

and hence, by Corollary 3.29,

$$(F_{u^*B})_\omega = (F_B)_\omega + 2(\partial\bar{\partial}\theta)_\omega = (F_B)_\omega + \frac{i}{n}d^*d\theta.$$

This shows that $F^{0,2}_{u^*B} = 0$. Moreover, by Hodge theory, there exists a function θ such that the right hand side is equal to a constant, namely the mean value of $(F_B)_\omega$. With this choice $F_{u^*B} \in \mathcal{A}^\omega(E)$.

Next we prove that if $B_0, B_1 \in \mathcal{A}^\omega(E)$ are complex gauge equivalent then they are unitarily gauge equivalent. To see this suppose that there is a function $u : X \to \mathbb{C}^*$ such that $B_1 = u^*B_0$. Write u in the form $u = e^\theta u_0$ where $\theta : X \to \mathbb{R}$ and $u_0 : X \to S^1$. Then

$$b := B_1 - B_0 = u^{-1}\bar{\partial}u - \bar{u}^{-1}\partial\bar{u} = {u_0}^{-1}du_0 + \bar{\partial}\theta - \partial\theta.$$

Since $B_0, B_1 \in \mathcal{A}^\omega(E)$, we have $(db)^{0,2} = 0$ and $(db)_\omega = 0$. Hence, by Lemma 3.41,

$$0 = db = d(\bar{\partial}\theta - \partial\theta) = 2\partial\bar{\partial}\theta.$$

By Corollary 3.29, $d^*d\theta = 0$, hence θ is constant, and hence

$$B_1 = B_0 + {u_0}^{-1}du_0 = {u_0}^*B_0.$$

Thus we have proved that the inclusion $\mathcal{A}^\omega(E) \to \mathcal{A}^{0,2}(E)$ induces a natural bijection of quotient spaces

$$\frac{\mathcal{A}^\omega(E)}{\mathcal{G}(E)} \cong \frac{\mathcal{A}^{0,2}(E)}{\mathcal{G}(E)^c}.$$

If the first Chern class $c_1(E)$ is of type $(1,1)$ then $\mathcal{A}^{0,2}(E) \neq \emptyset$ and hence, by what we just proved, $\mathcal{A}^\omega(E) \neq \emptyset$.

It remains to prove that the quotient $\mathcal{A}^\omega(E)/\mathcal{G}(E)$ is diffeomorphic to the torus. To see this fix a base point $B_0 \in \mathcal{A}^\omega(E)$. By Lemma 3.41,

$$\mathcal{A}^\omega(E) = \left\{B_0 + b \,|\, b \in \Omega^1(X, i\mathbb{R}),\, db = 0\right\}.$$

Hence the map $B_0 + b \mapsto b$ determines is a bijection of quotient spaces

$$\frac{\mathcal{A}^\omega(E)}{\mathcal{G}(E)} \cong \frac{\ker(d : \Omega^1(X, i\mathbb{R}) \to \Omega^2(X, i\mathbb{R}))}{\{u^{-1}du \,|\, u : X \to S^1\}}.$$

By Proposition 5.30 in Section 5.4 below, a closed 1-form $b \in \Omega^1(X, i\mathbb{R})$ has the form $b = u^{-1}du$ for some smooth map $u : X \to S^1$ if and only if all its periods are integer multiples of $2\pi i$. Hence the quotient $\mathcal{A}^\omega(E)/\mathcal{G}(E)$ can be identified with the torus $H^1(X; i\mathbb{R})/H^1(X; 2\pi i\mathbb{Z})$ as claimed. □

Theorem 3.40 has interesting connections with symplectic geometry. The space $\mathcal{A}(E)$ carries a natural symplectic form Ω defined by

$$\Omega_B(b, b') = -\int_X b \wedge b' \wedge \omega^{n-1}.$$

This form is easily seen to be nondegenerate and closed. There is a compatible complex structure given by

$$b \mapsto * \left(b \wedge \frac{\omega^{n-1}}{(n-1)!}\right) = -b \circ J.$$

(See Lemma 3.27.) The submanifold $\mathcal{A}^{0,2}(E)$ is invariant under the complex structure and hence the restriction of the form Ω to $\mathcal{A}^{0,2}(E)$ is still nondegenerate. The action of the gauge group $\mathcal{G}(E)$ is a Hamiltonian one and the moment map is given by

$$B \mapsto n!((F_B)_\omega - \mu).$$

Here the number μ is given by (3.16) and is chosen such that the image of the moment map consists of all functions with mean value zero. Thus the space $\mathcal{A}^\omega(E)/\mathcal{G}(E)$ is the Marsden-Weinstein quotient. The proof of Theorem 3.40 shows that the quotient of the total space $\mathcal{A}^{0,2}(E)$ by the complexified gauge group can be identified with the symplectic reduction

$$\mathcal{A}^{0,2}(E)/\!\!/\mathcal{G}(E) = \mathcal{A}^\omega(E)/\mathcal{G}(E)$$

at the zero set of the moment map. This is an infinite dimensional analogue of the GIT quotient for Hamiltonian group actions on finite dimenaional Kähler manifolds.

In the 4-dimensional case $\mathcal{A}^\omega(E)$ is the space of anti-self-dual connections and hence there is a one-to-one correspondence between gauge equivalence classes of anti-self-dual S^1-connections and isomorphism classes of holomorphic line bundles. This is a simple model case for the much deeper theorem of Donaldson relating stable holomorphic rank-2 bundles over Kähler surfaces to anti-self-dual U(2)-connections (cf. [21, Chapter 6]).

3.6 The Hirzebruch-Riemann-Roch theorem

Let (X, J, ω) be a Kähler manifold of real dimension $2n$. Fix a holomorphic vector bundle $E \to X$ and consider the $\bar\partial$-complex

$$\Omega^{0,0}(X,E) \xrightarrow{\bar\partial} \Omega^{0,1}(X,E) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \Omega^{0,n}(X,E).$$

In the integrable case we have

$$\bar\partial \circ \bar\partial = 0$$

and the cohomology groups are denoted by

$$H^{0,k}_{\bar\partial}(X,E) = \frac{\ker \bar\partial}{\operatorname{im} \bar\partial}.$$

The alternating sum of the dimensions is an invariant of the tangent bundle TX and the vector bundle E. It is called the **holomorphic Euler characteristic** of the pair (X, E) and is denoted by

$$\chi(X,E) = \sum_{k=0}^{n} (-1)^k \dim^c H^{0,k}_{\bar\partial}(X,E).$$

In the almost complex case there is no canonical $\bar\partial$-operator on a nontrivial complex vector bundle. However, given a connection B on E one can consider the associated operator $\bar\partial_B : \Omega^{0,k}(X,E) \to \Omega^{0,k+1}(X,E)$ as above. In general, the composition $\bar\partial_B \circ \bar\partial_B$ will be nonzero and then there is no corresponding chain complex. Note, however, that in the integrable case the holomorphic Euler characteristic can be expressed as the complex Fredholm index of the operator

$$\bar\partial_B + \bar\partial_B^* : \Omega^{0,\mathrm{ev}}(X,E) \to \Omega^{0,\mathrm{odd}}(X,E).$$

Thus in the almost complex case it is natural to extend the definition of the holomorphic Euler-characteristic via

$$\chi(X,E) = \operatorname{index}^c(\bar\partial_B + \bar\partial_B^*).$$

This number is independent of the connection B and depends only on the homotopy class of the almost complex structure J. The Hirzebruch-Riemann-Roch theorem expresses this number in terms of the Chern classes of (TX, J) and E.

Theorem 3.42. (Hirzebruch-Riemann-Roch) *Let $E \to X$ be a complex vector bundle over an almost complex manifold (X, J). Then the twisted J-holomorphic Euler characteristic is given by*

$$\chi(X, E) = \int_X \mathrm{ch}(E) \wedge \mathrm{td}(TX)$$

where $\mathrm{ch}(E)$ denotes the Chern character and $\mathrm{td}(TX)$ denotes the Todd class of the tangent bundle with the almost complex structure J.

The **Todd class** and **Chern character** of a complex vector bundle $E \to X$ are integral cohomology classes given by the formulae

$$\mathrm{td}(E) = \prod_{j=1}^{m} \frac{x_j}{1 - e^{-x_j}}, \qquad \mathrm{ch}(E) = \sum_{j=1}^{m} e^{x_j}$$

where the x_j are to be understood as formal variables of degree 2 representing the first Chern classes of line bundles L_j in a decomposition

$$E = L_1 \oplus L_2 \oplus \cdots \oplus L_m$$

if such a decomposition exists. The **total Chern class** of E can be expressed in the form

$$c(E) = 1 + c_1(E) + \cdots + c_m(E) = \prod_{j=1}^{m} (1 + x_j).$$

The individual Chern classes $c_j(E)$ are the elementary symmetric functions in the variables $x_1, \ldots, x_m$. Conversely, every symmetric function in the x_j can be expressed in terms of the elementary symmetric functions and hence in terms of the Chern classes of E. This is how the above formulae for $\mathrm{td}(E)$ and $\mathrm{ch}(E)$ should be interpreted when E does not decompose. In particular, when X is a 4-manifold, we have

$$\mathrm{td}(E) = 1 + \frac{1}{2} c_1(E) + \frac{1}{12} (c_1(E)^2 + c_2(E))$$

and, with $m = \operatorname{rank} E$,

$$\mathrm{ch}(E) = m + c_1(E) + \frac{1}{2} (c_1(E)^2 - 2c_2(E)).$$

This leads to the following index formula in dimension 4.

Corollary 3.43 *The holomorphic Euler characteristic of a compact connected almost complex manifold (X, J) is given by*

$$\chi(X, \mathcal{O}) = \frac{1}{4}\sigma(X) + \frac{1}{4}\chi(X) = \frac{1 - b_1 + b^+}{2}$$

where $\sigma(X)$ denotes the signature, $\chi(X)$ the ordinary Euler characteristic, and b_1, $b_2 = b^+ + b^-$ the Betti numbers. After twisting by a line bundle E we have

$$\chi(X, E) = \frac{1}{8}\langle c_1(K^* \otimes E^2)^2, [X]\rangle - \frac{1}{8}\sigma(X).$$

where $K = \det(T^*X) = \Lambda^{2,0}T^*X$ is the canonical bundle.

Proof 1: If $E = \mathbb{C}$ is the trivial bundle then the formula of the Hirzebruch-Riemann-Roch theorem gives

$$\begin{aligned}
\chi(X, \mathcal{O}) &= \langle \mathrm{td}(TX), [X]\rangle \\
&= \frac{1}{12}\langle c_1(TX)^2 + c_2(TX), [X]\rangle \\
&= \frac{1}{12}\langle c_1(TX)^2 - 2c_2(TX), [X]\rangle + \frac{1}{4}\langle c_2(TX), [X]\rangle \\
&= \frac{1}{4}\sigma(X) + \frac{1}{4}\chi(X).
\end{aligned}$$

Here we have used $\chi(X) = \langle c_2(TX), [X]\rangle$ (see Remark 1.37) and the Hirzebruch signature formula (1.9). Note that, in terms of the Betti numbers, $\chi(X) = 2 - 2b_1 + b^+ + b^-$ and $\sigma(X) = b^+ - b^-$. It follows that for compact Kähler surfaces the number $b_0 - b_1 + b^+$ must be even and we have $\chi(X, \mathcal{O}) = \frac{1}{2}(1 - b_1 + b^+)$ as claimed. In general

$$\begin{aligned}
\chi(X, E) &= \frac{1}{2}\langle c_1(E)c_1(TX) + c_1(E)^2, [X]\rangle + \frac{1}{12}\langle c_1(TX)^2 + c_2(TX), [X]\rangle \\
&= \frac{1}{8}\langle (c_1(TX) + 2c_1(E))^2, [X]\rangle - \frac{1}{24}\langle c_1(TX)^2 - 2c_2(TX), [X]\rangle \\
&= \frac{1}{8}\langle c_1(K^* \otimes E^2)^2, [X]\rangle - \frac{1}{8}\sigma(X).
\end{aligned}$$

Here we have again used the Hirzebruch signature theorem. This proves the corollary. □

Proof 2: Here is a proof of Corollary 3.43 in the case $E = \mathbb{C}$, which does not rely on Theorem 3.42. Assume that (X, ω) is a symplectic manifold with compatible almost complex structure $J \in \mathcal{J}(X, \omega)$. Consider the operator

$$D : \Omega^{0,1}(X) \to \Omega^{0,0}(X) \oplus \Omega^{0,2}(X)$$

defined by

$$D\tau_1 = (\bar\partial^*\tau_1, \bar\partial\tau_1 - \bar\tau_1 \circ N_J/4)$$

where $N_J : TX \otimes TX \to TX$ denotes the Nijenhuis tensor of J. This is a compact (but not necessarily complex linear) perturbation of $\bar\partial + \bar\partial^* : \Omega^{0,1} \to \Omega^{0,0} \oplus \Omega^{0,2}$. Hence the real Fredholm index of D is

$$\operatorname{index} D = -2\chi(X, \mathcal{O})$$

Moreover, D is isomorphic to the self-duality operator $D^+ = d^* \oplus d^+$ (see Lemma 8.15). There is a commutative diagram

$$\begin{array}{ccc} \Omega^{0,1}(X) & \xrightarrow{D} & \Omega^{0,0}(X) \oplus \Omega^{0,2}(X) \\ \downarrow & & \downarrow \\ \Omega^1(X, i\mathbb{R}) & \xrightarrow{D^+} & \Omega^0(X, i\mathbb{R}) \oplus \Omega^{2,+}(X, i\mathbb{R}) \end{array}$$

where the vertical isomorphisms are given by

$$\tau_1 \mapsto \tau_1 - \bar\tau_1, \qquad (\tau_0, \tau_2) \mapsto (2i\mathrm{Im}\,\tau_0, i(\mathrm{Re}\,\tau_0)\omega + \tau_2 - \bar\tau_2).$$

The commutativity of the diagram can be expressed in the explicit form

$$d^*\beta = 2i\mathrm{Im}\,\bar\partial^*\beta^{0,1}, \quad (d\beta)_\omega = i\mathrm{Re}\,(\bar\partial^*\beta^{0,1}), \quad (d\beta)^{0,2} = \bar\partial\beta^{0,1} + \frac{1}{4}\beta^{1,0} \circ N_J.$$

for $\beta \in \Omega^1(X, i\mathbb{R})$ with $\tau_1 = \beta^{0,1}$. These identities follow from Corollary 3.28 and Proposition 3.16. But the kernel and cokernel of D^+ are $\ker D^+ = H^1(X; i\mathbb{R})$ and $\operatorname{coker} D^+ = H^0(X; i\mathbb{R}) \oplus H^{2,+}(X; i\mathbb{R})$. Hence

$$\chi(X, \mathcal{O}) = -\frac{1}{2}\operatorname{index} D^+ = \frac{1 + b^+ - b_1}{2} = \frac{\chi(X) + \sigma(X)}{4}.$$

This proves the result in the case of the trivial bundle. □

The proof of Corollary 3.43 shows how the Cauchy-Riemann operator can be used to construct a natural orientation

$$\varepsilon_J \in \mathrm{Or}(H^0(X) \oplus H^1(X) \oplus H^{2,+}(X))$$

on the cohomology of an almost complex manifold (X, J). To see this consider the operator family $D_t : \Omega^{0,1}(X) \to \Omega^{0,0}(X) \to \Omega^{0,2}(X)$ defined by $D_t\tau_1 = (\bar\partial^*\tau_1, \bar\partial\tau_1 - t\bar\tau_1 \circ N_J/4)$ for $0 \le t \le 1$. For $t = 0$ this operator is complex linear and hence its kernel and cokernel carry natural complex structures. Trivializing the determinant line bundle over the path $t \mapsto D_t$ of Fredholm operators gives rise to an orientation of $\det(D_1) \cong \det(D^+)$

where the last isomorphism is as in the proof of Corollary 3.43 (see Appendix A). The orientation of $\det(D^+)$ can be interpreted as an orientation of $\ker D^+ \oplus \operatorname{coker} D^+ \cong H^0(X) \oplus H^1(X) \oplus H^{2,+}(X)$ which will be denoted by ε_J.

Let us denote by $\mathcal{J}(X)$ the set of almost complex structures on X. In [16, 17] Donaldson proved that there is an involution

$$\mathcal{J}(X) \to \mathcal{J}(X) : J \mapsto \tilde{J}$$

such that

$$c_1(TX, \tilde{J}) = c_1(TX, J), \qquad \varepsilon_{\tilde{J}} = -\varepsilon_J.$$

This shows, in particular, that $\tilde{J}$ and J lie in different components of $\mathcal{J}(X)$ even though the complex vector bundles (TX, J) and $(TX, \tilde{J})$ are isomorphic. (See Exercises 1.43 and 1.44.) In [15] it was proved by Connolly, Lé Hông, and Ono that if J is compatible with some Kähler form ω then $\tilde{J}$ is not compatible with any Kähler form. Their proof uses the Seiberg-Witten invariants.

3.7 Kähler-Einstein metrics

Let (X, J, ω) be a Kähler manifold and $R \in \Omega^2(X, \operatorname{End}(TX))$ denote the curvature tensor of the Kähler metric. The first statement in Lemma 3.33 asserts that, for all $u, v \in T_xX$, the endomorphism $R(u, v)$ is complex linear and hence is a skew-Hermitian transformation of T_xX. This is an example of the general fact that the curvature of a G-connection is a 2-form with values in the Lie algebra $\mathfrak{g} = \mathrm{Lie(G)}$. The equation $R(u, v) = R(Ju, Jv)$ asserts that the curvature is a form of type $(1, 1)$. (See Remark 3.7.) The condition $S(u, v) = S(Ju, Jv)$ asserts that the **Ricci-form**

$$\rho(v, w) = \rho_\omega(v, w) = S(Jv, w)$$

is skew-symmetric. Moreover, it turns out that this form is always closed and in fact represents the first Chern class of the tangent bundle TX. To see this note that

$$\rho(v, w) = \frac{1}{2}\operatorname{trace}(JR(v, w)) = i\operatorname{trace}^c(R(u, v)).$$

Here trace denotes the real trace of $JR(v, w) \in \operatorname{End}(TX)$. This endomorphism is Hermitian and hence half the real trace agrees with the complex trace of $JR(v, w)$ and hence with $i\operatorname{trace}^c(R(u, v))$. Now we know from the general theory of connections that the 2-form $(i/2\pi)\operatorname{trace}^c(F^\nabla)$ is closed and represents the first Chern class of the bundle for any Hermitian connection ∇. Thus we obtain the following.

Lemma 3.44 *If (X, J, ω) is a Kähler manifold with Ricci-form ρ_ω then $d\rho_\omega = 0$ and the first Chern class of TX is given by $c_1(TX) = [\rho_\omega]/2\pi$.*

A Kähler manifold (X, J, ω) is called a **Kähler-Einstein manifold** if the Kähler metric g is an Einstein metric. This means that the Ricci tensor S is a multiple of the metric tensor g, i.e.

$$S = \lambda g$$

(see Lemma 2.7). Moreover, the constant is given by $\lambda = s/2n$ where X has real dimension $2n$ and s is the scalar curvature, which by Lemma 2.7 is constant in the case $n \geq 2$. Since $\rho_\omega(v, w) = S(Jv, w)$ and $\omega(v, w) = g(Jv, w)$ the identity $S = \lambda g$ can be expressed in the form

$$\rho_\omega = \frac{s}{2n}\omega. \tag{3.17}$$

Example 3.45. (Riemann surfaces) Let Σ be a compact oriented Riemann surface with a complex structure J which is compatible with the orientation. Let g be a Riemannian metric on Σ which is compatible with J and let ω_g be the corresponding volume form. Then (Σ, J, ω_g) is a Kähler manifold of complex dimension 1. By Lemma 3.33, the Ricci tensor S_g satisfies $S_g(u, v) = S_g(Ju, Jv)$ and Example 3.1 shows that S_g is, at each point of Σ, a scalar multiple of the Riemannian metric g. The proof of Lemma 2.7 shows that the factor is half the scalar curvature s_g or, equivalently, the Gauss curvature K_g:

$$S_g = \frac{1}{2}s_g g = K_g g.$$

Note, however, that in dimension 2 this does not imply that the scalar curvature is constant. Example 3.1 also shows that any other metric which is compatible with J lies in the same conformal class as g and, by Theorem 2.20, there is up to scaling a unique metric of constant scalar curvature in each conformal class. Hence every compact oriented Riemann surface admits a Kähler-Einstein metric with constant scalar curvature. □

Example 3.46. (Euclidean space) The standard Kähler structure on $\mathbb{C}^n$ with coordinates $z_1 = x_1 + iy_1, \ldots, z_n = x_n + iy_n$ is given by the Euclidean metric and the standard complex structure i. The corresponding symplectic form

$$\omega_0 = \sum_{j=1}^n dx_j \wedge dy_j$$

can also be expressed in the form

$$\omega_0 = \frac{1}{2i}\sum_{j=1}^n d\bar{z}_j \wedge dz_j.$$

or, equivalently,

$$\omega_0 = \frac{1}{2i}\bar{\partial}\partial f, \qquad f(z) = \sum_{\nu=1}^{n} |z_\nu|^2$$

where

$$\partial = \sum_{j=1}^{n} \frac{\partial}{\partial z_j} dz_j, \qquad \bar{\partial} = \sum_{j=1}^{n} \frac{\partial}{\partial \bar{z}_j} d\bar{z}_j$$

with

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \qquad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

and $dz_j = dx_j + i dy_j$, $d\bar{z}_j = dx_j - i dy_j$. □

Let $f : \mathbb{C}^n \to \mathbb{R}$ be a smooth function. Then the 2-form

$$\omega = \frac{1}{2i}\bar{\partial}\partial f \tag{3.18}$$

on $\mathbb{C}^n$ is real valued and closed. Moreover, it satisfies $\omega(Jv, Jw) = \omega(v, w)$ and hence is of type $(1, 1)$. It is compatible with the standard complex structure if and only if the Hermitian matrix

$$H = \left(\frac{\partial^2 f}{\partial \bar{z}^j \partial z^k}\right)$$

is positive definite (for all $z \in \mathbb{C}^n$). In this case the Kähler metric associated to ω is given by

$$g_\omega(v, w) = \mathrm{Re} \sum_{j,k=1}^{n} \bar{v}^j \frac{\partial^2 f}{\partial \bar{z}^j \partial z^k} w^k$$

for $v, w \in \mathbb{C}^n$.

Lemma 3.47 *The Ricci-form of the Kähler metric g_ω is given by*

$$\rho_\omega = \frac{1}{2i}\bar{\partial}\partial\varphi, \qquad \varphi = -2\log\det\left(\frac{\partial^2 f}{\partial \bar{z}^j \partial z^k}\right).$$

Proof: Kobayashi-Nomizu [57], Volume II, pp.155–158. □

That the sign and the constant are correct in this equation can be checked in the example of the standard 2-sphere (see Example 3.48 below). It follows from Lemma 3.47 that the Kähler form (3.18) gives rise to a Kähler-Einstein metric with factor $\lambda = s/2n$ if and only if

$$2\log\det\left(\frac{\partial^2 f}{\partial \bar{z}^j \partial z^k}\right) + \lambda f = h \tag{3.19}$$

where $h : \mathbb{C}^n \to \mathbb{R}$ satisfies $\bar{\partial}\partial h = 0$ (i.e. h is an affine map). Hence the scalar curvature of the Kähler metric g_ω is given by

$$s_\omega = 2n\lambda \tag{3.20}$$

whenever ω is given by (3.18) and f satisfies (3.19).

Example 3.48. (2-sphere) Identify S^2 with $\mathbb{C} \cup \{\infty\}$ via stereographic projection

$$\mathbb{C} \to S^2 : z \mapsto \left(\frac{2\mathrm{Re}\, z}{|z|^2+1}, \frac{2\mathrm{Im}\, z}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right).$$

The pullback of the standard area form (with total area 4π) under this map is given by

$$\omega = \frac{4dx \wedge dy}{(1+x^2+y^2)^2} = \frac{2d\bar{z} \wedge dz}{i(1+|z|^2)^2} = \frac{1}{2i}\bar{\partial}\partial f$$

where

$$\frac{\partial^2 f}{\partial \bar{z} \partial z} = \frac{4}{(1+|z|^2)^2}$$

A function $f : \mathbb{C} \to \mathbb{R}$ which satisfies this equation is given by

$$f(z) = 4\log(1+|z|^2).$$

Hence

$$2\log\left(\frac{\partial^2 f}{\partial \bar{z} \partial z}\right) + f = 4\log 2$$

and so f satisfies (3.19) with $\lambda = 1$. Hence $s_\omega = 2$. □

Example 3.49. (Complex projective space) A point in $\mathbb{C}P^n$ is a complex line $\ell \subset \mathbb{C}^{n+1}$ or an equivalence class $[z_0 : z_1 : \cdots : z_n]$ of nonzero vectors in $\mathbb{C}^{n+1}$ under the equivalence relation $[z_0 : \cdots : z_n] \equiv [\lambda z_0 : \cdots : \lambda z_n]$ for $\lambda \in \mathbb{C} - \{0\}$. This manifold can be described by $n+1$ coordinate patches

$$U_j = \{[z_0 : z_1 : \cdots : z_n] \,|\, z_j = 1\}.$$

The transition maps are obviously holomorphic and hence the manifold carries a complex structure. A compatible Kähler form is given by the formula

$$\omega_j = \frac{1}{2\pi i}\bar{\partial}\partial f_j, \qquad f_j(z) = \log\left(1 + \sum_{\nu \neq j} |z_\nu|^2\right)$$

over the coordinate patch U_j. ω_j is the restriction of the form

$$\omega = \frac{1}{2\pi i|z|^2}\sum_{j=0}^n d\bar{z}_j \wedge dz_j - \frac{1}{2\pi i|z|^4}\sum_{j=0}^n z_k d\bar{z}_k \wedge \bar{z}_j dz_j$$

on $\mathbb{C}^{n+1} - \{0\}$ to the affine subspace $z_j = 1$. The cohomology class of ω is an integral class with area 1 over the standard 2-sphere $\mathbb{C}P^1 \subset \mathbb{C}P^2$. The corresponding metric on $\mathbb{C}P^n$ is called the **Fubini-Study metric**. The reader may check that the function f_j satisfies the condition (3.19) with $\lambda = 2n+2$. Hence $\mathbb{C}P^n$ is a Kähler-Einstein manifold with positive Ricci tensor. It has scalar curvature

$$s_\omega = 4\pi n(n+1).$$

The additional factor π arises from the factor $1/\pi$ in $\omega_j = \bar{\partial}\partial f_j/2\pi i$. Note that in the case $n = 1$ the standard metric differs by a factor 4π from the Fubini-Study metric on $S^2 \cong \mathbb{C}P^1$. □

For any Kähler manifold (X, J, ω) the Ricci-form ρ_ω is a closed 2-form of type $(1, 1)$ such that $(2\pi)^{-1}\rho_\omega$ represents the first Chern class of TX. In 1957 Calabi first posed the question whether any such form ρ appears as the Ricci form of some Kähler metric. An affirmative answer to this question is a deep theorem in Kähler geometry which is due to Yau.

Theorem 3.50. (Yau) *Let (X, J) be a complex manifold which admits a Kähler metric in the cohomology class $a \in H^2(X;\mathbb{Z})$. Then for every closed 2-form $\rho \in \Omega^2(X)$ such that*

$$\rho(v, w) = \rho(Jv, Jw), \qquad \frac{1}{2\pi}[\rho] = c_1(TX, J)$$

there exists a unique Kähler form ω such that

$$\rho = \rho_\omega, \qquad [\omega] = a.$$

In other words, this theorem asserts that the map $\omega \mapsto \rho_\omega$ is a bijection from the space of Kähler forms representing the class a to the space of $(1, 1)$-forms representing the class c_1. The question of the existence of a Kähler-Einstein metric can now be rephrased as the existence of a fixed point of the projectivization of this map. An obvious necessary condition for the existence of such a metric is that the cohomology class $c_1 = c_1(TX, J)$ is a multiple of the Kähler class $[\omega]$ for some Kähler form. Yau proved that this condition is also sufficient, provided that the factor is nonpositive.

Theorem 3.51. (Yau) *Let (X, J, ω) be a Kähler manifold such that*

$$c_1(TX, J) = \lambda[\omega], \qquad \lambda \le 0.$$

Then X admits a Kähler-Einstein metric.

In particular, this shows that every Kähler manifold with vanishing first Chern class admits a Ricci-flat Kähler metric.

Remark 3.52 The condition $\lambda \le 0$ in Theorem 3.51 cannot be removed. The Kähler surfaces which satisfy $c_1(TX, J) = \lambda[\omega]$ for some $\lambda > 0$ are $\mathbb{C}P^1 \times \mathbb{C}P^1$ and $\mathbb{C}P^2$ with up to eight points blown up. Tian [122] proved that among these only $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ and $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2 \# \overline{\mathbb{C}P}^2$ do not admit Kähler-Einstein metrics. □

3.8 Minimal Kähler surfaces

Let X be a compact smooth 4-manifold. An **exceptional sphere in** X is an embedded 2-sphere with self-intersection number $S \cdot S = -1$. If (X, ω) is a symplectic manifold then a submanifold $S \subset X$ is called an **exceptional symplectic sphere in** X if it is an exceptional sphere and a symplectic submanifold. If (X, J) is a complex surface, then a submanifold $S \subset X$ is called an **exceptional divisor** if it is an exceptional sphere and a holomorphic curve. A complex surface (X, J) is called **minimal** if it does not contain any exceptional divisor. Likewise, a symplectic 4-manifold (X, ω) is called **minimal** if it does not contain any exceptional symplectic spheres.

The significance of these definitions lies in the fact that if $S \subset X$ is an exceptional sphere then there exists a 4-manifold X' and a diffeomorphism

$$X \cong X' \# \overline{\mathbb{C}P}^2$$

which identitfies S with $\mathbb{C}P^1$. Moreover, if X is symplectic (respectively Kähler) and S is symplectic (respectively complex) then X' can be chosen to be symplectic (respectively Kähler). Here is how this works.

Blowing up a point

We describe a construction called **blowing up a point**. The data required for this construction are a smooth 4-manifold X, a point $x_0 \in X$, an open neighbourhood U_r of x_0, and a diffeomorphism $\varphi : U_r \to B_r$, where

$$B_r = \{w \in \mathbb{C}^2 \,|\, |w| < r\}.$$

Denote by $W_r \subset B_r \times \mathbb{C}P^1$ the submanifold

$$W_r = \left\{(w_0, w_1, [z_0 : z_1]) \in B_r \times \mathbb{C}P^1 \,|\, w_0 z_1 = w_1 z_0\right\}.$$

By Exercise 3.55 below, this is a disc bundle in a line bundle $L \to \mathbb{C}P^1$ with Chern number -1. Hence the zero section $E = \{0\} \times \mathbb{C}P^1$ is an exceptional sphere. The projection $\pi_r : W_r \to B_r$ restricts to a diffeomorphism from $W_r - E \to B_r - \{0\}$. The blowup $\widetilde{X} = \widetilde{X}(\varphi)$ of X at x_0 is defined as the quotient

$$\widetilde{X} = (X - \{x_0\}) \cup W_r / \sim,$$

where $x \in X - \{x_0\}$ is equivalent to $(w, \ell) \in W_r$ if and only if $x \in U_r$ and $\varphi(x) = w$. By Exercise 3.54, W_r is diffeomorphic to the complement of a ball in $\mathbb{C}P^2$ and hence

$$\widetilde{X} \cong X \# \overline{\mathbb{C}P}^2.$$

Moreover, there is a natural projection $\pi : \widetilde{X} \to X$ which maps the exceptional sphere $E \subset W_r$ to x_0 and is a diffeomorphism on $\widetilde{X} - E$.

Exercise 3.53 Show that the diffeomorphism type of $\widetilde{X}(\varphi)$ is independent of the choice of φ. □

Exercise 3.54 Prove that the map

$$W_r \to \mathbb{C}P^2 : (w_0, w_1, [z_0 : z_1]) \mapsto [z_0 : z_1 : z_0\bar{w}_0 + z_1\bar{w}_1]$$

is an orientation reversing embedding and that the image is the complement of a ball. **Hint:** $[z_0 : z_1 : z_0\bar{w}_0 + z_1\bar{w}_1] = [w_0 : w_1 : |w_0|^2 + |w_1|^2]$. □

Exercise 3.55 Let $L \to \mathbb{C}P^1$ be the line bundle whose fibre over $\ell \in \mathbb{C}P^1$ is the line ℓ itself. Prove that $\langle c_1(L), [\mathbb{C}P^1] \rangle = -1$. **Hint:** Find a section with precisely one nondegenerate zero. □

Proposition 3.56 **(i)** *Let (X, J) be a complex surface and $\varphi : U_r \to B_r$ be a holomorphic coordinate chart. Then $\widetilde{X}(\varphi)$ admits a unique complex structure $\widetilde{J}$ such that E is an exceptional divisor and the projection $\pi : \widetilde{X} \to X$ is holomorphic.*

(ii) *Let (X, ω) be a symplectic 4-manifold and $\varphi : U_r \to B_r$ be a Darboux chart. Fix a real number $\lambda \in (0, r)$. Then $\widetilde{X}(\varphi)$ admits a symplectic structure $\tilde{\omega}_\lambda$ such that E is an exceptional symplectic sphere of area $\pi\lambda^2$ and the projection $\pi : \widetilde{X} - \widetilde{U}_r \to X - U_r$ is a symplectomorphism.*

(iii) *Let (X, J, ω) be a Kähler surface and $\varphi : U_r \to B_r$ be a holomorphic coordinate chart. Then, for $\lambda \in (0, r)$ sufficiently small, $\widetilde{X}(\varphi)$ admits a Kähler form $\tilde{\omega}_\lambda$ which is compatible with $\widetilde{J}$ such that E has area $\pi\lambda^2$ and the projection $\pi : \widetilde{X} - \widetilde{U}_r \to X - U_r$ is a Kähler isomorphism.*

Proof 1: To prove (i) note that W_r is a complex submanifold of $B_r \times \mathbb{C}P^1$ and the projection $\pi_r : W_r \to B_r$ is holomorphic. Hence $\widetilde{X}(\varphi)$ admits a complex structure $\widetilde{J}$ which is equal to J on $X - \{x_0\}$ and equal to i on W_r.

We prove (ii) for $\lambda > 0$ sufficiently small. Let (X, ω) be a symplectic manifold and $\pi : \widetilde{X} \to X$ be a smooth blowup of X at x_0. Choose a tubular neighbourhood $\widetilde{U} \subset \widetilde{X}$ of the exceptional divisor E and denote by $p : \widetilde{U} \to E$ the projection. Let τ be a symplectic form on E of area π. Since

$\widetilde{U} - E$ is diffeomorphic to the punctured 4-ball, the restriction of $p^*\tau$ to $\widetilde{U} - E$ is exact. Hence there exists a 1-form $\sigma \in \Omega^1(\widetilde{U} - E)$ such that

$$p^*\tau|_{\widetilde{U}-E} = d\sigma.$$

Let $\beta : \widetilde{U} \to [0, 1]$ be a smooth cutoff function which is equal to 1 near E and equal to 0 near $\partial\widetilde{U}$. Then the 2-form

$$\widetilde{\omega} = \pi^*\omega + \lambda^2 d(\beta\sigma)$$

is nondegenerate for $\lambda > 0$ sufficiently small.

To prove (iii) let (X, J, ω) be a Kähler surface and $\pi : (\widetilde{X}, \widetilde{J}) \to (X, J)$ a complex blowup of X at x_0. Let $p : \widetilde{U} \to E$ be a holomorphic projection of a tubular neighbourhood and $\tau \in \Omega^2(E)$ be a symplectic form of area π. In dimension 2 any symplectic form is compatible with any complex structure. Hence $p^*\tau$ is of type (1,1). Hence there exists a function $h : \widetilde{U} - E \to \mathbb{R}$ such that $p^*\tau|_{\widetilde{U}-E} = i\partial\bar{\partial}h$. Let β be as above. Then the 2-form

$$\widetilde{\omega} = \pi^*\omega + \lambda^2 i\partial\bar{\partial}(\beta h)$$

is nondegenerate and compatible with $\widetilde{J}$ for $\lambda > 0$ sufficiently small. □

Proof 2: Following McDuff et al [83] we prove (ii) for every $\lambda < r$. Let (X, ω) be a symplectic manifold and suppose that $\varphi : (U_r, \omega) \to (B_r, \omega_0)$ is a Darboux chart. Fix a positive real number $\lambda < r$ and consider the symplectic form ω_λ on $\mathbb{C}^2 - \{0\}$ given by

$$\begin{aligned}\omega_\lambda &= \frac{i}{2}\partial\bar{\partial}(|w|^2 + \lambda^2 \log |w|^2)\\ &= \frac{i}{2}\left(\left(1 + \frac{\lambda^2}{|w|^2}\right)\sum_{j=0}^{1} dw_j \wedge d\bar{w}_j - \frac{\lambda^2}{|w|^4}\sum_{j,k=0}^{1} \bar{w}_j w_k dw_j \wedge d\bar{w}_k\right).\end{aligned}$$

We leave it as an exercise to prove that ${\pi_\delta}^*\omega_\lambda$ extends to a symplectic form on W_δ for any $\delta > 0$. Choose $\delta = \sqrt{r^2 - \lambda^2}$ and define the diffeomorphism $f_\lambda : B_\delta - \{0\} \to B_r - \mathrm{cl}(B_\lambda)$ by $f_\lambda(w) = \sqrt{1 + \lambda^2/|w|^2}w$. Then ${f_\lambda}^*\omega_0 = \omega_\lambda$ (Exercise 3.57). Consider the manifold

$$\widetilde{X}_\lambda = (X - \mathrm{cl}(U_\lambda)) \cup W_\delta/\sim,$$

where $x \in X - \mathrm{cl}(U_\lambda)$ is equivalent to $(w, \ell) \in W_\delta$ if and only if $\lambda < |\varphi(x)| < r$ and $\varphi(x) = f_\lambda(w)$. This manifold admits a symplectic form $\widetilde{\tau}_\lambda$ which is equal to ω on $X - \mathrm{cl}(U_\lambda)$ and equal to ω_λ on W_δ. Geometrically, $\widetilde{X}_\lambda$ is obtained from X by removing an open ball of radius λ and forming the quotient of the boundary by the standard circle action.

We construct a diffeomorphism $\widetilde{f} : \widetilde{X} \to \widetilde{X}_\lambda$ such that composition $\pi \circ \widetilde{f}^{-1} : \widetilde{X}_\lambda \to X$ is a symplectomorphism on $X - U_r$. Let $0 < \varepsilon < \delta$ and $\alpha : [0, r] \to [\lambda, r]$ be a smooth function such that

$$\alpha(t) = \begin{cases} \sqrt{\lambda^2 + t^2}, & \text{for } 0 \le t \le \varepsilon, \\ t, & \text{for } t \text{ near } r, \end{cases} \qquad \alpha'(t) > 0,$$

and define $f : B_r - \{0\} \to B_r - \mathrm{cl}(B_\lambda)$ by $f(w) := \alpha(|w|)w/|w|$. The required diffeomorphism $\widetilde{f} : \widetilde{X} \to \widetilde{X}_\lambda$ is given by

$$\widetilde{f}(x) := \begin{cases} x, & \text{for } x \in X - U_r, \\ \varphi^{-1} \circ f \circ \varphi(x), & \text{for } x \in U_r - \{x_0\}, \end{cases}$$

for $x \in X - \{x_0\}$ and by $\widetilde{f}(w, \ell) = ({f_\lambda}^{-1} \circ f(w), \ell)$ for $(w, \ell) \in W_\delta$. The symplectic structure on $\widetilde{X}$ is given by

$$\widetilde{\omega}_\lambda := \widetilde{f}^* \widetilde{\tau}_\lambda \in \Omega^2(\widetilde{X}).$$

On $X - U_r$ this form agrees with ω and on $U_r - \{x_0\}$ it is given by $\widetilde{\omega}_\lambda = \varphi^* f^* \omega_0$. Since $f|_{B_\varepsilon} = f_\lambda$ it follows that $\widetilde{\omega} = \varphi^* \omega_\lambda$ on U_ε and hence $\widetilde{\omega}$ agrees with ω_λ on W_ε. Hence $E = \{0\} \times \mathbb{C}P^1$ is an exceptional symplectic sphere.

Now suppose that (X, J, ω) is a Kähler manifold. Choose holomorphic coordinates $\varphi : (U, J) \to (B, i)$ near x_0 such that ω agrees with $\varphi^* \omega_0$ at x_0. By Exercise 3.59, ω can be deformed within its cohomology class to a Kähler form (still denoted by ω) which agrees with $\varphi^* \omega_0$ near x_0 and with ω outside a small neighbourhood of x_0. Hence we may assume that $\varphi : (U_r, \omega, J) \to (B_r, \omega_0, i)$ is a Kähler isomorphism for some $r > 0$. Under this assumption it follows from Exercise 3.58 that the above form $\widetilde{\omega}_\lambda$ is compatible with $\widetilde{J}$. Hence $(\widetilde{X}, \widetilde{J}, \widetilde{\omega}_\lambda)$ is a Kähler surface and E is an exceptional divisor. □

Exercise 3.57 Prove that ω_λ is compatible with i. Prove that $\pi^* \omega_\lambda$ extends to a symplectic form on W and that ${f_\lambda}^* \omega_0 = \omega_\lambda$. □

Exercise 3.58 Let $\alpha : (0, \infty) \to (0, \infty)$ be a smooth function such that $\alpha'(t) > 0$ for every $t > 0$. Consider the function $f : \mathbb{R}^{2n} - \{0\} \to \mathbb{R}^{2n} - \{0\}$, defined by $f(z) = \alpha(|z|)z/|z|$. Prove that $f^* \omega_0$ is compatible with J_0. □

Exercise 3.59 Let $\omega \in \Omega^2(\mathbb{C}^n)$ be a symplectic form that is compatible with i and agrees with ω_0 at the origin. Prove that there exists a smooth function $h : \mathbb{C}^n \to \mathbb{R}$ which vanishes up to second order at $z = 0$ and satisfies $\omega = \omega_0 + i\partial\bar{\partial}h$. Let $\beta : \mathbb{C}^n \to \mathbb{R}$ be a smooth cutoff function which vanishes in the unit ball and is equal to 1 outside the ball of radius 2. For $r > 0$ define $\beta_r(z) = \beta(z/r)$. Prove that, for $r > 0$ sufficiently small, the 2-form $\omega_r = \omega_0 + i\partial\bar{\partial}(\beta_r h)$ is nondegenerate, compatible with i, and agrees with ω_0 on the ball of radius r and with ω outside the ball of radius $2r$. □

Blowing down an exceptional divisor

The next result is the converse of Proposition 3.56. It asserts that every smooth 4-manifold which contains an exceptional sphere is diffeomorphic to the blowup of some 4-manifold X'. Moreover, if X carries a complex, symplectic, or Kähler structure then so does X'. In the smooth case the result is obvious and in the symplectic case it was noted by Gromov [46]. The proof in the complex case relies on the *Grauert criterion* [42, 43, 8]. Grauert's theorem extends the *Castelnuovo–Enriques criterion* which only applies to the algebraic case [45, page 476].

Theorem 3.60 *Let X be a compact smooth 4-manifold and $S \subset X$ be an exceptional sphere.*

(i) *There exists a smooth 4-manifold X', a point $x_0' \in X'$, and a projection $\pi : X \to X'$ such that $\pi^{-1}(x_0') = S$ and the restriction $\pi : X - S \to X' - \{x_0'\}$ is a diffeomorphism.*

(ii) *If (X, J) is a complex manifold and S is a complex submanifold then X' admits a complex structure J' and π can be chosen holomorphic.*

(iii) *If (X, ω) is a symplectic manifold and S is an exceptional symplectic sphere then X' admits a symplectic form ω' and π can be chosen to be a symplectomorphism outside an arbitrarily small neighbourhood of S.*

(iv) *Let (X, J, ω) be a Kähler surface, $S \subset X$ be an exceptional divisor, and $\pi : (X, J) \to (X', J')$ be a holomorphic projection as in (ii). Then there exists a Kähler form ω' on (X', J') such that π is a Kähler isomorphism outside an arbitrarily small neighbourhood of S.*

Proof: The normal bundle of S has Euler number -1. Hence, by Exercise 3.55, there exists a tubular neighbourhood V_δ of S and a diffeomorphism $\psi : V_\delta \to W_\delta$ which identifies $S \subset V_\delta$ with $E = \{0\} \times \mathbb{C}P^1 \subset W_\delta$. We define

$$X' = (X - S) \cup B_\delta / \sim,$$

where $x \in X - S$ is equivalent to $w \in B_\delta$ if and only if $x \in V_\delta$ and $\pi_\delta(\psi(x)) = w$. The manifold X' is obtained from X by cutting out the tubular neighbourhood V_δ of S and replacing it with a copy of the open unit ball $B_\delta \subset \mathbb{C}^2$. The original manifold X can be reconstructed by cutting out the ball B_δ and replacing it with a copy of W_δ. Moreover, there is an obvious projection $\pi : X \to X'$ which maps $X - S$ diffeomorphically to $X - \{x_0'\}$, where $x_0' = 0 \in B_\delta \subset X'$. This proves (i).

We prove (ii). It follows from the Grauert criterion [8, Thms 2.1 and 4.1] that for every exceptional divisor in a complex surface there exist a neighbourhood V_δ and a holomorphic diffeomorphism $\psi : (V_\delta, J) \to (W_\delta, i)$ which sends S to E. Hence X' admits a complex structure J' which agrees with J on $X - S$ and with i on B_δ. With this complex structure the projection $\pi : X \to X'$ is holomorphic.

The proof of (iii) relies on the symplectic neighbourhood theorem [85, Theorem 3.30]. In the present case the theorem asserts that, for every exceptional symplectic sphere S of area $\pi\lambda^2$, there exist a $\delta > 0$, a neighbourhood V_δ of S, and a symplectomorphism $\psi : (V_\delta, \omega) \to (W_\delta, \omega_\lambda)$ which identifies S with E. Let $r = \sqrt{\lambda^2 + \delta^2}$ and define

$$X'_\lambda = (X - S) \cup B_r / \sim,$$

where $x \in X - S$ is equivalent to $z \in B_r$ if and only if $x \in V_\delta$, $\lambda < |z| < r$, and $f_\lambda \circ \pi_\delta \circ \psi(x) = z$. Then X'_λ admits a symplectic form τ'_λ which is equal to ω on $X - S$ and to ω_0 on B_r. Note that X'_λ is obtained from X by replacing the exceptional sphere S with the ball B_λ while X' is obtained from X by replacing S with a single point x'_0.

We construct a diffeomorphism $f' : X' \to X'_\lambda$ such that the composition $f' \circ \pi : X \to X'_\lambda$ is a symplectomorphism on $X - V_\delta$. Let $0 < \varepsilon < \delta$ and $\beta : [0, \delta] \to [0, r]$ be a smooth function such that

$$\beta(t) = \begin{cases} \lambda t/\varepsilon, & \text{for } 0 \le t \le \varepsilon, \\ \sqrt{\lambda^2 + t^2}, & \text{for } t \text{ near } \delta, \end{cases} \qquad \beta'(t) > 0.$$

Define $g : B_\delta \to B_r$ by

$$g(w) := \frac{\beta(|w|)}{|w|} w.$$

The required diffeomorphism $f' : X' \to X'_\lambda$ is then given by

$$f'(x) := \begin{cases} x, & \text{for } x \in X - V_\delta, \\ (\pi_\delta \circ \psi)^{-1} \circ {f_\lambda}^{-1} \circ g \circ (\pi_\delta \circ \psi)(x), & \text{for } x \in V_\delta - V_\varepsilon, \end{cases}$$

for $x \in X - V_\varepsilon$ and by $f'(w) := g(w)$ for $w \in B_\delta$. This diffeomorphism identifies the ball $B_\varepsilon \subset X'$ with $B_\lambda \subset X'_\lambda$. The symplectic structure on X' is given by

$$\omega'_\lambda := {f'}^* \tau'_\lambda \in \Omega^2(X').$$

This form agrees with ω on $X - V_\delta$ and with $(\pi_\delta \circ \psi)^* g^* \omega_0$ on $V_\delta - V_\varepsilon$. On $B_\delta - B_\varepsilon$ it is given by $g^*\omega_0$ and on B_ε by $\lambda\omega_0/\varepsilon$.

If (X, J, ω) is a Kähler manifold and S is an exceptional divisor then, by (ii), X' admits a complex structure J' such that the projection $\pi : X \to X'$ is holomorphic. In [94] Miyaoka proved that for every Kähler form ω on the punctured ball $B - \{0\}$ there exists a Kähler form ω' on B which agrees with ω near the boundary. Hence there exists a Kähler form ω' on (X', J') such that π is a Kähler isomorphism outside a neighbourhood of the exceptional divisor. This proves the theorem. □

Remark 3.61 In [42] Grauert proved that a system of rational curves $C_1, \ldots, C_n$ in a complex surface X can be blown down if and only if the intersection matrix with entries $C_i \cdot C_j$ is negative definite. In general the reduced variety is singular, but in the case $m = 1$ and $C_1 \cdot C_1 = -1$ it is smooth. □

Remark 3.62 The work of Kodaira [58], Miyaoka [95], and Siu [112] shows that a complex surface (X, J) admits a Kähler metric if and only if $b_1(X)$ is even. By Proposition 3.56 it suffices to consider minimal complex surfaces. Kodaira's classification theorem [58] implies that every minimal complex surface with even first Betti number is either elliptic or a K3-surface or a complex torus or algebraic. Miyaoka [95] proved that every elliptic surface with even first Betti number is Kähler and Siu [112] proved that every complex K3-surface is Kähler. Complex tori and algebraic surfaces obviously admit Kähler forms. Putting these result together one obtains that every minimal complex surface with even first Betti number admits a Kähler form. □

Minimal Kähler surfaces

Let us now examine minimal Kähler surfaces. Note, for example, that every spin Kähler surface is minimal. Throughout we denote by $K = \Lambda^{2,0}T^*X$ the **canonical bundle** of X. Note that $c_1(K) = -c_1(TX)$.

Proposition 3.63 Every minimal Kähler surface X with $b^+ > 1$ satisfies

$$c_1(K) \cdot c_1(K) \geq 0.$$

Proof: Assume, by contradiction, that $c_1(K) \cdot c_1(K) < 0$. Since $b^+ > 1$ there is a non-zero holomorphic 2-form and hence the canonical bundle K has a non-zero holomorphic section. Let $s : X \to K$ be such a section and consider its divisor $D = \sum_i m_i V_i$. Each irreducible component V_i is the image of some nonconstant holomorphic curve $u_i : \Sigma_i \to X$ defined on a compact connected Riemann surface (see Appendix F). The first Chern class of K is given by

$$c_1(K) = \sum_i m_i \mathrm{PD}([V_i]).$$

By assumption, $c_1(K) \cdot c_1(K) < 0$ and hence the bundle is nontrivial. This implies that at least one of the curves V_i is nonempty. Suppose first that each curve is immersed with d_i regular double points. Any such curve satisfies the adjunction formula

$$c_1(K) \cdot V_i = 2g_i - 2 - V_i \cdot V_i + 2d_i \tag{3.21}$$

where g_i is the genus of Σ_i. To see this note that the pullback bundle ${u_i}^*TX$ splits into the direct sum of the normal bundle and the tangent bundle. The first Chern number of the normal bundle is given by $c_1(\nu_j) = V_j \cdot V_j - 2d_j$ and the Chern class of the tangent bundle by $2-2g_i$. Their sum is the Chern number of $u_i{}^*TX$ and, since $c_1(TX) = -c_1(K)$, this number is $-c_1(K) \cdot V_i$. This proves (3.21).

Since $c_1(K) = \sum_i m_i \mathrm{PD}([V_i])$ and $c_1(K) \cdot c_1(K) < 0$ it follows that $c_1(K) \cdot V_i < 0$ for some i. Since $V_i \cdot V_j \geq 0$ for $i \neq j$ this implies $V_i \cdot V_i < 0$. For this value of i the right hand side of (3.21) can only be negative if

$$g_i = 0, \qquad V_i \cdot V_i = -1, \qquad d_i = 0.$$

This means that V_i is an *exceptional divisor* and hence X is not minimal in contradiction to our assumption. This proves the proposition in the case where the V_i are all immersed. The general case can be reduced to this by a generic perturbation of the complex structure to an almost complex structure. By a theorem of Nijenhuis and Wolf there exists such a perturbation after which the classes $[V_i]$ are represented by immersed pseudo-holomorphic curves with regular double points. The important point to note is that each singularity will contribute a positive number of double points and hence the number d_i can only be zero if the original curve V_i was already embedded. The details of this argument will not be carried out. (See [89] and [79] for singularities on pseudo-holomorphic curves.) □

Example 3.64 The assumptions in Proposition 3.63 that X be minimal and $b^+ > 1$ cannot be removed. Consider for example the product

$$X = \Sigma_1 \times \Sigma_2$$

of two Riemann surfaces of genus g_1 and g_2, respectively. This manifold has Betti-numbers $b_1 = 2(g_1 + g_2)$ and $b_2 = 2 + 4g_1 g_2$ and signature zero. The intersection form is even and hence the manifold is spin. By the Hirzebruch signature theorem, $c_1 \cdot c_1 = 2\chi + 3\sigma = 2(2 - 2g_1)(2 - 2g_2)$. This number is negative whenever $g_1 = 0$ and $g_2 > 1$ and in this case we have $b^+ = b^- = 1$. That the assumption of minimality is necessary follows by considering the formulae

$$\chi(X \# \overline{\mathbb{C}P}^2) = \chi(X) + 1, \qquad \sigma(X \# \overline{\mathbb{C}P}^2) = \sigma(X) - 1.$$

This shows that blowing up a point decreases the number $c_1(K) \cdot c_1(K) = 2\chi(X) + 3\sigma(X)$ by 1 and so, after blowing up sufficiently many points, this number will eventually become negative. For surfaces with $c_1(K) = 0$, such as the 4-torus and the K3-surface, blowing up a single point suffices. □

Remark 3.65 A minimal Kähler surface (X, J, ω) is said to be of general type if the canonical class $K = -c_1(TX, J)$ satisfies

$$K \cdot K > 0, \qquad K \cdot \omega > 0.$$

(compare with Proposition 3.63). For such surfaces the following are equivalent.

(i) There is no embedded holomorphic sphere S with $S \cdot S = -2$.

(ii) (X, J) admits a Kähler form ω such that $c_1 = \lambda[\omega]$ for some $\lambda < 0$.

(iii) (X, J) admits a Kähler-Einstein metric.

That (i) implies (iii) was proved by Yau and the implications (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) are easy exercises. □

3.9 Hypersurfaces in projective space

Consider the hypersurface of degree d in $\mathbb{C}P^3$

$$X_d = \left\{ [z_0 : \cdots : z_3] \in \mathbb{C}P^3 \,\Big|\, \sum_{j=0}^{3} {z_j}^d = 0 \right\}.$$

This is a smooth manifold and it follows from the Lefschetz hyperplane theorem that $X = X_d$ is simply connected. Hence the second homology is generated by $\pi_2(X)$. The second homology splits into the negative and positive parts under the intersection form and their dimensions are given in the following proposition. The proof is due to Milnor [91].

Proposition 3.66 *The second Betti number of the hypersurface $X_d \subset \mathbb{C}P^3$ is given by*

$$b_2 = d^3 - 4d^2 + 6d - 2 \tag{3.22}$$

and the intersection form Q_{X_d} has signature

$$\mathrm{sign}(Q_{X_d}) = \frac{1}{3}(4 - d^2)d. \tag{3.23}$$

Moreover, the first and second Chern classes of TX are given by

$$c_1(TX) = (4 - d)\iota^* h, \qquad c_2(TX) = (6 - 4d + d^2)\iota^* h^2. \tag{3.24}$$

where $h \in H^2(\mathbb{C}P^3; \mathbb{Z})$ is the canonical generator of H^2 and $\iota : X_d \to \mathbb{C}P^3$ denotes the inclusion.

Proof: The proof relies on the following observations.

(i) The canonical generator of $H^2(\mathbb{C}P^3; \mathbb{Z})$ is the first Chern class

$$h = c_1(H) \in H^2(\mathbb{C}P^3; \mathbb{Z})$$

of the canonical line bundle H whose fiber over $\ell \in \mathbb{C}P^3$ is the space $H_\ell = \ell^*$ of complex linear functionals on ℓ. To see this fix a nonzero vector $w \in \mathbb{C}^4$ and consider the section $s : \mathbb{C}P^3 \to H$ which assigns to $\ell \in \mathbb{C}P^3$ the restriction of the linear functional $v \mapsto \bar{w}^T v$ to ℓ. This section is transverse to the zero section and its zero set is a copy of $\mathbb{C}P^2$ in $\mathbb{C}P^3$. Hence h is the Poincaré dual of the hyperplane class

$$[\mathbb{C}P^2] = \mathrm{PD}(h) \in H_4(\mathbb{C}P^3; \mathbb{Z}).$$

Note also that h agrees, up to a **positive** factor, with the cohomology class of the standard symplectic structure on $\mathbb{C}P^3$, defined in Example 3.49.
(ii) The tangent bundle of $\mathbb{C}P^3$ satisfies

$$T\mathbb{C}P^3 \oplus \mathbb{C} \simeq H^4 = H \oplus H \oplus H \oplus H.$$

To see this note that $T_\ell \mathbb{C}P^3 \cong \mathrm{Hom}(\ell, \ell^\perp)$ and $\mathbb{C} \cong \mathrm{Hom}(\ell, \ell)$.
(iii) The normal bundle ν_X is a complex line bundle over X and its first Chern class satisfies

$$c_1(\nu_X) = d\iota^* h.$$

where $\iota : X_d \to \mathbb{C}P^3$ denotes the natural embedding. To see this let $\Sigma \subset X$ be a submanifold such that $\iota(\Sigma) \subset \mathbb{C}P^3$ represents a 2-dimensional homology class of degree k. Then the intersection number of X and Σ is $X \cdot \Sigma = dk$. Now the intersection number is also the oriented number of zeros of a generic section $s : X \to \nu_X$ when restricted to Σ. This means that $\langle c_1(\nu_X), [\Sigma] \rangle = dk = d\langle \iota^* h, [\Sigma] \rangle$.
(iv) The cohomology class $\iota^* h^2 \in H^4(X; \mathbb{Z})$ is given by

$$\langle \iota^* h^2, [X] \rangle = d.$$

To see this note that h^2 is the generator of $H^4(\mathbb{C}P^3; \mathbb{Z})$ and its Poincaré dual is a line. Any such line intersects X in d points, counted with multiplicity.

With these preparations in place we consider the tangent bundle of $\mathbb{C}P^3$ restricted to the submanifold $X = X_d$. This bundle splits as $T_X \mathbb{C}P^3 = TX \oplus \nu_X$ and hence

$$c(T_X \mathbb{C}P^3) = c(TX) c(\nu_X).$$

where $c = 1 + c_1 + c_2$ denotes the total Chern class. By (ii) we have $c(T_X \mathbb{C}P^3) = (1 + \iota^* h)^4$ and, by (iii), $c(\nu_X) = 1 + d\iota^* h$. Hence

$$(1 + \iota^* h)^4 = (1 + c_1(TX) + c_2(TX))(1 + d\iota^* h).$$

Solving this equation, first for c_1 and then for c_2, we obtain (3.24). Now use (iv) and the fact that $\langle c_2(TX), [X] \rangle = 2 + b_2$ is the Euler characteristic to obtain (3.22). Finally, (3.23) follows from the Hirzebruch signature formula (1.9). $\square$

The equations (3.22) and (3.23) together show that the positive and negative parts of Q_X are of dimension

$$b^+(X) = \frac{1}{3}(d^3 - 6d^2 + 11d - 3), \qquad b^-(X) = \frac{1}{3}(2d^3 - 6d^2 + 7d - 3).$$

Since both numbers are positive (unless $d = 1$) the form Q_X is indefinite. Now for any simply connected Kähler surface X the first Chern class of TX satisfies

$$\langle c_1(TX), \alpha \rangle = Q_X(\alpha, \alpha) \pmod 2$$

for $\alpha \in H_2(X; \mathbb{Z})$. In view of (3.24) this means that Q_X is **even** (i.e. $Q_X(\alpha, \alpha)$ is even for all $\alpha \in H_2(X)$) if and only if d is even. Hence the classification theorem of indefinite quadratic forms (cf. [92]) shows that the intersection form of X_d, in the odd case, is diagonalizable and so agrees with that of

$$X'_d = \ell\mathbb{C}P^2 \# m\overline{\mathbb{C}P}^2, \qquad \ell = b^+(X), \qquad m = b^-(X).$$

We shall see below that these manifolds have different Seiberg-Witten invariants for $d \geq 4$ and hence are not diffeomorphic. Moreover, these invariants show that the manifold X_d does not admit a metric of positive scalar curvature for $d \geq 4$ while, by Theorem 2.18, the manifold X'_d does admit such a metric. The case $d = 3$ is an exception. In this case one can prove that the manifold X_3 is in fact diffeomorphic to $\mathbb{C}P^2 \# 6\overline{\mathbb{C}P}^2$.

It is interesting to distinguish the cases $d \leq 3$, $d = 4$, and $d \geq 5$. The manifolds

$$X_1 = \mathbb{C}P^2, \qquad X_2 = S^2 \times S^2, \qquad X_3 = \mathbb{C}P^2 \# 6\overline{\mathbb{C}P}^2$$

are **positive** in the sense that the cohomology class $[\omega]$ of the restriction of the Fubini–Study form is a positive multiple of c_1. Such manifolds are also called **Fano varieties**. The manifold X_4 is a compact, connected, simply connected 4-dimensional Kähler manifold whose first Chern class vanishes. All 4-manifolds with these properties are diffeomorphic and they are called $K3$**-surfaces**. Their second Betti number is $b_2 = 22$. $K3$-surfaces have played an important role in 4-dimensional topology [21]. The manifolds X_d for $d \geq 5$ are surfaces of **general type**. For all these manifolds the first Chern class of TX_d is a negative multiple of the cohomology class $\iota^* h = [\omega]$ of the standard symplectic structure. Note, in particular, that for $d \geq 4$ the manifold X_d satisfies the assumptions of Yau's theorem 3.51 and hence admits a Kähler-Einstein metric. The manifolds X_1 and X_2 admit Kähler-Einstein metrics for obvious reasons. Kähler Einstein metrics on X_3 were found by Tian [122].

Part II

SPIN GEOMETRY
AND DIRAC OPERATORS

4

SPIN GEOMETRY

In this chapter the development of the theory that leads up to the Seiberg-Witten invariants begins in earnest with the discussion of the spin and spin^c groups of a real vector space. As a warmup Section 4.1 deals with the spin groups in dimensions 3 and 4. Section 4.2 puts these groups in the more general context of the real and complex Clifford algebras $C(V)$ and $C^c(V)$ of an oriented real inner product space V. In Section 4.3 the groups $\mathrm{Spin}(V)$ and $\mathrm{Spin}^c(V)$ are defined and their fundamental properties discussed. Section 4.4 introduces (irreducible) spin^c representations as certain linear maps $\Gamma : V \to \mathrm{End}(W)$ which extend to algebra isomorphisms from the complexified Clifford algebra to $\mathrm{End}(W)$. Section 4.5 discusses the canonical splitting $W = W^+ \oplus W^-$ of a spin^c representation. Section 4.6 introduces spin structures as spin^c structures together with a complex anti-linear automorphism of W which commutes with Γ and is either an involution or another complex structure, depending on the dimension of V. Section 4.7 discusses the canonical spin^c structure $W_{\mathrm{can}} = \Lambda^{0,*}V^*$ in the case where the underlying vector space V itself carries a complex structure. The final section 4.8 examines the action of the exterior algebra of V on the spin^c representation W via its identification with the Clifford algebra. Here a special emphasis is placed on the 4-dimensional case and on the canonical spin^c structure in the complex case.

To begin with let V be a finite dimensional oriented real Hilbert space of dimension $\dim V \geq 3$. Then the group $\mathrm{SO}(V)$ of orientation preserving orthogonal transformations has fundamental group $\pi_1(\mathrm{SO}(V)) = \mathbb{Z}_2$. Its universal cover is called $\mathrm{Spin}(V)$ and there is an exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(V) \longrightarrow \mathrm{SO}(V) \longrightarrow 1.$$

The group $\mathrm{Spin}^c(V)$ is defined by $\mathrm{Spin}^c(V) = \mathrm{Spin}(V) \times_{\mathbb{Z}_2} S^1$. This group is a circle extension of $\mathrm{SO}(V)$ and there is an exact sequence

$$1 \longrightarrow S^1 \longrightarrow \mathrm{Spin}^c(V) \longrightarrow \mathrm{SO}(V) \longrightarrow 1.$$

To understand these groups better it is useful to think of them as subgroups of the Clifford algebra $C(V)$ or the complexified Clifford algebra $C^c(V)$ of V. This will be discussed in Section 4.3.

4.1 Spin groups in dimensions three and four

The group SO(3) is naturally diffeomorphic to the unit tangent bundle of the 2-sphere. The diffeomorphism sends an orthogonal matrix to the first two columns. Now the unit tangent bundle of the 2-sphere is diffeomorphic to $\mathbb{R}P^3$ and hence the universal cover of SO(3) is the 3-sphere. Geometrically, the nontrivial element in $\pi_1(\mathrm{SO}(3))$ can be realized as a full rotation around one axis. Explicitly, the universal cover can be described in terms of the unit quaternions.

Quaternions

Identify $\mathbb{R}^4$ with the quaternions $\mathbb{H}$ via $x = x_0 + ix_1 + jx_2 + kx_3$ where the multiplication rules are

$$i^2 = j^2 = k^2 = -1, \quad jk = -kj = i, \quad ki = -ik = j, \quad ij = -ji = k.$$

The quaternions form an algebra over the reals with unit 1. Conjugation defines a natural involution $x \mapsto \bar{x} = x_0 - ix_1 - jx_2 - kx_3$ which satisfies

$$\overline{xy} = \bar{y}\bar{x}, \qquad \bar{x}x = |x|^2 = \sum_{\nu=0}^{3} {x_\nu}^2.$$

The unit quaternions form a group

$$\mathrm{Sp}(1) = \{x \in \mathbb{H} \mid |x| = 1\}$$

whose Lie algebra $\mathfrak{sp}(1) = \mathrm{Lie}(\mathrm{Sp}(1)) = \mathrm{Im}(\mathbb{H})$ consists of the imaginary quaternions. The standard orientation of Sp(1) is determined by the basis i, j, k of $\mathfrak{sp}(1)$.

There is a natural embedding $\gamma : \mathbb{H} \to \mathbb{C}^{2\times 2}$ given by

$$\gamma(x) = \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix}. \tag{4.1}$$

This map is obviously linear with $\gamma(1) = \mathbb{1}$ and the matrices $I = \gamma(i)$, $J = \gamma(j)$, $K = \gamma(k)$ are given by

$$I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{4.2}$$

These matrices satisfy the same commutation relations as i, j, k and hence

$$\gamma(xy) = \gamma(x)\gamma(y), \qquad \gamma(\bar{x}) = \gamma(x)^*. \tag{4.3}$$

This shows that γ is an algebra homomorphism.

Exercise 4.1 Prove that the center of $\mathbb{H}$ is given by $Z(\mathbb{H}) = \{1, -1\}$. □

Exercise 4.2 Prove that γ induces a Lie group isomorphism $\mathrm{Sp}(1) \to \mathrm{SU}(2)$ and a Lie algebra isomorphism $\mathfrak{sp}(1) = \mathrm{Im}(\mathbb{H}) \to \mathfrak{su}(2)$. □

Exercise 4.3 Prove that $\gamma(\mathbb{H}) = \mathfrak{su}(2) \oplus \mathbb{R}\mathbb{1} = \{tU \,|\, t \in \mathbb{R},\, U \in \mathrm{SU}(2)\}$. Prove that γ extends to an algebra isomorphism $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C}^{2\times 2}$. □

The spin group in dimension three

The next lemma shows that that $\mathrm{Sp}(1) \cong S^3$ is naturally isomorphic to $\mathrm{Spin}(3)$. The projection $\mathrm{Sp}(1) \to \mathrm{SO}(3)$ is given by the adjoint action of $\mathrm{Sp}(1)$ on the imaginary quaternions.

Lemma 4.4 *For every $u \in \mathrm{Sp}(1)$ there exists a unique orthogonal matrix $\Phi(u) \in \mathrm{SO}(3)$ such that*

$$\Phi(u)x = ux\bar{u}$$

for $x = x_1 i + x_2 j + x_3 k \in \mathrm{Im}(\mathbb{H}) = \mathbb{R}^3$. The map $\Phi : \mathrm{Sp}(1) \to \mathrm{SO}(3)$ is a surjective homomorphism with kernel $\{\pm 1\}$ and hence

$$\mathrm{SO}(3) \cong \mathrm{Sp}(1)/\mathbb{Z}_2, \qquad \mathrm{Spin}(3) \cong \mathrm{Sp}(1).$$

Proof: Fix an element $u \in \mathrm{Sp}(1)$. Then one checks easily, by direct calculation, that the real part of $ux\bar{u}$ is zero for every $x \in \mathrm{Im}(\mathbb{H})$. Hence there is a well defined matrix $\Phi(u) \in \mathbb{R}^{3\times 3}$ such that $\Phi(u)x = ux\bar{u}$ for all $x \in \mathrm{Im}(\mathbb{H})$. Since $|ux\bar{u}| = |x|$ it follows that $\Phi(u) \in \mathrm{O}(3)$ and, since $\mathrm{Sp}(1)$ is connected and $\Phi(1) = \mathbb{1}$ we have $\Phi(u) \in \mathrm{SO}(3)$ for all $u \in \mathrm{Sp}(1)$. The map $u \mapsto \Phi(u)$ is obviously a group homomorphism.

We prove that its kernel is ± 1. Suppose that $\Phi(u) = \mathbb{1}$. Then $ux = xu$ for all $x \in \mathrm{Im}(\mathbb{H})$. Inserting $x = i, j, k$ we obtain $u_1 = u_2 = u_3 = 0$ and hence $u = u_0 = \pm 1$.

We prove that Φ is surjective. Identify the Lie algebra $\mathfrak{sp}(1) = \mathrm{Im}(\mathbb{H})$ of $\mathrm{Sp}(1)$ with $\mathbb{R}^3$ via $\xi = \xi_1 i + \xi_2 j + \xi_3 k$. With this identification the Lie bracket on $\mathbb{R}^3$ is given by $[\xi, \eta] = 2\,\xi \times \eta$. The differential of the Lie group homomorphism $\Phi : \mathrm{Sp}(1) \to \mathrm{SO}(3)$ is the Lie algebra homomorphism $\dot{\Phi} : \mathbb{R}^3 \to \mathfrak{so}(3)$ given by

$$\dot{\Phi}(\xi)x = \xi x - x\xi,$$

or in matrix form

$$\dot{\Phi}(\xi) = \begin{pmatrix} 0 & -2\xi_3 & 2\xi_2 \\ 2\xi_3 & 0 & -2\xi_1 \\ -2\xi_2 & 2\xi_1 & 0 \end{pmatrix}.$$

Hence $\dot{\Phi}$ is an isomorphism. Since the exponential map of $\mathrm{SO}(3)$ is surjective there exists, for every $A \in \mathrm{SO}(3)$, a $\xi \in \mathrm{Im}(\mathbb{H})$ such that $A = \exp(\dot{\Phi}(\xi)) = \Phi(\exp(\xi))$. This proves the lemma. □

Exercise 4.5 **(i)** Prove that the homomorphism $\dot{\Phi} : \mathbb{R}^3 \to \mathfrak{so}(3)$ is equivariant with respect to the standard action of SO(3) on $\mathbb{R}^3$ and the adjoint action on its Lie algebra.
(ii) Prove that every equivariant map $\mathbb{R}^3 \to \mathfrak{so}(3)$ must be a constant multiple of $\dot{\Phi}$.
(iii) Prove that the homomorphism $\Phi : \mathrm{Sp}(1) \to \mathrm{SO}(3)$ is given by the explicit formula

$$\Phi(u) = \begin{pmatrix} u_0^2 + u_1^2 - u_2^2 - u_3^2 & 2(u_1u_2 - u_0u_3) & 2(u_1u_3 + u_0u_2) \\ 2(u_1u_2 + u_0u_3) & u_0^2 - u_1^2 + u_2^2 - u_3^2 & 2(u_2u_3 - u_0u_1) \\ 2(u_1u_3 - u_0u_2) & 2(u_2u_3 + u_0u_1) & u_0^2 - u_1^2 - u_2^2 + u_3^2 \end{pmatrix}.$$

(iv) Prove that for every unit vector $\xi \in \mathbb{R}^3$ the map

$$S^3 \to S^2 : u \mapsto \Phi(u)\xi$$

is a Hopf fibration. □

By Lemma 4.4 and Exercise 4.2, $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$. The covering homomorphism is still denoted by $\Phi : \mathrm{SU}(2) \to \mathrm{SO}(3)$ and is again given by the adjoint action of SU(2) on its Lie algebra $\mathfrak{su}(2)$. With this convention $\mathrm{Spin}^c(3) = \mathrm{Spin}(3) \times_{\mathbb{Z}_2} S^1$ can be identified with

$$\mathrm{Spin}^c(3) \cong \left\{ e^{i\theta}U \,|\, \theta \in \mathbb{R},\, U \in \mathrm{SU}(2) \right\} = \mathrm{U}(2).$$

The extended homomorphism

$$\Phi : \mathrm{U}(2) \to \mathrm{SO}(3)$$

is also given by the adjoint action. In fact, the Lie algebra of U(2) consists of the skew adjoint matrices and splits as

$$\mathfrak{u}(2) = \mathfrak{su}(2) \oplus i\mathbb{R}\mathbb{1}.$$

The adjoint action of U(2) on its Lie algebra preserves the subspace $\mathfrak{su}(2)$. Explicitly, if $U \in \mathrm{U}(2)$ then the matrix $\Phi(U) \in \mathbb{R}^{3\times 3}$ is given by

$$\gamma(\Phi(U)\xi) = U\gamma(\xi)U^{-1}$$

for $\xi \in \mathrm{Im}(\mathbb{H}) = \mathbb{R}^3$. Another way to describe the homomorphism Φ is as follows. Given a matrix $U \in \mathrm{U}(2)$ one can obtain a matrix with determinant 1 by dividing by a square root λ of $\det(U)$. Since there are two choices for the square root this gives rise to a map $\mathrm{U}(2) \to \mathrm{SU}(2)/\{\pm\mathbb{1}\}$. The homomorphism $\Phi : \mathrm{U}(2) \to \mathrm{SO}(3)$ is the composition of this map with the isomorphism $\mathrm{SU}(2)/\{\pm\mathbb{1}\} \cong \mathrm{Sp}(1)/\{\pm 1\} \to \mathrm{SO}(3)$ of Lemma 4.4.

The spin group in dimension four

The next lemma shows how the group $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ can be naturally identified with $\mathrm{Spin}(4)$.

Lemma 4.6 *For every pair $u, v \in \mathrm{Sp}(1)$ there exists a unique orthogonal matrix $\Psi(u, v) \in \mathrm{SO}(4)$ such that*

$$\Psi(u, v)x = ux\bar{v}$$

for $x \in \mathbb{H} = \mathbb{R}^4$. The map $\Psi : \mathrm{Sp}(1) \times \mathrm{Sp}(1) \to \mathrm{SO}(4)$ is a surjective homomorphism with kernel $\{\pm 1\}$ and hence

$$\mathrm{SO}(4) \cong \mathrm{Sp}(1) \times_{\mathbb{Z}_2} \mathrm{Sp}(1), \qquad \mathrm{Spin}(4) \cong \mathrm{Sp}(1) \times \mathrm{Sp}(1).$$

Proof: The linear map $\mathbb{H} \to \mathbb{H} : x \mapsto ux\bar{v}$ is obviously orthogonal for $(u, v) \in \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ and, since $\mathrm{Sp}(1)$ is connected, it has determinant 1. The map $\Psi : \mathrm{Sp}(1) \times \mathrm{Sp}(1) \to \mathrm{SO}(4)$ is obviously a homomorphism.

We prove that its kernel is ± 1. If $\Psi(u, v) = \mathbb{1}$ then $ux = xv$ for all $x \in \mathbb{H}$. With $x = 1$ we see that $u = v$ and with $x = i, j, k$ it follows that $u = v = \pm 1$.

We prove that Ψ is surjective. Its differential at the identity is the Lie algebra homomorphism $\dot{\Psi} : \mathrm{Im}(\mathbb{H}) \times \mathrm{Im}(\mathbb{H}) \to \mathfrak{so}(4)$ which assigns to a pair $\xi, \eta \in \mathrm{Im}(\mathbb{H})$ the matrix $\dot{\Psi}(\xi, \eta) \in \mathfrak{so}(4)$ defined by

$$\dot{\Psi}(\xi, \eta)x = \xi x - x\eta$$

for $x \in \mathbb{H}$. Writing this explicitly in matrix form one fiinds that $\dot{\Psi}$ is an isomorphism. Since the exponential map of $\mathrm{SO}(4)$ is surjective, it follows as in the proof of Lemma 4.4 that Ψ is surjective. $\square$

There are two natural homomorphisms

$$\rho^{\pm} : \mathrm{SO}(4) \to \mathrm{SO}(3)$$

given by the inverse of the isomorphism $\mathrm{Sp}(1) \times_{\mathbb{Z}_2} \mathrm{Sp}(1) \xrightarrow{\cong} \mathrm{SO}(4)$ of Lemma 4.6 followed by the projection of either of the two factors onto $\mathrm{SO}(3) \cong \mathrm{Sp}(1)/\mathbb{Z}_2$. In more explicit terms ρ^+ maps $\Psi(u, v)$ to $\Phi(u)$ and ρ^- maps $\Psi(u, v)$ to $\Phi(v)$. Hence there are two exact sequences

$$1 \longrightarrow \mathrm{Sp}(1) \xrightarrow{\iota^{\pm}} \mathrm{SO}(4) \xrightarrow{\rho^{\pm}} \mathrm{SO}(3) \longrightarrow 1$$

where the inclusions $\iota^{\pm} : \mathrm{Sp}(1) \to \mathrm{SO}(4)$ are given by $\iota^+(v) = \Psi(1, v)$ and $\iota^-(u) = \Psi(u, 1)$. We shall see below that these sequences are related to the action of $\mathrm{SO}(4)$ on the spaces $\Lambda^{\pm}$ of self-dual and anti-self-dual 2-forms.

By Lemma 4.6 and Exercise 4.2, $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$. The covering homomorphism is still denoted by $\Psi : \mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathrm{SO}(4)$ and is given by the action of $\mathrm{SU}(2)$ on $\mathfrak{su}(2) \oplus \mathbb{R}\mathbb{1} = \gamma(\mathbb{H})$ on the left and right. With this convention $\mathrm{Spin}^c(4) = \mathrm{Spin}(4) \times_{\mathbb{Z}_2} S^1$ can be identified with

$$\mathrm{Spin}^c(4) \cong \{(U, V) \in \mathrm{U}(2) \times \mathrm{U}(2) \,|\, \det U = \det V\}.$$

The extended homomorphism $\Psi : \mathrm{Spin}^c(4) \to \mathrm{SO}(4)$ is also given by the action on $\gamma(\mathbb{H})$. Explicitly, if $U, V \in \mathrm{U}(2)$ with $\det(U) = \det(V)$ then the matrix $\Psi(U, V) \in \mathbb{R}^{4\times 4}$ is given by

$$\gamma(\Psi(U, V)\xi) = U\gamma(\xi)V^* \tag{4.4}$$

for $\xi \in \mathbb{H}$.

Exercise 4.7 Prove that the map $\Psi : \mathrm{Sp}(1) \times \mathrm{Sp}(1) \to \mathrm{SO}(4)$ extends to an algebra isomorphism

$$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \to \mathbb{R}^{4\times 4}.$$

Hint: Ψ obviously extends to an algebra homomorphism and the dimensions are equal. Thus it remains to show that every 4×4 matrix is a linear combination of orthogonal ones. □

4.2 Clifford algebras

Let V be an n-dimensional real vector space with an inner product $\langle \cdot, \cdot \rangle$ and choose an orthonormal basis $e_1, \ldots, e_n$. Associated to V is the real **Clifford algebra** $C(V)$. This is a 2^n-dimensional real vector space and an algebra with unit 1. It is generated by the basis vectors $e_1, \ldots, e_n$ with multiplication rules

$$e_i{}^2 = -1, \qquad e_ie_j = -e_je_i$$

for $i \neq j$. For general vectors $v, w \in V$ this amounts to the multiplication rule

$$vw + wv = -2\langle v, w\rangle.$$

A basis of $C(V)$ as a real vector space is given by the elements

$$e_0 = 1, \qquad e_I = e_{i_1}e_{i_2}\cdots e_{i_k}$$

for $I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$ with $i_1 < i_2 < \cdots < i_k$. For such a multi-index denote $k = |I| = \deg(e_I)$. An element $x = \sum_I x_I e_I \in C(V)$ is said to be of **degree** k if $x_I = 0$ unless $|I| = k$. Denote by $C_k(V)$ the subset of elements of degree k and by $C^{\mathrm{ev}}(V)$ and $C^{\mathrm{odd}}(V)$ the subspace of all elements of even, respectively odd, degree. Note that $C^{\mathrm{ev}}(V)$ is a subalgebra of $C(V)$. Note also that $C_0(V) = \mathbb{R}$ and for any $x \in C(V)$ denote by $x_0 \in \mathbb{R}$ its degree-0 part.

Remark 4.8. (Naturality) There are different presentations of the Clifford algebra, one for each choice of an orthonormal basis, and for the basis $e_1, \dots, e_n$ this algebra should be denoted by $C(V, e_1, \dots, e_n)$. If $f_1, \dots, f_n$ is another orthonormal basis then there is a canonical isomorphism

$$C(V, e_1, \dots, e_n) \to C(V, f_1, \dots, f_n) : \sum_I x_I e_I \mapsto \sum_J y_J f_J$$

given by

$$y_J = \sum_I x_I a_{IJ}, \qquad a_{IJ} = \det(a_{i_\nu j_\mu})_{\nu,\mu=1}^k, \qquad a_{ij} = \langle e_i, f_j \rangle.$$

These isomorphisms preserve the grading and the degree-0 part. To be completely rigorous one should define the Clifford algebra as the set of all maps $\xi : \mathcal{B}(V) \to \mathbb{R}^{2^n}$, defined on the set $\mathcal{B}(V)$ of orthonormal bases of V, such that $\xi(e_1, \dots, e_n)$ and $\xi(f_1, \dots, f_n)$ are related by the above isomorphism for any two orthonormal bases $\{e_i\}$ and $\{f_j\}$. □

Remark 4.9. (Center) If $n = \dim V$ is even then the center of $C(V)$ is

$$Z(C(V)) = C_0(V).$$

If n is odd then the center is

$$Z(C(V)) = C_0(V) \oplus C_n(V).$$

To see this, consider a basis vector $e_I = e_{i_1} \cdots e_{i_k}$. Then

$$e_I e_{i_\nu} = (-1)^{k-1} e_{i_\nu} e_I, \qquad e_I e_j = (-1)^k e_j e_I, \quad j \notin I,$$

Hence e_I commutes with all elements of $C(V)$ if and only if either $I = \emptyset$ or $I = \{1, \dots, n\}$ with n odd. □

Remark 4.10. (Inner product) The Clifford algebra carries a natural inner product

$$\langle x, y \rangle = \sum_I x_I y_I$$

for $x, y \in C(V)$. This inner product is preserved by the canonical isomorphisms of Remark 4.8. □

Remark 4.11. (Involution) There is a natural involution

$$C(V) \to C(V) : x \mapsto \tilde{x}$$

defined by

$$\tilde{x} = \sum_I \varepsilon_I x_I e_I, \qquad \varepsilon_I = (-1)^{k(k+1)/2}, \qquad k = |I|,$$

for $x = \sum_I x_I e_I \in C(V)$. This involution is preserved by the canonical isomorphisms of Remark 4.8. Moreover,

$$\widetilde{xy} = \tilde{y}\tilde{x}, \qquad (\tilde{x}y)_0 = \langle x, y \rangle.$$

To prove the second identity just note that ${e_I}^2 = \varepsilon_I$. The first identity is an exercise. **Hint:** Consider first the case $x = e_I$ and $y = e_j$ and use the commutation relations of Remark 4.9. □

Remark 4.12. (Direct sum) For any two finite dimensional real Hilbert spaces V and W there is a natural isomorphism

$$C(V \oplus W) \cong C(V)\hat{\otimes}C(W)$$

of graded algebras. Here the tensor product is over the reals and the degree k part of $C(V)\hat{\otimes}C(W)$ is defined as usual as the direct sum of the tensor products $C_j(V)\hat{\otimes}C_{k-j}(W)$ over $j = 0, 1, \ldots, k$. As a vector space $C(V)\hat{\otimes}C(W)$ agrees with the ordinary tensor product $C(V) \otimes C(W)$; however, the product structure is defined in the graded sense as

$$(x\hat{\otimes}y)(x'\hat{\otimes}y') = (-1)^{\deg(y)\deg(x')}(xx'\hat{\otimes}yy')$$

for homogeneous elements $x, x' \in C(V)$ and $y, y' \in C(W)$. (For more details see Lawson-Michelsohn [67].) □

The Clifford algebra is characterized by the following universal property.

Proposition 4.13 *Let V be an n-dimensional real Hilbert space and A be a finite dimensional associative algebra over $\mathbb{R}$ with unit $\mathbb{1}$ and involution $a \mapsto a^*$. Then every linear map*

$$f : V \to A$$

which satisfies

$$f(v)^* + f(v) = 0, \qquad f(v)^* f(v) = |v|^2 \mathbb{1} \tag{4.5}$$

extends uniquely to an algebra homomorphism $C(V) \to A$, still denoted by f. If $\dim V$ is even then the extended homomorphism

$$f : C(V) \to A$$

is injective.

Proof: First consider the formula

$$f(v+w)^*f(v+w) = |v+w|^2\mathbb{1}$$

to obtain $f(v)^*f(w) + f(w)^*f(v) = 2\langle v, w\rangle\mathbb{1}$. This shows that

$$f(v)f(w) + f(w)f(v) = -2\langle v, w\rangle\mathbb{1}$$

for $v, w \in V$. Hence f extends naturally to an algebra homomorphism which is still denoted by $f : C(V) \to A$. An easy calculation shows that $f(\tilde{e}_I) = f(e_I)^*$ for all I and hence $f(\tilde{x}) = f(x)^*$ for all $x \in C(V)$.

Now suppose that V has even dimension. Consider, for each $a \in A$, the linear map $L_a : A \to A$ given by $L_a b = ab$ and define $\tau : A \to \mathbb{R}$ by

$$\tau(a) = \frac{1}{\dim A}\mathrm{trace}(L_a).$$

Then $\tau(ab) = \tau(ba)$. We prove that

$$\tau \circ f(x) = x_0 \tag{4.6}$$

for every $x \in C(V)$. It suffices to prove this for the basis vectors e_I. For $e_0 = 1$ we get $\tau(f(1)) = \tau(\mathbb{1}) = 1$. For $I \neq \emptyset$ choose an element $j \notin I$ when $|I|$ is odd, and an element $j \in I$ when $|I|$ is even. In either case $e_I e_j = -e_j e_I$ and hence $e_I = e_j e_I e_j$. This implies

$$\tau(f(e_I)) = \tau(f(e_j)f(e_I)e(e_j)) = \tau(f(e_I)f(e_j)e(e_j)) = -\tau(f(e_I))$$

and hence $\tau(e_I) = 0$. This proves (4.6). Injectivity now follows easily. If $f(x) = 0$ then

$$0 = \tau(f(\tilde{x}x)) = (\tilde{x}x)_0 = |x|^2$$

and hence $x = 0$. This proves the proposition. □

Exercise 4.14 Prove that the map $f : \mathbb{R}^k \times \mathbb{R}^\ell \to C(\mathbb{R}^k) \otimes C(\mathbb{R}^\ell)$ given by

$$f(v, w) = v \otimes \varepsilon + 1 \otimes w$$

satisfies (4.5) if and only if the element $\varepsilon \in C(\mathbb{R}^\ell)$ satisfies

$$\tilde{\varepsilon} = \varepsilon, \qquad \varepsilon^2 = 1, \qquad \varepsilon w + w\varepsilon = 0$$

for every $w \in \mathbb{R}^\ell$. In particular, $\varepsilon = 1$ never satisfies these conditions. □

Examples

Example 4.15 $C(\mathbb{R}) = \mathbb{C}$. The Clifford algebra of $\mathbb{R}$ can be identified with the complex numbers via $i = e_1$ and the involution is given by complex conjugation. □

Example 4.16 $C(\mathbb{R}^2) = \mathbb{H}$. The Clifford algebra of $\mathbb{R}^2$ can be identified with the quaternions via $j = e_1$, $k = e_2$, $i = e_1e_2$. □

Example 4.17 $C(\mathbb{R}^3) = \mathbb{H} \oplus \mathbb{H}$. The Clifford algebra of $\mathbb{R}^3$ can be identified with $\mathbb{H} \oplus \mathbb{H}$ via

$$\begin{aligned} e_0 &= (1,1), & e_1e_2e_3 &= (-1,1), \\ e_1 &= (i,i), & e_2e_3 &= (i,-i), \\ e_2 &= (j,-j), & e_3e_1 &= (j,j), \\ e_3 &= (k,k), & e_1e_2 &= (k,-k). \end{aligned}$$

If $\mathbb{R}^3$ is identified with $\mathrm{Im}(\mathbb{H})$ via $x = ix_1 + jx_2 + kx_3$, then the inclusion $\mathrm{Im}(\mathbb{H}) \to \mathbb{H} \oplus \mathbb{H}$ is given by $x \mapsto (x, jxj)$. □

Example 4.18 $C(\mathbb{R}^4) = \mathbb{H}^{2\times 2}$. The Clifford algebra of $\mathbb{R}^4$ can be identified with the 2×2 quaternion matrices via

$$e_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}.$$

The involution is given by $A \mapsto A^*$ where A^* denotes the conjugate transpose. □

Classification

Here is a complete list of the Clifford algebras of Euclidean spaces. Excellent references are the book by Lawson-Michelsohn [67] and the paper by Atiyah-Bott-Shapiro [6]. In our proof we follow the argument in [67].

Theorem 4.19 *The Clifford algebras of Euclidean spaces are given by the following table.*

$$\begin{array}{ccc} n & C(\mathbb{R}^n) & \mathrm{Spin}(\mathbb{R}^n) \\ & & \\ 0 & \mathbb{R} & 1 \\ 1 & \mathbb{C} & \mathbb{Z}_2 \\ 2 & \mathbb{H} & \mathrm{U}(1) \\ 3 & \mathbb{H} \oplus \mathbb{H} & \mathrm{Sp}(1) \\ 4 & \mathbb{H}^{2\times 2} & \mathrm{Sp}(1) \times \mathrm{Sp}(1) \\ 5 & \mathbb{C}^{4\times 4} & \mathrm{Sp}(2) \\ 6 & \mathbb{R}^{8\times 8} & \mathrm{SU}(4) \\ 7 & \mathbb{R}^{8\times 8} \oplus \mathbb{R}^{8\times 8} & \\ 8 & \mathbb{R}^{16\times 16}. & \end{array} \tag{4.7}$$

Moreover, for every n there is an algebra isomorphism

$$C(\mathbb{R}^{n+8}) \cong C(\mathbb{R}^n) \otimes \mathbb{R}^{16\times 16}$$

where the tensor product is to be understood over the reals.

Proof: For every $n \geq 0$ there is an algebra isomorphism

$$f : C(\mathbb{R}^n \oplus \mathbb{R}^4) \to C(\mathbb{R}^n) \otimes C(\mathbb{R}^4). \tag{4.8}$$

It is given by

$$f(v, w) = v \otimes \varepsilon + 1 \otimes w$$

for $v \in \mathbb{R}^n$ and $w \in \mathbb{R}^4$, where $\varepsilon = e_1 e_2 e_3 e_4 \in C(\mathbb{R}^4)$ for any orthonormal basis e_1, e_2, e_3, e_4 of $\mathbb{R}^4$. Since $\varepsilon w + w\varepsilon = 0$ and $\varepsilon^2 = 1$ it follows that f satisfies the requirements of Proposition 4.13 (see Exercise 4.14). Hence f extends to an algebra homomorphism $f : C(\mathbb{R}^{n+4}) \to C(\mathbb{R}^n) \otimes C(\mathbb{R}^4)$. If n is even it follows from Proposition 4.13 that f is injective and, for dimensional reasons, f is an isomorphism. If n is odd denote $\varepsilon' = e'_1 \cdots e'_n$ for some orthonormal basis $e'_1, \dots, e'_n$ of $\mathbb{R}^n$ and check directly that $f(\varepsilon'\varepsilon) = \varepsilon' \otimes 1$. With $\tau : C(\mathbb{R}^n) \otimes C(\mathbb{R}^4) \to \mathbb{R}$ defined by $\tau(x, y) = x_0 y_0$, the same argument as in the proof of Proposition 4.13 shows that f is bijective. This proves the existence of the isomorphism (4.8).

Now use the identities

$$\mathbb{H}^{2\times 2} = \mathbb{H} \otimes \mathbb{R}^{2\times 2}, \qquad \mathbb{R}^{k\times k} \otimes \mathbb{R}^{\ell\times\ell} = \mathbb{R}^{k\ell\times k\ell}, \qquad \mathbb{H} \otimes \mathbb{H} = \mathbb{R}^{4\times 4}$$

(see Exercise 4.7 for the last equation) to obtain the eightfold periodicity

$$\begin{aligned} C(\mathbb{R}^{n+8}) &= C(\mathbb{R}^n) \otimes \mathbb{H}^{2\times 2} \otimes \mathbb{H}^{2\times 2} \\ &= C(\mathbb{R}^n) \otimes \mathbb{H} \otimes \mathbb{H} \otimes \mathbb{R}^{2\times 2} \otimes \mathbb{R}^{2\times 2} \\ &= C(\mathbb{R}^n) \otimes \mathbb{R}^{4\times 4} \otimes \mathbb{R}^{4\times 4} \\ &= C(\mathbb{R}^n) \otimes \mathbb{R}^{16\times 16}. \end{aligned}$$

Moreover, by Exercise 4.3, $\mathbb{H} \otimes \mathbb{C} = \mathbb{C}^{2\times 2}$ and hence

$$C(\mathbb{R}^5) = C(\mathbb{R}) \otimes \mathbb{H}^{2\times 2} = \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{R}^{2\times 2} = \mathbb{C}^{2\times 2} \otimes \mathbb{R}^{2\times 2} = \mathbb{C}^{4\times 4}.$$

Similarly,

$$C(\mathbb{R}^6) = C(\mathbb{R}^2) \otimes \mathbb{H}^{2\times 2} = \mathbb{H} \otimes \mathbb{H} \otimes \mathbb{R}^{2\times 2} = \mathbb{R}^{4\times 4} \otimes \mathbb{R}^{2\times 2} = \mathbb{R}^{8\times 8},$$

$$C(\mathbb{R}^7) = C(\mathbb{R}^3) \otimes \mathbb{H}^{2\times 2} = (\mathbb{H} \oplus \mathbb{H}) \otimes \mathbb{H}^{2\times 2} = \mathbb{R}^{8\times 8} \oplus \mathbb{R}^{8\times 8},$$

and

$$C(\mathbb{R}^8) = C(\mathbb{R}^0) \otimes \mathbb{R}^{16\times 16} = \mathbb{R}^{16\times 16}.$$

This proves the theorem. $\square$

Example 4.20. (Cayley numbers) The Clifford algebra of $\mathbb{R}^8$ is related to the Cayley numbers as follows. There is a cross product on $\mathbb{R}^7$ given by

$$u \times v = A(u)v, \qquad A(u) = \begin{pmatrix} 0 & -u_3 & u_2 & -u_5 & u_4 & -u_7 & u_6 \\ u_3 & 0 & -u_1 & u_6 & -u_7 & -u_4 & u_5 \\ -u_2 & u_1 & 0 & -u_7 & -u_6 & u_5 & u_4 \\ u_5 & -u_6 & u_7 & 0 & -u_1 & u_2 & -u_3 \\ -u_4 & u_7 & u_6 & u_1 & 0 & -u_3 & -u_2 \\ u_7 & u_4 & -u_5 & -u_2 & u_3 & 0 & -u_1 \\ -u_6 & -u_5 & -u_4 & u_3 & u_2 & u_1 & 0 \end{pmatrix}$$

for $u, v \in \mathbb{R}^7$. The reader may check that this product structure is skew-symmetric and distributive and satisfies

$$\langle u \times v, w \rangle = \langle u, v \times w \rangle,$$

$$(u \times v) \times w + u \times (v \times w) = 2\langle u, w \rangle v - \langle u, v \rangle w - \langle v, w \rangle u.$$

Given these rules the product is characterized by the relations

$$e_1 \times e_2 = e_3, \qquad e_1 \times e_4 = e_5, \qquad e_1 \times e_6 = e_7, \qquad e_2 \times e_4 = -e_6.$$

This gives rise to a map $\gamma : \mathbb{R}^8 \to \mathbb{R}^{8\times 8}$ given by

$$\gamma(v) = v_0 \mathbb{1} + \begin{pmatrix} 0 & -\bar{v}^T \\ \bar{v} & A(\bar{v}) \end{pmatrix} = \begin{pmatrix} v_0 & -v_1 & -v_2 & -v_3 & -v_4 & -v_5 & -v_6 & -v_7 \\ v_1 & v_0 & -v_3 & v_2 & -v_5 & v_4 & -v_7 & v_6 \\ v_2 & v_3 & v_0 & -v_1 & v_6 & -v_7 & -v_4 & v_5 \\ v_3 & -v_2 & v_1 & v_0 & -v_7 & -v_6 & v_5 & v_4 \\ v_4 & v_5 & -v_6 & v_7 & v_0 & -v_1 & v_2 & -v_3 \\ v_5 & -v_4 & v_7 & v_6 & v_1 & v_0 & -v_3 & -v_2 \\ v_6 & v_7 & v_4 & -v_5 & -v_2 & v_3 & v_0 & -v_1 \\ v_7 & -v_6 & -v_5 & -v_4 & v_3 & v_2 & v_1 & v_0 \end{pmatrix} \tag{4.9}$$

for $v = (v_0, \ldots, v_7) \in \mathbb{R}^8$ where $\bar{v} = (v_1, \ldots, v_7)$. The restriction of γ to $\mathbb{R}^6$ satisfies the requirements of Proposition 4.13 and hence gives rise to an algebra isomorphism $C(\mathbb{R}^6) \to \mathbb{R}^{8\times 8}$. The reader may also check that the restriction of Γ to $\mathbb{R}^5$ determines a linear map $\mathbb{R}^5 \to \mathbb{C}^{4\times 4}$ which satisfies (4.18), where the complex structure on $\mathbb{R}^8$ is given by

$$x = u_0 + iu_1, \qquad y = u_2 - iu_3, \qquad z = u_4 - iu_5, \qquad w = u_6 + iu_7.$$

The isomorphism $C(\mathbb{R}^8) \to \mathbb{R}^{16\times 16}$ is induced by the map $\Gamma : \mathbb{R}^8 \to \mathbb{R}^{16\times 16}$ given by

$$\Gamma(v) = \begin{pmatrix} 0 & \gamma(v) \\ -\gamma(v)^T & 0 \end{pmatrix}$$

for $v \in \mathbb{R}^8$, where $\gamma(v)$ is given by (4.9). $\square$

The complexified Clifford algebra

Denote by

$$C^c(V) = C(V) \otimes_{\mathbb{R}} \mathbb{C}$$

the complexified Clifford algebra of V. Thus, with a given orthonormal basis $e_1, \ldots, e_n$ of V the elements of $C(V)$ can be written in the form

$$x = \sum_I x_I e_I$$

with $x_I \in \mathbb{C}$. In this case the involution $x \mapsto \tilde{x}$ is given by

$$\tilde{x} = \sum_I \varepsilon_I \bar{x}_I e_I.$$

As before $\widetilde{xy} = \tilde{y}\tilde{x}$ and now there is a Hermitian structure

$$\langle x, y \rangle = (\tilde{x}y)_0 = \sum_I \bar{x}_I y_I.$$

Example 4.21 In Example 4.15 it was shown that the $C(\mathbb{R}) \cong \mathbb{C}$ and hence the complexified clifford algebra of $\mathbb{R}$ is

$$C^c(\mathbb{R}) = \mathbb{C} \oplus \mathbb{C}.$$

The inclusion $\mathbb{R} \to \mathbb{C} \oplus \mathbb{C}$ is given by $e_1 \mapsto (i, -i)$. □

Example 4.22 In Example 4.16 it was shown that $C(\mathbb{R}^2) \cong \mathbb{H}$ and hence the complexified Clifford algebra of $\mathbb{R}^2$ is

$$C^c(\mathbb{R}^2) = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}^{2\times 2}.$$

The inclusion $\mathbb{R}^2 \to \mathbb{C}^{2\times 2}$ is given by $e_1 \mapsto J$ and $e_2 \mapsto K$. (See (4.1) and Exercise 4.3.) □

Theorem 4.23 *For every n there are algebra isomorphisms*

$$C^c(\mathbb{R}^{2n}) \cong \mathbb{C}^{2^n \times 2^n}, \qquad C^c(\mathbb{R}^{2n+1}) \cong \mathbb{C}^{2^n \times 2^n} \oplus \mathbb{C}^{2^n \times 2^n}.$$

Proof: This result follows directly from Theorem 4.19. Alternatively, one can prove that, for every n, there is an algebra isomorphism

$$C^c(\mathbb{R}^n \oplus \mathbb{R}^2) \to C^c(\mathbb{R}^n) \otimes C^c(\mathbb{R}^2) : (v, w) \mapsto v \otimes \varepsilon + 1 \otimes w,$$

where $\varepsilon = ie_2e_1 \in C^c(\mathbb{R}^2)$. The details are as in Theorem 4.19 and are left to the reader. Note in particular that $\varepsilon^2 = 1$ because of the factor i. □

4.3 The groups Spin and Spin^c

The group Spin

An element $x \in C(V)$ is called a **unit** if it has an inverse (denoted by x^{-1}). Note here that x has a right inverse if and only if it has a left inverse, that both inverses are unique, and that they are equal. These are general facts about finite dimensional algebras.

Exercise 4.24 Let $x \in C(V)$. Prove that x has a right inverse if and only if x has a left inverse and that both inverses agree. **Hint:** Consider the linear operator $L_x : C(V) \to C(V)$ defined by $L_x y = xy$ for $y \in C(V)$. Prove that $L_x^* = L_{\tilde{x}}$. Prove that x has a right inverse iff L_x is surjective, and has a left inverse iff $L_{\tilde{x}}$ is surjective.

Consider the **twisted adjoint action** of the units on $C(V)$. For a unit x the action $\mathrm{ad}(x) : C(V) \to C(V)$ is given by

$$\mathrm{ad}(x)\xi = (x^{\mathrm{ev}} - x^{\mathrm{odd}})\xi\tilde{x} \tag{4.10}$$

for $\xi \in C(V)$, where $x^{\mathrm{ev}} \in C^{\mathrm{ev}}(V)$ and $x^{\mathrm{odd}} \in C^{\mathrm{odd}}(V)$ denote the even and odd parts of x. One checks easily that the map $x \mapsto x^{\mathrm{ev}} - x^{\mathrm{odd}}$ is an algebra automorphism of $C(V)$. This implies that ad is a group homomorphism from the units in $C(V)$ to the automorphisms of $C(V)$. Namely, for any two units $x, y \in C(V)$,

$$\mathrm{ad}(xy) = \mathrm{ad}(x)\mathrm{ad}(y), \qquad \mathrm{ad}(\tilde{x}) = \mathrm{ad}(x)^*, \qquad \mathrm{ad}(1) = \mathbb{1}. \tag{4.11}$$

Note that the adjoint action is orthogonal whenever $\tilde{x}x = 1$. We define

$$\mathrm{Spin}(V) = \{x \in C^{\mathrm{ev}}(V) \,|\, \tilde{x}x = 1,\, xV\tilde{x} = V\}.$$

The next lemma shows that $\mathrm{Spin}(V)$ is the universal cover of $\mathrm{SO}(V)$.

Lemma 4.25 *Assume* $\dim V \geq 3$. *Then the group* $\mathrm{Spin}(V)$ *is compact, connected, and simply connected. There is an exact sequence*

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(V) \xrightarrow{\mathrm{ad}} \mathrm{SO}(V) \longrightarrow 1.$$

Proof: We follow the exposition in Atiyah-Bott-Shapiro [6]. They introduce the group

$$\mathrm{Pin}(V) = \left\{x \in C(V) \,|\, \tilde{x}x = 1,\, (x^{\mathrm{ev}} - x^{\mathrm{odd}})V\tilde{x} = V\right\}.$$

This is a subgroup of the units of $C(V)$. Since $|x|^2 = (\tilde{x}x)_0 = 1$ for $x \in \mathrm{Pin}(V)$, this subgroup is compact. The twisted adjoint action determines a homomorphism

$$\mathrm{ad} : \mathrm{Pin}(V) \to \mathrm{O}(V)$$

which assigns to every $x \in \mathrm{Pin}(V)$ the restriction of $\mathrm{ad}(x)$ to V. A simple calculation shows that every $w \in V$ with $|w| = 1$ belongs to $\mathrm{Pin}(V)$ and satisfies

$$\mathrm{ad}(w)v = v - 2\langle v, w\rangle w$$

for $v \in V$. Thus $\mathrm{ad}(w)$ is the reflection at the hyperplane orthogonal to w and, since every orthogonal transformation is a composition of reflections, it follows that $\mathrm{ad} : \mathrm{Pin}(V) \to \mathrm{O}(V)$ is surjective.

Next we prove that the kernel of ad is $\{\pm 1\}$. To see this note that $\mathrm{ad}(x) = \mathbb{1}$ if and only if

$$(x^{\mathrm{ev}} - x^{\mathrm{odd}})v = vx$$

for all $v \in V$. Equivalently x^{ev} commutes with v and x^{odd} anticommutes with v for all $v \in V$, and this is the case if and only if $x^{\mathrm{odd}} = 0$ and $x^{\mathrm{ev}} = \pm 1$. Thus there is an exact sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Pin}(V) \xrightarrow{\mathrm{ad}} \mathrm{O}(V) \longrightarrow 1$$

and it follows that $\mathrm{Pin}(V)$ is a double cover of $\mathrm{O}(V)$.

Next we prove that $x \in \mathrm{Pin}(V)$ is even if and only if $\mathrm{ad}(x) \in \mathrm{SO}(V)$. To see this let $x \in \mathrm{Pin}(V)$, choose reflections $R_1, \dots, R_m \in \mathrm{O}(V)$ such that $\mathrm{ad}(x) = R_1 \cdots R_m$, and choose unit vectors $w_i \in V$ such that $\mathrm{ad}(w_i) = R_i$. Then

$$\mathrm{ad}(x) = \mathrm{ad}(w_1) \cdots \mathrm{ad}(w_m)$$

and hence $x = \pm w_1 \cdots w_m$. Hence x is even if and only if m is even if and only if $\mathrm{ad}(x)$ is orientation preserving. This means that

$$\mathrm{Spin}(V) = \mathrm{ad}^{-1}(\mathrm{SO}(V)).$$

Hence there is an exact sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}(V) \xrightarrow{\mathrm{ad}} \mathrm{SO}(V) \longrightarrow 1$$

as claimed.

We prove that $\mathrm{Spin}(V)$ is connected. Given $x_0 \in \mathrm{Spin}(V)$ choose a path $[0, 1] \to \mathrm{SO}(V) : t \mapsto A_t$ such that $A_0 = \mathrm{ad}(x_0)$ and $A_1 = \mathbb{1}$. Let $[0, 1] \to \mathrm{Spin}(V) : t \mapsto x_t$ be the unique continuous lift and note that $x_1 = \pm 1$. If $x_1 = -1$ choose a path $\gamma : [0, \pi] \to \mathrm{Spin}(V)$ such that $\gamma(0) = 1$ and $\gamma(\pi) = -1$. An explicit formula is given by

$$\gamma(t) = \exp(te_1e_2) = \cos(t) + \sin(t)e_1e_2.$$

This shows that $\mathrm{Spin}(V)$ is connected. That the group is simply connected follows from the fact that $\pi_1(\mathrm{SO}(V)) = \mathbb{Z}_2$. This proves the lemma. □

Exercise 4.26 Prove that the Lie algebra of $\mathrm{Spin}(V)$ agrees with that of $\mathrm{Pin}(V)$ and is given by the degree-2-part of the Clifford algebra:

$$\mathrm{Lie}(\mathrm{Pin}(V)) = \mathrm{Lie}(\mathrm{Spin}(V)) = C_2(V).$$

Show that the linear map $\mathrm{Ad} : C_2(V) \to \mathfrak{so}(V)$ which sends $\xi \in C_2(V)$ to the endomorphism $\mathrm{Ad}(\xi) : V \to V$ given by

$$\mathrm{Ad}(\xi)v = [\xi, v] = \xi v - v\xi \tag{4.12}$$

is a Lie algebra isomorphism. Show that this isomorphism is the differential of the Lie algebra homomorphism $\mathrm{ad} : \mathrm{Pin}(V) \to \mathrm{O}(V)$ at $x = 1$. Deduce that the identity component of $\mathrm{Pin}(V)$ is equal to $\mathrm{Spin}(V) = \mathrm{Pin}(V) \cap C^{\mathrm{ev}}(V)$. □

Lemma 4.27 *If* $\dim V \geq 3$ *then there is no nontrivial homomorphism* $\mathrm{Spin}(V) \to S^1$.

Proof 1: Since $\mathrm{Spin}(V)$ is simply connected, any such homomorphism lifts to a homomorphism $\mathrm{Spin}(V) \to \mathbb{R}$. The image of the lift is a compact subgroup of $\mathbb{R}$. The only such subgroup is $\{1\}$. □

Proof 2: Let $\rho : \mathrm{Spin}(V) \to S^1$ be a homomorphism. Then the kernel of ρ is a normal subgroup of $\mathrm{Spin}(V)$. Since $\mathrm{Spin}(V)$ is a simple group there is no nontrivial such subgroup and hence $\rho = 1$. □

Proof 3: Consider the induced Lie algebra homomorphism $\dot{\rho} : C_2(V) \to i\mathbb{R}$. Since $\dim V \geq 3$ every basis vector $e_i e_j \in C_2(V)$ can be expressed as a Lie bracket of two other vectors

$$e_i e_j = \frac{1}{2}[e_i e_k, e_j e_k], \qquad k \neq i,\ k \neq j.$$

Hence $\dot{\rho}(e_i e_j) = 0$ for all i and j. This implies that ρ is locally constant. Since $\mathrm{Spin}(V)$ is connected it follows that $\rho(x) = 1$ for all $x \in \mathrm{Spin}(V)$. □

Exercise 4.28 Fix an orientation of V and let $e_1, \ldots, e_n$ be a positively oriented orthonormal basis of V. Prove that the element

$$\varepsilon = e_n \cdots e_1 \in C_n(V)$$

is independent of the choice of the basis used to define it. Prove that

$$\varepsilon\tilde{\varepsilon} = 1, \qquad \varepsilon^2 = (-1)^{n(n+1)/2},$$

and $\varepsilon v \tilde{\varepsilon} = (-1)^{n-1} v$ for every $v \in V$. Deduce that

$$\varepsilon \in \mathrm{Spin}(V), \qquad \mathrm{ad}(\varepsilon) = -\mathbb{1} \in \mathrm{SO}(V),$$

whenever n is even. □

The group Spin^c

As in the real case the group $\mathrm{Spin}^c(V)$ is defined as the set of even elements $x \in C^{\mathrm{ev}}(V) \otimes \mathbb{C}$ which satisfy $\tilde{x}x = 1$ and $xV\tilde{x} = V$. Its Lie algebra is given by

$$\mathrm{Lie}(\mathrm{Spin}^c(V)) = C_2(V) \oplus i\mathbb{R}.$$

Hence $\mathrm{Spin}^c(V)$ is a central extension of $\mathrm{SO}(V)$ with center S^1 and it can be written in the form

$$\mathrm{Spin}^c(V) = \left\{ e^{i\theta}x \,|\, \theta \in \mathbb{R},\, x \in \mathrm{Spin}(V) \right\}. \tag{4.13}$$

Thus one can identify

$$\mathrm{Spin}^c(V) = \mathrm{Spin}(V) \times_{\mathbb{Z}_2} S^1$$

where the action of $\mathbb{Z}_2 = \{\pm 1\}$ is the obvious diagonal one.

There is an exact sequence

$$1 \longrightarrow S^1 \longrightarrow \mathrm{Spin}^c(V) \xrightarrow{\mathrm{ad}} \mathrm{SO}(V) \longrightarrow 1 \tag{4.14}$$

where the second map is the obvious inclusion of S^1 into $\mathrm{Spin}^c(V)$ and the map $\mathrm{ad} : \mathrm{Spin}^c(V) \to \mathrm{SO}(V)$ is defined by (4.10) as before. There is another exact sequence

$$1 \longrightarrow \mathrm{Spin}(V) \longrightarrow \mathrm{Spin}^c(V) \xrightarrow{\delta} S^1 \longrightarrow 1 \tag{4.15}$$

where the map $\delta : \mathrm{Spin}^c(V) \to S^1$ is given by

$$\delta(e^{i\theta}x) = e^{2i\theta} \tag{4.16}$$

for $\theta \in \mathbb{R}$ and $x \in \mathrm{Spin}(V)$ or, equivalently, $\delta(x) = \sum_I {x_I}^2$ for $x \in \mathrm{Spin}^c(V)$. Define the **degree** of a loop $\gamma : S^1 \to \mathrm{Spin}^c(V)$ as the degree of the map $\delta \circ \gamma : S^1 \to S^1$.

Remark 4.29 In analogy to the real case one can introduce the group

$$\mathrm{Pin}^c(V) = \left\{ x \in C^c(V) \,|\, \tilde{x}x = 1,\, (x^{\mathrm{ev}} - x^{\mathrm{odd}})V\tilde{x} = V \right\}$$

with the obvious representation $\mathrm{ad} : \mathrm{Pin}^c(V) \to \mathrm{O}(V)$. There is an exact sequence

$$1 \longrightarrow S^1 \longrightarrow \mathrm{Pin}^c(V) \xrightarrow{\mathrm{ad}} \mathrm{O}(V) \longrightarrow 1.$$

Hence $\mathrm{Pin}^c(V)$ has two components and the identity component agrees with $\mathrm{Spin}^c(V)$. □

Lemma 4.30 *Assume* $\dim V = n \geq 3$.
(i) *The homomorphism* $\delta : \mathrm{Spin}^c(V) \to S^1$ *induces an ismomorphism of fundamental groups*

$$\pi_1(\mathrm{Spin}^c(V)) \cong \pi_1(S^1) \cong \mathbb{Z}.$$

(ii) *The homomorphism* $\mathrm{ad} : \mathrm{Spin}^c(V) \to \mathrm{SO}(V)$ *induces a surjective homomorphism of fundamental groups*

$$\pi_1(\mathrm{Spin}^c(V)) \cong \mathbb{Z} \to \pi_1(\mathrm{SO}(V)) \cong \mathbb{Z}_2.$$

Explicitly, given $\gamma : S^1 \to \mathrm{Spin}^c(V)$*, the loop* $\mathrm{ad} \circ \gamma : S^1 \to \mathrm{SO}(V)$ *is contractible if and only if* $\delta \circ \gamma : S^1 \to S^1$ *has even degree.*
(iii) *For* $\gamma : S^1 \to \mathrm{Spin}^c(V)$ *and* $\lambda : S^1 \to S^1$,

$$\deg(\lambda \cdot \gamma) = \deg(\gamma) + 2\deg(\lambda).$$

Proof: The first assertion follows from the exact sequence (4.15) since $\mathrm{Spin}(V)$ is simply connected. Here is an alternative proof. Let e_1, e_2 be orthonormal vectors in V. Then the loop

$$\gamma_1(t) = \exp(te_1e_2 + it) = e^{it}(\cos(t) + \sin(t)e_1e_2), \qquad 0 \leq t \leq \pi$$

has degree 1. Hence the homomorphism $\pi_1(\mathrm{Spin}^c(V)) \to \pi_1(S^1)$ is surjective. Moreover, the loop $\mathrm{ad}\circ\gamma_1 : \mathbb{R}/\pi\mathbb{Z} \to \mathrm{SO}(V)$ is not contractible because it lifts to a path $t \mapsto e^{-it}\gamma_1(t)$ in $\mathrm{Spin}(V)$ which connects 1 to -1. This shows that the homomorphism $\pi_1(\mathrm{Spin}^c(V) \to \pi_1(\mathrm{SO}(V))$ is surjective.

To examine the kernel of both homomorphisms, observe that, if $\gamma : S^1 \to \mathrm{Spin}^c(V)$ is a loop such that $\delta \circ \gamma : S^1 \to S^1$ has even degree, then there exists a loop $\lambda : S^1 \to S^1$ such that

$$\lambda(t)^2 = \delta(\gamma(t))$$

for $t \in S^1$. Consider the loop $\gamma_0 : S^1 \to \mathrm{Spin}(V)$ defined by

$$\gamma_0(t) = \lambda(t)^{-1}\gamma(t).$$

By Lemma 4.25, this loop is contractible. Hence the loop $\mathrm{ad} \circ \gamma = \mathrm{ad} \circ \gamma_0 : S^1 \to \mathrm{SO}(V)$ is contractible. If, moreover, $\deg(\delta \circ \gamma) = 0$ then the loop $\lambda : S^1 \to S^1$ is contractible as well and hence so is $\gamma = \lambda\gamma_0$. This proves (i). To complete the proof of (ii) just note that, if $\deg(\delta\circ\gamma)$ is odd, then $\gamma_1 \cdot \gamma^{-1}$ has even degree and hence the loop $\mathrm{ad} \circ \gamma \sim \mathrm{ad} \circ \gamma_1$ is not contractible. This proves (ii). Statement (iii) is obvious from the definition of δ because $\delta(\lambda \cdot \gamma) = \lambda^2 \cdot \delta(\gamma)$. □

Exercise 4.31 Suppose that V has even dimension and let $x \in C^c(V)$ such that $\tilde{x}x = 1$ and $xV\tilde{x} = V$. Prove that $x \in \mathrm{Spin}^c(V)$ if and only if the transformation

$$V \to V : v \mapsto xv\tilde{x}$$

is orientation preserving. **Hint:** Show that the Lie algebra of the group $\mathrm{G} = \{x \in C^c(V) \,|\, \tilde{x}x = 1,\, xV\tilde{x} = V\}$ agrees with that of $\mathrm{Spin}^c(V)$. Show that the identity component of G agrees with $\mathrm{Spin}^c(V)$. □

4.4 Spin^c representations

If V has even dimension $2n$ then, by Theorem 4.23, the complexified Clifford algebra $C^c(V)$ can be identified with the algebra of endomorphisms of a 2^n-dimensional complex Hermitian vector space W. More precisely, such an identification is an algebra isomorphism

$$\Gamma : C^c(V) \to \mathrm{End}(W)$$

which satisfies

$$\Gamma(x+y) = \Gamma(x) + \Gamma(y), \quad \Gamma(xy) = \Gamma(x)\Gamma(y), \quad \Gamma(\tilde{x}) = \Gamma(x)^* \qquad (4.17)$$

for all $x, y \in C^c(V)$. In particular, this implies $\Gamma(0) = 0$ and $\Gamma(1) = \mathbb{1}$. Note that, firstly, for any W the isomorphism $\Gamma : C^c(V) \to \mathrm{End}(W)$ is not unique and, secondly, any such Γ is uniquely determined by its restriction to $V \subset C^c(V)$. This gives rise to the following definition.

Definition 4.32 *Let V be a real inner product space of dimension $2n$ or $2n+1$. A* **spin^c structure** *on V is a pair (W, Γ), where W is a 2^n-dimensional complex Hermitian vector space and*

$$\Gamma : V \to \mathrm{End}(W)$$

is a linear map which satisfies

$$\Gamma(v)^* + \Gamma(v) = 0, \qquad \Gamma(v)^*\Gamma(v) = |v|^2\mathbb{1} \qquad (4.18)$$

for every $v \in V$. A **spin^c isomorphism** *from (V_0, W_0, Γ_0) to (V_1, W_1, Γ_1) is a pair (A, Φ), where $A : V_0 \to V_1$ is an orientation preserving orthogonal transformation, $\Phi : W_0 \to W_1$ is a unitary isomorphism, and*

$$\Phi\Gamma_0(v_0)\Phi^{-1} = \Gamma_1(Av_0) \qquad (4.19)$$

for $v_0 \in V_0$. The set of spin^c isomorphisms is denoted by

$$\mathrm{Hom}^{\mathrm{spin}^c}(W_0, W_1) = \{(A, \Phi) \,|\, (4.19)\}.$$

Definition 4.32 can be rephrased in the form that a spin^c structure on a real Hilbert space V is an irreducible representation of the complexified Clifford algebra $C^c(V)$. One can think of this as a category with objects (V, W, Γ) and morphisms (A, Φ). Theorem 4.23 shows that the set of objects is nonempty. In the even dimensional case Proposition 4.33 below shows that the set of morphisms between any two objects is nonempty and that each spin^c structure $\Gamma : V \to \mathrm{End}(W)$ indeed extends to an algebra isomorphism $C^c(V) \to \mathrm{End}(W)$. Proposition 4.36 then shows that this extended isomorphism identifies the group $\mathrm{Spin}^c(V)$ with the group of spin^c automorphisms of (V, W, Γ). Thus there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Spin}^c(V) \arrow[rr, "\Gamma"] \arrow[dr, "\mathrm{ad}"'] & & \mathrm{Hom}^{\mathrm{spin}^c}(W, W) \arrow[dl] \\
& \mathrm{SO}(V) &
\end{tikzcd}.$$

where the map $\mathrm{Hom}^{\mathrm{spin}^c}(W, W) \to \mathrm{SO}(V)$ is the projection $(A, \Phi) \mapsto A$.

Proposition 4.33 *Assume* $\dim V = 2n$ *and let* $\Gamma : V \to \mathrm{End}(W)$ *be a spin*^{c} *structure on* V*. Then* Γ *extends uniquely to an algebra isomorphism (denoted by the same letter)*

$$\Gamma : C^c(V) \to \mathrm{End}(W)$$

which satisfies $\Gamma(\tilde{x}) = \Gamma(x)^*$ *for* $x \in C^c(V)$*. If*

$$\Gamma_0 : V \to \mathrm{End}(W_0), \qquad \Gamma_1 : V \to \mathrm{End}(W_1)$$

are two spin^c *structures on* V *then there exists a unitary isomorphism* $\Phi : W_0 \to W_1$ *such that*

$$\Gamma_1(v) = \Phi \Gamma_0(v) \Phi^{-1}$$

for every $v \in V$*.*

Proof: By Proposition 4.13, Γ extends uniquely to an algebra homomorphism, which is still denoted by $\Gamma : C^c(V) \to \mathrm{End}(W)$. Proposition 4.13 also asserts that the extended homomorphism is injective. For dimensional reasons Γ must be bijective.

Now assume that $\Gamma_0 : V \to \mathrm{End}(W_0)$ and $\Gamma_1 : V \to \mathrm{End}(W_1)$ are two linear maps which satisfy (4.18). By the first part of the proof, both maps extend to algebra isomorphisms $\Gamma_i : C^c(V) \to \mathrm{End}(W_i)$. Hence the map

$$f = \Gamma_1 \circ {\Gamma_0}^{-1} : \mathrm{End}(W_0) \to \mathrm{End}(W_1)$$

is an algebra isomorphism, i.e.

$$f(A+B)=f(A)+f(B),\qquad f(AB)=f(A)f(B),\qquad f(A^*)=f(A)^*$$

for all $A,B\in\mathrm{End}(W_0)$. We claim that any such isomorphism has the form

$$f(A)=\Phi A\Phi^*$$

for some unitary transformation $\Phi:W_0\to W_1$. To see this note that for any 1-dimensional complex subspace $\ell\subset W_0$ the set of endomorphisms $A\in\mathrm{End}(W_0)$ with $\mathrm{im}\,A\subset\ell$ is a minimal right ideal in $\mathrm{End}(W_0)$. Since $f:\mathrm{End}(W_0)\to\mathrm{End}(W_1)$ maps minimal right ideals to minimal right ideals it follows that f preserves the set of rank-1 endomorphisms. Now every rank-1 endomorphism $A\in\mathrm{End}(W_0)$ has the form

$$A=xy^*=x\langle y,\cdot\rangle$$

for some $x,y\in W_0$. Fix vectors $x_0,y_0\in W_0$ and $x_1,y_1\in W_1$ such that

$$f(x_0{y_0}^*)=x_1{y_1}^*,\qquad |x_1|=|x_0|=1.$$

Then f identifies the two minimal right ideals determined by x_0 and x_1:

$$f:\{A_0\in\mathrm{End}(W_0)\,|\,\mathrm{im}\,A_0\subset\mathbb{C}\,x_0\}\longrightarrow\{A_1\in\mathrm{End}(W_1)\,|\,\mathrm{im}\,A_1\subset\mathbb{C}\,x_1\}.$$

Hence there exists a function $\varphi:W_0\to W_1$ such that

$$f(x_0y^*)=x_1\varphi(y)^*$$

for $y\in W_0$. Since $f(A^*)=f(A)^*$,

$$f(z{x_0}^*)=\varphi(z){x_1}^*$$

for $z\in W_0$. Now use the condition $f(AB)=f(A)f(B)$ with $A=z{x_0}^*$ and $B=x_0y^*$ to obtain

$$f(zy^*)=\varphi(z)\varphi(y)^*$$

for $y,z\in W_0$. Since f is complex linear so is φ and hence

$$f(zy^*)=\Phi zy^*\Phi^*$$

for $y,z\in W_0$ and some $\Phi\in\mathrm{Hom}(W_0,W_1)$. Since f is bijective $\Phi\neq0$. Choose $z\in W_0$ such that $\Phi z\neq0$ and use the condition $f(A)f(B)=f(AB)$ for $A=zy^*$ and $B=yz^*$ to obtain

$$y^*\Phi^*\Phi y=|y|^2$$

for all $y\in W_0$. This shows that $\Phi^*\Phi=\mathbb{1}$ as claimed. □

Proposition 4.34 *Assume* $\dim V = 2n+1$ *and let* $\Gamma : V \to \mathrm{End}(W)$ *be a spin*^{c} *structure on* V*. Then the following holds.*

(i) Γ *extends to a surjective algebra homomorphism* $\Gamma : C^c(V) \to \mathrm{End}(W)$ *which satisfies* $\Gamma(\tilde{x}) = \Gamma(x)^*$ *for* $x \in C^c(V)$*.*

(ii) *There exists a unique vector* $\varepsilon_\Gamma \in C_{2n+1}(V)$ *such that*

$$\Gamma(\varepsilon_\Gamma) = -i^{n+1}\mathbb{1}. \tag{4.20}$$

(iii) *If* ε_Γ *is defined by (4.20) then, for every* $x \in C^c(V)$*,*

$$\Gamma(x) = 0 \qquad \Longleftrightarrow \qquad \varepsilon_\Gamma x = i^{n+1}x. \tag{4.21}$$

(iv) *Two spin*^{c} *structures* $\Gamma_0 : V \to \mathrm{End}(W_0)$ *and* $\Gamma_1 : V \to \mathrm{End}(W_1)$ *are isomorphic if and only if* $\varepsilon_{\Gamma_0} = \varepsilon_{\Gamma_1}$*.*

Proof: We prove (i). By Proposition 4.13, Γ extends to an algebra homomorphism $C^c(V) \to \mathrm{End}(W)$. The restriction of Γ to any codimension-1 subspace of V is a spin^c structure on an even dimensional vector space. Hence it follows from Proposition 4.33 that Γ is surjective.

We prove (ii). Let $e_1, \dots, e_{2n+1}$ be an orthonormal basis of V and define

$$\varepsilon = e_{2n+1} \cdots e_1 \in C_{2n+1}(V).$$

Then $\varepsilon^2 = (-1)^{n+1}$ and $\varepsilon v = v\varepsilon$ for all $v \in V$. Hence $\Gamma(\varepsilon)$ is a scalar multiple of the identity and $\Gamma(\varepsilon)^2 = (-1)^{n+1}\mathbb{1}$. Changing the ordering of the basis, if necessary, we obtain $\Gamma(\varepsilon) = -i^{n+1}\mathbb{1}$. Since $C_{2n+1}(V)$ is a one-dimensional real vector space, this equation determines $\varepsilon = \varepsilon_\Gamma$ uniquely.

We prove (iii). Denote $\varepsilon = \varepsilon_\Gamma$. If $\varepsilon x = i^{n+1}x$ then

$$i^{n+1}\Gamma(x) = \Gamma(\varepsilon x) = \Gamma(\varepsilon)\Gamma(x) = -i^{n+1}\Gamma(x)$$

and hence $\Gamma(x) = 0$. This proves the 'if' part of (4.21). The 'only if' part follows from dimensional considerations. Namely, since $\Gamma : C^c(V) \to \mathrm{End}(W)$ is surjective $\dim \ker \Gamma = 2^{2n}$. On the other hand the operator $x \mapsto x^{\mathrm{ev}} - x^{\mathrm{odd}}$ anti-commutes with $x \mapsto \varepsilon x$ and hence interchanges its eigenspaces $E^\pm = \{x \in C^c(V) \,|\, \varepsilon x = \pm i^{n+1}x\}$. Hence E^+ and E^- have the same dimension 2^{2n}. This proves (iii).

We prove (iv). Two spin^c structures Γ_0 and Γ_1 on V which satisfy $\varepsilon_{\Gamma_0} \neq \varepsilon_{\Gamma_1}$ are obviously not isomorphic. If on the other hand $\varepsilon_{\Gamma_0} = \varepsilon_{\Gamma_1}$ then the two extended operators $\Gamma_i : C^c(V) \to \mathrm{End}(W_i)$ have the same kernel. Hence there exists an algebra isomorphism $f : \mathrm{End}(W_0) \to \mathrm{End}(W_1)$ such that $\Gamma_1 = f \circ \Gamma_0$. It follows as in the proof of Proposition 4.33 that $f(A) = \Phi A \Phi^*$ for some unitary automorphism $\Phi : W_0 \to W_1$. This proves the proposition. □

Let V be an oriented real inner product space of dimension $2n+1$. A spin^c structure $\Gamma : V \to \mathrm{End}(W)$ is said to be **compatible with the orientation** if

$$\Gamma(e_{2n+1}) \cdots \Gamma(e_1) = -i^{n+1}\mathbb{1} \tag{4.22}$$

for every positively oriented orthonormal basis $e_1, \dots, e_{2n+1}$ of V. This means that

$$\varepsilon_\Gamma = e_{2n+1} \cdots e_1$$

for such a basis.

Exercise 4.35 Let $\Gamma : V \to \mathrm{End}(W)$ be a spin^c structure on an odd dimensional real vector space. Prove that the restriction of the extended homomorphism $\Gamma : C^c(V) \to \mathrm{End}(W)$ to $C^{\mathrm{ev}}(V) \otimes_{\mathbb{R}} \mathbb{C}$ is an algebra isomorphism. Prove that the restriction of Γ to $C^{\mathrm{odd}}(V) \otimes_{\mathbb{R}} \mathbb{C}$ is injective. □

Proposition 4.36 *Let V be a real inner product space of dimension $2n$ or $2n+1$. Let $\Gamma : V \to \mathrm{End}(W)$ be a spin^c structure on V.*

(i) *Let $A \in \mathrm{SO}(V)$ and $\Phi \in \mathrm{Aut}(W)$ be a unitary automorphism. Then A and Φ satisfy (4.19) if and only if there exists an $x \in \mathrm{Spin}^c(V)$ such that*

$$\Gamma(x) = \Phi, \qquad \mathrm{ad}(x) = A.$$

(ii) *If $x \in \mathrm{Spin}^c(V)$ then*

$$\det(\Gamma(x)) = \delta(x)^{2^{n-1}},$$

where $\delta : \mathrm{Spin}^c(V) \to S^1$ is given by (4.16).

(iii) *Let $\Phi \in \mathrm{End}(W)$ be a complex linear endomorphism. The Φ commutes with $\Gamma(v)$ for every $v \in V$ if and only if $\Phi = z\mathbb{1}$ for some $z \in \mathbb{C}$.*

Proof: If $\Phi = \Gamma(x)$ for some $x \in \mathrm{Spin}^c(V)$ then

$$\Phi\Gamma(v) = \Gamma(xv) = \Gamma(xv\tilde{x})\Gamma(x) = \Gamma(\mathrm{ad}(x)v)\Phi.$$

for $v \in V$ and hence (4.19) holds with $A = \mathrm{ad}(x)$. Conversely, suppose that $\Phi \in \mathrm{Aut}(W)$ and $A \in \mathrm{SO}(V)$ satisfy (4.19) and $\Phi^*\Phi = \mathbb{1}$. By Propositions 4.33 and 4.34, there exists an $x \in C^c(V)$ such that $\Phi = \Gamma(x)$. If V is odd dimensional then, by Exercise 4.35, we may choose $x \in C^{\mathrm{ev}}(V)$. In either case it follows from $\Phi^*\Phi = \mathbb{1}$ that

$$\tilde{x}x = 1.$$

By (4.19), we have

$$\Gamma(Av) = \Phi\Gamma(v)\Phi^* = \Gamma(xv\tilde{x})$$

for all $v \in V$. Since both Av and $xv\tilde{x}$ are odd elements of $C^c(V)$, and the restriction of Γ to $C^{\mathrm{odd}}(V) \otimes_{\mathbb{R}} \mathbb{C}$ is always injective, it follows that

$$xV\tilde{x} = V.$$

If $\dim V$ is odd this implies $x \in \mathrm{Spin}^c(V)$. If $\dim V$ is even it remains to prove that x is even. But this follows from Exercise 4.31 and the fact that the map $V \to V : x \mapsto xv\tilde{x}$ is orientation preserving. This proves (i).

To prove (ii) note that, by Lemma 4.27,

$$\det(\Gamma(y)) = \delta(y) = 1$$

for $y \in \mathrm{Spin}(V)$. Now write $x \in \mathrm{Spin}^c(V)$ in the form

$$x = e^{i\theta}y, \qquad y \in \mathrm{Spin}(V).$$

Then $\Gamma(x) = e^{i\theta}\Gamma(y)$ and $\delta(x) = e^{2i\theta}$. Since dim $W = 2^n$ it follows that

$$\det(\Gamma(x)) = e^{2^n i\theta}\det(\Gamma(y)) = e^{2^n i\theta} = \delta(x)^{2^{n-1}}$$

for $y \in \mathrm{Spin}(V)$ and $e^{i\theta} \in S^1$. This proves (ii). (iii) follows immediately from the fact that $\Gamma : C^c(V) \to \mathrm{End}(W)$ is surjective. □

Example 4.37 Recall from Example 4.16 that the real Clifford algebra of $\mathbb{C} = \mathbb{R}^2$ can be identified with the quaternions $\mathbb{H}$ via $e_1 \mapsto j$ and $e_2 \mapsto k$. Now the map $\gamma : \mathbb{H} \to \mathbb{C}^{2\times 2}$ given by (4.1) satisfies (4.17) and hence determines a spin^c structure on $\mathbb{R}^2$. The composition of γ with the inclusion $\mathbb{R}^2 = \mathbb{C} \to \mathbb{H}$ is the map

$$\mathbb{C} \to \mathbb{C}^{2\times 2} : z \mapsto \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix}$$

This is a spin^c structure on $\mathbb{R}^2$. □

Example 4.38 Let Λ be an oriented 3-dimensional real Hilbert space. A spin^c structure on Λ is a pair (W, γ), where W is a 2-dimensional Hermitian vector space and $\gamma : \Lambda \to \mathrm{End}(W)$ is a linear map which satisfies (4.17). It is compatible with the orientation if

$$\gamma(e_3)\gamma(e_2)\gamma(e_1) = \mathbb{1}$$

for every positively oriented orthonormal basis e_1, e_2, e_3 of Λ. These two conditions can be expressed in the equivalent form

$$\gamma(v) + \gamma(v)^* = 0, \qquad \gamma(v)\gamma(w) = \gamma(v \times w) - \langle v, w\rangle \mathbb{1}, \tag{4.23}$$

where $(v, w) \mapsto v \times w$ denotes the cross product, i.e. $v \times w$ is the unique vector which is orthogonal to v and w and satisfies

$$|v \times w|^2 = |v|^2|w|^2 - \langle v, w\rangle^2, \qquad \det(v, w, v \times w) = 1.$$

The map $\gamma : \mathrm{Im}(\mathbb{H}) \to \mathbb{C}^{2\times 2}$ given by (4.1) is an example. □

Example 4.39 Consider the map $\Gamma : \mathbb{H} \to \mathbb{C}^{4\times 4}$ given by

$$\Gamma(x) = \begin{pmatrix} 0 & \gamma(x) \\ -\gamma(x)^* & 0 \end{pmatrix}$$

where $\gamma : \mathbb{H} \to \mathbb{C}^{2\times 2}$ is the algebra homomorphism (4.1). The reader may check that this map satisfies (4.18). Hence, by Proposition 4.33, it extends to an isomorphism $\Gamma : C^c(\mathbb{H}) \to \mathbb{C}^{4\times 4}$. It is interesting to note that

$$\Gamma(e_0e_1) = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \Gamma(e_0e_2) = \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}, \quad \Gamma(e_0e_3) = \begin{pmatrix} K & 0 \\ 0 & -K \end{pmatrix},$$

$$\Gamma(e_2e_3) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad \Gamma(e_3e_1) = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}, \quad \Gamma(e_1e_2) = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}.$$

This shows that Γ identifies the Lie algebra of $\mathrm{Spin}(4)$ with $\mathfrak{su}(2) \times \mathfrak{su}(2)$ and it follows again that $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$. □

The formula in Example 4.39 shows that there is a splitting of the 4-dimensional spin^c representation into

$$W = \mathbb{C}^4 = \mathbb{C}^2 \oplus \mathbb{C}^2$$

which is preserved under the even elements of the Clifford algebra. Such a splitting exists in every spin^c representation. However, in 4 dimensions it is related to the splitting

$$\Lambda^2 = \Lambda^{2,+} \oplus \Lambda^{2,-}$$

of the space of 2-forms into the self-dual and anti-self-dual ones. More precisely, identify $\Lambda^2 V^*$ with $C_2(V)$ in the obvious way via $e_i^* \wedge e_j^* \mapsto e_ie_j$ for $i \neq j$. Then Γ induces a map $\Lambda^2 \to \mathrm{End}(\mathbb{C}^4)$ and the formulae for $\Gamma(e_ie_j)$ in Example 4.39 show that the self-dual forms only act on $\mathbb{C}^2 \oplus \{0\}$ while the anti-self-dual ones act on $\{0\} \oplus \mathbb{C}^2$. This is discussed in more detail in Section 4.8 below.

4.5 Splitting and orientation

Let (W, Γ) be a spin^c structure on V. Fix an orientation of V and a positively oriented orthonormal basis $e_1, \ldots, e_{2n}$, and denote

$$\varepsilon = e_{2n} \cdots e_2 e_1 \in C(V).$$

This element is independent of the choice of the basis used to define it. By Exercise 4.28,

$$\varepsilon^2 = (-1)^n.$$

Hence there is a splitting

$$W = W^+ \oplus W^-$$

into the eigenspaces of $\Gamma(\varepsilon)$:

$$W^\pm = \{\theta \in W \,|\, \Gamma(\varepsilon)\theta = \pm i^n \theta\}.$$

Note that changing the orientation of V interchanges the spaces W^+ and W^-. Note also that $\Gamma(v)$ maps W^- to W^+ (and W^+ to W^-) for every $v \in V$ and hence

$$\dim_{\mathbb{C}} W^+ = \dim_{\mathbb{C}} W^- = 2^{n-1}.$$

It follows that the subspaces $W^\pm$ are invariant under the even elements of the Clifford algebra. Since $C^c(V) \cong \mathrm{End}(W)$ it is easy to see that $C^{\mathrm{ev}}(V) \otimes \mathbb{C}$ acts transitively on $W^\pm - \{0\}$.

Lemma 4.40 *Up to interchanging $+$ and $-$ the splitting $W = W^+ \oplus W^-$ is uniquely determined by the condition*

$$\Gamma(v) W^\pm \subset W^\mp$$

for every $v \in V$.

Proof: Let $W = W_1 \oplus W_2$ be any splitting such that $\Gamma(v)$ interchanges W_1 and W_2. It remains to prove that either $W_1 = W^+$ or $W_1 = W^-$. By assumption, W_1 and W_2 are invariant under $\Gamma(x)$ for all even elements $x \in C^{\mathrm{ev}}(V)$ of the Clifford algebra and hence under $\Gamma(\varepsilon)$. Hence

$$W_1 = W_1^+ \oplus W_1^-, \qquad W_1^\pm = W_1 \cap W^\pm.$$

Assume without loss of generality that $W_1^+ \neq \{0\}$ and choose a nonzero vector $\theta \in W_1^+$. Then $\Gamma(x)\theta \in W_1^+$ for all even elements $x \in C^{\mathrm{ev}}(V) \otimes \mathbb{C}$. Since the even part of the Clifford algebra acts transitively on $W^+ - \{0\}$ it follows that $W^+ \subset W_1$. Hence, for dimensional reasons, $W^+ = W_1$. This proves the lemma. □

The restrictions of $\Gamma(v)$ to W^- for $v \in V$ determine a linear map $\gamma : V \to \mathrm{Hom}(W^-, W^+)$ which satisfies

$$\gamma(v)^*\gamma(v) = |v|^2\mathbb{1} \tag{4.24}$$

for every $v \in V$. The spin^c structure $\Gamma : V \to \mathrm{End}(W)$ can be recovered from γ via $W = W^+ \oplus W^-$ and

$$\Gamma(v) = \begin{pmatrix} 0 & \gamma(v) \\ -\gamma(v)^* & 0 \end{pmatrix}. \tag{4.25}$$

Obviously, γ satisfies (4.24) if and only if Γ satisfies (4.18). By Proposition 4.33, this shows that every linear map $\gamma : V \to \mathrm{Hom}(W^-, W^+)$ which satisfies (4.24) determines a natural isomorphism $\Gamma : C^c(V) \to \mathrm{End}(W^+ \oplus W^-)$ via (4.25).

Lemma 4.41 *Assume that $\gamma : V \to \mathrm{Hom}(W^-, W^+)$ satisfies (4.24) with $\dim_{\mathbb{C}} W^\pm = 2^{n-1}$ and $\dim_{\mathbb{R}} V = 2n \geq 4$. Then*

$$\det(\gamma(v)^*\gamma(w)) = |v|^{2^{n-1}}|w|^{2^{n-1}}$$

for $v, w \in V$.

Proof: We prove first that

$$\mathrm{trace}(\gamma(v)^*\gamma(w)) = 2^{n-1}\langle v, w\rangle. \tag{4.26}$$

To see this note that W^- is invariant under $\Gamma(\xi)$ for $\xi \in C_2(V)$, and that the Lie algebra homomorphism $C_2(V) \to i\mathbb{R} : \xi \mapsto \mathrm{trace}(\Gamma(\xi)|_{W^-})$ is the differential of the Lie group homomorphism $\mathrm{Spin}(V) \to S^1 : x \mapsto \det(\Gamma(x)|_{W^-})$. By Lemma 4.27, this homomorphism is trivial for $\dim V = 2n \geq 4$. Hence $\mathrm{trace}(\Gamma(\xi)|_{W^-}) = 0$ for all $\xi \in C_2(V)$. Apply this to the element $\xi = vw + \langle v, w\rangle \in C_2(V)$ with

$$\Gamma(vw + \langle v, w\rangle)|_{W^-} = -\gamma(v)^*\gamma(w) + \langle v, w\rangle\mathbb{1}.$$

to obtain (4.26).

Now consider the path $[0, 1] \to V : t \mapsto v(t) = w + t(v - w)$. Then, by (4.26), the functions

$$f_1(t) = \det(\gamma(v(t))^*\gamma(w)), \qquad f_2(t) = |v(t)|^{2^{n-1}}|w|^{2^{n-1}}$$

both satisfy the differential equation

$$\dot{f} = 2^{n-1}|v|^{-2}\langle v, \dot{v}\rangle f, \qquad f(0) = |w|^{2^n},$$

and hence must be equal for all t. This proves the lemma. □

Exercise 4.42 Let V be a 4-dimensional real Hilbert space.

(i) Show that a spin^c structure on V can be defined as a triple (W^+, W^-, γ) where $W^\pm$ are 2-dimensional Hermitian vector spaces and

$$\gamma : V \to \mathrm{Hom}(W^-, W^+)$$

is a linear map which satisfies (4.24) and

$$\gamma(e_0)^*\gamma(e_1)\gamma(e_2)^*\gamma(e_3) = \mathbb{1} \tag{4.27}$$

for every positively oriented orthonormal basis of V. More precisely, if $W = W^+ \oplus W^-$ and $\Gamma : V \to \mathrm{End}(W)$ is given by (4.25), show that W^+ is the -1-eigenspace and W^- the $+1$-eigenspace of $\Gamma(e_3e_2e_1e_0)$.

(ii) Show that the map $\gamma : \mathbb{H} \to \mathbb{C}^{2\times 2}$, defined by (4.1), satisfies (4.27).

(iii) Let $W^\pm$, γ, and Γ be as in (i). Prove that $\Gamma : C^c(V) \to \mathrm{End}(W)$ identifies $\mathrm{Spin}^c(V)$ with the endomorphisms of the form

$$\Gamma(x) = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}, \quad U \in \mathrm{U}(W^+), \quad V \in \mathrm{U}(W^-), \quad \det(U) = \det(V).$$

Prove that the maps $\delta : \mathrm{Spin}^c(V) \to S^1$ and $\mathrm{ad} : \mathrm{Spin}^c(V) \to \mathrm{SO}(V)$ are given by

$$\delta(x) = \det(U) = \det(V), \qquad \gamma(\mathrm{ad}(x)v) = U\gamma(v)V^*$$

for $v \in V$, $x \in \mathrm{Spin}^c(V)$, $U \in \mathrm{U}(S^+)$, $V \in \mathrm{U}(S^-)$ with $\Gamma(x) = \mathrm{diag}(U, V)$.
Hint: Use the formulae for $\Gamma(e_ie_j)$ in Example 4.39. □

The previous exercise shows that there is a Lie group isomorphism

$$\mathrm{SU}(W^+) \times_{\mathbb{Z}_2} \mathrm{SU}(W^-) \xrightarrow{\cong} \mathrm{SO}(V).$$

This gives rise to an action of $\mathrm{SO}(V)$ on the Lie algebras $\mathfrak{su}(W^+)$ and $\mathfrak{su}(W^-)$ via the adjoint action of $\mathrm{SU}(W^\pm)$ on their respective Lie algebras. In Section 4.8 it will be shown that these Lie algebras can be naturally identified with the spaces $\Lambda^\pm$ of self-dual, respectively anti-self-dual, 2-forms. This gives rise to exact sequences

$$1 \longrightarrow \mathrm{SU}(W^\mp) \longrightarrow \mathrm{SO}(V) \longrightarrow \mathrm{SO}(\Lambda^\pm) \longrightarrow 1$$

where the inclusions $\mathrm{SU}(W^\mp) \to \mathrm{SO}(V)$ are given by the above identification of $\mathrm{Spin}(V)$ with $\mathrm{SU}(W^+) \times \mathrm{SU}(W^-)$ and the homomorphisms $\mathrm{SO}(V) \to \mathrm{SO}(\Lambda^\pm)$ are given by the obvious action of $\mathrm{SO}(V)$ on $\Lambda^\pm$.

4.6 Spin representations

Recall from (4.7) that the Clifford algebra of a finite dimensional real Hilbert space V has a quaternionic structure if $\dim V \equiv 2, 3, 4 (\mathrm{mod}\ 8)$, a real structure if $\dim V \equiv 0, -1, -2 (\mathrm{mod}\ 8)$ and a complex structure if $\dim V \equiv 1 (\mathrm{mod}\ 4)$. In the quaternionic case this leads to the following definition of a spin structure.

Definition 4.43 *Let V be a real inner product space of dimension $2n \equiv 2, 4 (\mathrm{mod}\ 8)$ or $2n+1 \equiv 3 (\mathrm{mod}\ 8)$. A* **spin structure** *on V is a quadruple (S, I, J, Γ) where S is a 2^{n+1}-dimensional real Hilbert space, I and J are two anti-commuting orthogonal complex structures, i.e.*

$$I^{-1} = I^* = -I, \qquad J^{-1} = J^* = -J, \qquad IJ = -JI,$$

and $\Gamma : V \to \mathrm{End}(S)$ is a real linear map which satisfies (4.18) and commutes with both I and J, i.e.

$$\Gamma(v)I = I\Gamma(v), \qquad \Gamma(v)J = J\Gamma(v).$$

for $v \in V$. A **spin isomorphism**

$$(V_0, S_0, I_0, J_0, \Gamma_0) \to (V_1, S_1, I_1, J_1, \Gamma_1)$$

is a pair (A, Φ) where $A : V_0 \to V_1$ and $\Phi : S_0 \to S_1$ are orientation preserving orthogonal transformations such that (4.19) holds and Φ commutes with both I and J. The set of spin isomorphisms is denoted by

$$\mathrm{Hom}^{\mathrm{spin}}(S_0, S_1) = \{(A, \Phi) \,|\, (4.19),\ I\Phi = \Phi I,\ J\Phi = \Phi J\}.$$

Definition 4.43 can be rephrased in the form that a spin structure on a real Hilbert space V of dimension 2, 3, or 4 modulo 8 is an irreducible representation of the real Clifford algebra $C(V)$. As in the complex case the group $\mathrm{Hom}^{\mathrm{spin}}(S, S)$ of automorphisms of such a structure is isomorphic to $\mathrm{Spin}(V)$ and there is a commutative diagram

$$\begin{array}{ccc}
C(V) & \xrightarrow{\Gamma} & \mathrm{End}_{\mathbb{H}}(S) \\
\cup & & \cup \\
\mathrm{Spin}(V) & \xrightarrow{\Gamma} & \mathrm{Hom}^{\mathrm{spin}}(W, W) \\
& {}_{\mathrm{ad}}\searrow \quad \swarrow & \\
& \mathrm{SO}(V) &
\end{array}\cdot$$

Here $\mathrm{End}_{\mathbb{H}}(S)$ denotes the set of all real linear isomorphisms of S which commute with both I and J. As before the map $\mathrm{Hom}^{\mathrm{spin}}(S, S) \to \mathrm{SO}(V)$ is the projection $(A, \Phi) \mapsto A$.

Note that the existence of two anti-commuting complex structures I and J on S is equivalent to the existence of an algebra homomorphism $R : \mathbb{H} \to \mathrm{End}(S)$ which satisfies

$$R(ab) = R(a)R(b), \qquad R(\bar{a}) = R(a)^*, \qquad R(1) = 1\!\!1.$$

Any such homomorphism is uniquely determined by the complex structures

$$I = R(i), \qquad J = R(j), \qquad K = R(k).$$

These satisfy the usual quaternionic commutation rules.

Exercise 4.44 Let S be any real vector space with two anti-commuting complex structures I and J. Prove that the dimension of S is divisible by 4. **Hint:** Prove that there exists an inner product on S with respect to which both I and J are orthogonal. Then show that if a linear subspace $E \subset S$ is invariant under I and J then so is its orthogonal complement. □

Any two anti-commuting complex structures I and J give rise to a 2-sphere of complex structures $J_a = R(a)$, parametrized by the imaginary unit quaternions $a \in \mathrm{Im}(\mathbb{H})$, $|a| = 1$. In this 2-sphere of complex structures the equator orthogonal to I forms a circle of complex structures which anti-commute with I. If, moreover, I and J commute with Γ then all the complex structures on this circle also commute with Γ. This relates to spin^c structures as follows. If (W, Γ) is a spin^c structure on V as in Definition 4.32 denote by S the underlying real Hilbert space obtained from W by forgetting the complex structure. The complex structure then becomes a real linear transformation $I \in \mathrm{End}(S)$ which satisfies

$$I^* = I^{-1} = -I, \qquad I\Gamma = \Gamma I.$$

Consider the set

$$Q(S, I, \Gamma) = \left\{ J \in \mathrm{End}(S) \,|\, J^* = J^{-1} = -J,\ IJ = -JI,\ J\Gamma = \Gamma J \right\}.$$

This set is a circle which for any given $J_0 \in Q(S, I, \Gamma)$ can be parametrized by the function

$$S^1 \to Q(S, I, \Gamma) : e^{i\theta} \mapsto J_\theta = \cos\theta J_0 + \sin\theta I J_0.$$

To see this just note that if $J \in Q(S, I, \Gamma)$ then JJ_0 commutes with I and with $\Gamma(v)$ for every $v \in V$. Hence, by Proposition 4.36 (iii), JJ_0 is given by multiplication with a unit complex number and this implies $J = J_\theta$ for some θ. This shows that for every spin^c structure (W, Γ) on V there is a circle of complex structures on W which convert the spin^c structure into a spin structure. This works only in dimensions 2, 3, or 4 modulo 8 because for all other dimensions there simply is no complex structure J which anti-commutes with I and commutes with Γ. That such a complex structure does exist in dimensions 2, 3, or 4 modulo 8 is a consequence of the classification theorem 4.19 for Clifford algebras. In dimensions 3 and 4 there are the following natural examples.

Example 4.45 Assume $\dim V = 3$, identify $V = \mathrm{Im}(\mathbb{H})$ and define $S = \mathbb{H}$. Consider the maps $\Gamma_0 : \mathrm{Im}(\mathbb{H}) \to \mathrm{End}(\mathbb{H})$ and $R_0 : \mathbb{H} \to \mathrm{End}(\mathbb{H})$ given by

$$\Gamma_0(v)\xi = v\xi, \qquad R_0(a)\xi = \xi\bar{a},$$

for $v \in \mathrm{Im}(\mathbb{H})$ and $a, \xi \in \mathbb{H}$. Then Γ_0, $I_0 = R_0(i)$, and $J_0 = R_0(j)$ satisfy the requirements of Definition 4.43. □

Example 4.46 If $\dim V = 4$ identify $V = \mathbb{H}$ and define $S = \mathbb{H} \oplus \mathbb{H}$. Consider the maps $\Gamma_1 : \mathbb{H} \to \mathrm{End}(\mathbb{H} \oplus \mathbb{H})$ and $R_1 : \mathbb{H} \to \mathrm{End}(\mathbb{H} \oplus \mathbb{H})$ given by

$$\Gamma_1(v)\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} v\eta \\ -\bar{v}\xi \end{pmatrix}, \qquad R_1(a)\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \xi\bar{a} \\ \eta\bar{a} \end{pmatrix},$$

for $v, a, \xi, \eta \in \mathbb{H}$. Then Γ_1, $I_1 = R_1(i)$, and $J_1 = R_1(j)$ satisfy the requirements of Definition 4.43. □

Lemma 4.47 *Let V be a real Hilbert space of dimension 2, 3, or 4 modulo 8 and (S, I, J, Γ) be a spin structure on V. Let $R : \mathbb{H} \to \mathrm{End}(S)$ be the unique algebra homomorphism which satisfies $R(i) = I$ and $R(j) = J$.*
(i) *If $\Phi \in \mathrm{End}(S)$ commutes with $\Gamma(v)$ for every $v \in V$ then $\Phi = R(a)$ for some $a \in \mathbb{H}$.*
(ii) *If $\Phi \in \mathrm{End}(S)$ commutes with $R(a)$ for every $a \in \mathbb{H}$ then $\Phi = \Gamma(x)$ for some $x \in C(V)$.*

Proof: Assume that $\dim V = 8k+2$ or $\dim V = 8k+4$. To prove (i) suppose that Φ commutes with $\Gamma(v)$ for every $v \in V$. Fix a complex structure $I = R(i)$ and write $\Phi = \Phi' + \Phi''$, where

$$\Phi' = \frac{1}{2}(\Phi - I\Phi I), \qquad \Phi'' = \frac{1}{2}(\Phi + I\Phi I)$$

denote the complex linear and complex anti-linear parts of Φ. They both commute with $\Gamma(v)$ for every v. Moreover, by Proposition 4.33, every complex linear transformation of S has the form $\Phi' = \Gamma(x) + I\Gamma(y)$ for some $x, y \in C(V)$. Since Φ' commutes with $\Gamma(v)$ for every $v \in V$ it follows from Proposition 4.36 that $x, y \in C_0(V) = \mathbb{R}$ and hence $\Phi' = x\mathbb{1} + yI$ for some $x, y \in \mathbb{R}$. A similar argument for the complex linear map $J\Phi''$ shows that $\Phi'' = zJ + wK$ for some $w, z \in \mathbb{R}$ where $J = R(j)$ and $K = R(k)$. Hence

$$\Phi = R(x + iy + jz + kw).$$

To prove (ii) suppose that Φ commutes with $R(a)$ for every $a \in \mathbb{H}$. Since Φ commutes with $I = R(i)$ it follows as above that $\Phi = \Gamma(x) + I\Gamma(y)$ for some $x, y \in C(V)$. Since $\Phi J = J\Phi$ for $J = R(j)$ it follows that $\Gamma(y) = 0$. This proves the lemma in the case where V has even rank. The case $\operatorname{rank} V = 8k+3$ is left to the reader. □

Exercise 4.48 Assume that $\dim V \equiv 0, 6 (\bmod\ 8)$ and let (S, I, Γ) be a spin^c structure on V. Prove that there exists a complex anti-linear orthogonal involution $T \in \operatorname{End}(S)$ which commutes with Γ:

$$T^* = T^{-1} = T, \qquad IT = -TI, \qquad \Gamma T = T\Gamma.$$

Prove that these involutions form a circle $Q(S, I, \Gamma)$. Define a spin structure on V as a puadruple (S, I, T, Γ) where (S, I, Γ) is a spin^c structure and $T \in Q(S, I, \Gamma)$. Prove that the group of automorphisms of such a spin structure is naturally isomorphic to $\operatorname{Spin}(V)$. Extend this to the case $\dim V \equiv 7 (\bmod\ 8)$. Interprete the Cayley numbers as an example (see Example 4.20). □

Exercise 4.49 Assume that $\dim V \equiv 5 (\bmod\ 8)$ and let (S, I, Γ) be a spin^c structure on V. Prove that for every unit vector $v \in V$ there exists an automorphism $J \in \operatorname{End}(S)$ such that

$$J^* = J^{-1} = -J, \qquad JI = -IJ, \tag{4.28}$$

and

$$J\Gamma(w) = \Gamma(w - 2\langle w, v\rangle v)J \tag{4.29}$$

for every $w \in V$. Denote the set of such pairs (v, J) by $R(S, I, \Gamma)$. Prove that the group $\operatorname{Spin}^c(V)$ acts on $R(S, I, \Gamma)$ via

$$(v, J) \mapsto (\operatorname{ad}(x)v, \Gamma(x)J\Gamma(\tilde{x})).$$

Prove that the quotient $Q(S, I, \Gamma) = R(S, I, \Gamma)/\operatorname{Spin}^c(V)$ is a circle. Define a spin structure on V as a spin^c structure (S, I, Γ) together with an element of the circle $Q(S, I, \Gamma)$. Prove that the group of automorphisms of such a spin structure is naturally isomorphic to $\operatorname{Spin}(V)$. **Hint:** Denote $\varepsilon = e_n \cdots e_1 \in C(V)$ as in Exercise 4.28. Show that $\Gamma(\varepsilon) = \pm i\mathbb{1}$ whenever $\dim V \equiv 1 (\bmod\ 4)$. Prove that Γ defines a spin^c structure on the orthogonal complement of v. □

Exercise 4.50 Assume that $\dim V \equiv 1 (\bmod\ 8)$. Define a spin structure on V as in Exercise 4.49, but with J replaced by an orthogonal involution $T \in \operatorname{End}(S)$ which anti-commutes with $\Gamma(v)$ and commutes with $\Gamma(w)$ whenever w is perpendicular to v. □

4.7 Complex vector spaces

Let V be an oriented real Hilbert space of dimension $2n$. A complex structure $J : V \to V$ is called **compatible with the orientation and metric** if $J^* = J^{-1} = -J$ and every basis of the form $e_1, Je_1, \ldots, e_n, Je_n$ is positively oriented. The space of such complex structures will be denoted by $\mathcal{J}^+(V)$. The space of complex structures which are compatible with the metric and the opposite orientation will be denoted by $\mathcal{J}^-(V)$.

Let $\Gamma : V \to \mathrm{End}(W)$ be a spin^c structure. Every unit vector $\tau \in W$ determines an isometric embedding $V \hookrightarrow W : v \mapsto \Gamma(v)\tau$. The image of this embedding is a real linear subspace which we shall denote by

$$V_\tau = \{\Gamma(v)\tau \,|\, v \in V\} \subset W. \tag{4.30}$$

If V_τ is a complex subspace, then the embedding determines a complex structure $J_\tau : V \to V$ which under the embedding corresponds to multiplication by i. Thus J_τ is given by

$$\Gamma(J_\tau v)\tau = i\Gamma(v)\tau \tag{4.31}$$

for $v \in V$. The next lemma shows that $J_\tau \in \mathcal{J}^+(V)$ whenever $\tau \in W^+$ and that every $J \in \mathcal{J}^+(V)$ has the form J_τ for some $\tau \in W^+$.

Lemma 4.51 *Let $\Gamma : V \to \mathrm{End}(W)$ be a spin^c structure on an oriented real Hilbert space V of dimension $2n$. Let $\tau \in W$ be a unit vector. If V_τ is a complex subspace of W then $\tau \in W^+ \cup W^-$ and*

$$J_\tau \in \mathcal{J}^\pm(V) \qquad \Longleftrightarrow \qquad \tau \in W^\pm. \tag{4.32}$$

Conversely, if $J \in \mathcal{J}^\pm(V)$ then

$$E_J = E_{J,\Gamma} = \bigcap_{v \in V} \ker\left(\Gamma(Jv) - i\Gamma(v)\right) \tag{4.33}$$

is a 1-dimensional complex subspace of $W^\pm$.

Proof: Fix a unit vector $\tau \in W$ such that V_τ is a complex subspace of W. Since $V \to V_\tau : v \mapsto \Gamma(v)\tau$ is an isometric embedding the complex structure $J = J_\tau$ on V is orthogonal. Hence it remains to examine the orientation induced by J. We first observe that, by (4.31),

$$v \in V, \quad |v| = 1 \qquad \Longrightarrow \qquad \Gamma(Jv)\Gamma(v)\tau = i\tau.$$

Choose a basis of V of the form $e_1, Je_1, \ldots, e_n, Je_n$. Then

$$\Gamma(Je_n)\Gamma(e_n)\cdots\Gamma(Je_1)\Gamma(e_1)\tau = i^n\tau.$$

If this basis is positively oriented then $\tau \in W^+$, and if it is negatively oriented we obtain $\tau \in W^-$. This proves (4.32).

Now let $J \in \mathcal{J}^+(V)$ and $E_J \subset W$ be defined by (4.33). It follows from (4.32) that $E_J \subset W^+$. Choose an orthonormal basis of V of the form $e_1, Je_1, \ldots, e_n, Je_n$ and consider the operators

$$T_\nu = \Gamma(Je_\nu) - i\Gamma(e_\nu) \in \mathrm{End}(W)$$

for $\nu = 1, \ldots, n$. These operators anti-commute and are nilpotent:

$$T_\nu T_\mu + T_\mu T_\nu = 0, \qquad {T_\nu}^2 = 0$$

for $\mu \neq \nu$. We prove by induction that

$$\dim \bigcap_{\nu=k+1}^{n} \ker T_\nu = 2^k \tag{4.34}$$

for $k = 0, \ldots, n-1$. For $k = n-1$ this follows from the fact that

$${T_n}^* T_n = 2\mathbb{1} - 2i\Gamma(e_n)\Gamma(Je_n).$$

Now, for any two orthogonal vectors $v, w \in V$, we have $(\Gamma(v)\Gamma(w))^2 = -\mathbb{1}$ and the eigenspaces of $\Gamma(v)\Gamma(w)$ corresponding to the two eigenvalues $\pm i$ are isomorphic via $\Gamma(v)$. Hence they both have the same dimension 2^{n-1}. This shows that $\dim \ker T_n = 2^{n-1}$. Now suppose that (4.34) has been proved for any $k \leq n-1$. Denote

$$V_k = \mathrm{span}\{e_1, Je_1, \ldots, e_k, Je_k\}, \qquad W_k = \bigcap_{\nu=k+1}^{n} \ker T_\nu.$$

Then Γ determines a spin^c structure $\Gamma_k : V_k \to \mathrm{End}(W_k)$. Hence the first step of the induction shows that $\dim W_{k-1} = \dim (W_k \cap \ker T_k) = 2^{k-1}$. This proves (4.34) with k replaced by $k-1$. With $k = 0$ it follows that $E_J = \bigcap_{\nu=1}^{n} \ker T_\nu$ has complex dimension one. This proves the lemma. □

The one dimensional subspace $E_J \subset W^+$, defined by (4.33), is called the space of **pure spinors**. Note that, for every unit vector $\tau \in W^+$, $\tau \in E_J$ if and only if V_τ is a complex subspace of W^- and $J_\tau = J$. Hence, by Lemma 4.51, the pure spinors determine a natural embedding

$$\mathcal{J}^+(V) \to \mathbb{P}W^+ : J \mapsto E_J.$$

This situation is particularly simple and beautiful in the 4-dimensional case. In this case, for dimensional reasons, $V_\tau = W^-$ for every unit vector $\tau \in W^+$ and hence $\mathcal{J}^+(V) \cong \mathbb{P}W^+$. This observation will play an important role in distinguishing homotopy classes of almost complex structures on 4-manifolds which induce isomorphic spin^c structure.

The canonical spin^c structure

Every Hermitian vector space (V, J, ω) admits a canonical spin^c structure. To describe this structure let us recall some notation from Section 3.1. The inner product on V, induced by J and ω, will be denoted by $g : V \times V \to \mathbb{R}$ and the Hermitian form by $\langle \cdot, \cdot \rangle$. As before, denote by V^* the real dual space of V, by $\bar{V}$ the vector space with the reversed complex structure, by $\mathrm{Hom}(V, \mathbb{C}) = \Lambda^{1,0}V^*$ the space of complex linear functionals, and by $\mathrm{Hom}(\bar{V}, \mathbb{C}) = \Lambda^{0,1}V^*$ the space of complex anti-linear functionals. Recall that there are natural isomorphisms

$$\bar{V} \to \mathrm{Hom}(V, \mathbb{C}) : v \mapsto v', \qquad V \to \mathrm{Hom}(\bar{V}, \mathbb{C}) : v \mapsto v''$$

given by

$$v' = v^* + i(Jv)^* = \langle v, \cdot \rangle, \qquad v'' = v^* - i(Jv)^* = \langle \cdot, v \rangle.$$

These satisfy $(Jv)' = -iv'$ and $(Jv)'' = iv''$.

Consider the complex vector space

$$W_{\mathrm{can}} = \Lambda^{0,*}V^*$$

of all alternating forms on V which are complex anti-linear (with respect to J) in all variables. This space is of complex dimension 2^n and carries a natural Hermitian structure as defined in Section 3.1. The homomorphism $\Gamma_{\mathrm{can}} : V \to \mathrm{End}(W_{\mathrm{can}})$ is defined by

$$\Gamma_{\mathrm{can}}(v)\tau = \frac{1}{\sqrt{2}} v'' \wedge \tau - \sqrt{2}\iota(v)\tau \tag{4.35}$$

for $v \in V$ and $\tau \in W$. Sometimes it will be convenient to stress the dependence on the complex structure J and then we will write $\Gamma_J : V \to \mathrm{End}(W_J)$ instead of $\Gamma_{\mathrm{can}} : V \to \mathrm{End}(W_{\mathrm{can}})$.

Lemma 4.52 *The operator $\Gamma_{\mathrm{can}}(v) : W_{\mathrm{can}} \to W_{\mathrm{can}}$ satisfies (4.18) and*

$$W_{\mathrm{can}}^+ = \bigoplus_{k \text{ even}} \Lambda^{0,k}V^*, \qquad W_{\mathrm{can}}^- = \bigoplus_{k \text{ odd}} \Lambda^{0,k}V^*.$$

The subspace of pure spinors is given by

$$E_{\mathrm{can}} = E_{J,\Gamma_{\mathrm{can}}} = \Lambda^{0,0}V^* = \mathbb{C}.$$

Proof: Recall from Lemma 3.4 that

$$\langle \tau, v'' \wedge \sigma \rangle = 2\langle \iota(v)\tau, \sigma \rangle.$$

for $\tau \in \Lambda^{0,k}V^*$, $\sigma \in \Lambda^{0,k-1}V^*$, and $v \in V$. This immediately implies

$$\Gamma_{\mathrm{can}}(v)^* = -\Gamma_{\mathrm{can}}(v).$$

To prove the second formula in (4.18) note that

$$\iota(v)v'' = \langle v, v\rangle = |v|^2.$$

Hence

$$\begin{aligned}
\Gamma_{\mathrm{can}}(v)\Gamma_{\mathrm{can}}(v)\tau &= \frac{1}{\sqrt{2}}v'' \wedge \left(\frac{1}{\sqrt{2}}v'' \wedge \tau - \sqrt{2}\iota(v)\tau\right) \\
&\qquad - \sqrt{2}\iota(v)\left(\frac{1}{\sqrt{2}}v'' \wedge \tau - \sqrt{2}\iota(v)\tau\right) \\
&= -\, v'' \wedge \iota(v)\tau - \iota(v)(v'' \wedge \tau) \\
&= -\,(\iota(v)v'')\tau \\
&= -\,|v|^2\tau
\end{aligned}$$

for $v \in V$ and $\tau \in \Lambda^{0,*}V^*$. This proves (4.18). Now the operator $\Gamma(v)$ obviously interchanges the subspaces W^+_{can} and W^-_{can}. Hence Lemma 4.40 shows that $W^\pm_{\mathrm{can}}$ are the two eigenspaces of $\Gamma_{\mathrm{can}}(\varepsilon)$. It remains to prove that $W^+_{\mathrm{can}} = \Lambda^{0,\mathrm{ev}}V^*$ is the eigenspace with eigenvalue i^n. To see this consider $1 \in \Lambda^{0,0}V^* \subset W^+_{\mathrm{can}}$ and note that

$$\Gamma_{\mathrm{can}}(Jv)\Gamma_{\mathrm{can}}(v)1 = \frac{1}{\sqrt{2}}\Gamma_{\mathrm{can}}(Jv)v'' = -\iota(Jv)v'' = i\iota(v)v'' = i$$

for $v \in V$ with $|v| = 1$. Hence $\Gamma_{\mathrm{can}}(\varepsilon)1 = i^n$ as required. If $\tau \in \Lambda^{0,0}V^*$ then, by definition,

$$\Gamma_{\mathrm{can}}(Jv)\tau = \frac{1}{\sqrt{2}}(Jv)'' \wedge \tau = i\Gamma_{\mathrm{can}}(v)\tau.$$

Hence the last assertion follows from Lemma 4.51. □

The unitary spin group

Let (W, Γ) be any spin^c structure on a Hermitian vector space (V, J, ω) of real dimension $2n$. Denote by $\mathrm{U}(V, J) \subset \mathrm{SO}(V)$ the subgroup of unitary transformation and define

$$\mathrm{U}^c(V, J) = \{x \in \mathrm{Spin}^c(V) \,|\, \mathrm{ad}(x) \in \mathrm{U}(V, J)\}.$$

Here $\mathrm{ad} : \mathrm{Spin}^c(V) \to \mathrm{SO}(V)$ denotes the homomorphism (4.10). The group $\mathrm{U}^c(V, J)$ has two natural characters:

$$
\begin{array}{ccccc}
\mathrm{U}^c(V,J) & \hookrightarrow & \mathrm{Spin}^c(V) & \stackrel{\delta}{\longrightarrow} & S^1 \\
\mathrm{ad}\downarrow & & & & . \\
\mathrm{U}(V,J) & & \stackrel{\det^c}{\longrightarrow} & & S^1
\end{array}
$$

The next lemma asserts that the subspace $E_{J,\Gamma}$ of pure spinors is invariant under $\mathrm{U}^c(V,J)$ and that the action of $\mathrm{U}^c(V,J)$ on $E_{J,\Gamma}$ is given by a square root of the quotient of the above two characters. The existence of such a square root implies that $\mathrm{U}^c(V,J)$ is isomorphic to the product $\mathrm{U}(V,J)\times S^1$.

Lemma 4.53 **(i)** *$E_{J,\Gamma}$ is invariant under $\mathrm{U}^c(V,J)$. Let $\Theta : \mathrm{U}^c(V,J) \to S^1$ denote the corresponding character, given by $\Theta(x)\tau = \Gamma(x)\tau$ for $x \in \mathrm{U}^c(V,J)$ and $\tau \in E_{J,\Gamma}$.*
(ii) *If $x \in \mathrm{U}^c(V,J)$ then*

$$\Theta(x)^2 \mathrm{det}^c(\mathrm{ad}(x)) = \delta(x), \qquad x \in \mathrm{U}^c(V,J).$$

(iii) *There is a natural isomorphism $\mathrm{U}^c(V,J) \to \mathrm{U}(V,J) \times S^1$.*

Proof: We examine the action of $\mathrm{U}^c(V,J)$ on W in the standard model with $W_{\mathrm{can}} = \Lambda^{0,*}V^*$ and $E_{\mathrm{can}} = \Lambda^{0,0}V^*$. Choose an orthonormal basis of V of the form $e_1, Je_1, \dots, e_n, Je_n$. Then the Lie algebra $\mathrm{Lie}(\mathrm{U}^c(V,J))$ consists of all elements $\xi \in C^c(V)$ of the complexified Clifford algebra which have the form

$$\xi = i\theta + \sum_{j<k} a_{jk}(e_je_k + (Je_j)(Je_k)) + \sum_{j,k} b_{jk}e_j(Je_k) \tag{4.36}$$

with real coefficients $a_{jk} = -a_{kj}$ and $b_{jk} = b_{kj}$. The action of $\mathrm{Lie}(\mathrm{U}^c(V,J))$ on W_{can} is determined by the formula (4.35). For $\tau \in \Lambda^{0,0}V^*$ this gives

$$\Gamma_{\mathrm{can}}(v)\Gamma_{\mathrm{can}}(w)\tau = \frac{1}{2}v'' \wedge w'' \wedge \tau - \langle v,w\rangle\tau.$$

With $\xi \in \mathrm{Lie}(\mathrm{U}^c(V,J))$ given by (4.36) it is easy to check that

$$\Gamma_{\mathrm{can}}(\xi)\tau = i\theta\tau - i\sum_{j=1}^n b_{jj}\tau$$

for $\tau \in \Lambda^{0,0}V^*$. This shows that E_{can} is invariant under $\mathrm{U}^c(V,J)$ and that the infinitesimal character $\dot{\Theta} : \mathrm{Lie}(\mathrm{U}^c(V,J)) \to i\mathbb{R}$ is given by

$$\dot{\Theta}(\xi) = i\theta - i\sum_j b_{jj} \tag{4.37}$$

for $\xi \in \mathrm{Lie}(\mathrm{U}^c(V,J))$. This proves (i).

We prove (ii). The two infinitesimal characters of $\mathrm{U}^c(V,J)$ are given by

$$\dot{\delta}(\xi) = 2i\theta, \qquad \mathrm{trace}^c(\mathrm{Ad}(\xi)) = 2i\sum_{j=1}^{n} b_{jj}. \tag{4.38}$$

To prove the second formula note that the adjoint action of $\mathrm{Lie}(\mathrm{U}^c(V,J))$ on V is described by the complex matrix $\mathrm{Ad}(\xi) \cong -2A + 2iB$ in the given complex basis $e_1, \ldots, e_n$ of V. Since A is skew symmetric this proves (4.38). Combining this with (4.37) we find that

$$2\dot{\Theta} + \mathrm{trace}^c \circ \mathrm{Ad} = \dot{\delta}.$$

This proves (ii) for the canonical spin^c structure. The general case follows from the observation that Θ is independent of Γ. The isomorphism in (iii) is given by

$$\mathrm{U}^c(V) \to \mathrm{U}(V) \times S^1 : x \mapsto (\mathrm{ad}(x), \Theta(x)).$$

This proves the lemma. □

4.8 Spin^c representations and exterior algebra

Let V be a real inner product space and $\Gamma : V \to \mathrm{End}(W)$ be a spin^c structure on V. Any such structure gives rise to an action of the space $\Lambda^2 V^*$ on W which is induced by Clifford multiplication. To see this identify $\Lambda^2 V^*$ with $C_2(V)$ in the obvious way via the map

$$\Lambda^2 V^* \to C_2(V) : \eta = \sum_{i<j} \eta_{ij} e_i{}^* \wedge e_j{}^* \mapsto \sum_{i<j} \eta_{ij} e_i e_j.$$

Compose this map with Γ to obtain a map $\rho : \Lambda^2 V^* \to \mathrm{End}(W)$ given by

$$\rho\left(\sum_{i<j} \eta_{ij} e_i{}^* \wedge e_j{}^*\right) = \sum_{i<j} \eta_{ij} \Gamma(e_i)\Gamma(e_j) \tag{4.39}$$

for any orthonormal basis $e_1, \ldots, e_{2n}$ of V. The reader may check that this map is independent of the choice of the orthonormal basis used to define it. It makes the following diagram commute.

$$\begin{tikzcd} \Lambda^2 V^* \arrow[r, "\rho"] & \mathrm{End}(W) \\ C_2(V) \arrow[u] \arrow[ur, "\Gamma"'] & \end{tikzcd}$$

The image of the map $\rho : \Lambda^2 V^* \to \mathrm{End}(W)$ corresponds to the Lie algebra of $\mathrm{Spin}(V)$ under Γ. The map ρ extends in an obvious way to a map

$$\rho : \Lambda^2 V^* \otimes \mathbb{C} \to \mathrm{End}(W)$$

on the space of complex valued 2-forms. Note that if η is a real valued 2-form then $\rho(\eta)$ is skew-Hermitian and if η is imaginary valued then $\rho(\eta)$ is Hermitian. If V is even dimensional then the spaces $W^\pm$ are invariant under $\rho(\eta)$ for every 2-form $\eta \in \Lambda^2 V^*$. In this case we denote $\rho^\pm(\eta) = \rho(\eta)|_{W^\pm}$ for $\eta \in \Lambda^2 V^*$.

Lemma 4.54 *Let $\rho : \Lambda^2 V^* \to \mathrm{End}(W)$ be given by (4.39). Then*

$$\eta(v, w) = \frac{1}{2^n}\mathrm{trace}(\Gamma(v)\rho(\eta)\Gamma(w)) \tag{4.40}$$

for $\eta \in \Lambda^2 V^*$ and $v, w \in V$. Moreover, the map ρ is equivariant with respect to the action of $\mathrm{Spin}^c(V)$, namely

$$\rho(\mathrm{ad}(x)^*\eta) = \Gamma(x)^{-1}\rho(\eta)\Gamma(x) \tag{4.41}$$

for $x \in \mathrm{Spin}^c(V)$ and $\eta \in \Lambda^2 V^*$.

Proof: It suffices to prove the formula (4.40) for $\eta = {e_i}^* \wedge {e_j}^*$ with $i < j$. Recall from the proof of Proposition 4.33 that

$$\mathrm{trace}(\Gamma(x)) = 2^n x_0$$

for every $x \in C_2(V)$. Now any two vectors $v = \sum_\nu v_\nu e_\nu$ and $w = \sum_\nu w_\nu e_\nu$ satisfy

$$(ve_ie_jw)_0 = v_iw_je_ie_ie_je_j + v_jw_ie_je_ie_je_i = v_iw_j - v_jw_i$$

and hence

$$\mathrm{trace}(\Gamma(v)\Gamma(e_i)\Gamma(e_j)\Gamma(w)) = 2^n(v_iw_j - v_jw_i).$$

This shows that η can be obtained from $\rho(\eta)$ via (4.40). To prove the equivariance note first that if $x \in \mathrm{Spin}^c(V)$ and $\Psi = \mathrm{ad}(x) \in \mathrm{SO}(V)$ then $\Gamma(\Psi v) = \Gamma(x)\Gamma(v)\Gamma(x)^{-1}$. Hence

$$\begin{aligned}
\Psi^*\eta(v, w) &= \eta(\Psi v, \Psi w) \\
&= 2^{-n}\mathrm{trace}(\Gamma(\Psi v)\rho(\eta)\Gamma(\Psi w)) \\
&= 2^{-n}\mathrm{trace}\Big(\Gamma(x)\Gamma(v)\Gamma(x)^{-1}\rho(\eta)\Gamma(x)\Gamma(w)\Gamma(x)^{-1}\Big) \\
&= 2^{-n}\mathrm{trace}\Big(\Gamma(v)\Gamma(x)^{-1}\rho(\eta)\Gamma(x)\Gamma(w)\Big).
\end{aligned}$$

This shows that $\rho(\Psi^*\eta) = \Gamma(x)^{-1}\rho(\eta)\Gamma(x)$ as claimed. □

The four dimensional case

Let V be an oriented four-dimensional reainner product space. Consider the Hodge-$*$-operator on the space $\Lambda^2 V^*$. For any positively oriented orthonormal basis e_0, e_1, e_2, e_3 of V this operator is given by

$$*\omega_{01} = \omega_{23}, \qquad *\omega_{02} = \omega_{31}, \qquad *\omega_{03} = \omega_{12}$$

where $\omega_{jk} = {e_j}^* \wedge {e_k}^*$. Denote by

$$\Lambda^{2,\pm} = \{\omega \in \Lambda^2 \mid *\,\omega = \pm\omega\}$$

the subspaces of self-dual, respectively anti-self-dual, forms. For any form $\omega \in \Lambda^2 V^*$ denote the self-dual and anti-self-dual parts of ω by

$$\omega^{\pm} = \frac{1}{2}(\omega \pm *\omega).$$

The next lemma shows that the splitting

$$\Lambda^2 V^* = \Lambda^{2,+} \oplus \Lambda^{2,-}$$

of the space of 2-forms into the self-dual and anti-self-dual ones corresponds under the map ρ to the splitting

$$\Gamma(\mathrm{Spin}(V)) = \mathrm{SU}(W^+) \times \mathrm{SU}(W^-).$$

Lemma 4.55 *Assume* $\dim V = 4$ *and let* $\rho : \Lambda^2 V \to \mathrm{End}(W)$ *be given by (4.39). Let* $\eta \in \Lambda^2 V^*$. *Then*

$$\rho^{\pm}(\eta) = 0 \qquad \Longleftrightarrow \qquad \eta^{\pm} = 0.$$

Proof: Consider the standard example $\gamma : V \to \mathbb{C}^{2\times 2}$ with

$$\gamma(e_0) = \mathbb{1}, \qquad \gamma(e_1) = I, \qquad \gamma(e_2) = J, \qquad \gamma(e_3) = K.$$

If $\Gamma : V \to \mathbb{C}^{4\times 4}$ is given by (4.25) then, as in Example 4.39,

$$\rho(\omega_{01}) = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \rho(\omega_{02}) = \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}, \quad \rho(\omega_{03}) = \begin{pmatrix} K & 0 \\ 0 & -K \end{pmatrix}$$

$$\rho(\omega_{23}) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad \rho(\omega_{31}) = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}, \quad \rho(\omega_{12}) = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}.$$

Since the left upper block represents $\rho^+(\eta)$ and the right lower block represents $\rho^-(\eta)$ it follows that $\rho^{\mp}(\eta) = 0$ if and only if $\eta \in \Lambda^{2,\pm}$. This proves the lemma. $\square$

The previous lemma shows that the map ρ gives rise to isomorphisms $\rho^\pm : \Lambda^{2,\pm} \to \mathfrak{su}(W^\pm)$. Here $\mathfrak{su}(W^\pm)$ denotes the space of traceless skew-Hermitian endmorphisms of $W^\pm$. In other words, every spin^c structure $\Gamma : V \to \mathrm{End}(W)$ on a 4-dimensional real Hilbert space induces spin^c structures $\rho^\pm : \Lambda^{2,\pm} \to \mathrm{End}(W^\pm)$ of the 3-dimensional real Hilbert spaces of self-dual and anti-self-dual 2-forms on V. The complexified map

$$\rho^\pm : \Lambda^{2,\pm} \otimes_{\mathbb{R}} \mathbb{C} \to \mathrm{End}_0(W^\pm)$$

is an isomorphism from the space of complex valued (anti-)self-dual 2-forms to the space $\mathrm{End}_0(W^\pm)$ of traceless endomorphisms of $W^\pm$. Note that $\rho^\pm(\eta)$ is Hermitian iff η is imaginary valued and skew-Hermitian iff η is real valued. Lemma 4.54 shows that the inverse map

$$\sigma^\pm = (\rho^\pm)^{-1} : \mathrm{End}_0(W^\pm) \to \Lambda^{2,\pm} \otimes_{\mathbb{R}} \mathbb{C}$$

is given by $\sigma^\pm(T)(v, w) = \frac{1}{4}\mathrm{trace}(\Gamma(v)T\Gamma(w))$ for $T \in \mathrm{End}_0(W^\pm)$. All these observations readily carry over to 4-dimensional vector bundles.

Exercise 4.56 Assume $\dim_{\mathbb{R}} V = 6$. Prove that a 2-form $\eta \in \Lambda^2 V^*$ satisfies $\rho^+(\eta) = \rho(\eta)|_{W^+} = 0$ if and only $\eta = 0$. □

Exercise 4.57 Let V be a 3-dimensional oriented real inner product space and $\gamma : V \to \mathrm{End}(W)$ be a spin^c structure which is compatible with the orientation. Identify V with V^*. Prove that, for $\eta \in \Lambda^2 V$,

$$\rho(\eta) = \gamma(*\eta). \qquad \square$$

Exercise 4.58 Let V be a 2-dimensional oriented real inner product space and $\gamma : V \to \mathrm{End}(W)$ be a spin^c structure. Prove that, for $\eta \in \Lambda^2 V$ and $\theta \in W^\pm$,

$$\rho(\eta)\theta = \mp * i\eta\theta. \qquad \square$$

The complex case

Let (V, J, ω) be a Hermitian vector space (see Section 3.1) of real dimension $2n$ and $\Gamma_{\mathrm{can}} : V \to \mathrm{End}(W_{\mathrm{can}})$ be the canonical spin^c structure with $W_{\mathrm{can}} = \Lambda^{0,*}V^*$ (see (4.35)). Consider the map $\rho_{\mathrm{can}} : \Lambda^2 V^* \otimes \mathbb{C} \to \mathrm{End}(W_{\mathrm{can}})$ defined by (4.39).

Lemma 4.59 *Let $e_1, Je_1, \ldots, e_n, Je_n$ be an orthonormal basis of V. Then, for $\tau \in \Lambda^{0,*}V^*$,*

$$\frac{1}{8}\rho_{\mathrm{can}}(e_i{}' \wedge e_j{}')\tau = \iota(e_i)\iota(e_j)\tau, \qquad \frac{1}{2}\rho_{\mathrm{can}}(e_i{}'' \wedge e_j{}'')\tau = e_i{}'' \wedge e_j{}'' \wedge \tau,$$

$$\frac{1}{4}\rho_{\mathrm{can}}(e_i{}' \wedge e_j{}'')\tau = e_j{}'' \wedge \iota(e_i)\tau - \frac{1}{2}\delta_{ij}\tau.$$

Proof: Recall that ${e_j}' = {e_i}^* + i(Je_j)^*$ and ${e_j}'' = {e_j}^* - i(Je_j)^*$. Hence

$$\begin{aligned}
{e_i}' \wedge {e_j}' &= {e_i}^* \wedge {e_j}^* - (Je_i)^* \wedge (Je_j)^* + i(Je_i)^* \wedge {e_j}^* + i{e_i}^* \wedge (Je_j)^*, \\
{e_i}'' \wedge {e_j}'' &= {e_i}^* \wedge {e_j}^* - (Je_i)^* \wedge (Je_j)^* - i(Je_i)^* \wedge {e_j}^* - i{e_i}^* \wedge (Je_j)^*, \\
{e_i}' \wedge {e_j}'' &= {e_i}^* \wedge {e_j}^* + (Je_i)^* \wedge (Je_j)^* + i(Je_i)^* \wedge {e_j}^* - i{e_i}^* \wedge (Je_j)^*.
\end{aligned}$$

(Note here that $i = \sqrt{-1}$ whenever i does not appear as a subscript.) It follows from the definition of Γ that

$$\begin{aligned}
\rho_{\mathrm{can}}({e_i}^* \wedge {e_j}^*)\tau &= \frac{1}{2}{e_i}'' \wedge {e_j}'' \wedge \tau + 2\iota(e_i)\iota(e_j)\tau \\
&\quad + {e_j}'' \wedge \iota(e_i)\tau - {e_i}'' \wedge \iota(e_j)\tau, \\
\rho_{\mathrm{can}}((Je_i)^* \wedge (Je_j)^*)\tau &= -\frac{1}{2}{e_i}'' \wedge {e_j}'' \wedge \tau - 2\iota(e_i)\iota(e_j)\tau \\
&\quad + {e_j}'' \wedge \iota(e_i)\tau - {e_i}'' \wedge \iota(e_j)\tau, \\
i\rho_{\mathrm{can}}((Je_i)^* \wedge {e_j}^*)\tau &= -\frac{1}{2}{e_i}'' \wedge {e_j}'' \wedge \tau + 2\iota(e_i)\iota(e_j)\tau \\
&\quad + {e_j}'' \wedge \iota(e_i)\tau + {e_i}'' \wedge \iota(e_j)\tau - \delta_{ij}\tau, \\
i\rho_{\mathrm{can}}({e_i}^* \wedge (Je_j)^*)\tau &= -\frac{1}{2}{e_i}'' \wedge {e_j}'' \wedge \tau + 2\iota(e_i)\iota(e_j)\tau \\
&\quad - {e_j}'' \wedge \iota(e_i)\tau - {e_i}'' \wedge \iota(e_j)\tau + \delta_{ij}\tau.
\end{aligned}$$

The lemma follows by combining these equations. □

Assume that (V, J, ω) is a Hermitian vector space of 2 complex dimensions and consider the space $\Lambda^{2,+}$ of self-dual 2-forms.

Lemma 4.60 *There is a natural isomorphism*

$$\Lambda^{2,+} \xrightarrow{\cong} \mathbb{R}\omega \oplus \Lambda^{0,2}$$

given by $\eta \mapsto (\eta^{1,1}, \eta^{0,2})$.

Proof: Consider $\mathbb{C}^2$ with its standard coordinates $z_j = x_j + iy_j$ and its standard Hermitian structure. The standard orientation is given by x_1, y_1, x_2, y_2. Hence the forms

$$\begin{aligned}
\operatorname{Re} dz_1 \wedge dz_2 &= dx_1 \wedge dx_2 - dy_1 \wedge dy_2, \\
\operatorname{Im} dz_1 \wedge dz_2 &= dx_1 \wedge dy_2 + dy_1 \wedge dx_2, \\
\omega &= dx_1 \wedge dy_1 + dx_2 \wedge dy_2
\end{aligned}$$

form a basis of the space $\Lambda^{2,+}$ of self-dual 2-forms. Since the $(2,0)$-part of a real-valued 2-form is determined by its $(0,2)$-part (see Remark 3.7) this proves the lemma. □

Consider the map $\rho^+_{\mathrm{can}} : \Lambda^{2,+} \otimes i\mathbb{R} \to \mathrm{End}(W^+_{\mathrm{can}})$ defined by (4.39) in the complex 2-dimensional case. This map is an isomorphism between imaginary valued self-dual 2-forms on V and traceless Hermitian endomorphisms of W^+.

Lemma 4.61 *If $\eta \in \Lambda^{2,+} \otimes i\mathbb{R}$ and $\tau = (\tau_0, \tau_2) \in W^+_{\mathrm{can}} = \Lambda^{0,0} \oplus \Lambda^{0,2}$ then*

$$\rho^+_{\mathrm{can}}(\eta) : \begin{pmatrix} \tau_0 \\ \tau_2 \end{pmatrix} \mapsto 2 \begin{pmatrix} \eta_0\tau_0 + \langle \eta_2, \tau_2 \rangle \\ \tau_0\eta_2 - \eta_0\tau_2 \end{pmatrix}$$

where $\eta^{1,1} = i\eta_0\omega$ and $\eta^{0,2} = \eta_2$.

Proof: By Lemma 4.60 the form $\eta \in \Lambda^+ \otimes i\mathbb{R}$ can be written as

$$\eta = a{e_1}'' \wedge {e_2}'' - \bar{a}{e_1}' \wedge {e_2}' + i\eta_0\omega,$$

where

$$\eta_2 = a{e_1}'' \wedge {e_2}'', \qquad \omega = \frac{i}{2}\left({e_1}' \wedge {e_1}'' + {e_2}' \wedge {e_2}''\right).$$

Let $\tau_2 = b{e_1}'' \wedge {e_2}''$. Then $\langle \eta_2, \tau_2 \rangle = 4\bar{a}b = -4\bar{a}\iota(e_1)\iota(e_2)\tau_2$. Now use Lemma 4.59 to obtain $\rho_{\mathrm{can}}({e_j}' \wedge {e_j}'')\tau_0 = -2\tau_0$ and $\rho_{\mathrm{can}}({e_j}' \wedge {e_j}'')\tau_2 = 2\tau_2$. Hence $\sigma(\omega)\tau_0 = -2i\tau_0$, $\sigma(\omega)\tau_2 = 2i\tau_2$, and so

$$\rho_{\mathrm{can}}(\eta)\tau_0 = 2a{e_1}'' \wedge {e_2}''\tau_0 + 2\eta_0\tau_0 = 2\tau_0\eta_2 + 2\eta_0\tau_0,$$

$$\rho_{\mathrm{can}}(\eta)\tau_2 = -8\bar{a}\iota(e_1)\iota(e_2)\tau_2 - 2\eta_0\tau_2 = 2\langle \eta_2, \tau_2 \rangle - 2\eta_0\tau_2.$$

This proves the lemma. □

The previous lemma is concerned with imaginary valued 2-forms η because such forms arise as (the self-dual parts of) curvature forms of connections on line bundles. Recall that in this case the endomorphism $\rho^+_{\mathrm{can}}(\eta)$ is Hermitian and has zero trace. It is useful to rephrase Lemma 4.61 in terms of the inverse map

$$\sigma^+_{\mathrm{can}} = (\rho^+_{\mathrm{can}})^{-1} : \mathrm{End}_0(W^+_{\mathrm{can}}) \to \Lambda^{2,+} \otimes_{\mathbb{R}} \mathbb{C}.$$

Note in particular that, if $T \in \mathrm{End}_0(W^+_{\mathrm{can}})$ is a traceless Hermitian endomorphism then $\sigma^+_{\mathrm{can}}(T)$ is an imaginary valued anti-self-dual 2-form. The next lemma gives an explicit formula for $\sigma^+_{\mathrm{can}}((\Phi\Phi^*)_0)$ for $\Phi \in W^+_{\mathrm{can}}$. Here $\Phi\Phi^* \in \mathrm{End}(W^+_{\mathrm{can}})$ denotes the Hermitian endomorphism

$$\Phi\Phi^*\tau = \langle \Phi, \tau \rangle \Phi$$

and $T_0 = T - \frac{1}{2}\mathrm{trace}(T)\mathbb{1} \in \mathrm{End}_0(W^+_{\mathrm{can}})$ denotes the traceless part of $T \in \mathrm{End}(W^+_{\mathrm{can}})$. Thus, for $\tau \in W^+_{\mathrm{can}}$,

$$(\Phi\Phi^*)_0\tau = \langle \Phi, \tau \rangle \Phi - \frac{1}{2}|\Phi|^2\tau.$$

Lemma 4.62 *In the case of the canonical spin^c structure on a 2-dimensional complex vector space (V, J)*

$$2\sigma^+_{\mathrm{can}}((\Phi\Phi^*)_0) = i\frac{|\varphi_0|^2 - |\varphi_2|^2}{2}\omega + \bar{\varphi}_0\varphi_2 - \varphi_0\bar{\varphi}_2$$

for $\Phi = (\varphi_0, \varphi_2) \in W^+_{\mathrm{can}} = \Lambda^{0,0} \oplus \Lambda^{0,2}$.

Proof: The 2-form $\eta = \sigma^+_{\mathrm{can}}((\Phi\Phi^*)_0) \in \Lambda^{2,+}V^* \otimes i\mathbb{R}$ decomposes as

$$\eta = \eta_2 + i\eta_0\omega - \bar{\eta}_2$$

where $\eta_0 \in \mathbb{R}$ and $\eta^{0,2} = \eta_2$ and $\eta^{2,0} = -\bar{\eta}_2$. Moreover,

$$\rho^+_{\mathrm{can}}(\eta)\tau = (\Phi\Phi^*)_0\tau = (\bar{\varphi}_0\tau_0 + \langle\varphi_2, \tau_2\rangle)\,\varphi - \frac{|\varphi_0|^2 + |\varphi_2|^2}{2}\tau$$

for $\tau = (\tau_0, \tau_2) \in W^+_{\mathrm{can}}$ with $\tau_0 \in \Lambda^{0,0} = \mathbb{C}$ and $\tau_2 \in \Lambda^{0,2}$. Comparing this with the formula of Lemma 4.61 for $\rho^+_{\mathrm{can}}(\eta)$ one finds

$$2\eta_0\tau_0 + 2\langle\eta_2, \tau_2\rangle = (\bar{\varphi}_0\tau_0 + \langle\varphi_2, \tau_2\rangle)\,\varphi_0 - \frac{|\varphi_0|^2 + |\varphi_2|^2}{2}\tau_0,$$

$$2\tau_0\eta_2 - 2\eta_0\tau_2 = (\bar{\varphi}_0\tau_0 + \langle\varphi_2, \tau_2\rangle)\,\varphi_2 - \frac{|\varphi_0|^2 + |\varphi_2|^2}{2}\tau_2.$$

The formula $\langle\varphi_2, \tau_2\rangle\varphi_2 = |\varphi_2|^2\tau_2$ shows that these two equations can be written in the form

$$2\eta_0\tau_0 + 2\langle\eta_2, \tau_2\rangle = \frac{|\varphi_0|^2 - |\varphi_2|^2}{2}\tau_0 + \langle\bar{\varphi}_0\varphi_2, \tau_2\rangle,$$

$$2\tau_0\eta_2 - 2\eta_0\tau_2 = \frac{|\varphi_2|^2 - |\varphi_0|^2}{2}\tau_2 + \tau_0\bar{\varphi}_0\varphi_2.$$

These last two equations hold for all $\tau_0 \in \mathbb{C}$ and all $\tau_2 \in \Lambda^{0,2}$ if and only if η is given by

$$2\eta_2 = \bar{\varphi}_0\varphi_2, \qquad 2\eta_0 = \frac{|\varphi_0|^2 - |\varphi_2|^2}{2}.$$

Since $\eta = \eta_2 + i\eta_0\omega - \bar{\eta}_2$ it follows that

$$2\eta = i\frac{|\varphi_0|^2 - |\varphi_2|^2}{2}\omega + \bar{\varphi}_0\varphi_2 - \varphi_0\bar{\varphi}_2$$

as claimed. □

Remark 4.63 The proof of Lemma 4.62 shows that $\eta \in \Lambda^{2,+}V^* \otimes i\mathbb{R}$ and $\Phi \in W^+_{\mathrm{can}}$ satisfy

$$\rho^+_{\mathrm{can}}(\eta) = (\Phi\Phi^*)_0$$

if and only if

$$2\eta^{0,2} = \bar{\varphi}_0\varphi_2, \qquad 2i\eta \wedge \omega = \frac{|\varphi_2|^2 - |\varphi_0|^2}{2}\omega \wedge \omega.$$

To see this just note that ω is of type $(1,1)$ and hence $\eta^{0,2} \wedge \omega = 0$. $\square$

5

SPIN STRUCTURES ON VECTOR BUNDLES

The goal of this chapter is to discuss spin structures and spin^c structures on vector bundles $V \to X$. The approach taken here is to define spin and spin^c structures as representations on spinor bundles $S \to X$ (in the spin case) and $W \to X$ (in the spin^c case). The reader may have noted that the Definitions 4.32 and 4.43 carry over directly to the vector bundle situation. That this is equivalent to the perhaps more familiar principal bundle approach is shown in the next section. The results for general vector bundles are then adapted to tangent bundles. Section 5.2 deals with the classification of spin^c structures in terms of integral lifts of $\mathrm{w}_2(TX)$. It is proved that every orientable smooth 4-manifold admits a spin^c structure. Section 5.3 examines spin^c structures on complex vector bundles and Section 5.4 deals with the classification of spin structures.

5.1 Basic definitions

Let X be a smooth manifold and $V \to X$ be an m-dimensional oriented real vector bundle with an inner product.

Definition 5.1 *A* **spin structure** *on V is a principal bundle $P \to X$ with structure group $\mathrm{G} = \mathrm{Spin}(m) = \mathrm{Spin}(\mathbb{R}^m)$ together with an isomorphism*

$$P \times_{\mathrm{ad}} \mathbb{R}^m \to V$$

of oriented Riemannian vector bundles. A bundle V is called **spin** *if it admits a spin structure.*

It is shown in Theorem 5.28 below that an oriented Riemannian vector bundle $V \to X$ admits a spin structure if and only if its second Stiefel-Whitney class $\mathrm{w}_2(V) \in H^2(X;\mathbb{Z}_2)$ vanishes. This means that the restriction of V to every embedded surface $\Sigma \subset X$ admits a trivialization. A manifold X is called **spin** if its tangent bundle TX admits a spin structure.

Let us examine this condition for compact oriented smooth 4-manifolds. Any such manifold X carries a **mod-2 intersection form**

$$Q_{X,2} : H_2(X;\mathbb{Z}_2) \times H_2(X;\mathbb{Z}_2) \to \mathbb{Z}_2.$$

The footnote on page 28 can be adapted to show that every $\mathbb{Z}_2$-homology class α can be represented by a smoothly embedded, but not necessarily oriented, surface $\Sigma \subset X$. The intersection form is given by the mod-2 intersection numbers of such embedded surfaces. The second Stiefel-Whitney class of TX satisfies

$$\langle \mathrm{w}_2(TX), \alpha \rangle = Q_{X,2}(\alpha, \alpha) \tag{5.1}$$

for $\alpha \in H_2(X; \mathbb{Z}_2)$. This follows from the multiplicativity of the Stiefel-Whitney classes for the splitting $T_\Sigma X = T\Sigma + \nu_\Sigma$. The proof of (5.1) uses the identities $\mathrm{w}_2(T\Sigma) = \chi(\Sigma)(\mathrm{mod}\ 2)$, $\mathrm{w}_2(\nu_\Sigma) = \Sigma \cdot \Sigma(\mathrm{mod}\ 2)$, and

$$\chi(\Sigma) = \mathrm{w}_1(T\Sigma)^2 \ (\mathrm{mod}\ 2) \tag{5.2}$$

for every 2-manifold Σ (compare this with Lemma 1.45). Equation (5.1) shows that a 4-manifold X is spin if and only if the mod-2 self-intersection number of every mod-2 homology class is zero. In the simply connected case this means that the integral intersection form $Q_X : H_2(X; \mathbb{Z}) \times H_2(X; \mathbb{Z}) \to \mathbb{Z}$ is even. However, care must be taken if X is not simply connected. In this case $\mathrm{w}_2(TX)$ may be a nonzero torsion class and any such class satisfies $\langle \mathrm{w}_2(TX), \alpha \rangle = 0$ for all $\alpha \in H_2(X; \mathbb{Z})$ (but the pairing is nonzero for some $\alpha \in H_2(X; \mathbb{Z}_2)$ which does not admit an integral lift). For an example of a non-spin 4-manifold with even intersection form see Example 6.28 below.

Definition 5.2 *A* **spin$^{\mathrm{c}}$ structure** *on an m-dimensional oriented Riemannian vector bundle $V \to X$ is a principal bundle $P \to X$ with structure group* $\mathrm{G} = \mathrm{Spin}^c(m) = \mathrm{Spin}^c(\mathbb{R}^m)$ *together with an isomorphism*

$$P \times_{\mathrm{ad}} \mathbb{R}^m \to V$$

of Riemannian vector bundles.

It is shown in Theorem 5.8 below that an oriented Riemannian vector bundle admits a spin^c structure if and only if its second Stiefel-Whitney class $\mathrm{w}_2(V) \in H^2(X; \mathbb{Z}_2)$ admits an integral lift. Such an integral lift is in fact given by the first Chern class $c = c_1(L)$ of the characteristic line bundle

$$L = P \times_\delta \mathbb{C}.$$

Here the action of $\mathrm{Spin}^c(m)$ on $\mathbb{C}$ is given by the homomorphism $\delta : \mathrm{Spin}^c(m) \to S^1$ in (4.16). The class $\mathrm{w}_2(V)$ admits an integral lift if and only if it maps to zero under the Bockstein homomorphism $H^2(X; \mathbb{Z}_2) \to H^3(X; \mathbb{Z})$. This is the case, for example, if there is no 2-torsion in $H^2(X; \mathbb{Z})$. In Theorem 5.8 it is also shown that a spin^c structure on a bundle V need not be determined uniquely (up to spin^c isomorphism) by its associated line bundle L. The difference of two such line bundles L_0 and L_1 arising from spin^c structures P_0 and P_1 is necessarily even, i.e. the mod-2 reduction of the difference of first Chern classes $c_1(L_1) - c_1(L_0) \in H^2(X; \mathbb{Z})$ is zero. In general spin^c structures are determined by *square roots* of these classes. In other words, isomorphism classes of spin^c structures form a principal space $\mathcal{S}^c(V)$ with structure group $H^2(X;\mathbb{Z})$ but in general there is no natural base point. The map $\mathcal{S}^c(V) \to H^2(X;\mathbb{Z})$ which assigns to a spin^c structure the first Chern class $c_1(L)$ is a bijection if and only if there is no 2-torsion in $H^2(X;\mathbb{Z})$.

A **spin^c structure** on an oriented Riemannian manifold X is defined as a spin^c structure on its tangent bundle. The above assertions state that such a structure exists if and only if the second Stiefel-Whitney class $\mathrm{w}_2(TX) \in H^2(X;\mathbb{Z}_2)$ of its tangent bundle admits an integral lift. In particular, it is shown in Theorem 5.10 below that every orientable smooth 4-manifold admits a spin^c structure, but it only admits a spin structure if $\mathrm{w}_2(TX) = 0$ (i.e. in the simply connected case if its intersection form is even).

Let us now return to the case of spin^c structures on general vector bundles $V \to X$ of rank $m = 2n$ or $m = 2n+1$. Then, by Theorem 4.23, there is a representation $\Gamma_0 : \mathrm{Spin}^c(\mathbb{R}^m) \to \mathrm{End}(\mathbb{C}^{2^n})$ and this gives rise to a Hermitian vector bundle

$$W = P \times_{\Gamma_0} \mathbb{C}^{2^n}.$$

Clifford multiplication determines a homomorphism $\Gamma : V \to \mathrm{End}(W)$ which satisfies (4.18). This gives rise to the following alternative definitions.

Definition 5.3 *A* **spin^c structure** *on an oriented Riemannian vector bundle $V \to X$ of dimension $2n$ or $2n+1$ is a pair (W, Γ) where $W \to X$ is a Hermitian vector bundle of rank 2^n and $\Gamma : V \to \mathrm{End}(W)$ is a homomorphism which satisfies (4.18) and, in the odd dimensional case, (4.22). A* **spin^c isomorphism** *from (W_0, Γ_0) to (W_1, Γ_1) is a unitary bundle isomorphism $\Phi : W_0 \to W_1$ which satisfies*

$$\Phi\Gamma_0 = \Gamma_1\Phi$$

Denote by $\mathcal{S}^c(V)$ the set of isomorphism classes of spin^c structures on V.

Definition 5.4 *A* **spin structure** *on an oriented Riemannian vector bundle $V \to X$ of rank $2n \equiv 2, 4 (\mathrm{mod}\ 8)$ or $2n+1 \equiv 3 (\mathrm{mod}\ 8)$ is a quadruple (S, I, J, Γ) where $S \to X$ is a Riemannian vector bundle of (real) rank 2^{n+1}, $I, J \in C^\infty(X, \mathrm{End}(S))$ are orthogonal anti-commuting complex structures, and $\Gamma : V \to \mathrm{End}(S)$ is a homomorphism which and commutes with I and J, and satisfies (4.18) and, in the odd dimensional case, (4.22). A* **spin isomorphism** *from $(S_0, I_0, J_0, \Gamma_0)$ to $(S_1, I_1, J_1, \Gamma_1)$ is an orthogonal bundle isomorphism $\Phi : S_0 \to S_1$ which satisfies*

$$\Phi\Gamma_0 = \Gamma_1\Phi, \qquad \Phi I_0 = I_1\Phi, \qquad \Phi J_0 = J_1\Phi.$$

Denote by $\mathcal{S}(V)$ the set of isomorphism classes of spin structures on V.

If (W, Γ) is a spin^c structure on an even dimensional vector bundle $V \to X$ then, by Proposition 4.33, Γ extends to an isomorphism $C^c(V) \to \mathrm{End}(W)$ which is still denoted by Γ. By Lemma 4.40, the Hermitian vector bundle W admits a natural splitting

$$W = W^+ \oplus W^-$$

into the bundles of eigenspaces of $\Gamma(\varepsilon)$ with eigenvalues $\pm i^n$ where $\varepsilon \in C_{2n}(V)$ is determined by a positively oriented orthonormal basis as in Lemma 4.40.

Remark 5.5 The definitions 5.2 and 5.3 are equivalent. To see this choose model space V_0, of dimension $2n$ or $2n+1$, and a model spin^c structure $\Gamma_0 : V_0 \to \mathrm{End}(W_0)$. Let $\Gamma : V \to \mathrm{End}(W)$ be a spin^c structure as in Definition 5.3. Then there is a principle frame bundle $P_\Gamma \to X$ with structure group $\mathrm{Spin}^c(V_0)$ such that the fibre of P_Γ over $x \in X$ is the space of all spin^c isomorphisms from the model space W_0 to the fibre W_x. Thus

$$P_\Gamma = \left\{ (x, A, \Phi) \,|\, x \in X,\ (A, \Phi) \in \mathrm{Hom}^{\mathrm{spin}^c}(W_0, W_x) \right\}.$$

Thus a point in P_Γ is a triple (x, A, Φ) where $x \in X$, $A : V_0 \to V_x$ is an orientation preserving orthogonal transformation, and $\Phi : W_0 \to W_x$ is a unitary isomorphism such that $\Phi\Gamma_0(v_0)\Phi^* = \Gamma(Av_0)$ for $v_0 \in V_0$. The group $\mathrm{Spin}^c(V_0)$ acts on P_Γ by

$$(x, A, \Phi) \mapsto (x, A \circ \mathrm{ad}(a), \Phi \circ \Gamma_0(a))$$

for $a \in \mathrm{Spin}^c(V_0)$. The bundle $P \times_{\mathrm{ad}} V_0$ is the space of equivalence classes $[x, A, \Phi, v_0] \in P \times_{\mathrm{ad}} V_0$ with equivalence relation

$$[x, A, \Phi, v_0] \equiv [x, A \circ \mathrm{ad}(a), \Phi \circ \Gamma_0(a), \mathrm{ad}(a)^{-1} v_0]$$

for $a \in \mathrm{Spin}^c(V_0)$. There is a canonical isomorphism

$$P_\Gamma \times_{\mathrm{ad}} V_0 \xrightarrow{\cong} V : [x, A, \Phi, v_0] \mapsto Av_0. \qquad \square$$

Conversely, if $P \to X$ is a principal $\mathrm{Spin}^c(V_0)$-bundle over X, then there is an obvious homomorphism $P \times_{\mathrm{ad}} V_0 \to \mathrm{End}(P \times_{\Gamma_0} W_0)$. If we assume $V \cong P \times_{\mathrm{ad}} V_0$ and define $W = P \times_{\Gamma_0} W_0$ This gives rise to a homomorphism $\Gamma : V \to \mathrm{End}(W)$ as required. $\square$

Exercise 5.6 Show that the the two definitions of spin structures in 5.1 and 5.4 are equivalent. Define a spin structure on a vector bundle of rank 0 or 6 mod 8 along the lines of 5.4 and show that your definition is equivalent to 5.1. **Hint:** See Exercise 4.48. $\square$

It is interesting to examine the associated line bundle

$$L_\Gamma = P_\Gamma \times_\delta \mathbb{C}.$$

The elements of L_Γ are equivalence classes of quadruples $[x, A, \Phi, z] \in P \times \mathbb{C}$ under the equivalence relation

$$[x, A, \Phi, z] \equiv [x, A \circ \mathrm{ad}(a), \Phi \circ \Gamma_0(a), \delta(a)^{-1} z] \tag{5.3}$$

for $a \in \mathrm{Spin}^c(V_0)$. The next lemma shows that, when V has rank 2, 3, or 4 modulo 8, then the unit sphere bundle in L_Γ is naturally isomorphic to the bundle $Q = Q_\Gamma \to X$ whose fiber over x consists of all orthogonal complex structures J on W_x which anti-commute with $I = i$ and commute with Γ:

$$Q_\Gamma = \left\{ (x, J) \,|\, J \in \mathrm{End}_\mathbb{R}(W_x),\ J^{-1} = J^* = -J,\ JI = -IJ,\ J\Gamma = \Gamma J \right\}.$$

Lemma 5.7 *Assume that V has rank 2, 3, or 4 modulo 8. Then the circle bundle $Q_\Gamma \to X$ is naturally isomorphic to the unit sphere bundle in L_Γ.*

Proof: Fix an orthogonal complex structure $J_0 : W_0 \to W_0$ which anti-commutes with $I_0 = i$ and commutes with Γ_0. Consider the map $P_\Gamma \times S^1 \to Q_\Gamma : (x, A, \Phi, z) \mapsto (x, J(\Phi, z))$, where $J(\Phi, z)$ is given by

$$J(\Phi, z) = \mathrm{Re}\, z \Phi J_0 \Phi^{-1} + \mathrm{Im}\, z \Phi I_0 J_0 \Phi^{-1}.$$

We prove that this map is invariant under the action of $\mathrm{Spin}^c(V_0)$ as in (5.3). Note first that J_0 commutes with $\Gamma_0(v_0)$ for every $v_0 \in V_0$ and hence with $\Gamma_0(a_0)$ for every $a_0 \in \mathrm{Spin}(V_0) \subset C(V_0)$. Now write $a \in \mathrm{Spin}^c(V_0)$ in the form $a = e^{i\theta} a_0$ where $a_0 \in \mathrm{Spin}(V_0)$. Then

$$\begin{aligned} J(\Phi \circ \Gamma_0(a), 1) &= \Phi \Gamma_0(a) J_0 \Gamma_0(a)^{-1} \Phi^{-1} \\ &= \Phi(\cos\theta + \sin\theta I_0) J_0 (\cos\theta - \sin\theta I_0) \Phi^{-1} \\ &= \cos(2\theta) \Phi J_0 \Phi^{-1} + \sin(2\theta) I_0 \Phi J_0 \Phi^{-1} \\ &= J(\Phi, e^{2i\theta}) \\ &= J(\Phi, \delta(a)). \end{aligned}$$

This proves the lemma. □

The previous lemma shows that the line bundle L_Γ associated to a spin^c structure (W, Γ) has first Chern class $c_1(L_\Gamma) = 0$ if and only the bundle Q_Γ admits a section. Any such section is precisely an orthogonal complex structure J on W which commutes with Γ and anti-commutes with the standard complex structure $I = i$. Hence a spin structure on an oriented $2n$-dimensional Riemannian vector bundle $V \to X$ can also be defined as a triple (W, Γ, θ) where (W, Γ) is a spin^c structure with $c_1(X, L_\Gamma) = 0$ and $\theta : X \to L_\Gamma$ is a section which satisfies $|\theta(x)| = 1$ for all $x \in X$.

5.2 Classification of spin^c structures

With the definitions in place we shall now examine the fundamental question of the existence of spin^c structures and their uniqueness up to natural spin^c isomorphisms. It is convenient to label spin^c structures by the map $\Gamma : V \to \mathrm{End}(W)$ and denote the associated principal and line bundles by P_Γ and L_Γ respectively. The first Chern class of the line bundle L_Γ is the fundamental object in the classification of spin^c structures. As a warm-up consider its relation with the determinant line bundles of W^+ and W^-. Note first that these determinant bundles are canonically isomorphic. The isomorphism is induced by $\Gamma(v)$ with $|v| = 1$ and, by Lemma 4.41, is independent of the choice of v. Moreover, it follows from Proposition 4.36 (ii) that the line bundle $L = L_\Gamma$ is related to these determinant bundles by

$$L_\Gamma{}^{\otimes 2^{n-2}} \cong \det(W^+) \cong \det(W^-). \tag{5.4}$$

The spin^c structure arises from a spin structure of X precisely if the line bundle L_Γ can be trivialized. The situation is particularly simple in the 4-dimensional case when $L \cong \det(W^+) \cong \det(W^-)$.

The next theorem is the fundamental result of this section. It answers the questions about the existence and uniqueness of spin^c structures. In particular, it asserts that, given a spin^c structure $\Gamma : V \to \mathrm{End}(W)$ on V, any other spin^c structure can be obtained from it by tensoring with a Hermitian line bundle and that the two structures are isomorphic if and only if the line bundle admits a trivialization. Thus the set $\mathcal{S}^c(V)$, if it is nonempty, is a principal space structure group $H^2(X;\mathbb{Z})$

Theorem 5.8 *Let $V \to X$ be an oriented Riemannian vector bundle of rank $2n$.*

(i) *If $\Gamma : V \to \mathrm{End}(W)$ is a spin^c structure on V then the first Chern class $c_1(L_\Gamma) \in H^2(X;\mathbb{Z})$ is an integral lift of $\mathrm{w}_2(V) \in H^2(X;\mathbb{Z}_2)$.*

(ii) *For every integral lift $c \in H^2(X;\mathbb{Z})$ of $\mathrm{w}_2(V)$ there exists a spin^c structure $\Gamma : V \to \mathrm{End}(W)$ with $c_1(L_\Gamma) = c$.*

(iii) *If $\Gamma : V \to \mathrm{End}(W)$ is a spin^c structures on V and $E \to X$ is a Hermitian line bundle then the characteristic line bundle of the twisted spin^c structure $\tilde{\Gamma} = \Gamma \otimes \mathbb{1} : V \to \mathrm{End}(W \otimes E)$ is given by*

$$L_{\tilde{\Gamma}} = L_\Gamma \otimes E^{\otimes 2}.$$

(iv) *If $\Gamma_1 : V \to \mathrm{End}(W_1)$ and $\Gamma_2 : V \to \mathrm{End}(W_2)$ are spin^c structures on V then there exists a Hermitian line bundle $E \to X$ such that*

$$W_2 \cong W_1 \otimes E, \qquad \Gamma_2 \cong \Gamma_1 \otimes \mathbb{1}.$$

These two spin^c structures are isomorphic if and only if $c_1(E) = 0$.

Proof: We prove (i). Let $\alpha \in H_2(X;\mathbb{Z}_2)$ and choose a compact embedded (but not necessarily oriented) 2-manifold $\Sigma \subset X$ such that

$$\alpha = [\Sigma] \in H_2(X;\mathbb{Z}_2)$$

(see the footnote on page 28). Write

$$\Sigma = \Sigma_1 \cup_C \Sigma_2,$$

where $\Sigma_1 \subset \Sigma$ is a closed disc, $\Sigma_2 = \mathrm{cl}(\Sigma - \Sigma_1)$, and $C = \partial\Sigma_1$ is the common boundary. Since $\mathrm{Spin}^c(V_0)$ is connected, there exist sections of P_Γ over Σ_1 and Σ_2. Write these in the form

$$\Sigma_i \to P_\Gamma : z \mapsto (z, A_i(z), \Phi_i(z)),$$

where $A_i(z) \in \mathrm{Hom}(V_0, V_z)$ and $\Phi_i(z) \in \mathrm{Hom}(W_0, W_z)$ satisfy

$$\Phi_i(z)\Gamma_0(v_0)\Phi_i(z)^* = \Gamma(A_i(z)v_0)$$

for $z \in \Sigma_i$, $v_0 \in V_0$, and $i = 1, 2$. By Proposition 4.36 (i), the two sections over C give rise to a transition map $\alpha : C \to \mathrm{Spin}^c(V_0)$ such that

$$\Phi_2(z) = \Phi_1(z) \circ \Gamma_0(\alpha(z)), \qquad A_2(z) = A_1(z) \circ \mathrm{ad}(\alpha(z))$$

for $z \in C$. Hence

$$\langle \mathrm{w}_2(V), [\Sigma]\rangle = \begin{cases} 0, & \text{if } \mathrm{ad} \circ \alpha \text{ is contractible,} \\ 1, & \text{otherwise.} \end{cases}$$

Now the trivializations of P_Γ also give rise to sections $s_i : \Sigma_i \to L_\Gamma$ given by

$$s_i(z) = [z, A_i(z), \Phi_i(z), 1].$$

These sections satisfy

$$s_2(z) = \delta(\alpha(z))s_1(z).$$

If Σ is oriented, then Proposition 1.34 shows that $\langle c_1(L_\Gamma), [\Sigma]\rangle = \deg(\delta\circ\alpha)$. With similar arguments one can show also in the nonorientable case that the number of zeros of a generic section of L_Γ over Σ is equal to the degree of $\delta \circ \alpha$ modulo 2. Hence

$$\langle \mathrm{w}_2(L_\Gamma), [\Sigma]\rangle = \deg_2(\delta \circ \alpha) = \langle \mathrm{w}_2(V), [\Sigma]\rangle.$$

The last identity follows from the fact that the degree of $\delta \circ \alpha$ is even if and only if $\mathrm{ad} \circ \alpha$ is contractible (Lemma 4.30). This proves (i).

We prove (ii). Triangulate X and denote by $X_k \subset X$ the k-skeleton. Think of the simplices in the triangulation as submanifolds with corners and write them in the form Δ_α, Δ_β, etc. The indices belong to a finite set $\mathcal{S} = \bigcup_{k=0}^{2n} \mathcal{S}_k$, and indices in $\mathcal{S}_k$ correspond to k-simplices. For each simplex $\alpha \in \mathcal{S}$ choose a trivialization of $V|_{\Delta_\alpha}$ and write it in the form $A_\alpha(x) : V_0 \to V_x$ for $x \in \Delta_\alpha$. Let us denote the transition functions by

$$A_{\beta\alpha}(x) = A_\beta(x)^{-1} A_\alpha(x) \in \mathrm{SO}(V_0), \qquad x \in \Delta_\alpha \cap \Delta_\beta.$$

Since V is oriented there exists a trivialization of V over the 1-skeleton. This translates into the condition

$$\alpha, \beta \in \mathcal{S}_0 \cup \mathcal{S}_1,\ x \in \Delta_\alpha \cap \Delta_\beta \qquad \Longrightarrow \qquad A_{\beta\alpha}(x) = 1\!\!1. \tag{5.5}$$

We shall construct the bundle W as the union of the trivial bundles $\Delta_\alpha \times W_0$ with spin^c structures $\Gamma_\alpha = \Gamma_0 \circ A_\alpha(x)^{-1} : V_x \to \mathrm{End}(W_0)$ for $x \in \Delta_\alpha$. Thus

$$W = \bigcup_\alpha \{\alpha\} \times \Delta_\alpha \times W_0 / \equiv$$

The equivalence relation, for $x \in \Delta_\alpha \cap \Delta_\beta$, has the form

$$[\alpha, x, \theta_0] \equiv [\beta, x, \Gamma_0(a_{\beta\alpha}(x))\theta_0],$$

where the transition functions $a_{\beta\alpha} : \Delta_\alpha \cap \Delta_\beta \to \mathrm{Spin}^c(V_0)$ satisfy

$$a_{\gamma\beta} a_{\beta\alpha} = a_{\gamma\alpha}, \qquad a_{\alpha\alpha}(x) = 1, \tag{5.6}$$

$$\mathrm{ad} \circ a_{\beta\alpha} = A_{\beta\alpha}. \tag{5.7}$$

We shall construct the transition functions over the k-skeleton, by induction over k. For $\alpha, \beta \in \mathcal{S}_0 \cup \mathcal{S}_1$ define $a_{\beta\alpha}(x) = 1$. Then (5.7) follows from (5.5). It also follows from (5.5) that, for every 2-simplex $\beta \in \mathcal{S}_2$ and every $x \in \partial\Delta_\beta$, the transition matrix $A_{\gamma\beta}(x) \in \mathrm{SO}(V_0)$ is independent of the 1-simplex γ (with $x \in \Delta_\gamma$) used to define it. Hence these transition matrices give rise to a loop $\rho_\beta : \partial\Delta_\beta \to \mathrm{SO}(V_0)$ defined by $\rho_\beta(x) = A_{\gamma\beta}(x)$ for $\gamma \in \mathcal{S}_1$ with $x \in \Delta_\gamma \subset \Delta_\beta$. Define $\ell_2 : \mathcal{S}_2 \to \mathbb{Z}_2$ by

$$\ell_2(\beta) = \begin{cases} 0, & \text{if } \rho_\beta \text{ is contractible,} \\ 1, & \text{if } \rho_\beta \text{ is not contractible.} \end{cases}$$

This function is a cocycle and represents the second Stiefel-Whitney class of V. Since c is an integral lift of $\mathrm{w}_2(V)$ there exists a cocycle $\ell : \mathcal{S}_2 \to \mathbb{Z}$ which represents the class c and whose mod-2 reduction agrees with ℓ_2. In particular, the sum of the labels over each boundary is zero and the sum of the labels over a cycle σ agrees with $\langle c, \sigma \rangle$.

For every 2-simplex $\beta \in \mathcal{S}_2$, choose a loop $a_\beta : \partial\Delta_\beta \to \mathrm{Spin}^c(V_0)$ such that

$$\mathrm{ad} \circ a_\beta = \rho_\beta, \qquad \deg(\delta \circ a_\beta) = \ell(\beta).$$

Such loops exist by Lemma 4.30. Now, for every $\gamma \in \mathcal{S}_0 \cup \mathcal{S}_1$ with $\Delta_\gamma \subset \Delta_\beta$, define

$$a_{\gamma\beta}(x) := a_\beta(x)$$

for $x \in \Delta_\alpha \cap \Delta_\beta$. These functions satisfy (5.7). If $\beta, \beta' \in \mathcal{S}_2$ are 2-simplices with nonempty intersection then we must define

$$a_{\beta'\beta}(x) = a_{\gamma\beta'}(x)^{-1} a_{\gamma\beta}(x) = a_{\beta'}(x)^{-1} a_\beta(x),$$

where $\gamma \in \mathcal{S}_1$ is chosen such that $x \in \Delta_\gamma \subset \Delta_\beta \cap \Delta_{\beta'}$. This defines W and Γ over the 2-skeleton such that the first Chern class of L_Γ agrees with c.

Now let $\alpha \in \mathcal{S}_3$. Consider the circle bundle

$$Q_\alpha \to \partial\Delta_\alpha$$

whose fibre over $x \in \Delta_\alpha \cap \Delta_\beta$, with $\beta \in \mathcal{S}_2$, consists of all $a \in \mathrm{Spin}^c(V_0)$ such that $\mathrm{ad}(a) = A_{\beta\alpha}(x)$. If $x \in \Delta_\beta \cap \Delta_{\beta'}$ then the two corresponding circles can be naturally identified via $a \mapsto a_{\beta'\beta}(x)a$. The square $Q_\alpha \times_{S^1} Q_\alpha$ of this bundle has transition functions $\delta \circ a_{\beta'\beta} = (\delta \circ a_{\beta'})^{-1}(\delta \circ a_\beta)$ and hence its Euler number is given by

$$e(Q_\alpha \times_{S^1} Q_\alpha) = \sum_{\substack{\beta \in \mathcal{S}_2 \\ \Delta_\beta \subset \Delta_\alpha}} \varepsilon(\beta, \alpha) \deg(\delta \circ a_\beta) = \sum_{\substack{\beta \in \mathcal{S}_2 \\ \Delta_\beta \subset \Delta_\alpha}} \varepsilon(\beta, \alpha) \ell(\beta) = 0.$$

Here the sign $\varepsilon(\beta, \alpha) \in \{\pm 1\}$ is determined by comparing the orientation of Δ_β with that of $\partial\Delta_\alpha$. The last equation follows from the fact that ℓ is a cocycle. Since $\partial\Delta_\alpha$ is a 2-sphere it follows that $e(Q_\alpha) = 0$ and hence Q_α has a section. By construction of Q_α, such a section determines the required transition functions $a_{\alpha\beta}$ for $\beta \in \mathcal{S}_2$ with $\Delta_\beta \subset \Delta_\alpha$.

Now suppose, by induction, that the $a_{\beta\alpha}$ have been constructed for all $\alpha, \beta \in \mathcal{S}_0 \cup \cdots \cup \mathcal{S}_k$, where $k \geq 3$. Let $\alpha \in \mathcal{S}_{k+1}$ and consider the circle bundle $Q_\alpha \to \partial\Delta_\alpha$, defined as above. In this case this is a circle bundle over a k-sphere with $k \geq 3$ and every such bundle has a section.* This proves the existence of the transition functions over the $(k+1)$-skeleton, and hence the existence of a spin^c structure Γ with $c_1(L_\Gamma) = c$.

* Any smooth map $u : \partial B^k \to S^1$ extends to B^k when $k \geq 3$. To see this note that, because ∂B^k is simply connected there exists a lift $\xi : \partial B^k \to \mathbb{R}$ such that $u(x) = \exp(i\xi(x))$ for $x \in \partial B^k$. The map ξ obviously extends over B^k via $\xi(tx) = t\xi(x)$ for $t \in [0, 1]$ and $x \in \partial B^k$. This extension is only continuous but a smooth extension can be obtained by a standard approximation argument.

We prove (iii). Assume that $\Gamma : V \to \mathrm{End}(W)$ is a spin^c structure and $E \to X$ a Hermitian line bundle. Consider the twisted spin^c structure

$$\tilde{W} = W \otimes E, \qquad \tilde{\Gamma} = \Gamma \otimes \mathbb{1}.$$

The corresponding principal bundle $\tilde{P}$ is given by

$$\tilde{P} = P \otimes_{S^1} P_E$$

where P_E denotes the unit sphere bundle of E. Hence the fiber over $x \in X$ of the line bundle $L_{\tilde{\Gamma}}$ consists of equivalence classes of triples $[\Phi, \lambda, z] \in P_x \times P_{Ex} \times \mathbb{C}$ under the equivalence relation

$$[\Phi, \lambda, z] \equiv [\Phi \circ \Gamma_0(a), \lambda, \delta(a)^{-1}z] \equiv [e^{i\theta}\Phi, e^{-i\theta}\lambda, z] \equiv [\Phi, e^{-i\theta}\lambda, e^{2i\theta}z]$$

for $a \in \mathrm{Spin}^c(V_0)$ and $e^{i\theta} \in S^1$. In particular, with $\theta = \pi$, one obtains $[\Phi, \lambda, z] \equiv [\Phi, -\lambda, z]$. The required isomorphism $L_{\tilde{\Gamma}} \to L_\Gamma \otimes E \otimes E$ is given by $[\Phi, \lambda, z] \mapsto [\Phi, z] \otimes \lambda \otimes \lambda$.

To prove (iv) assume that $\Gamma_1 : V \to \mathrm{End}(W_1)$ and $\Gamma_2 : V \to \mathrm{End}(W_2)$ are two spin^c structures and consider the circle bundle $Q \to X$ whose fiber over x consists of all spin^c isomorphisms $\Phi : W_{1x} \to W_{2x}$ which lift the identity isomorphism of V_x, i.e.

$$\Phi\Gamma_1(v) = \Gamma_2(v)\Phi$$

for all $v \in V_x$. Indeed, any two such spin^c isomorphisms are related by multiplication with a complex number of modulus 1. Now the two spin^c structures are isomorphic if and only if the bundle $Q \to X$ admits a section. Hence consider the tensor product of W_1 with a line bundle E. Then the corresponding modification $Q_E = Q \otimes_{S^1} P_{E^*}$ of Q is given by the tensor product with the unit circle bundle of E^*. With a suitable choice of the line bundle E the resulting circle bundle Q_E of spin^c isomorphisms will have a section and so give rise to an isomorphism $W_2 \cong W_1 \otimes E$. Obviously, the isomorphism class of E is uniquely determined by Γ_1 and Γ_2. This proves the theorem. □

Exercise 5.9 Let $\Gamma : V \to \mathrm{End}(W)$ be a spin^c structures on a real vector bundle $V \to X$ of rank $2n$ and $A \in C^\infty(X, \mathrm{SO}(V))$ be an automorphism.

(i) Show that there is a natural homomorphism

$$\rho_A : \pi_1(X) \to \mathbb{Z}_2$$

defined as follows. Given a loop $\gamma : S^1 \to X$ trivialize the bundle γ^*V and define $\rho_A(\gamma) = 1$ whenever the resulting loop in $\mathrm{SO}(2n)$ determined by $A \circ \gamma$ is not contractible and $\rho_A(\gamma) = 0$ if this loop is contractible.

(ii) Let $e_A \in H^2(X;\mathbb{Z})$ be the image of the class $\rho_A \in H^1(X;\mathbb{Z}_2)$ under the boundary homomorphism $H^1(X;\mathbb{Z}_2) \to H^2(X;\mathbb{Z})$ in the coefficient exact sequence. Let $E \to X$ be a complex line bundle with first Chern class $c_1(E) = e_A$. Prove that the spin^c structure $\Gamma \circ A : V \to \mathrm{End}(W)$ is isomorphic to $\Gamma \otimes \mathbb{1} : V \to \mathrm{End}(W \otimes E)$.

(iii) Deduce that for two spin^c structures Γ_1 and Γ_2 on V there exist a unitary bundle isomorphism $\Phi : W_1 \to W_2$ and an orientation preserving orthogonal bundle automorphism $A : V \to V$ with

$$\Phi\Gamma_1\Phi^{-1} = \Gamma_2 \circ A$$

if and only if $c_1(L_{\Gamma_1}) = c_1(L_{\Gamma_2})$. □

Theorem 5.10 *Let X be a compact oriented smooth 4-manifold. Then X admits a spin^c structure. Moreover, if $c \in H^2(X;\mathbb{Z})$ satisfies*

$$\langle c, \alpha\rangle \equiv \alpha \cdot \alpha (\mathrm{mod}\ 2) \tag{5.8}$$

for all $\alpha \in H_2(X;\mathbb{Z})$ then there exists a torsion class $c_0 \in H^2(X;\mathbb{Z})$ such that $c + c_0$ is an integral lift of $\mathrm{w}_2(TX)$.

Lemma 5.11 *Let C be a free $\mathbb{Z}$-module and $\partial : C \to C$ be a boundary operator. Denote by $R_2 : H^*(C,\partial;\mathbb{Z}) \to H^*(C,\partial;\mathbb{Z}_2)$ the homomorphism induced by the projection $r_2 : \mathbb{Z} \to \mathbb{Z}_2$. If $\mathrm{w}_0 \in H^*(C,\partial;\mathbb{Z}_2)$ satisfies*

$$\langle \mathrm{w}_0, \alpha\rangle = 0 \tag{5.9}$$

for every $\alpha \in H_*(C,\partial;\mathbb{Z})$, then there exists a torsion class $c_0 \in H^*(C,\partial;\mathbb{Z})$ such that $\mathrm{w}_0 = R_2(c_0)$.

Proof: The proof is based on the following two facts.

(i) Every submodule of a free $\mathbb{Z}$-module is free.

(ii) If B is a free $\mathbb{Z}$-module then, for every homomorphism $\varphi : B \to \mathbb{Z}_2$, there exists a homomorphism $\psi : B \to \mathbb{Z}$ such that $\varphi = r_2 \circ \psi$.

Let $\mathrm{w}_0 \in H^*(C,\partial;\mathbb{Z}_2)$ satisfy (5.9) and choose a cocycle $\varphi : C \to \mathbb{Z}_2$ that represents w_0. By (5.9), φ vanishes on $\ker\partial \subset C$ and hence descends to a homomorphism $\bar\varphi : C/\ker\partial \to \mathbb{Z}_2$. By (i), $C/\ker\partial \cong \mathrm{im}\partial \subset C$ is a free $\mathbb{Z}$-module. By (ii), there exists a homomorphism $\bar\psi : C/\ker\partial \to \mathbb{Z}$ such that $\bar\varphi = r_2 \circ \bar\psi$. Denote by ψ the composition of the projection $C \to C/\ker\partial$ with $\bar\psi$. Then

$$\partial\sigma = 0 \qquad \Longrightarrow \qquad \psi(\sigma) = 0.$$

and $\varphi = r_2 \circ \psi$. The first condition asserts that $c_0 = [\psi] \in H^*(C,\partial;\mathbb{Z})$ is a torsion class and the second condition implies $R_2(c_0) = [r_2 \circ \psi] = [\varphi] = \mathrm{w}_0$. This proves the lemma. □

Proof of Theorem 5.10: Recall from (5.1) that

$$\langle \mathrm{w}_2(TX), \alpha \rangle = \alpha \cdot \alpha \,(\mathrm{mod}\ 2)$$

for $\alpha \in H_2(X;\mathbb{Z})$. This implies that $\alpha \mapsto \langle \mathrm{w}_2(TX), \alpha\rangle$ descends to a homomorphism $H_2(X;\mathbb{Z})/\mathrm{torsion} \to \mathbb{Z}_2$. Since $H_2(X;\mathbb{Z})/\mathrm{torsion} \cong \mathbb{Z}^m$ for some integer m, any such homomorphism lifts to a homomorphism $H_2(X;\mathbb{Z})/\mathrm{torsion} \to \mathbb{Z}$. Composing this with the projection $H_2(X;\mathbb{Z}) \to H_2(X;\mathbb{Z})/\mathrm{torsion}$, we obtain a homomorphism $\varphi : H_2(X;\mathbb{Z}) \to \mathbb{Z}$ such that $\varphi(\alpha) = \alpha \cdot \alpha(\mathrm{mod}\ 2)$ for every $\alpha \in H_2(X;\mathbb{Z})$. Since the map $H^2(X;\mathbb{Z}) \to \mathrm{Hom}(H_2(X;\mathbb{Z}), \mathbb{Z})$ is surjective, there exists a class $c \in H^2(X;\mathbb{Z})$ which satisfies (5.8).

If $c \in H^2(X;\mathbb{Z})$ satisfies (5.8) then $\mathrm{w}_0 = \mathrm{w}_2(TX) - R_2(c) \in H^2(X;\mathbb{Z}_2)$ satisfies (5.9). Hence, by Lemma 5.11, there exists a torsion class $c_0 \in H^2(X;\mathbb{Z})$ such that $\mathrm{w}_2(TX) - R_2(c) = R_2(c_0)$. By Theorem 5.8 (ii), there exists a spin^c structure Γ on TX such that $c_1(L_\Gamma) = c + c_0$. This proves the theorem. □

Exercise 5.12 Let X be a compact oriented smooth 4-manifold. Show that $\mathrm{w} \in H^2(X;\mathbb{Z}_2)$ is in the image of $R_2 : H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{Z}_2)$ if and only if $\mathrm{w} \cdot \alpha = 0(\mathrm{mod}\ 2)$ for every torsion class $\alpha \in \mathrm{Tor}(H_2(X;\mathbb{Z}))$ (use Lemma 5.11). Deduce that the cohomology of X has the form

$$H^2(X;\mathbb{Z}_2) \cong H \oplus T \oplus T'$$

where $H \oplus T \cong H^2(X;\mathbb{Z}) \otimes \mathbb{Z}_2$, $T = \mathrm{Tor}(H^2(X;\mathbb{Z})) \otimes \mathbb{Z}_2$ and $T \cong T'$. The isomorphism between T and T' is given by Poincaré duality with $\mathbb{Z}_2$ coefficients. □

Example 5.13 Consider the manifold

$$X = \mathbb{R}P^3 \times S^1.$$

Its second homology is given by

$$H_2(X;\mathbb{Z}) = \mathbb{Z}_2, \qquad H_2(X;\mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

If $\gamma \subset \mathbb{R}P^3$ denotes the nontrivial loop then $[\gamma \times S^1]$ is a nontrivial integral homology class while the nonorientable submanifold $\mathbb{R}P^2 \times \{\mathrm{pt}\}$ represents a class in $H_2(X;\mathbb{Z}_2)$ but not in $H_2(X;\mathbb{Z})$. Note, however, that the tangent bundle of $\mathbb{R}P^3$ admits a trivialization and hence X admits a spin structure. A more interesting example is given by the Enriques surface $X = X_4/\mathbb{Z}_2$ where $X_4 \subset \mathbb{C}P^3$ denotes the K3-surface (a hypersurface of degree 4). This manifold has even intersection form but does not admit a spin structure (see Example 6.28). Thus the class $c = 0$ satisfies (5.8) and the torsion class c_0 in Theorem 5.10 has to be chosen nonzero. □

Lemma 5.14 *Let* $\Gamma : TX \to \mathrm{End}(W)$ *be a spin*^{c} *structure on a compact oriented smooth* 4*-manifold. Then*

$$2\chi(X) + 3\sigma(X) = \langle c_1(W^+)^2 - 4c_2(W^+), [X]\rangle.$$

Proof: The Hirzebruch signature formula takes the form

$$\sigma(X) = \frac{p_1(X)}{3} = -\frac{\langle c_2(TX \otimes_{\mathbb{R}} \mathbb{C}), [X]\rangle}{3}. \tag{5.10}$$

Here $p_1(X) = \langle p_1(TX), [X]\rangle$ denotes the first Pontryagin number (see page 212). Now Γ defines a bundle isomorphism

$$TX \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{Hom}(W^-, W^+).$$

Since $c_1(W^+) = c_1(W^-)$ this gives

$$c_2(TX \otimes_{\mathbb{R}} \mathbb{C}) = 2c_2(W^+) + 2c_2(W^-) - c_1(W^+)^2. \tag{5.11}$$

Moreover, the second Chern classes of W^+ and W^- are related by

$$\langle c_2(W^-) - c_2(W^+), [X]\rangle = \chi(X). \tag{5.12}$$

To see this choose a section $s^- : X \to W^-$ and a vector field $v : X \to TX$ which are both transverse to the zero section and have no common zeros. Then (5.12) follows by counting the zeros of

$$s^+ = \Gamma(v)s^- : X \to W^+$$

and noting that, for every nonzero vector $s^- \in W_x^-$ the isomorphism $T_xX \to W_x^+ : v \mapsto \Gamma(v)s^-$ is orientation reversing. Combining (5.10) with (5.11) and (5.12) we obtain

$$\begin{aligned} 2\chi(X) + 3\sigma(X) &= 2\chi(X) - \langle c_2(TX \otimes_{\mathbb{R}} \mathbb{C}), [X]\rangle \\ &= \langle c_1(W^+)^2 - 4c_2(W^+), [X]\rangle. \end{aligned}$$

This proves the lemma. □

Exercise 5.15 Prove that

$$p_1(X) = \langle c_1(TX)^2 - 2c_2(TX), [X]\rangle,$$

whenever X is an almost complex 4-manifold. Hence in this case (1.9) and (5.10) coincide. □

The three dimensional case

Recall from Section 4.1 that the group $\mathrm{Spin}^c(3)$ is naturally isomorphic to U(2) and that the projection $\Phi : \mathrm{U}(2) \to \mathrm{SO}(3)$ is given by the adjoint action of U(2) on $\mathfrak{su}(2)$. Let $W \to X$ be a Hermitian rank-2 bundle and consider the associated principal frame bundle $P \to X$ with structure group $\mathrm{G} = \mathrm{U}(2)$. The homomorphism $\Phi : \mathrm{U}(2) \to \mathrm{SO}(3)$ gives rise to an associated $\mathbb{R}^3$-bundle $\Lambda = P \times_\Phi \mathbb{R}^3$. This bundle is naturally isomorphic to

$$\mathfrak{su}(W) = \{(x, A) \,|\, A \in \mathrm{End}(W_x),\ A + A^* = 0,\ \mathrm{trace}(A) = 0\}.$$

If one thinks of P_x as the set of unitary isomorphisms $p : \mathbb{C}^2 \to W_x$ then Λ_x is the set of equivalence classes $[p, \xi] \in P_x \times \mathbb{R}^3$ under the equivalence relation $[p, \xi] \equiv [p \circ U, \Phi(U)^{-1}\xi]$ and the isomorphism $\Lambda_x \to \mathfrak{su}(W_x)$ is given by $[p, \xi] \mapsto p \circ \gamma(\xi) \circ p^{-1}$. Note that

$$\mathfrak{su}(W) \cong \mathfrak{su}(W \otimes E), \qquad \det(W \otimes E) \cong \det(W) \otimes E^{\otimes 2}$$

for every Hermitian line bundle $E \to X$.

Now let $\Lambda \to X$ be a 3-dimensional oriented real vector bundle equipped with an inner product. A **spin^c structure** on Λ is a Hermitian rank-2 bundle $W \to X$ together with an orientation preserving isomorphism $\mathfrak{su}(W) \to \Lambda$. Any such isomorphism can be expressed as a fiberwise linear map $\gamma : \Lambda \to \mathrm{End}(W)$ which satisfies

$$\gamma(v)^* + \gamma(v) = 0, \qquad \gamma(v)^*\gamma(v) = |v|^2 \mathbb{1} \tag{5.13}$$

for $v \in \Lambda$. The reader should note that any such map γ satisfies

$$\gamma(v)\gamma(w) + \gamma(w)\gamma(v) = -2\langle v, w\rangle \mathbb{1}$$

for $(v, w) \in \Lambda \oplus \Lambda$. In particular, $\gamma(v) = -\gamma(w)\gamma(v)\gamma(w)^{-1}$ whenever $v \perp w$ and $w \neq 0$. Thus $\mathrm{trace}(\gamma(v)) = 0$ for all $v \in \Lambda$ and hence γ is indeed an isomorphism from Λ to $\mathfrak{su}(W)$ (see Proposition 4.13). This isomorphism is orientation preserving if and only if

$$\gamma(e_3)\gamma(e_2)\gamma(e_1) = \mathbb{1} \tag{5.14}$$

for every positively oriented orthonormal frame e_1, e_2, e_3 of Λ. The conditions (5.13) and (5.14) together can be expressed in the form

$$\gamma(v)^* + \gamma(v) = 0, \qquad \gamma(v)\gamma(w) = \gamma(v \times w) - \langle v, w\rangle \mathbb{1} \tag{5.15}$$

for $(v, w) \in \Lambda \oplus \Lambda$ (see Example 4.38).

Theorem 5.16 *Let $\Lambda \to X$ be an oriented Riemannian vector bundle of real dimension* 3.

(i) *If $\gamma : \Lambda \to \mathrm{End}(W)$ is a spin^c structure on Λ then the first Chern class $c_1(W) \in H^2(X;\mathbb{Z})$ is an integral lift of $\mathrm{w}_2(\Lambda) \in H^2(X;\mathbb{Z}_2)$.*

(ii) *For every integral lift $c \in H^2(X;\mathbb{Z})$ of $\mathrm{w}_2(\Lambda)$ there exists a spin^c structure $\gamma : \Lambda \to \mathrm{End}(W)$ with $c_1(W) = c$.*

(iii) *If $\gamma_1 : \Lambda \to \mathrm{End}(W_1)$ and $\gamma_2 : \Lambda \to \mathrm{End}(W_2)$ are spin^c structures on Λ then there exists a Hermitian line bundle $E \to X$ such that*

$$W_2 \cong W_1 \otimes E, \qquad \gamma_2 \cong \gamma_1 \otimes 1\!\!1, \qquad c_1(W_2) = c_1(W_1) + 2c_1(E).$$

These two spin^c structures are isomorphic if and only if $c_1(E) = 0$.

Proof: There is a one-to-one correspondence between isomorphism classes of spin^c structures on Λ and isomorphism classes of spin^c structures on the 4-dimensional Riemannian vector bundle

$$V = \mathbb{R} \oplus \Lambda.$$

To see this define $\widetilde{W} = W \oplus W$ and let $\Gamma : \mathbb{R} \oplus \Lambda \to \mathrm{End}(\widetilde{W})$ be given by

$$\Gamma(v_0, v) = \begin{pmatrix} 0 & \gamma(v) + v_0 1\!\!1 \\ \gamma(v) - v_0 1\!\!1 & 0 \end{pmatrix}$$

for $v_0 \in \mathbb{R}$ and $v \in \Lambda$. It is an easy exercise to show that Γ defines a spin^c structure on V. It follows from (5.14) that

$$\widetilde{W}^+ = W \oplus \{0\}, \qquad \widetilde{W}^- = \{0\} \oplus W$$

are the ∓ 1 eigenspaces of

$$\Gamma(e_3)\Gamma(e_2)\Gamma(e_1)\Gamma(e_0) = \begin{pmatrix} -1\!\!1 & 0 \\ 0 & 1\!\!1 \end{pmatrix}$$

for $e_0 = (1, 0) \in \mathbb{R} \oplus \Lambda$ and a positively oriented orthonormal frame e_1, e_2, e_3 of Λ. Conversely, suppose that $\Gamma : V \to \mathrm{End}(\widetilde{W})$ is a spin^c structure on V. Then there is a natural isomorphism $\Gamma(1, 0) : \widetilde{W}^- \to \widetilde{W}^+$. Define $W = \widetilde{W}^+$ and $\gamma : \Lambda \to \mathrm{End}(W)$ by

$$\gamma(v) = \Gamma(1, 0)\Gamma(0, v)|_{\widetilde{W}^+} = -\,\Gamma(0, v)\Gamma(1, 0)|_{\widetilde{W}^+}$$

for $v \in \Lambda$. This map satisfies (5.13). Hence there is a one-to-one correspondence between isomorphism classes of spin^c structures on Λ and isomorphism classes of spin^c structures on $\mathbb{R} \oplus \Lambda$. Hence the assertions follow from Theorem 5.8. □

Corollary 5.17 *The tangent bundle of an orientable smooth 3-manifold Y admits a spin^c structure.*

Proof: By Theorem 5.10 choose a spin^c structure on $S^1 \times Y$. Restrict this structure to $\{e^{i\theta}\} \times Y$ to obtain a spin^c structure on the bundle $\mathbb{R} \oplus TY$. Now use the proof of Theorem 5.16 to obtain a spin^c structure on TY. □

Exercise 5.18 Give a direct proof of the fact that the first Chern class of W is an integral lift of the second Stiefel-Whitney class of $\mathfrak{su}(W)$. □

Exercise 5.19 Generalize Theorem 5.16 to arbitrary odd dimensional real vector bundles. □

5.3 Spin^c structures on complex vector bundles

The existence and uniqueness problem for spin^c structures appears in a new light in the case of complex vector bundles. Let $V \to X$ be an oriented real vector bundle of rank $2n$ equipped with a Riemannian metric $g : V \otimes V \to \mathbb{R}$ and denote by $\mathcal{J}(V) \subset C^\infty(X, \mathrm{Aut}(V))$ the set of complex structures on V which are compatible with metric and orientation. For each $J \in \mathcal{J}(V)$ denote by

$$\langle v, w\rangle = \langle v, w\rangle_J = g(v, w) + i\omega(v, w)$$

the corresponding Hermitian structure with $\omega(v, w) = \omega_J(v, w) = g(Jv, w)$. The **canonical bundle** $K = K_J$ is the complex line bundle over X defined as the highest complex exterior power of the dual bundle (V^*, J^*):

$$K = K_J = \Lambda_J^{n,0} V^*.$$

If $\Gamma : V \to \mathrm{End}(W)$ is a spin^c structure on V then there is a natural line subbundle $E_J = E_{J,\Gamma} \subset W$ of *pure spinors* defined by

$$E_J = \{(x, \theta) \,|\, x \in X,\, \theta \in W_x,\, \Gamma(Jv)\theta = i\Gamma(v)\theta \ \text{ for all } v \in V_x\}.$$

This bundle is not invariant under spin^c automorphisms of W but it is fiberwise invariant under the action of the lift $\mathrm{U}^c(V_x)$ of the unitary group of V_x.

Lemma 5.20 *For a spin^c structure $\Gamma : V \to \mathrm{End}(W)$ on a complex vector bundle (V, J) the line bundle $L_\Gamma = P_\Gamma \times_\delta \mathbb{C}$ is related to the bundle of pure spinors by*

$$L_\Gamma \cong E_{J,\Gamma} \otimes E_{J,\Gamma} \otimes {K_J}^*.$$

Moreover, the bundle of pure spinors of the twisted spin^c structure $\Gamma \otimes \mathbb{1}_E : V \to \mathrm{End}(W \otimes E)$ is given by

$$E_{J,\Gamma\otimes\mathbb{1}_E} = E_{J,\Gamma} \otimes E$$

for any Hermitian line bundle $E \to X$.

Proof: Choose a unitary frame bundle $P \to X$ with structure group $\mathrm{U}^c(V_0, J_0)$ (instead of $\mathrm{Spin}^c(V_0)$) and recover E_J as the bundle associated to the homomorphism $\Theta : \mathrm{U}^c(V_0, J_0) \to S^1$ in Lemma 4.53. Recall the formula $\Theta^2 \cdot \det^c \circ \mathrm{ad} = \delta$. Since

$$L_\Gamma \cong P \times_\delta \mathbb{C}, \qquad {K_J}^* \cong P \times_{\det^c \circ \mathrm{ad}} \mathbb{C}, \qquad E_J \cong P \times_\Theta \mathbb{C}$$

it follows that $E_J \otimes E_J \otimes {K_J}^* \cong L_\Gamma$ as claimed. The second statement is obvious. $\square$

The discussion of Section 4.7 shows that every $J \in \mathcal{J}(V, g)$ determines a canonical spin^c structure given by

$$W_{\mathrm{can}} = \Lambda^{0,*}V^*$$

with $\Gamma_{\mathrm{can}} : V \to \mathrm{End}(W_{\mathrm{can}})$ given by

$$\Gamma_{\mathrm{can}}(v)\tau = \frac{1}{\sqrt{2}} v'' \wedge \tau - \sqrt{2}\iota(v)\tau \tag{5.16}$$

for $v \in T_xX$ and $\tau \in \Lambda^{0,\mathrm{odd}}T_x^*X$ with $v'' = \langle \cdot, v \rangle$. Lemma 4.52 shows that in this case the splitting is given by

$$W_{\mathrm{can}}^- = \Lambda^{0,\mathrm{odd}}V^*, \qquad W_{\mathrm{can}}^+ = \Lambda^{0,\mathrm{ev}}V^*.$$

Note that reversing the complex structure results in an alternative spin^c structure with L replaced by L^* and complex anti-linear forms replaced by complex linear ones. Sometimes we shall use the notation $W_J = \Lambda_J^{0,*}V^*$ and $\Gamma_J : V \to \mathrm{End}(W_J)$ to stress the dependence on the almost complex structure. The following corollary shows that the line bundle $L = L_{\Gamma_J}$ is isomorphic to the anti-canonical bundle ${K_J}^* = \Lambda^{0,n}V^*$.

Corollary 5.21 *For a complex vector bundle (V, J) the line bundle $L_{\Gamma_{\mathrm{can}}}$ associated to the canonical spin^c structure (5.16) is isomorphic to the anti-canonical bundle:*

$$L_{\Gamma_{\mathrm{can}}} \cong K^*.$$

Proof: By Lemma 4.53, $E_{J,\Gamma_{\mathrm{can}}} \cong \Lambda^{0,0}V^* \cong \mathbb{C}$ and hence the result follows from Lemma 5.20. $\square$

Any spin^c structure on a complex vector bundle (V, J) can be obtained from the canonical spin^c structure $\Gamma_{\mathrm{can}} : V \to \mathrm{End}(W_{\mathrm{can}})$ by twisting with a line bundle $E \to X$. Denote the resulting spin^c structure by $\Gamma_E : V \to \mathrm{End}(W_E)$ where

$$W_E = W_{\mathrm{can}} \otimes E.$$

By Lemma 5.20, $L_{\Gamma_E} = K^* \otimes E^{\otimes 2}$ and $E_{J,\Gamma_E} \cong E$. This proves the following.

Corollary 5.22 *For complex vector bundles (V, J) the correspondence*

$$(W, \Gamma) \mapsto E_{J,\Gamma}$$

determines a bijection between isomorphism classes of spin^c structures and isomorphism classes of complex line bundles over X.

Exercise 5.23 If two almost complex structures $J_0, J_1 \in \mathcal{J}(V)$ are homotopic prove that the corresponding canonical spin^c structures are isomorphic. □

Exercise 5.24 Let $\Gamma : V \to \mathrm{End}(W)$ be any spin^c structure on a $2n$-dimensional real Riemannian vector bundle and $J \in \mathcal{J}(V)$ be any almost complex structure. Prove that the following are equivalent

(i) Γ is isomorphic to the canonical spin^c structure of J.

(ii) There exists a section $s : X \to W^+$ such that $|s(x)| = 1$ and

$$\Gamma(J(x)v)s(x) = i\Gamma(v)s(x)$$

for all $x \in X$ and $v \in V_x$.

Hint: Suppose that $W = W_J \otimes E$ and use Lemma 5.20 to show that E admits a nonzero section. □

Consider the map

$$\pi_0(\mathcal{J}(V)) \to \mathcal{S}^c(V) : [J] \mapsto [\Gamma_J]$$

which assigns to a homotopy class of almost complex structures the isomorphism class of the corresponding canonical spin^c structures. This map is neither onto nor injective. For any spin^c structure $\Gamma : V \to \mathrm{End}(W)$ denote the set of orthogonal almost complex structures J on V whose canonical spin^c structure Γ_J is isomorphic to Γ by

$$\mathcal{J}(V, \Gamma) = \{J \in \mathcal{J}(V) \,|\, \Gamma_J \cong \Gamma\}.$$

If X is a smooth 4-manifold and V a rank-4 bundle then this set is nonempty if and only if $c_2(W^+) = 0$. If in addition V is the tangent bundle of X then the components of $\mathcal{J}(TX, \Gamma)$ are characterized in the next proposition.

Proposition 5.25 *For every compact oriented smooth 4-manifold X and every spin^c structure $\Gamma : TX \to \mathrm{End}(W)$ with $c_2(W^+) = 0$ there is a natural isomorphism*

$$\pi_0(\mathcal{J}(TX, \Gamma)) \cong \mathbb{Z}_2 \oplus \frac{H^3(X; \mathbb{Z})}{H^1(X; \mathbb{Z}) \cup c_1(W^+)}.$$

Proof: By Exercise 5.23, the canonical spin^c structure Γ_J is isomorphic to Γ if and only if there exists a unit section $s : X \to W^+$ such that $\Gamma(J\cdot)s = i\Gamma(\cdot)s$. The almost complex structure J is uniquely determined by s and, conversely, the section s is determined by J up to multiplication with a function $u : X \to S^1$. Hence there is a one-to-one correpondence between homotopy classes of almost complex structures $J \in \mathcal{J}(V)$ with $\Gamma_J \cong \Gamma$ and homotopy classes of unit sections of W^+ up to S^1-gauge equivalence. These homotopy classes of unit sections can be understood in terms of the Pontryagin-Thom constructions as follows (compare Milnor [90], Chapter 7).

Denote by W_1^+ the unit sphere bundle in W^+ and fix a reference section $\bar{s} : X \to W_1^+$. For any section $s : X \to W_1^+$ which is transverse to $\bar{s}$ consider the 1-dimensional submanifold

$$C = C_s = \{x \in X \,|\, s(x) = \bar{s}(x)\}.$$

There are two real rank-3 vector bundles over C, namely the normal bundle $N_C \to C$ and the vertical tangent bundle $V_C \to C$ of W_1^+:

$$N_C = \{(x, \xi) \,|\, x \in C,\, \xi \in T_xX,\, \xi \perp T_xC\},$$

$$V_C = \left\{(x, \tau) \,|\, x \in C,\, \tau \in W_x^+,\, \tau \perp \bar{s}(x)\right\}.$$

Transversality implies that the vertical differential $D(s - \bar{s})(x) : T_xX \to W_x^+$ determines an isomorphism

$$\rho_s : N_C \to V_C,$$

called the **vertical framing**. The pair (C_s, ρ_s) is called the **Pontryagin manifold** of the section $s : X \to W_1^+$. Note that the framing $\rho : N_C \to V_C$ determines an orientation of the normal bundle N_C and hence of the tangent bundle TC via $T_CX = TC \oplus N_C$. Thus a framed 1-manifold (C, ρ) carries a natural orientation.

Two (vertically) framed 1-manifolds (C_0, ρ_0) and (C_1, ρ_1) are called **(vertically) framed cobordant** if there exists an oriented cobordism $\Sigma \subset X \times [0, 1]$ from C_0 to C_1 together with a framing

$$\rho : N_\Sigma \to V_\Sigma$$

such that

$$\rho|_{C_0} = \rho_0, \qquad \rho|_{C_1} = \rho_1.$$

Here N_Σ is the normal bundle of Σ in $X \times [0, 1]$ and V_Σ is the vertical tangent bundle of W_1^+ along $\bar{s}|_\Sigma$. With this understood the result is proved in five steps.

Step 1 *Every smooth section $s : X \to W_1^+$ is smoothly homotopic to one which is transverse to $\bar{s}$.*

Step 2 *Every vertically framed 1-manifold (C, ρ) arises as the Pontryagin manifold of a section $s : X \to W_1^+$ which is transverse to $\bar{s}$.*

Step 3 *Two sections $s_0, s_1 : X \to W_1^+$ which are both transverse to $\bar{s}$ are smoothly homotopic if and only if the corresponding Pontryagin manifolds (C_{s_0}, ρ_{s_0}) and (C_{s_1}, ρ_{s_1}) are framed cobordant.*

Step 4 *For each homology class $\alpha \in H_1(X; \mathbb{Z})$ there are precisely two cobordism classes of vertically framed 1-manifolds (C, ρ) with $[C] = \alpha$.*

Step 5 *For every section $s : X \to W_1^+$ and every smooth map $u : X \to S^1$ such that both s and us are transverse to $\bar{s}$ we have*

$$\mathrm{PD}([C_{us}]) = \mathrm{PD}([C_s]) + \frac{[u^{-1}du]}{2\pi i} \cup c_1(W^+).$$

Moreover, if $[C_s] = [C_{us}]$ then the two framings ρ_s and ρ_{us} are homotopic.

The first three steps are adapted from Chapter 7 in Milnor [90] and can be proved with the same techniques. To prove Step 4, represent α by a connected 1-manifold $C \subset X$ and note that, since $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}_2$, there are precisely two framings ρ^+ and ρ^- along C up to homotopy. We must show that these framings are not cobordant. To see this, let $\Sigma \subset X \times S^1$ be a compact oriented embedded surface and note that the second Stiefel-Whitney classes of TX and W^+ agree over Σ. Since the normal bundle N_Σ and the vertical bundle V_Σ are obtained from $TX \oplus \mathbb{R}$ and W^+ by splitting off a summand with $\mathrm{w}_2 = 0$ it follows that

$$\mathrm{w}_2(V_\Sigma) = \mathrm{w}_2(N_\Sigma).$$

With this established it is easy to see that the two framings over a 1-manifold $C \subset X$ are not cobordant and this proves Step 4.

We prove Step 5. Consider a splitting $W^+ = \mathbb{C} \oplus L$ for some line bundle $L \to X$ and choose the constant section $\bar{s}(x) = (1, 0)$. Suppose that

$$s(x) = (f(x), \sigma(x))$$

for some section $\sigma : X \to L$ and some function $f : X \to \mathbb{C}$ with $|f(x)|^2 + |\sigma(x)|^2 = 1$. Then

$$C_s = f^{-1}(1) \cap \sigma^{-1}(0), \qquad C_{us} = (uf)^{-1}(1) \cap \sigma^{-1}(0).$$

Suppose that σ is transverse to the zero section. Then the embedded surface $\Sigma = \sigma^{-1}(0) \subset X$ is Poincaré dual to the first Chern class $c = c_1(L) = c_1(W^+)$. Suppose that 1 is a regular value of both functions $f : \Sigma \to S^1$ and $fu : \Sigma \to S^1$. Then the 1-manifold $f^{-1}(1) \cap \Sigma$ is Poincaré dual (relative

Σ) to the cohomology class $(2\pi i)^{-1}[f^{-1}df]$. Similarly for $fu : \Sigma \to S^1$ and $u : X \to S^1$. Hence, with $\iota : \Sigma \to X$ denoting the obvious embedding,

$$\begin{aligned}
[C_{us}]_X - [C_s]_X &= \iota_* \mathrm{PD}_\Sigma \left(\frac{(uf)^{-1}d(uf) - f^{-1}df}{2\pi i} \right) \\
&= \iota_* \mathrm{PD}_\Sigma \left(\frac{u^{-1}du}{2\pi i} \right) \\
&= [u^{-1}(1) \cap \Sigma]_X \\
&= \mathrm{PD}_X \left(\frac{[u^{-1}du]}{2\pi i} \cup c_1(W^+) \right).
\end{aligned}$$

The proof that the two framings ρ_s and ρ_{us} are homotopic in the case $[C_s] = [C_{us}]$ is left as an exercise. This proves Step 5 and the five steps obviously prove the proposition. □

Example 5.26 The condition $V = TX$ cannot be removed in Proposition 5.25. Consider for example the trivial rank-4 bundle

$$V_0 = \mathbb{C}P^2 \times \mathbb{R}^4$$

over $\mathbb{C}P^2$ equipped with the trivial spin^c structure Γ_0. Then the space $\mathcal{J}(V_0, \Gamma_0)$ is connected. To see this note that homotopy classes of maps $\mathbb{C}P^2 \to S^3$ are characterized by framed cobordism classes of 1-manifolds in $\mathbb{C}P^2$ and there is only one such class. (The two framings along a loop are related by a cobordism along $\mathbb{C}P^1$.) This shows that for each choice of first Chern class there is precisely one almost complex structure on the trivial $\mathbb{R}^4$-bundle over $\mathbb{C}P^2$. The geometric reason for the absence of the $\mathbb{Z}_2$-summand is the fact that $\mathbb{C}P^2$ is not spin while the bundle V_0 admits a spin structure. (See Step 4 in the proof of Proposition 5.25.) □

5.4 Classification of spin structures

Theorem 5.8 shows that a vector bundle $V \to X$ admits a spin^c structure if and only if its second Stiefel-Whitney class $\mathrm{w}_2(V) \in H^2(X; \mathbb{Z}_2)$ admits an integral lift $c \in H^2(X; \mathbb{Z})$. If $\mathrm{w}_2(V) = 0$ then this integral lift can be chosen to be zero and the resulting spin^c structure (W, Γ) has a line bundle L_Γ which admits a trivialization, that is a section

$$\theta : X \to L_\Gamma$$

of norm 1. Recall from Section 5.1 that, when the rank of V is 2, 3, or 4 modulo 8, then any such section precisely determines a spin structure on V in the sense of Definition 5.4. This shows that the bundle $V \to X$ admits a spin structure if and only if $\mathrm{w}_2(V) = 0$. This gives rise to a third defintion of spin structures which is equivalent to Definitions 5.1 and 5.4.

Definition 5.27 *Let $V \to X$ be an oriented Riemannian vector bundle. A* **spin structure** *on V is a triple (W, Γ, θ) where (W, Γ) is a spin^c structure on V with $c_1(L_\Gamma) = 0$ and $\theta : X \to L_\Gamma$ is a section with $|\theta(x)| = 1$ for all $x \in X$. A* **spin isomorphism** *from $(W_0, \Gamma_0, \theta_0)$ to $(W_1, \Gamma_1, \theta_1)$ is a spin^c isomorphism $\Phi : W_0 \to W_1$ such that the induced isomorphism $\Phi_L : L_0 \to L_1$ satisfies*

$$\Phi_L \theta_0 = \theta_1.$$

Denote by $\mathcal{S}(V)$ the set of isomorphism classes of spin structures on V.

Theorem 5.28 *Let $V \to X$ be an oriented Riemannian vector bundle.*

(i) *V admits a spin structure if and only if $\mathrm{w}_2(V) = 0$.*

(ii) *If $(W_1, \Gamma_1, \theta_1)$ and $(W_2, \Gamma_2, \theta_2)$ are two spin structures on V then there exists a Hermitian line bundle $E \to X$ with $2c_1(E) = 0$ and a unitary section $\psi : X \to E \otimes E$ such that*

$$W_2 \cong W_1 \otimes E, \qquad \theta_2 \cong \theta_1 \otimes \psi.$$

These two spin structures are isomorphic if and only if $c_1(E) = 0$ and E admits a unitary section $\varphi : X \to E$ such that

$$\psi = \varphi \otimes \varphi.$$

(iii) *The set $\mathcal{S}(V)$ of isomorphism classes of spin structures on V is a principal space with structure group $H^1(X; \mathbb{Z}_2)$.*

Proof: It $\mathrm{w}_2(V) = 0$ then, by Theorem 5.8, there exists a spin^c structure $\Gamma : V \to \mathrm{End}(W)$ with

$$c_1(L_\Gamma) = 0.$$

Since line bundles are classified by their first Chern class the bundle L_Γ admits a nonzero section θ. This proves (i). To prove (ii) recall first from Theorem 5.8 that for any two spin^c structures (W_1, Γ_1) and (W_2, Γ_2) on V there exists a Hermitian line bundle $E \to X$ such that

$$W_2 \cong W_1 \otimes E, \qquad \Gamma_2 \cong \Gamma_1 \otimes \mathbb{1}, \qquad L_2 \cong L_1 \otimes E^{\otimes 2}$$

where $L_i = L_{\Gamma_i}$ for $i = 1, 2$. Now for any two unit sections $\theta_1 : X \to L_1$ and $\theta_2 : X \to L_2$ there exists a unique unit section $\psi : X \to E^{\otimes 2}$ such that $\theta_2 \cong \theta_1 \otimes \psi$. This proves the first part of (ii). To prove the second part note first that if E admits a trivialization φ with $\psi = \varphi \otimes \varphi$ then there is an obvious isomorphism $\Phi : W_1 \to W_2 = W_1 \otimes E$ (given by the tensor product with φ) which intertwines the two spin structures. Conversely, if two spin structures are isomorphic then the underlying spin^c structures are necessarily isomorphic. Hence consider two spin structures of the form

(W,Γ,θ_1) and (W,Γ,θ_2). These are isomorphic if and only if there exists a spin^c isomorphism $\Phi \in C^\infty(X,\mathrm{End}(W))$ which commutes with Γ and satisfies $\theta_2 = \delta(\Phi)\theta_1$. Now, by Proposition 4.36 (iii), every isomorphism $\Phi \in C^\infty(X,\mathrm{End}(W))$ which commutes with Γ is of the form $\Phi = u\mathbb{1}$ where $u : X \to S^1$. Since $\delta(u\mathbb{1}) = u^2$ it follows that the two spin structures (W,Γ,θ_1) and (W,Γ,θ_2) are isomorphic if and only if there exists a smooth map $u : X \to S^1$ such that

$$\theta_2 = u^2\theta_1.$$

This proves (ii). To prove (iii) fix a spin^c structure (W,Γ,θ). Then, by (ii), $\mathcal{S}(V)$ can be identified with the space of equivalence classes of pairs (E,ψ) where $E \to X$ is a Hermitian line bundle and $\psi : X \to E \otimes E$ is a unit section. Two such pairs (E,ψ) and (E',ψ') are equivalent if there exists a unitary isomorphism $u : E \to E'$ such that $(u\otimes u)\circ\psi = \psi'$. Now every pair (E,ψ) determines a homomorphism $\rho_{E,\psi} : \pi_1(X) \to \mathbb{Z}_2$ as follows. Given a loop $\gamma : S^1 \to X$, define $\rho_{E,\psi}(\gamma) = 0$ if the pullback bundle γ^*E admits a section whose square is $\gamma^*\psi$, and $\rho_{E,\psi}(\gamma) = 1$ otherwise. It is easy to check that (E,ψ) is equivalent to (E',ψ') if and only if $\rho_{E,\psi} = \rho_{E',\psi'}$.

Here is an alternative proof of (iii). By Theorem 5.8, isomorphism classes of spin^c structures (W,Γ) with $c_1(L_\Gamma) = 0$ form a principal space with structure group

$$\mathrm{Tor}_2(H^2(X;\mathbb{Z})) = \{c \in H^2(X;\mathbb{Z}) \,|\, 2c = 0\}.$$

By (ii), the isomorphism classes of spin structures of the form (W,Γ,θ) with fixed spin^c structure Γ can be identified with equivalence classes of unitary sections $\theta : X \to L_\Gamma$ under the equivalence relation $\theta \sim u^2\theta$ for $u : X \to S^1$. These equivalence classes form a principal space with structure group

$$\mathrm{Map}(X,S^1)/\mathrm{Map}^{\mathrm{ev}}(X,S^1) \cong H^1(X;\mathbb{Z})/H^1(X;2\mathbb{Z}).$$

Here $\mathrm{Map}^{\mathrm{ev}}(X,S^1)$ denotes the space of *even* maps from X to S^1. This notion and the isomorphism of the quotient spaces are explained in Proposition 5.30 below. It follows that $\mathcal{S}(V)$ is a principal space with structure group $H^1(X;\mathbb{Z})/H^1(X;2\mathbb{Z}) \times \mathrm{Tor}_2(H^2(X;\mathbb{Z}))$. The exact sequence

$$H^1(X;\mathbb{Z}) \xrightarrow{2} H^1(X;\mathbb{Z}) \to H^1(X;\mathbb{Z}_2) \to H^2(X;\mathbb{Z}) \xrightarrow{2} H^2(X;\mathbb{Z})$$

shows that this group is isomorphic to $H^1(X;\mathbb{Z}_2)$. □

Exercise 5.29 Let $E \to X$ be a Hermitian line bundle and $\psi : X \to E \otimes E$ be a unit section. Let $\rho_{E,\psi} : \pi_1(X) \to \mathbb{Z}_2$ be defined as in the proof of Theorem 5.28. Prove that the image of $\rho_{E,\psi}$ under the Bockstein homomorphism $H^1(X;\mathbb{Z}_2) \to H^2(X;\mathbb{Z})$ is the first Chern class of E. □

For the understanding of spin^c structures it is sometimes interesting to compare them with spin structures, if these exist. Thus let $V \to X$ be a real Riemannian vector bundle of rank 2, 3, or 4 modulo 8 with $\mathrm{w}_2(V) = 0$. Then V admits a spin structure (S, I, J, Γ) as in Definition 5.4. Think of (S, I) as a complex vector bundle. Then (S, I, Γ) is a spin^c structure whose corresponding line bundle has first Chern class zero. In fact the complex structure J corresponds, by Lemma 5.7, to a nonzero section of this line bundle. Now Theorem 5.8 shows that every other spin^c structure on V is of the form

$$W = S \otimes L^{1/2}$$

for some line bundle $L^{1/2} \to X$. Theorem 5.8 also shows that the characteristic line bundle of this spin^c structure is given by $L = L^{1/2} \otimes L^{1/2}$. Note that if this bundle has Chern class zero then W is again a spin structure.

Physicists sometimes use the notation $W = S \otimes L^{1/2}$ even when the bundle V does not admit a spin structure. Then neither S nor $L^{1/2}$ are meaningful, only their tensor product is well defined. Nevertheless, this notation gives a good intuition of what spin^c structures are. This is particularly enlightening when it comes to spin^c connections because the Riemannian metric on X determines a natural spin connection on S and thus every connection on $L^{1/2}$ determines a spin^c connection on W. This is explained in detail in Chapter 6 below.

Appendix to Section 5.4: Maps to the circle

A smooth map $u : X \to S^1$ is called **even** if it admits a square root. It is easy to see that this is only a condition on the homotopy class of u. Hence the quotient $\mathrm{Map}(X, S^1)/\mathrm{Map}^{\mathrm{ev}}(X, S^1)$ can be identified with the corresponding quotient $\pi_0(\mathrm{Map}(X, S^1))/\pi_0(\mathrm{Map}^{\mathrm{ev}}(X, S^1))$ of the groups of components. The next proposition shows that the components of $\mathrm{Map}(X, S^1)$ are in one-to-one correspondence with the homomorphism $\pi_1(X) \to \mathbb{Z}$ and the components of $\mathrm{Map}^{\mathrm{ev}}(X, S^1)$ are in one-to-one correspondence with the homomorphism $\pi_1(X) \to 2\mathbb{Z}$. Thus

$$\pi_0(\mathrm{Map}(X, S^1)) \cong H^1(X; \mathbb{Z}), \qquad \pi_0(\mathrm{Map}^{\mathrm{ev}}(X, S^1)) \cong H^1(X; 2\mathbb{Z})$$

and this shows that $\mathrm{Map}(X, S^1)/\mathrm{Map}^{\mathrm{ev}}(X, S^1)$ is naturally isomorphic to $H^1(X; \mathbb{Z})/H^1(X; 2\mathbb{Z})$ as required. Moreover, the proposition shows that every homotopy class of maps $X \to S^1$ has a harmonic representative. Here a map $u : X \to S^1$ is called **harmonic** if

$$d^*(u^{-1}du) = 0.$$

Since $d(u^{-1}du) = 0$ for every smooth map $u : X \to S^1$ this means that $u^{-1}du \in \Omega^1(X, i\mathbb{R})$ is a harmonic 1-form.

Proposition 5.30 *Let X be a compact connected manifold.*
(i) *Every component of the group* $\mathrm{Map}(X, S^1)$ *contains a harmonic representative which is unique up to multiplication by a constant.*
(ii) *The map* $\mathrm{Map}(X, S^1) \to \mathrm{Hom}(\pi_1(X), \mathbb{Z}) : u \mapsto \rho_u$ *given by*

$$\rho_u(\gamma) = \deg(u \circ \gamma) = \frac{1}{2\pi i} \int_\gamma u^{-1} du$$

induces an isomorphism $\pi_0(\mathrm{Map}(X, S^1)) \to \mathrm{Hom}(\pi_1(X); \mathbb{Z})$.
(iii) *A map* $u : X \to S^1$ *is even if and only if* $\rho_u \in \mathrm{Hom}(\pi_1(X), 2\mathbb{Z})$.

Proof: We prove (i). Given $u : X \to S^1$ there exists, by Hodge theory, a smooth function $\xi : X \to i\mathbb{R}$ such that

$$d^*(u^{-1}du + d\xi) = 0.$$

The function

$$u_0(x) = e^{\xi(x)} u(x)$$

is harmonic and is homotopic to u via $u_t(x) = e^{(1-t)\xi(x)}u(x)$. To prove uniqueness, suppose that u and v are homotopic and are both harmonic. Then the 1-form $v^{-1}dv - u^{-1}du$ is harmonic and exact (differentiate along a homotopy connecting u to v). Hence $u^{-1}du = v^{-1}dv$. Fix a point $x_0 \in X$ and choose $\theta \in \mathbb{R}$ such that $v(x_0) = e^{i\theta}u(x_0)$. For any path $t \mapsto x(t)$ with $x(0) = x_0$ the circle valued functions $\alpha(t) = e^{i\theta}u(x(t))$ and $\beta(t) = v(x(t))$ satisfy $\alpha(0) = \beta(0)$ and $\dot\alpha/\alpha = \dot\beta/\beta$. Hence $\alpha(t) = \beta(t)$ for all t. Since X is connected, this proves that $v(x) = e^{i\theta}u(x)$ for all $x \in X$.

We prove (ii). We show first that the map

$$\pi_0(\mathrm{Map}(X, S^1)) \to \mathrm{Hom}(\pi_1(X), \mathbb{Z}) : [u] \mapsto \rho_u$$

is injective. This means that two maps $u, v : X \to S^1$ are homotopic if and only if $\rho_u = \rho_v$. The *only if* statement follows from the homotopy invariance of the degree. Hence assume that $\rho_u = \rho_v$. By (i), we may assume without loss of generality that u and v are harmonic. Thus $v^{-1}dv - u^{-1}du$ is a harmonic 1-form whose integral over every loop is zero. Hence $u^{-1}du = v^{-1}dv$ and, as in the proof of (i), there exists a constant $\theta \in \mathbb{R}$ such that $v(x) = e^{i\theta}u(x)$ for all $x \in X$. Hence u and v are homotopic. This shows that the map $[u] \mapsto \rho_u$ is injective. It remains to show that this map is onto, i.e. that for every homomorphism $\rho : \pi_1(X) \to \mathbb{Z}$ there exists a smooth map $u : X \to S^1$ such that

$$\deg(u \circ \gamma) = \rho(\gamma)$$

for every loop $\gamma : S^1 \to X$. That such a map u exists can be seen by triangulating X and then constructing u as follows. First define $u(x) = 1$

for each vertex x. Then choose a map $\ell : \{\text{edges}\} \to \mathbb{Z}$ such that the oriented sum of the labels over a loop γ agrees with $\rho(\gamma)$. Such a lift exists whenever ρ is a homomorphism. Now extend u over the 1-skeleton such that the degree along each oriented edge e agrees with $\ell(e)$. By construction the degree around the boundary of any 2-simplex is zero and hence u extends over the 2-skeleton. Finally, u extends over X because every smooth map $\partial B^m \to S^1$ extends to B^m for $m \geq 3$. (See the footnote on page 160.)

Here is an alternative argument. By deRham's theorem, choose a closed 1-form $\alpha \in \Omega^1(X)$ such that

$$\int_\gamma \alpha = \rho(\gamma)$$

for every loop $\gamma : S^1 \to X$. Next define $\theta : \widetilde{X} \to \mathbb{R}$ by integrating α along paths, so that $d\theta = \alpha$. Since $\rho(\gamma) \in \mathbb{Z}$ for all γ, the function $u(x) = e^{2\pi i\theta(x)}$ is well defined on X. It satisfies $u^{-1}du = 2\pi i\alpha$ and hence $\rho_u = \rho$. This proves (ii).

To prove (iii) note first that if $u = v^2$ is even then $\rho_u = 2\rho_v \in \mathrm{Hom}(\pi_1(X), 2\mathbb{Z})$. Conversely, if $\rho_u \in \mathrm{Hom}(\pi_1(X), 2\mathbb{Z})$ then the existence of a square root of u follows by a simple lifting argument along paths in X which is left to the reader. □

5.5 Euler and Spin^c structures on three-manifolds

Spin structures

Throughout let Y be a compact oriented smooth 3-manifold. We begin by establishing the existence of spin structures on TY.

Proposition 5.31 *Every compact oriented smooth 3-manifold Y admits a spin structure.*

Proof: Let $\Sigma \subset Y$ be a (not necessarily oriented) 2-dimensional submanifold and denote by ν_Σ the normal bundle. Then, by (5.2),

$$\langle \mathrm{w}_2(TY), [\Sigma]\rangle = \mathrm{w}_2(T\Sigma) + \mathrm{w}_1(T\Sigma)\cdot \mathrm{w}_1(\nu_\Sigma) = \chi_2(\Sigma) + \mathrm{w}_1(T\Sigma)^2 = 0,$$

where χ_2 denotes the mod-2 Euler characteristic. Hence $\mathrm{w}_2(TY) = 0$. □

Lemma 5.32 *Let $V \to Y$ be an oriented real vector bundle over a compact oriented smooth 3-manifold. Suppose that* $\operatorname{rank} V \geq 3$ *and* $\mathrm{w}_2(V) = 0$. *Then V admits a trivialization.*

Proof: Here is a sketch of the argument. Triangulate Y and trivialize V over the 1-skeleton. Use the fact that $\mathrm{w}_2(V) = 0$ and $\operatorname{rank} V \geq 3$ to modify this trivialization in such a way that it extends over the 2-skeleton. Since $\pi_2(\mathrm{SO}(n)) = 0$ the trivialization then extends over the 3-skeleton.

The details are similar to the proofs of Theorem 5.8 and Proposition 5.30 and are left to the reader. □

It follows from Proposition 5.31 and Lemma 5.32 that the tangent bundle of every compact oriented smooth 3-manifold Y admits a trivialization. Suppose that the vector fields $e_1, e_2, e_3 : Y \to TY$ form a global positively oriented orthonormal frame and consider the function $\gamma_e : TY \to \mathbb{C}^{2\times 2}$, defined by

$$\gamma_e(\xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3) = \xi_1 I + \xi_2 J + \xi_3 K, \tag{5.17}$$

where $I, J, K \in \mathfrak{su}(2)$ are given by (4.2). It follows from (4.3) that γ_e is a spin^c structure, and evidently $c_1(L_{\gamma_e}) = 0$. To obtain a spin structure we must specify a family $j : Y \to \mathrm{End}_{\mathbb{R}}(\mathbb{C}^2)$ of orthogonal complex structures on $\mathbb{C}^2$ which anti-commute with i. An example is the automorphism $j_0 : \mathbb{C}^2 \to \mathbb{C}^2$ given by

$$j_0(z_1, z_2) = (-\bar{z}_2, \bar{z}_1).$$

Any other such map has the form

$$j_\lambda(y) = \lambda(y) j_0,$$

where $\lambda : Y \to S^1$. It follows by straightforward computation that j_λ commutes with $\mathfrak{su}(2)$ and hence with γ_e for every e.

Exercise 5.33 **(i)** Prove that every spin structure on Y is isomorphic to one of the form (γ_e, j_λ). If $H^1(Y;\mathbb{Z}) = 0$ prove that every spin structure on Y is isomorphic to one of the form (γ_e, j_0). **Hint:** Use Lemma 5.7 and Exercise 1.43.

(ii) Let $A : Y \to \mathrm{SO}(3)$ be a gauge transformation with matrix entries $a_{ij} : Y \to \mathbb{R}$ and consider the action on frames via

$$(Ae)_i = \sum_i a_{ij} e_j.$$

Prove that (γ_e, j_0) and (γ_{Ae}, j_0) are isomorphic spin structures if and only if A admits a lift $\Phi : Y \to \mathrm{SU}(2)$

(iii) Prove that γ_e and γ_{Ae} are isomorphic spin^c structures if and only if A admits a lift $\Phi : Y \to \mathrm{U}(2)$.

Exercise 5.34 Find a trivialization of TS^3. □

Exercise 5.35 Prove that every smooth map $A : S^3 \to \mathrm{SO}(3)$ lifts to a smooth map $\Phi : S^3 \to \mathrm{SU}(2)$. □

Exercise 5.36 Prove that a smooth map $A : \mathbb{R}P^3 \to \mathrm{SO}(3)$ lifts to a smooth map $\Phi : \mathbb{R}P^3 \to \mathrm{SU}(2)$ if and only if it has even degree. **Hint:** The Borsuk-Ulam theorem asserts that a smooth map $f : S^3 \to S^3$ which satisfies $f(-x) = -f(x)$ has odd degree. □

Exercise 5.37 Let $e = (e_1, e_2, e_3)$ be a global positively oriented orthonormal frame of $T\mathbb{R}P^3$ and suppose that $A : \mathbb{R}P^3 \to \mathrm{SO}(3)$ is a map of odd degree. Prove that γ_e and γ_{Ae} are not isomorphic as spin^c structures on $\mathbb{R}P^3$. **Hint:** By the Borsuk-Ulam theorem, there is no smooth map $\lambda : S^3 \to S^1$ which satisfies $\lambda(-x) = -\lambda(x)$. Use Exercise 5.33 (iii). □

Euler structures

Let $\pi : SY \to Y$ denote the unit sphere bundle in TY and, for $y \in Y$, denote by $\iota_y : S_yY \to SY$ the inclusion of the fibre. The vertical tangent space $T_vS_yY = v^\perp$ carries a natural complex structure

$$\eta \mapsto v \times \eta.$$

Hence the fibres of SY carry a natural orientation as complex manifolds. However, they do not carry a natural complex line bundle representing the generator of $H^2(S_yY;\mathbb{Z})$ over each fiber.

Exercise 5.38 The unit sphere bundle always admits a natural orientation even if Y is not orientable. To see this, regard SY as a submanifold of the cotangent bundle $T^*Y \cong TY$ with its standard symplectic structure. Call a basis $e_1, \dots, e_5$ of $T_{(y,v)}SY$ positively oriented if the vectors $v, e_1, \dots, e_5$ form a positively oriented basis of $T_{(y,v)}TY$. Check that this orientation agrees with the above, when e_1, e_2, e_3 are chosen as a positively oriented horizontal basis and e_4, e_5 as a positively oriented vertical basis. □

An **Euler structure** on Y is a cohomology class $e \in H^2(SY;\mathbb{Z})$ such that the restriction to each fibre S_yY is the canonical generator of $H^2(S_yY;\mathbb{Z})$ (cf. Turaev [123]). Denote by $\mathcal{E}(Y)$ the set of all Euler structures on Y. This space carries a natural involution

$$\mathcal{E}(Y) \to \mathcal{E}(Y) : e \mapsto \tilde{e},$$

given by

$$\tilde{e} = -\tau^* e,$$

where the diffeomorphism $\tau : SY \mapsto SY$ is defined by $\tau(y, v) = (y, -v)$. The space $\mathcal{E}(Y)$ also carries a natural action of $H^2(Y;\mathbb{Z})$ via

$$\mathcal{E}(Y) \times H^2(Y;\mathbb{Z}) \to \mathcal{E}(Y) : (e, a) \mapsto e + \pi^* a.$$

The next lemma shows that this action is transitive and free. Hence there exists a unique function $h : \mathcal{E}(Y) \times \mathcal{E}(Y) \to H^2(Y;\mathbb{Z})$ such that

$$e_1 - e_0 = \pi^* h(\tilde{e}_0, e_1) \tag{5.18}$$

for all $e_0, e_1 \in \mathcal{E}(Y)$.

Lemma 5.39 *The action of $H^2(Y;\mathbb{Z})$ on $\mathcal{E}(Y)$ is transitive and free.*

Proof: Consider the Gysin sequence

$$0 \to H^2(Y;\mathbb{Z}) \xrightarrow{\pi^*} H^2(SY;\mathbb{Z}) \xrightarrow{\pi_*} H^0(Y;\mathbb{Z}) \xrightarrow{\wedge \mathrm{Eul}} H^3(Y;\mathbb{Z}) \to \cdots$$

(cf. Bott-Tu [11, pp.177–179]). The third map is given by integration over the fibre. If $e, e' \in \mathcal{E}(Y)$ then $\iota_y{}^*(e'-e) = 0$ for $y \in Y$ and hence $\pi_*(e'-e) = 0$. By exactness, this implies that $e' - e = \pi^* a$ for some $a \in H^2(Y;\mathbb{Z})$. This shows that the action is transitive. That it is free follows from the fact that $\pi^* : H^2(Y;\mathbb{Z}) \to H^2(SY;\mathbb{Z})$ is injective. □

Exercise 5.40 The Gysin sequence for the product $Y \times S^2$ is en easy exercise in homological algebra. The homology of $Y \times S^2$ is generated by a chain complex of the form

$$D_* = C_* \otimes (\mathbb{Z} \oplus u\mathbb{Z}),$$

where $C_* = C_*(Y)$, $\deg(u) = 2$, and $\partial(\sigma + \tau u) = (\partial\sigma) + (\partial\tau)u$ for $\sigma, \tau \in C_*$. Assume, without loss of generality, that $C_0 = \mathbb{Z}$ and denote by $1 \in C_0$ the generator. The projection induced map $\pi_* : D_* \to C_*$ is given by $\pi_*(\sigma + \tau u) = \sigma$ and the condition $\iota_y{}^*[\varphi] = 0$ for $[\varphi] \in H^2(Y \times S^2) = H^2(D)$ translates into $\varphi(1 \otimes u) = 0$. Use these observations to show that

$$H^i(D) \cong H^i(C) \oplus H^{i-2}(C)$$

and establish the Gysin sequence. □

For every unit vector field $v : Y \to SY$ denote by

$$[v] = v_*[Y] \in H_3(SY;\mathbb{Z})$$

the homology class represented by the image. If $E \to SY$ is a complex line bundle with first Chern class $c_1(E) = \mathrm{PD}([v])$ then its restriction to each fiber S_yY is isomorphic to the canonical bundle and hence $e = \mathrm{PD}([v])$ is an Euler structure. The dual structure is given by

$$\tilde{e} = \mathrm{PD}([-v]).$$

Proposition 5.41 below shows that every Euler structure can be expressed in this form. This gives rise to a more geometric definition of Euler structures on Y. More precisely, two unit vector fields $v_0, v_1 : Y \to SY$ are called **homologous** if $[v_0] = [v_1] \in H_3(SY;\mathbb{Z})$. An Euler structure on Y can now be defined as an equivalence class of homologous unit vector fields. Before proving that this agrees with the original definition we shall examine the function $h : \mathcal{E}(Y) \times \mathcal{E}(Y) \to H^2(Y;\mathbb{Z})$ in terms of unit vector fields.

Two unit vector fields $v_0, v_1 : Y \to SY$ are called **transverse** if their images $v_0(Y)$ and $v_1(Y)$ are transverse as submanifolds of SY. In this case the set

$$C(v_0, v_1) = \{y \in Y \mid v_0(y) = v_1(y)\}$$

is a 1-dimensional submanifold of Y, called the **Pontryagin manifold** of the pair (v_0, v_1). There are two real rank-2 bundles over C, namely the normal bundle $N_C \to C$ and the vertical tangent bundle $V_C \to C$ of SY:

$$N_C = \{(y, v) \mid y \in C,\, v \in T_yY,\, v \perp T_yC\},$$

$$V_C = \{(y, \eta) \mid y \in C,\, \eta \in T_yY,\, \eta \perp v_0(y)\}.$$

The vertical differential $D(v_1 - v_0)(y) : T_yY \to T_yY$ (the ordinary differential followed by projection onto the vertical subspace of TSY) determines, in the transverse case, an isomorphism $\rho(v_0, v_1) : N_C \to V_C$, hence an orientation of N_C, and hence an orientation of C. We define

$$h(v_0, v_1) := \mathrm{PD}([C(v_0, v_1)]) \in H^2(Y; \mathbb{Z}) \tag{5.19}$$

for two transverse unit vector fields $v_0, v_1 : Y \to SY$. One proves as in [90] that the right hand side depends only on the homotopy classes of v_0 and v_1 and hence is well defined for all pairs of unit vector fields, transverse or not. Note that

$$h(v_0, v_1) = h(v_1, v_0) = -h(-v_0, -v_1) \tag{5.20}$$

for any two unit vector fields $v_0, v_1 : Y \to SY$. The crucial point is that the orientation of the vertical bundle of $C(v_0, v_1)$ is determined by the vector product with $v_0(y) = v_1(y)$ for $y \in Y$.

Proposition 5.41. (Turaev) *Let Y be a compact oriented smooth 3-manifold.*

(i) *For any two unit vector fields $v_0, v_1 : Y \to SY$,*

$$\mathrm{PD}([v_1]) - \mathrm{PD}([v_0]) = \pi^* h(-v_0, v_1).$$

(ii) *For any three unit vector fields $v, v_0, v_1 : Y \to SY$,*

$$h(-v_0, v_1) = h(v, v_1) - h(v, v_0).$$

(iii) *For every Euler structure $e \in \mathcal{E}(Y)$ there exists a unit vector field $v : Y \to SY$ such that $e = \mathrm{PD}([v])$.*

(iv) *Two unit vector fields $v_0, v_1 : Y \to SY$ are homologous if and only if they are homotopic over $Y - \mathrm{int}(B)$ for any embedded closed 3-ball $B \subset Y$.*

Proof: To prove (i) it suffices to assume that $-v_0$ and v_1 are transverse. Choose a trivialization of TY and let $f : Y \times S^2 \to SY$ be the corresponding trivialization of the unit sphere bundle. Let us define $u_0 : Y \to S^2$ and $u_1 : Y \to S^2$ by

$$v_0(y) = f(y, u_0(y)), \qquad v_1(y) = f(y, u_1(y)).$$

Fix an arbitrarily small tubular neighbourhood N_C of C. The Pontryagin-Thom construction shows that the homotopy class of u_1 is uniquely determined by the framed Pontryagin manifold $C = C(-u_0, u_1) = C(-v_0, v_1)$. Hence, without changing the submanifold C (nor the function u_0 nor the homotopy class of u_1), we may assume that $u_1(y) = u_0(y)$ for $y \notin N_C$, $u_1(y) = -u_0(y)$ for $y \in C$, and that u_1 maps each normal slice in N_C onto S^2 with degree 1. We claim that under these assumptions

$$[\mathrm{graph}\,(u_1)] = [\mathrm{graph}\,(u_0)] + [C \times S^2]. \tag{5.21}$$

To see this, triangulate $Y \times S^2$ in such a way that the 1-simplices are disjoint from graph (u_0) and $N_C \times S^2$. Then the intersection numbers of all three submanifolds graph (u_1), graph (u_0), and $C \times S^2$, with 2-simplices are well defined, and hence they determine integral simpicial cocycles. With this construction one checks easily that (5.21) holds on the chain level. With (5.21) established, it follows that

$$\mathrm{PD}([\mathrm{graph}\,(u_1)]) = \mathrm{PD}([\mathrm{graph}\,(u_0)]) + \mathrm{PD}([C \times S^2]).$$

Hence the vector fields $v_i = f \circ \mathrm{graph}\,(u_i) : Y \to SY$ satisfy

$$\mathrm{PD}([v_1]) = \mathrm{PD}([v_0]) + \pi^*\mathrm{PD}([C]).$$

This proves (i).

Assertion (ii) follows immediately from (i). (iii) follows from the fact that for every cohomology class $a \in H^2(Y; \mathbb{Z})$ and every vector field $v_0 : Y \to SY$ there exists a vector field $v_1 : Y \to SY$ which is transverse to $-v_0$ and satisfies $a = h(-v_0, v_1)$. In fact one can prescribe the Pontryagin manifold $C(-v_0, v_1)$ (see Milnor [90]).

We prove (iv). As before, let $f : Y \times S^2 \to SY$ be a trivialization of the sphere bundle and choose $u_0, u_1 : Y \to S^2$ such that $v_i(y) = f(y, u_i(y))$ for $i = 1, 2$. Let $x_0 \in S^2$ be a common regular value of u_0 and u_1, and denote by $C_i = {u_i}^{-1}(x_0)$ the framed Pontryagin manifolds. The framings are trivializations of the normal bundle determined by a fixed framing of $T_{x_0}S^2$. Then u_0 and u_1 are homotopic if and only if C_0 and C_1 are framed cobordant (see Milnor [90]). By (i) and (ii), they are homologous if and only if C_0 and C_1 are homologous.

Assume first that C_0 and C_1 are homologous. Fix a connected oriented 1-dimensional submanifold $C \subset Y$ which is homologous to C_0 (and hence to C_1). Then, with two suitable framings, this submanifold is framed cobordant to C_0 and C_1, respectively. Hence u_0 and u_1 are homotopic to smooth maps $u_0', u_1' : Y \to S^2$ which both have C as their Pontryagin manifolds, but with possibly different framings. Moreover, we may assume without loss of generality that C passes through the interior of our ball B and that the two framings, and the functions u_0' and u_1' themselves, agree outside B. Removing this ball, we obtain a homotopy from u_0 to u_1.

Conversely, suppose that there exists an embedded 3-ball $B \subset Y$ and a homotopy $(Y - \mathrm{int}(B)) \times [0,1] : (y, \lambda) \to u_\lambda(y)$ from u_0 to u_1. Choose a common regular value $x_0 \in S^2$ of u_0, u_1, the homotopy $(Y - \mathrm{int}(B)) \times [0,1] \to S^2$, and of its restriction to the boundary $\partial B \times [0,1] \to S^2$. Examining the preimage of x_0 we see that $C_0 = {u_0}^{-1}(x_0)$ and $C_1 = {u_1}^{-1}(x_0)$ are homologus as relative classes in $H_1(Y, B; \mathbb{Z})$. Collapsing B to a point we see that they are homologous in $H_1(Y; \mathbb{Z})$. This proves the proposition. □

Exercise 5.42 Let Σ be a compact oriented Riemann surface and $u = (u_0, u_1) : \Sigma \to S^2 \times S^2$ be a smooth function which is transverse to the diagonal $\Delta \subset S^2 \times S^2$. Prove that

$$u \cdot \Delta = \deg(u_0) + \deg(u_1).$$

Hint: Δ is homologous to $\{\mathrm{pt}\} \times S^2 \cup S^2 \times \{\mathrm{pt}\}$. □

Exercise 5.43 Let Σ be a compact oriented Riemann surface. Prove that, for any two smooth functions $u : \Sigma \to S^2$ and $A : \Sigma \to \mathrm{SO}(3)$,

$$\deg(Au) = \deg(u).$$

Hint: Prove this first in the case $u(z) \equiv \mathrm{const}$, using $H_2(\mathrm{SO}(3); \mathbb{Z}) = 0$. Reduce the general case to the case $\deg(u) = 0$ by using Exercise 5.42. □

Exercise 5.44 Let $f : Y \times S^2 \to SY$ be a trivialization of the unit sphere bundle and $\Sigma \subset Y$ be a compact oriented embedded 2-manifold. For every unit vector field $v : Y \to SY$ define $f^*v : Y \to S^2$ by $f(y, f^*v(y)) = v(y)$ for $y \in Y$. Prove that the integer

$$\deg\left(v|_\Sigma\right) := \deg\left(f^*v|_\Sigma\right) = \langle [v], f_*[\Sigma \times \{\mathrm{pt}\}] \rangle$$

is independent of the choice of f. **Hint:** Use Exercise 5.43. □

Exercise 5.45 Let $v_0, v_1 : Y \to SY$ be unit vector fields and $\Sigma \subset Y$ be a compact oriented embedded 2-manifold. Prove that

$$\langle h(v_0, v_1), [\Sigma] \rangle = \deg\left(v_0|_\Sigma\right) + \deg\left(v_1|_\Sigma\right).$$

Hint: Use Exercises 5.42 and 5.44. □

Spin^c structures

For every unit vector field $v : Y \to SY$ denote by $L_v \to Y$ the complex line bundle

$$L_v = v^\perp = \{(y, \eta) \,|\, y \in Y,\, \eta \in T_yY,\, \eta \perp v(y)\}\,.$$

This is the pullback of the vertical tangent bundle of SY under the map $v : Y \to SY$. The decomposition $TY \cong \mathbb{R}v \oplus L_v$ shows that $\mathrm{w}_2(L_v) = 0$ and so $c_1(L_v)$ is an integral lift of $\mathrm{w}_2(TY) = 0$. Hence there exists a spin^c structure $\gamma : TY \to \mathrm{End}(W)$ with $\det(W) \cong L_v$. An explicit spin^c structure with this property is the map $\gamma_v : TY \to \mathrm{End}(\mathbb{C} \oplus L_v)$, given by

$$\gamma_v(\eta)\begin{pmatrix} z \\ \zeta \end{pmatrix} = \begin{pmatrix} -\,i\langle \eta, v\rangle z - \langle \eta, \zeta\rangle - i\langle v \times \eta, \zeta\rangle \\ \langle \eta, v\rangle v \times \zeta + x(\eta - \langle \eta, v\rangle v) + yv \times \eta \end{pmatrix} \tag{5.22}$$

for $z \in \mathbb{C}$, $\zeta \in v^\perp$, and $\eta \in TY$. Note that (5.22) is isomorphic to (5.17) whenever $v = e_1$ (and hence e_2 and e_3 trivialize L_v).

Exercise 5.46 Prove that (5.22) is a spin^c structure. Prove that $c_1(L_{v_0}) = c_1(L_{v_1})$ if and only if there exists a gauge transformation $A : Y \to \mathrm{SO}(TY)$ such that $Av_0 = v_1$. Prove that a complex line bundle $L \to Y$ is isomorphic to L_v for some v if and only if $\mathrm{w}_2(L) = 0$. **Hint:** Trivialize $L \oplus \mathbb{R}$. □

Proposition 5.47. (Turaev) *Let Y be a compact oriented smooth 3-manifold.*

(i) *If $v : Y \to SY$ is a unit vector field then $c_1(L_v) = h(v, v)$.*

(ii) *Every spin^c structure on Y is isomorphic to one of the form γ_v for some unit vector field $v : Y \to SY$.*

(iii) *Two unit vector fields $v_0, v_1 : Y \to SY$ are homologous if and only if the spin^c structures γ_{v_0} and γ_{v_1} are isomorphic.*

Proof: Choose a vector field $w : Y \to SY$ which is close to v and transverse to v. Consider the section $s : Y \to L_v$ given by

$$s(y) = w(y) - \langle w(y), v(y)\rangle v(y)$$

for $y \in Y$. Since w is close to v we have $w(y) \neq -v(y)$ for all $y \in Y$. Hence $s(y) = 0$ if and only if $w(y) = v(y)$, and so the zero set of s is given by

$$s^{-1}(0) = C(v, w).$$

Moreover, for every $y \in C(v, w)$,

$$Ds(y) = D(w - v)(y) : T_yY \to v(y)^\perp.$$

Hence s is transverse to the zero section and the orientation of C as the zero set of s agrees with its orientation as a Pontryagin manifold of the pair (v, w). This proves (i).

Now suppose that v_0 and v_1 are any two unit vector fields and consider the line bundle $E \to Y$ whose fiber over $y \in Y$ consists of all spin^c isomorphisms $\Phi : \mathbb{C} \oplus v_0(y)^\perp \to \mathbb{C} \oplus v_1(y)^\perp$. Thus

$$E = \left\{(y, \Phi) \mid \Phi \in \mathrm{Hom}^c(\mathbb{C} \oplus v_0(y)^\perp, \mathbb{C} \oplus v_1(y)^\perp),\ \Phi\gamma_{v_0} = \gamma_{v_1}\Phi\right\}$$

We shall prove that the first Chern class of this line bundle is given by

$$c_1(E) = h(-v_0, v_1). \tag{5.23}$$

The proof of Theorem 5.8 shows that $\gamma_{v_0} \otimes E \cong \gamma_{v_1}$. Hence it follows from (5.23) that γ_{v_0} and γ_{v_1} are isomorphic if and only if $h(-v_0, v_1) = 0$ and this means that v_0 and v_1 are homologous. That every spin^c structure is isomorphic to one of the form γ_v follows immediately from Theorem 5.8.

It remains to establish (5.23). We assume that $-v_0$ is transverse to v_1. A general complex linear homomorphism

$$\mathbb{C} \oplus {v_0}^\perp \to \mathbb{C} \oplus {v_1}^\perp : (z_0, \zeta_0) \mapsto (z_1, \zeta_1) = \Phi(z_0, \zeta_0),$$

has the form

$$\begin{aligned} z_1 &= (a + ib)z_0 + \langle w_0, \zeta_0\rangle + i\langle v_0 \times w_0, \zeta_0\rangle, \\ \zeta_1 &= x_0 w_1 + y_0 v_1 \times w_1 + \varphi(\zeta_0), \end{aligned} \tag{5.24}$$

where $a + ib \in \mathbb{C}$, $w_0 \in {v_0}^\perp$, $w_1 \in {v_1}^\perp$, and $\varphi \in \mathrm{Hom}^c({v_0}^\perp, {v_1}^\perp)$. Suppose that $v_0 \neq v_1$. Then one checks by direct calculation that $\Phi\gamma_{v_0} = \gamma_{v_1}\Phi$ if and only if

$$a = -\frac{\langle v_0 \times v_1, w_1\rangle}{1 - \langle v_0, v_1\rangle}, \quad b = -\frac{\langle v_0, w_1\rangle}{1 - \langle v_0, v_1\rangle}, \quad w_0 = -w_1 + b(v_1 - v_0),$$

$$\varphi(\zeta_0) = \frac{\langle v_1, v_0 \times \zeta_0\rangle}{1 - \langle v_0, v_1\rangle} w_1 + \frac{\langle v_1, \zeta_0\rangle}{1 - \langle v_0, v_1\rangle} v_1 \times w_1.$$

Hence there is an isomorphism $L_{v_1} \to E : (y, w_1) \mapsto (y, \Phi)$ over $Y - C(v_0, v_1)$. In particular, this isomorphism is defined over a neighbourhood of $C(-v_0, v_1)$. Now there is a smooth section

$$s : Y \to E$$

where $s(y) = \Phi_y \in E_y$ is the map (5.24) determined by

$$w_1(y) = v_1(y) \times v_0(y).$$

If $v_0(y) \neq v_1(y)$ then this determines $a + ib$, w_0, and φ uniquely. Moreover, one checks easily that s extends to a smooth section over all of Y. Namely,

$$a(y) = 1 + \langle v_0, v_1 \rangle, \qquad b(y) = 0, \qquad w_0(y) = v_0 \times v_1$$

obviously extend, and for

$$\varphi_y(\zeta_0) = \frac{\langle v_0 \times v_1, \zeta_0 \rangle}{1 - \langle v_0, v_1 \rangle} v_0 \times v_1 + \frac{\langle v_1, \zeta_0 \rangle}{1 - \langle v_0, v_1 \rangle} (v_0 - \langle v_0, v_1 \rangle v_1)$$

this is an easy exercise. One obtains $\varphi_y(\zeta_0) = 2\zeta_0$ whenever $v_0(y) = v_1(y)$. Evidently, $s(y) = 0$ if and only if $y \in C(-v_0, v_1)$. We must prove that s is transverse to the zero section and that the natural orientation of $C(-v_0, v_1)$ agrees with the one induced by s. The isomorphism between E and $L_{v_1} = {v_1}^\perp$ near $C(-v_0, v_1)$ shows that it suffices to examine $C(-v_0, v_1)$ as a zero set of the section

$$Y \to L_{v_1} : y \mapsto v_1(y) \times v_0(y).$$

Multiplication by $-\sqrt{-1}$ does not change orientations and hence we may consider instead the section

$$Y \to L_{v_1} : y \mapsto v_0(y) - \langle v_0(y), v_1(y) \rangle v_1(y).$$

The vertical differential of this section at $y \in C(-v_0, v_1)$ is given by $D(v_0 + v_1)(y) : T_yY \to v_1(y)^\perp$. This determines the correct orientation of $C(-v_0, v_1)$. This proves (5.23) and the proposition. □

Both Propositions 5.41 and 5.47 are due to Turaev [123]. They show that there is a natural one-to-one correspondence between isomorphism classes of Euler structures and spin^c structure, given by

$$\mathcal{E}(Y) \to \mathcal{S}^c(Y) : \mathrm{PD}([v]) \mapsto [\gamma_v].$$

In the case $b_1(Y) \geq 1$ Turaev defines a function

$$\mathcal{T} : \mathcal{E}(Y) \to \mathbb{Z}$$

which assigns to each Euler structure on Y a torsion invariant, called the **Turaev-Milnor torsion**. This is a kind of refinement of the Reidemeister torsion. On the other hand the Seiberg-Witten invariants have the form of a function

$$\mathrm{SW} : \mathcal{S}^c(Y) \to \mathbb{Z},$$

also defined in the case $b_1(Y) \geq 1$. In [123] Turaev conjectures that these two invariants agree. This is a refinement of a result by Meng–Taubes [87]. A special case of this (for mapping tori) is proved in [108].

6

DIRAC OPERATORS

This chapter is devoted to the study of the Dirac operator and their fundamental properties. Section 6.1 gives an introduction to spin^c connections and Section 6.2 to Dirac operators. Section 6.3 deals with Dirac operators on symplectic manifolds with their canonical spin^c structure (associated to a compatible almost complex structure) and, in particular, it is proved that the Dirac operator agrees with the Cauchy-Riemann operator $\bar{\partial} + \bar{\partial}^*$. The Weitzenböck formula is proved in Section 6.4 and the Fredholm index is discussed (without proof) in Section 6.5. The chapter closes with two applications to 4-manifold topology, namely Rohlin's theorem and the theorem of Lichnerowicz which asserts that spin 4-manifolds with positive scalar curvature have zero signature. This result was one of the starting points for the work of Gromov and Lawson on positive scalar curvature manifolds and, as a vanishing theorem involving the Dirac operator, it can be viewed as a kind of prelude to the Seiberg-Witten invariants.

6.1 Spin^c connections

Let X be a compact Riemannian manifold of dimension $2n$ or $2n+1$ and suppose that $\Gamma : TX \to \mathrm{End}(W)$ is a spin^c structure. Recall that, in the even dimensional case, there is a canonical splitting $W = W^+ \oplus W^-$. A Hermitian connection ∇ on W is called a **spin$^{\mathbf{c}}$ connection** if there exists a connection on TX, also denoted by ∇, such that

$$\nabla_v(\Gamma(w)\Phi) = \Gamma(w)\nabla_v\Phi + \Gamma(\nabla_v w)\Phi \tag{6.1}$$

for $\Phi \in C^\infty(X, W)$ and $v, w \in \mathrm{Vect}(X)$. Note that the connection on TX is uniquely determined by the spin^c connection on W but not vice versa. It is left to the reader to prove that the induced connection on TX is necessarily Riemannian. However, it need not be torsion free.

Lemma 6.1 *Let ∇^1 and ∇^2 be two spin^c connections on W. Then there exists a 1-form $\alpha \in \Omega^1(X, C_2(TX) \oplus i\mathbb{R})$ such that*

$$\nabla_v^2\Phi - \nabla_v^1\Phi = \Gamma(\alpha(v))\Phi$$

for $\Phi \in C^\infty(X, W)$ and $v \in \mathrm{Vect}(X)$. Conversely, if ∇ is a spin^c connection on W and $\alpha \in \Omega^1(X, C_2(X) \oplus i\mathbb{R})$ then $\nabla + \Gamma(\alpha)$ is also a spin^c connection.

Proof: The map $\nabla^2 - \nabla^1 : C^\infty(X,W) \to \Omega^1(X,W)$ is linear over the functions and is therefore given by an endomorphism valued 1-form $A \in \Omega^1(X,\mathrm{End}(W))$. Since ∇^1 and ∇^2 are both spin^c connections there exists a homomorphism $a \in C^\infty(X,\mathrm{End}(TX))$ which represents the difference of the induced connections on TX. Taking the difference of the formulae (6.1) for ∇^1 and ∇^2 one obtains

$$A(v)\Gamma(w) - \Gamma(w)A(v) = \Gamma(a(v)w)$$

for $v,w \in \mathrm{Vect}(X)$. This implies $A(v) \in \Gamma(C_2(V) \oplus i\mathbb{R})$. Thus there is a unique 1-form $\alpha \in \Omega^1(X, C_2(TX) \oplus i\mathbb{R})$ such that $[\alpha(v), w] = a(v)w$ and $A(v) = \Gamma(\alpha(v))$ for $v,w \in \mathrm{Vect}(X)$. This proves the lemma. □

Lemma 6.2 *Assume* $\dim X = 2n$.

(i) *Every spin^c connection ∇ on W preserves the subbundles W^+ and W^-.*

(ii) *If X carries an almost complex structure J then ∇ preserves the subbundle $E_{J,\Gamma}$ if and only if the induced connection on TX satisfies $\nabla J = 0$.*

Proof: To prove (i) fix a path $\beta : \mathbb{R} \to X$ and choose parallel vector fields $v_1, \dots, v_{2n}$ along β which form a positively oriented orthonormal frame. Then $\Phi \in C^\infty(X,W^+)$ satisfies $\Gamma(v_{2n}) \cdots \Gamma(v_1)\Phi = i^n\Phi$ and hence

$$\Gamma(v_{2n}) \cdots \Gamma(v_1)\nabla_{\dot\beta}\Phi = \nabla_{\dot\beta}\left(\Gamma(v_{2n}) \cdots \Gamma(v_1)\Phi\right) = i^n \nabla_{\dot\beta}\Phi$$

and hence $\nabla\Phi \in \Omega^1(X,W^+)$. Similarly for W^-. This proves the first assertion. To prove (ii) assume that $\Phi \in C^\infty(X,E_{J,\Gamma})$ and hence $\Gamma(Jv)\Phi = i\Gamma(v)\Phi$ for every vector field $v : X \to TX$. Then

$$\begin{aligned} 0 &= \nabla_v\left(\Gamma(Jw)\Phi - i\Gamma(w)\Phi\right) \\ &= \Gamma(J(\nabla_v w))\Phi - i\Gamma(\nabla_v w)\Phi \\ &\quad + \Gamma(Jw)\nabla_v\Phi - i\Gamma(w)\nabla_v\Phi + \Gamma((\nabla_v J)w)\Phi \\ &= \Gamma(Jw)\nabla_v\Phi - i\Gamma(w)\nabla_v\Phi + \Gamma((\nabla_v J)w)\Phi. \end{aligned}$$

Hence $\nabla_v\Phi \in C^\infty(X,E_{J,\Gamma})$ if and only if $\Gamma((\nabla_v J)w)\Phi = 0$ for all $w \in \mathrm{Vect}(X)$. This shows that ∇ preserves $E_{J,\Gamma}$ if and only if $\nabla J = 0$. □

A spin^c connection ∇ on W is said to be **compatible** with the Levi-Civita connection if it satisfies the formula (6.1) with $\nabla_v w$ denoting the Levi-Civita connection on X. Any two such connections ∇^1, ∇^2 differ by an imaginary valued 1-form $a \in \Omega^1(X, i\mathbb{R})$. Moreover, the group $\mathrm{Map}(X,S^1)$ acts on the space of connections by

$$(u^*\nabla)\Phi = u^{-1}\nabla(u\Phi) = u^{-1}du \otimes \Phi + \nabla\Phi$$

for $\Phi \in C^\infty(X,W)$ and $u \in \mathrm{Map}(X,S^1)$.

Curvature

Recall that the curvature tensor of a spin^c connection is an endomorphism valued 2-form $F^\nabla \in \Omega^2(X, \mathrm{End}(W))$ defined by

$$F^\nabla(v,w)\Phi = \nabla_v \nabla_w \Phi - \nabla_w \nabla_v \Phi + \nabla_{[v,w]}\Phi$$

for $v, w \in \mathrm{Vect}(X)$ and $\Phi \in C^\infty(X, W)$. Since ∇ is compatible with the Levi-Civita connection the traceless part of F^∇ is given by the Riemannian curvature tensor of X. This means that

$$F^\nabla(v,w) - \frac{1}{2^n}\mathrm{trace}(F^\nabla(v,w)) = \rho(R(v,w)) \tag{6.2}$$

where the homomorphism $\rho : \mathfrak{so}(TX) \to \mathrm{End}(W)$ is defined by the formula $\rho \circ \mathrm{Ad} = \Gamma : C_2(V) \to \mathrm{End}(W)$. This means that the map ρ makes the following diagram commute

$$\begin{tikzcd} \mathfrak{so}(TX) \arrow[r, "\rho"] & \mathrm{End}(S) \\ C_2(TX) \arrow[u, "Ad"] \arrow[ur, "\Gamma"'] & \end{tikzcd}$$

Here the map $\mathrm{Ad} : C_2(TX) \to \mathfrak{so}(TX)$ is defined by (4.12). The trace of the curvature (times 2^{1-n}) is the curvature of the induced connection on the line bundle L_Γ. To see this it is convenient to reformulate spin^c connections in terms of principal bundles.

Spin^c connections on principal bundles

As in Remark 5.5 denote by $P = P_\Gamma \to X$ the principal frame bundle of W, based on some model structure $\Gamma_0 : V_0 \to \mathrm{End}(W_0)$. It has structure group $\mathrm{Spin}^c(V_0)$, and there are natural isomorphisms

$$W \cong P \times_{\Gamma_0} W_0, \qquad TX \cong P \times_{\mathrm{ad}} V_0, \qquad L_\Gamma \cong P \times_\delta \mathbb{C}.$$

In particular, a section $\Phi : X \to W$ can be identified with an equivariant map $\Phi_0 : P_\Gamma \to W_0$ via $\Phi(x) = p\Phi_0(p)$ for $p \in P_x = \mathrm{Hom}^{\mathrm{spin}^c}(W_0, W_x)$. Abbreviate

$$\mathrm{G} = \Gamma_0(\mathrm{Spin}^c(V_0)) \subset \mathrm{Aut}(W_0),$$
$$\mathfrak{g} = \mathrm{Lie}(\mathrm{G}) = \Gamma_0(C_2(V_0) \oplus i\mathbb{R}) \subset \mathrm{End}(W_0).$$

With this terminology in place a **spin$^\mathbf{c}$ connection** on P can be defined as in Chapter 1 as a Lie algebra valued connection 1-form $\widehat{A} \in \Omega^1(P, \mathfrak{g})$ which is equivariant and canonical in the vertical directions. Every such connection induces covariant derivative operators on the associated bundles W, TX, and L_Γ. Moreover, by Exercise 1.18, there is a one-to-one correspondence between spin^c connections on W and connection 1-forms on P.

Exercise 6.3 The curvature 2-form of the spin^c connection $\widehat{A} \in \mathcal{A}(P) \subset \Omega^1(P, \mathfrak{g})$ is the Lie algebra valued 2-form

$$F_{\widehat{A}} = d\widehat{A} + \frac{1}{2}[\widehat{A} \wedge \widehat{A}] \in \Omega^2(X, \mathfrak{g}_P)$$

as in Chapter 1. This 2-form can be identified with an endomorphism valued 2-form via the spin^c isomorphism $p \in P_x = \mathrm{Hom}^{\mathrm{spin}^c}(W_0, W_x)$. In fact, given $v, w \in T_xX$ choose $p \in P_x$ and vectors $\hat{v}, \hat{w} \in T_pP$ which descend to v, w. Prove that the endomorphism $p \circ F_{\widehat{A}}(\hat{v}, \hat{w}) \circ p^{-1} : W_x \to W_x$ is independent of the choices and agrees with $F^{\nabla}(v, w)$, where ∇ is the covariant derivative operator on W induced by $\widehat{A}$. □

There is a splitting of the Lie algebra

$$\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathbb{R}$$

with $\mathfrak{g}_0 = \Gamma_0(C_2(V_0))$. Correspondingly every spin^c connection $\widehat{A} \in \mathcal{A}(P) \subset \Omega^1(P, \mathfrak{g})$ decomposes into the traceless part $\widehat{A}_0 \in \Omega^1(P, \mathfrak{g}_0)$ and the trace:

$$\widehat{A} = \widehat{A}_0 + \frac{1}{2^n}\mathrm{trace}(\widehat{A}).$$

Since $\mathfrak{g}_0$ is isomorphic to $\mathfrak{so}(V_0)$ via $\mathfrak{g}_0 \to \mathfrak{so}(V_0) : \Gamma_0(\xi) \mapsto \mathrm{Ad}(\xi)$ it follows that $\widehat{A}_0$ induces a connection on $TX \cong P \times_{\mathrm{ad}} V_0$. The reader may check that the corresponding covariant derivative operator on TX is precisely the one that appears in (6.1).

Throughout denote the trace of $\widehat{A}$ by

$$A = \frac{1}{2^n}\mathrm{trace}(\widehat{A}).$$

This is an imaginary valued 1-form $A \in \Omega^1(P, i\mathbb{R})$ which satisfies

$$A_{pg}(vg) = A_p(v), \qquad A_p(p \cdot \xi) = \frac{1}{2^n}\mathrm{trace}(\xi). \tag{6.3}$$

for $v \in T_pP$, $g \in \mathrm{G}$, and $\xi \in \mathfrak{g} \subset \mathrm{End}(W_0)$. Denote

$$\mathcal{A}(\Gamma) = \left\{A \in \Omega^1(P, i\mathbb{R}) \,|\, A \text{ satisfies } (6.3)\right\}.$$

A spin^c connection $\widehat{A} \in \mathcal{A}(P)$ is uniquely determined by the induced connection on TX and the 1-form $A = 2^{-n}\mathrm{trace}(\widehat{A}) \in \mathcal{A}(\Gamma)$ via $\widehat{A} = \widehat{A}_0 + A\mathbb{1}$. Hence there is a one-to-one correspondence of 1-forms $A \in \mathcal{A}(\Gamma)$ with spin^c connections on W (in the sense of covariant derivative operators) which are compatible with the Levi-Civita connection. For every $A \in \mathcal{A}(\Gamma)$ denote the associated covariant derivative operator by

$$\nabla_A : C^\infty(X, W) \to \Omega^1(X, W).$$

One must be careful to distinguish the curvature of $\nabla = \nabla_A$, which is an endomorphism valued 2-form $F^\nabla \in \Omega^2(X, \mathrm{End}(W))$, from the curvature of the corresponding 1-form A, which is a scalar 2-form $F_A \in \Omega^2(X, i\mathbb{R})$. They are related by

$$F_A(v, w) = \frac{1}{2^n}\mathrm{trace}(F^\nabla(v, w))$$

for $v, w \in T_xX$.

Remark 6.4 The space $\mathcal{A}(\Gamma)$ is an affine space with parallel vector space $\Omega^1(X, i\mathbb{R})$. If $A \in \mathcal{A}(\Gamma)$ and $a \in \Omega^1(X, i\mathbb{R})$ then

$$F_{A+a} = F_A + da.$$

Moreover, the covariant derivative operator $\nabla_A : C^\infty(X, W) \to \Omega^1(X, W)$ is uniquely characterized by the compatibility condition with the Levi-Civita connection and the fact that the induced connection on the line bundle $\det(W) = L_\Gamma{}^{\otimes 2^{n-1}}$ is given by $2^n A$.

Remark 6.5 Note that $\delta(e^{i\theta}\mathbb{1}) = e^{2i\theta}$ and so $\dot{\delta}(i\theta\mathbb{1}) = 2i\theta$. Hence it follows from Exercise 1.19 that for every $A \in \mathcal{A}(\Gamma)$ the 1-form $2A \in \Omega^1(P, i\mathbb{R})$ represents a connection on the line bundle L_Γ. This shows that the first Chern class of L_Γ is represented by the 2-form F_A via

$$c_1(L_\Gamma) = \left[\frac{i}{\pi}F_A\right].$$

If the bundle L_Γ admits a square root $\mathcal{L} \to X$ with

$$\mathcal{L} \otimes \mathcal{L} = L_\Gamma$$

then the 1-form A can be interpreted as a connection on $\mathcal{L}$. However, such a square root only exists if X admits a spin structure. In general, the 1-form A may not be a connection on any bundle over X and thus the 2-form $iF_A/2\pi$ may not represent an integral cohomology class. However, for notational purposes it will be more convenient to work with 1-forms A which satisfy (6.3) rather than the actual corresponding connection $2A$ on L_Γ. In the following A is sometimes called a **virtual connection** on the **virtual line bundle** $L_\Gamma{}^{1/2}$. □

Gauge transformations

The group $\mathcal{G} = \mathrm{Map}(X, S^1)$ acts on the space $\mathcal{A}(\Gamma)$ by

$$u^*A = u^{-1}du + A$$

for $A \in \mathcal{A}(\Gamma)$ and $u \in \mathcal{G}$. One can think of $\mathcal{G}$ as a subgroup of the automorphism group of the bundle W. The action of $\mathcal{G}$ on the space of covariant derivative operators on W corresponds to conjugation with this automorphism. Hence

$$\nabla_{u^*A}(u^{-1}\Phi) = u^{-1}\nabla_A \Phi$$

and

$$D_{u^*A}(u^{-1}\Phi) = u^{-1}D_A\Phi \tag{6.4}$$

for $A \in \mathcal{A}(\Gamma)$, $\Phi \in C^\infty(X, W)$ and $u \in \mathcal{G}$. Moreover, the 1-form $u^{-1}du$ is always closed and hence

$$F_{u^*A} = F_A. \tag{6.5}$$

It will follow from (6.4) and (6.5) that, in dimensions three and four, the space of solutions of the Seiberg-Witten equations is invariant under the action of the gauge group on $\mathcal{A}(\Gamma) \times C^\infty(X, W)$ via $(A, \Phi) \mapsto (u^*A, u^{-1}\Phi)$.

Components of the gauge group

Recall from the discussion on page 176 that a map $u : X \to S^1$ is called harmonic if the 1-form $u^{-1}du \in \Omega^1(X, i\mathbb{R})$ satisfies $d^*(u^{-1}du) = 0$. Denote by

$$\mathcal{G}_0 = \left\{ u : X \to S^1 \mid d^*(u^{-1}du) = 0 \right\}$$

the subgroup of harmonic gauge transformations. Proposition 5.30 shows that the inclusion $\mathcal{G}_0 \hookrightarrow \mathcal{G}$ induces an isomorphism of π_0. It also shows that there is a split exact sequence

$$1 \longrightarrow S^1 \longrightarrow \mathcal{G}_0 \longrightarrow H^1(X; \mathbb{Z}) \longrightarrow 0$$

where the third map is given by $u \mapsto (2\pi i)^{-1}[u^{-1}du]$. Given a basepoint $x_0 \in X$ there is a homomorphism

$$H^1(X; \mathbb{Z}) \cong \mathrm{Hom}(\pi_1(X), \mathbb{Z}) \to \mathcal{G}_0$$

which assigns to $\rho \in \mathrm{Hom}(\pi_1(X), \mathbb{Z})$ the unique harmonic gauge transformation $u \in \mathcal{G}_0$ with $u(x_0) = 1$ and $\deg(u \circ \gamma) = \rho(\gamma)$ for $\gamma \in \pi_1(X)$. This map induces an isomorphism

$$H^1(X; \mathbb{Z}) \cong \pi_0(\mathcal{G}).$$

Spin connections

Let X be a Riemannian manifold of dimension 2, 3, or 4 modulo 8 and (S, I, J, Γ) be a spin structure on TX. Thus

$$\Gamma : TX \to \mathrm{End}(S)$$

satisfies (4.18) and I, J are two orthogonal anti-commuting complex structures on S which both commute with Γ. A Riemannian connection ∇ on S is called a **spin connection** if it is compatible with the Levi-Civita connection on TX as in (6.1) and commutes with I and J.

Lemma 6.6 *Let X be a Riemannian spin-manifold of dimension* 2, 3, *or* 4 *modulo* 8 *and* (S, I, J, Γ) *be a spin structure on TX as in Definition 5.4. Then there exists a unique spin connection on S.*

Proof: A connection on S which satisfies (6.1) and commutes with I is a spin^c connection with respect to I. Let ∇ be such a connection. Then every other spin^c connection on (S, I) has the form

$$\nabla_{\alpha,u}\Phi = \nabla_u\Phi + I\alpha(u)\Phi$$

for some 1-form $\alpha \in \Omega^1(X)$. It remains to prove that there exists a unique such 1-form α such that

$$\nabla_{\alpha,u}J = \nabla_u J + 2\alpha(u)IJ = 0.$$

To see this note that $\nabla_u J$ commutes with $\Gamma(v)$ for every v. Hence it follows from Lemma 4.47 that $\nabla_u J = R(a)$ for some $a = a(u) \in \mathbb{H}$. Now the formulae $J^2 = -\mathbb{1}$ and $JI = -IJ$ show that $\nabla_u J$ anti-commutes with I and J. Hence $\nabla_u J$ must be a multiple of $K = IJ$. Hence there exists a (unique) 1-form α such that $\nabla_u J = -2\alpha(u)IJ$. This shows that $\nabla_\alpha = \nabla + I\alpha$ is the required spin connection on S. □

Exercise 6.7 Prove the analogue of Lemma 6.6 when X has dimension 0, 6, or 7 modulo 8. □

Exercise 6.8 Give an alternative proof of Lemma 6.6 using connections on principal bundles. □

6.2 The Dirac operator

Continue the above notation. Given any virtual connection $A \in \mathcal{A}(\Gamma)$ denote by

$$\nabla = \nabla_A : C^\infty(X, W) \to \Omega^1(X, W)$$

the corresponding covariant derivative operator which is compatible with the Levi-Civita connection and induces the connection $2^n A$ on $\det(W)$. The **Dirac operator**

$$\mathcal{D}_A : C^\infty(X, W) \to C^\infty(X, W)$$

is defined by

$$\mathcal{D}_A \Phi = \sum_\nu \Gamma(e_\nu) \nabla_{e_\nu} \Phi$$

for $\Phi \in C^\infty(X, W)$, where the vectors $e_1, \dots, e_{2n}$ form an orthonormal basis of TX. It is easy to see that the expression on the right is independent of the choice of this basis. If X has even dimension and $\Phi \in C^\infty(X, W^+)$ then $\mathcal{D}_A \Phi \in C^\infty(X, W^-)$ and vice versa. In this case we write

$${\mathcal{D}_A}^\pm : C^\infty(X, W^\pm) \to C^\infty(X, W^\mp).$$

The next Lemma asserts that the Dirac operator is self-adjoint. If X has even dimension this means that

$${\mathcal{D}_A}^- = ({\mathcal{D}_A}^+)^*.$$

Lemma 6.9 *The Dirac operator $\mathcal{D}_A : C^\infty(X, W) \to C^\infty(X, W)$ is formally self-adjoint. Moreover,*

$$d^* \langle \Gamma(\cdot)\Psi, \Phi \rangle = \langle \Psi, \mathcal{D}_A \Phi \rangle - \langle \mathcal{D}_A \Psi, \Phi \rangle \tag{6.6}$$

for $\Phi, \Psi \in C^\infty(X, W)$.

Proof 1: Think of $\langle \Psi, \Gamma(\cdot)\Phi \rangle$ as a complex valued 1-form on X and let $e_1, \dots, e_m$ be a local orthonormal frame of TX. Abbreviate $\nabla_i = \nabla_{e_i}$. Then, by Lemma 2.27

$$\begin{aligned}
d^* \langle \Gamma(\cdot)\Psi, \Phi \rangle &= -\sum_i \iota(e_i) \nabla_i \langle \Gamma(\cdot)\Psi, \Phi \rangle \\
&= -\sum_i \nabla_i \langle \Gamma(e_i)\Psi, \Phi \rangle + \sum_i \langle \Gamma(\nabla_i e_i)\Psi, \Phi \rangle \\
&= -\sum_i \langle \Gamma(e_i) \nabla_{A,e_i} \Psi, \Phi \rangle - \sum_i \langle \Gamma(e_i)\Psi, \nabla_{A,e_i} \Phi \rangle \\
&= \langle \Psi, \mathcal{D}_A \Phi \rangle - \langle \mathcal{D}_A \Psi, \Phi \rangle.
\end{aligned}$$

This proves (6.6). Since the integral of $d^*\alpha$ over X is zero for every $\alpha \in \Omega^1(X)$ it follows that

$$\int_X \langle \mathcal{D}_A \Psi, \Phi \rangle \,\mathrm{dvol} = \int_X \langle \Psi, \mathcal{D}_A \Phi \rangle \,\mathrm{dvol}$$

for all $\Phi, \Psi \in C^\infty(X, W)$. Hence $\mathcal{D}_A$ is formally self-adjoint. $\square$

Proof 2: By Lemma 2.23,

$$\sum_i (\nabla_{e_i} e_i + \mathrm{div}(e_i) e_i) = 0$$

for every orthonormal frame of TX. Moreover, by Lemma 2.22,

$$\nabla_{e_i}{}^* = -\nabla_{e_i} - \mathrm{div}(e_i)$$

and hence

$$\begin{aligned}
\mathcal{D}_A{}^*\Phi &= \sum_i \nabla_{e_i}{}^*(\Gamma(e_i)^*\Phi) \\
&= \sum_i \nabla_{e_i}(\Gamma(e_i)\Phi) + \sum_i \mathrm{div}(e_i)\Gamma(e_i)\Phi \\
&= \sum_i \Gamma(e_i)\nabla_{e_i}\Phi + \sum_i \Gamma(\nabla_{e_i} e_i + \mathrm{div}(e_i) e_i)\Phi \\
&= \mathcal{D}_A\Phi.
\end{aligned}$$

Note that this calculation can be simplified by choosing an orthonormal frame with $\nabla_{e_i} e_j = 0$ for all i and j at a given point $x_0 \in X$.

Now identify TX with T^*X and think of Γ as a bundle homomorphism $T^*X \to \mathrm{End}(W)$ whenever convenient. Then, by definition of the Dirac operator,

$$\mathcal{D}_A(f\Psi) - f\mathcal{D}_A\Psi = \Gamma(df)\Psi.$$

Take the L^2-inner product with Φ to obtain

$$\int_X (\langle \Psi, \mathcal{D}_A\Phi\rangle - \langle \mathcal{D}_A\Psi, \Phi\rangle)\, f \,\mathrm{dvol} = \int_X \langle \Gamma(df)\Psi, \Phi\rangle \,\mathrm{dvol}$$

for all $\Phi, \Psi \in C^\infty(X, W)$. This implies (6.6). □

In the subsequent chapters it will be conveniend to use the notation $D_A = \mathcal{D}_A{}^+$ and $D_A{}^* = \mathcal{D}_A{}^-$ in the even dimensional case, and $D_A = \mathcal{D}_A$ in the odd dimensional case.

Exercise 6.10 Let (S, I, J, Γ) be a spin structure on a Riemannian manifold X of dimension 2, 3, or 4 modulo 8. Let ∇ be the unique spin connection on W and denote by

$$\mathcal{D} : C^\infty(X, S) \to C^\infty(X, S)$$

the associated Dirac operator. Prove that Dirac operator commutes with I and J. Deduce that its kernel carries an action of the quaternions $\mathbb{H}$. □

Dirac operators on three-manifolds

Let Y be a compact oriented Riemannian 3-manifold and

$$\gamma : TY \to \mathrm{End}(W)$$

be a spin^c structure on Y which is compatible with the orientation. This means that $W \to Y$ is a Hermitian rank-2 bundle and γ satisfies (5.15). As in Lemma 3.2 define the metric on the endomorpism bundle $\mathrm{End}(W)$ as half the trace. For $\Phi, \Psi \in W_y$ let $\Phi\Psi^* \in \mathrm{End}(W_y)$ be given by $\Phi\Psi^*\theta = \Phi\langle\Psi, \theta\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the Hermitian inner product on W_y. The traceless part of $\Phi\Psi^*$ is given by

$$(\Phi\Psi^*)_0 = \Phi\Psi^* - \frac{1}{2}\langle\Psi, \Phi\rangle\mathbb{1}.$$

Compare this with the discussion preceding Lemma 4.62. Let us denote by $\mathrm{End}_0(W)$ the bundle of traceless complex linear endomorphisms of W. The next lemma contains some useful identities for the three dimensional case.

Lemma 6.11 *Let $\gamma : TY \to \mathrm{End}(W)$ be a spin^c structure on a compact oriented smooth 3-manifold and $A \in \mathcal{A}(\gamma)$. Then the following holds for $\Phi, \Psi \in C^\infty(Y, W)$, $\alpha \in \Omega^1(Y)$, $v \in \mathrm{Vect}(Y)$, and $T \in C^\infty(Y, \mathrm{End}_0(W))$.*

(i)

$$|v| = |\gamma(v)|.$$

(ii)

$$\gamma^{-1}((\Phi\Psi^*)_0) = \frac{1}{2}\langle\gamma(\cdot)\Psi, \Phi\rangle,$$
$$\gamma^{-1}((\Phi\Psi^* - \Psi\Phi^*)_0) = \mathrm{Re}\,\langle\gamma(\cdot)\Psi, \Phi\rangle,$$
$$\gamma^{-1}((\Phi\Psi^* + \Psi\Phi^*)_0) = i\mathrm{Im}\,\langle\gamma(\cdot)\Psi, \Phi\rangle.$$

(iii)

$$\gamma^{-1}((\gamma(\alpha)\Phi\Psi^* + \Phi\Psi^*\gamma(\alpha))_0) = \langle\Psi, \Phi\rangle\alpha.$$

(iv)

$$\gamma(*d\alpha) = \mathcal{D}_A\gamma(\alpha) + \gamma(\alpha)\mathcal{D}_A + \nabla_{A,\alpha} - (\nabla_{A,\alpha})^*.$$

(v)

$$*d\langle\gamma(\cdot)\Psi, \Phi\rangle = \langle\gamma(\cdot)\mathcal{D}_A\Psi + \nabla_A\Psi, \Phi\rangle - \langle\Psi, \gamma(\cdot)\mathcal{D}_A\Phi + \nabla_A\Phi\rangle.$$

Proof: Assertion (i) is obvious from the definitions. To prove (ii) take the inner product with α:

$$\langle\alpha, \gamma^{-1}((\Phi\Psi^*)_0)\rangle = \langle\gamma(\alpha), \Phi\Psi^*\rangle = \frac{1}{2}\mathrm{trace}(\gamma(\alpha)^*\Phi\Psi^*) = \frac{1}{2}\langle\gamma(\alpha)\Psi, \Phi\rangle.$$

To prove (iii) write $\alpha' = \gamma^{-1}((\gamma(\alpha)\Phi\Psi^* + \Phi\Psi^*\gamma(\alpha))_0)$. Then the Hermitian inner product of $\beta \in \Omega^1(Y)$ with α' is given by

$$\begin{aligned}
\langle \beta, \alpha' \rangle &= \langle \gamma(\beta), \gamma(\alpha)\Phi\Psi^* + \Phi\Psi^*\gamma(\alpha) \rangle \\
&= \frac{1}{2}\mathrm{trace}\left(\gamma(\beta)^*\gamma(\alpha)\Phi\Psi^* + \gamma(\beta)^*\Phi\Psi^*\gamma(\alpha)\right) \\
&= -\frac{1}{2}\mathrm{trace}\left((\gamma(\beta)\gamma(\alpha) + \gamma(\alpha)\gamma(\beta))\Phi\Psi^*\right) \\
&= \langle \beta, \alpha \rangle \mathrm{trace}\left(\Phi\Psi^*\right) \\
&= \langle \beta, \langle \Psi, \Phi \rangle \alpha \rangle.
\end{aligned}$$

This continues to hold for complex valued 1-forms $\beta \in \Omega^1(Y, \mathbb{C})$. The proof of (iv) relies on the identities

$$*d\alpha = \sum_{i=1}^{3} e_i{}^* \times \nabla_i \alpha, \qquad d^*\alpha = -\sum_{i=1}^{3} \iota(e_i)\nabla_i \alpha,$$

for an orthonormal frame e_1, e_2, e_3 of TY (see Lemma 2.27). We obtain

$$\begin{aligned}
&\mathcal{D}_A(\gamma(\alpha)\Phi) + \gamma(\alpha)\mathcal{D}_A\Phi \\
&\quad= \sum_i \Big(\gamma(e_i)\nabla_{A,e_i}(\gamma(\alpha)\Phi) + \gamma(\alpha)\gamma(e_i)\nabla_{A,e_i}\Phi\Big) \\
&\quad= \sum_i \Big(\gamma(e_i)\gamma(\nabla_i\alpha)\Phi + (\gamma(e_i)\gamma(\alpha) + \gamma(\alpha)\gamma(e_i))\nabla_{A,e_i}\Phi\Big) \\
&\quad= \sum_i \Big(\gamma(e_i^* \times \nabla_i\alpha)\Phi - (\iota(e_i)\nabla_i\alpha)\Phi - 2\sum_i \alpha(e_i)\nabla_{A,e_i}\Phi\Big) \\
&\quad= \gamma(*d\alpha)\Phi + d^*\alpha\Phi - 2\nabla_{A,\alpha}\Phi \\
&\quad= \gamma(*d\alpha)\Phi - \nabla_{A,\alpha}\Phi + (\nabla_{A,\alpha})^*\Phi.
\end{aligned}$$

This proves (iv). To prove (v) consider the Hermitian L^2-inner product of $\alpha \in \Omega^1(Y)$ and $*d\langle \Psi, \gamma(\cdot)\Phi \rangle$:

$$\langle \alpha, *d\langle \gamma(\cdot)\Psi, \Phi \rangle \rangle = 2\langle \gamma(*d\alpha), \Phi\Psi^* \rangle = \langle \gamma(*d\alpha)\Psi, \Phi \rangle.$$

Here the first identity follows from (i) and the fact that $*d$ is self-adjoint. The second identity follows as in the first line of the proof. Hence, by (iv),

$$\begin{aligned}
\langle \alpha, *d\langle \gamma(\cdot)\Psi, \Phi \rangle \rangle &= \langle \mathcal{D}_A(\gamma(\alpha)\Psi) + \gamma(\alpha)\mathcal{D}_A\Psi + \nabla_{A,\alpha}\Psi - (\nabla_{A,\alpha})^*\Psi, \Phi \rangle \\
&= \langle \gamma(\alpha)\mathcal{D}_A\Psi + \nabla_{A,\alpha}\Psi, \Phi \rangle - \langle \Psi, \gamma(\alpha)\mathcal{D}_A\Phi + \nabla_{A,\alpha}\Phi \rangle \\
&= \langle \alpha, \langle \gamma(\cdot)\mathcal{D}_A\Psi + \nabla_A\Psi, \Phi \rangle - \langle \Psi, \gamma(\cdot)\mathcal{D}_A\Phi + \nabla_A\Phi \rangle \rangle
\end{aligned}$$

This identity continues to hold for $\alpha \in \Omega^1(Y, \mathbb{C})$. This proves (v) and the lemma. □

6.3 Dirac operators on symplectic manifolds

Let (X, ω) be a symplectic manifold with a compatible almost complex structure J and a corresponding Hermitian structure

$$\langle v, w\rangle = g(v, w) + i\omega(v, w).$$

on TX, where $g(v, w) = \omega(v, Jw)$ is the Riemannian metric determined by ω and J. (See Section 3.3 for the relevant definitions.) Recall from Section 3.2 that for any Hermitian line bundle $E \to X$ with Hermitian connection B there is an operator

$$\bar{\partial}_B + \bar{\partial}_B^* : \Omega^{0,\mathrm{ev}}(X, E) \to \Omega^{0,\mathrm{odd}}(X, E).$$

The goal of this section is to compare this operator with the Dirac operator with respect to a suitable spin^c connection.

Spin^c connection on symplectic manifolds

The canonical spin^c bundle of X is the 2^n dimensional complex vector bundle $W_{\mathrm{can}} = \Lambda^{0,*}T^*X$ with its standard Hermitian structure and splitting into

$$W_{\mathrm{can}}^+ = \Lambda^{0,\mathrm{ev}}T^*X, \qquad W_{\mathrm{can}}^- = \Lambda^{0,\mathrm{odd}}T^*X.$$

The canonical spin^c representation $\Gamma_{\mathrm{can}} : TX \to \mathrm{End}(W_{\mathrm{can}})$ is given by

$$\Gamma_{\mathrm{can}}(v)\tau = \frac{1}{\sqrt{2}}v'' \wedge \tau - \sqrt{2}\iota(v)\tau.$$

The goal is to find a spin^c connection on W_{can} which is compatible with the Levi-Civita connection on TX. Recall from Section 3.3 that the Levi-Civita connection does not preserve the spaces $\Omega^{0,k}(X)$ unless J is integrable. However, there is a Hermitian connection on TX given by the formula

$$\widetilde{\nabla}_v w = \nabla_v w - \frac{1}{2}J(\nabla_v J)w$$

for $v, w \in \mathrm{Vect}(X)$ and the induced connection on $\Lambda^k T^*X \otimes \mathbb{C}$ is given by

$$\widetilde{\nabla}_v \tau = \nabla_v \tau + \frac{1}{2}\iota(J\nabla_v J)\tau$$

for $\tau \in \Omega^k(X)$ and $v \in \mathrm{Vect}(X)$ where $\iota(J\nabla_v J)\tau$ is defined by (2.19). (See (3.8) in Section 3.3.) This connection preserves the subspaces $\Omega^{0,k}(X)$ and hence defines a connection on the canonical spin^c bundle W_{can}. The next lemma shows that this is a spin^c connection. However, it is compatible with the Hermitian connection $\widetilde{\nabla}$ on TX instead of the Levi-Civita connection ∇.

Lemma 6.12 *For $\tau \in \Omega^{0,*}(X)$ and $u, v \in \mathrm{Vect}(X)$*

$$\widetilde{\nabla}_u(\Gamma(v)\tau) = \Gamma(\widetilde{\nabla}_u v)\tau + \Gamma(v)\widetilde{\nabla}_u\tau$$

This formula continues to hold for forms $\tau \in \Omega^{0,*}(X, E) = C^\infty(X, W_E)$ with values in a Hermitian line bundle E with Hermitian connection B provided that the connection $\widetilde{\nabla}$ on $\Lambda^{0,*}T^*X$ is replaced by the connection $\widetilde{\nabla}_B$ on $W_E = \Lambda^{0,*}T^*X \otimes E$ as in Exercise 3.24.

Proof: By Lemma 3.20,

$$\begin{aligned}
\widetilde{\nabla}_u(\Gamma_{\mathrm{can}}(v)\tau) &= \widetilde{\nabla}_u\left(\frac{1}{\sqrt{2}}v^* \wedge \tau + \frac{1}{\sqrt{2}}i(Jv)^* \wedge \tau - \sqrt{2}\iota(v)\tau\right) \\
&= \frac{1}{\sqrt{2}}(\widetilde{\nabla}_u v)^* \wedge \tau + \frac{1}{\sqrt{2}}i(J\widetilde{\nabla}_u v)^* \wedge \tau - \sqrt{2}\iota(\widetilde{\nabla}_u v)\tau \\
&\quad + \frac{1}{\sqrt{2}}v^* \wedge \widetilde{\nabla}_u\tau + \frac{1}{\sqrt{2}}i(Jv)^* \wedge \widetilde{\nabla}_u\tau - \sqrt{2}\iota(v)\widetilde{\nabla}_u\tau \\
&= \Gamma_{\mathrm{can}}(\widetilde{\nabla}_u v)\tau + \Gamma_{\mathrm{can}}(v)\widetilde{\nabla}_u\tau.
\end{aligned}$$

This proves the lemma in the untwisted case. The twisted case is left as an exercise. □

It is necessary to modify the connection $\widetilde{\nabla}$ on W_{can} by a suitable endomorphism valued 1-form in order to obtain a connection which is compatible with the Levi-Civita connection on TX. To find this endomorphism recall that there is a homomorphism $\mu : \mathfrak{so}(TX) \to \mathrm{End}(W_{\mathrm{can}})$ which makes the following diagram commute.

$$\begin{tikzcd}
\mathfrak{so}(TX) \arrow[r, "\mu"] & \mathrm{End}(W_{\mathrm{can}}) \\
C_2(TX) \arrow[u, "ad"] \arrow[ur, "\Gamma"'] &
\end{tikzcd}$$

Since $\mathrm{ad}(\xi)v = \xi v - v\xi$ for $\xi \in C_2(T_xX)$ and $v \in T_xX$ the homomorphism μ can be characterized by the identity

$$[\mu(A), \Gamma(v)] = \Gamma(Av). \tag{6.7}$$

Consider the connection $\nabla_{\mathrm{can}} : C^\infty(X, W_{\mathrm{can}}) \to \Omega^1(X, W_{\mathrm{can}})$ defined by

$$\nabla_{\mathrm{can},v}\tau = \widetilde{\nabla}_v\tau \ + \frac{1}{2}\mu(J(\nabla_v J))\tau \tag{6.8}$$

for $\tau \in \Omega^{0,*}(X)$ and $v \in \mathrm{Vect}(X)$. Note that in the Kähler case $\nabla_{\mathrm{can}} = \widetilde{\nabla} = \nabla$ is the Levi-Civita connection on forms.

Lemma 6.13 *The connection ∇_{can} on W_{can} is a spin^c connection which is compatible with the Levi-Civita connction on X.*

Proof: By Lemma 3.20 (iii) and 6.7),

$$\begin{aligned}\nabla_{\mathrm{can},v}(\Gamma(w)\tau) &= \widetilde{\nabla}_v(\Gamma(w)\tau) + \frac{1}{2}\mu(J(\nabla_v J))\Gamma(w)\tau \\ &= \Gamma(w)\widetilde{\nabla}_v\tau + \Gamma(\widetilde{\nabla}_v w)\tau + \frac{1}{2}\mu(J(\nabla_v J))\Gamma(w)\tau \\ &= \Gamma(w)\nabla_{\mathrm{can},v}\tau + \Gamma(\widetilde{\nabla}_v w)\tau + \frac{1}{2}[\mu(J(\nabla_v J)),\Gamma(w)]\tau \\ &= \Gamma(w)\nabla_{\mathrm{can},v}\tau + \Gamma(\widetilde{\nabla}_v w)\tau + \frac{1}{2}\Gamma(J(\nabla_v J)w)\tau \\ &= \Gamma(w)\nabla_{\mathrm{can},v}\tau + \Gamma(\nabla_v w)\tau.\end{aligned}$$

This proves the lemma. □

The discussion on page 190 shows that there is a one-to-one correspondence between spin^c connections $\nabla = \nabla_A$ on W and virtual connections $A \in \mathcal{A}(\Gamma)$ on the virtual bundle $L_\Gamma^{1/2}$. In the case of the canonical spin^c structure this shows that there is a unique connection

$$A_{\mathrm{can}} \in \mathcal{A}(\Gamma_{\mathrm{can}})$$

such that $\nabla_{\mathrm{can}} = \nabla_{A_{\mathrm{can}}}$. Note that $2A_{\mathrm{can}}$ is a connection on the anti-canonical bundle

$$L_{\Gamma_{\mathrm{can}}} = K^* = \Lambda^{0,n}T^*X.$$

Lemma 6.14 *Assume that (X, J, ω) is a Kähler manifold and so*

$$\nabla_{\mathrm{can}} = \widetilde{\nabla} = \nabla.$$

Then the connection $2A_{\mathrm{can}}$ on $L_{\Gamma_{\mathrm{can}}} = \Lambda^{0,n}T^*X$ agrees with the Levi-Civita connection.

Proof: Since $\nabla J = 0$, it follows that the connection on the principal bundle $P \to X$ is a $\mathrm{U}^c(V_0)$-connection. The induced connection on $TX \cong P \times_{\mathrm{ad}} V_0$ is the Levi-Civita connection and hence the induced connection on

$$\Lambda^{0,n}T^*X = P \times_{\det^c \circ \mathrm{ad}} \mathbb{C}$$

is also the Levi-Civita connection. Now recall from Lemma 4.53 that

$$\delta(x) = \det{}^c(\mathrm{ad}(x))$$

for $x \in \mathrm{U}^c(V_0)$ and hence ∇ also induces the Levi-Civita connection on $L_{\Gamma_{\mathrm{can}}} = P \times_\delta \mathbb{C} \cong \Lambda^{0,n}T^*X$. □

In order to understand the induced connection on the bundle $L_{\Gamma_{\mathrm{can}}} = \Lambda^{0,n}T^*X$ in the nonintegrable case we must examine the homomorphism $\mu : \mathfrak{so}(TX) \to \mathrm{End}(W_{\mathrm{can}})$ in more detail. The next lemma expresses this homomorphism in terms of the Nijenhuis tensor as defined in Section 3.2. Recall from (3.7) in Section 3.3 that, on a symplectic manifold, the Nijenhuis tensor can be interpreted as a linear map $\mathrm{Vect}(X) \to \Omega^{0,2}(X) : u \mapsto \Theta_u$ given by $\Theta_u(v,w) = \langle u, N(v,w)\rangle$. Recall also the notation $\iota(\bar{\Theta}_u)\tau$ as defined in (3.3) in Section 3.1.

Lemma 6.15 **(i)** *For* $v, w \in \mathrm{Vect}(X)$

$$\mu(vw^* - wv^*) = \frac{1}{2}\Big(g(v,w)\mathbb{1} - \Gamma(v)\Gamma(w)\Big).$$

(ii) *For* $v \in \mathrm{Vect}(X)$ *and* $\tau \in \Omega^{0,*}(X)$

$$\mu(J\nabla_v J)\tau = \frac{1}{4}\Theta_v \wedge \tau + \iota(\bar{\Theta}_v)\tau.$$

Proof: In a unitary frame $e_1, \dots, e_n, Je_1, \dots, Je_n$ of TX the following holds

$$e_ie_je_k - e_ke_ie_j = \begin{cases} 2e_j, & \text{if } k = i \neq j, \\ -2e_i, & \text{if } k = j \neq i, \\ 0, & \text{otherwise.} \end{cases}$$

Hence $\mathrm{Ad}(e_ie_j) = 2(e_je_i{}^* - e_ie_j{}^*)$ for $i \neq j$. This proves (i). To prove (ii) note first that for two vector fields $v, w \in \mathrm{Vect}(X)$

$$\Gamma(v)\Gamma(w)\tau - \Gamma(Jv)\Gamma(Jw)\tau = v'' \wedge w'' \wedge \tau + 4\iota(v)\iota(w)\tau,$$

$$\Gamma(v)\Gamma(Jw)\tau + \Gamma(v)\Gamma(Jw)\tau = \sqrt{-1}v'' \wedge w'' \wedge \tau - 4\sqrt{-1}\iota(v)\iota(w)\tau.$$

Now

$$\begin{aligned} J\nabla_v J &= \sum_{i,j} g(e_i, J(\nabla_v J)e_j)\Big(e_ie_j{}^* - (Je_i)(Je_j)^*\Big) \\ &\quad + \sum_{i,j} g(e_i, (\nabla_v J)e_j)\Big(e_i(Je_j)^* + (Je_i)e_j{}^*\Big) \\ &= \sum_{i<j} g(e_i, J(\nabla_v J)e_j)\Big(e_ie_j{}^* - e_je_i{}^* - (Je_i)(Je_j)^* + (Je_j)(Je_i)^*\Big) \\ &\quad + \sum_{i<j} g(e_i, (\nabla_v J)e_j)\Big(e_i(Je_j)^* - (Je_j)e_i{}^* + (Je_i)e_j{}^* - e_j(Je_i)^*\Big) \end{aligned}$$

and hence, using (i) and the formulae $\nabla_v J = J(\nabla_{Jv}J)$ and $g(v, N(e_i,e_j)) = 2g(J(\nabla_v J)e_i, e_j)$,

$$\begin{aligned}
\mu(J\nabla_v J)\tau = &-\frac{1}{2}\sum_{i<j} g(e_i, J(\nabla_v J)e_j)\Big(\Gamma(e_i)\Gamma(e_j) - \Gamma(Je_i)\Gamma(Je_j)\Big)\tau \\
&-\frac{1}{2}\sum_{i<j} g(e_i, J(\nabla_{Jv} J)e_j)\Big(\Gamma(e_i)\Gamma(Je_j) + \Gamma(Je_i)\Gamma(e_j)\Big)\tau \\
= &\frac{1}{4}\sum_{i,j} g(J(\nabla_v J)e_i, e_j)\Big(\Gamma(e_i)\Gamma(e_j) - \Gamma(Je_i)\Gamma(Je_j)\Big)\tau \\
&+\frac{1}{4}\sum_{i,j} g(J(\nabla_{Jv} J)e_i, e_j)\Big(\Gamma(e_i)\Gamma(Je_j) + \Gamma(Je_i)\Gamma(e_j)\Big)\tau \\
= &\sum_{i,j} g(v, N(e_i,e_j))\left(\frac{1}{8}e_i'' \wedge e_j'' \wedge \tau + \frac{1}{2}\iota(e_i)\iota(e_j)\tau\right) \\
&+\sqrt{-1}\sum_{i,j} g(Jv, N(e_i,e_j))\left(\frac{1}{8}e_i'' \wedge e_j'' \wedge \tau - \frac{1}{2}\iota(e_i)\iota(e_j)\tau\right) \\
= &\sum_{i,j}\left(\frac{\langle v, N(e_i,e_j)\rangle}{8}e_i'' \wedge e_j'' \wedge \tau + \frac{\langle N(e_i,e_j), v\rangle}{2}\iota(e_i)\iota(e_j)\tau\right).
\end{aligned}$$

Now recall from Section 3.3 that

$$\Theta_v = \frac{1}{2}\sum_{i,j}\langle v, N(e_i,e_j)\rangle e_i'' \wedge e_j''$$

and hence, by definition of $\iota(\bar{\Theta}_v)\tau$ in (3.3) in Section 3.1,

$$\iota(\bar{\Theta}_v)\tau = \frac{1}{2}\sum_{i,j}\langle N(e_i,e_j), v\rangle\iota(e_i)\iota(e_j)\tau.$$

This proves the lemma. □

Lemma 6.16 *Assume that (X,ω) is a 4-dimensional symplectic manifold with compatible almost complex structure J. Then the connection $2A_{\mathrm{can}}$ on the line bundle $L_{\Gamma_{\mathrm{can}}} = \det^c(W^+_{\mathrm{can}}) = \det^c(W^-_{\mathrm{can}}) = \Lambda^{0,2}T^*X$ agrees with $\widetilde{\nabla}$. Hence*

$$F_{A_{\mathrm{can}}} = \frac{1}{2}\mathrm{trace}^c(\widetilde{R})$$

where $\widetilde{R}$ denotes the full curvature tensor of $\widetilde{\nabla}$ on (TX,J).

Proof: In the 4-dimensional case $W^-_{\mathrm{can}} = \Lambda^{0,1}T^*X$ and the formula of Lemma 6.15 (ii) shows that $\mu(J\nabla_v J)\tau = 0$ for $\tau \in \Lambda^{0,1}T^*X$. Hence the spin^c connection ∇_{can} agrees with $\widetilde{\nabla}$ on W^-_{can} and this implies that the induced connection on $L_{\Gamma_{\mathrm{can}}} = \Lambda^2_{\mathbb{C}}W^-_{\mathrm{can}}$ is also given by $\widetilde{\nabla}$. □

The Dirac operator and the Cauchy-Riemann operator

Let (X, ω) be a $2n$-dimensional symplectic manifold with compatible almost complex structure $J \in \mathcal{J}(X, \omega)$ and corresponding canonical spin^c structure $W_{\mathrm{can}} = \Lambda^{0,*}T^*X$. Given a Hermitian line bundle $E \to X$ denote the twisted spin^c bundle by

$$W_E = \Lambda^{0,*}T^*X \otimes E$$

and its standard splitting by $W_E = W_E^+ \oplus W_E^-$ where $W_E^+ = \Lambda^{0,\mathrm{ev}}T^*X \otimes E$ and $W_E^- = \Lambda^{0,\mathrm{odd}}T^*X \otimes E$. Consider the spin^c connection ∇_{can} on $W_{\mathrm{can}} = \Lambda^{0,*}T^*X$, defined by (6.8), and fix a Hermitian connection B on E. Together these determine a spin^c connection $\nabla_A = \nabla_{A_{\mathrm{can}}+B}$ on W_E, which preserves the subbundles W_E^+ and W_E^-, and is defined by

$$\nabla_A(\tau \otimes s) = (\nabla_{\mathrm{can}}\tau) \otimes s + \tau \otimes d_B s$$

for $s \in C^\infty(X, E)$ and $\tau \in \Omega^{0,*}(X)$. The associated Dirac operator is denoted by $D_A : C^\infty(X, W_E^+) \to C^\infty(X, W_E^-)$. The space of sections of W_E can be identified with the space of $(0,*)$-forms on X with values in E and there is an operator $\bar\partial_B + \bar\partial_B^* : \Omega^{0,\mathrm{ev}}(X, E) \to \Omega^{0,\mathrm{odd}}(X, E)$ induced by the Riemannian metric on X and the connection B. The following theorem relates these two operators.

Theorem 6.17 *The Dirac operator $D_A : \Omega^{0,\mathrm{ev}}(X, E) \to \Omega^{0,\mathrm{odd}}(X, E)$ on a symplectic manifold has the form*

$$\frac{1}{\sqrt{2}} D_{A_{\mathrm{can}}+B} = \bar\partial_B + \bar\partial_B^*.$$

Proof: Fix a local unitary frame $e_1, Je_1, \ldots, e_n, Je_n$ of TX and recall that the homomorphism $\mu_E : \mathfrak{so}(TX) \to \mathrm{End}(W_E)$ is defined by $\mu_E \circ \mathrm{Ad} = \Gamma_E : C_2(TX) \to \mathrm{End}(W_E)$ as in (6.8) The first step is to prove the formula

$$\frac{1}{\sqrt{2}} D_{A_{\mathrm{can}}+B} = \bar\partial_B + \bar\partial_B^* + R \tag{6.9}$$

where the operator R given by

$$R = \frac{1}{2\sqrt{2}} \sum_k \Big(\Gamma_E(e_k)\mu_E(J(\nabla_{e_k}J)) + \Gamma_E(Je_k)\mu_E(J(\nabla_{Je_k}J)) \Big). \tag{6.10}$$

To see this note first that the formula (6.8) continues to hold with ∇_{can} and $\widetilde\nabla$ replaced by $\nabla_{A_{\mathrm{can}}+B}$ and $\widetilde\nabla_B$. Now combine the formulae for $\bar\partial_B\tau$ and $\bar\partial_B^*\tau$ in Proposition 3.23 (and Exercise 3.24) with the definition of the Dirac operator D_A with $A = A_{\mathrm{can}} + B$ to obtain

$$
\begin{aligned}
\frac{1}{\sqrt{2}} D_A \tau &= \frac{1}{\sqrt{2}} \sum_k \Big(\Gamma_E(e_k) \nabla_{A,e_k} \tau + \Gamma_E(Je_k) \nabla_{A,Je_k} \tau \Big) \\
&= \frac{1}{\sqrt{2}} \sum_k \Big(\Gamma_E(e_k) \widetilde{\nabla}_{B,e_k} \tau + \Gamma_E(Je_k) \widetilde{\nabla}_{B,Je_k} \tau \Big) \\
&\quad + \frac{1}{2\sqrt{2}} \sum_k \Big(\Gamma_E(e_k) \mu_E(J(\nabla_{e_k} J)) \tau + \Gamma_E(Je_k) \mu_E(J(\nabla_{Je_k} J)) \tau \Big) \\
&= \frac{1}{2} \sum_k \Big({e_k}'' \wedge \widetilde{\nabla}_{B,e_k} \tau + (Je_k)'' \wedge \widetilde{\nabla}_{B,Je_k} \tau \Big) \\
&\quad - \sum_k \Big(\iota(e_k) \widetilde{\nabla}_{B,e_k} \tau + \iota(Je_k) \widetilde{\nabla}_{B,Je_k} \tau \Big) + R\tau \\
&= \bar{\partial}_B \tau + \bar{\partial}_B^* \tau + R\tau.
\end{aligned}
$$

This proves the formula (6.9) with R given by (6.10). Now insert the formula for $\mu_E(J\nabla_v J)$ of Lemma 6.15 into (6.10) and use the identities

$$
\Theta_{Jv} = -i\Theta_v, \qquad \iota(\bar{\Theta}_{Jv})\tau = i\iota(\bar{\Theta}_v)\tau,
$$

with $\Theta_u \in \Omega^{0,2}(X)$ given by (3.7), to obtain

$$
\begin{aligned}
2R\tau &= \frac{1}{\sqrt{2}} \sum_k \Gamma_E(e_k) \left(\frac{1}{4} \Theta_{e_k} \wedge \tau + \iota(\bar{\Theta}_{e_k})\tau \right) \\
&\quad + \frac{1}{\sqrt{2}} \sum_k \Gamma_E(Je_k) \left(\frac{1}{4} \Theta_{Je_k} \wedge \tau + \iota(\bar{\Theta}_{Je_k})\tau \right) \\
&= \frac{1}{8} \sum_k {e_k}'' \wedge \Theta_{e_k} \wedge \tau - \frac{1}{4} \sum_k \iota(e_k)(\Theta_{e_k} \wedge \tau) \\
&\quad + \frac{1}{2} \sum_k {e_k}'' \wedge \iota(\bar{\Theta}_{e_k})\tau - \sum_k \iota(e_k)\iota(\bar{\Theta}_{e_k})\tau \\
&\quad + \frac{1}{8} \sum_k J{e_k}'' \wedge \Theta_{Je_k} \wedge \tau - \frac{1}{4} \sum_k \iota(Je_k)(\Theta_{Je_k} \wedge \tau) \\
&\quad + \frac{1}{2} \sum_k J{e_k}'' \wedge \iota(\bar{\Theta}_{Je_k})\tau - \sum_k \iota(Je_k)\iota(\bar{\Theta}_{Je_k})\tau \\
&= \frac{1}{4} \sum_k {e_k}'' \wedge \Theta_{e_k} \wedge \tau - 2 \sum_k \iota(e_k)\iota(\bar{\Theta}_{e_k})\tau \\
&= 0.
\end{aligned}
$$

The last equation follows from Exercise 3.19 in Section 3.3. (See also the definition of $\iota(\bar{\Theta}_u)\tau$ in (3.3) in Section 3.1.) This proves the theorem. □

6.4 The Weitzenböck formula

Let X be a Riemannian manifold of dimension $2n$ or $2n+1$ and $\Gamma : TX \to \mathrm{End}(W)$ be a spin^c structure. Denote by $P \to X$ the associated frame bundle with structure group $\mathrm{Spin}^c(V_0)$ and by $L_\Gamma = P \times_\delta \mathbb{C}$ the associated line bundle. Recall that $\mathcal{A}(\Gamma)$ denotes the space of *virtual connections* on the *virtual line bundle* ${L_\Gamma}^{1/2}$ and that every $A \in \mathcal{A}(\Gamma)$ determines a spin^c connection ∇_A on W and a Dirac operator $D_A : C^\infty(X,W) \to C^\infty(X,W)$ respectively, in the even dimensional case, $D_A : C^\infty(X,W^+) \to C^\infty(X,W^-)$. Recall also that $F_A = 2^{-n}\mathrm{trace}(F^{\nabla_A}) \in \Omega^2(X, i\mathbb{R})$. In Section 4.8 it was shown that there is a natural linear operator $\rho : \Lambda^2 T^*X \otimes \mathbb{C} \to \mathrm{End}(W)$ defined by

$$\rho\left(\sum_{i<j} \eta_{ij} {e_i}^* \wedge {e_j}^*\right) = \sum_{i<j} \eta_{ij}\Gamma(e_i)\Gamma(e_j) \tag{6.11}$$

for any orthonormal frame $e_1, \ldots, e_{2n}$.

Exercise 6.18 Denote by ${\nabla_A}^* : \Omega^1(X,W) \to C^\infty(X,W)$ the L^2 adjoint of the covariant derivative operator $\nabla = \nabla_A$. The composition

$${\nabla_A}^*\nabla_A : C^\infty(X,W) \to C^\infty(X,W)$$

is called the **Bochner Laplacian**. Prove that in an orthonormal frame this operator is given by ${\nabla_A}^*\nabla_A\Phi = \sum_i {\nabla_i}^*\nabla_i\Phi$ where $\nabla_i = \nabla_{A,e_i}\Phi$ denotes the covariant derivative in the direction e_i. $\Box$

Theorem 6.19. (Weitzenböck formula) *Let $s : X \to \mathbb{R}$ denote the scalar curvature. Then, for $A \in \mathcal{A}(\Gamma)$ and $\Phi \in C^\infty(X,W)$,*

$$\mathcal{D}_A\mathcal{D}_A\Phi = {\nabla_A}^*\nabla_A\Phi + \frac{1}{4}s\Phi + \rho(F_A)\Phi.$$

Proof: Choose a local orthonormal frame $e_1, \ldots, e_{2n}$ of TX and denote

$$a_{ij} = -\Gamma(e_i)\Gamma(e_j) \in \mathrm{End}(W^+)$$

for $i, j = 1, \ldots, 2n$. These endomorphisms satisfy (6.14) in Lemma 6.20 below. Hence, in particular,

$$a_{ij} + a_{ji} = 2\delta_{ij}\mathbb{1}.$$

Abbreviate $\nabla_i\Phi = \nabla_A\Phi(e_i)$ for $\Phi \in C^\infty(X,W^+)$. The curvature F^∇ is a 2-form on X with values in $\mathrm{End}(W)$. In the local frame $F^\nabla = \sum_{i<j} F_{ij}{e_i}^* \wedge {e_j}^*$, where

$$F_{ij}\Phi = \nabla_i\nabla_j\Phi - \nabla_j\nabla_i\Phi + \nabla_{[e_i,e_j]}\Phi$$

for $\Phi \in C^\infty(X,W)$. These endomorphisms F_{ij} preserve W^+ and W^-.

Recall that the Dirac operator $\mathcal{D}_A : C^\infty(X, W) \to C^\infty(X, W)$ is defined by $\mathcal{D}_A\Phi = \mathcal{D}_A{}^*\Phi = \sum_i \Gamma(e_i)\nabla_i\Phi$ for $\Phi \in C^\infty(X, W^+)$ and hence

$$\mathcal{D}_A\mathcal{D}_A\Phi = \sum_{i,j} \nabla_i{}^* \left(a_{ij}\nabla_j\Phi\right).$$

The diagonal terms with $i = j$ give the operator $\nabla_A{}^*\nabla_A\Phi = \sum_i \nabla_i{}^*\nabla_i\Phi$ and the remaining terms can be expressed as follows

$$\mathcal{D}_A\mathcal{D}_A\Phi = \nabla_A{}^*\nabla_A\Phi - \sum_{i<j} a_{ij}F_{ij}\Phi. \tag{6.12}$$

To see this note that, by Lemma 2.22,

$$\begin{aligned}
\mathcal{D}_A\mathcal{D}_A\Phi - \nabla_A{}^*\nabla_A\Phi &= \sum_{i\neq j} \nabla_i{}^* \left(a_{ij}\nabla_j\Phi\right) \\
&= -\sum_{i\neq j} \nabla_i \left(a_{ij}\nabla_j\Phi\right) - \sum_{i\neq j} \mathrm{div}(e_i)a_{ij}\nabla_j\Phi \\
&= -\sum_{i\neq j} a_{ij}\nabla_i\nabla_j\Phi - \sum_{i\neq j} \left(\nabla_i a_{ij} + \mathrm{div}(e_i)a_{ij}\right)\nabla_j\Phi \\
&= -\sum_{i<j} a_{ij}(\nabla_i\nabla_j\Phi - \nabla_j\nabla_i\Phi) \\
&\qquad - \sum_{i\neq j} \left(\nabla_i a_{ij} + \mathrm{div}(e_i)a_{ij}\right)\nabla_j\Phi \\
&= -\sum_{i<j} a_{ij}F_{ij}\Phi + \sum_{i<j} a_{ij}\nabla_{[e_i,e_j]}\Phi \\
&\qquad - \sum_{i\neq j} \left(\nabla_i a_{ij} + \mathrm{div}(e_i)a_{ij}\right)\nabla_j\Phi.
\end{aligned}$$

It remains to prove that the first order terms vanish, i.e.

$$\sum_{i<j} a_{ij}\nabla_{[e_i,e_j]}\Phi - \left(\nabla_i a_{ij} + \mathrm{div}(e_i)a_{ij}\right)\nabla_j\Phi = 0$$

This is obvious for the following reason. Firstly, the left hand side of this equation (as a local section of W^+) is independent of the choice of the orthonormal frame. This can either be proved directly or deduced from the fact that the other terms in the above equation are independent of the choice of the frame. Now fix a point $x_0 \in X$ and choose a frame $e_1, \ldots, e_n$ near x_0 such that all the covariant derivatives $\nabla_{e_i}e_j = 0$ vanish at x_0. Then $\nabla_i a_{ij} = 0$, $\mathrm{div}(e_i) = 0$ and $[e_i, e_j] = 0$ at x_0 and hence the first order terms all vanish at x_0. This proves (6.12).

Let us now examine the right hand side of (6.12). The curvature terms F_{ij} split up into the trace and the traceless part $F_{ij}^0 = F_{ij} - 2^{-n}\mathrm{trace}(F_{ij})\mathbb{1}$. This part is entirely determined by the Levi-Civita connection and it satisfies

$$\sum_{i<j} a_{ij} F_{ij}^0 = -\frac{1}{4} s\mathbb{1} \tag{6.13}$$

where $s : X \to \mathbb{R}$ denotes the scalar curvature. To see this recall the formula (6.2) which asserts that the traceless part of the curvature is given by

$$F_{ij}^0 = \rho(R(e_i, e_j))$$

where $R \in \Omega^2(X, \mathrm{End}(TX))$ denotes the Riemann curvature tensor and the homomorphism $\rho : \mathfrak{so}(TX) \to \mathrm{End}(W)$ is defined by $\rho(\mathrm{Ad}(\xi)) = \Gamma(\xi)$ for $\xi \in C_2(TX)$. The proof of Lemma 6.15 shows that $\mathrm{Ad}(e_\ell e_k) = 2(e_k {e_\ell}^* - e_\ell {e_k}^*)$ for $k \neq \ell$ and thus

$$\rho(e_k {e_\ell}^* - e_\ell {e_k}^*) = \frac{1}{2}\Gamma(e_\ell)\Gamma(e_k) = \frac{1}{2} a_{k\ell}$$

for $k \neq \ell$. Hence

$$\begin{aligned} F_{ij}^0 &= \rho(R(e_i, e_j)) \\ &= \sum_{k,\ell} R_{ijk\ell} \cdot \rho(e_k {e_\ell}^*) \\ &= \sum_{k<\ell} R_{ijk\ell} \cdot \rho(e_k {e_\ell}^* - e_\ell {e_k}^*) \\ &= \frac{1}{2} \sum_{k<\ell} R_{ijk\ell} \cdot a_{k\ell}. \end{aligned}$$

Now it follows from Lemma 6.20 below that

$$\sum_{i<j} a_{ij} F_{ij}^0 = \frac{1}{2} \sum_{i<j} \sum_{k<\ell} R_{ijk\ell} a_{ij} a_{k\ell} = \frac{1}{8} \sum_{i,j,k,\ell} R_{ijk\ell} a_{ij} a_{k\ell} = -\frac{1}{4} s\mathbb{1}.$$

This proves (6.13). Combining this with (6.12) gives

$$\begin{aligned} \mathcal{D}_A \mathcal{D}_A \Phi &= {\nabla_A}^* \nabla_A \Phi - \sum_{i<j} a_{ij} F_{ij} \Phi \\ &= {\nabla_A}^* \nabla_A \Phi - \sum_{i<j} a_{ij} F_{ij}^0 \Phi - \frac{1}{2^n} \sum_{i<j} a_{ij} \mathrm{trace}(F_{ij}) \Phi \\ &= {\nabla_A}^* \nabla_A \Phi + \frac{1}{4} s\Phi - \frac{1}{2^n} \sum_{i<j} a_{ij} \mathrm{trace}(F_{ij}) \Phi. \end{aligned}$$

Now the curvature of the virtual connection A is the 2-form

$$F_A = \frac{1}{2^n}\mathrm{trace}(F^\nabla) = \frac{1}{2^n}\sum_{i<j}\mathrm{trace}(F_{ij})e_i{}^* \wedge e_j{}^*.$$

Hence it follows from the definition of $\rho^+ : \Lambda^2 T^*X \to \mathrm{End}(W^+)$ in (6.11) that

$$\rho^+(F_A) = \frac{1}{2^n}\sum_{i<j}\mathrm{trace}(F_{ij})\Gamma(e_i)\Gamma(e_j) = -\frac{1}{2^n}\sum_{i<j} a_{ij}\mathrm{trace}(F_{ij}).$$

Hence

$$\begin{aligned}\mathcal{D}_A\mathcal{D}_A\Phi &= \nabla_A{}^*\nabla_A\Phi + \frac{1}{4}s\Phi - \frac{1}{2^n}\sum_{i<j} a_{ij}\mathrm{trace}(F_{ij})\Phi \\ &= \nabla_A{}^*\nabla_A\Phi + \frac{1}{4}s\Phi + \rho^+(F_A)\end{aligned}$$

and this proves the theorem. □

Lemma 6.20 *Let $R_{ijk\ell}$ denote the curvature coefficients in an orthonormal frame $e_1, \dots, e_{2n}$ of TX and assume that $a_{ij} \in \mathrm{End}(W)$ satisfy*

$$a_{ij}a_{jk} = a_{ik}, \qquad a_{ii} = \mathbb{1}, \qquad a_{ij}a_{k\ell} + a_{ik}a_{j\ell} = 2\delta_{jk}a_{i\ell} \tag{6.14}$$

for $i, j, k, \ell = 1 \dots, 2n$. Then

$$\sum_{i,j,k,\ell} R_{ijk\ell}a_{ij}a_{k\ell} = -2s\mathbb{1}$$

where $s = \sum_{i,j} R_{ijij}$ is the scalar curvature.

Proof: Equation (6.14) with $k = \ell = i$ implies $a_{ij} + a_{ji} = 2\delta_{ij}\mathbb{1}$. Moreover, recall from Section 2.1 that $R_{ijk\ell} = R_{k\ell ij} = -R_{jik\ell}$ and the Bianchi identity reads

$$R_{ijk\ell} + R_{ik\ell j} + R_{i\ell jk} = 0.$$

In particular, this shows that $R_{jijk} = R_{jkji}$ and hence

$$\sum_{i,j,k} R_{ijjk}a_{ik} = -\frac{1}{2}\sum_{i,j,k} R_{jijk}(a_{ik} + a_{ki}) = -\sum_{i,j,k} R_{jijk}\delta_{ik}\mathbb{1} = -s\mathbb{1}.$$

Abbreviating

$$a = \sum_{i,j,k,\ell} R_{ijk\ell}a_{ij}a_{k\ell}, \qquad b = \sum_{i,j,k,\ell} R_{ijk\ell}a_{ik}a_{j\ell}$$

one finds

$$a + b = 2 \sum_{i,j,k,\ell} R_{ijk\ell} \delta_{jk} a_{i\ell} = 2 \sum_{i,j,\ell} R_{ijj\ell} a_{i\ell} = -2s\mathbb{1}$$

and, by the Bianchi identity,

$$\begin{aligned}
a - 2b &= \sum_{i,j,k,\ell} R_{ijk\ell} a_{ij} a_{k\ell} - 2 \sum_{i,j,k,\ell} R_{ijk\ell} a_{ik} a_{j\ell} \\
&= - \sum_{i,j,k,\ell} R_{ijk\ell} \left(a_{ik} a_{\ell j} + a_{i\ell} a_{jk} \right) - 2 \sum_{i,j,k,\ell} R_{ijk\ell} a_{ik} a_{j\ell} \\
&= - \sum_{i,j,k,\ell} R_{ijk\ell} a_{ik} \left(a_{\ell j} + a_{j\ell} \right) \\
&= -2 \sum_{i,j,k} R_{ijkj} a_{ik} \\
&= - \sum_{i,j,k} R_{ijkj} (a_{ik} + a_{ki}) \\
&= -2 \sum_{i,j} R_{ijij} \mathbb{1} \\
&= -2s\mathbb{1}.
\end{aligned}$$

This implies $a = -2s\mathbb{1}$ and $b = 0$ as claimed. □

Recall from Exercise 2.31 a Weitzenböck formula for the Laplace-Beltrami operator on 1-forms which involves the Ricci tensor. Compare this with the formula in Theorem 6.19 which only involves the scalar curvature. To understand this consider the case of a symplectic 4-manifold X with the standard spin^c structure. Then $W^-_{\mathrm{can}} = \Lambda^{0,1}T^*X$ and $2^{-1/2}D_A$ is just the operator $\bar{\partial} + \bar{\partial}^*$ (see Theorem 6.17). The second formula in Theorem 6.19 becomes

$$\bar{\partial}^* \bar{\partial} \alpha + \bar{\partial} \bar{\partial}^* \alpha = \frac{1}{2} \widetilde{\nabla}^* \widetilde{\nabla} \alpha + \frac{1}{8} s\alpha + \frac{1}{2} \rho^-(\widetilde{F}) \alpha$$

for $\alpha \in \Omega^{0,1}(X)$. Here $\widetilde{\nabla}$ denotes the Hermitian connection on TX and $\widetilde{F} = 2F_{A_{\mathrm{can}}}$ denotes the complex trace of the curvature. The sum of the last two terms in this formula correspond to the Ricci term in Exercise 2.31. An interesting special case is that of a spin structure where the line bundle L_Γ is trivial and A is the zero connection. Then the last term in the Weitzenböck formula of Theorem 6.19 is zero and only the scalar curvature remains. Thus, as is shown in Section 6.6 below, the Dirac operator can be used in the spin case, in conjunction with the Atiyah-Singer index theorem, to obtain interesting obstructions to the existence of metrics with positive scalar curvature. Such obstructions cannot be obtained from the formula in Exercise 2.31 because the full Ricci tensor appears.

6.5 The Fredholm index

The Weitzenböck formula is a powerful tool for studying the Dirac operator and its relation with the geometry of the underlying manifold. In the first place it can be used to prove that the Dirac operator on a compact manifold is a Fredholm operator. It is a first order differential operator and can naturally be considered as an operator between Hilbert spaces

$$\mathcal{D}_A : W^{1,2}(X, W) \to L^2(X, W).$$

Here $L^2(X, E)$ denotes the space of L^2-sections of a vector bundle $E \to X$ with norm

$$\|\Phi\|_{L^2} = \sqrt{\int_X |\Phi|^2 \mathrm{dvol}}.$$

The Sobolev space $W^{1,2}(X, E)$ is defined as the completion of the space $C^\infty(X, E)$ with respect to the norm

$$\|\Phi\|_{W^{1,2}} = \sqrt{\int_X \left(|\Phi|^2 + |\nabla\Phi|^2\right) \mathrm{dvol}}.$$

Here $\nabla\Phi \in \Omega^1(X, E)$ denotes the covariant derivative of Φ with respect to some connection on E. Thus the $W^{1,2}$-norm depends on a choice of a connection, however, the space of $W^{1,2}$-sections of E is independent of this choice.

Proposition 6.21 *The Dirac operator $\mathcal{D}_A : W^{1,2}(X, W) \to L^2(X, W)$ is a Fredholm operator.*

Proof: The Weitzenböck formula of Theorem 6.19 shows that

$$\begin{aligned}
\|\mathcal{D}_A\Phi\|_{L^2}^2 &= \langle \Phi, \mathcal{D}_A\mathcal{D}_A\Phi \rangle_{L^2} \\
&= \left\langle \Phi, \nabla_A{}^*\nabla_A s + \frac{1}{4} s\Phi + \sigma^+(F_A)\Phi \right\rangle_{L^2} \\
&= \|\nabla_A\Phi\|_{L^2}^2 + \int_X \left(\frac{1}{4} s|\Phi|^2 + \langle \Phi, \sigma^+(F_A)\Phi \rangle \right) \mathrm{dvol} \\
&\geq \|\nabla_A\Phi\|_{L^2}^2 - c\,\|\Phi\|_{L^2}^2
\end{aligned}$$

for some constant $c > 0$ which is independent of Φ. Hence

$$\|\Phi\|_{W^{1,2}}^2 \leq \|\mathcal{D}_A\Phi\|_{L^2}^2 + (1 + c)\,\|\Phi\|_{L^2}^2$$

for every $\Phi \in C^\infty(X, W)$. Now the inclusion $W^{1,2}(X, W) \hookrightarrow L^2(X, W)$ is a compact operator, by Rellich's theorem, and hence it follows from

Lemma A.1 that $\mathcal{D}_A$ has a finite dimensional kernel and a closed range. Moreover, by elliptic regularity, the orthogonal complement of the image of $\mathcal{D}_A$ is contained in $W^{1,2}(X, W)$ and agrees with the kernel of the formal adjoint operator $\mathcal{D}_A{}^* = \mathcal{D}_A$ and hence is also finite dimensional. Hence $\mathcal{D}_A$ is a Fredholm operator. □

The Dirac operator $\mathcal{D}_A$ is always self-adjoint and hence has Fredholm index zero. However, in the even dimensional case, the index of the operator

$$D_A = \mathcal{D}_A{}^+ : W^{1,2}(X, W^+) \to L^2(X, W^-)$$

is an interesting topological invariant, given by the Atiyah-Singer index theorem. Here is the answer in the 4-dimensional case.

Theorem 6.22. (Atiyah-Singer) *Let X be a compact smooth 4-manifold with a spin^c structure $\Gamma : TX \to \mathrm{End}(W)$ and associated line bundle L_Γ. Then the real Fredholm index of the Dirac operator D_A is given by*

$$\mathrm{index}\, D_A = \frac{\langle c_1(L_\Gamma)^2, [X]\rangle - \sigma(X)}{4}.$$

Consider the Kähler case with the canonical spin^c structure twisted by a holomorphic line bundle $E \to X$. Then Theorem 6.17 shows that the Dirac operator D_A agrees with the Cauchy-Riemann operator $\bar\partial + \bar\partial^*$ and so its kernel and cokernel are given by

$$\ker\, D_A \cong H^{0,\mathrm{ev}}(X, E), \qquad \mathrm{coker}\, D_A \cong H^{0,\mathrm{odd}}(X, E).$$

Hence the real Fredholm index of D_A is twice the twisted holomorphic Euler characteristic

$$\mathrm{index}\, D_A = 2\chi(X, E) = \frac{\langle c_1(K^* \otimes E^{\otimes 2})^2, [X]\rangle - \sigma(X)}{4}$$

where $K = \Lambda^{2,0}T^*X$. The last identity follows from Corollary 3.43. Now recall from Lemma 5.20 and Corollary 5.21 that the line bundle L_Γ is given by $L_\Gamma = K^* \otimes E^{\otimes 2}$. Thus the index formula of Theorem 6.22 agrees with the Hirzebruch-Riemann-Roch formula.

Another interesting special case is that of a **spin structure** on a manifold X of dimension $4k$. Recall from Definition 5.1 that, when k is odd, a spin structure on X is a quadruple (S, I, J, Γ) where I and J are anti-commuting orthogonal complex structures on S and $\Gamma : TX \to \mathrm{End}(S)$ satisfies (4.18) and commutes with I and J. When k is even, a spin structure is a quadruple (S, I, T, Γ) where S, I, Γ are as above and T is an involution of S which anti-commutes with I and commutes with Γ. In either case the triple (S, I, Γ) is a spin^c structure whose canonical line bundle

L_Γ carries a natural trivialization. (See Exercise 5.6 and Lemma 5.7.) Recall from Lemma 6.6 that the Levi-Civita connection on X determines a unique spin connection on S and that this connection preserves the subbundles S^+ and S^-. Let $E \to X$ be any Hermitian vector bundle over X with connection A. Together with the spin connection on S this determines a connection $\nabla = \nabla_A$ on $S \otimes E$ and hence there is a twisted Dirac operator $D_A : C^\infty(X, S^+ \otimes E) \to C^\infty(X, S^- \otimes E)$.

Theorem 6.23. (Atiyah-Singer) *Let X be a spin manifold of dimension $4k$. Then the complex Fredholm index of the Dirac operator $D_A : C^\infty(X, S^+ \otimes E) \to C^\infty(X, S^- \otimes E)$ is given by*

$$\operatorname{index}^c D_A = \int_X \operatorname{ch}(E) \wedge \widehat{A}(TX)$$

where $\widehat{A}(TX) \in H^*(X;\mathbb{Z})$ denotes the $\widehat{A}$-genus. If the dimension of X is not divisible by 4 then $\operatorname{index} D_A = 0$.

The $\widehat{A}$-genus of a real vector bundle $E \to X$ of rank $2m$ is defined as the formal power series

$$\widehat{A}(E) = \prod_{i=1}^{m} \frac{x_i}{e^{x_i/2} - e^{-x_i/2}}$$

where the x_i are to be understood as formal variables in $H^2(X;\mathbb{Z})$. They are related to the Pontryagin classes

$$p_j(E) = (-1)^j c_{2j}(E \otimes_{\mathbb{R}} \mathbb{C}) \in H^{4j}(X;\mathbb{Z})$$

by the formula

$$p(E) = 1 + p_1(E) + \cdots + p_m(E) = \prod_{i=1}^{m} (1 + {x_i}^2).$$

Thus the classes $p_j(E) \in H^{4j}(X;\mathbb{Z})$ can be expressed as the elementary symmetric functions in the variables ${x_i}^2$. A moment's thought shows that the above power series for $\widehat{A}(E)$ is a symmetric function in the ${x_i}^2$ and hence can be expressed as a function of the Pontryagin classes. Consider the special case where E itself is a complex vector bundle. Then $E \otimes_{\mathbb{R}} \mathbb{C} \cong E \oplus \bar{E}$ where $\bar{E}$ denotes the bundle with the reversed complex structure. If E decomposes as a direct sum of line bundles $E = L_1 \oplus \cdots \oplus L_m$ then the total Chern class of $E \oplus \bar{E}$ is given by $c(E \otimes \bar{E}) = \prod_i (1 - {x_i}^2)$ and so the variables x_i represent the first Chern classes of the bundles L_i as in Section 3.6. Now consider the **canonical bundle**

$$K = \Lambda^{m,0}E^*.$$

If this bundle admits a square root $K^{1/2}$ then the Chern character of this square root is given by

$$\mathrm{ch}(K^{1/2}) = \prod_{i=1}^{m} e^{-x_i/2}.$$

Multiplying this class with the Todd class

$$\mathrm{td}(E) = \prod_{i=1}^{m} \frac{x_i}{1 - e^{-x_i}}$$

gives the $\widehat{A}$-genus. Thus

$$\widehat{A}(E) = \mathrm{ch}(K^{1/2}) \wedge \mathrm{td}(E) \tag{6.15}$$

for any complex vector bundle E. It is not obvious that the $\widehat{A}$-genus of the tangent bundle of a $4k$-dimensional spin manifold is an integral cohomology class. This, however, follows from the index theorem 6.23.

There is a natural generalization of Theorem 6.23 to families of Dirac operators. Assume as before that X is a $4k$-dimensional spin manifold with spin structure $\Gamma : TX \to \mathrm{End}(S)$. Let Z be a finite dimensional compact manifold (the parameter space) and

$$\mathbb{E} \to X \times Z$$

be a complex vector bundle with a Hermitian structure. Suppose that for each $z \in Z$ the pullback bundle $E_z = {\iota_z}^*\mathbb{E}$ under the obvious inclusion $\iota_z : X \to X \times Z$ is equipped with a connection A_z which varies continuously with z. Then there is a family of Dirac operators

$$D_{A_z} : C^\infty(X, S^+ \otimes E_z) \to C^\infty(X, S^- \otimes E_z).$$

Any such family of Fredholm operators determines a K-theory class

$$\mathcal{IND}(D_A) = \ker\, D_A \ominus \mathrm{coker}\, D_A \in K(Z).$$

This formula can be interpreted literally when the kernel and cokernel of the operators D_{A_z} are of constant dimension and hence form complex vector bundles over Z. (See Section 1.7 for the notation $\ominus$.) In general one has to stabilize to make sense of the index as a class in K-theory (see Section A.1). The Atiyah-Singer index theorem for families gives a formula for the Chern character of $\mathcal{IND}(D_A)$ which reduces to Theorem 6.23 when Z is a point.

Theorem 6.24. (Atiyah-Singer) *Let $\mathbb{E} \to X \times Z$ and D_{A_z} be as above. Then*

$$\mathrm{ch}(\mathcal{IND}(D_A)) = \int_X \mathrm{ch}(\mathbb{E}) \wedge \widehat{A}(TX) \in H^*(Z)$$

where the right hand is to be understood as integration over the fiber.

Let $\Gamma : X \to \mathrm{End}(W)$ be a spin^c structure with characteristic line bundle $L = L_\Gamma$ where X has dimension $4k$. Assume that A is a spin^c connection on W with corresponding Dirac operator D_A. As before let $\mathbb{E} \to X \times Z$ be a complex vector bundle equipped with a family of connections B_z on $E_z = \iota_z{}^*\mathbb{E}$ for $z \in Z$. Then there is a family of spin^c Dirac operators

$$D_{A+B_z} : C^\infty(X, W^+ \otimes E_z) \to C^\infty(X, W^- \otimes E_z)$$

with corresponding topological index $\mathcal{IND}(D_{A+B}) \in K(Z)$. If X admits a spin structure then Theorem 6.24 asserts that

$$\mathrm{ch}(\mathcal{IND}(D_{A+B})) = \int_X \mathrm{ch}(L^{1/2}) \wedge \mathrm{ch}(\mathbb{E}) \wedge \widehat{A}(TX) \in H^*(Z). \qquad (6.16)$$

This formula continues to hold when X does not admit a spin structure.

Remark 6.25 Let X be a smooth compact oriented Riemannian 4-manifold with a spin structure (S, I, J, Γ). Think of (S, I) as a complex vector bundle. Then any spin^c structure on X can be obtained from S by twisting with a line bundle $L^{1/2} \to X$, namely. $W = S \otimes L^{1/2}$. The canonical line bundle associated to this spin^c structure is L itself. In the 4-dimensional case direct computation shows that the $\widehat{A}$-genus of TX and the Chern character of $L^{1/2}$ are given by

$$\widehat{A}(TX) = 1 - \frac{1}{24} p_1(TX), \qquad \mathrm{ch}(L^{1/2}) = 1 + \frac{1}{2} c_1(L) + \frac{1}{8} c_1(L)^2.$$

Hence the Hirzebruch signature formula shows that

$$2 \int_X \mathrm{ch}(L^{1/2}) \wedge \widehat{A}(TX) = \frac{c_1(L)^2 - \sigma(X)}{4}.$$

The left hand side is the index formula of Theorem 6.23 while the right hand side is the index formula of Theorem 6.22. Thus the two formulae are consistent. In the case $c_1(L) = 0$ this computation shows that the $\widehat{A}$-genus of a smooth 4-manifold is given by

$$\int_X \widehat{A}(TX) = -\frac{1}{8} \sigma(X).$$

In the spin case the $\widehat{A}$-genus is an integer because, for purely algebraic reasons, the signature of an even unimodular quadratic form over the integers is divisible by 8. (Theorem 6.27 below shows in fact that it is divisible by 16.) However, $\widehat{A}(TX)$ may not be an integral class when X is not spin. $\square$

Remark 6.26 Let X be a complex spin manifold. Then its canonical bundle $K = \Lambda^{n,0}T^*X$ has a square root. In this case the formula (6.15) for the tangent bundle of X is consistent with the Hirzebruch-Riemann-Roch theorem 3.42. More precisely, the Cauchy-Riemann operator

$$D = \bar{\partial} + \bar{\partial}^* : \Omega^{0,\mathrm{ev}}(X, E) \to \Omega^{0,\mathrm{odd}}(X, E)$$

agrees with the Dirac operator corresponding to the spin^c representation

$$W_E = W_{\mathrm{can}} \otimes E = S \otimes K^{-1/2} \otimes E$$

where $W_{\mathrm{can}} = \Lambda^{0,*}T^*X$ is the canonical spin^c structure. Since $L_{\Gamma_{\mathrm{can}}} = K^*$ it follows that

$$S = W_{\mathrm{can}} \otimes K^{1/2}$$

is a spin structure on X and so, by Theorem 6.23,

$$\operatorname{index} D = 2 \int_X \mathrm{ch}(L^{1/2}) \wedge \widehat{A}(TX),$$

where

$$L^{1/2} = K^{-1/2} \otimes E.$$

Comparing this with the Hirzebruch-Riemann-Roch theorem one obtains

$$\int_X \mathrm{ch}(L^{1/2}) \wedge \widehat{A}(TX) = \int_X \mathrm{ch}(E) \wedge \mathrm{td}(TX),$$

for every complex vector bundle $E \to X$. This is consistent with (6.15). $\square$

Theorem 6.27. (Rohlin) *If X is a compact smooth 4 manifold which admits a spin structure then its signature is divisible by 16.*

Proof: Let (S, I, J, Γ) be a spin structure on TX as in Definition 5.4 Let ∇ be the corresponding spin connection on S defined in Lemma 6.6 and $D : C^\infty(X, S^+) \to C^\infty(X, S^-)$ be the associated Dirac operator. Since the line bundle L_Γ admits a trivialization it follows from Theorem 6.22 that this operator has Fredholm index

$$\operatorname{index} D = -\frac{1}{4}\sigma(X).$$

Moreover, by Lemma 6.6, the Dirac operator commutes with the action of $\mathbb{H}$ via $i \mapsto I$, $j \mapsto J$, and $k \mapsto K = IJ$. Hence $\mathbb{H}$ acts on the kernel and cokernel of D and this implies that the real Fredholm index is divisible by 4 (see Exercise 4.44). $\square$

Example 6.28 The hypersurface $X_d \subset \mathbb{C}P^3$ given by

$$X_d = \{z_0^d + z_1^d + z_2^d + z_3^d = 0\} \subset \mathbb{C}P^3$$

carries an involution $\tau : X_d \to X_d$ given by complex conjugation. If d is even then this map has no fixed points and hence determines a free $\mathbb{Z}_2$ action. The quotient $Z = X_4/\mathbb{Z}_2$ is called the **Enriques surface**. It has fundamental group $\pi_1(Z) = \mathbb{Z}_2$ and intersection form

$$Q_Z = E_8 \oplus H.$$

Hence $\sigma(Z) = -8$ and, by Theorem 6.27, this manifold is not spin. □

Exercise 6.29 Find a nonorientable 2-dimensional submanifold of the Enriques surface with mod-2 self-intersection number 1. □

6.6 Metrics with positive scalar curvature

Consider the trivial bundle $E = X \times \mathbb{C}$ and a spin structure $\Gamma : X \to \mathrm{End}(S)$ with the canonical spin connection. Then the Chern character of E is 1 and thus the index of the Dirac operator is

$$\mathrm{index}\, D = 2 \int_X \widehat{A}(TX).$$

This gives rise to the following theorem due to Lichnerowicz [75].

Theorem 6.30. (Lichnerowicz) *Let X be a compact spin manifold of dimension $4k$ and assume that X admits a metric of positive scalar curvature. Then*

$$\int_X \widehat{A}(TX) = 0.$$

Proof: The Weitzenböck formula for the Dirac operator $D : C^\infty(X, S^+) \to C^\infty(X, S^-)$ associated to a spin structure $\Gamma : TX \to \mathrm{End}(S)$ and a flat connection on the trivial bundle $L_\Gamma \cong X \times \mathbb{C}$ reads

$$D^* D\Phi = \nabla^* \nabla \Phi + \frac{1}{4} s\Phi$$

for $\Phi \in C^\infty(X, S^+)$ and hence

$$\|D\Phi\|_{L^2}^2 = \|\nabla\Phi\|_{L^2}^2 + \frac{1}{4}\int_X s|\Phi|^2 \mathrm{dvol}.$$

Since the scalar curvature s is positive it follows that $\ker D = \{0\}$. The same formula holds for the adjoint operator D^* and hence $\ker D^* = \{0\}$. this proves that D must be bijective and hence $\mathrm{index}\, D = 0$. Now the assertion follows from Theorem 6.23. □

The condition that X be a spin manifold cannot be removed in Theorem 6.30. The manifold $\mathbb{C}P^n$, for example, has nonzero $\widehat{A}$-genus but admits a metric of positive scalar curvature. Of course, $\mathbb{C}P^n$ does not admit a spin structure. The $K3$-surface $X_4 \subset \mathbb{C}P^3$, on the other hand, does admit a spin structure and has $\widehat{A}$-genus

$$\int_{X_4} \widehat{A}(TX_4) = -\frac{1}{8}\sigma(X_4) = 2.$$

Hence for any metric the Dirac operator must have a nontrivial kernel and so the $K3$-surface does not admit a metric of positive scalar curvature. A similar example is given by a hypersurfaces $X_d \subset \mathbb{C}P^3$ of even degree

$$d = 2k \geq 4.$$

This manifold has $\widehat{A}$-genus

$$\int_{X_d} \widehat{A}(TX_d) = -\frac{1}{8}\sigma(X_d) = \frac{(d^2-4)d}{24} > 0.$$

(See Proposition 3.66.) If d is even then X_d is spin and so, by Theorem 6.30, it does not admit a metric of positive scalar curvature.

In the late 70's Schoen and Yau proved, using minimal surfaces, that the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ does not admit a metric of positive scalar curvature for $n \leq 7$. Note, however, that the torus has $\widehat{A}$-genus zero and so Theorem 6.30 does not apply. In [48] Gromov and Lawson refined the techniques of Lichnerowicz to prove that, for any n, the n-torus does not admit a metric of positive scalar curvature. In fact, they proved that for any compact spin manifold Y of dimension n the connected sum

$$X = \mathbb{T}^n \# Y$$

does not admit a metric of positive scalar curvature. Moreover, they proved that if X admits a metric of nonnegative scalar curvature then this metric must be flat and X must be the standard n-torus.

In the odd case the hypersurface $X_d \subset \mathbb{C}P^3$ of degree d is not a spin manifold and hence in this case Theorem 6.30 does not apply. This is where spin^c structures come in and the Seiberg-Witten invariants can be used to show that, in the odd case, the hypersurface X_d does not admit a metric of positive scalar curvature for $d \geq 5$. Note that

$$X_3 \cong \mathbb{C}P^2 \# 6\overline{\mathbb{C}P}^2$$

and so, by Theorem 2.18, this manifold admits a metric of positive scalar curvature.

Part III

SEIBERG-WITTEN INVARIANTS

7

SEIBERG-WITTEN INVARIANTS OF FOUR-MANIFOLDS

The goal of this chapter is to discuss the Seiberg-Witten monopole equations, and to show how these give rise to invariants of smooth 4-manifold. The more technical parts in the proofs of the compactness and transversality theorems are deferred to the Chapter 8. Section 7.1 give an introduction to the Seiberg-Witten equations in dimension four. Section 7.2 lays the foundations for the constructions of the invariants with the discussion of the moduli spaces. It outlines the proofs of the compactness and regularity theorems, and the proofs that the moduli spaces form orientable smooth finite dimensional manifolds. Section 7.3 deals with cobordisms. The Seiberg-Witten invariants are discussed in Section 7.4. The final section discusses some basic properties of the Seiberg-Witten invariants of 4-manifolds such as finiteness (the invariant is nonzero for only finitely many isomorphism classes of spin^c structures), the behaviour under complex conjugation, the vanishing in the case of positive scalar curvature, LeBrun's generalization of the Miyaoka-Yau inequality, and Witten's fundamental conjecture about the relation between the Seiberg-Witten and the Donaldson invariants.

7.1 The Seiberg-Witten equations in dimension four

Let X be a compact connected oriented smooth 4-manifold and fix a spin^c structure $\Gamma : TX \to \mathrm{End}(W)$ (see Definition 5.3). Recall that there is a natural splitting

$$W = W^+ \oplus W^-$$

(see Section 4.4) and that the characteristic line bundle L_Γ is, in the 4-dimensional case, given by

$$L_\Gamma \cong \det(W^+) \cong \det(W^-).$$

(see (5.4)). As in Section 6.1 denote by $\mathcal{A}(\Gamma)$ the space of *virtual connections* on the *virtual line bundle* ${L_\Gamma}^{1/2}$. For $A \in \mathcal{A}(\Gamma)$, denote the corresponding spin^c connection by $\nabla_A : C^\infty(X, W) \to \Omega^1(X, W)$ and the corresponding Dirac operator by $D_A : C^\infty(X, W^+) \to C^\infty(X, W^-)$ (see Section 6.2). Recall that the curvature of A is a scalar 2-form

$$F_A = \frac{1}{4}\mathrm{trace}^c(F^{\nabla_A}) \in \Omega^2(X, i\mathbb{R}),$$

and that the 2-form iF_A/π represents the first Chern class of L_Γ.

The Seiberg-Witten monopole equations are a system of first order differential equations for a pair (A, Φ) where $A \in \mathcal{A}(\Gamma)$ and $\Phi \in C^\infty(X, W^+)$. They read

$$D_A\Phi = 0, \qquad F_A^+ = \sigma^+((\Phi\Phi^*)_0). \tag{7.1}$$

In the physics terminology F_A is the field strength and Φ is a **monopole** coupled to the dual gauge field A. Here $\Phi\Phi^* \in C^\infty(X, \mathrm{End}(W^+))$ is defined by $\Phi\Phi^*\tau = \langle\Phi, \tau\rangle\Phi$ for $\tau \in C^\infty(X, W^+)$. Its traceless part is given by

$$(\Phi\Phi^*)_0\tau = \langle\Phi, \tau\rangle\Phi - \frac{1}{2}|\Phi|^2\tau.$$

Let $\mathrm{End}_0(W^+)$ denote the bundle of traceless endomorphisms of W^+. The bundle isomorphism

$$\sigma^+ : \mathrm{End}_0(W^+) \to \Lambda^{2,+}T^*X \otimes \mathbb{C}$$

is the inverse of the map $\rho^+ : \Lambda^{2,+}T^*X \otimes \mathbb{C} \to \mathrm{End}_0(W^+)$ defined by (4.39) in Section 4.8. Recall from Lemma 4.55 that, in the 4-dimensional case, ρ^+ identifies the imaginary valued self-dual 2-forms on X with the traceless Hermitian endomorphisms of W^+ (and the real valued forms with the traceless skew-Hermitian endomorphisms). Thus $\sigma^+((\Phi\Phi^*)_0)$ is an imaginary valued self-dual 2-form and so is F_A^+. Sometimes it is useful to write the second equation in (7.1) in the form

$$\rho^+(F_A) = (\Phi\Phi^*)_0.$$

This can be read as an equation in the (real rank 3) bundle of traceless Hermitian endomorphisms of W^+. Note that the term $\rho^+(F_A)$ appears in the Weitzenböck formula. The reader who is interested in an explicit discussion of the monopoles on flat Euclidean 4-space may wish to consult Section 8.1 in the next chapter.

Remark 7.1 The Seiberg-Witten equations make sense on manifolds X of any even dimension if the second equation is written in the form $\rho^+(F_A) = (\Phi\Phi^*)_0$. However, they have interesting consequences only in dimension 4. The reason lies in the fact that $\rho^+(F_A)$ depends only on F_A^+ in the 4-dimensional case. In dimension $2n \geq 6$ the equation $\rho^+(F_A) = 0$ is equivalent to $F_A = 0$, but the bundle L_Γ does not carry any flat connections unless its Chern class is torsion. In other words, the equations are *overdetermined* in dimensions bigger than 4. If one assumes, however, that the manifold X carries some additional structure such as a Kähler structure or a symplectic form, then it may well be possible that a suitable modification of the Seiberg-Witten equations gives rise to interesting new invariants in higher dimensions. □

Vanishing

A first observation is that the Seiberg-Witten equations have no nontrivial solutions whenever X is a 4-manifold with positive scalar curvature. This is a kind of nonlinear analogue of Licherovicz' theorem 6.30.

Lemma 7.2 *If X is a Riemannian 4-manifold with nonnegative scalar curvature then $\Phi = 0$ for every solution of (7.1).*

Proof: Let (A, Φ) be a solution of (7.1). Then, by the Weitzenböck formula,

$$0 = D_A{}^* D_A \Phi = \nabla_A{}^* \nabla_A \Phi + \frac{s}{4}\Phi + \rho^+(F_A)\Phi = \nabla_A{}^* \nabla_A \Phi + \frac{s}{4}\Phi + \frac{1}{2}|\Phi|^2\Phi$$

Taking the L^2-inner product with Φ we obtain

$$0 = \int_X \left(|\nabla_A \Phi|^2 + \frac{s}{4} |\Phi|^2 + \frac{1}{2} |\Phi|^4 \right) \mathrm{dvol}.$$

Hence $\Phi = 0$. □

Energy

The **energy** or **action** of a pair $(A, \Phi) \in \mathcal{A}(\Gamma) \times C^\infty(X, W^+)$ is defined by

$$E(A, \Phi) = \int_X \left(|\nabla_A \Phi|^2 + \frac{s}{4}|\Phi|^2 + \frac{1}{4}|\Phi|^4 + |F_A|^2 \right) \mathrm{dvol}. \tag{7.2}$$

This integral is not necessarily positive because the scalar curvature term may be negative. However, the following proposition shows that the action integral has a universal lower bound which is attained by the solutions of the Seiberg-Witten (7.1) equations if such solutions exist.

Proposition 7.3 *The energy satisfies*

$$E(A, \Phi) = \int_X \left(|D_A \Phi|^2 + 2 \left| F_A^+ - \sigma^+((\Phi\Phi^*)_0) \right|^2 \right) - \pi^2 \langle c_1(L_\Gamma)^2, [X] \rangle.$$

As in Lemma 3.2 we define the norm on $\mathrm{End}(W^\pm)$ as half the trace. The proof of Proposition 7.3 relies on the following rules.

Lemma 7.4 *Let $\Gamma : TX \to \mathrm{End}(W)$ be a spin^c structure on a compact oriented smooth 4-manifold. Then the following holds for $\eta \in \Omega^{2,+}(X, \mathbb{C})$, $T \in C^\infty(X, \mathrm{End}_0(W^+))$, and $\Phi \in C^\infty(X, W^+)$.*

$$\left|\rho^+(\eta)\right|^2 = 2 \left|\eta^+\right|^2, \qquad \left|\sigma^+(T)\right|^2 = \frac{1}{2} |T|^2,$$

$$|(\Phi\Phi^*)_0|^2 = \frac{1}{4} |\Phi|^4, \qquad \langle T, (\Phi\Phi^*)_0 \rangle = \frac{1}{2} \langle T\Phi, \Phi \rangle.$$

Proof: Any 2-form $\eta = \sum_{i<j} \eta_{ij} e_i{}^* \wedge e_j{}^* \in \Omega^2(X, \mathbb{C})$ satisfies

$$\begin{aligned}
|\rho(\eta)|^2 &= |\sum_{i<j} \eta_{ij} \Gamma(e_i)\Gamma(e_j)|^2 \\
&= \frac{1}{2} \mathrm{trace} \left(\sum_{i<j} \sum_{k<\ell} \bar{\eta}_{ij} \eta_{k\ell} \Gamma(e_j)\Gamma(e_i)\Gamma(e_k)\Gamma(e_\ell) \right) \\
&= \frac{1}{2} \mathrm{trace} \left(\sum_{i<j} |\eta_{ij}|^2 \mathbb{1} \right) \\
&= 2|\eta|^2.
\end{aligned}$$

The last equation uses the fact that here $\mathbb{1}$ denotes the identity on a 4-dimensional complex vector space. Hence $|\rho^+(\eta)|^2 = |\rho(\eta^+)|^2 = 2|\eta^+|^2$ and this proves the first two assertions. The remaining assertions are statements about complex vector spaces and follow from Lemma 3.2. □

Proof of Proposition 7.3: Any 2-form $\eta \in \Omega^2(X, i\mathbb{R})$ satisfies $\|\eta\|^2 = -\int_X \eta \wedge *\eta$ where $\|.\|$ denotes the L^2-norm. This implies

$$\left\|F_A^+\right\|^2 = -\int_X F_A^+ \wedge F_A^+, \qquad \left\|F_A^-\right\|^2 = \int_X F_A^- \wedge F_A^-$$

and hence

$$\begin{aligned}
2\left\|F_A^+\right\|^2 - \|F_A\|^2 &= \left\|F_A^+\right\|^2 - \left\|F_A^-\right\|^2 \\
&= -\int_X F_A^+ \wedge F_A^+ - \int_X F_A^- \wedge F_A^- \\
&= -\int_X F_A \wedge F_A \\
&= \pi^2 \langle c_1(L_\Gamma)^2, [X] \rangle.
\end{aligned}$$

The last equality uses the fact that the 2-form $(i/\pi)F_A$ represents the first Chern class of the line bundle L_Γ. Now Lemma 7.4 shows that

$$\begin{aligned}
2\left|F_A^+ - \sigma^+((\Phi\Phi^*)_0)\right|^2 &= 2\left|F_A^+\right|^2 + 2\left|\sigma^+((\Phi\Phi^*)_0)\right|^2 \\
&\quad - 4\left\langle F_A^+, \sigma^+((\Phi\Phi^*)_0)\right\rangle \\
&= 2\left|F_A^+\right|^2 + \frac{1}{4}|\Phi|^4 - 2\left\langle \rho^+(F_A), (\Phi\Phi^*)_0 \right\rangle \\
&= 2\left|F_A^+\right|^2 + \frac{1}{4}|\Phi|^4 - \left\langle \Phi, \rho^+(F_A)\Phi \right\rangle.
\end{aligned}$$

Note here that since $\rho^+(F_A)$ and $(\Phi\Phi^*)_0$ are both Hermitian endomorphisms their Hermitian inner product is real. The last two equations together give

$$\begin{aligned}
2\left\|F_A^+ - \sigma^+((\Phi\Phi^*)_0)\right\|^2 - \pi^2\langle c_1(L_\Gamma)^2, [X]\rangle \\
&= \int_X \left(2\left|F_A^+\right|^2 + \frac{1}{4}|\Phi|^4 - \left\langle \Phi, \rho^+(F_A)\Phi\right\rangle\right)\mathrm{dvol} - \pi^2\langle c_1(L_\Gamma)^2, [X]\rangle \\
&= \int_X \left(|F_A|^2 + \frac{1}{4}|\Phi|^4 - \left\langle \Phi, \sigma^+(F_A)\Phi\right\rangle\right)\mathrm{dvol}.
\end{aligned}$$

Now the Weitzenböck formula shows that

$$\|D_A\Phi\|^2 = \|\nabla_A\Phi\|^2 + \int_X \frac{s}{4}|\Phi|^2\mathrm{dvol} + \left\langle \Phi, \sigma^+(F_A)\Phi\right\rangle_{L^2}$$

where $\|.\|$ denotes the L^2-norm. Take the sum with the previous identity to obtain the required formula for the action. □

Remark 7.5 The action integral has another universal lower bound which depends only on the metric g, namely

$$E(A,\Phi) \geq \frac{1}{4}\int_X (s|\Phi|^2 + |\Phi|^4)\mathrm{dvol} \geq -\frac{1}{16}\int_X s^2\mathrm{dvol}.$$

By Proposition 7.3, this shows that the Seiberg-Witten equations (7.1) can only have a solution (A,Φ) if

$$\langle c_1(L_\Gamma)^2, [X]\rangle \leq \frac{1}{16\pi^2}\int_X s^2\mathrm{dvol}.$$

This is a rather crude estimate which can only be attained if $F_A = 0$, $\nabla_A\Phi = 0$, and $2|\Phi|^2 + s = 0$. □

In the physics terminology the integrand of the action integral (7.2) is the *Lagrangian* and plays a more fundamental role than the equations (7.1). This Lagrangian led Seiberg and Witten to the discovery of their monopole equations.

Exercise 7.6 The action integral (7.2) makes sense for any section of the spinor bundle. Prove that the restriction of E to $\mathcal{A}(\Gamma) \times C^\infty(X, W^-)$ is minimized by the solutions of the *negative Seiberg-Witten equations*

$$D_A{}^*\Psi = 0, \qquad F_A^- = \sigma^-((\Psi\Psi^*)_0). \tag{7.3}$$

Prove that changing the orientation of X interchanges (7.1) and (7.3). □

Perturbations

The solutions of the Seiberg-Witten equations can be used to define invariants of 4-manifolds. The basic idea is quite similar to the way in which the degree of a map $f : X \to Y$ between compact manifolds of the same dimension can be defined by counting the preimages of a regular value. The solutions of the Seiberg-Witten equations can be thought of as the zeros of a map between Banach manifolds. Roughly speaking, the number of zeros, counted with appropriate signs, determine an invariant of the underlying 4-manifold. To make this idea work one has to choose the number of preimages of a regular value, rather than the zeros, if zero is not a regular value. This can be reformulated as a perturbation of the Seiberg-Witten equations. Fix a self-dual 2-form $\eta \in \Omega^{2,+}(X, i\mathbb{R})$ and consider the perturbed equation

$$D_A\Phi = 0, \qquad F_A^+ + \eta = \sigma^+((\Phi\Phi^*)_0). \tag{7.4}$$

The solutions of these equations minimize the action

$$E(A,\Phi;\eta) = \int_X \left(|\nabla_A\Phi|^2 + \frac{s}{4}|\Phi|^2 + 2|\sigma^+((\Phi\Phi^*)_0) - \eta|^2 + |F_A + 2\eta|^2\right).$$

Exercise 7.7 Prove the energy identity

$$\begin{aligned} E_\eta(A,\Phi) &= \int_X \left(|D_A\Phi|^2 + 2\left|F_A^+ + \eta - \sigma^+((\Phi\Phi^*)_0)\right|^2\right)\mathrm{dvol} \\ &\quad + 4\int_X |\eta|^2\mathrm{dvol} - \pi^2\langle c_1(L_\Gamma)^2, [X]\rangle. \end{aligned}$$

Deduce that the solutions of (7.4) minimize the action E. Moreover, prove that

$$E_\eta(A,\Phi) \geq -\frac{1}{16}\int_X \left(4\sqrt{2}|\eta| - s\right)^2 \mathrm{dvol}$$

for every pair $(A,\Phi) \in \mathcal{A}(\Gamma) \times C^\infty(X, W^+)$. □

Scale invariance

The next proposition shows how the solutions of the Seiberg-Witten equations behave under rescaling of the metric by a constant factor. Denote by $E(A,\Phi;g,\eta)$ the action functional with respect to the metric g.

Proposition 7.8 *The pair (A,Φ) satisfies (7.4) with the metric g if and only if the pair $(A,\lambda^{-1}\Phi)$ satisfies (7.4) with the metric $\lambda^2 g$. Moreover,*

$$E_\eta(A,\Phi;g) = E_\eta(A,\lambda^{-1}\Phi;\lambda^2 g).$$

Proof: The spin^c structure for the rescaled metric $\tilde{g} = \lambda^2 g$ is given by

$$\widetilde{\Gamma}(v) = \lambda\Gamma(v), \qquad \tilde{\rho}^+(\eta) = \lambda^{-2}\rho^+(\eta)$$

for $v \in TX$ and $\eta \in \Lambda^2T^*X$. To prove the second equation choose an orthonormal frame $e_1, \ldots, e_4$ of TX with respect to g and note that the vectors $\tilde{e}_\nu = \lambda^{-1}e_\nu$ form an orthonormal frame with respect to $\tilde{g}$. Hence $\tilde{e}_\nu^* = \lambda e_\nu^*$ and $\widetilde{\Gamma}(\tilde{e}_\nu) = \Gamma(e_\nu)$, and this implies the formula for $\tilde{\rho}^+$. It follows that $\rho^+(F_A+\eta) = (\Phi\Phi^*)_0$ if and only if $\tilde{\rho}^+(F_A+\eta) = (\widetilde{\Phi}\widetilde{\Phi}^*)_0$, where $\widetilde{\Phi} = \lambda^{-1}\Phi$. Moreover, note that $\widetilde{D}_A\widetilde{\Phi} = \lambda^{-2}D_A\Phi$.

Now consider the action integral with respect to the rescaled metric. The pointwise norm of a 1-form gets multiplied by λ^{-1} and that of a 2-form by λ^{-2}. Hence

$$\left|\nabla_A\widetilde{\Phi}\right|_{\tilde{g}}^2 = \lambda^{-4}\left|\nabla_A\Phi\right|_g^2, \qquad \left|F_A + 2\eta\right|_{\tilde{g}}^2 = \lambda^{-4}\left|F_A+2\eta\right|_g^2.$$

Recall also from Lemma 2.16 that the scalar curvature of the rescaled metric is given by $\tilde{s} = \lambda^{-2}s$. This shows that the integrand of the action functional scales with the factor λ^{-4}. Since the volume form of the rescaled metric is given by $\mathrm{dvol}_{\lambda^2 g} = \lambda^4\mathrm{dvol}_g$ it follows that the action integral remains unchanged. This proves the proposition. □

Symmetry

Let $W^* \to X$ denote the bundle $W^* = \mathrm{Hom}(W, \mathbb{C})$. This corresponds to reversing the complex structure. Then

$$\Gamma^* : TX \to \mathrm{End}(W^*)$$

defines a spin^c structure on X. If $\Phi \in C^\infty(X, W)$ then we shall denote by $\Phi^* = \langle\Phi, \cdot\rangle$ the corresponding section of W^*. Note that $W^{*+} = W^{+*}$. If ∇_A is a spin^c connection on W then the induced connection on W^* will be denoted by ∇_{A^*}. This corresponds to the fact that $L_{\Gamma^*} = \Lambda^{2,0}W^{+*} \cong \mathrm{Hom}(L_\Gamma, \mathbb{C})$ and hence

$$L_{\Gamma^*} \cong {L_\Gamma}^*.$$

If A is a connection on ${L_\Gamma}^{1/2}$ then A^* denotes the corresponding connection on ${L_{\Gamma^*}}^{1/2}$.

Exercise 7.9 Prove that

$$F_{A^*} = -F_A, \qquad \sigma^{*+}((\Phi^*\Phi^{**})_0) = -\sigma^+((\Phi\Phi^*)_0)$$

for $A \in \mathcal{A}(\Gamma)$ and $\Phi \in C^\infty(X, W^+)$. Deduce that A, Φ, and η satisfy (7.4) if and only if A^*, Φ^*, and $\eta^* = -\eta$ satisfy (7.4). □

The bundle W^* can be naturally identified with

$$W^* \cong W \otimes {L_\Gamma}^*.$$

To see this note that for every rank-2 complex vector bundle E there is a natural isomorphism $E \otimes \det(E)^* \to \mathrm{Hom}(E, \mathbb{C}) : (\Phi, \beta) \mapsto \iota(\Phi)\beta$ where $\beta \in \det(E)^* = \Lambda^{2,0}E^*$. Apply this isomorphism to both W^+ and W^- to obtain the isomorphism $W \otimes {L_\Gamma}^* \to W^*$.

The Γ*-wall*

Consider the action of the gauge group

$$\mathcal{G} = \mathrm{Map}(X, S^1)$$

on $\mathcal{A}(\Gamma) \times C^\infty(X, W^+)$ by $(A, \Phi) \mapsto (u^*A, u^{-1}\Phi)$ (see page 192). By (6.4) and (6.5), the space of solutions of the Seiberg-Witten equations is invariant under this action. Now the isotropy subgroup of a connection A is the group of constant gauge transformations. Hence $\mathcal{G}$ acts freely on the space of all those pairs $(A, \Phi) \in \mathcal{A}(\Gamma) \times C^\infty(X, W^+)$ with $\Phi \neq 0$. To obtain a smooth moduli space it will therefore be important to avoid solutions of the form (A, Φ) with $\Phi = 0$. In this case the connection $A \in \mathcal{A}(\Gamma)$ satisfies

$$F_A^+ + \eta = 0. \tag{7.5}$$

Such connections give rise to singular points in the moduli space. The next proposition shows that if $b^+(X) > 0$ then, for a generic choice of η, there are no solutions of (7.5).

Proposition 7.10 *The set*

$$\Omega_\Gamma^{2,+}(X, i\mathbb{R}) = \left\{\eta \in \Omega^{2,+}(X, i\mathbb{R}) \,|\, \exists\, A \in \mathcal{A}(\Gamma) \ni F_A^+ + \eta = 0\right\}$$

is an affine subspace of codimension b^+ *whose parallel vector space is the image of the operator* $d^+ : \Omega^1(X, i\mathbb{R}) \to \Omega^{2,+}(X, i\mathbb{R})$.

Proof: Fix an element $\eta_0 \in \Omega_\Gamma^{2,+}(X, i\mathbb{R})$ and a connection $A_0 \in \mathcal{A}(\Gamma)$ such that $F_{A_0}^+ + \eta_0 = 0$. Then

$$\Omega_\Gamma^{2,+}(X, i\mathbb{R}) = \eta_0 + \mathrm{im}\, d^+.$$

Namely, if $F_A^+ + \eta = 0$ then $\eta - \eta_0 = d^+(A_0 - A)$ and if $\eta = \eta_0 + d^+\alpha$ then $F_{A_0-\alpha}^+ + \eta = 0$. Now the result follows from the fact that $\Omega^{2,+}(X, i\mathbb{R})$ decomposes as a direct sum

$$\Omega^{2,+}(X, i\mathbb{R}) = H^{2,+}(X, i\mathbb{R}) \oplus \mathrm{im}\, d^+. \tag{7.6}$$

To see this let $\tau \in \Omega^{2,+}(X, i\mathbb{R})$ and, by Hodge theory, write

$$\tau = \tau_0 + d\alpha + *d\beta,$$

where τ_0 is harmonic and $\alpha, \beta \in \Omega^1(X, i\mathbb{R})$. Then

$$\tau = *\tau = *\tau_0 + d\beta + *d\alpha$$

and hence $\tau_0 = *\tau_0$ and $d\alpha = d\beta$. This shows that

$$\tau = \tau_0 + d\alpha + *d\alpha = \tau_0 + 2d^+\alpha,$$

where τ_0 is a self-dual harmonic 2-form. Since every self-dual harmonic 2-form is orthogonal to the image of d^+, this proves (7.6). □

Consider the cases $b^+ > 1$, $b^+ = 1$, and $b^+ = 0$. In the first case the Γ-wall $\Omega^{2,+}_\Gamma(X, g)$ has codimension at least 2 and hence its complement

$$\Omega^{2,+}(X, i\mathbb{R}) - \Omega^{2,+}_\Gamma(X, i\mathbb{R})$$

is connected. In the case $b^+ = 1$ this complement has two components which can be distinguished as follows. For every metric g on X and every orientation of $H^{2,+}(X; i\mathbb{R})$ there exists a unique self-dual harmonic 2-form

$$\omega_g \in H^{2,+}(X; i\mathbb{R})$$

which has L^2-norm 1 and represents the given orientation of $H^{2,+}(X; i\mathbb{R})$. With this notation we obtain

$$\eta \in \Omega^{2,+}_\Gamma(X, i\mathbb{R}) \qquad \Longleftrightarrow \qquad \varepsilon(g, \eta) = 0$$

where

$$\varepsilon(g, \eta) = \varepsilon_\Gamma(g, \eta) = -\int_X \langle i\eta, \omega_g \rangle \mathrm{dvol}_g - \pi[\omega_g] \cdot c_1(L_\Gamma). \tag{7.7}$$

This follows from the fact that $\eta \in \Omega^{2,+}_\Gamma(X, i\mathbb{R})$ if and only if there exists a closed 2-form $\tau \in \Omega^2(X)$ which represents the class $[\tau] = c_1(L_\Gamma)$ and satisfies $\tau^+ + i\eta/\pi = 0$. When $b^+ = 1$ this is equivalent to $\varepsilon(g, \eta) = 0$.

The two components of the complement of the Γ-wall can be distinguished by the sign of $\varepsilon(g, \eta)$. The minus sign in (7.7) is introduced so that $\varepsilon(g, \eta)$ is positive whenever η is a large multiple of $i\omega_g$. Note that the choice of the basis vector ω_g depends on an orientation of $H^{2,+}(X; i\mathbb{R})$. In the case $b^+ = 0$ the Γ-wall is the entire space $\Omega^{2,+}(X, i\mathbb{R})$ and hence in this case there exists, for every metric g and every perturbation η, a solution of the Seiberg-Witten equations (7.4) with $\Phi = 0$.

7.2 Moduli spaces

Fix a smooth compact Riemannian 4-manifold X equipped with a spin^c structure $\Gamma : TX \to \mathrm{End}(W)$. The goal of this section is to discuss the basic properties of the moduli space

$$\mathcal{M}(X,\Gamma,g,\eta) = \frac{\{(A,\Phi) \in \mathcal{A}(\Gamma) \times C^\infty(X,W^+) \,|\, (7.4)\}}{\mathcal{G}}$$

of gauge equivalence classes of solutions of the Seiberg-Witten equations. We shall prove that, whenever $b^+(X) > 0$, the moduli space $\mathcal{M}(X,\Gamma,g,\eta)$ is a finite dimensional compact oriented smooth manifold for a generic choice of the perturbation η. The Seiberg-Witten invariant of (X,Γ) will be defined as the integral of a certain characteristic class over this moduli space. We shall begin by formulating the basic regularity and compactness theorems, then examine why the moduli space is a smooth manifold, and finally discuss orientability.

Compactness and Regularity

The basic regularity theorem asserts that every weak solution (A,Φ) of (7.4) of class $W^{1,p}$ with $p > 2$ is gauge equivalent to a smooth solution by a gauge transformation of class $W^{2,p}$. To be more precise we fix a smooth reference connection $A_0 \in \mathcal{A}(\Gamma)$ and consider the space

$$\mathcal{A}^{1,p}(\Gamma) = \left\{A_0 + \alpha \,|\, \alpha \in W^{1,p}(X, T^*X \otimes i\mathbb{R})\right\}.$$

The group

$$\mathcal{G}^{2,p} = W^{2,p}(X,S^1) = \left\{u : X \to S^1 \,|\, u^{-1}du \in W^{1,p}\right\}$$

acts naturally on this space. Note that, by the Sobolev embedding theorem, every function $u : X \to S^1$ of class $W^{2,p}$ is continuous. The next theorem asserts that $\mathcal{M}(X,\Gamma,g,\eta)$ can be naturally identified with the $W^{1,p}$ quotient space

$$\mathcal{M}(X,\Gamma,g,\eta) \cong \frac{\left\{(A,\Phi) \in \mathcal{A}^{1,p}(\Gamma) \times W^{1,p}(X,W^+) \,|\, (7.4)\right\}}{\mathcal{G}^{2,p}}.$$

Theorem 7.11. (Regularity) *If $A \in \mathcal{A}^{1,p}(\Gamma)$ and $\Phi \in W^{1,p}(X,W^+)$ with $p > 2$ satisfy (7.4) then there exists a gauge transformation $u \in \mathcal{G}^{2,p}$ such that the pair $(u^*A, u^{-1}\Phi)$ is C^∞ smooth.*

It is convenient, for later reference, to formulate the compactness theorem for a convergent sequence of Riemannian metrics on X. Note that the spin^c representation $\Gamma : TX \to \mathrm{End}(W)$ depends on the Riemannian metric. However, it is possible to fix the bundle W and the splitting

$W = W^+ \oplus W^-$ with $\det(W^+) \cong \det(W^-)$ and choose, for each metric g, a spin^c structure on W which respects the given splitting. All these spin^c structures have the same characteristic line bundle $L_\Gamma = \det(W^+)$ and hence the same space of spin^c connections $\mathcal{A}(\Gamma) = \mathcal{A}({L_\Gamma}^{1/2})$.

Theorem 7.12. (Compactness) *Let g_ν be a sequence of Riemannian metrics on X which converges in the C^∞-topology to a metric g and let $\Gamma_\nu : TX \to \mathrm{End}(W)$ be a corresponding convergent sequence of spin^c structures which respects the splitting $W = W^+ \oplus W^-$. Let $\eta_\nu \in \Omega^{2,+}(X, i\mathbb{R}; g_\nu)$ be a sequence of g_ν-self-dual 2-forms converging in the C^∞-topology to η. Then for every sequence $(A_\nu, \Phi_\nu) \in \mathcal{A}^{1,p}(\Gamma) \times W^{1,p}(X, W^+)$ of solutions of (7.4) with $p > 2$ there exists a subsequence ν' and a sequence of gauge transformations $u_{\nu'} \in W^{2,p}(X, S^1)$ such that the sequence $({u_{\nu'}}^* A_{\nu'}, {u_{\nu'}}^{-1}\Phi_{\nu'})$ converges uniformly with all derivatives.*

In particular, Theorem 7.12 asserts that the moduli space $\mathcal{M}(X, \Gamma, g, \eta)$ is compact The proof is due to Kronheimer and Mrowka. It is based on the following crucial lemma which gives a universal upper bound for the sup-norm of the monopole Φ for any solution of (7.4).

Lemma 7.13 *Let (A, Φ) be a smooth solution of (7.4). Then either $\Phi \equiv 0$ or*

$$\sup_X |\Phi|^2 \le \frac{1}{2} \sup_X \left(4\sqrt{2}|\eta| - s \right).$$

In particular $\Phi \equiv 0$ whenever the metric on X has positive scalar curvature and η is sufficiently small.

Proof: The proof relies on the identity

$$\Delta_g |\Phi|^2 = -2|\nabla_A \Phi|^2 + 2\mathrm{Re}\, \langle \Phi, {\nabla_A}^* \nabla_A \Phi \rangle \tag{7.8}$$

where $\Delta_g = d^*d$ denotes the positive definite Laplace-Beltrami operator of the metric g. To prove (7.8) recall from Exercise 2.30 that the Laplace-Beltrami operator is, in an orthonormal frame e_0, e_1, e_2, e_3 of TX, given by

$$\begin{aligned}
\Delta_g |\Phi|^2 &= -\sum_i \Big(\partial_i \partial_i |\Phi|^2 + \mathrm{div}(e_i) \partial_i |\Phi|^2 \Big) \\
&= -2 \sum_i \Big(\partial_i \mathrm{Re}\, \langle \Phi, \nabla_i \Phi \rangle + \mathrm{div}(e_i) \mathrm{Re}\, \langle \Phi, \nabla_i \Phi \rangle \Big) \\
&= -2 \sum_i |\nabla_i \Phi|^2 - 2 \sum_i \mathrm{Re}\, \langle \Phi, \nabla_i \nabla_i \Phi + \mathrm{div}(e_i) \nabla_i \Phi \rangle \\
&= -2 |\nabla_A \Phi|^2 + 2\mathrm{Re}\, \langle \Phi, {\nabla_A}^* \nabla_A \Phi \rangle .
\end{aligned}$$

The last equality holds because ${\nabla_i}^* = -\nabla_i - \mathrm{div}(e_i)$. This proves (7.8). Now use the Weitzenböck formula of Theorem 6.19 with $D_A \Phi = 0$ to obtain

$$\begin{aligned}
\Delta_g |\Phi|^2 &\le 2\mathrm{Re}\,\langle \Phi, \nabla_A{}^*\nabla_A\Phi\rangle \\
&= -\,2\left\langle \Phi, \rho^+(F_A)\Phi\right\rangle - \frac{1}{2}s\,|\Phi|^2 \\
&= 2\left\langle \Phi, \rho^+(\eta)\Phi - (\Phi\Phi^*)_0\Phi\right\rangle - \frac{1}{2}s\,|\Phi|^2 \\
&= 4\left\langle \rho^+(\eta), (\Phi\Phi^*)_0\right\rangle - |\Phi|^4 - \frac{1}{2}s\,|\Phi|^2 \\
&\le 4\left|\rho^+(\eta)\right| |(\Phi\Phi^*)_0| - |\Phi|^4 - \frac{1}{2}s\,|\Phi|^2 \\
&= 2\sqrt{2}\,|\eta|\,|\Phi|^2 - |\Phi|^4 - \frac{1}{2}s\,|\Phi|^2 .
\end{aligned}$$

Note that this inequality relies on the identity $\rho^+(F_A) = (\Phi\Phi^*)_0 - \rho^+(\eta)$ from (7.4), and on the formulae of Lemma 7.4. This proves that every solution (A,Φ) of (7.4) satisfies

$$|\Phi|^2\left(2\sqrt{2}|\eta| - |\Phi|^2 - \frac{1}{2}s\right) \ge \Delta_g|\Phi|^2.$$

Let $x_0 \in X$ be a point at which the function $x \mapsto |\Phi(x)|^2$ attains its maximum. At such a point

$$\Delta_g|\Phi|^2 = -\sum_i \partial_i\partial_i|\Phi|^2 \ge 0$$

and hence either $\Phi(x_0) = 0$ or

$$|\Phi(x_0)|^2 \le 2\sqrt{2}|\eta(x_0)| - \frac{1}{2}s(x_0).$$

This proves the lemma. □

Uhlenbeck's theorem

The general result which deals with the compactness problem for connections is **Uhlenbeck's theorem** and it applies to any principal G-bundle P over a compact manifold X, where G is a compact Lie group. It asserts that a connection with an L^p bound on the curvature is gauge equivalent to a connection which satisfies an L^p bound on all its first derivatives whenever $2p > \dim X$. (cf. [125]). As a result, every sequence of connections with a uniform L^p-bound on the curvature is gauge equivalent to a sequence which has a weakly convergent subsequence. For general compact Lie groups this is a deep analytical theorem in gauge theory. However, in the case $\mathrm{G} = S^1$ the proof is an elementary consequence of Hodge theory and will be discussed next.

Theorem 7.14. (Uhlenbeck) *Fix a connection $A_0 \in \mathcal{A}(\Gamma)$ and a constant $p > n = \frac{1}{2}\dim X$. Then there exists a constant $c > 0$ such that for every $A \in \mathcal{A}^{1,p}(\Gamma)$ there exists a $u \in W^{2,p}(X, S^1)$ such that*

$$d^*(u^*A - A_0) = 0, \qquad \|u^*A - A_0\|_{W^{1,p}} \le c(1 + \|F_A\|_{L^p}). \tag{7.9}$$

Proof: Denote by $H^1(X; i\mathbb{R})$ the space of imaginary valued harmonic 1-forms on X and consider the lattice $\Lambda = H^1(X; 2\pi i\mathbb{Z}) \subset H^1(X; i\mathbb{R})$. It consists of all harmonic 1-forms $\alpha \in H^1(X; \mathbb{R})$ whose integral over every loop is an integer multiple of $2\pi i$. By Proposition 5.30, Λ can also be characterized as the set of harmonic 1-forms of the form $\alpha = u^{-1}du$ where $u : X \to S^1$ satisfies $d^*(u^{-1}du) = 0$.

Now let $A = A_0 + \alpha$ with $\alpha \in W^{1,p}(X, T^*X \otimes i\mathbb{R})$. By Hodge theory, the 1-form α decomposes as

$$\alpha = \alpha_0 + d\xi + *d\eta$$

where $\alpha_0 \in H^1(X; i\mathbb{R})$ is a harmonic 1-form and $\xi, \eta \in W^{2,p}(X, i\mathbb{R})$. Consider the operator

$$\alpha \mapsto (\alpha_0, d\alpha, d^*\alpha)$$

from $W^{1,p}$ to the appropriate L^p spaces. By the Calderón-Zygmund inequality, Rellich's theorem, and Lemma A.1, this operator has a closed range. Moreover, it is obviously injective. Hence it follows from the open mapping theorem that there is an estimate

$$\|\alpha\|_{W^{1,p}} \le c\left(\|\alpha_0\|_{L^p} + \|d\alpha\|_{L^p} + \|d^*\alpha\|_{L^p}\right) \tag{7.10}$$

where the constant c is independent of α. Now choose a bounded fundamental domain in $H^1(X; i\mathbb{R})$ with respect to the action of the subgroup Λ. Given a 1-form $\alpha \in W^{1,p}(X, T^*X \otimes i\mathbb{R})$ with harmonic part α_0 there exists a function $u_0 : X \to S^1$ such that ${u_0}^{-1}du_0 \in \Lambda$ is harmonic and $\alpha_0 + {u_0}^{-1}du_0$ lies in the given fundamental domain. Since this domain is bounded there is a constant $c_0 > 0$ which is independent of α_0 such that

$$\left\|\alpha_0 + {u_0}^{-1}du_0\right\|_{L^p} \le c_0.$$

If $\alpha = \alpha_0 + d\xi + *d\eta$ as above define

$$u(x) = e^{-\xi(x)}u_0(x).$$

Then $u \in W^{2,p}(X, S^1)$ and

$$\alpha + u^{-1}du = \alpha_0 + {u_0}^{-1}du_0 + *d\eta.$$

Hence $d^*(\alpha + u^{-1}du) = 0$ and $d(\alpha + u^{-1}du) = d\alpha$. So the inequality (7.10) shows that

$$\left\|\alpha + u^{-1}du\right\|_{W^{1,p}} \le c(c_0 + \|d\alpha\|_{L^p}).$$

This proves the theorem. □

To prove Theorem 7.11 one chooses a gauge transformation $u \in \mathcal{G}^{2,p}$ such that $d^*(u^*A - A_0) = 0$ and then proves that the pair $(u^*A, u^{-1}\Phi)$ is smooth. Thus it remains to prove that every $W^{1,p}$-solution (A, Φ) of the equation

$$D_A\Phi = 0, \qquad F_A^+ + \eta = \sigma^+((\Phi\Phi^*)_0), \qquad d^*(A - A_0) = 0 \tag{7.11}$$

is smooth. This is based on the fact that the 1-form $\alpha = A - A_0$ and the section Φ satisfy the elliptic equations

$$D_{A_0}\Phi = -\Gamma(\alpha)\Phi, \qquad d^*d\alpha + dd^*\alpha = 2d^*F_A^+. \tag{7.12}$$

Here we identify TX with T^*X and think of Γ as a bundle homomorphism $T^*X \to \mathrm{End}(W)$. Elliptic regularity now tells us that every $W^{k,p}$-solution of (7.11) is in fact of class $W^{k+1,p}$ provided that $kp > 4$. Hence every $W^{1,p}$-solution is smooth, at least if $p > 4$. If $p > 2$ a slightly more subtle version of this argument gives smoothness.

On can use a similar argument to prove that every sequence of solutions $(A_0 + \alpha, \Phi)$ of (7.11) which also satisfy the estimate

$$\|\alpha\|_{W^{1,p}} \le c(1 + \|d\alpha\|_{L^p}) \tag{7.13}$$

of Theorem 7.14 has a convergent subsequence. The starting point is the uniform L^∞ bound of Φ and hence F_A^+. The second equation in (7.12) then gives a uniform L^p bound on $d\alpha$ and, by (7.13), a uniform $W^{1,p}$-bound on α. Now a standard elliptic bootstrapping argument gives uniform $W^{k,p}$-bounds on α and Φ for all k. The Sobolev embedding theorem then gives a uniform C^k-bound on α and Φ for every k. Finally, it follows from the Arzéla-Ascoli theorem that every sequence (A^ν, Φ^ν) of solutions of (7.11) and (7.13) has a subsequence which converges in the C^k-norm for every k. The details of these arguments will be carried out in the next chapter.

Remark 7.15 Both Theorems 7.11 and 7.12 extend to the C^ℓ category. Thus if η and the Riemannian metric are of class C^ℓ then for every solution (A, Φ) of (7.4) of class $W^{1,p}$ with $p > 2$ there exists a gauge transformation $u = \exp(\xi)$ of class $W^{2,p}$ such that the pair $(u^*A, u^{-1}\Phi)$ is of class C^ℓ. Similarly, if η_ν and the Riemannian metrics g_ν are of class C^ℓ and converge in the C^ℓ-norm then the solutions of (7.4) are, up to gauge equivalence, uniformly bounded in the C^ℓ-norm and hence the subsequence in the assertion of Theorem 7.12 converges in the $C^{\ell-1}$-norm. □

Transversality

Our next goal is to prove that for a generic perturbation η the moduli spaces $\mathcal{M}(X, \Gamma, g, \eta)$ are all smooth manifolds. There is, however, the problem that the circle does not act freely on those solutions (A, Φ) of (7.4) which satisfy $\Phi = 0$. It is convenient to introduce the spaces

$$\widetilde{\mathcal{M}}(X, \Gamma, g, \eta) = \{(A, \Phi) \mid (7.4),\ d^*(A - A_0) = 0, \},$$

$$\widetilde{\mathcal{M}}^*(X, \Gamma, g, \eta) = \left\{(A, \Phi) \in \widetilde{\mathcal{M}}(X, \Gamma, g, \eta) \mid \Phi \neq 0\right\}.$$

The moduli spaces of Seiberg-Witten monopoles can be identified with the quotient spaces

$$\mathcal{M}(X, \Gamma, g, \eta) = \frac{\widetilde{\mathcal{M}}(X, \Gamma, g, \eta)}{\mathcal{G}_0}, \qquad \mathcal{M}^*(X, \Gamma, g, \eta) = \frac{\widetilde{\mathcal{M}}^*(X, \Gamma, g, \eta)}{\mathcal{G}_0}$$

where $\mathcal{G}_0$ denotes the 1-dimensional group of harmonic gauge transformations $u : X \to S^1$ such that $d^*(u^{-1}du) = 0$. Recall from page 192 that $\mathcal{G}_0$ is a central extension of $H^1(X; \mathbb{Z})$ with

$$S^1 \to \mathcal{G}_0 \to H^1(X; \mathbb{Z}).$$

Since $\mathcal{G}_0$ acts freely and properly on $\widetilde{\mathcal{M}}^*$ it suffices to show that $\widetilde{\mathcal{M}}^*$ is a smooth finite dimensional manifold for a generic perturbation η.

Theorem 7.16 *There exists a set $\Omega^{2,+}_{\mathrm{reg}}(X, i\mathbb{R}) \subset \Omega^{2,+}(X, i\mathbb{R})$, which is of the second category in the sense of Baire with respect to the C^∞-topology (it is a countable intersection of open and dense sets), such that for every $\eta \in \Omega^{2,+}_{\mathrm{reg}}(X, i\mathbb{R})$ the space $\mathcal{M}^*(X, \Gamma, g, \eta)$ is a smooth finite dimensional manifold of real dimension*

$$\dim\, \mathcal{M}^*(X, \Gamma, g, \eta) = \frac{\langle c_1(L_\Gamma)^2, [X]\rangle}{4} - \frac{2\chi(X) + 3\sigma(X)}{4}.$$

By Lemma 5.14, this number coincides with $\langle c_2(W^+), [X]\rangle$.

Exercise 7.17 Let (X, J) be an almost complex 4-manifold and $\Gamma_E : TX \to \mathrm{End}(W_E)$ be the canonical spin^c structure twisted by a Hermitian line bundle $E \to X$. Prove that in this case

$$\dim\, \mathcal{M}^*(X, \Gamma_E, g, \eta) = \langle c_1(W_E^+), [X]\rangle = \langle c_1(E)^2 - c_1(E) \cup c_1(K), [X]\rangle,$$

where $c_1(K) = -c_1(TX, J)$ denotes the canonical class. **Hint:** Use the Hirzebruch signature formula (1.9). □

Exercise 7.18 Baire's category theorem asserts that every countable intersection of open and dense sets in a complete metric space is dense. Prove that the distance function

$$d(f,g) = \sum_{k=0}^{\infty} 2^{-k} \frac{\|f-g\|_{C^k}}{1+\|f-g\|_{C^k}}$$

makes the space $C^\infty(X)$ of smooth functions on a compact smooth manifold X into a complete metric space. Deduce that the set $\Omega^{2,+}_{\mathrm{reg}}(X,i\mathbb{R})$ is dense in $\Omega^{2,+}(X,i\mathbb{R})$. □

The proof of Theorem 7.16 consists essentially of two parts. The first part is to consider the linearized Seiberg-Witten equations, to show that the resulting operator is Fredholm, and to compute its index. The second part of the proof is to show that this linearized operator is onto for a generic perturbation η.

The space $\widetilde{\mathcal{M}}(X,\Gamma,g,\eta)$ of solutions of (7.11) can be expressed as the zero set of a map $\mathcal{F}_\eta : \mathcal{X} \to \mathcal{Y}$, where

$$\mathcal{X} = \mathcal{A}(\Gamma) \times C^\infty(X,W^+),$$

$$\mathcal{Y} = \Omega^0(X,i\mathbb{R}) \oplus \Omega^{2,+}(X,i\mathbb{R}) \oplus C^\infty(X,W^-),$$

$$\mathcal{F}_\eta \begin{pmatrix} A \\ \Phi \end{pmatrix} = \begin{pmatrix} d^*(A-A_0) \\ F_A^+ + \eta - \sigma^+((\Phi\Phi^*)_0) \\ D_A\Phi \end{pmatrix}. \tag{7.14}$$

By Theorem 7.11, the zero set of this map agrees with the zero set of the extended map $\mathcal{F}_\eta : \mathcal{X}^{1,p} \to \mathcal{Y}^p$ between the Sobolev completions

$$\mathcal{X}^{1,p} = \mathcal{A}^{1,p}(\Gamma) \times W^{1,p}(X,W^+),$$

$$\mathcal{Y}^p = L^p(X,i\mathbb{R}) \oplus L^p(X,\Lambda^{2,+}T^*X \otimes i\mathbb{R}) \oplus L^p(X,W^-).$$

We shall prove that $\mathcal{F}_\eta : \mathcal{Z}^{1,p} \to \mathcal{Y}^p$ is a Fredholm map of Fredholm index $\langle c_2(W^+),[X]\rangle$. To see this, note that

$$T_{(A,\Phi)}\mathcal{X}^{k,p} = W^{1,p}(X,i\mathbb{R}) \oplus W^{1,p}(X,W^+)$$

and consider the operator $\mathcal{D}_{A,\Phi} = d\mathcal{F}_\eta(A,\Phi) : T_{(A,\Phi)}\mathcal{X}^{1,p} \to \mathcal{Y}^p$. It is given by

$$\mathcal{D}_{A,\Phi} \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} = \begin{pmatrix} d^*\alpha \\ d^+\alpha \\ D_A\varphi \end{pmatrix} + \begin{pmatrix} 0 \\ -\sigma^+((\Phi\varphi^* + \varphi\Phi^*)_0) \\ \Gamma(\alpha)\Phi \end{pmatrix}. \tag{7.15}$$

This operator is a zeroth order perturbation of $D^+ \oplus D_A$, where

$$D^+ : \Omega^1(X, i\mathbb{R}) \to \Omega^0(X, i\mathbb{R}) \oplus \Omega^{2,+}(X, i\mathbb{R})$$

is given by $D^+\alpha = d^*\alpha \oplus d^+\alpha$. This is a Fredholm operator of index

$$\operatorname{index} D^+ = b_1 - 1 - b^+ = -\frac{\chi(X) + \sigma(X)}{2}.$$

(See Lemma 8.15 in the next chapter.) Moreover, by Proposition 6.21 and Theorem 6.22, D_A is a Fredholm operator of index

$$\operatorname{index} D_A = \frac{c \cdot c - \sigma}{4},$$

where $c = c_1(L_\Gamma)$. Hence the $\mathcal{D}_{A,\Phi}$ is Fredholm operator of index

$$\operatorname{index} \mathcal{D}_{A,\Phi} = \frac{\langle c_1(L_\Gamma)^2, [X]\rangle}{4} - \frac{2\chi(X) + 3\sigma(X)}{4}. \tag{7.16}$$

The L^2-orthogonal complement of $\operatorname{im} \mathcal{D}_{A,\Phi}$ always contains the space of constant functions in $L^p(X, i\mathbb{R})$ and so is at least 1-dimensional. A perturbation $\eta \in \Omega^{2,+}(X, i\mathbb{R})$ is called **regular** if the cokernel of $D_{A,\Phi}$ has dimension 1, i.e. $\operatorname{coker} \mathcal{D}_{A,\Phi} \cong H^0(X; i\mathbb{R})$, for all $(A, \Phi) \in \widetilde{\mathcal{M}}^*(X, \Gamma, g, \eta)$. The set of regular perturbations will be denoted by

$$\Omega^{2,+}_{\mathrm{reg}}(X, i\mathbb{R}) \subset \Omega^{2,+}(X, i\mathbb{R}).$$

It is a simple consequence of the implicit function theorem B.3 that the moduli space $\widetilde{\mathcal{M}}^*(X, \Gamma, g, \eta)$ is a smooth manifold of dimension

$$\dim \widetilde{\mathcal{M}}^*(X, \Gamma, g, \eta) = \operatorname{index} \mathcal{D}_{A,\Phi} + 1$$

for all $\eta \in \Omega^{2,+}_{\mathrm{reg}}(X, i\mathbb{R})$.

Remark 7.19 Recall that the moduli space $\widetilde{\mathcal{M}}^*(X, \Gamma, g, \eta)$ carries an S^1-action $(A, \Phi) \mapsto (A, e^{i\theta}\Phi)$. Since $\Phi \neq 0$ for all $(A, \Phi) \in \widetilde{\mathcal{M}}^*(X, \Gamma, g, \eta)$ this action is free. The generator of the S^1-action is the vector field

$$\widetilde{\mathcal{M}}^*(X, \Gamma, g, \eta) \longrightarrow T\widetilde{\mathcal{M}}^*(X, \Gamma, g, \eta) : (A, \Phi) \mapsto (0, i\Phi).$$

Since the tangent space of $\widetilde{\mathcal{M}}^*(X, \Gamma, g, \eta)$ at (A, Φ) is the kernel of the operator $\mathcal{D}_{A,\Phi}$ it follows that

$$\mathbb{R}(0, i\Phi) \subset \ker\, \mathcal{D}_{A,\Phi}$$

whenever (A, Φ) is a solution of (7.4). Hence $\mathcal{D}_{A,\Phi}$ always has at least a 1-dimensional kernel. Note that the tangent space of the quotient $\mathcal{M}^* = \widetilde{\mathcal{M}}^*/\mathcal{G}_0$ at (A, Φ) is the quotient space $\ker\, \mathcal{D}_{A,\Phi}/\mathbb{R}(0, i\Phi)$. □

It remains to prove that the set $\Omega^{2,+}_{\mathrm{reg}}(X,g)$ is of the second category in the sense of Baire. The main idea is to prove first that the space

$$\mathcal{N}^{k,p} = \left\{(A,\Phi) \in \mathcal{X}^{k,p} \,|\, D_A\Phi = 0,\ d^*(A - A_0),\ \Phi \not\equiv 0\right\}$$

is always a smooth Banach manifold. Then it follows from the Sard-Smale theorem B.13 that the set of smooth perturbations $\eta \in \Omega^{2,+}(X, i\mathbb{R})$ which are regular values of the maps

$$\mathcal{N}^{k,p} \to W^{k-1,p}(X, \Lambda^{2,+}T^*X \otimes i\mathbb{R}) : (A,\Phi) \mapsto \sigma^+((\Phi\Phi^*)_0) - F_A^+$$

for all k form an open and dense subset of $\Omega^{2,+}(X, i\mathbb{R})$. This completes the sketch of the proof of Theorem 7.16. Details will be carried out in the next chapter.

Orientation

The next aim is to prove that the moduli spaces $\mathcal{M}^*(X,\Gamma,g,\eta)$ carry a natural orientation. To see this note first that the tangent space at a point $(A,\Phi) \in \widetilde{\mathcal{M}}^*(X,\Gamma,g,\eta)$ is given by the kernel of the operator $\mathcal{D}_{A,\Phi}$. Hence an orientation of $\widetilde{\mathcal{M}}^*(X,\Gamma,g,\eta)$ is equivalent to a trivialization of the real line bundle $\mathcal{D}et \to \widetilde{\mathcal{M}}^*(X,\Gamma,g,\eta)$ with fibers

$$\mathcal{D}et_{A,\Phi} = \det(\mathcal{D}_{A,\Phi}) = \Lambda^{\max} \ker \mathcal{D}_{A,\Phi}.$$

(See Appendix A.) Here the highest exterior power of the kernel can be identified with the determinant line because the cokernel $H^0(X, i\mathbb{R})$ is naturally isomorphic to $\mathbb{R}$. By Theorem A.6 of Appendix A the bundle $\mathcal{D}et$ extends to a locally trivial bundle

$$\mathcal{D}et \to \mathcal{A}(\Gamma) \times C^\infty(X, W^+)$$

over the space of all pairs (A,Φ). The latter space is contractible and hence the bundle $\mathcal{D}et$ can be trivialized over the entire space $\mathcal{A}(\Gamma) \times C^\infty(X, W^+)$.

A natural trivialization of this bundle can be described as follows. The operator $\mathcal{D}_{A,\Phi}$ is a compact perturbation of

$$\mathcal{D}_{A,0} = D^+ \oplus D_A.$$

Hence an orientation of $\det(D^+ \oplus D_A)$ determines an orientation of $\mathcal{D}_{A,\Phi}$ by considering the family of operators $\mathcal{D}_{A,t\Phi}$ for $0 \le t \le 1$. Now the Dirac operator $D_A : C^\infty(X, W^+) \to C^\infty(X, W^-)$ is a complex linear operator between complex vector spaces and so its kernel and cokernel are complex vector spaces. Hence they carry natural orientations induced by the complex structure and this determines an orientation of the line

$\det(D_A)$. By Exercise A.14, this orientation is invariant under trivializations of the determinant line bundle. In other words, a trivialization of the line bundle $\bigcup_t \det(D_{A_t})$ over the unit interval, corresponding to the path $[0,1] \to \mathcal{A}(\Gamma) : t \mapsto A_t$, identifies the orientations which arise from the complex structure. Now recall that

$$\ker D^+ = H^1(X; i\mathbb{R}), \qquad \operatorname{coker} D^+ = H^0(X; i\mathbb{R}) \oplus H^{2,+}(X; i\mathbb{R}).$$

The space $H^0(X; i\mathbb{R}) = \mathbb{R}$ carries a natural orientation. Let us fix once and for all orientations of the real vector spaces $H^1(X; i\mathbb{R})$ and $H^{2,+}(X; i\mathbb{R})$. This determines an orientation of the line $\det(D^+)$ and hence of

$$\det(\mathcal{D}_{A,0}) = \det(D^+) \otimes \det(D_A).$$

Now use the path $\lambda \mapsto \det(\mathcal{D}_{A,\lambda\Phi})$ of 1-dimensional vector spaces to obtain the required orientation of $\det(\mathcal{D}_{A,\Phi})$. In fact, any path in $\mathcal{A}(\Gamma) \times C^\infty(X, W^+)$ starting at some pair $(A_0, 0)$ and ending at (A, Φ) gives rise to an orientation of the line $\det(\mathcal{D}_{A,\Phi})$. That this orientation is independent of the choice of the path follows from Theorem A.6 in Appendix A and the fact that the space $\mathcal{A}(\Gamma) \times C^\infty(X, W^+)$ is simply connected. That it is also independent of the choice of the base point $(A_0, 0)$ follows from Exercise A.14.

Recall that if $H^1(X; i\mathbb{R}) \neq 0$ then the action of the disconnected group $\mathcal{G}_0 = \{u : X \to S^1 \,|\, d^*(u^{-1}du) = 0\}$ acts on $\widetilde{\mathcal{M}}^*(X, \Gamma, g, \eta)$. The next proposition asserts that $\mathcal{G}_0$ acts by orientation preserving diffeomorphisms, and hence the quotient $\mathcal{M}^*(X, \Gamma, g, \eta)$ is orientable.

Proposition 7.20 **(i)** *Orientations of $H^1(X)$ and $H^{2,+}(X)$ determine natural orientations*

$$\varepsilon(\Gamma, g, \eta) \in \operatorname{Or}(\widetilde{\mathcal{M}}^*(X, \Gamma, g, \eta)),$$

one for every spin^c structure Γ and every 2-form $\eta \in \Omega^{2,+}_{\mathrm{reg}}(X, i\mathbb{R})$.
(ii) *The group $\mathcal{G}_0$ acts on $\widetilde{\mathcal{M}}^*(X, \Gamma, g, \eta)$ by orientation preserving diffeomorphisms.*
(iii) *Reversing the complex structure of W results in diffeomorphic moduli spaces $\widetilde{\mathcal{M}}^*(X, \Gamma^*, g, \eta) \cong \widetilde{\mathcal{M}}^*(X, \Gamma, g, \eta)$. If these are 1-dimensional then $(\chi + \sigma)/2 = 1 + b^+ - b_1 \in 2\mathbb{Z}$ and*

$$\varepsilon(\Gamma^*, g, \eta) = (-1)^{\frac{\chi+\sigma}{4}} \varepsilon(\Gamma, g, \eta).$$

Proof: Statement (i) was proved above. To prove (ii) choose two paths $t \mapsto A_t$ and $t \mapsto \Phi_t$ with

$$A_1 = u^* A_0, \qquad \Phi_1 = u^{-1} \Phi_0$$

for $u \in \mathcal{G}_0$. Then a trivialization of the determinant line bundle over the path $t \mapsto \det(\mathcal{D}_{A_t,\Phi_t})$ gives rise to an isomorphism

$$\det(\mathcal{D}_{A_0,\Phi_0}) \to \det(\mathcal{D}_{A_1,\Phi_1}).$$

On the other hand, linearizing the action of the gauge group $(A, \Phi) \mapsto (u^*A, u^{-1}\Phi)$ gives rise to isomorphisms

$$\ker\, \mathcal{D}_{A_0,\Phi_0} \to \ker\, \mathcal{D}_{A_1,\Phi_1} : (\alpha, \varphi) \mapsto (\alpha, u^{-1}\varphi),$$

$$\operatorname{coker} \mathcal{D}_{A_0,\Phi_0} \to \operatorname{coker} \mathcal{D}_{A_1,\Phi_1} : (\xi, \tau, \psi) \mapsto (\xi, \tau, u^{-1}\psi).$$

We prove that the induced isomorphism $\det(\mathcal{D}_{A_0,\Phi_0}) \to \det(\mathcal{D}_{A_1,\Phi_1})$ of determinant lines agrees with the above. To see this consider first the case $\Phi_t = 0$ for every t. Then

$$\det(\mathcal{D}_{A_t,0}) = \det(D^+) \otimes \det(D_{A_t}).$$

This line bundle over the interval has the obvious natural trivialization because the operators D_{A_t} are all complex linear and the resulting map $\det(D_{A_0}) \to \det(D_{A_1})$ identifies the two orientations arising from the complex structures (see Exercise A.14). On the other hand the isomorphisms

$$\ker\, D_{A_0} \to \ker\, D_{A_1} : \varphi \mapsto u^{-1}\varphi$$

and

$$\ker\, {D_{A_0}}^* \to \ker\, {D_{A_1}}^* : \psi \mapsto u^{-1}\psi$$

are complex linear and hence induce the same map $\det(D_{A_0}) \to \det(D_{A_1})$. This proves the assertion in the case $\Phi_t \equiv 0$. The general case now follows from a standard homotopy argument.

It remains to prove the assertion about the reversal of the complex structure. The only thing that changes are the orientations of the kernel and cokernel of the Dirac operator. The factor is -1 if and only if the complex dimension is odd. Hence

$$\varepsilon(\Gamma^*, g, \eta) = (-1)^\lambda \varepsilon(\Gamma, g, \eta), \qquad \lambda = \frac{\langle c_1(L_\Gamma)^2, [X]\rangle}{8} - \frac{\sigma(X)}{8},$$

where λ is the complex index of the Dirac operator D_A. Now suppose that the moduli space $\widetilde{\mathcal{M}}^*(X, \Gamma, g, \eta)$ is 1-dimensional. Then the dimension formula of Theorem 7.16 shows that $\langle c_1(L_\Gamma)^2, [X]\rangle = 2\chi(X) + 3\sigma(X)$ and hence

$$\lambda = \frac{\chi(X) + \sigma(X)}{4}.$$

This proves the proposition. □

7.3 Cobordisms

Recall that a spin^c structure on X is a bundle homomorphism $\Gamma : TX \to \mathrm{End}(W)$ which assigns to each tangent vector $v \in T_xX$ a skew Hermitian endomorphism $\Gamma(v) \in \mathrm{End}(W_x)$ such that

$$\Gamma(v)^*\Gamma(v) = |v|^2 \mathbb{1}.$$

The right hand side depends on the Riemannian metric. Given two Riemannian metrics g_0 and g_1, two corresponding spin^c structures Γ_0 and Γ_1 on W are called **equivalent** if

$$\frac{1}{|v|_{g_0}}\Gamma_0(v) = \frac{1}{|v|_{g_1}}\Gamma_1(v).$$

Given a spin^c structure Γ_0 for g_0 there is a unique spin^c structure Γ for any other metric g which is equivalent to Γ_0.

Assume that g_0 and g_1 are two Riemannian metrics on X with equivalent spin^c structures $\Gamma_0, \Gamma_1 : TX \to \mathrm{End}(W)$. Assume also that

$$\eta_0 \in \Omega^{2,+}_{\mathrm{reg}}(X, i\mathbb{R}; \Gamma_0, g_0), \qquad \eta_1 \in \Omega^{2,+}_{\mathrm{reg}}(X, i\mathbb{R}; \Gamma_1, g_1).$$

The goal of this section is to prove that the corresponding moduli spaces $\mathcal{M}(X, \Gamma_0, g_0, \eta_0)$ and $\mathcal{M}(X, \Gamma_1, g_1, \eta_1)$ are cobordant. The proof goes along the lines of Proposition B.17 in Appendix B. Fix a path $[0, 1] \longrightarrow \mathcal{M}et(X) : t \mapsto g_t$ of Riemannian metrics connecting g_0 to g_1 with a corresponding path of equivalent spin^c structures $\Gamma_t : TX \to \mathrm{End}(W)$. Denote by

$$\mathcal{Z} = \Omega^{2,+}(X, i\mathbb{R}; \{g_t\}, \eta_0, \eta_1)$$

the space of all smooth paths $[0, 1] \longrightarrow \Omega^2(X, i\mathbb{R}) : t \mapsto \eta_t$ connecting η_0 to η_1 such that $\eta_t \in \Omega^{2,+}(X, i\mathbb{R}; g_t)$ for every t. For $\{\eta_t\} \in \mathcal{Z}$ consider the moduli space

$$\mathcal{W}^* = \{[t, A, \Phi] \,|\, t \in [0, 1],\ [A, \Phi] \in \mathcal{M}^*(X, \Gamma, g_t, \eta_t)\}.$$

In general it will not be possible to find a path $t \mapsto \eta_t$ such that $\eta_t \in \Omega^{2,+}_{\mathrm{reg}}(X, i\mathbb{R}; g_t)$ for every t. The complement of of the set $\Omega^{2,+}_{\mathrm{reg}}(X, i\mathbb{R}; g)$ is, roughly speaking, of codimension 1 for every Riemannian metric g. This is the anlogue of the observation, in finite dimensional differential topology, that the set of regular values of a smooth map $f : X \to Y$ is in general not connected. However, any two regular values y_0 and y_1 can be connected by a path whose preimage under f is a smooth manifold with boundary. Similarly, in the present context, there exists a path $t \mapsto \eta_t$ from η_0 to η_1 such that $\mathcal{W}^*$ is a smooth manifold with boundary.

Theorem 7.21 *Assume $b^+ \geq 1$. There exists a set*

$$\mathcal{Z}_{\mathrm{reg}} = \Omega^{2,+}_{\mathrm{reg}}(X, i\mathbb{R}; \{g_t\}, \eta_0, \eta_1) \subset \mathcal{Z},$$

which is of the second category in the sense of Baire with respect to the C^∞-topology and satisfies the following. For every path $\{\eta_t\} \in \mathcal{Z}_{\mathrm{reg}}$ the space $\mathcal{W}^* = \mathcal{W}^*(X, \{\Gamma_t\}, \{g_t\}, \{\eta_t\})$ is a smooth oriented finite dimensional manifold with boundary. It has real dimension

$$\dim \mathcal{W}^* = \frac{\langle c_1(L_\Gamma)^2, [X]\rangle}{4} - \frac{2\chi(X) + 3\sigma(X)}{4} + 1$$

and its boundary is given by

$$\partial \mathcal{W}^* = \mathcal{M}^*(X, \Gamma_1, g_1, \eta_1) - \mathcal{M}^*(X, \Gamma_0, g_0, \eta_0).$$

Here the minus sign stands for the reversal of orientation

Proof: The proof is a combination of the arguments in the proofs of Theorem 7.16 and Proposition B.17. Here is a sketch of the main points. Consider the space

$$\mathcal{X} = \mathcal{A}(\Gamma) \times C^\infty(X, W^+)$$

and the infinite dimensional vector bundle $\mathcal{E} \to [0,1] \times \mathcal{X}$ whose fiber over the point (t, A, Φ) depends only on t and is given by

$$\mathcal{E}_t = \Omega^0_0(X, i\mathbb{R}; g_t) \oplus \Omega^{2,+}(X, i\mathbb{R}; g_t) \oplus C^\infty(X, W^-).$$

Here $\Omega^0_0(X, i\mathbb{R}; g_t)$ denotes the space of smooth functions $\xi : X \to i\mathbb{R}$ which have mean value zero with respect to the metric g_t and $\Omega^{2,+}(X, i\mathbb{R}; g_t)$ denotes the space of imaginary valued 2-forms which are self-dual with respect to g_t. Consider the section $\mathcal{F} : \mathcal{X} \to \mathcal{E}$ of this bundle given by

$$\mathcal{F}(t, A, \Phi) = \begin{pmatrix} d^{*_t}(A - A_0) \\ F_A^+ + \eta_t - {\sigma_t^+}^{-1}((\Phi\Phi^*)_0) \\ D_{A;t}\Phi \end{pmatrix}.$$

Here the Hodge-$*$-operator, the maps $A \mapsto F_A^+$ and $\sigma_t^+ : \Lambda^{2,+}T^*X \to \mathrm{End}(W^+)$, and the Dirac operator $D_{A;t}$ depend on the metric g_t. Denote by

$$D\mathcal{F}(t, A, \Phi) : \mathbb{R} \times T_{(A,\Phi)}\mathcal{X} \to \mathcal{E}_t$$

the differential of $\mathcal{F}$ (as a function with values in $\Omega^0 \oplus \Omega^2 \oplus C^\infty(X, S^-)$) followed by the L^2-orthogonal projection onto $\mathcal{E}_t$, denoted by Π_t. This operator is given by

$$D\mathcal{F}(t,A,\Phi)(\tau,\alpha,\varphi) = \mathcal{D}_{A,\Phi;t}(\alpha,\varphi) + \tau\Pi_t\frac{\partial}{\partial t}\mathcal{F}(t,A,\Phi). \tag{7.17}$$

where $\mathcal{D}_{A,\Phi;t} : T_{(A,\Phi)}\mathcal{X} \to \mathcal{E}_t$ denotes the linearized operator defined in (7.15) for the metric g_t. Recall that $\mathcal{D}_{A,\Phi;t}$ is a Fredholm operator and hence $\mathcal{F}$ is a Fredholm section of the infinite dimensional vector bundle $\mathcal{E}$ (modified with appropriate completions and Sobolev norms). A path $\{\eta_t\} \in \Omega^{2,+}(X,\{g_t\},\eta_0,\eta_1)$ is called **regular** if the operator $D\mathcal{F}(t,A,\Phi)$: $\mathbb{R}\times T_{(A,\Phi)}\mathcal{X} \to \mathcal{E}_t$ is onto whenever $\mathcal{F}(t,A,\Phi) = 0$. The set of regular paths is denoted by

$$\mathcal{Z}_{\text{reg}} = \Omega^{2,+}_{\text{reg}}(X,i\mathbb{R};\{g_t\},\eta_0,\eta_1)$$

For every regular path $\{\eta_t\}$ the section $\mathcal{F}$ is transverse to the zero section and hence, by the implicit function theorem B.3, its zero set is a smooth manifold whose dimension agrees with the index of the map $\mathcal{F}$. Now this zero set is precisely given by

$$\mathcal{F}^{-1}(0) = \widetilde{\mathcal{W}}^* = \widetilde{\mathcal{W}}^*(X,\{\Gamma_t\},\{g_t\},\{\eta_t\}).$$

For $\{\eta_t\} \in \mathcal{Z}_{\text{reg}}$ it is a manifold of dimension

$$\dim\widetilde{\mathcal{W}}^* = \text{index}(\mathcal{F}) = \frac{\langle c_1(L_\Gamma)^2,[X]\rangle}{4} - \frac{2\chi(X)+3\sigma(X)}{4} + 2.$$

As in the finite dimensional analogue the boundary of this manifold is given by its intersection with the boundary of $[0,1]\times\mathcal{X}$ and thus

$$\partial\widetilde{\mathcal{W}}^* = \widetilde{\mathcal{M}}^*(\eta_1) - \widetilde{\mathcal{M}}^*(\eta_0)$$

where $\widetilde{\mathcal{M}}^*(\eta_t) = \widetilde{\mathcal{M}}^*(X,\Gamma_t,g_t,\eta_t)$. That this assertion is correct with orientations will be proved below. There is a natural action of the group

$$\mathcal{G}_0 = S^1 \times H^1(X;2\pi i\mathbb{Z})$$

on $\widetilde{\mathcal{W}}^*$. To see this fix a point $x_0 \in X$ and denote by

$$\mathcal{G}_t = \left\{u : X \to S^1 \,|\, d^{*_t}(u^{-1}du) = 0\right\}$$

the group of g_t-harmonic gauge transformations. Then, for every $(\lambda,\alpha) \in S^1 \times H^1(X;2\pi i\mathbb{Z})$ and every $t \in [0,1]$, there is a unique g_t-harmonic gauge transformation $u_t = u_{t,\lambda,\alpha} \in \mathcal{G}_t$ such that $u(x_0) = \lambda$ and $u^{-1}du = \alpha$. The action of (λ,α) on $\widetilde{\mathcal{W}}^*$ is then given by $(t,A,\Phi) \mapsto (t,{u_t}^*A,{u_t}^{-1}\Phi)$. The quotient $\mathcal{W}^* = \widetilde{\mathcal{W}}^*/\mathcal{G}_0$ is the required cobordism with

$$\partial\mathcal{W}^* = \mathcal{M}^*(\eta_1) - \mathcal{M}^*(\eta_0).$$

It remains to prove that the set $\mathcal{Z}_{\text{reg}} \subset \mathcal{Z}$ is of the second category and to verify the assertion about the orientations. The proof of the former is strictly analogous to the corresponding arguments in the proof of Theorem 7.16. One first proves that the *universal moduli space* $\widetilde{\mathcal{U}}^*$ of all quadruples $(t, A, \Phi, \{\eta_t\})$ with $\{\eta_t\} \in \mathcal{Z}^\ell$ and $(t, A, \Phi) \in \widetilde{\mathcal{W}}^*(\{\eta_t\})$ is a smooth Banach manifold (with the C^ℓ-norm on $\mathcal{Z}^\ell$) and then considers the obvious projection $\pi : \widetilde{\mathcal{U}}^* \to \mathcal{Z}^\ell$. A path $\{\eta_t\} \in \mathcal{Z}^\ell$ is a regular value of this projection if and only if $\{\eta_t\} \in \mathcal{Z}^\ell_{\text{reg}}$ and it thus follows from the Sard-Smale theorem B.13 that the set $\mathcal{Z}^\ell_{\text{reg}} \subset \mathcal{Z}^\ell$ is of the second category in the sense of Baire. The details of this argument, as well as the reduction of the C^∞-case to the C^ℓ-case, are exactly the same as in Theorem 7.16 and Proposition B.17 and will be omitted.

Let us now turn to the question of orientations. The tangent space of the manifold $\widetilde{\mathcal{W}}^* = \widetilde{\mathcal{W}}^*(\{\eta_t\})$ at a triple (t, A, Φ) is the kernel of the operator $D\mathcal{F}(t, A, \Phi)$:

$$T_{(t,A,\Phi)}\widetilde{\mathcal{W}}^* = \ker\left(D\mathcal{F}(t, A, \Phi) : \mathbb{R} \times T_{(A,\Phi)}\mathcal{X} \to \mathcal{E}_t\right).$$

By assumption on $\{\eta_t\} \in \mathcal{Z}_{\text{reg}}$, this operator is surjective and hence an orientation of its kernel corresponds to an orientation of its determinant line. Now recall from (7.17) that this operator has the form

$$D\mathcal{F}(t, A, \Phi)(\tau, \alpha, \varphi) = \tau\zeta + \mathcal{D}_{A,\Phi;t}(\alpha, \varphi)$$

for some vector $\zeta \in \mathcal{E}_t$. In the notation of Section A.2 this operator can be written as $D\mathcal{F}(t, A, \Phi) = \zeta \oplus \mathcal{D}_{A,\Phi;t}$ where ζ is to be understood as the operator $\mathbb{R} \to \mathcal{E}_t : \tau \mapsto \tau\zeta$. Operators of this form are discussed in detail in the proof of Theorem A.6 and in Exercise A.7 where it is shown that there is a natural isomorphism

$$\det(\mathcal{D}_{A,\Phi;t}) \to \det(D\mathcal{F}(t, A, \Phi)).$$

Hence an orientation of the determinant line $\det(\mathcal{D}_{A,\Phi;t})$ induces naturally an orientation of the determinant line $\det(D\mathcal{F}(t, A, \Phi))$ and hence of the tangent space of $\widetilde{\mathcal{W}}^*$. In Section 7.2 above it is proved that the determinant line of $\mathcal{D}_{A,\Phi;t}$ carries a natural orientation, given orientations of $H^1(X; i\mathbb{R})$ and $H^{2,+}(X; i\mathbb{R})$. It follows that this orientation carries over to the determinant line of $D\mathcal{F}(t, A, \Phi)$. Hence the manifold $\widetilde{\mathcal{W}}^*$ is orientable and in fact carries a natural orientation. That the group $\{\mathcal{G}_t\}$ acts by orientation preserving diffeomorphisms is proved as in Proposition 7.20 and it follows that the orientation of $\widetilde{\mathcal{W}}^*$ descends to the quotient $\mathcal{W}^*$. It remains to compare the orientation of $\widetilde{\mathcal{W}}^*$ with that of $\widetilde{\mathcal{M}}^*(\eta_0)$ and $\widetilde{\mathcal{M}}^*(\eta_1)$ near the boundary. For this it is useful to give a more explicit description of the orientation of the cobordism $\widetilde{\mathcal{W}}^*$.

Consider the obvious projection $\pi : \widetilde{\mathcal{W}}^* \to [0,1]$. It is easy to see that $(t, A, \Phi) \in \widetilde{\mathcal{W}}^*$ is regular for π (i.e. $d\pi(t, A, \Phi)$ is surjective) if and only if the operator $\mathcal{D}_{A,\Phi;t} : T_{(A,\Phi)}\mathcal{X} \to \mathcal{E}_t$ is onto. If (t, A, Φ) is such a regular point for π then Exercise A.8 shows that a positively oriented basis of $\ker D\mathcal{F}(t, A, \Phi) = T_{(t,A,\Phi)}\widetilde{\mathcal{W}}^*$ is of the form

$$(1, \xi_0), (0, \xi_1), \ldots, (0, \xi_k)$$

where the vectors $\xi_1, \ldots, \xi_k$ form a positively oriented basis of $\ker \mathcal{D}_{A,\Phi;t}$. The standard convention for orienting the boundary is given by choosing the outward unit normal as the first basis vector and then completing to a positively oriented basis by adding a basis of the tangent space to the boundary. With this convention the boundary components at $t = 1$ inherit their original orientation as the boundary of $\widetilde{\mathcal{W}}^*$ while those at $t = 0$ inherit the opposite orientation. Thus

$$\partial\widetilde{\mathcal{W}}^* = \widetilde{\mathcal{M}}^*(\eta_1) - \widetilde{\mathcal{M}}^*(\eta_0).$$

To obtain the same formula for the quotient $\mathcal{W}^*$ we use the convention that the tangent vector to the S^1-action comes last, i.e. that a basis of the quotient space is called positively oriented if after completing it by adding the generator of the S^1-action at the end, it is a positively oriented basis of the total space. This proves the theorem. □

In order for $\mathcal{W}^*$ to be compact it is necessary to assume that the Seiberg-Witten equations (7.4) have no solution with $\Phi = 0$ for any pair (g_t, η_t). This means that

$$\eta_t \notin \Omega^{2,+}_{\Gamma_t}(X, i\mathbb{R}; g_t) \tag{7.18}$$

for every t. By Proposition 7.10, the set $\Omega^{2,+}_{\Gamma_t}(X, i\mathbb{R}; g_t)$ is a hyperplane of codimension $b^+ = b^+(X)$ and hence, if $b^+ \geq 2$, there always is a path $\{\eta_t\}$ connecting η_0 and η_1 which satisfies (7.18). If $b^+ = 1$ then $\Omega^{2,+}_{\Gamma_t}(X, g_t)$ is a codimension-1 hyperplane. If η_0 and η_1 lie *on opposite sides* of this hyperplane, in the sense that the numbers $\varepsilon(\eta_0, g_0)$ and $\varepsilon(\eta_1, g_1)$ defined by (7.7) have opposite sign, then every path from η_0 to η_1 must pass through the Γ-wall for some value of t and hence there is no path from η_0 to η_1 which satisfies (7.18). Thus a compact cobordism will only exist if either $b^+ \geq 2$ or $b^+(X) = 1$ and in addition $\varepsilon(\eta_0, g_0)$ and $\varepsilon(\eta_1, g_1)$ have the same sign. If either of these conditions are satisfied there exists a regular path $\{\eta_t\}$ from η_0 to η_1 which satisfies (7.18) and for such a path the space

$$\mathcal{W} = \mathcal{W}(X, \{\Gamma_t\}, \{g_t\}, \{\eta_t\}) = \mathcal{W}^*(X, \{\Gamma_t\}, \{g_t\}, \{\eta_t\})$$

is a compact smooth oriented cobordism with $\partial\mathcal{W} = \mathcal{M}(\eta_1) - \mathcal{M}(\eta_0)$. These cobordisms give rise to the Seiberg-Witten invariants.

7.4 Invariants of smooth four-manifolds

The zero dimensional case

Consider the case where the moduli space $\mathcal{M}(X, \Gamma, g, \eta)$ has dimension zero, that is

$$\frac{c \cdot c - \sigma}{4} = \frac{\chi + \sigma}{2}. \tag{7.19}$$

This is the real index of the Dirac operator and is therefore an even number. The right hand side is equal to $1 + b^+ - b_1$ and hence $b^+ - b_1$ must be odd. A zero-dimensional compact manifold consists of finitely many points and hence, under the assumption (7.19), the moduli space $\mathcal{M}(X, \Gamma, g, \eta)$ is a finite set whenever $\eta \in \Omega^{2,+}_{\mathrm{reg}}(X, i\mathbb{R}; c, g)$ where $c = c_1(L_\Gamma) \in H^2(X; \mathbb{Z})$. The simplest version of the Seiberg-Witten invariant is the number of points in the moduli space, counted modulo 2. That this is an invariant, i.e. is independent of the choices of g and η, is a consequence of Theorem 7.22 below.

An orientation, in the zero-dimensional case, consists of attaching a sign ± 1 to each point of the manifold. Here is an explicit description of this sign. Let $(A, \Phi) \in \widetilde{\mathcal{M}}(X, \Gamma, g, \eta)$ represent a point in $\mathcal{M}(X, \Gamma, g, \eta)$, that is, the corresponding equivalence class under the action of $\mathcal{G}_0$, denoted by

$$[A, \Phi] = \left\{ (u^*A, u^{-1}\Phi) \,|\, d^*(u^{-1}du) = 0 \right\}.$$

The transversality condition $\eta \in \Omega^{2,+}_{\mathrm{reg}}(X, i\mathbb{R}; g)$ means, in the zero-dimensional case, that

$$\ker \mathcal{D}_{A,\Phi} = i\mathbb{R}\Phi, \qquad \operatorname{coker} \mathcal{D}_{A,\Phi} = H^0(X; i\mathbb{R}) = i\mathbb{R}$$

for all $(A, \Phi) \in \widetilde{\mathcal{M}}^*(X, \Gamma, g, \eta)$. (See Remark 7.19.) Hence the determinant $\det(\mathcal{D}_{A,\Phi})$ is naturally isomorphic to $\mathbb{R}$. Now recall that the determinant line of $\mathcal{D}_{A,0}$ has a canonical orientation. A trivialization of the 1-dimensional real vector bundle

$$\bigcup_{0 \le \lambda \le 1} \det(\mathcal{D}_{A,\lambda\Phi})$$

over the unit interval gives rise to an isomorphism

$$\det(\mathcal{D}_{A,0}) \to \det(\mathcal{D}_{A,\Phi})$$

and hence to an orientation of the line $\det(\mathcal{D}_{A,\Phi}) \cong \mathbb{R}$. It is interesting to consult Propositions A.9 and A.10 in Appendix A for a more precise discussion of such trivializations in the case of index zero. Define $\nu(A, \Phi) = 1$ if the resulting orientation of $\det(\mathcal{D}_{A,\Phi}) \cong \mathbb{R}$ agrees with the standard orientation of $\mathbb{R}$ and $\nu(A, \Phi) = -1$ otherwise.

The zero-dimensional **Seiberg-Witten invariant** can now be defined as follows. Assume that $b^+ \geq 2$ and $b^+ - b_1$ is odd and fix a spin^c structure Γ. For a Riemannian metric g and a regular self-dual 2-form $\eta \in \Omega^{2,+}_{\mathrm{reg}}(X, i\mathbb{R}; g)$ define

$$SW(X, \Gamma; g, \eta) = \sum_{[A,\Phi]} \nu(A, \Phi). \tag{7.20}$$

Here the sum runs over the finite set of all equivalence classes $[A, \Phi] \in \mathcal{M}(X, \Gamma, g, \eta)$.

Theorem 7.22. (Seiberg-Witten) *Assume that $b^+ > 1$ and $b^+ - b_1$ is odd. Then the number*

$$SW(X, \Gamma) = SW(X, \Gamma; g, \eta)$$

is independent of the choice of g and η and depends only on the isomorphism class of the spin^c structure Γ.

Choose $\{\eta_t\} \in \Omega^{2,+}_{\mathrm{reg}}(X, i\mathbb{R}; \{g_t\}, \eta_0, \eta_1)$ such that

$$\eta_t \in \Omega^{2,+}_{\Gamma_t}(X, i\mathbb{R}; g_t)$$

for all t. This is possible because $b^+ > 1$ and it follows that all solutions (A, Φ) of (7.4) with $g = g_t$ and $\eta = \eta_t$ satisfy $\Phi \neq 0$. Hence the moduli space $\mathcal{W} = \mathcal{W}(X, \{\Gamma_t\}, \{g_t\}, \{\eta_t\})$ constructed in Theorem 7.21 is a compact oriented cobordism with

$$\partial \mathcal{W} = \mathcal{M}(\eta_1) - \mathcal{M}(\eta_0).$$

Lemma 7.23 *Assume $\mathcal{W}$ has dimension 1 and let $[t, A, \Phi]$ be a regular point of the projection*

$$\pi : \mathcal{W} \to [0, 1]$$

given by $\pi([t, A, \Phi]) = t$. Then π is orientation preserving at $[t, A, \Phi]$ if and only if $\nu(A, \Phi) = 1$.

Proof: Consider the operator

$$\mathcal{D}_{A,\Phi,t} : T_{(A,\Phi)}\mathcal{X} \to \mathcal{E}_t$$

as in (7.17) in the proof of Theorem 7.21. A point $[t, A, \Phi] \in \mathcal{W}$ is regular for $\pi : \mathcal{W} \to [0, 1]$ if and only if this operator is surjective. In this case the kernel of $\mathcal{D}_{A,\Phi,t}$ is 1-dimensional and spanned by the standard basis vector $\xi_1 = (0, i\Phi)$. This is the tangent vector to the S^1-action on $\widetilde{\mathcal{W}}$ and it is positively oriented if and only if $\nu(A, \Phi) = 1$. In this case $\{(1, \xi_0), (0, \xi_1)\}$ is a positively oriented basis of $\ker D\mathcal{F}(t, A, \Phi)$. (See equation (7.17).) Hence the equivalence class of the vector $[1, \xi_0]$ is a positively oriented basis vector of the tangent space to the quotient $\mathcal{W} = \widetilde{\mathcal{W}}/\{\mathcal{G}_t\}$ at $[t, A, \Phi]$ and it is mapped to 1 under $d\pi$. Conversely, if $\nu(A, \Phi) = -1$ then the vector $\xi_1 = (0, i\Phi)$ is a negatively oriented basis of $\ker \mathcal{D}_{A,\Phi;t}$, thus $\{(1, \xi_0), (0, \xi_1)\}$ is a negatively oriented basis of $\ker D\mathcal{F}(t, A, \Phi)$, and thus $[1, \xi_0]$ is a negatively oriented tangent vector of $\mathcal{W}$. This proves the Lemma. □

Proof of Theorem 7.22: The cobordism $\mathcal{W}$ determines finitely many paths

$$[0, 1] \to \mathcal{W} : s \mapsto [t_j(s), A_j(s), \Phi_j(s)],$$

which parametrize the components of $\mathcal{W}$ that are diffeomorphic to the unit interval. Their endpoints lie on $\partial\mathcal{W}$ and the signs of these are denoted by

$$\nu_j(0) = \nu(A_j(0), \Phi_j(0)), \qquad \nu_j(1) = \nu(A_j(1), \Phi_j(1)).$$

Note that $t_j(0) \in \{0, 1\}$ and $t_j(1) \in \{0, 1\}$. Now consider the projection $\pi : \mathcal{W} \to [0, 1]$. This map is a diffeomorphism near each boundary point (A, Φ) and Lemma 7.23 asserts that $\nu(A, \Phi) = 1$ if and only if this diffeomorphism is orientation preserving near (A, Φ) and $\nu(A, \Phi) = -1$ otherwise. Now if both ends of the path $s \mapsto (t_j(s), A_j(s), \Phi_j(s))$ lie on the same side of the boundary of $\mathcal{W}$ (i.e. $t_j(0) = t_j(1)$) then the projection π has opposite parity at the two ends of the path and hence

$$t_j(0) = t_j(1) \qquad \Longrightarrow \qquad \nu_j(0) + \nu_j(1) = 0.$$

(In this case the crossing number of the operator family $\mathcal{D}_{A_j(s),\Phi_j(s)}$ as in (A.4) is odd.) On the other hand, if the path $s \mapsto (t_j(s), A_j(s), \Phi_j(s))$ runs from $t = 0$ to $t = 1$ or vice versa then π has the same parity at the two ends of the path and hence

$$t_j(0) \neq t_j(1) \qquad \Longrightarrow \qquad \nu_j(0) = \nu_j(1).$$

(In this case the crossing number of the operator family $\mathcal{D}_{A_j(s),\Phi_j(s)}$ is even.) Hence

$$\begin{aligned} SW(X, \Gamma_0, \eta_0, g_0) &= \sum_{t_j(0)=0} \nu_j(0) + \sum_{t_j(1)=0} \nu_j(1) \\ &= \sum_{t_j(0)=1} \nu_j(0) + \sum_{t_j(1)=1} \nu_j(1) \\ &= SW(X, \Gamma_1, \eta_1, g_1). \end{aligned}$$

This proves the theorem. □

Higher dimensional moduli spaces

For every metric g and every perturbation $\eta \notin \Omega^{2,+}_\Gamma(X,g)$ the moduli space $\mathcal{M}(X,\Gamma,g,\eta)$ is a compact subset of the infinite dimensional configuration space

$$\mathcal{C}(\Gamma) = \frac{\mathcal{A}(\Gamma)\times C^\infty(X,W^+)^*}{\mathcal{G}}$$

where $C^\infty(X,W^+)^* = C^\infty(X,W^+) - \{0\}$. If $\eta \in \Omega^{2,+}_{\mathrm{reg}}(X,i\mathbb{R};g)$ then this moduli space is a finite dimensional oriented submanifold of $\mathcal{C}(\Gamma)$ and a numerical invariant of X can be obtained by integrating a suitable cohomology class in $H^*(\mathcal{C}(\Gamma);\mathbb{Z})$ over this moduli space.

There is a natural 2-dimensional cohomology class $\tau \in H^2(\mathcal{C}(\Gamma);\mathbb{Z})$ which, in the simply connected case, is in fact the generator of H^2. To describe this class fix a point $x_0 \in X$ and consider the **based gauge group**

$$\mathcal{G}(x_0) = \{u\in\mathcal{G}\,|\,u(x_0)=1\}\,.$$

This group acts freely on $\mathcal{A}(\Gamma)$ and the quotient

$$\mathcal{C}(\Gamma,x_0) = \frac{\mathcal{A}(\Gamma)\times C^\infty(X,W^+)^*}{\mathcal{G}(x_0)}$$

is a circle bundle over $\mathcal{C}(\Gamma)$ where the circle acts on a pair $[A,\Phi]$ by rotating Φ and leaving A unchanged, i.e. $[A,\Phi]\mapsto[A,e^{i\theta}\Phi]$. Denote by

$$\tau \in H^2(\mathcal{C}(\Gamma);\mathbb{Z})$$

the Euler class of this circle bundle. The following exercise shows that this class is independent of the choice of the base point x_0 when X is connected.

Exercise 7.24 Assume that X is connected. Prove that the circle bundles $\mathcal{C}(\Gamma,x_i)\to\mathcal{C}(\Gamma)$ are isomorphic for two different points $x_0,x_1\in X$. **Hint:** Choose a smooth path $\gamma:[0,1]\to X$ with $\gamma(0)=x_0$ and $\gamma(1)=x_1$. Define a map $\rho_\gamma:\mathcal{A}(\Gamma)\to S^1$ by

$$\rho_\gamma(A_0+\alpha) = \exp\left(\int_0^1 \alpha_{\gamma(t)}(\dot\gamma(t))\,dt\right)$$

and show that

$$\rho_\gamma(u^*A) = u(x_1)\rho_\gamma(A)u(x_0)^{-1}$$

for all $A\in\mathcal{A}(\Gamma)$ and $u\in\mathcal{G}(\Gamma)$. Prove that the map

$$\mathcal{C}(\Gamma,x_0)\to\mathcal{C}(\Gamma,x_1):[A,\Phi]_0\mapsto[A,\rho_\gamma(A)\Phi]_1$$

is the required bundle isomorphism. □

The above construction of a cohomology class in the configuration space $\mathcal{C}(\Gamma)$ is a particularly simple example of Donaldson's μ-map in gauge theory. The general construction of this μ-map is based on the **universal bundle** over the product $X \times \mathcal{C}(\Gamma)$. Then every characteristic class of this universal bundle gives rise to a map $H_*(X) \to H^*(\mathcal{C}(\Gamma))$ given by the slant product. In the case at hand this universal bundle is a line bundle

$$\mathcal{L} \to X \times \mathcal{C}(\Gamma)$$

which in explicit terms can be expressed as the quotient

$$\mathcal{L} = \mathcal{L}(\Gamma) = \frac{X \times \mathcal{A}(\Gamma) \times C^\infty(X, W^+)^* \times \mathbb{C}}{\mathcal{G}}.$$

Here the action of $u \in \mathcal{G}$ on (x, A, Φ, z) is given by

$$u^*(x, A, \Phi, z) = \left(x, u^*A, u^{-1}\Phi, u(x)^{-1}z\right).$$

The above circle bundle $\mathcal{C}(\Gamma, x_0) \to \mathcal{C}(\Gamma)$ can evidently be identified with the unit circle bundle of the restriction $\mathcal{L}(x_0) = \mathcal{L}(\Gamma, x_0) = {\iota_{x_0}}^*\mathcal{L}$ to the submanifold $\{x_0\} \times \mathcal{C}(\Gamma)$. Hence the Euler class of the circle bundle $\mathcal{C}(\Gamma, x_0)$ agrees with the first Chern class of $\mathcal{L}(x_0)$:

$$\tau = c_1(\mathcal{L}(x_0)) \in H^2(\mathcal{C}(\Gamma); \mathbb{Z}).$$

From this point of view it is obvious that this class is independent of the choice of x_0 when X is connected. More generally, as in Donaldson theory, one can consider the μ-map

$$\mu : H_j(X; \mathbb{Z}) \to H^{2-j}(\mathcal{C}(\Gamma); \mathbb{Z})$$

given by the slant product with $c_1(\mathcal{L}) \in H^2(X \times \mathcal{C}(\Gamma); \mathbb{Z})$. The class $\tau \in H^2(\mathcal{C}(\Gamma); \mathbb{Z})$ is the image of the generator $1 \in H_0(X; \mathbb{Z})$. It is easy to see that the bundle $\mathcal{L}$ admits a trivialization over $\Sigma \times \{\mathrm{pt}\} \subset X \times \mathcal{C}(\Gamma)$ for every 2-dimensional submanifold $\Sigma \subset X$. Hence $\mu(\alpha) = 0$ for every $\alpha \in H_2(X; \mathbb{Z})$.

Remark 7.25 The configuration space $\mathcal{C}(\Gamma)$ fibers over the quotient

$$\mathcal{B}(\Gamma) = \frac{\mathcal{A}(\Gamma)}{\mathcal{G}} \sim \frac{H^1(X; \mathbb{R})}{H^1(X; 2\pi i\mathbb{Z})}$$

with projection $\pi : \mathcal{C}(\Gamma) \to \mathcal{B}(\Gamma)$ given by $\pi([A, \Phi]) = [A]$. The fibers are given by

$$\mathcal{F}(\Gamma) = \frac{C^\infty(X, W^+)^*}{S^1} \sim \mathbb{C}P^\infty.$$

For each $A \in \mathcal{A}(\Gamma)$ the inclusion of the fiber is the map $\iota_A : \mathcal{F}(\Gamma) \to \mathcal{C}(\Gamma)$ given by $\iota_A([\Phi]) = [A, \Phi]$. In other words $\mathcal{C}(\Gamma)$ fibers over the torus of dimension b_1 with fiber the infinite dimensional complex projective space.

$$\begin{array}{ccc} \mathcal{F}(\Gamma) & \hookrightarrow & \mathcal{C}(\Gamma) \\ & & \downarrow \pi \\ & & \mathcal{B}(\Gamma) \end{array}$$

The restriction of $\mathcal{L}(\Gamma, x_0)$ to the fiber $\mathcal{F}(\Gamma)$ is the canonical line bundle and thus the class ${\iota_A}^*\tau \in H^2(\mathcal{F}(\Gamma);\mathbb{Z})$ is the canonical generator of $H^2(\mathbb{C}P^\infty;\mathbb{Z})$. This property determines the class τ uniquely whenever X is simply connected, but in general one can add to the class τ the pullback of any 2-dimensional class on the base. □

Exercise 7.26 A generator of $H_{b_1}(\mathcal{B}(\Gamma);\mathbb{Z})$ is given by the moduli space $\mathcal{T} = \mathcal{T}(\eta)$ of reducible solutions of the perturbed Seiberg-Witten equations, i.e. the space of gauge equivalence classes of connections $A \in \mathcal{A}(\Gamma)$ which satisfy $F_A^+ + \eta = 0$. This space can be identified with the torus $H^1(X;i\mathbb{R})/H^1(X;2\pi i\mathbb{Z})$. (See page 305 below for more details.) Fix an orientation of $H^1(X;\mathbb{R})$ and denote by

$$\mathrm{dvol}_{\mathcal{T}} \in H^{b_1}(\mathcal{B}(\Gamma);\mathbb{Z})$$

the positive generator which evaluates to 1 on the fundamental class of $\mathcal{T}$. If

$$\gamma_1, \dots, \gamma_{b_1} \in H_1(X;\mathbb{Z}).$$

is a positively oriented set of integral generators prove that

$$\mu(\gamma_1) \wedge \cdots \wedge \mu(\gamma_{b_1}) = \pi^* \mathrm{dvol}_{\mathcal{T}} \in H^{b_1}(\mathcal{C}(\Gamma);\mathbb{Z}).$$ □

Exercise 7.27 For every 1-form $\alpha \in \Omega^1(X, i\mathbb{R})$ and every smooth path $\gamma : [0,1] \to X$ consider the *holonomy* $\rho_\alpha(\gamma) \in S^1$ defined by

$$\rho_\alpha(\gamma) = \exp\left(\int_\gamma \alpha\right).$$

Fix a point $x_0 \in X$ and for each point $x \in X$ near x_0 denote by $\gamma_x : [0,1] \to X$ the path running from x to x_0 in a straight line in some fixed local chart. Fix a reference connection A_0 and a and a nonzero section $\Psi \in C^\infty(X, W^+)$ with support in the given neighbourhood of x_0 and consider the map $h : \mathcal{A}(\Gamma) \to C^\infty(X, W^+)^*$ defined by

$$h(A)(x) = \rho_{A-A_0}(\gamma_x)\Psi(x)$$

Prove that this map satisfies

$$h(u^*A) = u(x_0)u^{-1}h(A)$$

for $u \in \mathcal{G}$. Prove that for any such map the function $(A, \Phi) \mapsto \langle h(A), \Phi \rangle_{L^2}$ can be interpreted as a section of the line bundle $\mathcal{L}(\Gamma, x_0) \to \mathcal{C}(\Gamma)$ which is transverse to the zero section. Deduce that the codimension-2 submanifold

$$\mathcal{N}_h = \left\{ [A, \Phi] \,\Big|\, \int_X \langle h(A), \Phi \rangle \mathrm{dvol} = 0 \right\} \subset \mathcal{C}(\Gamma)$$

admits a natural coorientation and represents the class τ in the sense that

$$\int_\Sigma \tau = \mathcal{N}_h \cdot \Sigma$$

for every 2-dimensional oriented submanifold $\Sigma \subset \mathcal{C}(\Gamma)$. □

Exercise 7.28 A connection on the bundle $\mathcal{L}(\Gamma, x_0)$ can be defined as an imaginary valued 1-form Θ on the total space $\mathcal{A}(\Gamma) \times C^\infty(X, W^+)^*$ which satisfies

$$\begin{aligned} \Theta_{(u^*A, u^{-1}\Phi)}(\alpha, u^{-1}\varphi) &= \Theta_{(A,\Phi)}(\alpha, \varphi), \\ \Theta_{(A,\Phi)}(d\xi, -\xi\Phi) &= \xi(x_0) \end{aligned} \tag{7.21}$$

for $A \in \mathcal{A}(\Gamma)$, $\Phi \in C^\infty(X, W^+)^*$, $u \in \mathcal{G}$, and $\xi \in \Omega^0(X, i\mathbb{R}) = \mathrm{Lie}(\mathcal{G})$. Prove that an example of such a connection is given by the formula

$$\Theta_{(A,\Phi)}(\alpha, \varphi) = -\frac{i}{\|\Phi\|^2} \int_X \mathrm{Im} \langle \Phi, \varphi - R_0(d^*\alpha)\Phi \rangle \mathrm{dvol}. \tag{7.22}$$

Here $\|\Phi\|$ denotes the L^2-norm and the linear operator $R_0 : \Omega^0(X, i\mathbb{R}) \to \Omega^0(X, i\mathbb{R})$ is defined by

$$R_0(\zeta) = \xi \qquad \Longleftrightarrow \qquad d^*d\xi = \zeta - \frac{1}{\mathrm{Vol}(X)} \int_X \zeta \mathrm{dvol}, \quad \xi(x_0) = 0.$$

Prove that the curvature 2-form $\tau = id\Theta/2\pi$ on $\mathcal{A}(\Gamma) \times C^\infty(X, W^+)^*$ descends to a closed form on $\mathcal{C}(\Gamma)$ which represents the first Chern class of the line bundle $\mathcal{L}(\Gamma, x_0)$. Prove that the pull back of Θ to the fiber $C^\infty(X, W^+)^*$ under the map $\iota_A(\Phi) = (A, \Phi)$ is given by

$$\iota_A{}^*\Theta = \frac{1}{2}\left(\bar{\partial} f - \partial f\right)$$

where $f : C^\infty(X, W^+)^* \to \mathbb{R}$ is the function

$$f(\Phi) = \log\left(\int_X |\Phi|^2 \mathrm{dvol}\right).$$

Deduce that
$$\tau = {\iota_A}^* \frac{i}{2\pi} d\Theta = \frac{1}{2\pi i} \bar{\partial}\partial f.$$
Find an explicit formula for this 2-form and prove that it represents the standard generator of $H^2(C^\infty(X, W^+)^*/S^1; \mathbb{Z}) = \mathbb{Z}$. Identify S^3 with the unit sphere in $\mathbb{C}^2$ and consider the map

$$S^2 = S^3/S^1 \to C^\infty(X, W^+)^*/S^1 : [z_0 : z_1] \mapsto [z_0\Phi_0 + z_1\Phi_1].$$

Prove that the integral of τ over this map is 1. **Hint:** Note that the form τ descends to $C^\infty(X, W^+)^*/\mathbb{C}^*$ and use coordinates $(1, z) \in \mathbb{C}^2$. Show that tha pullback form on $\mathbb{C}$ is a constant multiple of the standard form in Example 3.48. The form ${\iota_A}^*\tau$ is an infinite dimensional version of the Fubini-Study Kähler form. Compare this with the proof of the normalization axiom on page 22. □

Now assume that the moduli space $\mathcal{M}(X, \Gamma, g, \eta)$ has dimension $2d$. This is equivalent to the condition

$$\frac{c \cdot c}{4} - \frac{2\chi(X) + 3\sigma(X)}{4} = 2d \tag{7.23}$$

where $c = c_1(L_\Gamma)$ and it follows that $b^+ - b_1$ is odd. If $\eta \in \Omega^{2,+}_{\mathrm{reg}}(X, i\mathbb{R}; \Gamma, g)$ then the moduli space $\mathcal{M}(X, \Gamma, g, \eta) \subset \mathcal{C}(\Gamma)$ is a smooth compact submanifold which carries a natural orientation. Thus it represents a homology class
$$[\mathcal{M}(X, \Gamma, g, \eta)] \in H_{2d}(\mathcal{C}(\Gamma); \mathbb{Z}).$$
Evaluating the d-th power of the class $\tau \in H^2(\mathcal{C}(\Gamma); \mathbb{Z})$ on the fundamental class of $\mathcal{M}(X, \Gamma, g, \eta)$ gives rise to the **Seiberg-Witten invariant**

$$SW(X, \Gamma; g, \eta) = \int_{\mathcal{M}(X,\Gamma,g,\eta)} \tau^d. \tag{7.24}$$

Note that the cohomology class τ in this context can be expressed in purely finite dimensional terms, namely, as the first Chern class of the restriction of the line bundle $\mathcal{L}(\Gamma, x_0) \to \mathcal{C}(\Gamma)$ to the moduli space $\mathcal{M}(X, \Gamma, g, \eta)$.

Theorem 7.29. (Seiberg-Witten) *Assume that $b^+ - b_1$ is odd and $b^+ > 1$. Then the number*

$$SW(X, \Gamma) = SW(X, \Gamma; g, \eta)$$

is independent of the choice of g and η and depends only on the isomorphism class of the spin^c structure Γ.

Proof: Let g_0 and g_1 be two Riemannian metrics on X with corresponding equivalent spin^c structures Γ_0 and Γ_1. Assume $\eta_j \in \Omega^{2,+}_{\mathrm{reg}}(X, i\mathbb{R}; \Gamma_j, g_j)$ and abbreviate $\mathcal{M}(\eta_j) = \mathcal{M}(X, \Gamma_j, g_j, \eta_j)$ for $j = 0, 1$. Fix a path of metrics $t \mapsto g_t$ from g_0 to g_1 with corresponding spin^c structures Γ_t. By Theorem 7.21 and Proposition 7.10, choose a generic path $\{\eta_t\} \in \Omega^{2,+}_{\mathrm{reg}}(X, i\mathbb{R}; \{g_t\}, \eta_0, \eta_1)$ such that $\eta_t \in \Omega^{2,+}(X; \Gamma, g_t)$ for every t. This is possible whenever $b^+ \geq 2$. Then the moduli space $\mathcal{M}(\{\eta_t\}) = \mathcal{M}(X, \{\Gamma_t\}, \{g_t\}, \{\eta_t\})$, constructed in Theorem 7.21, has dimension $2d + 1$ and is a compact oriented cobordism with oriented boundary $\partial\mathcal{M}(\{\eta_t\}) = \mathcal{M}(\eta_1) - \mathcal{M}(\eta_0)$. Hence, by Stokes' theorem,

$$\int\limits_{\mathcal{M}(\eta_1)} \tau^d - \int\limits_{\mathcal{M}(\eta_0)} \tau^d = \int\limits_{\partial\mathcal{M}(\{\eta_t\})} \tau^d = \int\limits_{\mathcal{M}(\{\eta_t\})} d\tau^d = 0.$$

This proves the theorem. $\square$

Four-manifolds with $b^+ = 1$

Let X be a compact oriented smooth 4-manifold with

$$b^+ = 1, \qquad b_1 \in 2\mathbb{Z}.$$

Fix an orientation of $H^{2,+}(X)$. Then for every Riemannian metric g on X there exists a unique self-dual harmonic 2-form

$$\omega_g \in H^{2,+}(X)$$

which has L^2-norm 1 and determines the given orientation of $H^{2,+}$. Recall that the Γ-wall

$$\Omega^{2,+}_{\Gamma}(X, g) \subset \Omega^{2,+}(X, g)$$

is defined as the set of those perturbations $\eta \in \Omega^{2,+}(X, g)$ for which the Seiberg-Witten equations (7.4) have solutions of the form $(A, 0)$, i.e. for which there exists a connection $A \in \mathcal{A}(\Gamma)$ with $F_A^+ + \eta = 0$. By Proposition 7.10 the Γ-wall has codimension $b^+ = 1$. If η is a regular perturbation in the complement of the Γ-wall, i.e.

$$\eta \in \Omega^{2,+}_{\mathrm{reg}}(X, i\mathbb{R}; g) - \Omega^{2,+}_{\Gamma}(X, i\mathbb{R}; g),$$

then there is a compact moduli space $\mathcal{M}(X, \Gamma, g, \eta)$ of the correct dimension and the Seiberg-Witten invariant $SW(X, \Gamma; g, \eta)$ can be defined as before by (7.20) in the zero-dimensional case and (7.24) for higher dimensional moduli spaces. Moreover, the proofs of Theorems 7.22 and 7.29 show that these invariants are independent of the pair (g, η) along any path which does not cross the Γ-wall. Now recall that the Γ-wall is characterized by the condition $\varepsilon(g,\eta)=0$, where

$$\varepsilon(g,\eta)=\varepsilon_\Gamma(g,\eta)=-\int_X\langle i\eta,\omega_g\rangle \mathrm{dvol}_g-\pi[\omega_g]\cdot c_1(L_\Gamma)$$

as in (7.7). Hence for every spin^c structure Γ there are two invariants $SW^\pm(X,\Gamma)$ defined by

$$SW^\pm(X,\Gamma)=SW(X,\Gamma;g,\eta),\qquad \pm\varepsilon(g,\eta)>0,$$

The relation between these invariants can be studied by examining how the moduli space $\mathcal{M}(X,\Gamma,g,\eta)$ changes as the parameter η crosses the Γ-wall. In the simply connected case one obtains the formula

$$SW^+(X,\Gamma)-SW^-(X,\Gamma)=1$$

whenever the moduli spaces have nonnegative dimension. In the case $b_1>0$ the relation between the two invariants is more complicated but there is still an explicit wall-crossing formula. This will be discussed in detail in Chapter 9.

7.5 Basic properties of the invariants

Finiteness

There are natural restrictions on a spin^c structure Γ with nontrivial Seiberg–Witten invariants, firstly from the a priori estimate of Lemma 7.13 and secondly from the requirement that the dimension of the moduli space $\mathcal{M}(X,\Gamma,g,\eta)$ be nonnegative.

Proposition 7.30. (Seiberg-Witten) *Let X be a compact oriented 4-manifold with $b^+>0$ and suppose that $\eta\in\Omega^{2,+}_{\mathrm{reg}}(X,i\mathbb{R};g)-\Omega^{2,+}_\Gamma(X,i\mathbb{R};g)$ for every spin^c structure Γ. Then there are only finitely many (isomorphism classes of) spin^c structures Γ with a nonempty moduli space $\mathcal{M}(X,\Gamma,g,\eta)$. In particular, when $b^+>1$ the Seiberg-Witten invariants $SW(X,\Gamma)$ are zero for all but finitely many spin^c structures Γ.*

Proof: Denote $c=c_1(L_\Gamma)$. The formula

$$\dim\mathcal{M}(X,\Gamma,g,\eta)=\frac{c\cdot c}{4}-\frac{2\chi(X)+3\sigma(X)}{4}$$

shows that the moduli space can only be nonempty if

$$c\cdot c\geq 2\chi(X)+3\sigma(X).$$

Now the class $c\in H^2(X;\mathbb{Z})$ is represented by the 2-form $\frac{i}{\pi}F_A$ and hence

$$\pi^2 c \cdot c = -\int_X F_A \wedge F_A = \left\| F_A^+ \right\|^2 - \left\| F_A^- \right\|^2 .$$

Here $\|.\|$ denotes the L^2-norm. It follows that there is a universal constant $K_0 = K_0(X) = -\pi^2(2\chi(X) + 3\sigma(X))$ such that

$$\left\| F_A^- \right\|^2 - \left\| F_A^+ \right\|^2 \le K_0$$

for every $(A, \Phi) \in \mathcal{M}(X, \Gamma, g, \eta)$. Now recall from Lemma 7.13 that there exists a constant $K_1 = K_1(X, g, \eta)$, again independent of the spin^c structure Γ, such that

$$\sup_X |\Phi| \le K_1$$

for every $(A, \Phi) \in \mathcal{M}(X, \Gamma, g, \eta)$ and every Γ. Since

$$F_A^+ = \sigma^+((\Phi\Phi^*)_0) - \eta$$

there is an estimate

$$\left\| F_A^+ \right\|^2 \le K_2$$

for a suitable constant $K_2 = K_2(X, g, \eta)$ and every $(A, \Phi) \in \mathcal{M}(X, \Gamma, g, \eta)$. Hence both F_A^+ and F_A^- are uniformly bounded and so

$$\|F_A\|^2 \le K_0 + 2K_2.$$

Thus there are only finitely many values of $c = c_1(L_\Gamma) \in H^2(X; \mathbb{Z})$ for which the moduli space $\mathcal{M}(X, \Gamma, g, \eta)$ is nonempty provided that η is regular. Since for every c there are only finitely many spin^c structures with $c_1(L_\Gamma) = c$ the proposition is proved. □

The bound on $\|F_A\|^2$ in Proposition 7.30 depends on the choice of the perturbation η. This is especially relevant in the case $b^+ = 1$. For example, if X admits a metric of positive scalar curvature (such as $\mathbb{C}P^2$ or $S^2 \times \Sigma$ for any Riemann surface Σ) then Proposition 7.32 shows that the Seiberg-Witten invariant must vanish on the side of the wall which contains the perturbation $\eta = 0$. Now there is a crossing-of-the-wall formula (discussed in Section 9.2 below) which in many cases asserts that the invariant must be nontrivial on the other side of the wall. Hence in this case there are infinitely many chambers with nontrivial Seiberg-Witten invariants. Proposition 7.30 asserts in this case that each particular perturbation parameter η can only lie in finitely many chambers with nontrivial invariants, or in other words, that the walls corresponding to different spin^c structures Γ move further and further away from the origin as $c_1(L_\Gamma) \to \infty$. (See Proposition 7.10.)

Symmetry

Recall from the discussion on page 227 that spin^c structures come in pairs $\Gamma : TX \to \mathrm{End}(W)$ and $\Gamma^* : TX \to \mathrm{End}(W^*)$ where $W^* = \mathrm{Hom}(W, \mathbb{C})$e. Thus

$$c_1(L_{\Gamma^*}) = -c_1(L_\Gamma).$$

The following proposition shows how the Seiberg-Witten invariants for Γ and Γ^* are related.

Proposition 7.31. (Seiberg-Witten) *Let X be a compact smooth oriented 4-manifold with $b^+ - b_1$ odd and $\Gamma : TX \to \mathrm{End}(W)$ be a spin^c structure. If $b^+ > 1$ then*

$$SW(X, \Gamma^*) = (-1)^{\frac{\chi+\sigma}{4}} SW(X, \Gamma)$$

and if $b^+ = 1$ then

$$SW^+(X, \Gamma^*) = (-1)^{\frac{\chi+\sigma}{4}} SW^-(X, \Gamma).$$

Proof: By Exercise 7.9 there is a natural bijection

$$\mathcal{M}(X, \Gamma, g, \eta) \to \mathcal{M}(X, \Gamma^*, g, -\eta) : [A, \Phi] \mapsto [A^*, \Phi^*]$$

where A^* denotes the virtual connection on ${L_{\Gamma^*}}^{1/2} = {L_\Gamma}^{-1/2}$ induced by A. In fact one can think of this bijection simply as a change of notation in the definition of the complex numbers, replacing i by $-i$ and it is then obvious that η is regular for Γ if and only if $\eta^* = -\eta$ is regular for Γ^*:

$$\eta \in \Omega^{2,+}_{\mathrm{reg}}(X, i\mathbb{R}; \Gamma, g) \qquad \Longleftrightarrow \qquad -\eta \in \Omega^{2,+}_{\mathrm{reg}}(X, i\mathbb{R}; \Gamma^*, g).$$

Moreover, in the regular case the above map is a diffeomorphism which, however, need not be orientation preserving. It relates the two orientations by the sign which is determined by the complex index of the Dirac operator. This diffeomorphism also reverses the sign of the first Chern class of the canonical line bundle $\mathcal{L} \to \mathcal{M}$. Hence the net change in the sign of the Seiberg-Witten invariant is

$$(-1)^{\mathrm{index}\, D_A/2 - \dim \mathcal{M}/2} = (-1)^{\frac{\sigma+\chi}{4}}.$$

In the case $b^+ = 1$ note that

$$\varepsilon_{\Gamma^*}(g, -\eta) = -\varepsilon_\Gamma(g, \eta).$$

and hence the correspondence $\Gamma \mapsto \Gamma^*$ interchanges the invariants SW^+ and SW^-. This proves the proposition. □

Scalar curvature

Proposition 7.32 *Let X be a compact oriented smooth 4-manifold of positive scalar curvature with $b^+ - b_1$ odd and $b^+ \geq 2$. Then all the Seiberg-Witten invariants are zero.*

Proof: First assume that g is a metric with positive scalar curvature and choose $\eta \in \Omega^{2,+}_{\mathrm{reg}}(X, i\mathbb{R}; \Gamma, g)$ so small that

$$2\sqrt{2} \sup_X |\eta| - \frac{1}{2} \inf_X s < 0.$$

Then the a piori estimate of Lemma 7.13 shows that $\mathcal{M}(X, \Gamma, g, \eta) = \emptyset$. Hence $SW(X, \Gamma, g, \eta) = 0$ for this choice of g and η and all spin^c structures Γ. □

The previous result can be interpreted as a nonlinear version of Lichnerowicz' theorem 6.30. It can be used to prove that hypersurfaces in $\mathbb{C}P^3$ of odd degree do not admit metrics of positive scalar curvature and hence cannot be diffeomorphic to connected sums of the form $\ell\mathbb{C}P^2 \# m\overline{\mathbb{C}P}^2$. This last assertion was previously proved by Donaldson via his polynomial invariants. There is a more sophisticated vanishing theorem for connected sums, which is due to Morgan, Taubes and others, and will be discussed in Section 11.2. Another interesting consequence of the nontriviality of the invariants arises for manifolds with constant scalar curvature.

Proposition 7.33. (Witten) *Let X be a compact smooth 4-manifold with a metric g of constant scalar curvature s. Fix a spin^c structure $\Gamma : X \to \mathrm{End}(W)$ and denote $c = c_1(L_\Gamma) \in H^2(X; \mathbb{Z})$. Suppose that one of the following conditions is satisfied.*

(i) *$b^+ \geq 2$ and $SW(X, \Gamma) \neq 0$.*

(ii) *$b^+ = 1$, $c \cdot [\omega_g] \leq 0$, and $SW^+(X, \Gamma) \neq 0$.*

Then

$$c \cdot c \leq \frac{s^2 \mathrm{Vol}(X)}{32\pi^2}$$

with equality if and only if there exists a pair $(A, \Phi) \in \mathcal{A}(\Gamma) \times C^\infty(X, W^+)$ which satisfies

$$|F_A^+|^2 = \frac{s^2}{32}, \qquad F_A^- = 0, \qquad \nabla_A \Phi = 0, \qquad |\Phi|^2 = -\frac{s}{2}.$$

Proof: Each of the conditions implies that the unperturbed moduli space $\mathcal{M}(X, \Gamma, g, 0)$ is nonempty. In the case (ii) the condition $c \cdot [\omega_g] \leq 0$ is equivalent to $\varepsilon_\Gamma(g, 0) \geq 0$ and hence $\mathcal{M}(X, \Gamma, g, i\lambda\omega_g)$ is nonempty for every $\lambda > 0$ and the assertion follows by taking the limit $\lambda \to 0$. Hence in both cases there exists a solution (A,Φ) of the unperturbed Seiberg-Witten equations (7.1). Assume first that that $\Phi \neq 0$. Then, by Lemma 7.13, the monopole Φ satisfies the pointwise inequality

$$2|\Phi| + s \leq 0$$

and, using the rules of Lemma 7.4, one finds that

$$\left|F_A^+\right|^2 = \left|\sigma^+((\Phi\Phi^*)_0)\right|^2 = \frac{1}{2}\left|(\Phi\Phi^*)_0\right|^2 = \frac{1}{8}\left|\Phi\right|^4 \leq \frac{s^2}{32}.$$

This inequality is obviously satisfied when $\Phi = 0$ and hence $F_A^+ = 0$. It follows that

$$c \cdot c = \frac{1}{\pi^2}\int_X \left(\left|F_A^+\right|^2 - \left|F_A^-\right|^2\right)\mathrm{dvol} \leq \frac{s^2\mathrm{Vol}(X)}{32\pi^2}$$

with equality if and only if $F_A^- = 0$ and $|F_A^+| \equiv s^2/32$. By Proposition 7.3, the energy of (A,Φ) is given by

$$E(A,\Phi) = -\pi^2 c\cdot c \geq -\frac{1}{32}\int_X s^2\mathrm{dvol}$$

Suppose now that $c \cdot c = s^2\mathrm{Vol}(X)/32\pi^2$ so that $|F_A| = |F_A^+| = s^2/32$. Then we find

$$\begin{aligned}
0 &= E(A,\Phi) + \frac{1}{32}\int_X s^2\mathrm{dvol} \\
&= \int_X \left(|\nabla_A\Phi|^2 + \frac{s}{4}|\Phi|^2 + \frac{1}{4}|\Phi|^4 + |F_A|^2 + \frac{s^2}{32}\right)\mathrm{dvol} \\
&= \int_X \left(|\nabla_A\Phi|^2 + \frac{s}{4}|\Phi|^2 + \frac{1}{4}|\Phi|^4 + \frac{s^2}{16}\right)\mathrm{dvol} \\
&= \int_X \left(|\nabla_A\Phi|^2 + \frac{1}{16}\left(2|\Phi|^2 + s\right)^2\right)\mathrm{dvol}.
\end{aligned}$$

This shows that $\nabla_A\Phi = 0$ and $2|\Phi|^2 = -s$ as claimed. □

This proposition has particularly interesting implications when X admits an Einstein metric and the moduli space is zero-dimensional. The following theorem is due to LeBrun (cf [71]). It is a generalization of the Miyaoka-Yau inequality for Kähler surfaces (cf [128]). LeBrun also used these techniques to prove that Einstein metrics are essentially unique on manifolds whose universal cover is the complex hyperbolic space.

Theorem 7.34. (LeBrun) *Let X be a compact oriented smooth 4-manifold with an Einstein metric g. Let $\Gamma : TX \to \mathrm{End}(W)$ be a spin^c structure such that the class $c = c_1(L_\Gamma)$ satisfies*

$$c \cdot c = 2\chi + 3\sigma$$

Suppose that one of the following conditions is satisfied.
(i) $b^+ \geq 2$ *and* $SW(X, \Gamma) \neq 0$.
(ii) $b^+ = 1$, $c \cdot [\omega_g] \leq 0$, *and* $SW^+(X, \Gamma) \neq 0$.
Then

$$-2\chi \leq 3\sigma \leq \chi.$$

Moreover, $3\sigma = \chi$ if and only if the universal cover of X is either $\mathbb{R}^4$ or the complex hyperbolic 2-space $\mathbb{CH}^2 = \mathrm{SU}(2,1)/\mathrm{U}(2)$.

Proof: The Hitchin-Thorpe formulae for the Euler characteristic and signature of a Riemannian 4-manifold are

$$\chi = \frac{1}{8\pi^2} \int_X \left(|W^+|^2 + |W^-|^2 + \frac{s^2}{24} - \frac{1}{2}|R_0|^2 \right) \mathrm{dvol}$$

and

$$\sigma = \frac{1}{12\pi^2} \int_X \left(|W^+|^2 - |W^-|^2 \right) \mathrm{dvol}.$$

Here s is the scalar curvature, R_0 is the traceless part of the Ricci tensor, and $W^\pm$ denote the self-dual and anti-self-dual parts of the Weyl tensor. The Einstein condition is precisely that the traceless part of the Ricci tensor is zero. Hence for Einstein manifolds we have

$$2\chi \pm 3\sigma = \frac{1}{4\pi^2} \int_X \left(2|W^\pm|^2 + \frac{s^2}{24} \right) \mathrm{dvol} \geq \frac{s^2 \mathrm{Vol}(X)}{96\pi^2}.$$

With the $+$ sign this is the **Hitchin-Thorpe inequality** $2\chi + 3\sigma \geq 0$. Now it follows from Proposition 7.33 that under the assumptions of the theorem

$$2\chi + 3\sigma = c \cdot c \leq \frac{s^2 \mathrm{Vol}(X)}{32\pi^2}.$$

The two inequalities together give

$$2\chi + 3\sigma \leq 3(2\chi - 3\sigma).$$

This is equivalent to LeBrun's generalization of the **Miyaoka-Yau inequality** $3\sigma \leq \chi$. If equality holds then $W^- = 0$ and $c \cdot c = s^2 \mathrm{Vol}(X)/32\pi^2$. This can be used to prove that X is Kähler and locally symmetric. It then follows that the exponential map induces an isometry between either $\mathbb{R}^4$ or complex hyperbolic space with the universal cover of X. For more details of this argument see [71]. □

Simple type

For all simply connected smooth 4-manifolds with $b^+ > 1$ for which the Seiberg-Witten invariants are known (at the time of writing) the higher dimensional invariants are zero. Such manifolds are said to be of simple type.

Definition 7.35 *Let X be a compact oriented smooth 4-manifold with $b^+ - b_1$ odd and $b^+ \geq 2$. A cohomology class $c \in H^2(X;\mathbb{Z})$ is called a* **basic class**, *or* **SW-basic class**, *if there exists a spin^c structure Γ on TX with $c_1(L_\Gamma) = c$ and $SW(X,\Gamma) \neq 0$. The manifold X is said to be of* **simple type**, *or* **SW-simple type**, *if*

$$c \cdot c = 2\chi(X) + 3\sigma(X) \tag{7.25}$$

for every basic class $c \in H^2(X;\mathbb{Z})$. This means that the spin^c structures with nonzero Seiberg-Witten invariants all have zero dimensional moduli spaces.

Remark 7.36 The cohomology classes $c \in H^2(X;\mathbb{Z})$ which are integral lifts of the second Stiefel-Whitney class $\mathrm{w}_2(TX)$ and satisfy (7.25) are in one-to-one correspondence with isomorphism classes of almost complex structures on TX. (See Proposition 13.1 below.) Hence, for 4-manifolds without 2-torsion in $H_1(X;\mathbb{Z})$, the simple type condition can be expressed in the form that the only possible spin^c structures with nontrivial Seiberg-Witten invariants are the canonical spin^c structures of almost complex structures on TX. □

Proposition 7.31 shows that the basic classes come in pairs $\pm c$. Evidently, the basic classes play a fundamental role in the topology of the manifold X. Any diffeomorphism of X must preserve these classes. Note, however, that there is no such restriction on homeomorphisms.

It is known that $\mathbb{C}P^2$ is not of simple type. However, care must be taken here since $b^+ = 1$ and so the *invariants* change as η crosses the Γ-walls. So far all known 4-manifolds with $b^+ > 1$ are either of simple type or it is not known whether they are. Witten conjectured that there should be 4-manifolds with $b^+ > 1$ which are not of simple type.

Exercise 7.37 Prove that every orientation preserving diffeomorphism $f : X \to X$ preserves the basic classes and

$$SW(X, f^*\Gamma) = SW(X,\Gamma)$$

for every spin^c structure Γ. Here $f^*\Gamma : TX \to \mathrm{End}(f^*W)$ denotes the obvious pullback structure. Show that $(A,\Phi) \in \widetilde{\mathcal{M}}(X,\Gamma,g,\eta)$ if and only if then $(f^*A, \Phi \circ f) \in \widetilde{\mathcal{M}}(X, f^*\Gamma, f^*g, f^*\eta)$. □

Relation to Donaldson's invariants

In their work [63, 64, 65] on the structure of Donaldson's invariants Kronheimer and Mrowka introduced another notion of simple type which will henceforth be called *D-simple type*. In [126] Witten conjectured that both notions of simple type should agree and that, for manifolds of simple type, the Donaldson invariants should be completely determined by the Seiberg-Witten invariants.

Here is a brief review of the definition of Donaldson's invariants. Let X be a compact connected simply connected smooth 4-manifold. Assume throughout that b^+ is odd and

$$b^+ \geq 3.$$

Given an integer k choose a principal SU(2)-bundle $P \to X$ with Chern number $c_2(P) = k$ and denote by M_k the moduli space of gauge equivalence classes of anti-self-dual connections (*instantons* in the physics terminology) on P. For a generic metric this space is a smooth manifold of dimension

$$\dim M_k = 2d = 8k - 3(1 + b^+).$$

In contrast to the Seiberg-Witten case this moduli space is never compact (unless it is empty) and there is a nontrivial compactification problem. Moreover, the topology of these moduli spaces (and of the ambient configuration space $\mathcal{B}(P)$ of gauge equivalence classes of connections on P) is much richer than in the Seiberg-Witten case. There is a universal SO(3)-bundle

$$\mathbb{P} \to X \times M_k$$

whose first Pontryagin class gives rise to a correspondence between the homology of X and the cohomology of M_k. This is Donaldson's μ-map

$$\mu : H_i(X) \to H^{4-i}(M_k).$$

(All homology and cohomology groups in this section are to be understood with integer coefficients.) This map extends naturally, via exterior products, to the symmetric algebra

$$\mathbb{A}(X) = S^*(H_0(X) \oplus H_2(X)).$$

This is to be understood as a graded algebra where the homology classes in $H_2(X)$ have degree 2 and those in $H_0(X)$ have degree 4. With this convention the map $\mu : \mathbb{A}(X) \to H^*(M_k)$ preserves the degree. Note that all elements of $\mathbb{A}(X)$ have even degree and, for any integer d, denote by $\mathbb{A}_{2d}(X)$ the subspace of elements of degree $2d$. Briefly, Donaldson's polynomial invariants of X are defined by evaluating the cohomology classes

$\mu(z) \in H^{2d}(M_k)$ on the fundamental class of the moduli space M_k of dimension $2d$, namely

$$\mathbb{D}_X(z) = \langle \mu(z), [M_k] \rangle$$

for $z \in \mathbb{A}_{2d}(X)$ where $2d = 8k - 3(1+b^+)$. To make this idea work requires a lot of sophisticated analysis. Although the construction is similar in spirit to that of the Seiberg-Witten invariants, it is technically much more difficult. (For details see [17, 18, 21].)

There is a distinguished homology class $u \in H_0(X)$, the generator, which plays an important role in the structure of the Donaldson invariants. This class can be used to formally relate moduli spaces of different dimensions, corresponding to bundles with different Chern numbers. For example, if $z \in \mathbb{A}_{2d}(X)$ with $2d = 8k - 3(1 + b^+)$ then $u^2 z \in \mathbb{A}_{2d+8}(X)$ and there are invariants

$$\mathbb{D}_X(z) = \langle \mu(z), [M_k] \rangle, \qquad \mathbb{D}_X(u^2 z) = \langle \mu(u^2 z), [M_{k+1}] \rangle.$$

Of course, a priori there is no guarantee that there should be any relation between these two numbers. However, such relations between the Donaldson invariants corresponding to moduli spaces of different dimensions were discovered by Kronheimer and Mrowka [65]. They introduced the concept of *D-simple type* and proved that a large class of simply connected 4-manifolds with $b^+ \geq 3$ possess this property.

Definition 7.38 *Let X be a compact connected simply connected smooth 4-manifold with b^+ odd and greater than or equal to 3. Then X is said to be of* **D-simple type** *if for every $z \in \mathbb{A}(X)$*

$$\mathbb{D}_X(u^2 z) = 4\,\mathbb{D}_X(z).$$

Exercise 7.39 The condition of D-simple type can be rephrased in the form that $\mathbb{D}_X$ annihilates the ideal in $\mathbb{A}(X)$ generated by $u^2 - 4$. Show that this is equivalent to the formula

$$\mathbb{D}_X(e^{\lambda u} z) = e^{2\lambda} \mathbb{D}_X\left(\left(1 + \frac{u}{2}\right) z\right) + e^{-2\lambda} \mathbb{D}_X\left(\left(1 - \frac{u}{2}\right) z\right)$$

for $z \in \mathbb{A}(X)$ and $\lambda \in \mathbb{Z}$.

In 1993 Kronheimer and Mrowka proved the following structure theorem for Donaldson's invariants [63, 64, 65].

Theorem 7.40. (Kronheimer-Mrowka) *Let X be a compact connected simply connected smooth 4-manifold with b^+ odd and greater than or equal to 3. Assume that X has D-simple type. Then there exist cohomology classes $K_1, \ldots, K_s \in H^2(X)$ and rational numbers $a_1, \ldots, a_s$ such that*

$$\mathbb{D}_X\left(\left(1+\frac{u}{2}\right)e^h\right) = e^{h\cdot h/2}\sum_{i=1}^{s} a_i e^{K_i\cdot h}$$

for every $h \in H_2(X)$. The cohomology classes K_i are all integral lifts of $\mathrm{w}_2(TX)$. *Moreover, every oriented connected smoothly embedded 2-manifold $\Sigma \subset X$ with $\Sigma\cdot\Sigma \geq 0$ (which represents a nontrivial homology class in the case of genus 0) satisfies*

$$2g(\Sigma) - 2 \geq \Sigma\cdot\Sigma + |K_i\cdot\Sigma|$$

where $g(\Sigma)$ denotes the genus.

The cohomology classes K_i are called the **KM-basic classes**. In [126] Witten conjectured that these agree with the basic classes of Definition 7.35 and that, moreover, the numbers a_i agree, up to a universal factor, with the corresponding Seiberg-Witten invariants.

Conjecture 7.41. (Witten) *Let X be a compact connected simply connected smooth 4-manifold with b^+ odd and greater than or equal to 3. Then X has D-simple type if and only if it has SW-simple type. Moreover, if X has simple type then the KM-basic classes agree with the SW-basic classes and*

$$\mathbb{D}_X\left(\left(1+\frac{u}{2}\right)e^h\right) = 2^{2+\frac{7\chi+11\sigma}{4}}e^{h\cdot h/2}\sum_{\Gamma} SW(X,\Gamma)e^{c_1(L_\Gamma)\cdot h}, \tag{7.26}$$

$$\mathbb{D}_X\left(\left(1-\frac{u}{2}\right)e^h\right) = 2^{2+\frac{7\chi+11\sigma}{4}}i^{\frac{\chi+\sigma}{4}}e^{-h\cdot h/2}\sum_{\Gamma} SW(X,\Gamma)e^{-ic_1(L_\Gamma)\cdot h} \tag{7.27}$$

for $h \in H_2(X)$. In both cases the sum runs over all isomorphism classes of spin^c structures on X.

Exercise 7.42 Use the formula of Proposition 7.31 to show that the right hand side of (7.27) is real. Moreover, show that the numbers $(7\chi + 11\sigma)/4$ and $(\chi + \sigma)/4$ are integers whenever $b_1 = 0$ and b^+ is odd. □

In [126] Witten gave a heuristic *"proof"* of his conjecture based on physical considerations. In his lectures in December 1994 Pidstrigach outlined a geometric approach for a mathematically rigorous proof which he developed jointly with Tyurin (cf. [104]). Their basic idea is to use analogues of the Seiberg-Witten monopole equations, for spin^c structures on X twisted by rank-2-bundles rather than line bundles, to obtain a moduli space which contains both the Seiberg-Witten moduli spaces $\mathcal{M}(X,\Gamma,g,\eta)$ and the ASD instanton moduli spaces M_k as reductions.

Example 7.43 It is a consequence of Theorem 12.9 below that $c = 0$ is the only SW-basic class of the $K3$ surface. Moreover, with $\chi = 24$ and $\sigma = -16$ one finds $2 + (7\chi + 11\sigma)/4 = 0$. Hence the right hand side of (7.26) gives

$$\mathbb{D}_{\mathrm{K3}}\left(\left(1 + \frac{u}{2}\right) e^h\right) = e^{h \cdot h/2}$$

in agreement with known computations of Donaldson's invariants. More generally, both invariants have been computed for general elliptic surfaces and these computations confirm Witten's conjecture in this case. The computation of the Seiberg-Witten invariants for elliptic surfaces will be discussed in Section 12.9 □

8

TRANSVERSALITY AND COMPACTNESS

This chapter contains the proofs of the fundamental theorems 7.12 and 7.16 concerning the compactness of the space of solutions of the Seiberg-Witten equations and the fact that these solutions form finite dimensional manifolds. The first section gives an explicit discussion of the Seiberg-Witten equations on flat $\mathbb{R}^4$. This is used in Section 8.2 for the proof of the removable singularity theorem. Moreover it may be a good starting point for readers to familiarize themselves with the Seiberg-Witten monopole equations.

8.1 Monopoles on flat Euclidean space

Consider the Seiberg-Witten equations (7.1) on flat Euclidean space $\mathbb{R}^4 = \mathbb{H}$ with coordinates x_0, x_1, x_2, x_3. Fix the constant spin^c structure $\Gamma : \mathbb{H} = T_x\mathbb{H} \to \mathbb{C}^{4\times 4}$ given by

$$\Gamma(\xi) = \begin{pmatrix} 0 & \gamma(\xi) \\ -\gamma(\xi)^* & 0 \end{pmatrix}, \qquad \gamma(\xi) = \begin{pmatrix} \xi_0 + i\xi_1 & \xi_2 + i\xi_3 \\ -\xi_2 + i\xi_3 & \xi_0 - i\xi_1 \end{pmatrix}. \tag{8.1}$$

Thus $\gamma(e_0) = \mathbb{1}$, $\gamma(e_1) = I$, $\gamma(e_2) = J$, and $\gamma(e_3) = K$ with

$$I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Consider the spin^c connection $\nabla = \nabla_A$ given by

$$\nabla_j \Phi = \frac{\partial \Phi}{\partial x_j} + A_j \Phi$$

where $A_j : \mathbb{H} \to i\mathbb{R}$ and $\Phi : \mathbb{H} \to \mathbb{C}^2$. The associated connection on the line bundle ${L_\Gamma}^{1/2} = \mathbb{H} \times \mathbb{C}$ is the connection 1-form

$$A = \sum_{i=0}^{3} A_i dx_i \in \Omega^1(\mathbb{H}, i\mathbb{R}).$$

Its curvature 2-form is given by

$$F_A = dA = \sum_{i<j} F_{ij} dx_i \wedge dx_j \in \Omega^2(\mathbb{H}, i\mathbb{R})$$

where
$$F_{ij} = \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}$$
for $i, j = 0, \ldots, 3$.

Lemma 8.1 *The Seiberg-Witten equations (7.1) for $A \in \Omega^1(X, i\mathbb{R})$ and $\Phi \in C^\infty(\mathbb{H}, \mathbb{C}^2)$ are equivalent to*
$$\nabla_0 \Phi = I\nabla_1 \Phi + J\nabla_2 \Phi + K\nabla_3 \Phi \tag{8.2}$$
and
$$\begin{aligned} F_{01} + F_{23} &= -2^{-1}\Phi^* I\Phi, \\ F_{02} + F_{31} &= -2^{-1}\Phi^* J\Phi, \\ F_{03} + F_{12} &= -2^{-1}\Phi^* K\Phi. \end{aligned} \tag{8.3}$$

Proof: The Dirac operator $D_A : C^\infty(\mathbb{H}, \mathbb{C}^2) \to C^\infty(\mathbb{H}, \mathbb{C}^2)$ on the space of positive spinors is given by
$$D_A\Phi = -\nabla_0 \Phi + I\nabla_1 \Phi + J\nabla_2 \Phi + K\nabla_3 \Phi$$
and hence the equation $D_A\Phi = 0$ is equivalent to (8.2). Now the formulae for $\rho(\omega_{ij})$ with $\omega_{ij} = dx_i \wedge dx_j$ in the proof of Lemma 4.55 show that the matrix $\rho^+(F_A) \in \mathbb{C}^{2\times 2}$ is given by
$$\rho^+(F_A) = (F_{01} + F_{23})I + (F_{02} + F_{31})J + (F_{03} + F_{12})K.$$
The traceless part
$$T_0 = T - \frac{1}{2}\mathrm{trace}T\mathbb{1}$$
of a complex 2×2-matrix A can be expressed in the form
$$T_0 = -\frac{1}{2}\mathrm{trace}(IT)I - \frac{1}{2}\mathrm{trace}(JT)J - \frac{1}{2}\mathrm{trace}(KT)K.$$
To see this just note that the formula holds for the matrices $T = \mathbb{1}, I, J, K$ and that every complex 2×2-matrix is a linear combination of these four. Apply this formula to $A = \Phi\Phi^*$ to obtain
$$(\Phi\Phi^*)_0 = -\frac{1}{2}(\Phi^* I\Phi)I - \frac{1}{2}(\Phi^* J\Phi)J - \frac{1}{2}(\Phi^* K\Phi)K.$$
This shows that the formula $\rho^+(F_A) = (\Phi\Phi^*)_0$ is equivalent to (8.3). $\square$

Exercise 8.2 In the case of the standard spin^c structure on $\mathbb{R}^4 = \mathbb{H}$ the adjoint of the Dirac operator D_A is given by

$$D_A{}^*\Psi = \nabla_0\Psi + I\nabla_1\Psi + J\nabla_2\Psi + K\nabla_3\Psi.$$

The Weitzenböck formula takes the form

$$\begin{aligned} &D_A{}^*D_A\Phi + \sum_{i=0}^{3}\nabla_i\nabla_i\Phi \\ &\quad = (F_{01}+F_{23})I\Phi + (F_{02}+F_{31})J\Phi + (F_{03}+F_{12})K\Phi. \end{aligned}$$

Give a direct proof of this formula. □

In the case of the standard spin^c structure on flat $\mathbb{R}^4$ the action of the pair (A,Φ) is given by

$$E(A,\Phi) = \int_{\mathbb{R}^4}\left(\sum_{i=0}^{3}|\nabla_i\Phi|^2 + \frac{1}{4}|\Phi|^4 + \sum_{i<j}|F_{ij}|^2\right). \tag{8.4}$$

For later reference we prove here a local version of the energy identity in Proposition 7.3. For any open set $\Omega \subset \mathbb{R}^4$ denote by $E(A,\Phi;\Omega)$ the action of (A,Φ) on Ω.

Lemma 8.3 *Let $\Omega \subset \mathbb{R}^4$ be a bounded open domain with smooth boundary and let $A \in \Omega^1(\mathbb{R}^4, i\mathbb{R})$ and $\Phi \in C^\infty(\mathbb{R}^4,\mathbb{C}^2)$. Then*

$$\begin{aligned} E(A,\Phi;\Omega) &= \int_\Omega\left(|D_A\Phi|^2 + 2\left|F_A^+ - \sigma^+((\Phi\Phi^*)_0)\right|^2\right) \\ &\quad + \int_{\partial\Omega} A\wedge dA + \int_{\partial\Omega}\langle\Phi, \nabla_{A,\nu}\Phi + \Gamma(\nu)D_A\Phi\rangle\,\mathrm{dvol}_{\partial\Omega} \end{aligned}$$

where $\nu : \partial\Omega \to \mathbb{R}^4$ denotes the outward unit normal vector field, $\nabla_{A,\nu}\Phi = \sum_i \nu_i\nabla_i\Phi$, and $\Gamma(\nu) = -\nu_0\mathbb{1} + \nu_1 I + \nu_2 J + \nu_3 K$.

Proof: As in the proof of Proposition 7.3 one finds

$$\int_\Omega\left(|F_A|^2 - 2|F_A^+|^2\right) = \int_\Omega F_A\wedge F_A = \int_{\partial\Omega} A\wedge dA.$$

The last equality follows from the fact that $F_A\wedge F_A = d(A\wedge dA)$. Moreover, a simple calculation shows that

$$\langle D_A\Phi, D_A\Phi\rangle - \langle\Phi, D_A{}^*D_A\Phi\rangle = \sum_{i=0}^{3}\frac{\partial}{\partial x_i}\langle\Gamma(e_i)\Phi, D_A\Phi\rangle$$

where $\Gamma(e_0) = -\mathbb{1}$, $\Gamma(e_1) = I$, $\Gamma(e_2) = J$, $\Gamma(e_3) = K$ (recall that $\Gamma(v)\Phi = -\gamma(v)^*\Phi$ for $\Phi \in W^+$). Similarly,

$$\langle \nabla_A \Phi, \nabla_A \Phi \rangle - \langle \Phi, {\nabla_A}^* \nabla_A \Phi \rangle = \sum_{i=0}^{3} \frac{\partial}{\partial x_i} \langle \Phi, \nabla_i \Phi \rangle.$$

These two equations show that

$$\begin{aligned}
\int_\Omega \left(|\nabla_A \Phi|^2 - |D_A \Phi|^2 \right) &= \int_\Omega \langle \Phi, {\nabla_A}^* \nabla_A \Phi - {D_A}^* D_A \Phi \rangle \\
&\quad + \int_{\partial\Omega} \langle \Phi, \nabla_{A,\nu} \Phi + \Gamma(\nu) D_A \Phi \rangle \, \mathrm{dvol}_{\partial\Omega} \\
&= - \int_\Omega \langle \Phi, \rho^+(F_A) \Phi \rangle \\
&\quad + \int_{\partial\Omega} \langle \Phi, \nabla_{A,\nu} \Phi + \Gamma(\nu) D_A \Phi \rangle \, \mathrm{dvol}_{\partial\Omega} \\
&= - 4 \int_\Omega \langle F_A^+, \sigma^+((\Phi\Phi^*)_0) \rangle \\
&\quad + \int_{\partial\Omega} \langle \Phi, \nabla_{A,\nu} \Phi + \Gamma(\nu) D_A \Phi \rangle \, \mathrm{dvol}_{\partial\Omega}.
\end{aligned}$$

The second equation follows from Exercise 8.2 and the last from Lemma 7.4. The rest of the proof is obvious. □

The next proposition is the main result of this section. It shows that there are no nontrivial finite energy solutions of the Seiberg-Witten equations (8.2) and (8.3) on $\mathbb{R}^4$.

Proposition 8.4 *Let* $A \in \Omega^1(\mathbb{H}, i\mathbb{R})$ *and* $\Phi \in C^\infty(\mathbb{H}, \mathbb{C}^2)$ *satisfy (8.2) and (8.3) with*

$$E(A, \Phi) < \infty.$$

Then $E(A, \Phi) = 0$, *i.e.* $\Phi = 0$ *and* $F_A = 0$.

Proof: Denote by

$$\Delta = - \sum_{i=0}^{3} \frac{\partial^2}{\partial {x_i}^2}$$

the positive definite Laplacian on $\mathbb{R}^4$. We shall prove that every solution (A, Φ) of (8.2) and (8.3) satisfies

$$\Delta |\Phi|^2 = -2 \sum_{i=0}^{3} |\nabla_i \Phi|^2 - |\Phi|^4. \tag{8.5}$$

This implies that the function $\mathbb{R}^4 \to \mathbb{R} : x \mapsto |\Phi(x)|^2$ is subharmonic and hence cannot have finite L^2-norm unless it vanishes. To prove (8.5) use the Weitzenböck formula of Exercise 8.2 and compute

$$\begin{aligned}
\Delta|\Phi|^2 &= -2\sum_i \frac{\partial}{\partial x_i}\mathrm{Re}\,\langle \Phi, \nabla_i\Phi\rangle \\
&= -2\sum_i |\nabla_i\Phi|^2 - 2\sum_i \mathrm{Re}\,\langle \Phi, \nabla_i\nabla_i\Phi\rangle \\
&= -2\sum_i |\nabla_i\Phi|^2 - 2\mathrm{Re}\,\langle \Phi, (F_{01}+F_{23})I\Phi\rangle \\
&\quad - 2\mathrm{Re}\,\langle \Phi, (F_{02}+F_{31})J\Phi\rangle - 2\mathrm{Re}\,\langle \Phi, (F_{03}+F_{12})K\Phi\rangle \\
&= -2\sum_i |\nabla_i\Phi|^2 - |\Phi^*I\Phi|^2 - |\Phi^*J\Phi|^2 - |\Phi^*K\Phi|^2 \\
&= -2\sum_i |\nabla_i\Phi|^2 - |\Phi|^4.
\end{aligned}$$

Here all inner products are real. The first two equalities are standard calculations with Riemannian connections. The third equality follows from the Weitzenböck formula in Exercise 8.2. The last but one equality uses the formula (8.3) of Lemma 8.1 and the fact that $\mathrm{Re}\,(\Phi^*I\Phi)^2 = -|\Phi^*I\Phi|^2$ etc. The last equality is equivalent to

$$|\Phi|^4 = |\Phi^*I\Phi|^2 + |\Phi^*J\Phi|^2 + |\Phi^*K\Phi|^2$$

and this can be proved by direct computation with

$$\Phi^*I\Phi = i(|\Phi_1|^2 - |\Phi_2|^2), \quad \Phi^*J\Phi = 2i\mathrm{Im}\,\overline{\Phi}_1\Phi_2, \quad \Phi^*K\Phi = 2i\mathrm{Re}\,\overline{\Phi}_1\Phi_2.$$

This proves (8.5) and it follows that

$$\Delta|\Phi|^4 = -2\sum_{i=0}^{3}\left|\frac{\partial}{\partial x_i}|\Phi|^2\right|^2 + 2|\Phi|^2\Delta|\Phi|^2 \le 0.$$

Here Δ is the positive definite Laplacian and so the function $x \mapsto |\Phi(x)|^4$ is subharmonic. Hence it satisfies the mean value inequality

$$|\Phi(x)|^4 \le \frac{2}{\pi^2 r^4}\int_{B_r(x)} |\Phi(y)|^4\,dy \tag{8.6}$$

for all $r > 0$ and all $x \in \mathbb{R}^4$. Since the L^4-norm of Φ is finite it follows, by taking the limit $r \to \infty$, that $\Phi(x) = 0$ for all x. The formula (8.3) now shows that the connection A is anti-self-dual. This means that

$$F_A = dA = - * dA$$

and hence

$$\Delta F_A = dd^* F_A + d^* dF_A = dd^* dA + d^* ddA = -dd^* * dA = 0.$$

This implies

$$\Delta |F_A|^2 = - \sum_{i,j,k} \left| \frac{\partial F_{ij}}{\partial x_k} \right|^2 \leq 0. \tag{8.7}$$

Since the L^2-norm of F_A is finite it follows from the mean value inequality, as above, that $F_A = 0$. This proves the proposition. □

Remark 8.5 It is easy to construct a 1-form A on $\mathbb{R}^4$ with an anti-self-dual differential $F_A = dA \neq 0$. The proof of Proposition 8.4 shows that such a 1-form must have infinite energy $\|dA\|^2 = \infty$. To construct A let $f : \mathbb{R}^3 \to \mathbb{R}$ be a nonconstant harmonic function and define

$$\omega = (\partial_1 f)\omega_1 + (\partial_2 f)\omega_2 + (\partial_3 f)\omega_3,$$

where $\omega_1 = dx_0 dx_1 - dx_2 dx_3$, $\omega_2 = dx_0 dx_2 - dx_3 dx_1$, $\omega_3 = dx_0 dx_3 - dx_1 dx_2$. This form is obviously anti-self-dual and one checks by direct calculation that ω is closed. By Poincaré's lemma, there exists a 1-form $A \in \Omega^1(\mathbb{R}^4)$ such that $dA = \omega$. □

8.2 Removal of singularities

The goal of this section is to prove the following removable singularity theorem for Seiberg-Witten monopoles defined on a punctured ball $B - \{0\}$ where $B = B^4 = \{x \in \mathbb{R}^4 \,|\, |x| \leq 1\}$. We consider here the standard flat metric even though the result continues to hold for any metric. If $\Phi = 0$ then the result reduces to Uhlenbeck's removable singularity theorem for ASD instantons in the case of the gauge group $\mathrm{G} = S^1$ (cf. Uhlenbeck [124] and Donaldson–Kronheimer [21], pp 58–72 and 166-170).

Theorem 8.6. (Removable singularities) *Let $A \in \Omega^1(B - \{0\}, i\mathbb{R})$ and $\Phi \in C^\infty(B - \{0\}, \mathbb{C}^2)$ satisfy (8.2) and (8.3) with $E(A, \Phi; B) < \infty$. Then there exists a gauge transformation $u : B - \{0\} \to S^1$ such that $u(x) = 1$ for $|x| = 1$ and $u^* A$ and $u^{-1}\Phi$ extend to a smooth solution of (8.2) and (8.3) over B.*

A crucial ingredient in the proof is the following weak removable singularity theorem for 1-forms on $\mathbb{R}^n$. The theorem asserts that if α is a 1-form on the punctured ball $B^n - \{0\}$ such that $d\alpha$ is of class L^2 then there exists a function $\xi : B^n - \{0\} \to \mathbb{R}$ such that $\alpha - d\xi$ is of class $W^{1,2}$ (and $d^*(\alpha - d\xi) = 0$). If $n = 4$ and α is anti-self-dual then it follows easily that

$\alpha - d\xi$ extends to a smooth 1-form on B^4. This is Uhlenbeck's removable singularity theorem for ASD instantons in the case $\mathrm{G} = S^1$. Note also that this is the special case $\Phi = 0$ in Theorem 8.6. The proof in the case $\mathrm{G} = S^1$ is quite simple compared to the nonabelian case. Throughout denote by $B^n(r) = \{x \in \mathbb{R}^n \,|\, |x| \le r\}$ the closed ball in $\mathbb{R}^n$ of radius r and abbreviate $B^n = B^n(1)$ and $A(r_0, r_1) = A^n(r_0, r_1) = \{x \in \mathbb{R}^n \,|\, r_0 \le |x| \le r_1\}$ for $r_0 < r_1$.

Proposition 8.7. (Uhlenbeck) *Assume $n \ge 4$ and let $\alpha \in \Omega^1(B^n - \{0\})$ be a smooth real valued 1-form which satisfies*

$$\int_{B^n} |d\alpha|^2 < \infty.$$

Then there exists a smooth function $\xi : B^n - \{0\} \to \mathbb{R}$ such that $\alpha - d\xi$ is of class $W^{1,2}$ on the (unpunctured) unit ball and satisfies

$$\int_{B^n} \left(|\nabla(\alpha - d\xi)|^2 + \frac{|\alpha - d\xi|^2}{|x|^2} \right) \le 4 \int_{B^n} |d\alpha|^2$$

as well as

$$d^*(\alpha - d\xi) = 0, \qquad \frac{\partial \xi}{\partial \nu} = \alpha(\nu).$$

Here $d\xi/\partial\nu$ denotes the normal derivative on ∂B^n and $\alpha(\nu) = \sum_i \alpha_i(x) x_i$ for $|x| = 1$.

Note that addition of any exact 1-form on $B^n - \{0\}$ does not alter the L^2-norm of $d\alpha$. Thus the behaviour of α near zero may be extremely singular. The proposition asserts that there exists an exact 1-form $d\xi$ on $B^n - \{0\}$ which *tames* the singularity at 0 in the sense that $\alpha - d\xi$ is of class $W^{1,2}$ on B^n. We shall construct ξ as a limit of functions $\xi_\varepsilon : B^n(1) - B^n(\varepsilon) \to \mathbb{R}$ defined by $d^*(\alpha - d\xi_\varepsilon) = 0$ with boundary condition $\partial\xi_\varepsilon/\partial\nu = \alpha(\nu)$ on $\partial(B_1 - B_\varepsilon)$. The convergence proof relies on the following three lemmata.

Lemma 8.8 *Assume $n \ge 4$. Then every smooth 1-form $\alpha \in \Omega^1(A^n(\varepsilon, 1))$ with $\alpha(\nu) = 0$ on $\partial A^n(\varepsilon, 1)$ satisfies the inequality*

$$\int_{A(\varepsilon,1)} \left(|\nabla\alpha|^2 + \frac{|\alpha|^2}{|x|^2} \right) \le 4 \int_{A(\varepsilon,1)} \left(|d\alpha|^2 + |d^*\alpha|^2 \right).$$

Proof: Let $\alpha = \sum_{i=1}^n \alpha_i dx_i$ be a smooth 1-form on a domain $\Omega \subset \mathbb{R}^n$ with smooth boundary. Suppose that

$$\langle \alpha, \nu \rangle = \sum_{i=1}^n \alpha_i \nu_i = 0$$

on $\partial\Omega$. This condition is equivalent to $*\alpha|_{\partial\Omega} = 0$. Note that $\Delta = d^*d + dd^*$ is the standard (positive definite) Laplace operator. Consider the identities

$$\begin{aligned}
\int_\Omega |\nabla\alpha|^2 &= \int_\Omega \langle \alpha, \Delta\alpha\rangle + \int_{\partial\Omega} \left\langle \alpha, \frac{\partial\alpha}{\partial\nu}\right\rangle \mathrm{dvol}_{\partial\Omega}, \\
\int_\Omega |d\alpha|^2 &= \int_\Omega \langle \alpha, d^*d\alpha\rangle + \int_{\partial\Omega} \alpha \wedge *d\alpha, \\
\int_\Omega |d^*\alpha|^2 &= \int_\Omega \langle \alpha, dd^*\alpha\rangle.
\end{aligned}$$

The last identity holds whenever $\langle\alpha, \nu\rangle = 0$ on $\partial\Omega$. Take the difference of these identities to obtain

$$\|\nabla\alpha\|^2 - \|d\alpha\|^2 - \|d^*\alpha\|^2 = \int_{\partial\Omega} \left\langle \alpha, \frac{\partial\alpha}{\partial\nu}\right\rangle \mathrm{dvol}_{\partial\Omega} - \int_{\partial\Omega} \alpha \wedge *d\alpha.$$

Here all norms on the left are L^2-norms on $A(\varepsilon, 1)$. Now use the formulae

$$\mathrm{dvol}_{\partial\Omega} = \sum_i \nu_i * dx_i, \qquad *dx_i|_{\partial\Omega} = \nu_i \mathrm{dvol}_{\partial\Omega}$$

and

$$dx_i \wedge *(dx_i \wedge dx_j) = - * dx_j, \qquad dx_j \wedge *(dx_i \wedge dx_j) = *dx_i$$

for $i < j$ and compute

$$\begin{aligned}
\int_{\partial\Omega} \alpha \wedge *d\alpha &= \sum_{i<j}\sum_k \int_{\partial\Omega} \alpha_k \left(\frac{\partial\alpha_j}{\partial x_i} - \frac{\partial\alpha_i}{\partial x_j}\right) dx_k \wedge *(dx_i \wedge dx_j) \\
&= \sum_{i<j} \int_{\partial\Omega} \alpha_i \left(\frac{\partial\alpha_j}{\partial x_i} - \frac{\partial\alpha_i}{\partial x_j}\right) dx_i \wedge *(dx_i \wedge dx_j) \\
&\quad + \sum_{i<j} \int_{\partial\Omega} \alpha_j \left(\frac{\partial\alpha_j}{\partial x_i} - \frac{\partial\alpha_i}{\partial x_j}\right) dx_j \wedge *(dx_i \wedge dx_j) \\
&= \sum_{i,j} \int_{\partial\Omega} \alpha_i \left(\frac{\partial\alpha_i}{\partial x_j} - \frac{\partial\alpha_j}{\partial x_i}\right) * dx_j \\
&= \sum_{i,j} \int_{\partial\Omega} \alpha_i \left(\frac{\partial\alpha_i}{\partial x_j} - \frac{\partial\alpha_j}{\partial x_i}\right) \nu_j \,\mathrm{dvol}_{\partial\Omega} \\
&= \int_{\partial\Omega} \left\langle \alpha, \frac{\partial\alpha}{\partial\nu}\right\rangle \mathrm{dvol}_{\partial\Omega} - \int_{\partial\Omega} \sum_{i,j} \alpha_i \frac{\partial\alpha_j}{\partial x_i} \nu_j \,\mathrm{dvol}_{\partial\Omega} \\
&= \int_{\partial\Omega} \left\langle \alpha, \frac{\partial\alpha}{\partial\nu}\right\rangle \mathrm{dvol}_{\partial\Omega} + \int_{\partial\Omega} \sum_{i,j} \alpha_i \alpha_j \frac{\partial\nu_j}{\partial x_i} \,\mathrm{dvol}_{\partial\Omega}.
\end{aligned}$$

The last equality follows from the fact that

$$\sum_i \alpha_i \frac{\partial}{\partial x_i}\left(\sum_j \alpha_j \nu_j\right) = 0$$

on $\partial\Omega$. This is because $\sum_i \alpha_i \nu_i = 0$ on $\partial\Omega$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ is tangent to $\partial\Omega$. Now consider the case $\Omega = A(\varepsilon, 1)$:

$$\begin{aligned}
\int_{\partial\Omega} \left\langle \alpha, \frac{\partial \alpha}{\partial \nu}\right\rangle \mathrm{dvol}_{\partial\Omega} - \int_{\partial\Omega} \alpha \wedge *d\alpha &= -\int_{\partial\Omega} \sum_{i,j} \alpha_i \alpha_j \frac{\partial \nu_j}{\partial x_i} \mathrm{dvol}_{\partial\Omega} \\
&= \frac{1}{\varepsilon}\int_{|x|=\varepsilon} |\alpha|^2 - \int_{|x|=1} |\alpha|^2.
\end{aligned}$$

Thus we have proved the identity

$$\|\nabla\alpha\|^2 = \|d\alpha\|^2 + \|d^*\alpha\|^2 + \frac{1}{\varepsilon}\int_{|x|=\varepsilon} |\alpha|^2 - \int_{|x|=1} |\alpha|^2 \tag{8.8}$$

for 1-forms on $A(\varepsilon, 1)$ which satisfy $\langle \alpha, \nu\rangle = 0$ on the boundary. Now consider the function $f : \mathbb{R}^n - \{0\} \to \mathbb{R}^n$ given by

$$f(x) = \frac{x}{|x|^2}, \qquad \mathrm{div}(f) = \frac{n-2}{|x|^2}.$$

Then for every smooth function $u : A(\varepsilon, 1) \to \mathbb{R}$

$$\begin{aligned}
\frac{1}{\varepsilon}\int_{|x|=\varepsilon} |u|^2 - \int_{|x|=1} |u|^2 &= -\int_{\partial A(\varepsilon,1)} \langle \nu, f\rangle |u|^2 \,\mathrm{dvol} \\
&= -\int_{A(\varepsilon,1)} \sum_{i=1}^n \frac{\partial}{\partial x_i}(f_i |u|^2) \\
&= -\int_{A(\varepsilon,1)} \sum_{i=1}^n \left(2 f_i u \frac{\partial u}{\partial x_i} + |u|^2 \frac{\partial f_i}{\partial x_i}\right) \\
&\le 2\int_{A(\varepsilon,1)} \frac{|u|\,|\nabla u|}{|x|} - \int_{A(\varepsilon,1)} \mathrm{div}(f)|u|^2 \\
&= 2\int_{A(\varepsilon,1)} \frac{|u|\,|\nabla u|}{|x|} - (n-2)\int_{A(\varepsilon,1)} \frac{|u|^2}{|x|^2} \\
&\le \delta \int_{A(\varepsilon,1)} |\nabla u|^2 - \left(n - 2 - \frac{1}{\delta}\right)\int_{A(\varepsilon,1)} \frac{|u|^2}{|x|^2}.
\end{aligned}$$

The last inequality holds for any constant $\delta > 0$. If $n \ge 4$ we can choose $1/(n-2) < \delta < 1$. For example, with $\delta = 3/4$ we obtain from (8.8)

$$\|\nabla\alpha\|^2 \leq \|d\alpha\|^2 + \|d^*\alpha\|^2 + \frac{3}{4}\|\nabla\alpha\|^2 - \left(n - 2 - \frac{4}{3}\right)\int_{A(\varepsilon,1)} \frac{|\alpha|^2}{|x|^2}.$$

This holds for all n. But for $n \geq 4$ the last term on the right is negative and the desired inequality follows. □

Lemma 8.9. (Poincaré inequality) *There is a constant $c = c(n) > 0$ such that every smooth function $\xi : A^n(1/2, 1) \to \mathbb{R}$ with*

$$\int_{A(1/2,1)} \xi = 0$$

satisfies the inequality

$$\int_{A(1/2,1)} |\xi|^2 \leq c \int_{A(1/2,1)} |d\xi|^2.$$

Proof: Suppoose otherwise that there exists a sequence of smooth functions $\xi_i : A(1/2, 1) \to \mathbb{R}$ which have mean value zero and satisfy

$$\int_{A(1/2,1)} |\xi_i|^2 = 1, \qquad \lim_{i\to\infty} \int_{A(1/2,1)} |d\xi_i|^2 = 0.$$

Then, by Rellich's theorem, there exists a subsequence (still denoted by ξ_i) which converges weakly in $W^{1,2}$ and strongly in L^2. The limit function $\xi = \lim_{i\to\infty} \xi_i$ lies in $W^{1,2}$, has L^2-norm 1, satisfies $d\xi = 0$, and has mean value zero. But $d\xi = 0$ implies that ξ is constant and the mean value zero condition shows that $\xi = 0$ contradicting the fact that the L^2-norm is 1. This proves the lemma. □

Lemma 8.10 *Every smooth function $\xi : A^n(r_0, r_1 + t) \to \mathbb{R}$ satisfies the inequality*

$$\int_{A(r_0,r_1)} |\xi|^2 \leq 2\int_{A(r_0+t,r_1+t)} |\xi|^2 + \int_{A(r_0,r_1+t)} |d\xi|^2$$

for $0 < r_0 < r_1 \leq 1$ and $0 \leq t \leq 1$.

Proof: Consider the identity

$$\xi(rx) = \xi((t + r)x) - \int_0^t \langle \nabla\xi((r + s)x), x\rangle\, ds$$

and use the Cauchy-Schwarz inequality to obtain

$$|\xi(rx)|^2 \leq 2\,|\xi((t + r)x)|^2 + \frac{2}{(n-2)r^{n-2}} \int_r^{r+t} s^{n-1}|d\xi(sx)|^2\, ds$$

for $|x| = 1$ and $n \geq 3$. In the case $n = 2$ there is a similar inequality with $1/(n-2)r^{n-2}$ replaced by $\log(r+t) - \log r \leq r - \log r$. Now multiply by r^{n-1} and integrate over S^{n-1} and over $r_0 \leq r \leq r_1$. □

Lemma 8.11 *Let $u : B^n - \{0\} \to \mathbb{R}$ be a smooth function such that*

$$\int_{B^n} |\nabla u(x)|^2 < \infty.$$

Then u is of class $W^{1,2}$ on B^n, i.e. its distributional derivatives exist and agree with the ordinary derivatives.

Proof: Note first that, by Lemma 8.10 with $r_1 = t = 1/2$ and $r_0 \to 0$, the function u is square integrable on B^n. Choose a test function $\varphi : \mathbb{R}^n \to \mathbb{R}$ with compact support in B^n. Then

$$\int_{\varepsilon \leq |x| \leq 1} \left(\frac{\partial \varphi}{\partial x_i} u + \varphi \frac{\partial u}{\partial x_i} \right) = - \int_{|x|=\varepsilon} \frac{x_i}{\varepsilon} \varphi u. \tag{8.9}$$

The integrand on the left is integrable on B^n and hence the limit as $\varepsilon \to 0$ exists. We must prove that this limit is zero. But the term on the right is bounded in absolute value by

$$f(\varepsilon) = c \int_{|x|=\varepsilon} |u| \, .$$

where $c = \sup |\varphi|$. By the Cauchy-Schwarz inequality,

$$f(\varepsilon)^2 \leq c^2 \omega_n \varepsilon^{n-1} \int_{|x|=\varepsilon} |u|^2$$

where ω_n denotes the area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$. Hence the condition on u shows that

$$\int_0^1 \frac{f(\varepsilon)^2}{\varepsilon^{n-1}} \, d\varepsilon \leq c^2 \omega_n \int_{0<|x|\leq 1} |u(x)|^2 \, dx < \infty.$$

This implies that $f(\varepsilon_k) \to 0$ for some sequence $\varepsilon_k \to 0$ and hence the limit in (8.9) as $\varepsilon \to 0$ is zero as required. □

Proof of Proposition 8.7: For every $\varepsilon > 0$ there exists a smooth function $\xi_\varepsilon : A^n(\varepsilon, 1) \to \mathbb{R}$ which satisfies

$$d^*(\alpha - d\xi_\varepsilon) = 0, \qquad \frac{\partial \xi_\varepsilon}{\partial \nu} = \langle \alpha, \nu \rangle$$

where the last equation holds on the boundary. To see this choose first a smooth function $\xi_0 : A^n(\varepsilon, 1) \to \mathbb{R}$ which satisfies $\partial \xi_0 / \partial \nu = \langle \alpha, \nu \rangle$ on $\partial A^n(\varepsilon, 1)$ and then choose $\xi_1 \in W^{2,2}(A^n(\varepsilon, 1))$ with

$$d^*d\xi_1 = d^*\alpha - d^*d\xi_0, \qquad \partial\xi_1/\partial\nu = 0$$

(the Neumann boundary value problem). Then $\xi_\varepsilon = \xi_0 + \xi_1$ is as required. Moreover, the function ξ_ε is only determined up to a constant which can be fixed by the normalization condition

$$\int_{1/2 \le |x| \le 1} \xi_\varepsilon(x)\, dx = 0.$$

It follows from Lemma 8.8 that

$$\|\nabla(\alpha - d\xi_\varepsilon)\|_{L^2}^2 + \int_{\varepsilon \le |x| \le 1} \frac{|\alpha - d\xi_\varepsilon|^2}{|x|^2} \le 4\, \|d\alpha\|_{L^2}^2 .$$

Fix some number $\delta > 0$. Then for $\varepsilon < \delta$

$$\|\nabla d\xi_\varepsilon\|_{L^2(A(\delta,1))} \le 2\, \|d\alpha\|_{L^2} + \|\nabla\alpha\|_{L^2(A(\delta,1))}$$

and

$$\|d\xi_\varepsilon\|_{L^2(A(\delta,1))} \le 2\, \|d\alpha\|_{L^2} + \|\alpha\|_{L^2(A(\delta,1))} .$$

Now use Lemma 8.9 and the mean value condition to control the L^2-norm of ξ_ε on $A(1/2,1)$ and Lemma 8.10 to control the this norm on $A(\delta,1/2)$. This shows that for every $\delta > 0$ there exists a constant $c_\delta > 0$ such that

$$\|\xi_\varepsilon\|_{W^{2,2}(A(\delta,1))} \le c_\delta$$

for every $\varepsilon \in (0,\delta)$. Now the usual diagonal sequence argument shows that there exists a sequence $\varepsilon_i \to 0$ such that ξ_{ε_i} converges strongly in $W^{1,2}(K)$ and weakly in $W^{2,2}(K)$ for every compact subset $K \subset B^n - \{0\}$. The limit function $\xi : B^n - \{0\} \to \mathbb{R}$ is of class $W^{2,2}$ on every compact subset away from 0 and satisfies

$$d^*(\alpha - d\xi) = 0, \qquad \langle \alpha - d\xi, \nu\rangle = 0.$$

Moreover,

$$\begin{aligned}\int_K \left(|\nabla(\alpha - d\xi)|^2 + \frac{|\alpha - d\xi|^2}{|x|^2} \right) &\le \lim_{i\to\infty} \int_K \left(|\nabla(\alpha - d\xi_i)|^2 + \frac{|\alpha - d\xi_i|^2}{|x|^2} \right) \\ &\le 4 \int_{B^n} |d\alpha|^2\end{aligned}$$

for every compact subset $K \subset B^n - \{0\}$. By Lemma 8.11, $\alpha - d\xi$ is of class $W^{1,2}$ on B^n. This proves the proposition. □

Proof of Theorem 8.6: By Proposition 8.7 there exists a smooth function $\xi : B^4 - \{0\} \to i\mathbb{R}$ such that $A - d\xi$ is of class $W^{1,2}$ on the closed ball B^4 and $d^*(A - d\xi) = 0$. Hence we may assume from now on that $A \in W^{1,2}$ and $d^*A = 0$. Moreover, by the finite energy condition, we have $\Phi \in L^4$ and $\nabla_i\Phi \in L^2$. The Sobolev embedding theorem shows that $A \in L^4$ and hence

$$\partial_i\Phi = \nabla_i\Phi - A_i\Phi \in L^2$$

for $i = 0, 1, 2, 3$. By Lemma 8.11, this shows that $\Phi \in W^{1,2}$. Thus we have a solution (A, Φ) of (8.2) and (8.3) which is smooth on the punctured ball $B^4 - \{0\}$ and on the closed ball satisfies

$$A \in W^{1,2}, \qquad \Phi \in W^{1,2}, \qquad d^*A = 0.$$

We shall prove in three steps that

$$\int_{|x|\le 1} \frac{2|\nabla_A\Phi|^2 + |\Phi|^4}{|x|^2} < \infty. \tag{8.10}$$

and that there exists a constant $c > 0$ such that

$$E_0(A, \Phi; B_r) = \int_{|x|\le r} \left(2|\nabla_A\Phi|^2 + |\Phi|^4\right) \le cr^2. \tag{8.11}$$

Then it will follow easily that Φ is of class L^p for some $p > 4$ and the rest of the argument is by elliptic bootstrapping.

Step 1: *For every* $r \in (0, 1]$

$$E_0(A, \Phi; B_r) = 2\int_{|x|=r} \sum_i \langle\Phi, \nabla_i\Phi\rangle \frac{x_i}{r}.$$

Let $\Omega \subset \mathbb{R}^4$ be any open domain with smooth boundary such that A and Ψ are defined on its closure. (Thus $0 \notin \bar{\Omega}$.) Consider the energy

$$\begin{aligned} E_0(A, \Phi; \Omega) &= \int_\Omega \left(2|\nabla_A\Phi|^2 + |\Phi|^4\right) \\ &= \int_\Omega \left(2|\nabla_A\Phi|^2 + \frac{1}{2}|\Phi|^4 + 4|F_A^+|^2\right) \\ &= 2\int_{\partial\Omega} \langle\Phi, \nabla_{A,\nu}\Phi\rangle \,\mathrm{dvol}_{\partial\Omega}. \end{aligned}$$

The second equality follows from the fact that $|\Phi|^4 = 8|F_A^+|^2$ for solutions of (8.3) and the last equality follows from the proof of Lemma 8.3. Abbreviate

$$f(r) = 2\int_{|x|=r} \sum_i \langle \Phi, \nabla_i \Phi\rangle \frac{x_i}{r}.$$

Then $f : (0, 1] \to \mathbb{R}$ is a smooth function and the previous identity shows that

$$E_0(A, \Phi; B_r - B_\varepsilon) = f(r) - f(\varepsilon).$$

Hence f is monotonically increasing. Moreover, the energy is finite and hence, by taking the limit $\varepsilon \to 0$, we see that f is bounded below. This shows that the limit

$$f(0) := \lim_{\varepsilon \to 0} f(\varepsilon)$$

exists. Now it follows from the finiteness of the energy that $\Phi \in L^4$ and $\nabla_i \Phi \in L^2$ and hence $\langle \Phi, \nabla_i \Phi\rangle \in L^{4/3}$ for all i. Moreover, by Hölder's inequality,

$$\begin{aligned} |f(r)|^{4/3} &\le \left(\int_{|x|=r} 1\right)^{1/3} \int_{|x|=r} (|\Phi|\,|\nabla_A \Phi|)^{4/3} \\ &\le (2\pi^2)^{1/3} r \int_{|x|=r} (|\Phi|\,|\nabla_A \Phi|)^{4/3} \end{aligned}$$

and hence

$$\int_0^1 \frac{|f(r)|^{4/3}}{r}\,dr < \infty.$$

This shows that there must be a sequence $\varepsilon_i \to 0$ with $f(\varepsilon_i) \to 0$ and it follows that $f(0) = 0$. This implies $f(r) = E_0(A, \Phi; B_r)$ as claimed.

Step 2: *Every smooth function $u : \mathbb{R}^4 - \{0\} \to \mathbb{R}$ satisfies the identity*

$$-\int_{\rho \le |x| \le r} \frac{\Delta u}{|x|^2} = \int_{|x|=r} \frac{2u + \langle \nabla u, x\rangle}{r^3} - \int_{|x|=\rho} \frac{2u + \langle \nabla u, x\rangle}{\rho^3}.$$

Note the choice of sign in the definition of the Laplacian

$$\Delta = -\sum_i \frac{\partial}{\partial {x_i}^2}.$$

With this sign we have the familiar identity

$$\int_\Omega \Big(u\Delta v - v\Delta u\Big) = \int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu} v - u\frac{\partial v}{\partial \nu}\right).$$

Step 2 arises as the special case of the annulus

$$\Omega = \left\{ x \in \mathbb{R}^4 \mid \rho \le |x| \le r \right\}$$

with $v(x) = 1/|x|^2$. This is the fundamental solution of Laplace's equation and satisfies $\nabla v(x) = -2x/|x|^4$ and $\Delta v(x) = 0$ for $x \neq 0$.

Step 3: *Proof of (8.10) and (8.11).*

Recall from the proof of Proposition 8.4 that

$$\Delta |\Phi|^2 = -2|\nabla_A \Phi|^2 - |\Phi|^4.$$

Moreover, note that

$$\int_{|x|=r} \langle \nabla |\Phi|^2, x \rangle = 2 \int_{|x|=r} \sum_i \langle \Phi, \nabla_i \Phi \rangle x_i = r f(r).$$

Hence it follows from Step 2 with $u = |\Phi|^2$ that

$$\begin{aligned} \int_{\rho \le |x| \le r} \frac{2|\nabla_A \Phi|^2 + |\Phi|^4}{|x|^2} \, dx = \frac{2}{r^3} \int_{|x|=r} |\Phi|^2 + \frac{f(r)}{r^2} \\ - \frac{2}{\rho^3} \int_{|x|=\rho} |\Phi|^2 - \frac{f(\rho)}{\rho^2} \end{aligned}$$

Since the terms involving ρ have negative sign the inequality (8.10) follows by taking the limit $\rho \to 0$. Moreover,

$$\frac{f(\rho)}{\rho^2} \le \frac{f(r)}{r^2} + \frac{2}{r^3} \int_{|x|=r} |\Phi|^2$$

for $0 < \rho \le r$ and this proves (8.11).

Recall from the proof of Proposition 8.4 that the function $x \mapsto |\Phi(x)|^4$ is subharmonic and hence

$$|\Phi(x)|^4 \le \frac{2}{\pi^2 r^4} \int_{B_r(x)} |\Phi|^4 \le \frac{2}{\pi^2 r^4} E_0(A, \Phi; B_{2r}) \le \frac{8c}{\pi^2 r^2}$$

for $r = |x|$. The first inequality is the mean value inequality for subharmonic functions, the second follows from the definition of E_0, and the last follows from (8.11). Thus

$$|\Phi(x)|^4 \le \frac{8c}{\pi^2 |x|^2}$$

and, since the function $x \mapsto 1/|x|^\alpha$ is integrable in a neighbourhood of zero whenever $\alpha < 4$, it follows that $|\Phi|^p$ is integrable for every $p < 8$. Thus we have proved that $|\Phi|^2 \in L^p$ for any $p < 4$. Since $d^+A = \sigma^+((\Phi\Phi^*)_0)$ this shows that $d^+A \in L^p$ for any $p < 4$. Now recall that $d^*A = 0$ and hence

$$\Delta A = d^*dA = 2d^*d^+A = 2d^*\sigma^+((\Phi\Phi^*)_0).$$

Note that A is a weak solution of this equation on the closed (unpunctured) ball and hence it follows that $A \in W^{1,p}$ for any $p < 4$. Thus $A \in L^q$ for any $q < \infty$. The formula

$$0 = D_A\Phi = D\Phi - \Gamma(A)\Phi$$

with $\Gamma(A)\Phi \in L^p$ now shows that $\Phi \in W^{1,p}$ for any $p < 4$. Thus $\Phi \in L^q$ for some $q > 4$ and using the last equation again with $\Gamma(A)\Phi \in L^q$ we find that $\Phi \in W^{1,q}$ for some $q > 4$. This implies $d^*\sigma^+((\Phi\Phi^*)_0) \in L^q$ and, by the previous equation $A \in W^{2,q}$. Using the two equations alternatingly we conclude that A and Φ are smooth on B_1. This is the elliptic bootstrapping argument. More details are carried out in the next section. □

8.3 Compactness and Regularity

The goal of this section is to give detailed proofs of Theorems 7.11 and 7.12 about the compactness and regularity properties of the solutions of the Seiberg-Witten equations (7.4). The proofs require some preparation.

Estimates for the Dirac-operator

Fix a smooth compact Riemannian 4-manifold X equipped with a spin^c structure $\Gamma : TX \to \mathrm{End}(W)$. The Lemmata 8.14 and 8.13 below deal entirely with the linear problem of the regularity of a section Φ in the kernel of D_A for a given connection A. It is important here to observe that the connection A is only assumed to be of class $W^{k,p}$ and it is necessary to keep track of how the constants in the elliptic estimates depend on A. These lemmata also are the essential ingredients in the proof of the following proposition which deals with the Fredholm properties of the Dirac operator under weak regularity assumptions on the connection A.

Proposition 8.12 *Let $A \in \mathcal{A}^{k,q}(\Gamma)$ for some constant $q > 1$ and some integer $k \geq 0$. Let $j \in \mathbb{Z}$ and $p > 1$ such that*

$$0 \leq j \leq k, \qquad j - \frac{4}{p} \leq k - \frac{4}{q}, \qquad (k+1)q > 4.$$

Then the Dirac operator $D_A : W^{j+1,p}(X, W^+) \to W^{j,p}(X, W^-)$ is Fredholm with index

$$\operatorname{index} D_A = \frac{c_1(L_\Gamma) \cdot c_1(L_\Gamma) - \sigma(X)}{4}.$$

The next lemma gives the fundamental elliptic L^p estimate for the Dirac operator. In particular, it shows how the constant depends on the connection A.

Lemma 8.13 *Fix integers j, k and real numbers $p, q \geq 1$ such that*

$$0 \leq j \leq k, \qquad j - \frac{4}{p} \leq k - \frac{4}{q}, \qquad (k+1)q > 4.$$

Then the following holds.
(i) *For any two connections $A_0, A_1 \in \mathcal{A}^{k,q}(\Gamma)$ the linear operator*

$$D_{A_1} - D_{A_0} : W^{j+1,p}(X, W^+) \to W^{j,p}(X, W^-)$$

is compact.
(ii) *If $kq > 4$ then for every smooth reference connection $A_0 \in \mathcal{A}(\Gamma)$ there exists a constant $c = c(A_0, j, k, p, q) > 0$ such that*

$$\|\Phi\|_{W^{j+1,p}} \leq c\Big(\|D_A\Phi\|_{W^{j,p}} + (1 + \|A - A_0\|_{W^{k,q}}) \|\Phi\|_{W^{j,p}} \Big)$$

for every $A \in \mathcal{A}^{k,q}(\Gamma)$ and every $\Phi \in W^{j+1,p}(X, W^+)$.

Proof: Let $A_0, A_1 \in \mathcal{A}^{k,q}(\Gamma)$ and denote $\alpha = A_1 - A_0 \in \mathcal{A}^{k,q}(X)$. Then the operator $D_{A_1} - D_{A_0} : W^{j+1,p}(X, W^+) \to W^{j,p}(X, W^-)$ is given by $\Phi \mapsto \Gamma(\alpha)\Phi$. A priori it is not even clear that $\Gamma(\alpha)\Phi$ actually lies in $W^{j,p}$. This will be established below, and that the operator is compact will essentially follow from Rellich's theorem. The proof relies on the product estimates of Proposition C.19 in Appendix C. There are three cases to consider.

Case 1: *Assume $kq < 4$ and $j - 4/p < k - 4/q$. Then Proposition C.19 (i) shows that there is an estimate*

$$\|\Gamma(\alpha)\Phi\|_{W^{j,p}} \leq c \|\alpha\|_{W^{k,q}} \|\Phi\|_{W^{j,r}} .$$

where $r = 4pq/(4q - 4p + kpq) > p$.

This shows that the map $W^{j,r}(X, W^+) \to W^{j,p}(X, W^-) : \Phi \mapsto \Gamma(\alpha)\Phi$ is a bounded linear operator. Moreover, the number r satisfies

$$\frac{1}{r} = \frac{1}{p} - \frac{1}{q} + \frac{k}{4}.$$

It turns out that there is a compact inclusion $W^{j+1,p} \hookrightarrow W^{j,r}$ precisely when $(k+1)q > 4$. This follows from Rellich's theorem and the fact that

$$r < \frac{4p}{4-p} \quad \Longleftrightarrow \quad \frac{1}{r} > \frac{1}{p} - \frac{1}{4} \quad \Longleftrightarrow \quad (k+1)q > 4.$$

Hence the operator $W^{j+1,p} \to W^{j,p} : \Phi \mapsto \Gamma(a)\Phi$ is the composition of a compact operator and a bounded linear operator and is therefore compact.

Case 2: *Assume $kq < 4$ and $j - 4/p = k - 4/q$. Then Proposition C.19 (ii) shows that there is an estimate*

$$\|\Gamma(\alpha)\Phi\|_{W^{j,p}} \le c\,\|\alpha\|_{W^{k,q}}\left(\|\Phi\|_{W^{j,4/j}} + \|\Phi\|_{L^\infty}\right).$$

In this case the map $\Phi \mapsto \Gamma(a)\Phi$ is a bounded linear operator from $W^{j,4/j}(X, W^+) \cap L^\infty(X, W^+)$ to $W^{j,p}(X, W^-)$. By Rellich's theorem, there is a compact inclusion $W^{j+1,p} \hookrightarrow W^{j,4/j} \cap L^\infty$ if and only if $(j+1)p > 4$ and, since $j - 4/p = k - 4/q$, this is equivalent to $(k+1)q > 4$.

Case 3: *If $kq = 4$ then Proposition C.19 (iii) shows that for every $\varepsilon > 0$ there is an estimate*

$$\|\Gamma(\alpha)\Phi\|_{W^{j,p}} \le c_\varepsilon\,\|\alpha\|_{W^{k,q}}\,\|\Phi\|_{W^{j,p+\varepsilon}}.$$

If $kq > 4$ then, by Proposition C.19 (iv), this estimate holds with $\varepsilon = 0$.

In this case use the fact that the inclusion $W^{j+1,p} \hookrightarrow W^{j,p+\varepsilon}$ is compact for $0 \le \varepsilon < 4p/(4-p)$ when $p \le 4$ and for any $\varepsilon \ge 0$ when $p > 4$. This proves (i).

To prove (ii) assume first that A is smooth, denote $\Psi = D_A\Phi$, and consider the equation

$$\nabla_A{}^*\nabla_A\Phi = D_A{}^*\Psi - \frac{s}{4}\Phi - \rho^+(F_A)\Phi. \tag{8.12}$$

Now use the Calderón-Zygmund inequality for the Bochner Laplacian to obtain

$$\begin{aligned}\|\nabla_A\Phi\|_{L^p} &\le c\,\sup_\varphi \frac{\langle\nabla_A\varphi, \nabla_A\Phi\rangle}{\|\varphi\|_{W^{1,q}}}\\ &= c\,\sup_\varphi \frac{\langle D_A\varphi, \Psi\rangle - \left\langle \frac{s}{4}\Phi + \rho^+(F_A)\varphi, \Phi\right\rangle}{\|\varphi\|_{W^{1,q}}}\end{aligned}$$

where $1/p + 1/q = 1$ and the supremum is over all $\varphi \in C^\infty(X, W^+)$. Hence, with $\Psi = D_A\Phi$

$$\|\Phi\|_{W^{1,p}} \le c\left(\|\Phi\|_{L^p} + \|D_A\Phi\|_{L^p}\right).$$

More generally, the Calderón-Zygmund inequality implies that

$$\|\Phi\|_{W^{j+1,p}} \le c\left(\|\Phi\|_{W^{j,p}} + \|D_A\Phi\|_{W^{j,p}}\right)$$

for any j where the constant c depends of j, p, and A. This proves (ii) when $A = A_0$ is smooth. The general case follows from the smooth case and Case 3 above. This proves the lemma. $\square$

The next lemma addresses the question of the regularity of Φ given a connection A. It shows that every weak solution Φ of $D_A\Phi = \Psi$ with A and Ψ of class $W^{j,p}$ must be of class $W^{j+1,p}$ whenever $(j+1)p > 4$.

Lemma 8.14 *Let $A \in \mathcal{A}^{j,p}(\Gamma)$ and $\Psi \in W^{j,p}(X, W^-)$ for some constant $p \geq 1$ and integer $j \geq 1$ with $(j+1)p > 4$. Suppose that $\Phi \in L^q(X, W^+)$ with $1/p + 1/q = 1$ satisfies*

$$\int_X \langle {D_A}^* \psi, \Phi \rangle \mathrm{dvol} = \int_X \langle \psi, \Psi \rangle \mathrm{dvol}$$

for all $\psi \in C^\infty(X, W^-)$. Then $\Phi \in W^{j+1,p}(X, W^+)$ and $D_A\Phi = \Psi$.

Proof: Assume first that A is smooth. Then the result follows from standard elliptic regularity for the Bochner Laplacian ${\nabla_A}^*\nabla_A$. Namely, choose a *test function* $\psi = D_A\varphi$ and use the Weitzenböck formula to obtain

$$\int_X \langle {\nabla_A}^* \nabla_A \varphi, \Phi \rangle = \int_X \langle D_A\varphi, \Psi \rangle - \int_X \left\langle \varphi, \frac{s}{4}\Phi + \rho^+(F_A)\Phi \right\rangle$$

for every $\varphi \in C^\infty(X, W^+)$. Thus Φ is a weak solution of the equation (8.12) above with $\Psi \in W^{j,p}$ and this implies $\Phi \in W^{j+1,p}$. This proves the lemma in the case where A is smooth. In the general case fix a smooth reference connection A_0, denote $\alpha = A - A_0 \in W^{j,p}(X, T^*X \otimes i\mathbb{R})$, and consider the equation

$$D_{A_0}\Phi = \Psi - \Gamma(\alpha)\Phi.$$

Assume first that $j = 1$ and $p > 2$. Since $\alpha \in W^{1,p} \subset L^{4p/(4-p)}$ and $\Phi \in L^q$ with $q = p/(p-1)$ it follows from Hölder's inequality that

$$\Gamma(\alpha)\Phi - \Psi \in L^r, \qquad \frac{1}{r} = \frac{4-p}{4p} + \frac{1}{q}.$$

By the first part of the proof, this implies

$$\Phi \in W^{1,r} \subset L^{q_1}, \qquad q_1 = \frac{4r}{4-r} = \frac{2pq}{2p + 2q - pq} > q.$$

The inequality $q_1 > q$ follows uses the fact that $p > 2$. Now continue by induction with

$$q_{i+1} = \frac{2pq_i}{2p + 2q_i - pq_i} > q_i$$

until $q' = q_{i+1} \geq 2p/(p-2)$. This shows that $\Phi \in L^{2p/(p-2)}$ and hence as above $\Gamma(\alpha)\Phi - \Psi \in L^4$. This in turn implies $\Phi \in W^{1,4} \subset L^s$ for any $s < \infty$ and thus $\Gamma(\alpha)\Phi - \Psi \in L^r$ for some $r > 4$. This shows that $\Phi \in W^{1,r}$ and hence $\Gamma(\alpha)\Phi - \Psi \in W^{1,p}$. (Multiplication by a function in $W^{1,r}$ with

$r > 4$ preserves any Sobolev space $W^{1,p}$.) Finally, using the first part of the proof again, one obtains $\Phi \in W^{2,p}$. This proves the lemma for $j = 1$ and $p > 2$. Now suppose, by induction over j, that the lemma has been proved for $j \geq 1$ and any $p \geq 1$ with $(j+1)p > 4$. Assume that $A \in \mathcal{A}^{j+1,q}$ and $\Psi \in W^{j+1,q}$ with $(j+2)q > 4$. Then

$$A \in \mathcal{A}^{j,p}, \qquad \Phi \in W^{j,p}, \qquad p = \frac{4q}{4-q}$$

and the reader may check that $(j+1)p > 4$. By the induction hypothesis, this implies that $\Phi \in W^{j+1,p}$. Since $(j+1)p > 4$ it follows that multiplication by Φ preserves the Sobolev space $W^{j+1,q}$ and thus $\Gamma(\alpha)\Phi \in W^{j+1,q}$. Hence the equation $D_{A_0}\Phi = \Psi - \Gamma(\alpha)\Phi$ shows that $\Phi \in W^{j+2,q}$. This proves the lemma. □

Proof of Proposition 8.12: By Lemma 8.13 (i) the operator $D_A - D_{A_0} : W^{j+1,p}(X, W^+) \to W^{j,p}(X, W^-)$ is compact whenever $A \in \mathcal{A}^{k,q}$, $0 \leq j \leq k$, $j - 4/p \leq k - 4/q$, and $(k+1)q > 4$. Hence it suffices to prove that D_{A_0} is a Fredholm operator of the required index whenever A_0 is a smooth connection. That D_{A_0} has a closed range and a finite dimensional kernel follows immediately from Lemma 8.13 (ii), Lemma A.1, and Rellich's theorem. That D_{A_0} has a finite dimensional cokernel follows from Lemma 8.14 and the fact that the formal adjoint operator $D_A{}^* : W^{j+1,p}(X, W^-) \to W^{j,p}(X, W^+)$ also has a finite dimensional kernel. The index formula follows from Theorem 6.22 and the fact that the index is independent of j, and p. This in turn follows from the fact that, again by Lemma 8.14, the elements of the kernel and cokernel of D_{A_0} are smooth whenever A_0 is smooth. This proves the proposition. □

Elliptic bootstrapping

Proof of Theorem 7.11: By Theorem 7.14, every connection is gauge equivalent to one which satisfies $d^*(A - A_0) = 0$ for some fixed smooth connection $A_0 \in \mathcal{A}(\Gamma)$. Assume without loss of generality that the reference connection A_0 is Yang-Mills, i.e. $d^*F_{A_0} = 0$ and hence $d^*F_{A_0}^+ = 0$. The 1-form $\alpha = A - A_0 \in \Omega^1(X, i\mathbb{R})$ satisfies

$$F_A^+ = F_{A_0}^+ + d^+a = F_{A_0}^+ + \frac{1}{2}(d\alpha + *d\alpha).$$

and hence $d\alpha = 2F_A^+ - 2F_{A_0}^+ - *d\alpha$. Since $d^*\alpha = 0$ and $d^*F_{A_0}^+ = 0$ it follows that

$$\langle d\beta, d\alpha\rangle + \langle d^*\beta, d^*\alpha\rangle = 2\langle d\beta, F_A^+\rangle. \tag{8.13}$$

for every 1-form $\beta \in \Omega^1(X, i\mathbb{R})$. Here $\langle \cdot, \cdot \rangle$ denotes the L^2 inner product.. Note that (8.13) is a *weak version* of the equation

$$d^*d\alpha + dd^*\alpha = 2d^*F_A^+.$$

But it only makes sense in the strong form once it has been established that α is of class $W^{2,p}$. This, however, follows easily: By Lemma 8.14, we have $\Phi \in W^{2,p}(X, W^+)$ and the identity

$$F_A^+ + \eta = \sigma^+((\Phi\Phi^*)_0)$$

shows that $F_A^+ \in W^{2,p}$. (Since $2p > 4$ the Sobolev space $W^{2,p}$ is invariant under products.) It now follows from (8.13) with $d^*F_{A^+} \in W^{1,p}$ that $\alpha \in W^{3,p}$. Hence $A \in \mathcal{A}^{3,p}$ and, by Lemma 8.14, $\Phi \in W^{4,p}$. The rest of the proof is an easy induction argument. Once $\Phi \in W^{k,p}$ for some $k \geq 2$ it follows as above that $F_A^+ \in W^{k,p}$, hence $\alpha \in W^{k+1,p}$, and thus $\Phi \in W^{k+2,p}$. This holds for all integers k and thus α and Φ are smooth. □

Proof of Theorem 7.12: Fix a constant $p > 4$ and a smooth reference connection $A_0 \in \mathcal{A}(\Gamma)$. Then, by Theorem 7.14, every solution (A, Φ) of the perturbed Seiberg-Witten equations (7.4) is gauge equivalent to one which satisfies

$$d^*\alpha = 0, \qquad \|\alpha\|_{W^{1,p}} \leq c(1 + \|d\alpha\|_{L^p}) \tag{8.14}$$

where $\alpha = A - A_0$. Here the constant $c > 0$ is independent of A and Φ. The proof of Theorem 7.11 shows that under this assumption ($d^*\alpha = 0$) the pair (A, Φ) is smooth. Hence Lemma 7.13 shows that there is an estimate

$$\sup_X |\Phi|^2 \leq c_0$$

where the constant c_0 is independent of the pair (A, Φ). Combine this with the formula $F_A^+ + \eta = \sigma^+((\Phi\Phi^*)_0)$ to obtain a uniform upper bound

$$\sup_X |F_A^+| \leq c_0.$$

Now recall from the proof of Theorem 7.11 the equation

$$\langle d\beta, d\alpha\rangle + \langle d^*\beta, d^*\alpha\rangle = 2\left\langle d\beta, F_A^+ - F_{A_0}^+\right\rangle \tag{8.15}$$

for $\beta \in \Omega^1(X, i\mathbb{R})$. The Calderón-Zygmund inequality asserts that

$$\|d\alpha\|_{L^p} \leq c \sup_\beta \frac{\langle d\beta, d\alpha\rangle + \langle d^*\beta, d^*\alpha\rangle}{\|\beta\|_{W^{1,q}}} = c \sup_\beta \frac{2\left\langle d\beta, F_A^+ - F_{A_0}^+\right\rangle}{\|\beta\|_{W^{1,q}}}$$

where $1/p + 1/q = 1$. The estimate on F_A^+ shows that the right hand side is uniformly bounded. Hence, by (8.14),

$$\|\alpha\|_{W^{1,p}} \leq c_1.$$

Since $p > 4$ it follows from the Sobolev embedding theorem that α is uniformly bounded in the sup-norm. Now Lemma 8.14 shows that $\Phi \in W^{2,p}$ and hence, by Lemma 8.13 (i) with $D_A\Phi = 0$, $p = q$, $k = 1$, and $j = 0$,

$$\|\Phi\|_{W^{1,p}} \leq c\,(1 + \|\alpha\|_{W^{1,p}})\,\|\Phi\|_{L^4} \leq c_1'.$$

Thus both α and Φ satisfy a uniform $W^{1,p}$-estimate. Now it follows from the Calderón-Zygmund inequality for equation (8.15) and Lemma 8.13 that for every integer $k \geq 1$ there exists a constant $c = c(k, X) > 0$ such that

$$\|\alpha\|_{W^{k+1,p}} \leq c\,(1 + \|\Phi\Phi^*\|_{W^{k,p}})\,,$$

$$\|\Phi\|_{W^{k+1,p}} \leq c\,(1 + \|\alpha\|_{W^{k,p}})\,\|\Phi\|_{W^{k,p}}$$

Use these inequalities inductively for $k = 1, 2, 3, \ldots$ to obtain uniform estimates

$$\|\alpha\|_{W^{k,p}} + \|\Phi\|_{W^{k,p}} \leq c_k.$$

Rellich's theorem asserts that every sequence which is bounded in $W^{k+1,p}$ has a subsequence which converges in $W^{k,p}$. Hence any sequence of solutions (A_ν, Φ_ν) of (7.1) which satisfies (7.9) has a subsequence which converges in $W^{k,p}$ for every integer $k > 0$. By the Sobolev embedding theorem ($W^{k,p} \subset C^\ell$ for $kp > \ell p + n$) the subsequence converges in the C^∞-topology. The reader may check that the constants in the proof are also independent of the choice of the metric g_ν and the perturbation η_ν provided that these are converging sequences. □

The technique in the proof of the Theorem 7.12 is the *elliptic bootstrapping* method. The proof is significantly simpler than the compactness theorem for anti-self-dual instantons with nonabelian Lie groups. The reason lies, firstly, in the fact that Lemma 7.13 gives a uniform bound on the function Φ and hence on the curvature of A. Secondly, the existence of a Coulomb gauge (the condition $d^*(A - A_0) = 0$) simply reduces to Hodge theory and the result is global, rather than local in a neighbourhood of A_0. This significantly simplifies the proof of Uhlenbeck's theorem in the case of compact abelian Lie groups.

In the general case one first proves a compactness theorem for solutions with, say, bounded curvature. Then one observes that the energy formula gives only a uniform bound on the L^2-norm of the curvature (a borderline case for the Sobolev embedding theorem in dimension 4). Using the conformal invariance of the energy (Proposition 7.8) one can then show that, if the curvature tends to infinity, a nontrivial instanton on Euclidean space *splits off*. This argument can also be used for the Seiberg-Witten equations and then the nonexistence of nontrivial Seiberg-Witten monopoles on $\mathbb{R}^4$ (Proposition 8.4) would guarantee a uniform bound on the curvature. However, the above argument with the a priori estimate of Lemma 7.13 is simpler.

8.4 Transversality in dimension four

Our goal in this section is to provide the proof of Theorem 7.16 which asserts that for a generic perturbation η the moduli spaces $\mathcal{M}^*(X, \Gamma, g, \eta)$ are all smooth manifolds. The proof will be based on the next two crucial lemmata. The first deals with the fundamental Fredholm properties of the operator D^+ defined by $D^+\alpha = (d^*\alpha, d^+\alpha)$ for $\alpha \in \Omega^1(X, i\mathbb{R})$ (see page 237.) The second deals with the *universal space* of all pairs (A, Φ) where $A \in \mathcal{A}(\Gamma)$ is a spin^c connection and $\Phi \in \ker D_A$ is a nonzero harmonic spinor. Proposition 8.16 below asserts that this space is a smooth paracompact separable Banach manifold.

Lemma 8.15 *The operator*

$$D^+ : W^{1,p}(X, T^*X) \longrightarrow L^p(X) \oplus L^p(X, \Lambda^{2,+}T^*X)$$

is Fredholm with

$$\operatorname{index} D^+ = b_1 - 1 - b^+ = -\frac{\chi(X) + \sigma(X)}{2}.$$

Proof: For general p the proof that D^+ is Fredholm is based on the Calderón-Zygmund inequality and this will not be carried out here. (See Appendix C for more details about L^p-estimates.) However in the case $p = 2$ a simple argument shows that the operator D^+ satisfies the estimate

$$\|\alpha\|_{W^{1,2}} \le c\Big(\big\|D^+\alpha\big\|_{L^2} + \|\alpha\|_{L^2}\Big). \tag{8.16}$$

To see this consider the formal adjoint operator

$$(D^+)^* : \Omega^0(X) \oplus \Omega^{2,+}(X) \longrightarrow \Omega^1(X),$$

defined by $\langle (D^+)^*(\eta, \tau), \alpha\rangle = \langle (\eta, \tau), D^+\alpha\rangle$ for $\eta \in \Omega^0(X)$, $\tau \in \Omega^{2,+}(X)$, $\alpha \in \Omega^1(X, i\mathbb{R})$. The identity $\langle \eta, d^*\alpha\rangle + \langle \tau, d^+\alpha\rangle = \langle d\eta + d^*\tau, \alpha\rangle$ shows that

$$(D^+)^*(\eta, \tau) = d\eta + d^*\tau.$$

Hence

$$(D^+)^*D^+\alpha = dd^*\alpha + \frac{1}{2}d^*d\alpha$$

for $\alpha \in \Omega^1(X, i\mathbb{R})$. Now recall from Exercise 2.31 that

$$\|d\alpha\|_{L^2}^2 + \|d^*\alpha\|_{L^2}^2 = \|\nabla\alpha\|_{L^2}^2 + \int_X S(\alpha^*, \alpha^*)\mathrm{dvol}$$

where $S : S^2TX \to \mathbb{R}$ denotes the Ricci tensor. This implies

$$\begin{aligned}\left\|D^+\alpha\right\|_{L^2}^2 &= \left\langle \alpha, dd^*\alpha + \frac{1}{2}d^*d\alpha \right\rangle \\ &= \|d^*\alpha\|_{L^2}^2 + \frac{1}{2}\|d\alpha\|_{L^2}^2 \\ &\geq \frac{1}{2}\|\nabla\alpha\|_{L^2}^2 - c\|\alpha\|_{L^2}^2 \\ &= \frac{1}{2}\|\alpha\|_{W^{1,2}}^2 - \left(c + \frac{1}{2}\right)\|\alpha\|_{L^2}^2 .\end{aligned}$$

This proves the estimate (8.16). By Lemma A.1, the operator D^+ has a closed range and a finite dimensional kernel. Moreover, elliptic regularity asserts that if $\eta \in L^2(X)$ and $\tau \in L^2(X, \Lambda^{2,+}T^*X)$ satisfy

$$\langle \eta, d^*\alpha \rangle + \langle \tau, d^+\alpha \rangle = 0$$

for all $\alpha \in \Omega^1(X)$ then η and τ are smooth and $(D^+)^*(\eta, \tau) = d\eta + d^*\tau = 0$. This shows that the cokernel of D^+ agrees with the kernel of $(D^+)^*$ and a similar estimate as (8.16) for the operator $(D^+)^*$ shows that the kernel of $(D^+)^*$ is also finite dimensional. This proves that D^+ is a Fredholm operator.

Now consider the kernel of D^+. Firstly, by elliptic regularity, all 1-forms in the kernel of D^+ are smooth. Secondly, for every $\alpha \in \Omega^1(X)$,

$$D^+\alpha = 0 \quad \Longleftrightarrow \quad d^+\alpha = 0,\, d^*\alpha = 0 \quad \Longleftrightarrow \quad d\alpha = 0,\, d^*\alpha = 0.$$

Note here that $2d^*d^+ = d^*d$ and hence $d^+\alpha = 0$ implies $d^*d\alpha = 0$. Then take the inner product with α to obtain $d\alpha = 0$. Thus the kernel of D^+ is the space of harmonic 1-forms

$$\ker\, D^+ = H^1 = \ker\, d \cap \ker\, d^*.$$

Now consider the operator $(D^+)^*(\eta, \tau) = d\eta + d^*\tau$. Since $d \circ d = 0$ it follows that $\langle d\eta, d^*\tau \rangle = 0$ for all η and τ. Hence $(D^+)^*(\eta, \tau) = 0$ if and only if $d\eta = 0$ and $d^+\tau = 0$. But a self-dual 2-form $\tau = *\tau \in \Omega^{2,+}(X)$ satisfies $d\tau = 0$ if and only if $d^*\tau = 0$. Hence

$$\ker\,(D^+)^* = H^0 \oplus H^{2,+}$$

where $H^0 = \ker\,(d : \Omega^0(X) \to \Omega^1(X))$ is the space of constant functions and $H^{2,+}$ is the space of self-dual harmonic 2-forms. Since $\dim\, H^0 = 1$, $\dim\, H^1 = b_1$, and $\dim\, H^{2,+} = b^+$ this proves the index formula for D^+. □

Recall from Proposition 8.12 that $D_A : W^{1,p}(X, W^+) \to L^p(X, W^-)$ is a Fredholm operator with index $D_A = (c \cdot c - \sigma)/4$ where $c = c_1(L_\Gamma)$. Hence it follows from Lemma 8.15 that the operator

$$\mathcal{D}_{A,\Phi} : \begin{array}{c} W^{1,p}(X, T^*X \otimes i\mathbb{R}) \\ \oplus \\ W^{1,p}(X, W^+) \end{array} \longrightarrow \begin{array}{c} L^p(X, i\mathbb{R}) \\ \oplus \\ L^p(\Lambda^{2,+}T^*X \otimes i\mathbb{R}) \\ \oplus \\ L^p(X, W^-) \end{array}$$

defined by (7.15) is a Fredholm operator whose index is the sum of the indices of D_A and D^+ and hence is given by (7.16):

$$\operatorname{index} \mathcal{D}_{A,\Phi} = \frac{\langle c_1(L_\Gamma)^2, [X]\rangle}{4} - \frac{2\chi(X) + 3\sigma(X)}{4}.$$

Recall that the L^2-orthogonal complement of $\operatorname{im} \mathcal{D}_{A,\Phi}$ always contains the space $H^0(X, i\mathbb{R})$ and that a perturbation $\eta \in i\Omega^{2,+}(X)$ is called regular if $\operatorname{coker} \mathcal{D}_{A,\Phi} \cong H^0(X, i\mathbb{R})$ for all $(A, \Phi) \in \widetilde{\mathcal{M}}^*(X, \Gamma, g, \eta)$. The proof that the set $\Omega^{2,+}_{\mathrm{reg}}(X, i\mathbb{R}; g)$ of regular perturbations is of the second category in the sense of Baire relies on the Sard-Smale theorem B.13 and on the following auxiliary lemma. Note here the choice of the constant $p > 4$. The result should continue to hold for any $p > 2$, however, for such values of p the proof of the unique continuation theorem (see Theorem E.8 in Appendix E) becomes more difficult and the following result suffices for all the applications treated in this book.

Proposition 8.16 *For every $p > 4$ and every integer $k \geq 1$ the space*

$$\mathcal{N}^{k,p} = \mathcal{N}^{k,p}(X, \Gamma, g)$$

of all pairs $(A, \Phi) \in \mathcal{A}^{k,p}(\Gamma) \times W^{k,p}(X, W^+)$ which satisfy

$$D_A\Phi = 0, \qquad d^*(A - A_0) = 0, \qquad \Phi \neq 0$$

is a smooth paracompact separable Banach manifold. Its tangent space at $(A, \Phi) \in \mathcal{N}^{k,p}$ consists of all pairs $\alpha \in W^{k,p}(X, T^*X \otimes i\mathbb{R})$, $\varphi \in W^{k,p}(X, W^+)$ which satisfy*

$$D_A\varphi + \Gamma(\alpha)\Phi = 0, \qquad d^*\alpha = 0.$$

Proof: Consider the Banach manifolds

$$\begin{aligned} \mathcal{X} &= \mathcal{A}^{k,p}(\Gamma) \oplus W^{k,p}(X, W^+)^*, \\ \mathcal{Y}_0 &= W^{k-1}_0(X, i\mathbb{R}) \oplus W^{k-1,p}(X, W^-) \end{aligned} \tag{8.17}$$

*A topological space is called separable if it admits a countable dense subset. It is called paracompact if every open cover admits a locally finite refinement. Both conditions together imply that every open cover has a countable subcover.

where $W^{k,p}(X,W^+)^*$ denotes the set of nonzero $W^{k,p}$-sections of W^+ and

$$W_0^{k-1,p}(X,i\mathbb{R}) = \left\{\xi \in W^{k-1,p}(X,i\mathbb{R}) \mid \int_X \xi \mathrm{dvol} = 0\right\}.$$

Note that $T_{(A,\Phi)}\mathcal{X} = W^{k,p}(X,T^*X\otimes i\mathbb{R}) \oplus W^{k,p}(X,W^+)$. Consider the map $\mathcal{F}_0 : \mathcal{X} \to \mathcal{Y}_0$ given by

$$\mathcal{F}_0(A,\Phi) = (d^*(A-A_0), D_A\Phi).$$

The differential $d\mathcal{F}_0(A,\Phi) = D_{A,\Phi} : T_{(A,\Phi)}\mathcal{X} \to \mathcal{Y}_0$ given by

$$D_{A,\Phi}\begin{pmatrix}\alpha\\ \varphi\end{pmatrix} = \begin{pmatrix} d^*\alpha \\ D_A\varphi + \Gamma(\alpha)\Phi\end{pmatrix}. \tag{8.18}$$

The next lemma is the key to the proof of the proposition.

Lemma 8.17 *Assume $p > 4$ and $k \geq 1$. Suppose that $A \in \mathcal{A}^{k,p}(\Gamma)$ and $\Phi \in W^{k,p}(X,W^+)$ satisfy $D_A\Phi = 0$ and $\Phi \neq 0$. Then the operator*

$$D_{A,\Phi} : \begin{array}{c} W^{k,p}(X,T^*X\otimes i\mathbb{R}) \\ \oplus \\ W^{k,p}(X,W^+)\end{array} \longrightarrow \begin{array}{c} W_0^{k-1,p}(X,i\mathbb{R}) \\ \oplus \\ W^{k-1,p}(X,W^-)\end{array}$$

defined by (8.18) is onto and has a right inverse.

Proof: Consider first the case $k = 1$. By Proposition 8.12, the operator $D_{A,0}$ has a closed range and a finite dimensional cokernel and, by Lemma 8.13, the difference $D_{A,\Phi} - D_{A,0}$ is compact. By Corollary A.3 in Appendix A, this implies that $D_{A,\Phi}$ has a closed range and a finite dimensional cokernel. Hence it remains to prove that $D_{A,\Phi}$ has a dense range. To see this consider the formal adjoint operator

$${D_{A,\Phi}}^* : \begin{array}{c} W_0^{k+1,p}(X,i\mathbb{R}) \\ \oplus \\ W^{k+1,p}(X,W^-)\end{array} \longrightarrow \begin{array}{c} W^{k,p}(X,T^*X\otimes i\mathbb{R}) \\ \oplus \\ W^{k,p}(X,W^+)\end{array}.$$

The formula

$$\langle\psi,\Gamma(\alpha)\Phi\rangle = \sum_j \operatorname{Im}\alpha(e_j)\langle\psi, i\Gamma(e_j)\Phi\rangle = \langle\alpha, i\langle\psi, i\Gamma(\cdot)\Phi\rangle\rangle$$

for $\psi \in C^\infty(X,W^-)$ shows that the operator ${D_{A,\Phi}}^*$ is given by

$${D_{A,\Phi}}^*\begin{pmatrix}\xi\\ \psi\end{pmatrix} = \begin{pmatrix} d\xi + i\langle\psi, i\Gamma(\cdot)\Phi\rangle \\ {D_A}^*\psi\end{pmatrix}. \tag{8.19}$$

The first claim is that, by elliptic regularity, every pair $(\xi, \psi) \in L^q_0(X, i\mathbb{R}) \oplus L^q(X, W^-)$ with $1/p + 1/q = 1$ which annihilates the image of $D_{A,\Phi}$ is of class $W^{2,p}$ and lies in the kernel of $D_{A,\Phi}{}^*$. More precisely, assume that the pair (ξ, ψ) is of class L^q and satisfies

$$\int_X \Big(\langle \xi, d^*\alpha \rangle + \langle \psi, D_A\varphi + \Gamma(\alpha)\Phi \rangle \Big) \mathrm{dvol} = 0, \qquad \int_X \xi \, \mathrm{dvol} = 0,$$

for all $\alpha \in \Omega^1(X, i\mathbb{R})$ and $\varphi \in C^\infty(X, W^+)$. First consider this formula with $\alpha = 0$ and use Lemma 8.14 to obtain $\psi \in W^{2,p}(X, W^-)$ and $D_A{}^*\psi = 0$. Secondly, note that

$$\int_X \langle \xi, d^*\alpha \rangle \mathrm{dvol} = -\int_X \langle \psi, \Gamma(\alpha)\Phi \rangle = -\int_X \langle \alpha, i\langle \psi, i\Gamma(\cdot)\Phi \rangle \rangle \mathrm{dvol}$$

for all $\alpha \in \Omega^1(X, i\mathbb{R})$. Since $\psi \in W^{2,p}$ and $\Phi \in W^{1,p}$ it follows that $i\langle \psi, i\Gamma(\cdot)\Phi \rangle \in W^{1,p}$. By elliptic regularity for the Laplace-Beltrami operator, this implies $\xi \in W^{2,p}$ and $d\xi + i\langle \psi, i\Gamma(\cdot)\Phi \rangle = 0$. This shows that the pair (ξ, ψ) is indeed in the kernel of the adjoint operator.

It remains to prove that the kernel of $D_{A,\Phi}{}^*$ is zero whenever $D_A\Phi = 0$ and $\Phi \neq 0$. To see this it is convenient to first compute the operator $D_{A,\Phi} D_{A,\Phi}{}^*$. The formulae

$$\Gamma(i\langle \psi, i\Gamma(\cdot)\Phi \rangle)\Phi = |\Phi|^2 \psi$$

and

$$\begin{aligned} d^*(i\langle \psi, i\Gamma(\cdot)\Phi \rangle) &= -i \sum_j \partial_j \langle \psi, i\Gamma(e_j)\Phi \rangle - i \sum_j \mathrm{div}(e_j) \langle \psi, i\Gamma(e_j)\Phi \rangle \\ &= -i \sum_j \langle \nabla_j \psi, i\Gamma(e_j)\Phi \rangle - i \sum_j \langle \psi, i\Gamma(e_j)\nabla_j \Phi \rangle \\ &\quad - i \sum_j \langle \psi, i\Gamma(\nabla_j e_j + \mathrm{div}(e_j) e_j)\Phi \rangle \\ &= i \sum_j \langle \Gamma(e_j)\nabla_j \psi, i\Phi \rangle - i \sum_j \langle \psi, i\Gamma(e_j)\nabla_j \Phi \rangle \\ &= i\langle D_A^* \psi, i\Phi \rangle - i\langle \psi, iD_A\Phi \rangle \end{aligned}$$

show that

$$D_{A,\Phi} D_{A,\Phi}{}^* \begin{pmatrix} \xi \\ \psi \end{pmatrix} = \begin{pmatrix} d^*d\xi + i\langle D_A^*\psi, i\Phi \rangle - i\langle \psi, iD_A\Phi \rangle \\ D_A D_A{}^*\psi + |\Phi|^2\psi + \Gamma(d\xi)\Phi \end{pmatrix}. \tag{8.20}$$

Now suppose that (ξ, ψ) is of class $W^{2,p}$ and lies in the kernel of $D_{A,\Phi}{}^*$, i.e.

$$D_A{}^*\psi = 0, \qquad d\xi + i\langle\psi, i\Gamma(\cdot)\Phi\rangle = 0.$$

Assume $D_A\Phi = 0$ and $\Phi \neq 0$. Then, by (8.20),

$$d^*d\xi = i\langle\psi, iD_A\Phi\rangle - i\langle D_A^*\psi, i\Phi\rangle = 0.$$

Hence ξ is constant and, since it has mean value zero, it must vanish. Now the second component of (8.20) vanishes as well and since $\xi = 0$ this means

$$D_A D_A{}^*\psi + |\Phi|^2\psi = 0.$$

Take the inner product with ψ to obtain

$$\int_X \left(|D_A{}^*\psi|^2 + |\Phi|^2|\psi|^2\right) \mathrm{dvol} = 0.$$

The function Φ is continuous and nonzero. Hence ψ vanishes on some open set and it follows from the unique continuation theorem E.8 in Appendix E that ψ must vanish everywhere. (This is the only place in the proof where the condition $p > 4$ rather than $p > 2$ is required.) This argument also gives a formula for the right inverse of the operator $D_{A,\Phi}$, namely,

$$T = D_{A,\Phi}{}^* \left(D_{A,\Phi}D_{A,\Phi}{}^*\right)^{-1}.$$

Here the operator $D_{A,\Phi}D_{A,\Phi}{}^*$ is to be understood from $W^{2,p}$ to L^p. By Lemma 8.13, this operator is a compact perturbation of the Laplacian. Hence it is Fredholm and has index zero. The above argument shows that its kernel is zero and so the operator is invertible. This proves the lemma in the case $k = 1$. The case $k \geq 1$ is now an easy consequence. Just note that the operator $D_{A,\Phi}D_{A,\Phi}{}^*$ is still bijective when regarded as an operator from $W^{k+1,p}$ to $W^{k-1,p}$ provided that A and Φ are of class $W^{k,p}$. Injectivity is obvious and surjectivity follows from elliptic regularity.* But since $D_{A,\Phi}D_{A,\Phi}{}^*$ is bijective from $W^{k+1,p}$ to $W^{k-1,p}$ the above operator T is a right inverse of $D_{A,\Phi} : W^{k,p} \to W^{k-1,p}$. This proves the lemma. □

Proof of Proposition 8.16 continued: By Lemma 8.17, the linearized operator $D_{A,\Phi} = d\mathcal{F}_0(A,\Phi) : T_{(A,\Phi)}\mathcal{X} \to \mathcal{Y}_0$ is onto and has right inverse whenever $(A,\Phi) \in \mathcal{X}$. Hence 0 is a regular value of $\mathcal{F}_0$ and it follows from the implicit function theorem B.3 that $\mathcal{N}^{k,p} = \mathcal{F}_0{}^{-1}(0)$ is a Banach manifold. As a metric space this manifold is paracompact. That it is separable follows immediately from the fact that $\mathcal{X}$ is a separable Banach space. Namely, by Proposition B.14, cover $\mathcal{N}^{k,p}$ by countably many charts and choose a dense sequence in each chart. □

*Given $(\xi',\psi') \in W^{k-1,p} \subset L^p$ the equation $D_{A,\Phi}D_{A,\Phi}{}^*(\xi,\psi) = (\xi',\psi')$ has a solution $(\xi,\psi) \in W^{2,p}$ which, by elliptic regularity, is necessarily of class $W^{k+1,p}$.

Proof of Theorem 7.16: Consider the map

$$\mathcal{F}_1 : \mathcal{N}^{k,p} \to W^{k-1,p}(X, \Lambda^{2,+}T^*X \otimes i\mathbb{R})$$

defined by

$$\mathcal{F}_1(A, \Phi) = F_A^+ - \sigma^+((\Phi\Phi^*)_0)$$

for $(A, \Phi) \in \mathcal{N}^{k,p}$. For any $p > 4$ and any integer $k \geq 1$ this a smooth Fredholm map of index

$$\operatorname{index}(\mathcal{F}_1) = \frac{c \cdot c}{4} - \frac{2\chi + 3\sigma}{4} + 1$$

where $c = c_1(L_\Gamma) \in H^2(X; \mathbb{Z})$ and $\chi = \chi(X)$, $\sigma = \sigma(X)$. To see this note that the linearized map $d\mathcal{F}_1(A, \Phi) : T_{(A,\Phi)}\mathcal{N}^{k,p}$ is given by

$$d\mathcal{F}_1(A, \Phi)(\alpha, \varphi) = d^+\alpha - \sigma^+((\Phi\varphi^* + \varphi\Phi^*)_0)$$

for $(\alpha, \varphi) \in W^{k,p}(X, T^* \otimes i\mathbb{R}) \times W^{k,p}(X, W^+)$ which satisfy

$$d^*\alpha = 0, \qquad D_A\varphi + \Gamma(\alpha)\Phi = 0.$$

The kernel and cokernel of this operator agree with those of the operator $\mathcal{D}_{A,\Phi}$ and hence both operators have the same index.* Recall here that the additional $+1$ arises when the target space of $\mathcal{D}_{A,\Phi}$ is restricted to triples (ξ, ω, ψ) where ξ has mean value zero. Note that

$${\mathcal{F}_1}^{-1}(\eta) = \widetilde{\mathcal{M}}^*(X, \Gamma, g, \eta)$$

for every $\eta \in W^{k-1,p}(X, \Lambda^{2,+}T^*X)$ and the above formulae show that η is a regular value of $\mathcal{F}_1$ if and only if the operator $\mathcal{D}_{A,\Phi}$ has a one dimensional cokernel (consisting of the constant functions) for every pair $(A, \Phi) \in {\mathcal{F}_1}^{-1}(\eta)$. Hence the set

$$\mathcal{Z}^{k-1,p}_{\mathrm{reg}}$$

of regular values of $\mathcal{F}_1$ consists of all $\eta \in W^{k-1,p}(X, \Lambda^{2,+}T^*X)$ such that $\operatorname{coker} \mathcal{D}_{A,\Phi} = H^0(X, i\mathbb{R})$ for all $(A, \Phi) \in {\mathcal{F}_1}^{-1}(\eta)$. It follows from the Sard-Smale theorem B.13 that this set is of the second category in the sense of Baire with respect to the $W^{k-1,p}$ topology.

*There is a finite dimensional analogue. Suppose that X, Y_0, Y_1 are finite dimensional manifolds and $f_0 : X \to Y_0$, $f_1 : X \to Y_1$ are smooth maps. Let $y_0 \in Y_0$ be a regular value of f_0 and consider the submanifold $N = {f_0}^{-1}(y_0) \subset X$. Then the differential of the restriction of f_1 to N at a point $x \in N$ has the same kernel and cokernel as the differential of the product map $f = f_0 \times f_1 : X \to Y_0 \times Y_1$.

We shall use an argument of Taubes to reduce the C^∞-case to the $W^{k,p}$-case. Abbreviate

$$\mathcal{Z}_{\mathrm{reg}} = \Omega^{2,+}_{\mathrm{reg}}(X, i\mathbb{R}; g)$$

(the set of smooth regular self-dual 2-forms) and note that

$$\mathcal{Z}_{\mathrm{reg}} = \bigcap_{\varepsilon>0} \mathcal{Z}_{\varepsilon,\mathrm{reg}}.$$

where $\mathcal{Z}_{\varepsilon,\mathrm{reg}}$ is defined as the set of all $\eta \in \Omega^{2,+}(X, g)$ such that the operator $\mathcal{D}_{A,\Phi} : \mathcal{X} \to \mathcal{Y}$ is onto for all solutions (A, Φ) of (7.4) with $\max_x \|\Phi(x)\| \geq \varepsilon$. We shall prove that the set $\mathcal{Z}_{\varepsilon,\mathrm{reg}}$ is open and dense in $\Omega^{2,+}(X)$ with respect to the C^∞ topology.

We first prove that the complement of $\mathcal{Z}_{\varepsilon,\mathrm{reg}}$ is closed. Choose a sequence $\eta_\nu \in \Omega^{2,+}(X, g) - \mathcal{Z}_{\varepsilon,\mathrm{reg}}$ and assume that η_ν converges to η in the C^∞-topology. Then there exists a sequence of solutions (A_ν, Φ_ν) of (7.4) with $\eta = \eta_\nu$ such that $\max_x \|\Phi_\nu(x)\| \geq \varepsilon$ and the operator $\mathcal{D}_{A_\nu,\Phi_\nu} : \mathcal{X} \to \mathcal{Y}$ is not onto. By Theorem 7.12 the sequence (A_ν, Φ_ν) has a convergent subsequence and the limit pair (A, Φ) is a solution of (7.4) with $\max_x \|\Phi(x)\| \geq \varepsilon$. Moreover, the operator $\mathcal{D}_{A,\Phi} : \mathcal{X} \to \mathcal{Y}$ cannot be onto since otherwise the operators $\mathcal{D}_{A_\nu,\Phi_\nu}$ would be onto for ν sufficiently large. Hence $\eta \notin \mathcal{Z}_{\varepsilon,\mathrm{reg}}$ and this proves that $\mathcal{Z}_{\varepsilon,\mathrm{reg}}$ is open.

We prove that $\mathcal{Z}_{\varepsilon,\mathrm{reg}}$ is dense in $\Omega^{2,+}(X, g)$ with respect to the C^∞-topology. To see this let $\eta \in \Omega^{2,+}(X, g)$ and recall from the first part of the proof that $\mathcal{Z}^{k,p}_{\mathrm{reg}}$ is dense in the space $W^{k,p}(X, \Lambda^{2,+}T^*X)$ of self-dual 2-forms of class $W^{k,p}$. Define

$$\mathcal{Z}^{k,p}_{\varepsilon,\mathrm{reg}} \subset W^{k,p}(X, \Lambda^{2,+}T^*X)$$

in the obvious way and notice, as above, that this set is open with respect to the $W^{k,p}$-topology. Moreover,

$$\mathcal{Z}^{k,p}_{\mathrm{reg}} \subset \mathcal{Z}^{k,p}_{\varepsilon,\mathrm{reg}}$$

and so this set is also dense. Hence approximate η by a sequence $\eta_\nu \in \mathcal{Z}^{k,p}_{\varepsilon,\mathrm{reg}}$ with respect to the $W^{k,p}$-topology. Since $\mathcal{Z}^{k,p}_{\varepsilon,\mathrm{reg}}$ is open with respect to the $W^{k,p}$ topology, each η_ν can be approximated by a C^∞ smooth 2-form $\eta'_\nu \in \mathcal{Z}^{k,p}_{\varepsilon,\mathrm{reg}}$. Since η'_ν is smooth it follows that $\eta'_\nu \in \mathcal{Z}_{\varepsilon,\mathrm{reg}}$ and, by construction, η'_ν converges to η. This proves that $\mathcal{Z}_{\varepsilon,\mathrm{reg}}$ is dense in $\Omega^{2,+}(X)$. It follows that the space $\mathcal{Z}_{\mathrm{reg}} = \Omega^{2,+}_{\mathrm{reg}}(X, i\mathbb{R}; g)$ is a countable intersection of open and dense sets in $\Omega^{2,+}(X, i\mathbb{R}; g)$ and hence is of the second category in the sense of Baire. This proves the theorem. □

Remark 8.18 It is sometimes useful to use regular perturbations with support in some given open subset $\Omega \subset X$. The existence of such perturbations can be proved by a standard argument similar to that in the proof of Theorem 7.16. Namely, consider the space

$$\mathcal{Z}_\Omega^{k-1,p} = \left\{\eta \in W^{k-1,p}(X, \Lambda^{2,+}T^*X \otimes i\mathbb{R}) \,|\, \operatorname{supp}\eta \subset \Omega\right\}$$

and the map

$$\mathcal{F}_2 : \mathcal{N}^{k,p} \times \mathcal{Z}_\Omega^{k-1,p} \to W^{k-1,p}(X, \Lambda^{2,+}T^*X \otimes i\mathbb{R})$$

defined by

$$\mathcal{F}_2(A, \Phi, \eta) = F_A^+ + \eta - \sigma^+((\Phi\Phi^*)_0).$$

The key point is that 0 is a regular value of this map. To see this just note that if $\omega \in L^q(X, \Lambda^{2,+}T^*X \otimes i\mathbb{R})$ annihilates the image of $d\mathcal{F}_2(A, \Phi, \eta)$ then ω is a self-dual harmonic 2-form which vanishes in Ω and hence must vanish everywhere. (See Remark E.9.) This proves that 0 is a regular value in the case $k = 0$. In the general case surjectivity of $d\mathcal{F}_2(A, \Phi, \eta)$ can be easily reduced to the case $k = 0$ via elliptic regularity. With this established one considers the *universal moduli space*

$$\widetilde{\mathcal{M}}^* = \left\{(A, \Phi, \eta) \in \mathcal{N}^{k,p} \times \mathcal{Z}_\Omega^{k-1,p} \,|\, F_A^+ + \eta = \sigma^+((\Phi\Phi^*)_0)\right\}$$

and defines $\mathcal{Z}_{\Omega,\mathrm{reg}}^{k-1,p} \subset \mathcal{Z}_\Omega^{k-1,p}$ as the set of regular values of the projection

$$\widetilde{\mathcal{M}}^* \to \mathcal{Z}_\Omega^{k-1,p} : (A, \Phi, \eta) \mapsto \eta.$$

That this set is of the second category in the sense of Baire follows from the Sard-Smale theorem B.13. Moreover, a smooth parameter η is regular and supported in Ω if and only if $\eta \in \mathcal{Z}_{\Omega,\mathrm{reg}}^{k-1,p}$ for every k. That the intersection of these sets is of the second category with respect to the C^∞-topology can be proved by the same arguments as above. The details are left to the reader. $\square$

Remark 8.19 There is an alternative approach to the transversality problem which elminates the action of S^1 and the obvious 1-dimensional parts of the kernel and cokernel of $\mathcal{D}_{A,\Phi}$. In this approach the linearized operator has the form

$$\widetilde{\mathcal{D}}_{A,\Phi}\begin{pmatrix}\alpha \\ \varphi\end{pmatrix} = \begin{pmatrix} d^*\alpha \\ d^+\alpha \\ D_A\varphi \end{pmatrix} + \begin{pmatrix} -i\langle i\Phi, \varphi\rangle \\ -\sigma^+((\Phi\varphi^* + \varphi\Phi^*)_0) \\ \Gamma(\alpha)\Phi \end{pmatrix}.$$

The difference to $\mathcal{D}_{A,\Phi}$ lies in the first component in the second column. This arises from considering the action of the gauge group $\mathcal{G} = \mathrm{Map}(X, S^1)$

on the pair (A, Φ) and noting that a tangent vector (α, φ) is orthogonal to the tangent space of the orbit of (A, Φ) under this action if and only if

$$d^*\alpha - i\langle i\Phi, \varphi\rangle = 0. \tag{8.21}$$

The operator $\widetilde{\mathcal{D}}_{A,\Phi}$ is onto whenever $\operatorname{coker} \mathcal{D}_{A,\Phi} = H^0(X, i\mathbb{R})$ and there is a natural isomorphism

$$\frac{\ker\, \mathcal{D}_{A,\Phi}}{\mathbb{R}(0, i\Phi)} \longrightarrow \ker\, \widetilde{\mathcal{D}}_{A,\Phi}.$$

Note that every pair (A', Φ') near (A, Φ) is gauge equivalent to one of the form $A' = A + \alpha$, $\Phi' = \Phi + \varphi$ where α and φ satisfy (8.21). But this local representative may not be globally unique. □

Remark 8.20 It is interesting to rephrase the compactness theorem 7.12 in the notation of this section. Recall from the proof of Theorem 7.16 on page 294 that there is a Fredholm map

$$\mathcal{F}_1 : \mathcal{N}^{k,p} \to W^{k-1,p}(X, \Lambda^{2,+}T^*X \otimes i\mathbb{R})$$

given by

$$\mathcal{F}_1(A, \Phi) = F_A^+ - \sigma^+((\Phi\Phi^*)_0).$$

where $\mathcal{N}^{k,p}$ is the Banach manifold of Proposition 8.16. This map is invariant under the action of the group $\mathcal{G}_0$ of harmonic gauge transformations. Denote by $\mathcal{W}^{k-1,p} \subset W^{k-1,p}(X, \Lambda^{2,+}T^*X \otimes i\mathbb{R})$ the complement of the Γ-wall (see page 228). Then Theorem 7.12 asserts that the induced map

$$\overline{\mathcal{F}}_1 : \mathcal{N}^{k,p}/\mathcal{G}_0 \to \mathcal{W}^{k-1,p}$$

is proper. Actually, the compactness theorem as stated only asserts that if $\eta_\nu = \mathcal{F}_1(A_\nu, \Phi_\nu)$ is smooth and converges in the C^∞-topology then (A_ν, Φ_ν) has a subsequence which converges, modulo gauge equivalence, in the C^∞-topology. However, the argument carries over easily to the Sobolev space $W^{k,p}$ provided that $p > 4$ and $k \geq 3$. Under this assumption A_ν and Φ_ν are twice continuously differentiable and Lemma 7.13 asserts that the sequence Φ_ν is uniformly bounded in the L^∞-norm. Then the elliptic bootstrapping argument employed in the proof of Theorem 7.12 shows that A_ν and Φ_ν are uniformly bounded in the $W^{k,p}$-norm (after modification by a suitable sequence of gauge transformations). It then follows from Rellich's theorem that some subsequence converges in $W^{k-1,p}$. With $(k-1)p > 4$ it follows from the Seiberg-Witten equations with their quadratic zeroth order nonlinearities that $d^+(A_\nu - A_0)$ and $D_{A_0}\Phi_\nu$ converge in the $W^{k-1,p}$-norm and hence A_ν and Φ_ν converge in the $W^{k,p}$-norm. □

9

INTERSECTION FORMS AND WALL CROSSING

The goal of this chapter is, firstly, to give a proof of the wall crossing formula for Seiberg-Witten invariants in the case $b^+ = 1$, secondly, to give a Seiberg-Witten proof of Donaldson's theorem about the diagonalizability of definite intersection forms and, thirdly, to explain Furuta's proof of the 10/8-conjecture. The reason why all these results are collected in one chapter is that their proofs have many common features. The first section is devoted to some background material on intersection forms and the second section formulates the general wall crossing formula and gives some applications.

9.1 Intersection forms

Let X be a compact oriented smooth 4-manifold. Denote by $H_2(X)$ the integral homology modulo torsion and consider the intersection form

$$Q_X : H_2(X) \times H_2(X) \to \mathbb{Z}.$$

This is a unimodular quadratic form. The goal of this section is to discuss the question which unimodular quadratic forms Q can be realized as intersection forms of smooth 4-manifolds. To begin with here is a brief review of symmetric bilinear forms over the integers.

Unimodular quadratic forms

For a general exposition of the subject and proofs the reader is referred to Milnor-Husemoller [92]. Let Λ be a finitely generated free abelian group. Any such group is isomorphic to $\mathbb{Z}^n$ for some n. A symmetric bilinear form $Q : \Lambda \times \Lambda \to \mathbb{Z}$ is called **unimodular** if its matrix representation A with respect to some (and hence any) integral basis of Λ has determinant ± 1. Two forms Q_0 and Q_1 are called **equivalent** if there exists an isomorphism $T : \Lambda_0 \to \Lambda_1$ such that $Q_1(T\alpha, T\beta) = Q_0(\alpha, \beta)$ for all $\alpha, \beta \in \Lambda_0$. The important invariants under this equivalence relation are the rank, signature and type of Q. The rank of Q is, by definition, the rank of Λ, and the signature is the number of positive minus the number of negative diagonal entries in a diagonalization over the reals. A form Q is called **even** (or of **type II**) if $Q(\alpha, \alpha) \in 2\mathbb{Z}$ for all α and is called **odd** (or of **type I**) if it is not even. Note that even forms cannot be diagonalized over the integers. The simplest examples of even forms are

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad E_8 = \begin{pmatrix} 2&0&1&0&0&0&0&0 \\ 0&2&0&1&0&0&0&0 \\ 1&0&2&1&0&0&0&0 \\ 0&1&1&2&1&0&0&0 \\ 0&0&0&1&2&1&0&0 \\ 0&0&0&0&1&2&1&0 \\ 0&0&0&0&0&1&2&1 \\ 0&0&0&0&0&0&1&2 \end{pmatrix}.$$

The form E_8 is related to the Dynkin diagram of the same name where the vertices correspond to the generators of Λ and the edges give rise to the off-diagonal entries 1. The form E_8 is positive definite while the form H is indefinite with signature 0. A vector $\gamma \in \Lambda$ is called **characteristic** for Q if

$$Q(\gamma, \alpha) \equiv Q(\alpha, \alpha) (\mathrm{mod}\ 2)$$

for every $\alpha \in V$. Since the map $\alpha \mapsto Q(\alpha, \alpha)$ is a homomorphism over $\mathbb{Z}_2$ such a characteristic vector always exists. In particular, if Q is even then the zero vector is characteristic. Hence the next lemma shows that the signature of an even form is divisible by 8.

Lemma 9.1 [92] *If $\gamma \in \Lambda$ is a characteristic vector then*

$$Q(\gamma, \gamma) \equiv \operatorname{sign} Q\ (\mathrm{mod}\ 8).$$

If $Q = \ell(1) \oplus m(-1)$ then this lemma follows from the fact that $k^2 - 1$ is divisible by 8 for every odd integer k. The general case can be easily reduced to this case since odd indefinite forms are diagonalizable over the integers. The latter is the contents of the next theorem which asserts that indefinite unimodular quadratic forms over the integers are completely classified by rank, signature, and type. For a proof the reader is referred to [92].

Theorem 9.2. (Hasse-Minkowski) *Let Q be a unimodular quadratic form over the integers. If Q is odd and indefinite then it can be diagonalized over $\mathbb{Z}$ and thus*

$$Q \sim \ell(1) \oplus m(-1)$$

for some positive integers ℓ and m. If Q is even and indefinite then it is equivalent to the form

$$Q \sim \ell E_8 \oplus mH$$

for some integers ℓ and $m \geq 1$.

Exercise 9.3 According to Theorem 9.2 the form $E_8 \oplus (-E_8)$ is equivalent over the integers to $8H$. Find a corresponding change of basis. $\square$

Exercise 9.4 Find an integral change of basis relating the quadratic forms

$$Q_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

These are the intersection forms of the diffeomorphic manifolds

$$X_1 = (\mathbb{C}P^1 \times \mathbb{C}P^1)\#\overline{\mathbb{C}P}^2, \qquad X_2 = \mathbb{C}P^2\#\overline{\mathbb{C}P}^2\#\overline{\mathbb{C}P}^2.$$

(See Exercise 12.20.) □

For definite forms the situation is quite different. In dimension 8 every even positive definite form is equivalent to E_8. In dimension 16 there are two forms $E_8 \oplus E_8$ and E_{16}. In dimension 24 there are five forms including $3E_8$, $E_8 \oplus E_{16}$, and the Leech lattice. The definite forms cannot be classified and are often referred to as **exotic forms**. Assume that Q is positive definite and note that in this case the rank and signature of Q agree. Thus Lemma 9.1 asserts that the number $Q(\gamma, \gamma) - \operatorname{rank} Q$ is a multiple of 8. Since $Q(\gamma, \gamma) > 0$ for every characteristic vector γ one can ask the question what the minimum of these numbers is over all characteristic vectors. This question has only recently been addressed and the following result was proved by Noam Elkies in May 1995 (cf. [22]).

Theorem 9.5. (Elkies) *Let $Q : \Lambda \times \Lambda \to \mathbb{Z}$ be a positive definite unimodular quadratic form which is not diagonalizable over the integers. Then there exists a characteristic vector $\gamma \in \Lambda$ such that*

$$Q(\gamma, \gamma) \le \operatorname{rank} Q - 8. \tag{9.1}$$

Such a vector γ obviously does not exist when Q is diagonalizable. In that case the minimum of the numbers $Q(\gamma, \gamma)$ over all characteristic vectors is the rank of Q. For the quadratic form $Q = E_8 \oplus \ell(1)$, for example, this minimum is $\operatorname{rank} Q - 8$. If Q is even then 0 is a characteristic vector. Since $\operatorname{rank} Q$ is an integer multiple of 8 this vector obviously satisfies the inequality (9.1).

Intersection forms of smooth 4-manifolds

In his thesis in 1982 Donaldson proved the following theorem in the simply connected case. The extension to general 4-manifolds was later given by Fintushel-Stern, Furuta, and Donaldson himself. For a further discussion see Donaldson and Kronheimer [21].

Theorem 9.6. (Donaldson) *If X is a compact oriented smooth 4-manifold with definite intersection form then Q_X is diagonalizable over the integers.*

The diagonal forms $Q_1 = \ell(+1)$ and $Q_2 = \ell(-1)$ are realized by connected sums of ℓ copies of $\mathbb{C}P^2$ respectively $\overline{\mathbb{C}P}^2$. Thus for the existence question it remains to consider indefinite forms.

Example 9.7 The manifold $X = S^2 \times S^2$ has intersection form

$$Q_{S^2\times S^2} = H$$

It follows from Proposition 3.66 and Theorem 9.2 that the $K3$-surface (a hypersurface of $\mathbb{C}P^3$ of degree 4) has intersection form

$$Q_{\mathrm{K3}} = 2(-E_8) \oplus 3H.$$

Hence the 4-manifold $X = k\mathrm{K3}\#m(S^2 \times S^2)$ has intersection form

$$Q_X = 2k(-E_8) \oplus (3k+m)H.$$

Note that these are all spin manifolds. □

Recall that every spin 4-manifold X has an even intersection form. This is because $\mathrm{w}_2(TX) = 0$ and

$$Q_X(\alpha,\alpha) \equiv \mathrm{w}_2(TX)\cdot\alpha (\mathrm{mod}\ 2).$$

(See (5.1) in Section 5.1.) Moreover, by Rohlin's theorem 6.27, the signature is divisible by 16 and, by Theorem 9.6, the form is indefinite. Hence a smooth spin 4-manifold has intersection form

$$Q = 2kE_8 \oplus mH$$

with $m \geq 1$. Example 9.7 shows that every form in this family which satisfies $m \geq 3|k|$ can be realized as the intersection form of a smooth spin 4-manifold. Nobody has found an example with $m < 3|k|$. This gave rise to the following conjecture.

The 11/8 conjecture: *If X is a smooth compact oriented spin 4-manifold then its intersection form Q_X is equivalent to $2kE_8 \oplus mH$ with $m \geq 3|k|$.*

The reader may check that the condition $m \geq 3|k|$ is equivalent to

$$\frac{b_2(X)}{|\sigma(X)|} \geq \frac{11}{8}.$$

Since every simply connected smooth 4-manifold with even intersection form is spin, an affirmative answer to the 11/8-conjecture would completely settle the existence question in the simply connected case. For non simply connected manifolds the situation is quite different. Recall from Example 6.28 that the Enriques surface $X = \mathrm{K3}/\mathbb{Z}_2$ has intersection form $Q_X = E_8 \oplus H$. This manifold has fundamental group $\pi_1(X) = \mathbb{Z}_2$ and it is not spin.

For spin manifolds the 11/8-conjecture has been confirmed by Donaldson for the case $k = 1$ and more recently for $k = 2, 3$ by Kronheimer with the use of the Seiberg-Witten invariants. The reader should note that if the conjecture holds for some integer $k = k_0$ then it also holds for any integer $k < k_0$. (A counterexample for k would give rise to a counterexample for $k + 1$ by taking a connected sum with the $K3$-surface.) Recently Furuta proved the following theorem for all values of k. The result is often referred to as the 10/8-**conjecture**. I learned about the proof from a lecture by Dan Freed in Gökova in May 1995.

Theorem 9.8. (Furuta) *If X is a smooth spin 4-manifold with indefinite intersection form then $Q_X = 2kE_8 \oplus mH$ with $m \geq 2|k| + 1$.*

The result should be contrasted with the theorem of M.H. Freedman which asserts that every unimodular quadratic form can be realized as the intersection form of a compact simply connected oriented *topological* 4-manifold. He also proved that any two compact simply connected oriented *smooth* 4-manifolds with equivalent intersection forms are homeomorphic. (In the C^0 case there are two homeomorphism types for each odd intersection form. At most one of these has a smooth representative.)

9.2 The wall crossing formula

Let X be a compact oriented smooth 4-manifold with

$$b^+ = 1, \qquad b_1 \in 2\mathbb{Z}$$

and fix throughout an orientation of $H^{2,+}(X)$. Recall that in this case for every spin^c structure $\Gamma : TX \to \mathrm{End}(W)$ there are two Seiberg-Witten invariants $SW^{\pm}(X, \Gamma)$ depending on the sign of the number

$$\varepsilon(g, \eta) = \varepsilon_\Gamma(g, \eta) = -\int_X \langle i\eta, \omega_g \rangle \mathrm{dvol}_g - \pi[\omega_g] \cdot c_1(L_\Gamma)$$

associated to the metric g and the perturbation η. The goal of this chapter is to compute the **wall crossing number**

$$w(X, \Gamma) = SW^+(X, \Gamma) - SW^-(X, \Gamma).$$

This can be done by examining the structure of the moduli spaces near a perturbation parameter on the Γ-wall. Fix throughout a metric g and a reference connection $A_0 \in \mathcal{A}(\Gamma)$ which may be chosen with $F_{A_0}^+ = 0$. It is convenient to distinguish the cases $b_1 = 0$ and $b_1 > 0$.

The case $b_1 = 0$

Theorem 9.9 *Let X be a compact connected oriented smooth 4-manifold with $b^+ = 1$ and $b_1 = 0$. Let Γ be a spin^c structure on X with*

$$c_1(L_\Gamma) \cdot c_1(L_\Gamma) \geq 2\chi + 3\sigma.$$

Then

$$SW^+(X, \Gamma) - SW^-(X, \Gamma) = 1.$$

Remark 9.10 The formula in Theorem 9.9 is invariant under change of orientation of $H^{2,+}(X, i\mathbb{R})$. A change of orientation interchanges the two invariants $SW^+(X, \Gamma)$ and $SW^-(X, \Gamma)$ but it also changes the sign in the definition of these invariants. Moreover, note that if $b^+ = 1$ and $b_1 = 0$ then $\chi + \sigma = 2 + 2b^+ - 2b_1 = 4$ and hence, by Proposition 7.31,

$$SW^+(X, \bar{\Gamma}) = -SW^-(X, \Gamma).$$

This shows that the formula of Theorem 9.9 is also invariant under reversing the complex structure on W. □

Examples

Example 9.11 Consider the complex projective space $X = \mathbb{C}P^2$ with its standard orientation and the spin^c structure $\Gamma_E : T\mathbb{C}P^2 \to W_E$ with $W_E = W_{\mathrm{can}} \otimes E$. Let g denote the Fubini-Study metric on $\mathbb{C}P^2$ so that the corresponding self-dual harmonic 2-form $\omega = \omega_g$ is the Kähler form with respect to which $\mathbb{C}P^1$ has area 1. Thus $[\omega] = H \in H^2(\mathbb{C}P^2, \mathbb{Z})$ is the unique positive generator. Suppose that $c_1(E) = dH$ with $d \in \mathbb{Z}$. Then

$$\varepsilon_{\Gamma_E}(g, 0) = \pi(c_1(K) - 2c_1(E)) \cdot \omega = -\pi(2d + 3).$$

Since g has positive scalar curvature it follows that $SW^-(X, \Gamma_E) = 0$ whenever $d \geq -1$ and $SW^+(X, \Gamma_E) = 0$ whenever $d \leq -2$. Moreover, the virtual dimension of the moduli space is given by

$$\dim \mathcal{M}(\mathbb{C}P^2, \Gamma_E) = \frac{c_1(L_{\Gamma_E}) \cdot c_1(L_{\Gamma_E}) - 2\chi - 3\sigma}{4} = d(d+3).$$

Hence the moduli space is empty whenever $d = -1$ or $d = -2$ and in these cases both invariants are zero. This shows that

$$SW^-(\mathbb{C}P^2, \Gamma_E) = \begin{cases} 0, & \text{if } d \geq -2, \\ -1, & \text{if } d \leq -3, \end{cases}$$

$$SW^+(\mathbb{C}P^2, \Gamma_E) = \begin{cases} 1, & \text{if } d \geq 0, \\ 0, & \text{if } d \leq -1. \end{cases}$$

The minus sign is consistent with the fact that $(\chi + \sigma)/4 = 1$. □

Example 9.12 Consider the 4-manifold $X = S^2 \times S^2$ with $\chi = 4$ and $\sigma = 0$. In this case $K = -2a_1 - 2a_2$ where $a_1 = \mathrm{PD}(S^2 \times \{\mathrm{pt}\})$ and $a_2 = \mathrm{PD}(\{\mathrm{pt}\} \times S^2)$. Consider the spin^c structure Γ_E with first Chern class

$$c_1(E) = p_1 a_1 + p_2 a_2.$$

Then for the standard product metric one finds

$$\varepsilon_{\Gamma_E}(g, 0) = -2\pi(2 + p_1 + p_2)$$

while the *virtual* dimension of the moduli space is given by

$$\dim \mathcal{M}(S^2 \times S^2, \Gamma_E) = 2(p_1 + 1)(p_2 + 1) - 2.$$

Since $S^2 \times S^2$ again admits a metric of positive scalar curvature it follows that

$$SW^-(S^2 \times S^2, \Gamma_E) = \begin{cases} -1, & \text{if } p_1 \leq -2 \text{ and } p_2 \leq -2, \\ 0, & \text{otherwise,} \end{cases}$$

$$SW^+(S^2 \times S^2, \Gamma_E) = \begin{cases} 1, & \text{if } p_1 \geq 0 \text{ and } p_2 \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Note, in particular, that if $p_1 < 0$ and $p_1 + p_2 + 2 \geq 0$ then $(p_1+1)(p_2+1) \leq 0$ and hence the moduli space has negative dimension. □

The torus of reducible solutions

If $b_1 = 0$ then for every perturbation $\eta \in \Omega^{2,+}_\Gamma(X, g)$ there exists a unique connection $A_\eta \in \mathcal{A}(\Gamma)$ such that

$$F^+_{A_\eta} + \eta = 0, \qquad d^*(A_\eta - A_0) = 0.$$

The existence of A_η follows from the definition of the Γ-wall. Moreover, if $A_1, A_2 \in \mathcal{A}(\Gamma)$ with $F^+_{A_i} + \eta = 0$ then the 1-form $\alpha = A_2 - A_1 \in \Omega^1(X, i\mathbb{R})$ satisfies $d^*\alpha = 0$ and $d^+\alpha = 0$. Hence α is a harmonic 1-form and, since $b_1 = 0$, it follows that $\alpha = 0$. It is also useful to recall that $\chi + \sigma = 4$ and hence the condition $c \cdot c \geq 2\chi + 3\sigma$ in Theorem 9.9 is equivalent to index $D_{A_\eta} \geq 2$. These observations are specific to the case $b_1 = 0$. In general the set

$$\widetilde{\mathcal{T}} = \widetilde{\mathcal{T}}(\eta) = \left\{ A \in \mathcal{A}(\Gamma) \,|\, F^+_A + \eta = 0,\, d^*(A - A_0) = 0 \right\}$$

is an affine space parallel to $H^1(X, i\mathbb{R})$. Dividing by the group

$$\mathcal{G}_0(x_0) = \left\{ u : X \to S^1 \,|\, d^*(u^{-1}du) = 0,\, u(x_0) = 1 \right\}$$

one obtains a torus

$$\mathcal{T} = \frac{\widetilde{\mathcal{T}}}{\mathcal{G}_0(x_0)} \cong \frac{H^1(X, i\mathbb{R})}{H^1(X, 2\pi i\mathbb{Z})}.$$

Denote the elements of $\mathcal{T}$ by ρ. This notation is justified by the fact that, given a fixed reference connection $A_0 \in \widetilde{\mathcal{T}}$, the quotient $\mathcal{T}$ can be identified with $\mathrm{Hom}(\pi_1(X), S^1)$ via the holonomy

$$\rho_A(\gamma) = \exp\left(\int_\gamma (A - A_0)\right)$$

for $A \in \widetilde{\mathcal{T}}$. (Compare with Proposition 5.30.)

The universal line bundle

There is a natural line bundle

$$\mathbb{E} \longrightarrow X \times \mathcal{T} \tag{9.2}$$

which can be explicitly represented as the quotient $\mathbb{E} = X \times \widetilde{\mathcal{T}} \times \mathbb{C}/\mathcal{G}_0(x_0)$. Here the action of an element $u \in \mathcal{G}_0(x_0)$ on a triple (x, A, z) is given by

$$u^*(x, A, z) = (x, u^*A, u(x)^{-1}z).$$

For every $\rho \in \mathcal{T}$ consider the pullback bundle

$$E_\rho = {\iota_\rho}^*\mathbb{E} \to X$$

under the obvious inclusion $\iota_\rho : X \to X \times \mathcal{T}$. This bundle is the set of all equivalence classes of triples (x, A, z) with $\rho_A = \rho$ under the above equivalence relation. For every $A \in \widetilde{\mathcal{T}}$ with $\rho_A = \rho$ the bundle E_ρ admits a trivialization

$$\iota_A : X \times \mathbb{C} \to E_\rho, \qquad \iota_A(x, z) = [x, A, z].$$

If $\rho_A = \rho$ then any other connection in $\widetilde{\mathcal{T}}$ with this property is of the form u^*A for some $u \in \mathcal{G}_0(x_0)$ and, moreover, the two trivializations of E_ρ are related by the same gauge transformation u, i.e. $\iota_{u^*A}(x, z) = \iota_A(x, u(x)z)$ The formula $\nabla_{u^*A} = u^{-1} \circ \nabla_A \circ u$ now shows that there is a natural spin^c connection ∇_{A_ρ} on the bundle

$$W_\rho = W \otimes E_\rho$$

in the gauge equivalence class ρ. One can think of ρ as the holonomy of connection on the bundle E_ρ which in the trivialization determined by A is given by $A - A_0$. Twisting the connection ∇_{A_0} on W by ρ gives the resulting spin^c connection ∇_{A_ρ} on W_ρ. Note that this spin^c connection is independent of the choice of the base point A_0 in $\widetilde{\mathcal{T}}$. In summary, the bundle

$$\mathbb{W} = W \otimes \mathbb{E} \to X \times \mathcal{T}$$

has the universal property that *for every $\rho \in \mathcal{T}$ the restriction $W_\rho = W \otimes E_\rho$ of $\mathbb{W}$ to $X \times \rho$ carries a natural spin^c connection ∇_{A_ρ} in the gauge equivalence class ρ.* For the computation of the first Chern class of $\mathbb{E}$ the reader may wish to consult the following exercises.

Exercise 9.13 **(i)** Prove that the first Chern class of the line bundle $\mathbb{E}$ over $X \times \mathcal{T}$ is represented by the 2-form $\Omega \in \Omega^2(X \times \mathcal{T})$ given by

$$\Omega_{x,A}((v,\alpha),(w,\beta)) = \frac{1}{2\pi i}\Big(\beta(v) - \alpha(w)\Big)$$

for $v, w \in T_xX$ and $\alpha, \beta \in T_A\mathcal{T} = H^1(X, i\mathbb{R})$.

(ii) Choose an integral basis $\alpha_\nu = {u_\nu}^{-1}du_\nu$ of $H^1(X, 2\pi i\mathbb{Z})$ and show that the form Ω can be expressed in the form

$$\Omega = \frac{1}{2\pi i}\sum_{\nu=1}^{n} \alpha_\nu \wedge dt_\nu,$$

where $t_1, \ldots, t_n \in \mathbb{R}/\mathbb{Z}$ are coordinates on $\mathcal{T}$ via $A = A_0 + \sum_{\nu=1}^n t_\nu\alpha_\nu$.

(iii) Consider a complex line bundle $E \to \mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ with first Chern class $c_1(E) = [\omega] \in H^2(\mathbb{T}^{2n}, \mathbb{Z})$ where

$$\omega = \sum_{\nu=1}^{n} ds_\nu \wedge dt_\nu.$$

In Example 1.38 it is shown that such a bundle can be explicitly represented as the quotient $E = \mathbb{R}^{2n} \times \mathbb{C}/\mathbb{Z}^{2n}$ where the action of $\mathbb{Z}^{2n}$ on $\mathbb{R}^{2n} \times \mathbb{C}$ is given by $(k,\ell)\cdot(s,t,z) = \big(s+k, t+\ell, ze^{-2\pi i s\cdot\ell}\big)$. Prove that the bundle $\mathbb{E} \to X \times \mathcal{T}$ can be naturally identified with the pullback $\mathbb{E} = f^*E$ of E under the map $f : X \times \mathcal{T} \to \mathbb{T}^{2n}$ given by

$$f(x,[A]) = [s_1, \ldots, s_n, t_1, \ldots, t_n],$$

where $s_\nu \in \mathbb{R}/2\pi i\mathbb{Z}$ and $t_\nu \in \mathbb{R}/2\pi i\mathbb{Z}$ are defined by

$$u_\nu(x) = e^{2\pi i s_\nu}, \qquad A = A_0 + \sum_{\nu=1}^{n} t_\nu\alpha_\nu.$$

As in (ii), $A_0 \in \widetilde{\mathcal{T}}$ is a reference connection and the harmonic 1-forms $\alpha_\nu = {u_\nu}^{-1} du_\nu$, with $u_\nu : X \to S^1$, form an integral basis of $H^1(X, 2\pi i\mathbb{Z})$. Show that $f^* ds_\nu = \alpha_\nu / 2\pi i$ and $f^*\omega = \Omega$. □

The general wall crossing formula

The following wall crossing formula was first discovered by Li and Liu [72]. At about the same time (around May-June 1995) the formula was also worked out, independently, by Ohta-Ono [101]. The proof given below is original and independent of their work. I benefited in my understanding of the result from discussions with John Jones, Dusa McDuff, and Kaoru Ono.

Theorem 9.14. (Li-Liu,Ohta-Ono) *Let X be a compact connected oriented smooth 4-manifold with $b^+ = 1$ and $b_1 = 2k$. Let Γ be a spin^c structure on X whose characteristic class $c = c_1(L_\Gamma) \in H^2(X, \mathbb{Z})$ satisfies $c \cdot c \geq 2\chi + 3\sigma$. Then*

$$SW^+(X, \Gamma) - SW^-(X, \Gamma) = \frac{1}{k!} \int_{\mathcal{T}} \left(-\frac{1}{4} \int_X \Omega \wedge \Omega \wedge c \right)^k$$

where $\Omega \in \Omega^2(X \times \mathcal{T})$ denotes the 2-form in Exercise 9.13 which represents the first Chern class of the universal line bundle $\mathbb{E} \to X \times \mathcal{T}$.

Theorem 9.14 was used by Ono and Ohta [101] and Liu [74] to prove that every minimal symplectic 4-manifold which admits a metric of positive scalar curvature must be diffeomorphic to either $\mathbb{C}P^2$ or a 2-sphere bundle over some Riemann surface. The proof of both theorems will be given below.

The topological index

The wall crossing formula in Theorem 9.14 can be interpreted in terms of the topological index of the family $\rho \mapsto D_{A_\rho}$ of Dirac operators parametrized by the torus $\mathcal{T}$. Here A_ρ denotes the unique spin^c connection on $W_\rho = W \otimes E_\rho$ for $\rho \in \mathcal{T}$. The topological index is the K-theory class $\mathcal{IND} \in K(\mathcal{T})$ defined as the formal difference

$$\mathcal{IND} = \ker\, D_{A_\rho} \ominus \operatorname{coker} D_{A_\rho}$$

This definition can be taken literally when the kernels and cokernels are of constant dimension and hence form vector bundles over $\mathcal{T}$. In general one has to stabilize as is explained in Section A.1. The Atiyah-Singer index theorem for families asserts that the Chern character of $\mathcal{IND}$ is given by

$$\operatorname{ch}(\mathcal{IND}) = \int_X \operatorname{ch}({L_\Gamma}^{1/2}) \wedge \widehat{A}(TX) \wedge \operatorname{ch}(\mathbb{E}) \in H^*(\mathcal{T}, \mathbb{Z}) \qquad (9.3)$$

where the right hand side is to be understood as integration over the fiber. (See Theorem 6.24 and (6.16).)

Lemma 9.15 *Suppose that $b^+(X) \le 2$. Then the 3-dimensional cohomology class $\alpha_1 \cup \alpha_2 \cup \alpha_3$ is a torsion class for any three 1-dimensional classes $\alpha_i \in H^1(X, \mathbb{Z})$.*

Proof: If $\alpha_1 \cup \alpha_2 \cup \alpha_3$ is not a torsion class then there exists a class $\alpha_0 \in H^1(X, \mathbb{Z})$ with $\langle \alpha_0 \cup \alpha_1 \cup \alpha_2 \cup \alpha_3, [X] \rangle > 0$. Hence the classes $\omega_1 = \alpha_0 \cup \alpha_1 + \alpha_2 \cup \alpha_3$, $\omega_2 = \alpha_0 \cup \alpha_2 + \alpha_3 \cup \alpha_1$, and $\omega_3 = \alpha_0 \cup \alpha_3 + \alpha_1 \cup \alpha_2$ satisfy $\omega_i \cdot \omega_i > 0$ and $\omega_i \cdot \omega_j = 0$ for $i \ne j$. It follows that the ω_i are linearly independent and the intersection form is positive on the subspace spanned by these classes. Thus $b^+(X) \ge 3$. □

Lemma 9.16 *The k-th Chern class of $-\mathcal{IND}$ is given by*

$$c_k(-\mathcal{IND}) = \frac{1}{k!}\left(-\frac{1}{4}\int_X \Omega \wedge \Omega \wedge c_1(L_\Gamma)\right)^k \in H^{2k}(\mathcal{T}, \mathbb{Z}).$$

Proof: By Lemma 9.15, $\Omega^k = 0$ for $k \ge 3$. Hence the Chern character of the universal bundle $\mathbb{E} \to X \times \mathcal{T}$ is given by

$$\mathrm{ch}(\mathbb{E}) = 1 + \Omega + \frac{\Omega^2}{2}.$$

Moreover, recall from Remark 6.26 that

$$\mathrm{ch}({L_\Gamma}^{1/2}) = 1 + \frac{c}{2} + \frac{c^2}{8}, \qquad \widehat{A}(TX) = 1 - \frac{1}{24}p_1(TX)$$

where $c = c_1(L_\Gamma)$. Take the product and integrate over X to obtain

$$\mathrm{ch}(\mathcal{IND}) = \frac{c \cdot c - \sigma}{8} + \frac{1}{4}\int_X \Omega \wedge \Omega \wedge c.$$

Here we have used the Hirzebruch signature theorem which asserts that $3\sigma(X) = \int_X p_1(TX)$. Note that the constant term is the complex Fredholm index of the Dirac operator. Moreover, the first Chern class of $-\mathcal{IND}$ is given by

$$c_1(-\mathcal{IND}) = -\frac{1}{4}\int_X \Omega \wedge \Omega \wedge c.$$

Write $-\mathcal{IND}$ formally as a sum of line bundles with first Chern classes $y_1, \dots, y_\ell$. Then the vanishing of the degree-k term in $\mathrm{ch}(-\mathcal{IND})$ is equivalent to $\sum_i {y_i}^k = 0$ for all $k \ge 2$. This implies

$$c_k(-\mathcal{IND}) = \sum_{i_1 < \cdots < i_k} y_{i_1} \cdots y_{i_k} = \frac{1}{k!}\left(\sum_{i=1}^{\ell} y_i\right)^k = \frac{c_1(-\mathcal{IND})^k}{k!}.$$

This proves the lemma. □

Ruled surfaces

The computation in the following example is due to Ohta-Ono [101] and Li-Liu [72]. A **ruled surface** is compact smooth 4-manifold X which fibers over a Riemann surface Σ with fiber $\mathbb{C}P^1$:

$$\mathbb{C}P^1 \hookrightarrow X \xrightarrow{\pi} \Sigma.$$

Such a 4-manifold admits a Kähler structure with respect to which the projection π is holomorphic. Suppose that the surface Σ is of genus g. We denote by $F \subset X$ a fiber of the projection and by $S \subset X$ be a section. Both are complex curves and the restriction $\pi|_S : S \to \Sigma$ is a diffeomorphism.

Proposition 9.17. (Li-Liu,Ohta-Ono) *Let X be as above and consider a spin^c structure $W_E = W_{\mathrm{can}} \otimes E$ where*

$$c_1(E) \cdot [F] = p, \qquad c_1(E) \cdot [S] = q.$$

Then

$$SW^-(X, \Gamma_E) = \begin{cases} -(p+1)^g, & \text{if } p \le -2,\ q + \frac{1}{2} S \cdot S \le (g-1)p/(p+1), \\ 0, & \text{otherwise}, \end{cases}$$

$$SW^+(X, \Gamma_E) = \begin{cases} (p+1)^g, & \text{if } p \ge 0,\ q + \frac{1}{2} S \cdot S \ge (g-1)p/(p+1), \\ 0, & \text{otherwise.} \end{cases}$$

If $p = -1$ then both invariants are zero.

Proof: Denote $a_F = \mathrm{PD}([S] - S \cdot S[F])$ and $a_S = \mathrm{PD}([F])$ so that

$$a_S \cdot [S] = 1, \quad a_S \cdot [F] = 0, \quad a_F \cdot [S] = 0, \quad a_F \cdot [F] = 1.$$

The first Chern class of TX satisfies $c_1(TX) \cdot [F] = 2$ and $c_1(TX) \cdot [S] = 2 - 2g + S \cdot S$. Hence $c_1(K) = -c_1(TX)$ and $c_1(E)$ are given by

$$c_1(K) = -2a_F + (2g - 2 - S \cdot S)a_S, \qquad c_1(E) = pa_F + qa_S.$$

A simple calculation with $c = c_1(L) = 2c_1(E) - c_1(K)$, $\sigma = 0$, and $\chi = 4 - 4g$ shows that the moduli spaces have dimension

$$\dim \mathcal{M} = \frac{c \cdot c}{4} + 2g - 2 = (p+1)(2q + S \cdot S) - 2p(g-1).$$

Thus the invariants are zero unless this number is nonnegative. Let us now examine the position of the Γ-wall. There is a metric with positive scalar curvature. For this metric the curvature of the sphere F must dominate that of the section S and hence the sphere F will have a very small radius.

Thus the corresponding symplectic form (which is self-dual) lies in the cohomology class

$$[\omega] = \delta a_F + a_S$$

for some small number $\delta > 0$. The position of the Γ-wall is now determined by the number

$$\begin{aligned}\varepsilon_{\Gamma_E}(g, 0) &= \pi(c_1(K) - 2c_1(E)) \cdot [\omega] \\ &= -\pi(2p + 2 + \delta(2q + S \cdot S + 2 - 2g)).\end{aligned}$$

If $p \geq 0$ then this number is negative and $SW^-(X, \Gamma_E) = 0$ while in the case $p \leq -2$ the number is positive and $SW^+(X, \Gamma_E) = 0$. In the case $p = -1$ the wall crossing formula will show that both invariants are zero.

We will prove that

$$SW^+(X, \Gamma_E) - SW^-(X, \Gamma_E) = (p + 1)^g \tag{9.4}$$

whenever $(p + 1)(q + \frac{1}{2}S \cdot S) \geq p(g - 1)$. In the case $g = 0$ the manifold X is simply connected and so (9.4) follows immediately from Theorem 9.9. Hence assume $g \geq 1$ and choose generators

$$\alpha_\nu = u_\nu^{-1} du_\nu, \qquad \nu = 1, \ldots, 2g$$

of $H^1(X, 2\pi i\mathbb{Z}) = H^1(\Sigma, 2\pi i\mathbb{Z})$. Then, by Exercise 9.13, the first Chern class of the universal line bundle $\mathbb{E} \to X \times \mathcal{T}$ in (9.2) is represented by the 2-form

$$\Omega = \sum_{\nu=1}^{2g} ds_\nu \wedge dt_\nu \in \Omega^2(X \times \mathcal{T}),$$

where $ds_\nu = (1/2\pi i){u_\nu}^{-1} du_\nu \in \Omega^1(X)$ and the 1-form $dt_\nu \in \Omega^1(\mathcal{T})$ is determined by the coordinate system $A = A_0 + \sum_\nu t_\nu \alpha_\nu$. Suppose that the basis $[ds_\nu]$ of $H^1(X, \mathbb{Z}) = H^1(\Sigma, \mathbb{Z})$ is chosen such that

$$[ds_1 \wedge ds_2] = [ds_3 \wedge ds_4] = \cdots = [ds_{2g-1} \wedge ds_g] = a_S$$

and all the other products are zero. Then

$$\Omega \wedge \Omega = -2a_S \wedge \omega_{\mathcal{T}}$$

where $\omega_{\mathcal{T}} = dt_1 \wedge dt_2 + dt_3 \wedge dt_4 + \cdots + dt_{2g-1} \wedge dt_g$ denotes the symplectic form on the torus. With

$$c = 2c_1(E) - c_1(K) = 2(p + 1)a_F + (2q + S \cdot S + 2 - 2g)a_S$$

it follows that

$$\int_X \Omega \wedge \Omega \wedge c = -4(p+1)\omega_{\mathcal{T}}.$$

Hence it follows from Theorem 9.14 that the crossing index is given by

$$\begin{aligned} SW^+(X,\Gamma_E) - SW^-(X,\Gamma_E) &= \frac{1}{g!}\int_{\mathcal{T}} \left(-\frac{1}{4}\int_X \Omega\wedge\Omega\wedge c\right)^g \\ &= \frac{(p+1)^g}{g!}\int_{\mathcal{T}} {\omega_{\mathcal{T}}}^g \\ &= (p+1)^g. \end{aligned}$$

This proves the proposition. □

9.3 Regular crossings

Recall that, for every crossing parameter $\eta \in \mathcal{Z}^{k-1,p}$, $\widetilde{\mathcal{T}} = \widetilde{\mathcal{T}}(\eta)$ denotes the set of connections $A \in \mathcal{A}^{k,p}(\Gamma)$ with $F_A^+ + \eta = 0$ and $d^*(A - A_0) = 0$. If $b_1 = 0$ then this set consists of a single point A_η. For any $A \in \mathcal{A}^{k,p}(\Gamma)$ denote by $\operatorname{coker} D_A \subset W^{k+1,p}(X, W^-)$ the L^2-orthogonal complement of $\operatorname{im} D_A$ or, equivalently, the kernel of ${D_A}^*$. Moreover, denote by

$$\pi_A : L^2(X, W^-) \to \operatorname{coker} D_A$$

the L^2-orthogonal projection. Likewise, denote by

$$\pi^+ : \Omega^{2,+}(X, i\mathbb{R}) \to H^{2,+}(X, i\mathbb{R})$$

the L^2-orthogonal projection.

Definition 9.18 *Assume $b^+ = 1$ and fix a spin^c structure Γ on X. Then a perturbation parameter $\eta \in \Omega^{2,+}_\Gamma(X, g)$ on the Γ-wall is called* **regular***, or a* **regular crossing parameter***, if the following holds.*

(a) *Every $(A, \Phi) \in \widetilde{\mathcal{M}}^*(X, \Gamma, g, \eta)$ satisfies $\operatorname{coker} \mathcal{D}_{A,\Phi} = H^0(X, i\mathbb{R})$.*

(b) *For every $A \in \widetilde{\mathcal{T}}(\eta)$ and every $\Phi \in \ker D_A$ with $\Phi \neq 0$ the linear map*

$$H^1(X, i\mathbb{R}) \to \operatorname{coker} D_A : \alpha \mapsto \pi_A(\Gamma(\alpha)\Phi)$$

is surjective.

(c) *For every $A \in \widetilde{\mathcal{T}}(\eta)$ and every $\Phi \in \ker D_A$ with $\Phi \neq 0$ there exists a $\varphi \in C^\infty(X, W^+)$ such that*

$$\pi^+\sigma^+((\varphi\Phi^* + \Phi\varphi^*)_0) \neq 0, \qquad D_A\varphi \in \Gamma(H^1(X, i\mathbb{R}))\Phi.$$

Remark 9.19 Suppose that η is a regular crossing parameter and D_A is not injective for some $A \in \widetilde{\mathcal{T}}(\eta)$. Then $\dim \ker D_A \geq 2$ and condition (ii) in Definition 9.18 implies that $\dim \operatorname{coker} D_A \leq b_1$. Hence

$$\operatorname{index} D_A \geq 2 - b_1$$

and, since $b^+ = 1$, this is equivalent to

$$c_1(L_\Gamma) \cdot c_1(L_\Gamma) \geq 2\chi + 3\sigma. \tag{9.5}$$

Thus the Seiberg-Witten moduli spaces for Γ have nonnegative dimension. Conversely, if (9.5) does not hold, then η is a regular crossing parameter if and only if D_A is injective for every $A \in \widetilde{\mathcal{T}}(\eta)$ and $\mathcal{M}^*(X, \Gamma, g, \eta) = \emptyset$. Such crossing parameters do exist precisely when $\operatorname{index} D_A < 2 - b_1$. In general, if D_A is injective for some $A \in \widetilde{\mathcal{T}}$, then there are no solutions of the Seiberg-Witten equations near $(A, 0)$ (for any parameter η' near η). It is also interesting to consider the case $b_1 = 0$. In this case condition (b) in Definition 9.18 asserts that D_{A_η} is onto and (c) asserts that the quadratic form $\ker D_{A_\eta} \to H^{2,+}(X, i\mathbb{R}) : \varphi \mapsto \pi^+\sigma^+((\varphi\varphi^*)_0)$ is nondegenerate. □

Proposition 9.20 *The set of regular crossings is an open and dense set in $\Omega^{2,+}_\Gamma(X, g)$ with respect to the C^∞-topology. Moreover, for every regular crossing parameter η_0 and every $p > 2$ there exists a constant $\varepsilon > 0$ such that every $\eta \in \Omega^{2,+}(X, i\mathbb{R}) - \Omega^{2,+}_\Gamma(X, g)$ with $\|\eta - \eta_0\|_{L^p} \leq \varepsilon$ is regular in the sense of Theorem 7.16.*

The proof will occupy the remainder of this section. To begin with note that Definition 9.18 extends to the $W^{k,p}$ category. Throughout denote by

$$\mathcal{Z}^{k,p} = \left\{\eta \in W^{k,p}(X, \Lambda^{2,+}T^*X \otimes i\mathbb{R})x \,|\, \varepsilon(g, \eta) = 0\right\}$$

the Banach manifold of crossing parameters of class $W^{k,p}$. The set of regular crossing parameters of class $W^{k,p}$ will be denoted by $\mathcal{Z}^{k,p}_{\mathrm{reg}}$. Abbreviate

$$\mathcal{Z}^p = \mathcal{Z}^{0,p}, \qquad \mathcal{Z}^p_{\mathrm{reg}} = \mathcal{Z}^{0,p}_{\mathrm{reg}}.$$

Remark 9.21 The conditions (a) and (b) in Definition 9.18 can be restated in terms of the Fredholm operator

$$P_{A,\Phi} : H^1(X, i\mathbb{R}) \oplus W^{1,2}(X, W^+) \to H^{2,+}(X, i\mathbb{R}) \oplus L^2(X, W^-)$$

defined by

$$P_{A,\Phi} \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} = \begin{pmatrix} \pi^+\sigma^+((\varphi\Phi^* + \Phi\varphi^*)_0) \\ D_A\varphi + \Gamma(\alpha)\Phi \end{pmatrix}.$$

Its formal adjoint operator

$$P_{A,\Phi}{}^* : H^{2,+}(X, i\mathbb{R}) \oplus W^{1,2}(X, W^-) \to H^1(X, i\mathbb{R}) \oplus L^2(X, W^+)$$

is given by

$$P_{A,\Phi}{}^* \begin{pmatrix} \omega \\ \psi \end{pmatrix} = \begin{pmatrix} \pi(i\langle \psi, i\Gamma(\cdot)\Phi\rangle) \\ D_A{}^*\psi - \frac{1}{2}\rho^+(\omega)\Phi \end{pmatrix}.$$

Here $\pi : \Omega^1(X, i\mathbb{R}) \to H^1(X, i\mathbb{R})$ denotes the L^2-orthogonal projection onto the space of harmonic 1-forms. To check the formula for $P_{A,\Phi}{}^*$ use Lemma 7.4 and compare with the calculation for the adjoint operator in the proof of Lemma 8.17.

A connection $A \in \mathcal{A}^{k,p}(\Gamma)$ satisfies conditions (b) and (c) in Definition 9.18 if and only if the operator $P_{A,\Phi}$ is surjective for every nonzero $\Phi \in \ker D_A$. Equivalently, the adjoint operator satisfies an estimate

$$\|P_{A,\Phi}{}^*(\omega, \psi)\|_{L^2} \geq \delta \, \|(\omega, \psi)\|_{W^{1,2}}. \tag{9.6}$$

A simple compactness argument shows that this estimate holds with a uniform constant δ for all $\Phi \in \ker D_A$ with

$$\|\Phi\|_{L^2} = 1.$$

(Note that the kernel of D_A is a finite dimensional vector space and hence it does not matter here which norm is used for Φ.) □

Lemma 9.22 *Suppose that $A \in \mathcal{A}^{k,p}(\Gamma)$ satisfies (a) and (b) in Definition 9.18. Then there exists a constant $\varepsilon > 0$ such that*

$$\operatorname{coker} \mathcal{D}_{A,\Phi} = H^0(X, i\mathbb{R})$$

for every $\Phi \in \ker D_A$ with

$$0 < \|\Phi\|_{L^4} \leq \varepsilon.$$

Moreover, the constant $\varepsilon > 0$ depends only on δ in (9.6) but not on A itself.

Proof: The formal adjoint operator

$$\mathcal{D}_{A,\Phi}{}^* : \begin{array}{c} W^{1,2}(X, i\mathbb{R}) \\ \oplus \\ W^{1,2}(X, \Lambda^{2,+}T^*X \otimes i\mathbb{R}) \\ \oplus \\ W^{1,2}(X, W^-) \end{array} \longrightarrow \begin{array}{c} L^2(X, T^*X \otimes i\mathbb{R}) \\ \oplus \\ L^2(X, W^+) \end{array}$$

is given by

$$\mathcal{D}_{A,\Phi}{}^* \begin{pmatrix} \xi \\ \omega \\ \psi \end{pmatrix} = \begin{pmatrix} d\xi + d^*\omega + i\langle \psi, i\Gamma(\cdot)\Phi\rangle \\ D_A{}^*\psi - \frac{1}{2}\rho^+(\omega)\Phi \end{pmatrix}$$

We shall prove an estimate

$$\|\mathcal{D}_{A,\Phi}{}^*(\xi,\omega,\psi)\|_{0,\lambda} \geq \frac{\delta}{2}\,\|(\xi,\omega,\psi)\|_{1,\lambda}$$

where $\lambda = \|\Phi\|_{L^2}$, and

$$\|(\alpha,\varphi)\|_{0,\lambda}^2 = \frac{1}{\lambda^2}\,\|\pi(\alpha)\|_{L^2}^2 + \|\alpha-\pi(\alpha)\|_{L^2}^2 + \|\varphi\|_{L^2}^2\,,$$

$$\|(\xi,\omega,\psi)\|_{1,\lambda}^2 = \lambda^2\left\|\pi^+\omega\right\|_{L^2}^2 + \|d^*\omega\|_{L^2}^2 + \|d\xi\|_{L^2}^2 + \|\psi\|_{W^{1,2}}^2\,.$$

From now on we adopt the convention that all norms are L^2-norms unless otherwise indicated. In the following the inequality $(a+b)^2 \geq a^2/2 - b^2$ will be used in several places. Denote $(\alpha,\varphi) = \mathcal{D}_{A,\Phi}{}^*(\xi,\omega,\psi)$. Then

$$\begin{aligned}
\|(\alpha,\varphi)\|_{0,\lambda}^2 &= \|d\xi + d^*\omega + (\mathrm{id}-\pi)(i\langle\psi, i\Gamma(\cdot)\Phi\rangle)\|^2 \\
&\quad + \frac{1}{\lambda^2}\,\|\pi(i\langle\psi, i\Gamma(\cdot)\Phi\rangle)\|^2 + \left\|D_A{}^*\psi - \frac{1}{2}\rho^+(\omega)\Phi\right\|^2 \\
&\geq \frac{1}{2}\,\|d\xi + d^*\omega\|^2 - \|i\langle\psi, i\Gamma(\cdot)\Phi\rangle\|^2 + \frac{1}{2}\left\|\pi(i\langle\psi, i\Gamma(\cdot)\lambda^{-1}\Phi\rangle)\right\|^2 \\
&\quad + \frac{1}{2}\left\|D_A{}^*\psi - \frac{1}{2}\rho^+(\lambda\pi^+\omega)\lambda^{-1}\Phi\right\|^2 - \frac{1}{4}\left\|\rho^+(\omega-\pi^+\omega)\Phi\right\|^2 \\
&= \frac{1}{2}\left\|d\xi\right\|^2 + \frac{1}{2}\left\|d^*\omega\right\|^2 + \frac{1}{2}\left\|(P_{A,\lambda^{-1}\Phi})^*(\lambda\pi^+\omega,\psi)\right\|^2 \\
&\quad - \|i\langle\psi, i\Gamma(\cdot)\Phi\rangle\|^2 - \frac{1}{4}\left\|\rho^+(\omega-\pi^+\omega)\Phi\right\|^2 \\
&\geq \frac{1}{2}\left\|d\xi\right\|^2 + \frac{1}{2}\left\|d^*\omega\right\|^2 + \frac{1}{2}\left\|(P_{A,\lambda^{-1}\Phi})^*(\lambda\pi^+\omega,\psi)\right\|^2 \\
&\quad - \|\Phi\|_{L^4}^2\left(\|\psi\|_{L^4}^2 + \left\|\omega-\pi^+\omega\right\|_{L^4}^2\right) \\
&\geq \frac{1}{2}\left\|d\xi\right\|^2 + \frac{1}{2}\left\|d^*\omega\right\|^2 + \frac{\delta^2}{2}\left(\lambda^2\left\|\pi^+\omega\right\|^2 + \|\psi\|_{W^{1,2}}^2\right) \\
&\quad - c\,\|\Phi\|_{L^4}^2\left(\|\psi\|_{W^{1,2}}^2 + \|d^*\omega\|^2\right) \\
&\geq \frac{\delta^2}{4}\left(\lambda^2\left\|\pi^+\omega\right\|^2 + \|\psi\|_{W^{1,2}}^2 + \|d\xi\|^2 + \|d^*\omega\|^2\right) \\
&= \frac{\delta^2}{4}\,\|(\xi,\omega,\psi)\|_{1,\lambda}^2\,.
\end{aligned}$$

The last but one inequality holds whenever $c\,\|\Phi\|_{L^4}^2 \leq \delta/4$. Here the constant c stems from the previous inequality which is based on the Sobolev estimates and is independent of A. It follows that $\mathcal{D}_{A,\Phi}(\xi,\omega,\psi)$ can only be zero when $\|(\xi,\omega,\psi)\|_{1,\lambda}=0$ and this is only the case for $\xi=$ constant and $\omega=0$, $\psi=0$. Hence $\operatorname{coker}\mathcal{D}_{A,\Phi}=H^0(X,i\mathbb{R})$ whenever $\Phi\in\ker D_A$ is nonzero and $\|\Phi\|_{L^4}^2\le\delta^2/4c$. □

Lemma 9.23 *Assume $p>2$ and let $\eta\in\mathcal{Z}^p_{\mathrm{reg}}$ be a regular crossing parameter. Then there exists a constant $\varepsilon>0$ such that if $A\in\mathcal{A}^{1,p}(\Gamma)$ and $\Phi\in\ker D_A$ satisfy*

$$\|A-A_0\|_{L^4}\le\varepsilon,\qquad 0<\|\Phi\|_{L^4}\le\varepsilon,$$

for some $A_0\in\widetilde{\mathcal{T}}(\eta)$ then $\operatorname{coker}\mathcal{D}_{A,\Phi}=H^0(X,i\mathbb{R})$.

Proof: If the assertion holds for a connection A_0 it also holds for u^*A_0 for any $u\in\mathcal{G}_0$. Since the quotient $\mathcal{T}(\eta)=\widetilde{\mathcal{T}}(\eta)/\mathcal{G}_0$ is compact it suffices to prove the lemma for some fixed connection $A_0\in\widetilde{\mathcal{T}}(\eta)$. By Remark 9.21 there exists a constant $\delta>0$ such that

$$\|P_{A_0,\Phi_0}{}^*(\omega,\psi)\|_{L^2}\ge\delta\,\|(\omega,\psi)\|_{W^{1,2}}\tag{9.7}$$

for all ω,ψ and all sections $\Phi_0\in\ker D_{A_0}$ with

$$\frac{1}{2}\le\|\Phi_0\|_{L^2}\le 2.\tag{9.8}$$

We shall prove that if $A\in\mathcal{A}^{1,p}(\Gamma)$ and $\Phi\in\ker D_A$ with

$$\|A-A_0\|_{L^4}\le\varepsilon,\qquad \|\Phi\|_{L^2}=1,$$

and $\varepsilon>0$ sufficiently small then

$$\|P_{A,\Phi}-P_{A_0,\Phi_0}\|<\frac{\delta}{2}\tag{9.9}$$

for some $\Phi_0\in\ker D_{A_0}$ which satisfies (9.8). Then the estimate (9.7) continues to hold with A_0,Φ_0,δ replaced by $A,\Phi,\delta/2$ and so the result follows from Lemma 9.22.

To find a suitable Φ_0 for which (9.9) is satisfied choose some pseudo-inverse $T_0:L^2(X,W^-)\to W^{1,2}(X,W^+)$ of the operator D_{A_0} so that

$$D_{A_0}T_0D_{A_0}=D_{A_0},\qquad T_0D_{A_0}T_0=T_0.$$

(See Proposition B.7 in Appendix B.) Then the section $\Phi_0\in W^{1,2}(X,W^+)$ defined by

$$\Phi_0=\Phi+T_0\Gamma(A-A_0)\Phi.$$

lies in the kernel of D_{A_0}. The equation $D_A\Phi=0$ can be written in the form $D_{A_0}\Phi=-\Gamma(A-A_0)\Phi$ and the elliptic estimate for D_{A_0} implies

$$\|\Phi\|_{W^{1,2}}\le c_1\left(\|D_{A_0}\|_{L^2}+\|\Phi\|_{L^2}\right)$$

$$\begin{aligned}
&= c_1 \left(\|\Gamma(A - A_0)\Phi\|_{L^2} + \|\Phi\|_{L^2}\right) \\
&\le c_1 \left(\|A - A_0\|_{L^4} \|\Phi\|_{L^4} + \|\Phi\|_{L^2}\right) \\
&\le c_2\varepsilon \|\Phi\|_{W^{1,2}} + c_1 \|\Phi\|_{L^2}
\end{aligned}$$

The last but one inequality is Hölder's inequality and the last inequality uses the Sobolev embedding $W^{1,2} \hookrightarrow L^4$. With $c_2\varepsilon \le 1/2$ it follows that

$$\|\Phi\|_{W^{1,2}} \le 2c_1 \|\Phi\|_{L^2}.$$

Of course, this is obvious for elements in the kernel of an elliptic operator, but the point here is that the constant c_1 depends only on A_0 and not on A. Now the difference $\Phi - \Phi_0$ can be estimated by

$$\begin{aligned}
\|\Phi - \Phi_0\|_{W^{1,2}} &\le c_3 \|\Gamma(A - A_0)\Phi\|_{L^2} \\
&\le c_3 \|A - A_0\|_{L^4} \|\Phi\|_{L^4} \\
&\le c_4 \|A - A_0\|_{L^4} \|\Phi\|_{W^{1,2}} \\
&\le 2c_1c_4 \|A - A_0\|_{L^4} \|\Phi\|_{L^2} \\
&\le 2c_1c_4\varepsilon \|\Phi\|_{L^2} \\
&\le 2c_1c_4\varepsilon.
\end{aligned}$$

Here c_3 is the norm of the operator T_0 and the subsequent inequalities use Hölder's inequality and the Sobolev embedding $W^{1,2} \hookrightarrow L^4$ as before. With $2c_1c_4\varepsilon \le 1/2$ it follows that Φ_0 satisfies (9.8). Moreover,

$$\|\Gamma(\alpha)(\Phi - \Phi_0)\|_{L^2} \le c_5 \|\alpha\|_{L^4} \|\Phi - \Phi_0\|_{W^{1,2}} \le 2c_1c_4c_5\varepsilon \|\alpha\|_{W^{1,p}}$$

for $\alpha \in H^1(X, i\mathbb{R})$,

$$\|D_A\varphi - D_{A_0}\varphi\|_{L^2} \le c_6 \|A - A_0\|_{L^4} \|\varphi\|_{W^{1,2}} \le c_6\varepsilon \|\varphi\|_{W^{1,p}}$$

for $\varphi \in W^{1,2}(X, W^+)$, and

$$\left\|\pi^+\sigma^+((\varphi\Phi^* + \Phi\varphi^*)_0) - \pi^+\sigma^+((\varphi\Phi_0^* + \Phi_0\varphi^*)_0)\right\|_{L^p} \le c_7\varepsilon \|\varphi\|_{W^{1,p}}.$$

for $\varphi \in W^{1,2}(X, W^+)$. With ε sufficiently small this shows that A and Φ satisfy (9.9) as claimed. This proves the lemma. □

Lemma 9.24 *For every $p > 2$ and every integer $k \ge 0$ the set $\mathcal{Z}^{k,p}_{\mathrm{reg}}$ of regular crossing parameters is open is $\mathcal{Z}^{k,p}$.*

Proof: Suppose otherwise that there exists a sequence $\eta_\nu \in \mathcal{Z}^{k,p} - \mathcal{Z}^{k,p}_{\mathrm{reg}}$ converging to a regular crossing $\eta \in \mathcal{Z}^{k,p}_{\mathrm{reg}}$. The proof of Lemma 9.23 shows that η_ν satisfies the conditions (b) and (c) in Definition 9.18. Hence there exists a sequence

$$(A_\nu, \Phi_\nu) \in \widetilde{\mathcal{M}}^*(X, \Gamma, g, \eta_\nu), \qquad \operatorname{coker} \mathcal{D}_{A_\nu, \Phi_\nu} \neq H^0(X, i\mathbb{R}).$$

By Theorem 7.14, assume without loss of generality that

$$\|A_\nu - A_0\|_{W^{1,p}} \leq c \left(1 + \|d(A_\nu - A_0)\|_{L^p}\right).$$

Then the proof of Theorem 7.12 shows that the sequence A_ν is bounded in $W^{k+1,p}$ and Φ_ν is bounded in $W^{k+2,p}$. Passing to a subsequence, we may assume that A_ν converges strongly in L^4 and weakly in $W^{k+1,p}$ and Φ_ν converges strongly in L^4 and weakly in $W^{k+2,p}$. The limits

$$A = \lim_{\nu \to \infty} A_\nu, \qquad \Phi = \lim_{\nu \to \infty} \Phi_\nu$$

determine a point in the moduli space $\widetilde{\mathcal{M}}(X, \Gamma, g, \eta)$ such that

$$\operatorname{coker} \mathcal{D}_{A,\Phi} \neq H^0(X, i\mathbb{R}).$$

By condition (iii) of Definition 9.18, this implies that $\Phi = 0$. But since A_ν converges to A in the L^4-norm, Φ_ν converges to zero in the L^4-norm, and $D_{A_\nu}\Phi_\nu = 0$, it follows from Lemma 9.23 that $\operatorname{coker} \mathcal{D}_{A_\nu, \Phi_\nu} = H^0(X, i\mathbb{R})$ for ν sufficiently large. This is a contradiction, and hence the assumption that $\mathcal{Z}^{k,p}_{\mathrm{reg}}$ is not an open set in $\mathcal{Z}^{k,p}$ must have been false. □

Lemma 9.25 *For every $p > 4$ and every integer $k \geq 1$ the space*

$$\mathcal{N}_0^{k,p} = \mathcal{N}_0^{k,p}(X, \Gamma, g)$$

of all pairs $(A, \Phi) \in \mathcal{A}^{k,p}(\Gamma) \times W^{k,p}(X, W^+)$ which satisfy

$$D_A\Phi = 0, \qquad d^*(A - A_0) = 0, \qquad \Phi \neq 0, \qquad \pi^+\sigma^+((\Phi\Phi^*)_0) = 0$$

is a smooth paracompact separable Banach manifold.

Proof: Recall from Proposition 8.16 that the space $\mathcal{N} = \mathcal{N}^{k,p}$ of all pairs $(A, \Phi) \in \mathcal{A}^{k,p}(\Gamma) \times W^{k,p}(X, W^+)$ which satisfy $D_A\Phi = 0$, $d^*(A - A_0) = 0$, and $\Phi \neq 0$, is a smooth Banach manifold. Consider the smooth map

$$f : \mathcal{N} \to H^{2,+}(X, i\mathbb{R}), \qquad f(A, \Phi) = \pi^+\sigma^+((\Phi\Phi^*)_0) = 0.$$

Then $\mathcal{N}_0 = f^{-1}(0)$ and thus it remains to prove that 0 is a regular value of f. The linearized operator is given by

$$df(A, \Phi)(\alpha, \varphi) = \pi^+\sigma^+((\varphi\Phi^* + \Phi\varphi^*)_0).$$

Moreover, the tangent space $T_{A,\Phi}\mathcal{N}$ consists of all pairs (α, φ) which satisfy

$$d^*\alpha = 0, \qquad D_A\varphi + \Gamma(\alpha)\Phi = 0.$$

The proof that $df(A, \Phi)$ is surjective consists of three steps.

Step 1: *Consider the map $\gamma : \mathbb{H} \to \mathbb{C}^{2\times 2}$ defined by (4.1). Then for any two vectors $z, w \in \mathbb{C}^2$ with $z \neq 0$ there exists a unique $x \in \mathbb{H}$ with*

$$\gamma(x)z = w.$$

Denote $a = x_0 + ix_1$ and $b = x_2 + ix_3$. Then the equation $\gamma(x)z = w$ can be written in the form

$$az_1 + bz_2 = w_1, \qquad -\bar{b}z_1 + \bar{a}z_2 = w_2.$$

The unique solution (a, b) is given by

$$a = \frac{\bar{z}_1 w_1 + z_2 \bar{w}_2}{|z|^2}, \qquad b = \frac{\bar{z}_2 w_1 - z_1 \bar{w}_2}{|z|^2}.$$

Step 2: *For every pair $(A, \Phi) \in \mathcal{A}^{k,p}(\Gamma) \times W^{k,p}(X, W^+)$ and every $\varphi \in W^{k,p}(X, W^+)$ with* $\mathrm{supp}(\varphi) \subset \mathrm{supp}(\Phi)$ *there is a pair $(\alpha', \varphi') \in T_{A,\Phi}\mathcal{N}^{k,p}$ with*

$$df(A, \Phi)(\alpha', \varphi') = \pi^+\sigma^+((\varphi\Phi^* + \Phi\varphi^*)_0).$$

By Step 1, there exists a unique 1-form $\alpha \in W^{k,p}(X, T^*X \otimes i\mathbb{R})$ such that $\Gamma(\alpha)\Phi = -D_A\varphi$. Now choose a section $\xi \in W^{k+1,p}(X, i\mathbb{R})$ such that $d^*(\alpha - d\xi) = 0$ and define

$$\alpha' = \alpha - d\xi, \qquad \varphi' = \varphi + \xi\Phi.$$

Then α' and φ' are of class $W^{k,p}$ and, moreover,

$$D_A(\xi\Phi) - \Gamma(d\xi)\Phi = \xi D_A\Phi = 0, \qquad (\xi\Phi)\Phi^* + \Phi(\xi\Phi)^* = 0.$$

Hence $D_A\varphi' + \Gamma(\alpha')\Phi = 0$ and $d^*\alpha' = 0$, i.e. $(\alpha', \varphi') \in T_{A,\Phi}\mathcal{N}$, and

$$df(A, \Phi)(\alpha', \varphi') = \pi^+\sigma^+((\varphi'\Phi^* + \Phi\varphi'^*)_0) = \pi^+\sigma^+((\varphi\Phi^* + \Phi\varphi^*)_0).$$

This proves Step 2.

Step 3: *The linear operator $df(A, \Phi)$ is onto for all $(A, \Phi) \in \mathcal{N}^{k,p}$.*

Choose some nonempty open set $U \subset \mathrm{supp}(\Phi)$ and a smooth nonzero cutoff-function $\beta : X \to [0, 1]$ with support in U. By Step 2, it suffices to prove that the map

$$L^2(X, W^+) \to H^{2,+}(X, i\mathbb{R}) : \varphi \mapsto \pi^+\sigma^+(\beta(\varphi\Phi^* + \Phi\varphi^*)_0)$$

is surjective. We prove that the kernel of the dual operator is zero. Using the formulae of Lemma 7.4, one obtains for every self-dual harmonic 2-form $\omega \in H^{2,+}(X, i\mathbb{R})$,

$$\begin{aligned}\left\langle \omega, \pi^+\sigma^+(\beta(\varphi\Phi^* + \Phi\varphi^*)_0)\right\rangle &= \left\langle \omega, \sigma^+(\beta(\varphi\Phi^* + \Phi\varphi^*)_0)\right\rangle \\ &= \frac{1}{2}\left\langle \beta\rho^+(\omega), (\varphi\Phi^* + \Phi\varphi^*)_0\right\rangle \\ &= \frac{1}{2}\left\langle \rho^+(\beta\omega)\Phi, \varphi\right\rangle.\end{aligned}$$

Here all inner products are real L^2-inner products. Now suppose that the right hand side vanishes for all φ. Then $\rho^+(\beta\omega)\Phi = 0$. Since $\mathrm{supp}(\beta) \subset \mathrm{supp}(\Phi)$ it follows from Step 1 that that $\beta\omega = 0$. Hence ω vanishes on some open set and, by unique continuation, $\omega \equiv 0$. (See Remark E.9 in Appendix E.) This proves Step 3.

Thus we have proved that 0 is a regular value of f and hence $\mathcal{N}_0 = f^{-1}(0)$ is a submanifold of $\mathcal{N}$. Since $\mathcal{N}$ is paracompact and separable, so is $\mathcal{N}_0$. □

Lemma 9.26 *Assume $b^+ = 1$. Then the set $\mathcal{Z}^{k,p}_{\mathrm{reg}}$ of regular crossings is dense in $\mathcal{Z}^{k,p}$ for every $p > 4$ and every integer $k \geq 0$.*

Proof: The proof is based on the Sard-Smale theorem B.13 and on the following three observations.

Observation 1: *Consider the map $\mathcal{F}_1 : \mathcal{N}_0^{k+1,p} \to \mathcal{Z}^{k,p}$ defined by*

$$\mathcal{F}_1(A, \Phi) = \sigma^+((\Phi\Phi^*)_0) - F_A^+.$$

Then $\mathcal{F}_1$ is a Fredholm map and $\eta \in \mathcal{Z}^{k,p}$ is a regular value of $\mathcal{F}_1$ if and only if it satisfies condition (a) in Definition 9.18.

Let η be a regular value of $\mathcal{F}_1$ and suppose that $(A, \Phi) \in \mathcal{M}^*(X, \Gamma, g, \eta)$. Then $\mathcal{F}_1(A, \Phi) = \eta$. Recall that $(\xi, \omega, \psi) = \mathcal{D}_{A,\Phi}(\alpha, \varphi)$ if and only if

$$\omega = d^+\alpha + \sigma^+((\Phi\varphi^* + \varphi\Phi^*)_0), \quad \xi = d^*\alpha, \quad \psi = D_A\varphi + \Gamma(\alpha)\Phi.$$

That (A, Φ) is a regular point for $\mathcal{F}_1$ is equivalent to the condition that all triples of the form $(\omega, 0, 0)$ with $\omega \in \mathrm{im}\, d^+$ lie in the image of $\mathcal{D}_{A,\Phi}$. Now the proof of Lemma 9.25 shows that there exists a pair (α, φ) with $d^*\alpha = 0$, $D_A\varphi + \Gamma(\alpha)\Phi = 0$, and $\sigma^+((\Phi\varphi^* + \varphi\Phi^*)_0) \notin \mathrm{im}\, d^+$. Hence all triples of the form $(\omega, 0, 0)$ lie in the image of $\mathcal{D}_{A,\Phi}$. But Lemma 8.17 shows that the last two components of $\mathcal{D}_{A,\Phi}$ form an operator $D_{A,\Phi}$ with cokernel $H^0(X, i\mathbb{R})$. Hence for every pair (ξ, ψ) with ξ of mean value zero there exists an ω such that $(\omega, \xi, \psi) \in \mathrm{im}\, \mathcal{D}_{A,\Phi}$. This proves the first observation.

Observation 2: *Consider the map $\mathcal{F}_2 : \mathcal{N}^{k+1,p} \to \mathcal{Z}^{k,p}$ defined by*

$$\mathcal{F}_2(A, \Phi) = -F_A^+.$$

This is a Fredholm map with $\operatorname{index} \mathcal{F}_2 = \operatorname{index} D_A + b_1$. *Moreover, $\eta \in \mathcal{Z}^{k,p}$ is a regular value of $\mathcal{F}_2$ if and only if it satisfies condition (b) in Definition 9.18.*

Note first that $\mathcal{F}_2(A, \Phi) = \eta$ if and only $F_A^+ + \eta = 0$ and $0 \neq \Phi \in \ker\, D_A$. In particular, if η does not lie in the image of $\mathcal{F}_2$ then D_A is injective for every $A \in \widetilde{\mathcal{T}}(\eta)$ and hence the assertion holds vacuously. Secondly, recall that the tangent space of $\mathcal{N}^{k,p}$ at a point (A, Φ) with $D_A \Phi = 0$ and $\Phi \neq 0$ consists of all pairs $(\alpha, \varphi) \in W^{k,p}(X, T^*X \otimes i\mathbb{R}) \times W^{k,p}(X, W^+)$ which satisfy

$$d^*\alpha, \qquad D_A\varphi + \Gamma(\alpha)\Phi = 0.$$

The differential of $\mathcal{F}_2$ at (A, Φ) is obviously given by $d\mathcal{F}_2(A, \Phi)(\alpha, \varphi) = d^+\alpha$. This is evidently a Fredholm operator. Now the tangent space of $\mathcal{Z}^{k,p}$ is the image of d^+ and the operator d^+ identifies this with the image of d^* in $\Omega^1(X, i\mathbb{R})$, or here in $W^{k+1,p}(X, T^*X \otimes i\mathbb{R})$. Hence $d\mathcal{F}_2(A, \Phi)$ is onto if and only if for every $\alpha_1 \in \operatorname{im} d^*$ there exists a pair $(\alpha_0, \varphi) \in W^{k+1,p}$ with $\alpha_0 \in H^1(X, i\mathbb{R})$ such that

$$D_A\varphi + \Gamma(\alpha_0 + \alpha_1)\Phi = 0.$$

But recall from Lemma 8.17 that

$$\operatorname{im} D_A + \left\{\Gamma(\alpha)\Phi \,|\, \alpha \in W^{k,p},\, d^*\alpha = 0\right\} = W^{k,p}(X, W^-).$$

This shows that the above condition is equivalent to

$$\operatorname{im} D_A + \left\{\Gamma(\alpha)\Phi \,|\, \alpha \in H^1(X, i\mathbb{R})\right\} = W^{k,p}(X, W^-).$$

This proves that η is a regular value of $\mathcal{F}_2$ if and only if it satisfies condition (b) in Definition 9.18. The assertion about the Fredholm index follows by examining a regular point (A, Φ). This is left as an exercise.

Observation 3: *Consider the Fredholm map $\mathcal{F}_3 : \mathcal{N}_0^{k+1,p} \to \mathcal{Z}^{k,p}$ defined by $\mathcal{F}_3(A, \Phi) = -F_A^+$. Assume that $\eta \in \mathcal{Z}^{k,p}$ is a regular value of $\mathcal{F}_3$. Then η satisfies condition (c) in Definition 9.18.*

A tangent space of $\mathcal{N}_0^{k+1,p}$ consists of all pairs (α, φ) which satisfy

$$d^*\alpha = 0, \quad D_A\varphi + \Gamma(\alpha)\Phi = 0, \quad \pi^+\sigma^+((\varphi\Phi^* + \Phi\varphi^*)_0) = 0, \qquad (9.10)$$

and the differential of $\mathcal{F}_3$ is again given by $d\mathcal{F}_3(A, \Phi)(\alpha, \varphi) = d^+\alpha$. Let η be a regular value of $\mathcal{F}_3$ and suppose, by contradiction, that η does not satisfy (c) in Definition 9.18. Then there exists an $A \in \widetilde{\mathcal{T}}(\eta)$ and a nonzero section $\Phi \in \ker D_A$ such that

$$\begin{array}{c} \alpha \in H^1(X, i\mathbb{R}), \\ D_A\varphi + \Gamma(\alpha)\Phi = 0 \end{array} \qquad \Longrightarrow \qquad \pi^+\sigma^+((\varphi\Phi^* + \Phi\varphi^*)_0) = 0. \tag{9.11}$$

Then, in particular, $\pi^+\sigma^+((\Phi\Phi^*)_0) = 0$ and hence $(A, \Phi) \in \mathcal{N}_0^{k,p}$ and $\eta = \mathcal{F}_3(A, \Phi)$. Since η is a regular value of $\mathcal{F}_3$ it follows that for every $\alpha \in \operatorname{im} d^*$ there exists some $\varphi \in W^{k,p}(X, W^+)$ such that (9.10) is satisfied. In connection with (9.11) this shows that

$$\begin{array}{c} d^*\alpha = 0, \\ D_A\varphi + \Gamma(\alpha)\Phi = 0 \end{array} \qquad \Longrightarrow \qquad \pi^+\sigma^+((\varphi\Phi^* + \Phi\varphi^*)_0) = 0.$$

But this is impossible by Lemma 9.25. Hence (9.11) must have been false. This proves the third observation. It follows from these three observations that $\eta \in \mathcal{Z}^{k,p}$ is a regular crossing parameter if and only if it is a common regular value of $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$. Since all three maps are C^∞ smooth the lemma follows from the Sard-Smale theorem B.13. □

Proof of Proposition 9.20: Firstly, it follows from Lemma 9.24 that the set of regular crossings in $\Omega^{2,+}_\Gamma(X, g)$ is open in the L^p-topology for any $p > 2$. That the set of regular crossings is dense in the C^∞-topology follows from Lemma 9.24 and Lemma 9.26. More precisely, given $\eta \in \Omega^{2,+}_\Gamma(X, g) \subset \mathcal{Z}^{k,p}$ choose, by Lemma 9.26, a regular crossing $\eta' \in \mathcal{Z}^{k,p}$ with

$$\|\eta - \eta'\|_{W^{k,p}} \le 2^{-(k+1)}.$$

Now choose a smooth perturbation $\eta_k \in \Omega^{2,+}_\Gamma(X, g)$ which satisfies

$$\|\eta - \eta_k\|_{W^{k,p}} \le 2^{-k}.$$

Lemma 9.24 asserts that η_k is still regular if it is sufficiently close to η'. Thus η_k is a sequence of regular crossings which converges to η in the C^∞ topology. This proves that the set of regular crossings is open and dense in the C^∞-topology.

To prove the second assertion fix a number $p > 2$ and suppose, by contradiction, that there is a sequence $\eta_\nu \in W^{k,p}(X, \Lambda^{2,+}T^*X \otimes i\mathbb{R}) - \mathcal{Z}^{k,p}$ converging to a regular crossing parameter $\eta \in \mathcal{Z}^{k,p}_{\mathrm{reg}}$. Then there exists a sequence

$$(A_\nu, \Phi_\nu) \in \widetilde{\mathcal{M}}^*(X, \Gamma, g, \eta_\nu), \qquad \operatorname{coker} \mathcal{D}_{A_\nu, \Phi_\nu} \ne H^0(X, i\mathbb{R}).$$

A contradiction is now derived, word by word, as in Lemma 9.24. This proves the proposition. □

9.4 Proof of the wall crossing formula

Choose a regular crossing parameter

$$\eta \in \Omega^{2,+}_{\Gamma}(X, g)$$

and recall that $\omega_g \in H^{2,+}(X)$ denotes the unique self-dual harmonic 2-form with norm 1 which determines the given orientation of $H^{2,+}(X)$. Consider the Seiberg-Witten moduli space

$$\mathcal{M}^*(\{\eta_t\}) = \mathcal{M}^*(X, \Gamma, g, \{\eta_t\})$$

corresponding to the path of perturbations

$$\eta_t = \eta + ti\omega_g, \qquad -\varepsilon \leq t \leq \varepsilon.$$

It follows from the definition of regular crossing and Lemma 9.23 that $\operatorname{coker} \mathcal{D}_{A,\Phi} = H^0(X, i\mathbb{R})$ for all triples $(A, \Phi, t) \in \widetilde{\mathcal{M}}^*(\{\eta_t\})$ provided that $\varepsilon > 0$ is sufficiently small. Hence the perturbations $\{\eta_t\}$ form a regular path as defined in the proof of Theorem 7.21. In fact, each η_t is a regular perturbation and each individual moduli space $\mathcal{M}^*(\eta_t)$ is a smooth manifold for $-\varepsilon \leq t \leq \varepsilon$ which, for $t \neq 0$, is compact and agrees with $\mathcal{M}(\eta_t)$. Thus the parametrized moduli space $\mathcal{M}^*(\{\eta_t\})$ is a cobordism with

$$\partial \mathcal{M}^*(\{\eta_t\}) = \mathcal{M}(\eta_\varepsilon) - \mathcal{M}(\eta_{-\varepsilon}).$$

It has dimension

$$\dim \mathcal{M}^*(\{\eta_t\}) = \operatorname{index} D_A + b_1 - 1 \geq 1.$$

for $A \in \mathcal{A}(\Gamma)$. However, the cobordism will not be compact in general. If $(A_\nu, \Phi_\nu, t_\nu) \in \widetilde{\mathcal{M}}^*(\{\eta_t\})$ is a convergent sequence whose limit point does not lie in $\widetilde{\mathcal{M}}^*(\{\eta_t\})$ then $t_\nu \to 0$ and $\Phi_\nu \to 0$ and A_ν converges to a point in $\widetilde{\mathcal{T}}(\eta)$. Hence the difference of the invariants is determined by the structure of the moduli space near the torus $\mathcal{T}(\eta)$. Let us examine the Seiberg-Witten equations

$$\begin{aligned} d^*(A - A_0) &= 0, \\ F_A^+ + \eta + it\omega_g &= \sigma^+((\Phi\Phi^*)_0), \\ D_A\Phi &= 0 \end{aligned} \tag{9.12}$$

in a neighbourhood of a connection $A \in \widetilde{\mathcal{T}}(\eta)$. Note first that if D_A is injective then so is the operator $D_{A'}$ for every connection $A' \in \mathcal{A}(\Gamma)$ which is sufficiently close to A (in the L^4-norm). Hence (9.12) cannot have any solutions near such a connection A except for other connections in $\widetilde{\mathcal{T}}(\eta)$. Thus it is interesting to examine the solutions of (9.12) near connections

$A \in \widetilde{\mathcal{T}}(\eta)$ for which D_A is not injective. This can easily be done globally. Choose a number $\delta > 0$ such that

$$(A, \Phi) \in \widetilde{\mathcal{M}}(\eta_\varepsilon) \cup \widetilde{\mathcal{M}}(\eta_{-\varepsilon}) \qquad \Longrightarrow \qquad \|\Phi\|_{L^2}^2 > \delta$$

and consider the moduli space

$$\widetilde{\mathcal{M}}_\delta(\{\eta_t\}) = \left\{(A, \Phi, t) \in \widetilde{\mathcal{M}}(\{\eta_t\}) \mid \|\Phi\|_{L^2}^2 \geq \delta\right\}.$$

As usual, denote the quotient by

$$\mathcal{M}_\delta(\{\eta_t\}) = \frac{\widetilde{\mathcal{M}}_\delta(\{\eta_t\})}{\mathcal{G}_0}.$$

The following observation shows that this is a manifold with boundary for $\delta > 0$ sufficiently small.

Lemma 9.27 *For $\delta > 0$ sufficiently small the moduli space $\widetilde{\mathcal{M}}(\{\eta_t\})$ is transverse to the codimension-1-submanifold of all triples (A, Φ, t) with $\|\Phi\|_{L^2}^2 = \delta$.*

Proof: The tangent space of $\widetilde{\mathcal{M}}(\{\eta_t\})$ at a triple (A, Φ, t) consists of all (α, φ, τ) which satisfy

$$\begin{aligned} d^*\alpha &= 0, \\ d^+\alpha + i\tau\omega_g &= \sigma^+((\Phi\varphi^* + \Phi\varphi^*)_0), \\ D_A\varphi + \Gamma(\alpha)\Phi &= 0. \end{aligned} \tag{9.13}$$

Transversality is equivalent to the existence of a triple (α, φ, τ) in this tangent space which satisfies

$$\langle \varphi, \Phi \rangle \neq 0.$$

To see that such a triple exists consider the operator $\Pi^+\mathcal{D}_{A,\Phi}\Pi$ where

$$\Pi^+ \begin{pmatrix} \xi \\ \omega \\ \psi \end{pmatrix} = \begin{pmatrix} \xi \\ \omega - \pi^+\omega \\ \psi \end{pmatrix}$$

and Π denotes the projection onto the orthogonal complement of Φ:

$$\Pi \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} = \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} - \frac{\langle \Phi, \varphi \rangle}{\|\Phi\|_{L^2}^2} \begin{pmatrix} 0 \\ \Phi \end{pmatrix}.$$

We must prove that the cokernel of this operator is given by

$$\operatorname{coker}\left(\Pi^+\mathcal{D}_{A,\Phi}\Pi\right) = H^0(X, i\mathbb{R}) \oplus H^{2,+}(X, i\mathbb{R}). \tag{9.14}$$

If this holds then the dimension of the space of solutions (α, φ, τ) of (9.13) which satisfy $\langle\varphi, \Phi\rangle = 0$ is one less than the dimension of the space of all solutions of (9.13) and hence there must be one solution with $\langle\varphi, \Phi\rangle \neq 0$. To prove (9.14) consider the adjoint operator $\mathcal{D}_{A,\Phi}{}^*$. The formula in the proof of Lemma 9.22 shows that

$$\Pi\mathcal{D}_{A,\Phi}{}^*\Pi^+\begin{pmatrix}\xi\\ \omega\\ \psi\end{pmatrix} = \mathcal{D}_{A,\Phi}{}^*\Pi^+\begin{pmatrix}\xi\\ \omega\\ \psi\end{pmatrix} - \frac{1}{2}\Pi\begin{pmatrix}0\\ \rho^+(\omega - \pi^+\omega)\Phi\end{pmatrix}.$$

The proof of Lemma 9.22 also shows that not only does $\mathcal{D}_{A,\Phi}{}^*$ have kernel $H^0(X, i\mathbb{R})$ but also that there is a uniform estimate

$$\|(\xi, \omega, \psi)\|_{1,\lambda} \leq c\,\|\mathcal{D}_{A,\Phi}{}^*(\xi, \omega, \psi)\|_{0,\lambda}$$

whenever Φ is nonzero and $\|\Phi\|_{L^4}$ is sufficiently small where $\lambda = \|\Phi\|_{L^2}$ and

$$\|(\alpha, \varphi)\|_{0,\lambda}^2 = \frac{1}{\lambda^2}\|\pi(\alpha)\|_{L^2}^2 + \|\alpha - \pi(\alpha)\|_{L^2}^2 + \|\varphi\|_{L^2}^2,$$

$$\|(\xi, \omega, \psi)\|_{1,\lambda}^2 = \lambda^2\left\|\pi^+\omega\right\|_{L^2}^2 + \|d^*\omega\|_{L^2}^2 + \|d\xi\|_{L^2}^2 + \|\psi\|_{W^{1,2}}^2.$$

The above formula now shows that

$$\begin{aligned}\left\|\Pi\mathcal{D}_{A,\Phi}{}^*\Pi^+(\xi, \omega, \psi) - \mathcal{D}_{A,\Phi}{}^*\Pi^+(\xi, \omega, \psi)\right\|_{0,\lambda} &\leq \left\|\rho^+(\omega - \pi^+\omega)\Phi\right\|_{L^2}^2\\ &\leq \left\|\omega - \pi^+\omega\right\|_{L^4}^2\|\Phi\|_{L^4}^2\\ &\leq c'\|\Phi\|_{L^4}^2\|d^*\omega\|_{L^2}^2\\ &\leq c'\|\Phi\|_{L^4}^2\left\|\Pi^+(\xi, \omega, \psi)\right\|_{1,\lambda}.\end{aligned}$$

If $\|\Phi\|_{L^4}$ is sufficiently small we obtain an inequality

$$\left\|\Pi^+(\xi, \omega, \psi)\right\|_{1,\lambda} \leq 2c\left\|\Pi\mathcal{D}_{A,\Phi}{}^*\Pi^+(\xi, \omega, \psi)\right\|_{0,\lambda}.$$

This proves (9.14) and hence the lemma. □

It follows from Lemma 9.27 that for δ sufficiently small the moduli space $\mathcal{M}_\delta(\{\eta_t\})$ is a smooth compact manifold with boundary

$$\partial\mathcal{M}_\delta(\{\eta_t\}) = \mathcal{M}(\eta_\varepsilon) - \mathcal{M}(\eta_{-\varepsilon}) - \mathcal{M}_\delta(\eta).$$

The third part of the boundary is the quotient

$$\mathcal{M}_\delta(\eta) = \frac{\left\{(A, \Phi, t) \in \widetilde{\mathcal{M}}(\{\eta_t\}) \mid \|\Phi\|_{L^2}^2 = \delta\right\}}{\mathcal{G}_0}.$$

To prove the theorem we must evaluate the cohomology class $c_1(\mathcal{L})^d$ over $\mathcal{M}_\delta(\eta)$. To do this it is convenient to first simplify the equation.

Lemma 9.28 *For $\delta > 0$ sufficiently small the moduli space $\mathcal{M}_\delta(\eta)$ is cobordant to*

$$\mathcal{M}_\delta^0(\eta) = \frac{\left\{(A, \Phi) \,|\, A \in \widetilde{\mathcal{T}}(\eta),\, D_A\Phi = 0,\, \|\Phi\|_{L^2}^2 = \delta\right\}}{\mathcal{G}_0}.$$

Proof: Note first that the number t can be eliminated from the equations (9.12). Since

$$\pi^+\omega_g = \omega_g, \qquad \pi^+(F_A^+ + \eta) = 0$$

one finds

$$t = \langle i\omega_g, \sigma^+((\Phi\Phi^*)_0)\rangle. \tag{9.15}$$

Hence equation (9.12) is equivalent to

$$\begin{aligned} d^*(A - A_0) &= 0, \\ F_A^+ + \eta &= (\mathrm{id} - \pi^+)\sigma^+((\Phi\Phi^*)_0), \\ D_A\Phi &= 0 \end{aligned}$$

with t given by (9.15). The cobordism from $\mathcal{M}_\delta^0(\eta)$ to $\mathcal{M}_\delta(\eta)$ is simply obtained by a homotopy which drives the term on the right to zero. It is actually a product cobordism with $\mathcal{M}_\delta^\lambda(\eta)$ defined as the moduli space of solutions of the equations

$$\begin{aligned} d^*(A - A_0) &= 0, \\ F_A^+ + \eta &= \lambda(\mathrm{id} - \pi^+)\sigma^+((\Phi\Phi^*)_0), \qquad \|\Phi\|_{L^2}^2 = \delta, \\ D_A\Phi &= 0, \end{aligned}$$

for $0 \le \lambda \le 1$. That this space is a smooth manifold follows from the arguments in the proof of Lemmata 9.22 and 9.27. The details are left to the reader. □

It follows from Proposition 1.48 and Lemma 9.28 that the integral of $c_1(\mathcal{L})^d$ over the moduli space $\mathcal{M}_\delta(\eta)$ with

$$2d = \mathrm{index}(D_A) + b_1 - 2 \ge 0$$

is given by

$$\int_{\mathcal{M}_\delta(\eta)} c_1(\mathcal{L})^d = \int_{\mathcal{M}_\delta^0(\eta)} c_1(\mathcal{L})^d = \int_{\mathcal{T}} c_k(-\mathcal{IND}).$$

Here $2k = b_1$ and $\mathcal{IND} \in K(\mathcal{T})$ is the topological index of the operator family D_{A_ρ} over the torus $\mathcal{T}$. This proves Theorem 9.14. Theorem 9.9

follows by specializing to $b_1 = 0$. In this case $\widetilde{\mathcal{T}}(\eta)$ is just a single point A_η and $\mathcal{M}_\delta^0(\eta)$ is simply the projective space of ker D_{A_η}. Hence the integral is 1 and this proves Theorem 9.9.

Remark 9.29 Equation (9.15) is actually of some interest in its own right. It illustrates the geometry of the moduli space $\mathcal{M}^*(\{\eta_t\})$ near the singular set. When A is close to a connection $A_0 \in \widetilde{\mathcal{T}}(\eta)$ then Φ is close to an element in the kernel of D_{A_0} and hence the local behaviour of the parameter t (as a function of A and Φ) is determined by the quadratic form

$$\ker\, \mathcal{D}_{A_0} \to H^{2,+}(X, i\mathbb{R}) : \varphi \mapsto \pi^+\sigma^+((\varphi\varphi^*)_0).$$

If this form is indefinite then there are solutions in $\mathcal{M}^*(X, \Gamma, g, \eta)$ converging to the singular part of the moduli space. On the other hand if, say, $b_1 = 0$ and this form is definite then A_η is an isolated point and the moduli space $\mathcal{M}^*(X, \Gamma, g, \eta)$ is compact. □

Consider the case where $b^+ = 1$ and

$$\operatorname{index} D_A < 2 - b_1.$$

In this case the virtual dimension of the Seiberg-Witten moduli spaces is negative and hence the moduli spaces for regular perturbations are empty. A regular crossing can still be defined as in Definition 9.18 and Proposition 9.20 continues to hold. But a regular crossing is one where the Dirac operator D_A is injective for every $A \in \widetilde{\mathcal{T}}(\eta)$. The proof of Theorem 9.9 above shows that in this case there are no solutions of the Seiberg-Witten equations near $(A_\eta, 0)$ for any perturbation near η which does not lie on the Γ-wall (in accordance with the fact that regular moduli spaces are empty). The discussion in Lemma 9.16 shows, however, that the Chern class $c_k(-\mathcal{IND})$ need not be zero and thus Theorem 9.14 does not extend to the case $\operatorname{index} D_A < 2 - b_1$.

9.5 Proof of Donaldson's theorem

The following inequality for characteristic vectors is an immediate consequence of Donaldson's theorem. By Elkies' theorem, this inequality is in fact equivalent to Donaldson's theorem. A proof using the Seiberg-Witten equations was found by Kronheimer.

Proposition 9.30. (Kronheimer) *Let X be a compact oriented smooth 4-manifold with negative definite intersection form Q_X. Then every characteristic vector $\gamma \in H_2(X) = H_2(X, \mathbb{Z})/$torsion for Q_X satisfies*

$$|\gamma \cdot \gamma| \geq b_2(X). \tag{9.16}$$

Proof: Suppose, by contradiction, that there exists a characteristic vector $\gamma \in H_2(X)$ with with $|\gamma \cdot \gamma| < b_2(X) = \operatorname{rank} Q_X$. Since $\gamma \cdot \gamma$ is negative and the difference $|\gamma \cdot \gamma| - \operatorname{rank} Q_X$ is divisible by 8 we have

$$\gamma \cdot \gamma \geq 8 - b_2.$$

By Theorem 5.10, there exists an integral lift $c \in H^2(X, \mathbb{Z})$ of $\mathrm{w}_2(TX)$ such that

$$c \cdot c \geq 8 - b_2.$$

Let $\Gamma : TX \to \operatorname{End}(W)$ be a spin^c structure with $c = c_1(L_\Gamma)$. Then the Dirac operator D_A has real Fredholm index

$$\operatorname{index} D_A = \frac{c \cdot c + b_2}{4} = 2(k+1) \geq 2$$

for $A \in \mathcal{A}(\Gamma)$ and, for every regular perturbation $\eta \in \Omega^{2,+}_{\mathrm{reg}}(X, g)$, the moduli space $\mathcal{M}^*(\eta) = \mathcal{M}^*(X, \Gamma, g, \eta)$ has dimension

$$\dim \mathcal{M}^*(\eta) = \frac{c \cdot c + b_2}{4} + b_1 - 1 = 2k + b_1 + 1.$$

This moduli space is not compact and we examine its structure near the singular part. As before denote by

$$\mathcal{T} = \mathcal{T}(\eta) = \frac{\widetilde{\mathcal{T}}}{\mathcal{G}_0(x_0)}$$

the space of gauge equivalence classes of connections A with $F_A^+ + \eta = 0$. Call a perturbation η **regular** if it satisfies conditions (a) and (b) in Definition 9.18:

(a) Every $(A, \Phi) \in \widetilde{\mathcal{M}}^*(X, \Gamma, g, \eta)$ satisfies $\operatorname{coker} \mathcal{D}_{A,\Phi} = H^0(X, i\mathbb{R})$.

(b) For every $A \in \widetilde{\mathcal{T}}(\eta)$ and every $\Phi \in \ker D_A$ with $\Phi \neq 0$ the linear map

$$H^1(X, i\mathbb{R}) \to \operatorname{coker} D_A : \alpha \mapsto \pi_A(\Gamma(\alpha)\Phi)$$

is surjective.

We shall now digress with two results about regular perturbation parameters η.

Proposition 9.31 *The set of regular crossings is an open and dense set in $\Omega^{2,+}(X, i\mathbb{R})$ with respect to the C^∞-topology.*

Lemma 9.32 *Assume $p > 2$ and let $\eta \in L^p(X, \Lambda^{2,+}T^*X \otimes i\mathbb{R})$ be a regular parameter. Then there exists a constant $\varepsilon > 0$ such that if $A \in \mathcal{A}^{1,p}(\Gamma)$ and $\Phi \in \ker D_A$ satisfy*

$$\|A - A_0\|_{L^4} \leq \varepsilon, \qquad 0 < \|\Phi\|_{L^4} \leq \varepsilon,$$

for some $A_0 \in \widetilde{\mathcal{T}}(\eta)$ then $\operatorname{coker} \mathcal{D}_{A,\Phi} = H^0(X, i\mathbb{R})$.

The proofs of these results are similar in spirit and detail to those of Proposition 9.20 and Lemma 9.23, only much simpler because $b^+ = 0$. The arguments will not be repeated here. This is the end of the digression.

Proof of Proposition 9.30 continued: Choose a regular perturbation η and consider the moduli space

$$\mathcal{M}_\delta(\eta) = \frac{\left\{(A,\Phi) \in \widetilde{\mathcal{M}}(\eta) \,|\, \|\Phi\|_{L^2}^2 \geq \delta\right\}}{\mathcal{G}_0}.$$

As in the proof of Lemma 9.27 it can be shown that the manifold $\widetilde{\mathcal{M}}^*(\eta)$ is transverse to the manifold of all pairs (A, Φ) with $\|\Phi\|_{L^2}^2 = \delta$ whenever $\delta > 0$ is sufficiently small. Hence $\mathcal{M}_\delta(\eta)$ is a smooth manifold with boundary

$$\partial\mathcal{M}_\delta(\eta) = \frac{\left\{(A,\Phi) \in \widetilde{\mathcal{M}}(\eta) \,|\, \|\Phi\|_{L^2}^2 = \delta\right\}}{\mathcal{G}_0}.$$

As in Lemma 9.28 this boundary is cobordant to

$$\mathcal{M}_\delta^0(\eta) = \frac{\left\{(A,\Phi) \,|\, A \in \widetilde{\mathcal{T}}(\eta),\, D_A\Phi = 0,\, \|\Phi\|_{L^2}^2 = \delta\right\}}{\mathcal{G}_0}.$$

The cobordism is again a product cobordism with $\mathcal{M}_\delta^\lambda(\eta)$ defined as the moduli space of solutions of the equations

$$\begin{aligned} d^*(A - A_0) &= 0, \\ F_A^+ + \eta &= \lambda\sigma^+((\Phi\Phi^*)_0), \qquad \|\Phi\|_{L^2}^2 = \delta, \\ D_A\Phi &= 0, \end{aligned}$$

for $0 \leq \lambda \leq 1$. As before it follows from the arguments in the proof of Lemmata 9.22 and 9.27 that this space is a smooth manifold for each λ. These manifolds are all of dimension

$$\dim \mathcal{M}_\delta^\lambda(\eta) = 2k + b_1.$$

If $b_1 = 0$ then $\widetilde{\mathcal{T}}(\eta)$ is a single point A_η and

$$\mathcal{M}_\delta^0(\eta) = P \ker D_{A_\eta}.$$

If $\operatorname{index} D_{A_\eta} = 2$ then $\mathcal{M}_\delta^0(\eta)$ is a single point and so is $\mathcal{M}_\delta^1(\eta) = \partial\mathcal{M}_\delta(\eta)$. Hence $\mathcal{M}_\delta(\eta)$ is a compact 1-manifold whose boundary consists of a single point. Such an object cannot exist. Similarly, if $\operatorname{index} D_{A_\eta} = 2(k+1) > 2$ then $\partial\mathcal{M}_\delta(\eta)$ has dimension $2k$ and

$$\int_{\partial\mathcal{M}_\delta(\eta)} c_1(\mathcal{L})^k = \int_{\mathcal{M}_\delta^0(\eta)} c_1(\mathcal{L})^k = 1.$$

This is impossible by Stokes' theorem and it follows, in the case $b_1 = 0$, that the original assumption that Q_X not be diagonalizable must have been false. In the case $b_1 > 0$ recall from Remark 7.25 that the quotient $\mathcal{B}(\Gamma) = \mathcal{A}(\Gamma)/\mathcal{G}$ is homotopy equivalent to the torus $\mathcal{T} = \mathcal{T}(\eta) \cong H^1(X, i\mathbb{R})/H^1(X, 2\pi i\mathbb{Z})$ of reducible solutions (see page 305). Recall also that the orientation of H^1 which is required for orienting the moduli space also determines an orientation of this torus. Denote by

$$\mathrm{dvol}_\mathcal{T} \in H^{b_1}(\mathcal{B}(\Gamma), \mathbb{Z})$$

the positive generator which evaluates to 1 on the fundamental class of $\mathcal{T} \subset \mathcal{B}(\Gamma)$ (compare with Exercise 7.26). Let $\pi : \mathcal{C}(\Gamma) \to \mathcal{B}(\Gamma)$ denote the canonical projection $[A, \Phi] \mapsto [A]$. Then it follows from Proposition 1.49 that

$$\begin{aligned}\int_{\partial\mathcal{M}_\delta(\eta)} c_1(\mathcal{L})^k \wedge \pi^*\mathrm{dvol}_\mathcal{T} &= \int_{\mathcal{M}_\delta^0(\eta)} c_1(\mathcal{L})^k \wedge \pi^*\mathrm{dvol}_\mathcal{T} \\ &= \int_\mathcal{T} \mathrm{dvol}_\mathcal{T} \\ &= 1.\end{aligned}$$

This contradicts Stokes' theorem and hence Proposition 9.30 is proved. □

Proof of Theorem 9.6: Let X be a compact oriented smooth 4-manifold with negative definite intersection form Q_X. If Q_X were not diagonalizable then, by Elkies' theorem 9.5, there would exist a characteristic vector γ with $\gamma \cdot \gamma \geq 8 - b_2(X)$, in contradiction to Proposition 9.30. □

Remark 9.33 Recently, Froyshov generalized the inequality (9.16) to 4-manifolds with boundary. In [33] he proved that for every rational homology-3-sphere Y there is a nonnegative integer $\mathrm{Fr}(Y) \in \mathbb{Z}$ such that the following holds. If X is a compact oriented smooth 4-manifold with boundary $\partial X = Y$ and $\gamma \in H_2(X, \partial X) \cong H_2(X)$ is a characteristic vector*

*Here all homology groups are understood as integral homology modulo torsion. The exact sequence of the pair $(X, \partial X)$ shows that $H_i(X) \cong H_i(X, \partial X)$ for $i \neq 4$ whenever ∂X is a rational homology-3-sphere.

of the intersection form Q_X then

$$|\gamma\cdot\gamma| \geq b_2(X) - \mathrm{Fr}(Y). \tag{9.17}$$

Now the intersection form Q_X is unimodular and hence decomposes as

$$Q_X = m(-1) \oplus \tilde{Q}$$

where $\tilde{Q}$ does not have a vector of square -1. If $\tilde{Q}$ is even then $\tilde{\gamma} = 0$ is a characteristic vector of $\tilde{Q}$ and hence there exists a characteristic vector γ of Q_X with $|\gamma\cdot\gamma| = m = b_2(X) - \operatorname{rank}\tilde{Q}$. Hence the inequality (9.17) implies

$$\operatorname{rank}\tilde{Q} \leq \mathrm{Fr}(Y).$$

This shows that only finitely many even forms $\tilde{Q}$ can occur in the intersection form of a smooth 4-manifold with boundary Y. Froyshov also proves that the invariant of the Poincaré 3-sphere $Y = P$ is $\mathrm{Fr}(P) = 8$ and hence in this case the only even possibilities are $\tilde{Q} = 0$ and $\tilde{Q} = E_8$. Furthermore, in [34] he proves that if X is simply connected with $\partial X = P$ then $\tilde{Q}$ is necessarily even. Froyshov's proof of (9.17) uses the Seiberg-Witten equations on 4-manifolds with cylindrical ends. □

9.6 Proof of Furuta's theorem

Let X be a compact connected smooth spin 4-manifold. Then $\mathrm{w}_2(TX) = 0$ and (5.1) shows that the intersection form $Q_X : H_2(X,\mathbb{Z}) \times H_2(X,\mathbb{Z}) \to \mathbb{Z}$ is even. An even form is never diagonalizable over the integers and hence, by Donaldson's theorem 9.6, the form Q_X is indefinite. Moreover, by Rohlin's theorem 6.27, the signature is divisible by 16. Hence the Hasse-Minkowski theorem 9.2 shows that the intersection form of X is equivalent to

$$Q_X \sim 2k(-E_8) \oplus mH$$

for some integers $k \in \mathbb{Z}$ and $m \geq 1$. If $k = 0$ there is nothing to prove. Hence assume, without loss of generality, that $k \geq 1$. Moreover, assume with loss of generality that $b_1 = 0$. The general case can be reduced to this by surgery along loops (cf. [35]).

Exercise 9.34 Let X be a compact oriented smooth 4-manifold. Use surgery along (nontorsion) loops to prove that there exists a smooth 4-manifold X' which has the same intersection form as X and satisfies $b_1(X') = 0$. If X is spin prove that X' is spin. The Enriques surface shows that there need not be a simply connected smooth 4-manifold with the same intersection form (see Example 6.28). □

Fix a spin structure (S, I, J, Γ) on TX as in Definition 5.4. Thus $S \to X$ is a real Riemannian rank-8 bundle, I and J are two anticommuting orthogonal complex structures on S, and $\Gamma : TX \to \mathrm{End}(S)$ is a bundle homomorphism which satisfies (4.18) and commutes with both I and J. Recall that there is a splitting $S = S^+ \oplus S^-$ which is invariant under both I and J and is interchanged by the endomorphisms $\Gamma(v)$ for $v \in TX$. Recall also from Lemma 6.6 that there is a unique spin connection ∇ on S (which commutes with both I and J, preserves the subbundles $S^\pm$, and is compatible with the Levi-Civita connection on TX). Denote by

$$D : C^\infty(X, S^+) \to C^\infty(X, S^-)$$

the corresponding Dirac operator. Think of (S, I) as a spin^c structure. It is convenient to write the Seiberg-Witten equations for this spin^c structure in real notation and always express the dependence on the complex structure I explicitly. Thus denote by $\mathfrak{su}(S^+, I)$ the bundle of all endomorphisms $T : S^+_x \to S^+_x$ which satisfy

$$T^* + T = 0, \qquad TI = IT, \qquad \mathrm{trace}(IT) = 0. \tag{9.18}$$

Here T^* denotes the adjoint with respect to the real inner product. The first two conditions mean that T is skew-Hermitian with respect to I and the last condition says that the complex trace of T vanishes. It is important to note that every endomorphism $T \in \mathfrak{su}(S^+, I)$ also commutes with J.

Lemma 9.35 *If $T \in \mathrm{End}(S^+)$ satisfies (9.18) then $TJ = JT$.*

Proof: For any unit vector $\zeta \in S^+_x$ the vectors ζ, $I\zeta$, $J\zeta$, $K\zeta$ with $K = IJ$ form an orthonormal basis of S^+_x. Hence

$$\zeta\zeta^* + (I\zeta)(I\zeta)^* + (J\zeta)(J\zeta)^* + (K\zeta)(K\zeta)^* = \mathbb{1}$$

where $\zeta^* \in \mathrm{Hom}(S^+_x, \mathbb{R})$ is defined by $\varphi \mapsto \zeta^*\varphi = \langle \zeta, \varphi \rangle$ for $\varphi \in S^+_x$. This identity can be written in the form

$$A - IAI - JAJ - KAK = \mathrm{trace}(A)\mathbb{1} \tag{9.19}$$

for $A = \zeta\zeta^* \in \mathrm{End}(S^+_x)$. Now every symmetric endomorphism of S^+_x is a linear combination of those of the form $A = \zeta\zeta^*$ and hence (9.19) continues to hold for all symmetric endomorphisms $A = A^*$ of S^+_x. Apply this to $A = IT$ to obtain $T + JTJ = 0$ and thus $JT = TJ$ as claimed. □

The previous result can be expressed in the form $\mathfrak{su}(S^+, I) = \mathfrak{su}(S^+, J)$ and hence we shall write $\mathfrak{su}(S^+) = \mathfrak{su}(S^+, I) = \mathfrak{su}(S^+, J)$ from now on. Note also that $\Gamma : TX \to \mathrm{End}(S)$ is a spin^c structure with respect to both I and J and hence the induced map

$$\rho^+ : \Lambda^{2,+}T^*X \to \mathfrak{su}(S^+)$$

identifies the skew-Hermitian endomorphisms of S^+ (with respect to either I or J) with the real valued self-dual 2-forms on X. This gives rise to an alternative proof of Lemma 9.35. The reader should be warned that throughout this section the notation ζ^* and A^* is used for the real adjoint rather than the complex adjoint as before. Thus what used to be called $(\varphi\varphi^*)_0$ will now be $\varphi\varphi^* - I\varphi\varphi^* I - \frac{1}{2}|\varphi|^2\mathbb{1}$. What used to be the skew-Hermitian endomorphism $i(\varphi\varphi^*)_0$ will now be

$$\varphi\varphi^* I + I\varphi\varphi^* - \frac{1}{2}|\varphi|^2 I \in \mathfrak{su}(S^+).$$

Compose this with the bundle isomorphism ${\rho^+}^{-1} : \mathfrak{su}(S^+) \to \Lambda^{2,+}T^*X$ to obtain the quadratic map $\sigma_I : S^+ \to \Lambda^{2,+}T^*X$ defined by

$$\sigma_I(\varphi) = {\rho^+}^{-1}\left(\varphi\varphi^* I + I\varphi\varphi^* - \frac{1}{2}|\varphi|^2 I\right)$$

This is the explicit formula for what used to be called $\sigma^+(i(\varphi\varphi^*)_0)$. Now a spin^c connection on (S^+, I) is of the form $\nabla + \alpha I$ for some real valued 1-form $\alpha \in \Omega^1(X)$. The corresponding Dirac operator is given by $D + \Gamma(\alpha)I$ and hence the unperturbed Seiberg-Witten equations for the spin^c structure (S, I, Γ) take the form

$$D\varphi + \Gamma(\alpha)I\varphi = 0, \qquad d^+\alpha + \sigma_I(\varphi) = 0, \qquad d^*\alpha = 0. \tag{9.20}$$

Note, in particular, that the second equation can be written in the form $id^+\alpha = -i\sigma_I(\varphi)$ and this corresponds to $F_A^+ = -i\sigma^+(i(\Phi\Phi^*)_0)$.

Consider the subgroup $\mathrm{G} = \mathrm{Pin}(2) \subset \mathrm{Sp}(1)$ which is generated by j and S^1. Explicitly this group is given by

$$\mathrm{Pin}(2) = \{\cos t + i\sin t \,|\, t \in \mathbb{R}\} \cup \{j\cos t + k\sin t \,|\, t \in \mathbb{R}\}. \tag{9.21}$$

This group acts naturally on the space $\Omega^1(X) \times C^\infty(X, S^+)$. The action of j is given by

$$(\alpha, \varphi) \mapsto (-\alpha, J\varphi)$$

and the action of e^{it} by

$$(\alpha, \varphi) \mapsto (\alpha, e^{It}\varphi).$$

Note that $e^{It} = (\cos t)\mathbb{1} + (\sin t)I$ and that the automorphism $(\alpha, \varphi) \mapsto (-\alpha, J\varphi)$ is of order 4. It turns out that the space of solutions of (9.20) is invariant under the action of G. To see this let us introduce the spaces

$$\begin{aligned}\mathcal{X} &= \Omega^1(X) \oplus C^\infty(X, S^+),\\ \mathcal{Y} &= \Omega^0_0(X) \oplus \Omega^{2,+}(X) \oplus C^\infty(X, S^-),\end{aligned}$$

where $\Omega_0^0(X)$ denotes the space of smooth real valued functions on X with mean value zero, and consider the map $\mathcal{F} : \mathcal{X} \to \mathcal{Y}$ defined by

$$\mathcal{F}\begin{pmatrix}\alpha \\ \varphi\end{pmatrix} = \begin{pmatrix} d^*\alpha \\ d^+\alpha \\ D\varphi \end{pmatrix} + \begin{pmatrix} 0 \\ \sigma_I(\varphi) \\ \Gamma(\alpha)I\varphi \end{pmatrix}. \tag{9.22}$$

Then the solution space of (9.20) is the inverse image of zero under $\mathcal{F}$. Now the group $\mathrm{Pin}(2)$ acts on both spaces $\mathcal{X}$ and $\mathcal{Y}$ and $\mathcal{F}$ is equivariant under this action.

Lemma 9.36 *The map $\mathcal{F} : \mathcal{X} \to \mathcal{Y}$ is equivariant under the action of* $\mathrm{Pin}(2)$. *Hence, if (α, φ) satisfies (9.20) then so do the pairs $(-\alpha, J\varphi)$ and $(\alpha, e^{It}\varphi)$ for $t \in \mathbb{R}$.*

Proof: The Dirac operator D commutes with both I and J. Hence

$$DJ\varphi + \Gamma(\alpha)IJ\varphi = J(D\varphi + \Gamma(-\alpha)I\varphi).$$

Note here that $IJ = -JI$ and $\Gamma(\alpha)J = J\Gamma(\alpha)$. Moreover, denote

$$A_I(\varphi) = \varphi\varphi^* I + I\varphi\varphi^* - \frac{1}{2}|\varphi|^2 I$$

and observe that

$$\begin{aligned} A_I(J\varphi) &= (J\varphi)(J\varphi)^* I + I(J\varphi)(J\varphi)^* - \frac{1}{2}|\varphi|^2 I \\ &= -J\varphi\varphi^* JI - IJ\varphi\varphi^* J - \frac{1}{2}|\varphi|^2 I \\ &= J\varphi\varphi^* IJ + JI\varphi\varphi^* J - \frac{1}{2}|\varphi|^2 JIJ \\ &= JA_I(\varphi)J \\ &= -A_I(\varphi). \end{aligned}$$

The last identity follows from the fact that $A_I(\varphi) \in \mathfrak{su}(S^+)$ commutes with J. This implies $\sigma_I(J\varphi) = -\sigma_I(\varphi)$ and hence

$$d^+(-\alpha) + \sigma_I(J\varphi) = -\Big(d^+\alpha + \sigma_I(\varphi)\Big)$$

This proves the equivariance of $\mathcal{F}$ under the action of J. The equivariance under the action of e^{it} follows from the fact that D commutes with e^{It} and that

$$A_I(e^{It}\varphi) = A_I(\varphi).$$

This proves the lemma. □

Proof of Theorem 9.8: We shall only prove the result in the case $b_1 = 0$. The map $\mathcal{F} : \mathcal{X} \to \mathcal{Y}$ extends to a Fredholm map between the Sobolev completions

$$\begin{aligned} \mathcal{X}^{1,p} &= W^{1,p}(X, T^*X) \oplus W^{1,p}(X, S^+), \\ \mathcal{Y}^p &= L^p_0(X) \oplus L^p(X, \Lambda^{2,+}T^*X) \oplus L^p(X, S^-). \end{aligned}$$

We shall drop the superscripts and throughout this proof denote $\mathcal{X} = \mathcal{X}^{1,p}$ and $\mathcal{Y} = \mathcal{Y}^p$. The group $\mathrm{Pin}(2)$ acts on both spaces and, by Lemma 9.36, $\mathcal{F}$ is equivariant under this action. It was Furuta's idea to use a global Kuranishi model as follows. Consider the linear operator

$$\mathcal{D} = d\mathcal{F}(0) : \mathcal{X} \to \mathcal{Y}$$

which is given by the first term on the right in (9.22):

$$\mathcal{D}\begin{pmatrix} \alpha \\ \varphi \end{pmatrix} = \begin{pmatrix} d^*\alpha \\ d^+\alpha \\ D\varphi \end{pmatrix}.$$

Choose a sequence of splittings

$$\mathcal{X} = \mathcal{X}_n \oplus \mathcal{X}'_n, \qquad \mathcal{Y} = \mathcal{Y}_n \oplus \mathcal{Y}'_n,$$

both invariant under the action of I and J, such that $\ker \mathcal{D} \subset \mathcal{X}_n$ and the restriction

$$\mathcal{D}_n : \mathcal{X}'_n \to \mathcal{Y}'_n$$

is a Banach space isomorphism. Moreover, suppose that $\mathcal{X}_n$ and $\mathcal{Y}_n$ are finite dimensional and the projection operators

$$Q_n : \mathcal{Y} \to \mathcal{Y}'_n$$

converge to zero in the strong operator topology. Thus

$$\lim_{n\to\infty} Q_n(\xi, \omega, \psi) = 0$$

for all $(\xi, \omega, \psi) \in \mathcal{Y}$. For example, such splittings can be constructed by means of a splitting of $\mathcal{X}$ into eigenspaces of the operator $\mathcal{D}^*\mathcal{D}$ and a splitting of $\mathcal{Y}$ into corresponding eigenspaces of $\mathcal{D}\mathcal{D}^*$. Now consider the map $\psi_n : \mathcal{X} \to \mathcal{X}$ defined by

$$\psi_n = \mathrm{id}_{\mathcal{X}} + {\mathcal{D}_n}^{-1} Q_n \widehat{\mathcal{F}}$$

where $\widehat{\mathcal{F}} = \mathcal{F} - \mathcal{D} : \mathcal{X} \to \mathcal{Y}$ is given by the second column on the right in (9.22). With $b_1 = 0$ it follows from Theorem 7.12 that the space of solutions of (9.20) is compact.* Hence there exists a number $R > 0$ such that

$$\|(\alpha,\varphi)\|_{W^{1,p}} \geq R \qquad \Longrightarrow \qquad \mathcal{F}(\alpha,\varphi) \neq 0.$$

Consider the linear operators

$$d\psi_n(\alpha,\varphi) - \mathrm{id}_{\mathcal{X}} = \mathcal{D}_n{}^{-1} Q_n d\widehat{\mathcal{F}}(\alpha,\varphi) : \mathcal{X} \to \mathcal{X}.$$

It follows from Rellich's theorem that the operators $d\widehat{\mathcal{F}}(\alpha,\varphi) : \mathcal{X} \to \mathcal{Y}$ are uniformly compact in the ball $B^{\mathcal{X}}_{3R} \subset \mathcal{X}$ of radius $3R$. Hence, by Lemma B.12, the operators $d\psi_n(\alpha,\varphi) - \mathrm{id}_{\mathcal{X}}$ converge to zero in the norm topology, uniformly on the ball $B^{\mathcal{X}}_{3R}$. Hence there exists an integer n such that

$$\|d\psi_n(\alpha,\varphi) - \mathrm{id}_{\mathcal{X}}\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})} \leq \frac{1}{2} \qquad \text{for all} \qquad (\alpha,\varphi) \in B^{\mathcal{X}}_{3R}.$$

By Lemma B.2 this implies that the smooth map ψ_n is injective on the ball $B^{\mathcal{X}}_{3R}$ with a smooth inverse and

$$\psi_n\left(B^{\mathcal{X}}_{R}\right) \subset \mathcal{B}_{3R/2}x^{\mathcal{X}} \subset \psi_n\left(B^{\mathcal{X}}_{3R}\right).$$

Consider the map $f_n : B^{\mathcal{X}_n}_{3R} \to \mathcal{Y}_n$ defined by

$$f_n = (\mathrm{id}_{\mathcal{Y}} - Q_n) \circ \left(D + \widehat{\mathcal{F}} \circ \psi_n{}^{-1}\right).$$

This map is equivariant under the action of Pin(2) and it satisfies the equation

$$\mathcal{F} \circ \psi_n{}^{-1} = Q_n \mathcal{D} + f_n. \tag{9.23}$$

If $(\alpha,\varphi) \in \mathcal{X}_n$ satisfies $\|(\alpha,\varphi)\|_{W^{1,p}} \geq 3R/2$ then $\left\|\psi_n{}^{-1}(\alpha,\varphi)\right\|_{W^{1,p}} \geq R$ and hence $\mathcal{F} \circ \psi_n{}^{-1}(\alpha,\varphi) \neq 0$. Moreover, $Q_n\mathcal{D}(\alpha,\varphi) = 0$ and hence equation (9.23) shows that $f_n(\alpha,\varphi) \neq 0$. Thus f_n never vanishes in $B^{\mathcal{X}_n}_{2R} - B^{\mathcal{X}_n}_{3R/2}$ and so induces a smooth map

$$f : (B\mathcal{X}_n, S\mathcal{X}_n) \to (B\mathcal{Y}_n, S\mathcal{Y}_n)$$

defined by

$$f(\alpha,\varphi) = \beta\left(\|\alpha,\varphi\|\right) f_n(2R\alpha, 2R\varphi) + \Big(1 - \beta\left(\|\alpha,\varphi\|\right)\Big) \frac{f_n(2R\alpha, 2R\varphi)}{\|f_n(2R\alpha, 2R\varphi)\|}.$$

Here $\beta : \mathbb{R} \to \mathbb{R}$ is a cutoff function function satisfying $\beta(r) = 1$ for $r \leq 3/4$ and $\beta(r) = 0$ for $r \geq 7/8$. Here the $W^{1,p}$-norm can be chosen

*The condition $b_1 = 0$ implies that the group $\mathcal{G}_0$ of harmonic gauge transformations is compact, namely it agrees with S^1, and thus compactness of the quotient by $\mathcal{G}_0$ is equivalent to compactness of the total space.

invariant under Pin(2) and then the map f is still equivariant. It is now interesting to return to the definition of the spaces $\mathcal{X}_n$ and $\mathcal{Y}_n$. Recall that the operator $\mathcal{D}$ is the direct sum $\mathcal{D} = D \oplus D^+$ where

$$D : W^{1,p}(X, S^+) \to L^p(X, S^-)$$

is the spin Dirac operator and

$$D^+ : W^{1,p}(X, T^*X) \to L^p_0(X) \oplus L^p(X, \Lambda^{2,+}T^*X)$$

is the self-duality operator. Recall, moreover, that the operator D is equivariant under the action of the quaternions. Hence both spaces $\mathcal{X}_n$ and $\mathcal{Y}_n$ have the form

$$\mathcal{X}_n \cong \mathbb{H}^t \oplus \mathbb{R}^s, \qquad \mathcal{Y}_n \cong \mathbb{H}^r \oplus \mathbb{R}^q$$

with

$$\operatorname{index} D = 4(t - r), \qquad \operatorname{index} D^+ + 1 = s - q$$

This holds because $\mathcal{D}_n : \mathcal{X}'_n \to \mathcal{Y}'_n$ is a Banach space isomorphism. The index of D^+ is given by $-b_0 - b^+$ but the b_0-term has to be discarded because the space $\mathcal{Y}$ does not include the constant functions. Now recall that the intersection form of X is given by $Q_X = 2k(-E_8) + mH$ so that

$$b^+ = m, \qquad b^- = m + 16k.$$

Hence

$$\operatorname{index} D = -\frac{\sigma}{4} = 4k, \qquad \operatorname{index} D^+ + 1 = 1 - \frac{\chi + \sigma}{2} = -m.$$

This shows that the spaces $\mathcal{X}_n$ and $\mathcal{Y}_n$ can be identified with

$$\mathcal{X}_n \cong \mathbb{H}^{r+k} \oplus \mathbb{R}^s, \qquad \mathcal{Y}_n \cong \mathbb{H}^r \oplus \mathbb{R}^{s+m}.$$

Both spaces are representations of the group Pin(2). This group acts on $\mathbb{R}$ by $j \mapsto -1$ and $e^{it} \mapsto 1$ and on $\mathbb{H}$ in the obvious way using $\mathrm{Pin}(2) \subset \mathbb{H}$. The map f is equivariant under this action. There is an induced map between the balls in the corresponding complexified representations and the next Proposition shows that such a map can only exist if either $k = 0$ or $m \geq 2k + 1$. This proves the theorem. □

Proposition 9.37. (Furuta) *Let*

$$V = \mathbb{H}^{cr+k} \oplus \mathbb{C}^s, \qquad W = \mathbb{H}^{cr} \oplus \mathbb{C}^{s+m},$$

and suppose that there exists a smooth map $f : (BV, SV) \to (BW, SW)$ which is equivariant under the action of Pin(2).* *Then either $k = 0$ or*

*As before Pin(2) acts on $\mathbb{H}^c$ in the obvious way and acts on $\mathbb{C}$ by $j \mapsto -1$ and $e^{it} \mapsto 1$.

$$m \geq 2k + 1.$$

This result is reminiscent of the Borsuk-Ulam theorem. The proof was explained to me by Stefan Bauer. It relies on equivariant K-theory and we begin with explaining some necessary background. Let G be a compact Lie group and X be a compact Hausdorff space on which G acts. A G-equivariant (complex) vector bundle over X is a vector bundle $\pi : E \to X$ which carries a G-action such that π is equivariant. The group $K_{\mathrm{G}}(X)$ is defined as the set of equivalence classes $E \ominus F$ of pairs of equivariant vector bundles $E \to X$ and $F \to X$ under the equivalence relation $E \ominus F \equiv E' \ominus F'$ iff there exists an equivariant vector bundle $H \to X$ such that $E \oplus F' \oplus H \cong F \oplus E' \oplus H$. (Here *"$\cong$"* means *"equivariantly isomorphic"*.)

For a pair $A \subset X$ of compact G-spaces the relative K_{G}-group $K_{\mathrm{G}}(X, A)$ is defined as the kernel of the natural homomorphism $K_{\mathrm{G}}(X/A) \to K_{\mathrm{G}}(\mathrm{pt})$. In explicit terms $K_{\mathrm{G}}(X, A)$ is the set of equivalence classes of pairs $E \ominus_{\varphi} F$ of pairs of equivariant vector bundles $E \to X$ and $F \to X$ equipped with an isomorphism $\varphi : E|_A \to F|_A$. The equivalence relation is $E \ominus_{\varphi} F \equiv E' \ominus_{\varphi'} F'$ iff there exists an equivariant vector bundle $H \to X$ and an isomorphism $\psi : E \oplus F' \oplus H \to F \oplus E' \oplus H$ which restricts to the obvious isomorphism $\varphi \oplus {\varphi'}^{-1} \oplus \mathrm{id}$ over A.

For every unitary representation V of G there is a natural equivariant K-theory class

$$\tau_V \in K_{\mathrm{G}}(BV, SV)$$

called the **equivariant Thom class**. It is defined by

$$\tau_V = \Lambda^{0,\mathrm{ev}} V^* \ominus_{\Gamma} \Lambda^{0,\mathrm{odd}} V^*$$

where $\Gamma : SV \to \mathrm{Hom}(\Lambda^{0,\mathrm{ev}} V^*, \Lambda^{0,\mathrm{odd}} V^*)$ denotes the canonical spin^c structure as introduced in Section 4.7.

Exercise 9.38 Let V be a Hermitian vector space and

$$\Gamma : V \to \mathrm{End}(\Lambda^{0,*} V^*)$$

be the canonical spin^c structure defined by (4.35). If V carries a unitary G-action show that Γ is equivariant:

$$\Gamma(g^{-1} v) g^* \tau = g^* \Gamma(\tau)$$

for $v \in V$, $\tau \in \Lambda^{0,*} V^*$, and $g \in \mathrm{G}$. □

Theorem 9.39. (Bott) *Let V be a Hermitian vector space which carries a unitary* G*-action. Then $K_{\mathrm{G}}(BV, SV)$ is naturally isomorphic to the representation ring $\mathcal{R}(\mathrm{G})$ via the homomorphism*

$$\mathcal{R}(\mathrm{G}) \to K_{\mathrm{G}}(BV, SV) : \rho \mapsto \rho \otimes \tau_V.$$

A proof of this result can be found in Atiyah [4]. It is the only deep theorem of equivariant K-theory needed in the proof of Proposition 9.37.

Exercise 9.40 Suppose that the group $\mathrm{G}_0 = S^1$ acts trivially on the Hermitian vector spaces V and W and let $f : (BV, SV) \to (BW, SW)$ be any smooth map. Prove that the induced map

$$f^* : K_{S^1}(BW, SW) \to K_{S^1}(BV, SV)$$

satisfies

$$f^*\tau_W = \deg(f)\tau_V$$

where $\deg(f)$ denotes the degree of the induced map of spheres. In particular, $f^*\tau_W = 0$ whenever V and W do not have equal dimension. □

Exercise 9.41 Prove that the representation ring of $\mathrm{Pin}(2) = \langle j, e^{it}\rangle \subset \mathrm{Sp}(1)$ is naturally isomorphic to the quotient

$$\mathcal{R}(\mathrm{Pin}(2)) \cong \frac{\mathbb{Z}[d,h]}{\langle d^2 = 1, dh = h\rangle}.$$

Hint: Denote by d the representation $\mathbb{C}$ with $j \mapsto -1$ and $e^{it} \mapsto 1$ and by h the obvious representation $\mathbb{H}$. Show that the only 1-dimensional complex representations of $\mathrm{Pin}(2)$ are 1 and d and that every 2-dimensional representation has the form

$$j \mapsto \begin{pmatrix} 0 & (-1)^n \\ 1 & 0 \end{pmatrix}, \qquad e^{it} \mapsto \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}$$

for some integer n. Denote this representation h_n and show that $h_0 = 1+d$, $h_1 = h$, $h_{-n} = h_n$, and

$$h_n h_m = h_{n+m} + h_{n-m}.$$

Finally, show that every complex representation of $\mathrm{Pin}(2)$ has a 1- or 2-dimensional summand and deduce that $\mathcal{R}(\mathrm{Pin}(2))$ is generated by d and h where the only relations are $d^2 = 1$ and $dh = h$. □

Exercise 9.42 **(i)** Let V be a Hermitian vector space with a unitary G-action and suppose that the map $\mathrm{G} \times BV \to \mathrm{Aut}(E_0) : (g, x) \mapsto \psi_g(x)$ satisfies the cocycle condition

$$\psi_h(gx)\psi_g(x) = \psi_{hg}(x)$$

for $x \in BV$ and $g, h \in \mathrm{G}$. Prove that there is a map $\varphi : BV \to \mathrm{Aut}(E_0)$ such that

$$\psi_g(x) = \varphi(gx)^{-1}\psi_g(0)\varphi(x), \qquad \varphi(0) = 1\!\!1.$$

Hint: Consider the cocycles $\psi_g^\lambda(x) = \psi_g(\lambda x)$ for $0 \le \lambda \le 1$. Construct the maps $\varphi^\lambda : BV \to \mathrm{Aut}(E_0)$ via

$$\frac{d}{d\lambda}\varphi^\lambda = \varphi^\lambda A^\lambda$$

where $A^\lambda : BV \to \mathrm{End}(E_0)$ is defined by

$$A^\lambda(x) = \frac{1}{\mathrm{Vol(G)}}\int_{\mathrm{G}} B_g^\lambda(x)d\mu(g), \qquad B_g^\lambda(x) = \psi_g(\lambda x)^{-1}\frac{d}{d\lambda}\psi_g(\lambda x).$$

Here $d\mu$ is a Haar measure on G. Show that

$$A^\lambda(x) = B_g^\lambda(x) + \psi_g^\lambda(x)^{-1}A^\lambda(gx)\psi_g^\lambda(x).$$

(ii) Prove that every equivariant vector bundle $E \to BV$ admits an equivariant trivialization. **Hint:** Let $\Psi_x : E_0 \to E_x$ be any trivialization and define

$$\psi_g(x) = \Psi_{gx}^{-1} \circ g \circ \Psi_x.$$

Choose $\varphi : BV \to \mathrm{Aut}(E_0)$ as in (i) and define $\Phi_x = \psi_x \circ \varphi(x)^{-1} : E_0 \to E_x$.
(iii) Prove that there is a natural isomorphism $K_{\mathrm{G}}(BV) \cong \mathcal{R}(\mathrm{G})$. □

Lemma 9.43. (Furuta) *Let $f : (BV, SV) \to (BW, SW)$ be as in Proposition 9.37 and consider the induced map*

$$f^* : K_{\mathrm{G}}(BW, SW) \to K_{\mathrm{G}}(BV, SV).$$

Let $a_f \in \mathcal{R}(\mathrm{Pin}(2))$ be the unique representation which satisfies

$$f^*\tau_W = a_f \otimes \tau_V$$

(see Theorem 9.39). If $m \ge 1$ then the character $\theta_{a_f} : \mathrm{Pin}(2) \to \mathbb{R}$ satisfies

$$\theta_{a_f}(e^{it}) = 0$$

for every $t \in \mathbb{R}$.

Proof: Abbreviate $\mathrm{G} = \mathrm{Pin}(2)$ and denote by $\mathrm{G}_0 = S^1 \subset \mathrm{Pin}(2)$ the identity component. Note that a_f is a representation of both G and G_0. To avoid confusion we shall use the notation $a_f^0 \in \mathcal{R}(\mathrm{G}_0)$. For any Hermitian G-vector space V let us denote

$$\lambda_V = \Lambda^{0,\mathrm{ev}}V^* \ominus \Lambda^{0,\mathrm{odd}}V^* \in \mathcal{R}(\mathrm{G})$$

and write $\lambda^0_V \in \mathcal{R}(\mathrm{G}_0)$ for the induced representation of $\mathrm{G}_0 = S^1$. In the case at hand consider the splittings

$$V = V_0 \oplus V_1, \qquad W = W_0 \oplus W_1$$

where $V_0 = \mathbb{C}^s$ and $W_0 = \mathbb{C}^{s+m}$ denote the respective subspaces which are fixed under $\mathrm{G}_0 = S^1$ and consequently $V_1 = \mathbb{H}^{cr+k}$ and $W_1 = \mathbb{H}^{cr}$. Note that f preserves the fixed point sets of G_0 and hence there is an induced map

$$f_0 : (BV_0, SV_0) \to (BW_0, SW_0).$$

This shows that there is a commuting diagram

$$\begin{array}{ccccc} \tau_W & \in & K_{\mathrm{G}}(BW, SW) & \xrightarrow{f^*} & K_{\mathrm{G}}(BV, SV) & \ni & a_f \tau_V \\ & & \downarrow & & \downarrow & & \\ \tau^0_W & \in & K_{\mathrm{G}_0}(BW, SW) & \xrightarrow{f^*} & K_{\mathrm{G}_0}(BV, SV) & \ni & a^0_f \tau^0_V \\ & & \downarrow & & \downarrow & & \\ \lambda^0_{W_1}\tau^0_{W_0} & \in & K_{\mathrm{G}_0}(BW_0, SW_0) & \xrightarrow{f_0{}^*} & K_{\mathrm{G}_0}(BV_0, SV_0) & \ni & a^0_f \lambda^0_{V_1} \tau^0_{V_0} \end{array} .$$

By Exercise 9.40 the last map is given by multiplication with the degree of $f_0 : SV_0 \to SW_0$. Hence

$$\lambda^0_{W_1} \deg(f_0) = a^0_f \lambda^0_{V_1}.$$

If $m \geq 1$ then the spheres SV_0 and SW_0 have different dimensions and hence the degree of f_0 is zero. This shows that $a^0_f \lambda^0_{V_1} = 0$. Examining the character of λ_{V_1} one finds that $a^0_f = 0$ as claimed. □

Proof of Proposition 9.37: The representation $a_f \in \mathcal{R}(\mathrm{Pin}(2))$ of Lemma 9.43 satisfies

$$\lambda_W = a_f \lambda_V.$$

This is because $K_{\mathrm{G}}(BV)$ and $K_{\mathrm{G}}(BW)$ are both naturally isomorphic to $\mathcal{R}(\mathrm{G})$ (see Exercise 9.42) and the following diagram commutes

$$\begin{array}{ccccc} \tau_W \in K_{\mathrm{G}}(BW, SW) & & \xrightarrow{f^*} & & K_{\mathrm{G}}(BV, SV) \ni a_f \tau_V \\ \downarrow & & & & \downarrow \\ K_{\mathrm{G}_0}(BW) & & \xrightarrow{f^*} & & K_{\mathrm{G}_0}(BV) \\ & \searrow & & \swarrow & \\ \lambda_W \quad \in & & \mathcal{R}(G) & & \ni \quad a_f \lambda_V \end{array} .$$

Now one checks easily that

$$\lambda_d = 1 - d, \qquad \lambda_h = 2 - h.$$

Using the relations $d^2 = 1$ and $dh = h$ one finds

$$(1-d)^2 = (2-h)(1-d) = 2(1-d).$$

Since $\lambda_{V\oplus W} = \lambda_V \lambda_W$ this implies

$$\lambda_V = 2^{2r+2k+s-1}(1-d), \qquad \lambda_W = 2^{2r+s+m-1}(1-d).$$

Now let us consider the characters. A moment's thought shows that

$$\theta_d(j) = -1, \qquad \theta_h(j) = 0.$$

Hence the identity $\theta_{\lambda_W}(j) = \theta_{a_f}(j)\theta_{\lambda_V}(j)$. takes the form

$$2^{2r+s+m} = 2^{2r+2k+s}\theta_{a_f}(j).$$

Since $\theta_{a_f}(j)$ is an integer this implies $m \geq 2k$. Moreover, we claim that if $k \geq 1$ then $\theta_{a_f}(j) \geq 2$. Firstly, the last equation shows that $\theta_{a_f}(j) \geq 1$. But if $\theta_{a_f}(j) = 1$ then the constant term of a_f (as a polynomial in h) has the form $m(1-d)+1$. Since $\theta_h(i) = 0$ this implies that $\theta_{a_f}(i) = 1$ which, by Lemma 9.43, is only possible in the case $m = 0$. Thus we have proved that $\theta_{a_f}(j) \geq 2$ and this implies

$$2^{2r+s+m} = 2^{2r+2k+s}\theta_{a_f}(j) \geq 2^{2r+2k+s+1}.$$

Hence $m \geq 2k+1$ as claimed. □

Remark 9.44 It was proved by Stolz that if $k \geq 1$ then there exists a smooth map $f : S(\mathbb{H}^{r+k} \oplus \mathbb{R}^s) \to S(\mathbb{H}^r \oplus \mathbb{R}^{s+m})$ which is equivariant under the action of j if and only if

$$m \geq \begin{cases} 2k+1, & \text{if } k \equiv 0 (\text{mod } 4), \\ 2k+1, & \text{if } k \equiv 1 (\text{mod } 4), \\ 2k+2, & \text{if } k \equiv 2 (\text{mod } 4), \\ 2k+3, & \text{if } k \equiv 3 (\text{mod } 4). \end{cases}$$

In particular, this implies that $m \geq 3k$ when $k = 1, 2, 3$ and thus recovers Kronheimer's result about the 11/8 conjecture in these cases. □

10

SEIBERG-WITTEN INVARIANTS OF THREE-MANIFOLDS

10.1 The Seiberg-Witten equations in dimension three

Chern-Simons-Dirac functional

Let Y be a compact oriented smooth 3-manifold and $\gamma : TY \to \mathrm{End}(W)$ be a spin^c structure (see page 165). Fix a spin^c connection $A_0 \in \mathcal{A}(\gamma)$ and consider the **Chern-Simons-Dirac functional**

$$\mathcal{CSD} : \mathcal{A}(\gamma) \times C^\infty(Y, W) \to \mathbb{R}$$

defined by

$$\mathcal{CSD}(A, \Phi) = \frac{1}{2} \int_Y (A_0 - A) \wedge (F_A + F_{A_0}) - \frac{1}{2} \int_Y \langle D_A \Phi, \Phi \rangle \,\mathrm{dvol} \tag{10.1}$$

for $A \in \mathcal{A}(\gamma)$ and $\Phi \in C^\infty(Y, W)$.

Lemma 10.1 *The differential of* $\mathcal{CSD}$ *is given by*

$$\begin{aligned} d\mathcal{CSD}(A, \Phi)(\alpha, \varphi) &= -\int_Y F_A \wedge \alpha - \int_Y \left(\frac{1}{2} \langle \Phi, \gamma(\alpha)\Phi \rangle + \langle D_A \Phi, \varphi \rangle \right) \mathrm{dvol} \\ &= \int_Y \Big(\langle \gamma(*F_A) - (\Phi\Phi^*)_0, \gamma(\alpha) \rangle - \langle D_A \Phi, \varphi \rangle \Big) \,\mathrm{dvol}. \end{aligned}$$

for $\alpha \in \Omega^1(Y, i\mathbb{R})$ *and* $\varphi \in C^\infty(Y, W)$.

Proof: Every $\alpha \in \Omega^1(Y, i\mathbb{R})$ satisfies

$$\int_Y (A - A_0) \wedge d\alpha = \int_Y d(A - A_0) \wedge \alpha = \int_Y (F_A - F_{A_0}) \wedge \alpha.$$

Hence

$$\begin{aligned} d\mathcal{CSD}(A, \Phi)(\alpha, \varphi) = &- \frac{1}{2} \int_Y (F_A + F_{A_0}) \wedge \alpha - \frac{1}{2} \int_Y (A - A_0) \wedge d\alpha \\ &- \frac{1}{2} \int_Y (\langle D_A \Phi, \varphi \rangle - \langle \Phi, D_A \varphi \rangle) - \frac{1}{2} \int_Y \langle \Phi, \gamma(\alpha)\Phi \rangle \end{aligned}$$

$$\begin{aligned}
&= -\int_Y F_A \wedge \alpha - \int_Y \langle (\Phi\Phi^*)_0, \gamma(\alpha)\rangle - \int_Y \langle D_A\Phi, \varphi\rangle \\
&= \int_Y \langle *F_A, \alpha\rangle - \int_Y \langle (\Phi\Phi^*)_0, \gamma(\alpha)\rangle - \int_Y \langle D_A\Phi, \varphi\rangle \\
&= \int_Y \langle \gamma(*F_A) - (\Phi\Phi^*)_0, \gamma(\alpha)\rangle - \int_Y \langle D_A\Phi, \varphi\rangle .
\end{aligned}$$

The second equation uses the formula $\langle \Phi, \gamma(\alpha)\Phi\rangle = 2\langle \gamma(\alpha), (\Phi\Phi^*)_0\rangle$ of Lemma 7.4. The sign change in the third equation arises from the fact that the 1-forms α and $*F_A$ are imaginary valued. □

It follows from Lemma 10.1 that

$$d\mathcal{CSD}(u^*A, u^{-1}\Phi)(\alpha, u^{-1}\varphi) = d\mathcal{CSD}(A, \Phi)(\alpha, \varphi)$$

and

$$d\mathcal{CSD}(A, \Phi)(d\xi, -\xi\Phi) = 0$$

for $u \in \mathcal{G} = \mathrm{Map}(Y, S^1)$ and $\xi \in \Omega^0(Y, i\mathbb{R})$. Hence the 1-form $d\mathcal{CSD}$ descends to the quotient space

$$\mathcal{C}(\gamma) = \frac{\mathcal{A}(\gamma) \times C^\infty(Y, W)}{\mathrm{Map}(Y, S^1)}.$$

However, $\mathcal{CSD}$ is not invariant under the action of the gauge group but only under the identity component. Recall that every function $u : Y \to S^1$ determines a cohomology class $[\alpha_u] \in H^1(Y; \mathbb{Z})$ represented by the closed 1-form

$$\alpha_u = \frac{1}{2\pi i} u^{-1} du.$$

Lemma 10.2 *For $A \in \mathcal{A}(\Gamma)$, $\Phi \in C^\infty(Y, W)$, and $u \in \mathrm{Map}(Y, S^1)$*

$$\mathcal{CSD}(A, \Phi) - \mathcal{CSD}(u^*A, u^{-1}\Phi) = 2\pi^2 [\alpha_u] \cdot c_1(W)$$

where $\cdot$ denotes the cup-product followed by evaluation on the fundamental class of Y.

Proof: The integrand $\langle \Phi, D_A\Phi\rangle$ does not change under the transformation $(A, \Phi) \mapsto (u^*A, u^{-1}\Phi)$. Hence

$$\begin{aligned}
\mathcal{CSD}(A, \Phi) - \mathcal{CSD}(u^*A, u^{-1}\Phi) &= \frac{1}{2}\int_Y (u^*A - A_0) \wedge (F_A + F_{A_0}) \\
&\quad - \frac{1}{2}\int_Y (A - A_0) \wedge (F_A + F_{A_0}) \\
&= \frac{1}{2}\int_Y u^{-1}du \wedge (F_A + F_{A_0})
\end{aligned}$$

$$
\begin{aligned}
&= \int_Y u^{-1}du \wedge F_{A_0} \\
&= 2\pi^2 \int_Y \left(\frac{1}{2\pi i}u^{-1}du\right) \wedge \left(\frac{i}{\pi}F_{A_0}\right) \\
&= 2\pi^2[\alpha_u] \cdot c_1(W).
\end{aligned}
$$

The last equality follows from the fact that $2F_{A_0}$ is the curvature of a connection on the line bundle $\det(W)$ and hence iF_{A_0}/π represents the first Chern class of W. □

Critical points

Lemma 10.1 shows that a pair $(A, \Phi) \in \mathcal{A}(\gamma) \times C^\infty(Y, W)$ is a critical point of $\mathcal{CSD}$ if and only if it satisfies the **three dimensional Seiberg-Witten equations**

$$D_A\Phi = 0, \qquad *F_A = \gamma^{-1}((\Phi\Phi^*)_0).$$

Here we identify TY with T^*Y and think of γ as an isomorphism $T^*Y \otimes_{\mathbb{R}} \mathbb{C} \to \mathrm{End}_0(W)$. Let $\eta \in \Omega^2(Y, i\mathbb{R})$ and consider the perturbed equation

$$D_A\Phi = 0, \qquad *(F_A + \eta) = \gamma^{-1}((\Phi\Phi^*)_0). \tag{10.2}$$

These are the critical points of the perturbed functional $\mathcal{CSD}_\eta : \mathcal{A}(\gamma) \times C^\infty(Y, W) \to \mathbb{R}$ given by

$$\mathcal{CSD}_\eta(A, \Phi) = \mathcal{CSD}(A, \Phi) - \int_Y (A - A_0) \wedge \eta.$$

The perturbation $A \mapsto \int_Y (A - A_0) \wedge \eta$ descends to the confiduration space $\mathcal{C}(\gamma)$ if and only if η is exact. The next lemma shows that (10.2) can only have solutions if η is closed.

Lemma 10.3 *If (10.2) has a solution (A, Φ) then η is closed.*

Proof: By Lemma 6.11 (i), $\gamma^{-1}((\Phi\Phi^*)_0) = \langle\gamma(\cdot)\Phi, \Phi\rangle/2$ and hence, by Lemma 6.9,

$$d^*\gamma^{-1}((\Phi\Phi^*)_0) = \frac{1}{2}\langle\Phi, D_A\Phi\rangle - \frac{1}{2}\langle D_A\Phi, \Phi\rangle = i\mathrm{Im}\,\langle\Phi, D_A\Phi\rangle = 0.$$

Hence η is closed. □

The next lemma gives a universal a priori estimate for the critical points of $\mathcal{CSD}_\eta$.

Lemma 10.4 *Every solution (A, Φ) of (10.2) with $\Phi \not\equiv 0$ satisfies*

$$\sup_X |\Phi|^2 \le \sup_X \left(2|\eta| - \frac{s}{2}\right).$$

Proof: By Theorem 6.19 and Exercise 4.57 the Weitzenböck formula for Dirac operators on a 3-manifold has the form

$$D_A D_A \Phi = \nabla_A{}^* \nabla_A \Phi + \frac{s}{4}\Phi + \gamma(*F_A)\Phi,$$

Moreover, as in the proof of Lemma 7.13,

$$\Delta|\Phi|^2 = -2|\nabla_A \Phi|^2 + 2\mathrm{Re}\,\langle \Phi, \nabla_A{}^* \nabla_A \Phi\rangle\,.$$

Hence the Weitzenböck formula with $D_A\Phi = 0$ shows that

$$\begin{aligned}\Delta|\Phi|^2 &\leq 2\mathrm{Re}\,\langle \Phi, \nabla_A{}^*\nabla_A\Phi\rangle \\ &= -\,2\,\langle \Phi, \gamma(*F_A)\Phi\rangle - \frac{s}{2}|\Phi|^2 \\ &= 2\,\langle \Phi, \gamma(\eta)\Phi - (\Phi\Phi^*)_0\Phi\rangle - \frac{s}{2}|\Phi|^2 \\ &\leq \left(2|\eta| - \frac{s}{2}\right)|\Phi|^2 - |\Phi|^4.\end{aligned}$$

Now let $y_0 \in Y$ be a point at which the function $y \mapsto |\Phi(y)|^2$ attains its maximum. At such a point $\Delta|\Phi|^2 = -\sum_i \partial_i\partial_i|\Phi|^2 \geq 0$, and hence either $\Phi \equiv 0$ or

$$|\Phi(y_0)|^2 \leq 2|\eta(y_0)| - \frac{s(y_0)}{2}.$$

This proves the lemma. □

This result can be used to prove that the set of critical points of $\mathcal{CSD}_\eta$ is compact in the quotient space $\mathcal{C}(\gamma)$. If $b_1(Y) > 0$ we shall prove that $\mathcal{CSD}_\eta$ is a Morse function for a generic closed perturbation η. Then the Seiberg-Witten invariant of (Y, γ) can be defined by counting the critical points of $\mathcal{CSD}_\eta$.

We shall prove that for a generic exact perturbation the set of critical points with $\Phi \neq 0$

The Hessian

The augmented Hessian of the Chern Simons functional at a critical point (A, Φ) is the linear operator

$$\mathcal{H}_{A,\Phi}: \begin{array}{c}\Omega^1(Y, i\mathbb{R}) \\ \oplus \\ \Omega^0(Y, i\mathbb{R}) \\ \oplus \\ C^\infty(Y, W)\end{array} \to \begin{array}{c}\Omega^1(Y, i\mathbb{R}) \\ \oplus \\ \Omega^0(Y, i\mathbb{R}) \\ \oplus \\ C^\infty(Y, W)\end{array}$$

defined by

$$\mathcal{H}_{A,\Phi}\begin{pmatrix}\alpha\\ \psi\\ \varphi\end{pmatrix} = \begin{pmatrix} *d\alpha + d\psi - \gamma^{-1}((\varphi\Phi^* + \Phi\varphi^*)_0) \\ d^*\alpha - i\langle i\Phi, \varphi\rangle \\ -\gamma(\alpha)\Phi - \psi\Phi - \mathcal{D}_A\varphi \end{pmatrix}. \tag{10.3}$$

This is a self-adjoint first order elliptic operator.

Remark 10.5 **(i)** The differential of the gradient

$$(A,\Phi) \mapsto \operatorname{grad}\mathcal{CS}(A,\Phi)$$

is the linear operator

$$\begin{pmatrix}\alpha\\ \varphi\end{pmatrix} \mapsto \begin{pmatrix} *d\alpha - \gamma^{-1}((\varphi\Phi^* + \Phi\varphi^*)_0) \\ -\mathcal{D}_A\varphi - \gamma(\alpha)\Phi \end{pmatrix}.$$

This is the (non-augemted) Hessian of the Chern simons functional and corresponds to the operator (10.3) with $\psi = 0$. At a critical point of $\mathcal{CS}$, the kernel and the cokernel of this operator contain the tangent space of the gauge orbit of the pair (A,Φ), i.e. all pairs (α,φ) which have the form $\alpha = d\psi$, $\varphi = -\psi\Phi$ for some $\psi \in \Omega^0(Y, i\mathbb{R})$. Adding these terms to the Hessian has the effect of removing this part of the cokernel.
(ii) The additional second row in the definition of $\mathcal{H}_{A,\Phi}$ makes the extended Hessian self-adjoint. Its geometric significance is, that $d^*\alpha - i\langle i\Phi,\varphi\rangle = 0$ if and only if the pair (α,φ) is orthogonal to the tangent space of the gauge orbit of (A,Φ).
(iii) Let (A,Φ) be a critical point of $\mathcal{CS}$ with $\Phi \neq 0$. Then every triple $(\alpha,\psi,\varphi) \in \ker \mathcal{H}_{A,\Phi}$ satisfies $\psi = 0$. This follows from the identity

$$\mathcal{H}_{A,\Theta}\mathcal{H}_{A,\Theta}\begin{pmatrix}\alpha\\ \psi\\ \varphi\end{pmatrix} = \begin{pmatrix} \Delta\alpha + |\Phi|^2\alpha - 2i\langle i\nabla_A\Phi, \varphi\rangle \\ \Delta\psi + |\Phi|^2\psi \\ \mathcal{D}_A\mathcal{D}_A\varphi + |\Phi|^2\varphi - 2\nabla_{A,\alpha}\Phi \end{pmatrix}.$$

See Salamon [108] for a proof. It follows that the kernel of the augmented Hessian agrees with the kernel of the actual Hessian $d^2\mathcal{CS}(A,\Phi)$ on the quotient $\Omega^1(Y,i\mathbb{R}) \times C^\infty(Y,W)/\{(d\xi, -\xi\Phi) \,|\, \xi \in \Omega^0(Y,i\mathbb{R})\}$.
(iv) A critical point (A,Φ) of $\mathcal{CS}$ with $\Phi \neq 0$ is called **nondegenerate** if the augmented Hessian $\mathcal{H}_{A,\Phi}$ is injective.
(v) A pair $(A,0)$ (with $\Phi = 0$) is a critical point of $\mathcal{CS}$ if and only if A is a flat connection, i.e. $F_A = 0$. Note that such critical points only exist if $c_1(W)$ is a torsion class. In this case the augmented Hessian is the operator

$$\mathcal{H}_{A,0} = \begin{pmatrix} *d & d & 0 \\ d^* & 0 & 0 \\ 0 & 0 & \mathcal{D}_A \end{pmatrix}$$

The kernel of this operator is the space

$$\ker \mathcal{H}_{A,0} = H^1(Y, i\mathbb{R}) \oplus H^0(Y, i\mathbb{R}) \oplus \ker \mathcal{D}_A.$$

Note here that the term $H^0(Y, i\mathbb{R}) \cong i\mathbb{R}$ corresponds to the tangent space of the isotropy subgroup S^1. A critical point of the form $(A, 0)$ is called **nondegenerate** if $H^1(Y, i\mathbb{R}) = 0$ and $\ker \mathcal{D}_A = 0$.
(vi) Suppose that Y is a rational homology 3-sphere with positive scalar curvature. Then it follows from Lemma 10.4 that every critical point (A, Φ) of $\mathcal{CS}$ satisfies $\Phi = 0$. Moreover, $H^2(Y, \mathbb{Z})$ consists only of torsion classes and so every line bundle $L \to Y$ admits a flat connection. Since $H^1(Y, i\mathbb{R}) = 0$, this flat connection is unique up to gauge equivalence. Moreover, if $F_A = 0$, then the Weitzenböck formula reduces to $\mathcal{D}_A\mathcal{D}_A = \nabla_A{}^*\nabla_A + s/4$, and in the case $s > 0$ this implies that

$$\ker \mathcal{H}_{A,0} = H^0(Y, i\mathbb{R}).$$

In summary, if Y is a rational homology 3-sphere Y with positive scalar curvature, then for every spin^c structure $\gamma : TY \to \mathrm{End}(W)$ the Seiberg-Witten Chern-Simons functional has a unique critical point up to gauge equivalence, and this critical point is nondegenerate.

Gradient flow lines

It follows from Lemma 10.1 that

$$\operatorname{grad} \mathcal{CS}(A, \Phi) = \begin{pmatrix} *F_A - \gamma^{-1}((\Phi\Phi^*)_0) \\ -D_A\Phi \end{pmatrix} \tag{10.4}$$

where $\gamma^{-1} : \mathrm{End}_0(W) \to \Omega^1(Y, \mathbb{C})$ assigns to every traceless Hermitian endomorphism of W an imaginary valued 1-form on Y. Hence the (negative) gradient flow lines of $\mathcal{CS}$ are paths $\mathbb{R} \mapsto \mathcal{A}(\gamma) \times C^\infty(Y, W) : t \mapsto (A(t), \Phi(t))$ which satisfy

$$\dot{\Phi} = D_A\Phi, \qquad \gamma(\dot{A} + *F_A) = (\Phi\Phi^*)_0.$$

Monopoles on tubes

The purpose of this section is to examine the solutions of the Seiberg-Witten equations on tubes $Y \times \mathbb{R}$ where Y is a compact oriented 3-manifold without boundary. The results of this section will also be used in the proof of the Thom conjecture in Section 14.2.

Seiberg-Witten equations on tubes

Remark 10.6 Every spin^c structure $\gamma : Y \to \mathrm{End}(W)$ determines a spin^c structure on $X = \mathbb{R} \times Y$. This is the map $\Gamma : TX \to \mathrm{End}(S)$ with $S = W \oplus W$ defined by

$$\Gamma(\tau, v) = \begin{pmatrix} 0 & \gamma(v) + \tau\mathbb{1} \\ \gamma(v) - \tau\mathbb{1} & 0 \end{pmatrix}$$

for $v \in TY$ and $\tau \in \mathbb{R}$. (See the proof of Theorem 5.16.) □

Exercise 10.7 A 2-form $\eta \in \Lambda^2 T^*X$ on $X = \mathbb{R} \times Y$ has the form $\eta = \beta + \alpha \wedge dt$ where $\alpha \in T^*Y$ and $\beta \in \Lambda^2 T^*Y$. Consider the map $\rho : \Lambda^2 T^*X \to \mathrm{End}(S)$, as defined in (4.39) with the spin^c structure Γ of Remark 10.6. Prove that this map is given by

$$\rho(\beta + \alpha \wedge dt) = \begin{pmatrix} \gamma(*\beta - \alpha) & 0 \\ 0 & \gamma(*\beta + \alpha) \end{pmatrix}$$

for $\alpha \in T^*Y$ and $\beta \in \Lambda^2 T^*Y$. Deduce that

$$\rho^+(\beta + \alpha \wedge dt) = \gamma(*\beta - \alpha).$$

and hence $\rho^+(\beta + \alpha \wedge dt) = 0$ if and only if $\beta = *\alpha$. □

Exercise 10.8 Prove that

$$*_4(\beta + \alpha \wedge dt) = - *_3 \alpha - *_3\beta \wedge dt$$

for $\alpha \in T^*Y$ and $\beta \in \Lambda^2 T^*Y$. Here $*_4$ denotes the Hodge-$*$-operator on $X = \mathbb{R} \times Y$ and $*_3$ denotes the Hodge-$*$-operator on Y. Deduce that $\beta + \alpha \wedge dt$ is anti-self-dual if and only if $\beta = *\alpha$. □

Every spin^c structure on Y induces a spin^c structure on the 4-manifold $X = \mathbb{R} \times Y$ as in Remark 10.6. With this convention a spin^c connection on X has the form $A(t) + \Psi(t)dt$ where $A(t) \in \mathcal{A}(\gamma)$ and $\Psi(t) \in \Omega^0(Y, i\mathbb{R})$ for every t. Thus one can think of the map

$$t \mapsto A(t) + \Psi(t)dt$$

as a smooth path in $\mathcal{A}(\gamma) \times \Omega^0(Y, i\mathbb{R})$. The Dirac operator for this connection is given by

$$\Phi \mapsto -\nabla_t \Phi + D_A \Phi$$

for $\Phi \in C^\infty(\mathbb{R} \times Y, W)$ where

$$\nabla_t \Phi = \dot{\Phi} + \Psi\Phi.$$

Throughout $\dot{\Phi}$ abbreviates the t-derivative $\partial\Phi/\partial t = \dot{\Phi}$. Note that both Φ and A depend on t but this t-dependence is not mentioned explicitly in the notation. The curvature form of $A + \Psi dt$ is given by

$$F_{A+\Psi dt} = F_A + (d\Psi - \dot{A}) \wedge dt.$$

Hence it follows from Exercise 4.57 that the unperturbed Seiberg-Witten equations on the tube $X = \mathbb{R} \times Y$ with respect to the product metric take the form

$$\nabla_t\Phi = D_A\Phi, \qquad \gamma(\dot{A} - d\Psi + *F_A) = (\Phi\Phi^*)_0. \tag{10.5}$$

The reader may wish to compare (10.5) with the Seiberg-Witten equations on flat Euclidean space in Section 8.1.

Temporal gauge

The equation (10.5) is again invariant under gauge transformations. Note here that a gauge transformation on X is a smooth map $u : X \to S^1$ and hence can be thought of as a smooth 1-parameter family of gauge transformations of Y. The action of such paths

$$\mathbb{R} \to \mathrm{Map}(Y, S^1) : t \mapsto u(t)$$

is given by

$$u^*(A, \Psi, \Phi) = (u^*A, \Psi + u^{-1}\dot{u}, u^{-1}\Phi).$$

Here the notation u^* is used ambiguously. On the left it is to be understood as the action of the entire map $t \mapsto u(t)$ whereas on the right it is to be understood as the pointwise action for every t and could be written more precisely in the form $u(t)^*A(t)$ etc. However, in each case the meaning should be clear from the context. The formulae

$$\frac{d}{dt}u^*A = u^*\dot{A} + d(u^{-1}\dot{u}), \qquad \frac{d}{dt}u^{-1}\Phi = u^{-1}\left(\dot{\Phi} - u^{-1}\dot{u}\Phi\right)$$

show that if (A, Ψ, Φ) is a solution of (10.5) then so is $u^*(A, \Psi, \Phi)$. The gauge transformation can obviously be chosen such that $\Psi + u^{-1}\dot{u} = 0$. Hence assume $\Psi = 0$. Then the equations (10.5) take the form

$$\dot{\Phi} = D_A\Phi, \qquad \gamma(\dot{A} + *F_A) = (\Phi\Phi^*)_0. \tag{10.6}$$

These are the Seiberg-Witten equations on the tube in **temporal gauge**.

Energy

Let $t \mapsto (A(t), \Psi(t), \Phi(t))$ be a smooth path in $\mathcal{A}(\gamma) \times \Omega^0(Y, i\mathbb{R}) \times C^\infty(Y, W)$ defined on the interval $0 \le t \le T$. The energy of such a path is defined by

$$\begin{aligned}
E_{[0,T]}(A, \Psi, \Phi) &= \int_0^T \int_Y \left(|\nabla_t\Phi|^2 + |D_A\Phi|^2 + \left|\dot{A} - d\Psi\right|^2 \right) \\
&\quad + \int_0^T \int_Y |\gamma(*F_A) - (\Phi\Phi^*)_0|^2 \\
&= \int_0^T \int_Y \left(|\nabla_t\Phi|^2 + |\nabla_A\Phi|^2 + \frac{s}{4}|\Phi|^2 + \frac{1}{4}|\Phi|^4 \right) \\
&\quad + \int_0^T \int_Y \left(\left|\dot{A} - d\Psi\right|^2 + |F_A|^2 \right).
\end{aligned}$$

The equality of these two expresssions is a direct consequence of the Weitzenböck formula and the identity

$$\nabla_t D_A\Phi - D_A\nabla_t\Phi = \gamma(\dot{A} - d\Psi)\Phi.$$

The second expression is the Seiberg-Witten action functional. The next proposition shows that the solutions of (10.5) are the minima of the energy functional subject to fixed boundary conditions.

Proposition 10.9 *For every smooth path* $t \mapsto (A(t), \Psi(t), \Phi(t)$ *in* $\mathcal{A}(\gamma) \times \Omega^0(Y, i\mathbb{R}) \times C^\infty(Y, W)$

$$\begin{aligned} E_{[0,T]}(A, \Psi, \Phi) = & \int_0^T \int_Y |\nabla_t\Phi - D_A\Phi|^2 \\ & + \int_0^T \int_Y \left|\gamma(\dot{A} - d\Psi + *F_A) - (\Phi\Phi^*)_0\right|^2 \\ & + 2\mathcal{CS}(A(0), \Phi(0)) - 2\mathcal{CS}(A(T), \Phi(T)). \end{aligned}$$

Proof: The proof is by direct calculation:

$$\begin{aligned} & E_{[0,T]}(A, \Psi, \Phi) - \int_0^T \int_Y |\nabla_t\Phi - D_A\Phi|^2 \text{ dvol} \\ & - \int_0^T \int_Y \left|\gamma(\dot{A} - d\Psi + *F_A) - (\Phi\Phi^*)_0\right|^2 \text{ dvol} \\ & = 2\int_0^T \int_Y \langle\nabla_t\Phi, D_A\Phi\rangle \,\text{dvol} \\ & \quad + 2\int_0^T \int_Y \langle\gamma(\dot{A} - d\Psi), (\Phi\Phi^*)_0 - \gamma(*F_A)\rangle \,\text{dvol} \\ & = \int_0^T \int_Y \Big(2\langle\nabla_t\Phi, D_A\Phi\rangle + \langle\Phi, \gamma(\dot{A} - d\Psi)\Phi\rangle\Big) \,\text{dvol} \\ & \quad + 2\int_0^T \int_Y (\dot{A} - d\Psi) \wedge F_A \\ & = \int_0^T \int_Y \Big(2\,\langle\nabla_t\Phi, D_A\Phi\rangle + \langle\Phi, \nabla_t D_A\Phi - D_A\nabla_t\Phi\rangle\Big) \\ & \quad + 2\int_0^T \int_Y \dot{A} \wedge F_A \\ & = \int_0^T \int_Y \Big(\langle\nabla_t\Phi, D_A\Phi\rangle + \langle\Phi, \nabla_t D_A\Phi\rangle\Big) \,\text{dvol} \\ & \quad + \int_0^T \frac{d}{dt}\left(\int_Y (A - A_0) \wedge (F_A + F_{A_0})\right) \end{aligned}$$

$$
\begin{aligned}
&= -2\int_0^T \frac{d}{dt}\mathcal{CS}(A(t),\Phi(t))\,dt \\
&= 2\mathcal{CS}(A(0),\Phi(0)) - 2\mathcal{CS}(A(T),\Phi(T)).
\end{aligned}
$$

This proves the proposition. $\square$

Proposition 10.9 plays a crucial role in studying the properties of Seiberg-Witten monopoles on 4-manifolds with *long necks*. This analysis is the key step in the proof of the generalized Thom conjecture.

The perturbed energy is given by

$$
\begin{aligned}
E_{\eta,[0,T]}(A,\Psi,\Phi) = &\int_0^T\int_Y \left(|\nabla_t\Phi|^2 + |D_A\Phi|^2 + \left|\dot{A} - d\Psi\right|^2\right) \\
&+ \int_0^T\int_Y |(\Phi\Phi^*)_0 - \gamma(*(F_A+\eta))|^2 .
\end{aligned}
$$

There are two important identities. The first relates this energy to the Seiberg-Witten action functional (7.2) which on the cylinder $Y\times[0,T]$ is given by

$$
\begin{aligned}
E^{SW}_{\eta,[0,T]}(A,\Psi,\Phi) = &\int_0^T\int_Y \left(|\nabla_t\Phi|^2 + |\nabla_A\Phi|^2\right) \\
&+ \int_0^T\int_Y \left(\frac{s}{4}|\Phi|^2 + |(\Phi\Phi^*)_0 - \gamma(*\eta)|^2\right) \\
&+ \int_0^T\int_Y \left(\left|\dot{A} - d\Psi + *\eta\right|^2 + |F_A+\eta|^2\right).
\end{aligned}
$$

The second identity relates the energy $E_{\eta,[0,T]}(A,\Psi,\Phi)$ to the Chern-Simons functional $\mathcal{CS}_\eta$.

Exercise 10.10 Let $\eta\in\Omega^2(Y)$ be independent of t. Prove that for every smooth path $t\mapsto(A(t),\Psi(t),\Phi(t))$ in $\mathcal{A}(\gamma)\times\Omega^0(Y,i\mathbb{R})\times C^\infty(Y,W)$

$$
\begin{aligned}
&E^{SW}_{\eta,[0,T]}(A,\Psi,\Phi) - E_{\eta,[0,T]}(A,\Psi,\Phi) \\
&\quad = 2T\int_Y|\eta|^2 - 2\int_Y(A(T)-A(0))\wedge\eta + 2\int_0^T\int_Y d\Psi\wedge\eta
\end{aligned}
$$

and

$$
\begin{aligned}
&E_{\eta,[0,T]}(A,\Psi,\Phi) \\
&\quad = \int_0^T\int_Y|\nabla_t\Phi - D_A\Phi|^2 + \int_0^T\int_Y\left|\gamma(\dot{A} - d\Psi + *(F_A+\eta)) - (\Phi\Phi^*)_0\right|^2
\end{aligned}
$$

$$+2\mathcal{CS}_\eta(A(0),\Phi(0)) - 2\mathcal{CS}_\eta(A(T),\Phi(T)) - 2\int_0^T \int_Y d\Psi \wedge \eta.$$

If η is closed the last term vanishes and hence the solutions of the perturbed Seiberg-Witten equations

$$\nabla_t \Phi = D_A \Phi, \qquad \gamma(\dot{A} - d\Psi + *(F_A + \eta)) = (\Phi\Phi^*)_0$$

minimize the energy $E_{\eta,[0,T]}(A,\Psi,\Phi)$ with respect to variations with fixed boundary values. Prove that the stationary solutions (A,Φ) (with $\Psi = 0$ and $\dot{A} = 0$, $\dot{\Phi} = 0$) satisfy either

$$\sup_X |\Phi|^2 \le \sup_X \left(2|\eta| - \frac{s}{2}\right)$$

or $\Phi = 0$. □

10.2 Transversality in dimension three

10.3 Invariants of three-manifolds

11

GLUING THEOREMS

The purpose of this chapter is to give a proof of the connected sum and blowup axioms for the Seiberg-Witten invariants. Both results are based on studying the the limiting behaviour of solutions of the Seiberg-Witten equations for a sequence of metrics which pinch the neck to a point. The proof of the vanishing theorem is considerably simpler because it only uses compactness, while the proof of the blowup formula requires subtle estimates for the inverse of the linearized operator. These estimates are Seiberg-Witten analogues of the gluing theorems for ASD instantons by Taubes [114, 115] (see also Donaldson-Kronheimer [21]). The first section begins with the precise formulation of the theorems proved in this chapter. Section 11.2 contains the proof of the vanishing theorem for connected sums where both summands satisfy $b^+ \geq 1$. The next four sections are of a preparatory nature. Section 10.1 examines the Seiberg-Witten equations on tubes $Y \times \mathbb{R}$ where Y is a 3-manifold. Section 11.3 establishes the existence of limit connections for finite energy solutions on tubes, and Section 11.5 establishes the basic Fredholm theory for 4-manifolds with cylindrical ends. As a result one can define Seiberg-Witten invariants for smooth 4-manifolds with boundary (or cylindrical ends) provided that all the boundary components have metrics with positive scalar curvature. In the case where the boundary components are spheres these invariants agree with the invariants of the corresponding closed manifold. The proof relies on the gluing theorem established in Section 11.6. This gluing theorem also gives rise to a proof of the blowup formula which is carried out in Section 11.7. We point out that the techniques developed in this chapter will also play an important role in the definition of Seiberg-Witten Floer homology.

11.1 Seiberg-Witten invariants for connected sums

Theorem 11.1 *Suppose that X is a compact oriented 4-manifold diffeomorphic to the connected sum $X_1 \# X_2$ where*

$$b^+(X_1) \geq 1, \qquad b^+(X_2) \geq 1,$$

and $b^+(X) - b_1(X)$ is odd. Then the Seiberg-Witten invariants of X are all zero.

This result was announced by Taubes and others in [116] and was also proved by Witten in his lecture on 6 December 1994 at the Isaac Newton

Institute in Cambridge. The proof given below was sketched by Donaldson in [20].

Theorem 11.2 *Let X and N be compact oriented smooth 4-manifolds with $b^+(X) \geq 2$, $b^+(N) = 0$, $b_1(N) = 0$, and consider the oriented connected sum $X' = X\#N$ with spin^c structure $\Gamma' = \Gamma\#\Gamma_N$. Suppose that*

$$c \cdot c - 2\chi(X) - 3\sigma(X) + e \cdot e + b_2(N) \geq 0 \tag{11.1}$$

where $c = c_1(L_\Gamma) \in H^2(X, \mathbb{Z})$ and $e = c_1(L_{\Gamma_N}) \in H^2(N, \mathbb{Z})$. Then

$$\mathrm{SW}(X', \Gamma') = \mathrm{SW}(X, \Gamma).$$

In particular, the basic classes of X' have the form $c' = c + e$ where $c \in H^2(X, \mathbb{Z})$ is a basic class of X and $e \in H^2(N, \mathbb{Z})$ is a characteristic vector.

Recall from Donaldson's theorem 9.6 that the intersection form of N is diagonalizable over the integers and hence $H^2(N, \mathbb{Z})$ has an integral basis $e_1, \ldots, e_m$ with $e_i \cdot e_i = -1$. For such a basis the vector $e = \sum_i k_i e_i$ is characteristic if and only if all the integers k_i are odd. Note that any characteristic vector satisfies

$$e \cdot e + b_2(N) \leq 0. \tag{11.2}$$

The proof of Theorem 11.2 is based on choosing a sequence of metrics which pinch the neck. Consider a spin^c structure Γ' on $X' = X\#N$ which restricts to spin^c structures Γ on X and Γ_N on N. With c and e denoting the characteristic classes of the spin^c structures Γ and Γ_N, respectively, we obtain that the virtual dimensions of the moduli spaces $\mathcal{M}' = \mathcal{M}(X', \Gamma')$ and $\mathcal{M} = \mathcal{M}(X, \Gamma)$ are related by

$$\dim \mathcal{M}' = \dim \mathcal{M} + \frac{e \cdot e + b_2(N)}{4}.$$

The last term on the right is the real index of the Dirac operator on N. Since e satisfies (11.2) it follows that $\dim \mathcal{M} \geq \dim \mathcal{M}'$ and thus condition (11.1) asserts that both moduli spaces have nonnegative dimension. One can use a standard gluing argument to show that under this condition $\mathrm{SW}(X', \Gamma') = \mathrm{SW}(X, \Gamma)$. The gluing argument is due to Taubes [114, 115] in the case of anti-self-dual instantons with nonabelian structure groups and a proof for that case can also be found in Donaldson-Kronheimer [21], pp 287–295. For the Seiberg-Witten case the gluing argument will be described below.

11.2 Proof of the vanishing theorem

The goal of this section is to give a proof of Theorem 11.1 about the vanishing of the Seiberg-Witten invariants for connected sums

$$X = X_1 \# X_2$$

where $b^+(X) - b_1(X)$ is odd and $b^+(X_1) \geq 1$, $b^+(X_2) \geq 1$. The proof given here was outlined by Donaldson in [20]. It is based on choosing a sequence of metrics g_ν on the connected sum $X_1 \# X_2$ which *pinches* the neck to a point and has the property that the scalar curvature s_ν is bounded below by a constant independent of ν. Note, however, that the scalar curvature will diverge to $+\infty$ near the *pinched neck*. More precisely, the following remark shows how to construct a metric on the unit disc in $\mathbb{R}^4$ which agrees with the standard metric outside a ball of radius δ and with the pullback metric from $\mathbb{R} \times \rho S^3$ under the diffeomorphism $x \mapsto (\rho \log |x|, \rho x/|x|)$ inside a punctured ball of radius δ^m for some integer m.

Remark 11.3 Consider the diffeomorphism $f : \mathbb{R}^4 - \{0\} \to \mathbb{R} \times \rho S^3$ given by

$$f(x) = \left(\rho \log |x|, \rho \frac{x}{|x|} \right)$$

for $x \neq 0$. The pullback of the standard metric g on $\mathbb{R} \times \rho S^3$ under this diffeomorphism has the form

$$f^* g(\xi, \eta) = \frac{\rho^2}{|x|^2} \langle \xi, \eta \rangle.$$

for $|x| \leq \rho^2$ (see Exercise 2.13). Now choose a function $\lambda : (0,1] \to [1, \infty)$ which satisfies

$$\lambda(r) = \begin{cases} \rho/r & \text{if } r \leq \delta^m, \\ 1 & \text{if } r \geq \delta. \end{cases} \tag{11.3}$$

and consider the metric g_λ on $\mathbb{R}^4 - \{0\}$ given by

$$g_\lambda(\xi, \eta) = \lambda(|x|)^2 \langle \xi, \eta \rangle.$$

For $|x| \leq \delta^m$ this metric agrees with the above pullback metric $f^* g$. By Lemma 2.16, the scalar curvature of g_λ is given by

$$s_\lambda = 6 \frac{\Delta \lambda}{\lambda^3} = -6 \frac{\lambda'' + 3\lambda'/r}{\lambda^3}.$$

One can choose λ decreasing and thus $\lambda'(r) \leq 0$ for all r. It remains to prove that λ can be chosen such that (11.3) is satisfied and, say,

$$\frac{\lambda''(r)}{\lambda(r)} + 3 \frac{\lambda'(r)}{r\lambda(r)} \leq 1. \tag{11.4}$$

Here the constant 1 is an arbitrary choice and can be replaced by any positive number (at the expense of increasing m). We must prove that for every $\delta > 0$ there exists a function $\lambda : [0, 1] \to [0, \infty)$ which satisfies (11.3) and (11.4) for some constant $\rho > 0$. As in [88] and in the proof of Theorem 2.18, consider the function $\alpha = \alpha(r)$ defined by

$$\frac{\lambda'}{\lambda} = -\frac{\alpha}{r}, \qquad \frac{\lambda''}{\lambda} = -\frac{\alpha'}{r} + \frac{\alpha + \alpha^2}{r^2}.$$

Then the conditions (11.3) and (11.4) take the form

$$\frac{\alpha'}{r} + \frac{\alpha(2-\alpha)}{r^2} \geq -1, \qquad \alpha(r) = \begin{cases} 1, & \text{for } r \leq \delta^m, \\ 0, & \text{for } r \geq \delta. \end{cases} \tag{11.5}$$

Introduce the new variable $t \geq 0$ via $r = \delta e^{-t}$ and consider the curve $\gamma(t) = \alpha(\delta e^{-t})$. Then (11.5) translates into

$$\dot{\gamma} \leq (2-\gamma)\gamma + \delta^2 e^{-2t}$$

with $\gamma(t) = 1$ for $t \geq T = \log(\delta^{1-m})$ and $\gamma(t) = 0$ for $t \leq 0$. A solution of the differential equation $\dot{\gamma} = (2-\gamma)\gamma$ is given by the explicit formula

$$\gamma(t) = \frac{2\delta^{2m-2}e^{2t}}{1 + \delta^{2m-2}e^{2t}}.$$

This function satisfies

$$\gamma(0) = \frac{2\delta^{2m-2}}{1+\delta^{2m-2}} \ll 1, \qquad \gamma(T) = \gamma(\log(\delta^{1-m})) = 1.$$

Perturbing γ slightly near $t = 0$ and $t = T$ gives a smooth solution of the required differential inequality provided that m is sufficiently large. □

Exercise 11.4 Prove that if the metric is constructed with the unperturbed function $\gamma(t)$ in Remark 11.3 then

$$\lambda(r) = \frac{r^2 + \delta^{2m}}{r^2 + r^2\delta^{2m-2}}, \qquad \delta^m \leq r \leq \delta,$$

and hence $\rho = \delta^m \lambda(\delta^m) = 2\delta^m/(1 + \delta^{2m-2})$. □

It is convenient to think of the connected sum as follows. Fix two points $x_1 \in X_1$ and $x_2 \in X_2$ and choose a metric g_i on X_i which is flat in a neighbourhood of x_i. Now construct a sequence of manifolds $X_\nu = X_1 \#_\nu X_2$ by removing arbitrarily small discs from X_1 and X_2, centered at x_1 and x_2 respectively, modifying the metrics g_i as in Remark 11.3 above, and then identifying two annuli which are isometric to $[0, 1] \times \rho_\nu S^3$. Given two spin^c structures Γ_1 over X_1 and Γ_2 over X_2 one obtains a corresponding sequence of spin^c structures Γ_ν over X_ν by identifying Γ_1 and Γ_2 in suitable trivializations over the two annuli. Let us choose a sequence of perturbations η_ν on X_ν which vanish near the *neck* and are independent of ν on the complement of the neck. Any such sequence determines two fixed perturbations η_1 and η_2 on X_1 and X_2, respectively, which vanish in the given neighbourhoods of x_1 and x_2. By Theorem 7.16 and Remark 8.18, the perturbation can be chosen such that the moduli spaces $\mathcal{M}(X_1, \Gamma_1, g_1, \eta_1)$ and $\mathcal{M}(X_2, \Gamma_2, g_2, \eta_2)$ are regular.

Assume first that the moduli space $\mathcal{M}(X_\nu, \Gamma_\nu, g_\nu, \eta_\nu)$ is zero dimensional. We prove that this space must be empty for ν sufficiently large. Suppose otherwise that for every ν there exists a solution (A_ν, Φ_ν) of the Seiberg-Witten equations for the metric g_ν and the perturbation η_ν. By Lemma 7.13, the spinors Φ_ν satisfy the inequality

$$\sup_X |\Phi_\nu| \leq -\frac{1}{2} \inf_X s_\nu.$$

where s_ν denotes the scalar curvature of g_ν. The previous exercise shows that there exists a constant $c > 0$ such that $s_\nu(x) \geq -c$ for all $x \in X$ and all ν. Hence the Φ_ν are uniformly bounded. Now A_ν and Φ_ν restrict to solutions of the Seiberg-Witten equations on X_1 (for the metric g_1 and the perturbation η_1) outside any neighbourhood of x_1. Hence it follows from the compactness theorem 7.12 that there exists a subsequence which converges in the C^∞-topology on every compact subset of $X_1 - \{x_1\}$ to a solution (A_1, Φ_1) of the Seiberg-Witten equations which is defined on $X_1 - \{x_1\}$ and has finite energy. Since g_1 is flat and η_1 vanishes near x_1 the removable singularity theorem 8.6 asserts that A_1 and Φ_1 extend to a smooth solution over all of X_1. This shows that the moduli space $\mathcal{M}_1 = \mathcal{M}(X_1, \Gamma_1, g_1, \eta_1)$ is nonempty. Obviously, the same argument applies to X_2. Now the perturbation η was chosen such that η_1 and η_2 are regular for g_1 and g_2. But the dimension formula shows that

$$0 = \dim \mathcal{M} = \dim \mathcal{M}_1 + \dim \mathcal{M}_2 + 1.$$

Hence one of the moduli spaces must have negative dimension. Since both moduli spaces are regular it follows that one of them must be empty, a contradiction. This shows that the assumption that $\mathcal{M}(X_\nu, \Gamma_\nu, g_\nu, \eta_\nu)$ was nonempty for all ν must have been false. But if there is a metric for which the moduli space is empty then the Seiberg-Witten invariant is zero. Thus we have proved that the Seiberg-Witten invariant must vanish whenever the moduli space is zero dimensional.

A similar argument applies to the cut-down moduli spaces in the higher dimensional case. More precisely, consider the intersection of the moduli space $\mathcal{M}_1$ with suitable submanifolds of the form

$$\mathcal{N}_h = \left\{ [A, \Phi] \,\Big|\, \int_{X_1} \langle h(A), \Phi \rangle \mathrm{dvol} = 0 \right\} \subset \mathcal{C}(\Gamma_1)$$

where the map $h : \mathcal{A}(\Gamma_1) \to C^\infty(X, W_1^+)^*$ satisfies

$$h(u^* A) = u(y) u^{-1} h(A)$$

for every $u : X_1 \to S^1$ and some $y \in X_1$. The map h can be localized near y as in Exercise 7.27. Now, as before, $\dim \mathcal{M} = \dim \mathcal{M}_1 + \dim \mathcal{M}_2 + 1$ and hence one of the moduli spaces must have dimension strictly smaller than $\mathcal{M}$. Suppose without loss of generality that

$$\dim \mathcal{M}_1 < \dim \mathcal{M} = 2d$$

and choose d functions $h_1, \ldots, h_d : \mathcal{A}(\Gamma_1) \to C^\infty(X, W_1^+)^*$ as above which are localized somewhere on X_1 away from x_1. Then, for a generic perturbation η_1,

$$\mathcal{M}(X_1, \Gamma_1, g_1, \eta_1) \cap \mathcal{N}_{h_1} \cap \cdots \cap \mathcal{N}_{h_d} = \emptyset. \tag{11.6}$$

On the other hand the h_i determine functions $h_{i,\nu} : \mathcal{A}(\Gamma_\nu) \to C^\infty(X, W_\nu^+)^*$ (this is obvious from the explicit construction in Exercise 7.27) and one can examine the moduli spaces $\mathcal{M}(X_\nu, \Gamma_\nu, g_\nu, \eta_\nu) \cap \mathcal{N}_{h_{1,\nu}} \cap \cdots \cap \mathcal{N}_{h_{d,\nu}}$. If these were nonempty for all ν then, by taking the limit $\nu \to \infty$, we would obtain a contradiction to (11.6). Hence these moduli spaces are empty for large ν and thus the Seiberg-Witten invariants are zero.

11.3 Existence of limits

This section establishes the existence of limit connections for finite energy solutions on tubes. We shall assume throughout that Y is a rational homology 3-sphere, equipped with a Riemannian metric with positive scalar curvature, and $\gamma : TY \to \mathrm{End}(W)$ is a spin^c structure on Y compatible with the given metric. We shall consider finite energy solutions of the Seiberg-Witten equations on the half-tube $Y \times [0, \infty)$ in temporal gauge. These are smooth maps $[0, \infty) \to \mathcal{A}(\gamma) \times C^\infty(Y, W) : t \mapsto (A(t), \Phi(t))$ which satisfy (10.6) and have finite energy, i.e.

$$\dot{\Phi} = D_A \Phi, \quad \dot{A} + *F_A = \gamma^{-1}((\Phi\Phi^*)_0), \quad \int_0^\infty \int_Y \left(\left|\dot{\Phi}\right|^2 + \left|\dot{A}\right|^2 \right) < \infty. \tag{11.7}$$

The main theorem of this section asserts that any solution of (11.7) converges exponential to a critical point of the Chern-Simons functional as $t \to \infty$.

Theorem 11.5 *Let Y be a rational homology 3-sphere with a metric of positive scalar curvature, and $\gamma : TY \to \mathrm{End}(W)$ be a spin^c structure.*

Suppose that $[0,\infty) \to \mathcal{A}(\gamma) \times C^\infty(Y,W) : t \mapsto (A(t),\Phi(t))$ *is a smooth solution of (11.7). Then there exists a flat connection* $A_0 \in \mathcal{A}(\gamma)$ *such that*

$$\lim_{t\to\infty} A(t) = A_0, \qquad \lim_{t\to\infty} \Phi(t) = 0.$$

The convergence is exponential in the C^k*-norm for every* k*.*

Remark 11.6 Theorem 11.5 continuous to hold for the perturbed Seiberg-Witten equations and any spin^c structure on any compact Riemannian 3-manifold for which the perturbed Chern-Simons functional has only nondegenerate critical points. However, the case of rational homology 3-spheres with positive scalar curvature suffices for our applications. □

Proof of Theorem 11.5:

11.4 Fredholm theory on four-manifolds with cylindrical ends

This section establishes the basic Fredholm theory for 4-manifolds with cylindrical ends.

11.5 Seiberg-Witten invariants of four-manifolds with cylindrical ends

11.6 Gluing Seiberg-Witten monopoles

11.7 Proof of the blowup formula

Part IV

KÄHLER SURFACES AND SYMPLECTIC MANIFOLDS

12

KÄHLER SURFACES

The goal of this chapter is to explain some of the fundamental properties of the Seiberg-Witten invariants for Kähler surfaces. It was observed already by Seiberg and Witten that Kähler surfaces with $b^+ > 1$ have nontrivial invariants corresponding to the canonical class and hence do not admit metrics of positive scalar curvature. This gave rise to much simpler proofs of earlier theorems by Donaldson distinguishing diffeomorphism types of smooth 4-manifolds with the same intersection form. Witten also proved that all Kähler manifolds have simple type. In fact, it was soon realized by several mathematicians (Tian, Yau, Kronheimer, Mrowka, Morrison, Friedman, Morgan) that for minimal Kähler surfaces of general type the only basic classes are plus and minus the canonical class. This leads to the important conclusion that up to the sign the canonical class is a diffeomorphism invariant and thus settles one of the main conjectures by Friedman and Morgan [29]. Another new result is the theorem by Kotschick that Kähler surfaces are irreducible in the sense that in any connected sum decomposition one of the components is a homology 4-sphere (cf [59]). In fact, this result extends to the symplectic category and it will be discussed in Chapter 13. LeBrun observed that the only minimal Kähler surfaces which do admit metrics of positive scalar curvature are $\mathbb{C}P^2$ and ruled surfaces [70]. He also extended the Miyaoka-Yau inequality to Einstein manifolds with nontrivial Seiberg-Witten invariants [71]. Another link between the Seiberg-Witten invariants and algebraic geometry is the observation, made by Bradlow, Taubes and others, that the moduli space of unperturbed Seiberg-Witten monopoles can be identified with the space of effective divisors. Mrowka used this to compute the Seiberg-Witten invariants for elliptic surfaces. All these results, except for the irreducibility, will be discussed in this chapter. The first section gives a brief review of the Enriques-Kodaira classification. Section 12.1 deals with the special form of the Seiberg-Witten equations in the Kähler case and describes some fundamental properties of their solutions.

12.1 The Enriques-Kodaira classification

The classification of Kähler surfaces was established by Enriques and Kodaira between the mid thirties and the mid fifties of this century. Key ingredients are the canonical bundle $K = K_X = \Lambda^{2,0}T^*X$ and the **Kodaira dimension**

$$\mathrm{Kod}(X) = \limsup_{m\to\infty} \frac{\log \dim H^0(X, \mathcal{O}_X(mK_X))}{\log m}.$$

Here $\mathcal{O}_X(mK_X)$ denotes the sheaf of holomorphic sections of $K^{\otimes m}$ and thus $H^0(X, \mathcal{O}_X(mK_X))$ is the space of global holomorphic sections of $K^{\otimes m}$. Recall that a Kähler surface is called **minimal** if it does not contain any embedded holomorphic 2-sphere C with self-intersection number $C \cdot C = -1$. Recall also that a **ruled surface** is a 2-sphere bundle over a Riemann surface and that a complex surface X is called **elliptic** if there exists a holomorphic map $f : X \to \mathbb{C}P^1$ with generic fiber a 2-torus (and finitely many exceptional fibers). Kodaira and Enriques proved (different parts) of the following classification theorem. A proof can be found in [45], pp 572, or [8, 9].

Theorem 12.1 *Let X be a minimal Kähler surface. Then the Kodaira dimension of X is either $-\infty$, 0, 1, or 2. Moreover, the following holds.*

(i) $\mathrm{Kod}(X) = -\infty$ *if and only if X is either $\mathbb{C}P^2$ or ruled.*

(ii) $\mathrm{Kod}(X) = 0$ *if and only if X is finitely covered by either the 4-torus or the $K3$-surface. In this case $c_1(K)$ is a torsion class.*

(iii) *If* $\mathrm{Kod}(X) = 1$ *then X is elliptic. Moreover, $c_1(K)$ is not a torsion class and*

$$c_1(K) \cdot c_1(K) = 0.$$

(iv) *If* $\mathrm{Kod}(X) = 2$ *then*

$$c_1(K) \cdot c_1(K) > 0$$

Such surfaces are called of **general type**.

Here are some more details about the four cases.

The case $\mathrm{Kod}(X) = -\infty$

This corresponds to the case where the canonical bundle (and any power of it) has no holomorphic sections. Minimal Kähler surfaces with this property are either rational or ruled. Equivalently there exists a Kähler metric with

$$c_1(K) \cdot [\omega] < 0,$$

or a Kähler metric with positive scalar curvature (Yau). Recently, it was proved by LeBrun that if there is any metric with positive scalar curvature then X is rational or ruled (see Theorem 12.14 below). Note that there are three cases $c_1(K)^2 > 0$ ($\mathbb{C}P^2$, $S^2 \times S^2$), $c_1(K)^2 = 0$ (S^2 bundles over $\mathbb{T}^2$), and $c_1(K)^2 < 0$ (S^2 bundles over Riemann surfaces of higher genus). In the first two cases all Kähler structures satisfy $c_1(K) \cdot [\omega] < 0$ while in the last case there are also Kähler structures with $c_1(K) \cdot [\omega] > 0$.

The case $\mathrm{Kod}(X) = 0$

In this case the space $H^0(X, \mathcal{O}_X(mK_X))$ of holomorphic sections of the m-th tensor power of the canonical bundle has dimension either 0 or 1 for every m. This is the case when $c_1(K)$ is a torsion class and, in particular,

$$c_1(K) \cdot c_1(K) = 0, \qquad c_1(K) \cdot [\omega] = 0.$$

The only Kähler surfaces with this property are finite quotients of either the 4-torus or the $K3$-surface. Note that b^+ is either 3 ($\mathbb{T}^4$ or $K3$) or 1 (finite quotients of $\mathbb{T}^4$ or $K3$). A specific example is the **Enriques surface** which can be described as the quotient of the $K3$-surface $\{{z_0}^4 + {z_1}^4 + {z_2}^4 + {z_3}^4\} \subset \mathbb{C}P^3$ by the $\mathbb{Z}_2$-action $z \mapsto \bar{z}$ (see Example 6.28). It has intersection form $Q_X = H \oplus (-E_8)$ and K is a nonzero torsion class with $2K = 0$. For all other Kähler surfaces with $\mathrm{Kod}(X) = 0$ the canonical class is zero.* The finite quotients of $\mathbb{T}^4$ are called **hyperelliptic surfaces**. They are diffeomorphic to $\mathbb{T}^4/\mathbb{Z}_m$ with $m = 2, 3, 4, 6$. To obtain explicit examples think of $\mathbb{T}^4$ as a product of two elliptic curves $E = \mathbb{C}/\mathbb{Z} + i\mathbb{Z}$ and $F = \mathbb{C}/\Lambda$ and let $\mathbb{Z}_m$ act by a translation by $z \mapsto z + 1/m$ on E and by rotation $z \mapsto e^{2\pi i/m} z$ on F. This rotation preserves the lattice $\Lambda = \mathbb{Z} + i\mathbb{Z}$ in the cases $m = 2, 4$ and the lattice $\Lambda = \mathbb{Z} + e^{\pi i/3}\mathbb{Z}$ in the cases $m = 3, 6$. For more details see Beauville [9].

The case $\mathrm{Kod}(X) = 1$

In this case K is a non torsion cohomology class with

$$c_1(K) \cdot c_1(K) = 0, \qquad c_1(K) \cdot [\omega] > 0,$$

and X is an elliptic surface,† i.e. there exists a holomorphic map $f : X \to \mathbb{C}P^1$ whose generic fiber is a 2-torus and with finitely many exceptional fibers.

Examples with $b^+ = 1$ and $b_1 = 0$ can be constructed by adding multiple fibers to the Enriques surface and to the rational elliptic surface $\mathbb{C}P^2 \# 9\overline{\mathbb{C}P}^2$. The latter manifolds, when simply connected, are known as **Dolgachev surfaces**. Recall that simply connected smooth 4-manifolds with even intersection forms are spin and hence, by Rohlin's theorem their signature is divisible by 16. Hence the intersection form $Q_X = H \oplus (-E_8)$ cannot occur in the simply connected case.

*Here one must be careful to distinguish between the canonical *divisor* $\mathcal{K}$ in $\mathrm{Pic}(X)$ which classifies the holomorphic structure of the canonical bundle and the first Chern class $c_1(K)$. In fact $\mathcal{K}$ is a nonzero torsion element for all nontrivial finite Kähler quotients of $\mathbb{T}^4$ or $K3$ while $c_1(K) = 0$ in all cases except for the Enriques surface.

†The converse is not true. $\mathbb{T}^2 \times S^2$ is an elliptic surface with Kodaira dimension $-\infty$ and the $K3$-surface is elliptic with Kodaira dimension 0.

Examples with $b^+ = 1$ and $b_1 = 2$ can be constructed by adding multiple fibers to $S^2 \times \mathbb{T}^2$. Alternatively, one can obtain examples by considering quotients of the form $X = \mathbb{T}^2 \times \Sigma/\mathrm{G}$ where Σ is a Riemann surface of higher genus, and G is a finite group which acts on $\mathbb{T}^2$ by tranlations and on Σ by holomorphic maps such that Σ/G is homeomorphic to S^2. Such surfaces are called **sesquielliptic** (cf. [9]).

The case $\mathrm{Kod}(X) = 2$

Minimal Kähler surfaces with Kodaira dimension 2 satisfy

$$c_1(K) \cdot c_1(K) > 0, \qquad c_1(K) \cdot [\omega] > 0,$$

and they are called **of general type**. The only known simply connected examples with $b^+ = 1$ are the Barlow surfaces with $Q_X = (1) \oplus 8(-1)$ and $K^2 = 1$. It is not known whether the moduli space of Barlow surfaces is connected or indeed whether they are all diffeomorphic.

It is useful to summarize the classification of Kähler surfaces with

$$b^+ = 1, \qquad c_1(K)^2 = 0.$$

The Hirzebruch signature formula in this case takes the form

$$9 - 4b_1 - b^- = c_1(K)^2 = 0.$$

Since $b^+ - b_1$ is odd it follows that b_1 is either zero or 2. When $b_1 = 0$ we have $b^- = 9$ and thus Q_X is either $H \oplus (-E_8)$ or $1 \oplus 9(-1)$. When $b_1 = 2$ we have $b^- = 1$ and thus Q_X is either H or $(1) \oplus (-1)$. The following table summarizes the classification of Kähler surfaces with these properties.

	$b_1 = 0$	$b_1 = 2$
$c_1(K)$ torsion	Enriques surface $2c_1(K) = 0, c_1(K) \neq 0$ $Q_X = H \oplus (-E_8)$	hyperelliptic surfaces $c_1(K) = 0$ $Q_X = H$
$c_1(K)$ not torsion $c_1(K)^2 = 0$	Dolgachev surfaces et al $c_1(K) \cdot [\omega] > 0$	S^2-bundles over $\mathbb{T}^2$ sesquielliptic surfaces

At the time of writing the only known nonKähler symplectic 4-manifolds with $b^+ = 1$ and $c_1(K)^2 = 0$ satisfy $c_1(K) = 0$ and $b_1 = 2$ and thus belong into the box on the right upper corner. Such manifolds are discussed in [23, 24] (see also [86] for a survey). Thus one might ask, for example, the following

Question: *Is every compact symplectic 4-manifold with $b^+ = 1$, $b_1 = 0$, and $c_1(K)$ torsion diffeomorphic to the Enriques surface?*

12.2 The monopole equations in the Kähler case

Let (X, ω, J) be a Kähler surface with the corresponding Kähler metric $g(v, w) = \omega(v, Jw)$. Recall that the tangent bundle carries a canonical spin^c structure

$$W_{\mathrm{can}} = \Lambda^{0,*}T^*X, \qquad L_{\Gamma_{\mathrm{can}}} = K^* = \Lambda^{0,2}T^*X$$

with $\Gamma_{\mathrm{can}} : TX \to \mathrm{End}(W_{\mathrm{can}})$ given by (4.35). Consider the spin^c connection ∇_{can} on W_{can} given by (6.8). By Lemma 6.14, the induced connection on $K^* = L_{\Gamma_{\mathrm{can}}}$ agrees with the Levi-Civita connection of the Kähler metric and the corresponding *virtual connection* is denoted by $A_{\mathrm{can}} \in \mathcal{A}(\Gamma_{\mathrm{can}})$. Its curvature is the 2-form $F_{A_{\mathrm{can}}} = \frac{1}{2}\mathrm{trace}^c(R) \in \Omega^2(X, i\mathbb{R})$ where $R \in \Omega^2(X, \mathrm{End}(TX))$ denotes the Riemann curvature tensor. Recall from Lemma 3.21 that $F_{A_{\mathrm{can}}}$ is of type $(1,1)$ and represents the class $[iF_{A_{\mathrm{can}}}/\pi] = -c_1(K) = c_1(L_{\Gamma_{\mathrm{can}}})$. Now take the tensor product with a Hermitian line bundle $E \to X$ to obtain

$$W_E^+ = (\Lambda^{0,0} \oplus \Lambda^{0,2}) \otimes E, \quad W_E^- = \Lambda^{0,1} \otimes E, \quad L_{\Gamma_E} = K^* \otimes E^2$$

where $\Lambda^{p,q} = \Lambda^{p,q}T^*X$. A Hermitian connection $B \in \mathcal{A}(E)$ induces a spin^c connection $\nabla_A = \nabla_{\mathrm{can}} + B$ on W_E with corresponding *virtual connection* $A = A_{\mathrm{can}} + B \in \mathcal{A}(\Gamma_E)$ and curvature 2-form $F_A = F_{A_{\mathrm{can}}} + F_B$.

Proposition 12.2 *In the Kähler case the Seiberg-Witten equations for the pair $(A_{\mathrm{can}} + B, \Phi)$ and the perturbation $\eta \in i\Omega^{2,+}(X, g)$ take the form*

$$\begin{aligned} \bar{\partial}_B\varphi_0 + \bar{\partial}_B^*\varphi_2 &= 0, \\ 2(F_B + \eta)^{0,2} &= \bar{\varphi}_0\varphi_2, \\ 4i(F_{A_{\mathrm{can}}} + F_B + \eta)_\omega &= |\varphi_2|^2 - |\varphi_0|^2. \end{aligned} \tag{12.1}$$

where $\Phi = (\varphi_0, \varphi_2) \in \Omega^{0,0}(X, E) \times \Omega^{0,2}(X, E)$.

Recall from Section 3.3 that for every 2-form $\tau \in \Omega^2(X, \mathbb{C})$ the function $\tau_\omega : X \to \mathbb{C}$ is defined by

$$\omega \wedge \tau = \tau_\omega \omega \wedge \omega.$$

Thus $\tau_\omega : X \to \mathbb{C}$ is the component of τ in the direction ω. The notation $\bar{\varphi}_0$ has to be handled with care. The section φ_0 takes values in the line bundle E and there is no complex conjugation. However, one can either think of $\bar{\varphi}_0$ as a section of the bundle $\bar{E} = E^*$ with the reversed complex structure and interpret the product $\bar{\varphi}_0\varphi_2$ as the tensor product, or use the Hermitian structure on E and define

$$\bar{\varphi}_0\varphi_2 = \langle \varphi_0 \wedge \varphi_2 \rangle$$

for $\varphi_0 \in \Omega^0(X, E)$ and $\varphi_2 \in \Omega^2(X, E)$.

Proof of Proposition 12.2: Recall from Theorem 6.17 that the Dirac operator of the connection ∇_A is given by $2^{-1/2}D_{A_{\mathrm{can}}+B} = \bar\partial_B + \bar\partial_B^*$. Hence the first equation in (12.1) is equivalent to $D_A\Phi = 0$. That the last two equations are equivalent to $F_{A_{\mathrm{can}}} + F_B + \eta = \sigma^+((\Phi\Phi^*)_0)$ follows from Lemma 4.62. □

A first interesting observation is that for every solution of (12.1) one of the components φ_0 and φ_2 must vanish whenever $\eta \in \Omega^{1,1}(X)$.

Proposition 12.3 *Suppose that X is connected. Let $B \in \mathcal{A}(E)$, $\varphi_0 \in \Omega^{0,0}(X,E)$, and $\varphi_2 \in \Omega^{0,2}(X,E)$ satisfy (12.1) with $\eta \in \Omega^{1,1} \cap \Omega^{2,+}$. Then either $\varphi_0 = 0$ or $\varphi_2 = 0$.*

Proof: Apply the operator $\bar\partial_B$ to the first equation in (12.1) to obtain

$$\bar\partial_B\bar\partial_B^*\varphi_2 = -\bar\partial_B\bar\partial_B\varphi_0 = -F_B^{0,2}\varphi_0 = -\frac{1}{2}|\varphi_0|^2\varphi_2.$$

The last equality follows from the second equation in (12.1) and the fact that $\eta^{0,2} = 0$. Now take the L^2-inner product with φ_2 to obtain

$$\int_X \left(|\bar\partial_B^*\varphi_2|^2 + \frac{1}{2}|\varphi_0|^2|\varphi_2|^2\right) \mathrm{dvol} = 0.$$

This shows that

$$\bar\partial_B^*\varphi_2 = 0, \qquad \bar\partial_B\varphi_0 = 0, \qquad \bar\varphi_0\varphi_2 = 0.$$

Suppose that φ_2 does not vanish everywhere. Then φ_0 must vanish on some open set. But the pair $(\varphi_0, 0)$ is in the kernel of the Dirac operator and hence, by the unique continuation theorem E.8, φ_0 must vanish everywhere. This proves the proposition. □

Note that the last equation in (12.1) determines which of the two components φ_0 or φ_2 has to vanish. Integrating the equation over X and using the fact that $\mathrm{dvol} = \frac{1}{2}\omega\wedge\omega$ one finds that

$$\begin{aligned}
\frac{\|\varphi_2\|^2 - \|\varphi_0\|^2}{2} &= \int_X \frac{|\varphi_2|^2 - |\varphi_0|^2}{4}\omega\wedge\omega \\
&= \int_X i(F_{A_{\mathrm{can}}} + F_B + \eta)\wedge\omega \\
&= \pi(2c_1(E) - c_1(K))\cdot[\omega] + \int_X i\eta\wedge\omega,
\end{aligned}$$

where $c_1(K) = -c_1(TX,J)$. The right hand side is precisely the term $\varepsilon_{\Gamma_E}(g,\eta)$ defined by (7.7) and this is relevant in the case $b^+ = 1$. Now consider the unperturbed case $\eta = 0$.

Corollary 12.4 *Suppose that X is connected and let $(B, \varphi_0, \varphi_2)$ satisfy (12.1) with $\eta = 0$. Then*

$$2c_1(E) \cdot [\omega] < c_1(K) \cdot [\omega] \qquad \Longrightarrow \qquad \varphi_0 \neq 0,\ \varphi_2 = 0,$$

$$2c_1(E) \cdot [\omega] > c_1(K) \cdot [\omega] \qquad \Longrightarrow \qquad \varphi_0 = 0,\ \varphi_2 \neq 0.$$

Recall from Proposition 3.38 that $c_1(K) \cdot [\omega] \geq 0$ for every Kähler surface with $b^+ > 1$. Moreover, by Proposition 3.37, the bundle E can only have a nonzero holomorphic section if $c_1(E) \cdot [\omega] \geq 0$. Likewise, any 2-form $\varphi_2 \in \Omega^{0,2}(X, E)$ which satisfies $\bar{\partial}_B^* \varphi_2 = 0$ determines a holomorphic section of $K \otimes E^*$ and a nonzero such section can only exist if $c_1(E) \cdot [\omega] \leq c_1(K) \cdot [\omega]$. Thus the unperturbed moduli space $\mathcal{M}^*(X, \Gamma_E, g)$ is empty unless

$$0 \leq c_1(E) \cdot [\omega] \leq c_1(K) \cdot [\omega].$$

The midpoint is given by $c_1(E) \cdot [\omega] = \frac{1}{2} c_1(K) \cdot [\omega]$. If $b^+ = 1$ then this is precisely the case where E admits a connection B with $F_B^+ = 0$ and thus $(B, 0, 0)$ is a solution of (12.1) with $\eta = 0$. Corollary 12.4 asserts that on the left of this midpoint the solutions have the form $(B, \varphi_0, 0)$ and on the right they have the form $(B, 0, \varphi_2)$. Suppose for example that $c_1(E) \cdot [\omega] < \frac{1}{2} c_1(K) \cdot [\omega]$. Then $\varphi_2 = 0$ and hence the unperturbed Seiberg-Witten equations (12.1) reduce to the Vortex equations

$$\bar{\partial}_B \varphi_0 = 0, \qquad F_B^{0,2} = 0, \qquad 4i(F_B)_\omega = \tau - |\varphi_0|^2 \tag{12.2}$$

where $\tau = -4i(F_{A_{\mathrm{can}}})_\omega$. These equations and their higher rank analogues were extensively studied by Bradlow [12, 13], Garcia-Prada [36, 37] and many others, before the Seiberg-Witten equations were discovered. In fact, during the early work on his thesis Garcia-Prada wrote down the Seiberg-Witten equations in the general smooth case with spin^c structures, but unfortunately failed to realize the significance of these equations with gauge group U(1) for 4-manifold topology.

12.3 Duality

It is interesting to examine more closely the relation between the spin^c structure $\Gamma_E : TX \to \mathrm{End}(W_E)$ and its dual $\bar{\Gamma}_E : TX \to \mathrm{End}(\bar{W}_E)$ obtained by reversing the complex structure on W_E. This corresponds to replacing the line bundle E with $K \otimes E^*$. Namely, there is a natural isomorphism

$$\bar{W}_E \longrightarrow W_{K \otimes E^*}$$

furnished by the symplectic form ω. Abstractly, for every rank-2 bundle W with Hermitian structure there is an isomorphism $\bar{W} \to W \otimes \det(W)^*$ and this can be used with $W = W_E^+$ and $\det(W)^* = K \otimes E^{-2}$ (see page 227) . It is convenient to identify

$$\bar{W}_E \cong \Lambda^{*,0} T^* X \otimes E^*.$$

With $K = \Lambda^{2,0} T^* X = \Lambda^{2,0}$ the isomorphism $\bar{W}_{\mathrm{can}} \to W_{\mathrm{can}} \otimes K$ is the composition $\Lambda^{k,0} \to \Lambda^{2,2-k} \to \Lambda^{0,2-k} \otimes K$ for $k = 0, 1, 2$ where the first map is induced by the symplectic form $\omega \in \Omega^{1,1}(X)$. More explicitly, given $\varphi_k \in \Omega^{0,k}(X, E)$ denote $\bar{\varphi}_k \in \Omega^{k,0}(X, E^*)$ and then the isomorphism

$$\Omega^{k,0}(X, E^*) \to \Omega^{2,2-k}(X, E^*) \to \Omega^{0,2-k}(X, K \otimes E^*)$$

is given by

$$\bar{\varphi}_k \mapsto \bar{\varphi}_k \wedge \frac{(i\omega)^{2-k}}{(2-k)!} \mapsto \widetilde{\varphi}_k.$$

The last map is the obvious one, but it is important to distinguish notationally between $\Omega^{2,2-k}(X, E^*)$ and $\Omega^{0,2-k}(X, K \otimes E^*)$. Exercise 3.31 shows that this isomorphism respects the Hermitian structure.

Proposition 12.5 *There is a natural bijection*

$$\mathcal{M}(X, \Gamma_E, g, \eta) \to \mathcal{M}(X, \Gamma_{K \otimes E^*}, g, -\eta)$$

given by $(B, \varphi_0, \varphi_2) \mapsto (-B - 2A_{\mathrm{can}}, \widetilde{\varphi}_2, \widetilde{\varphi}_0)$. *Moreover,* η *is regular for* Γ_E *if and only if* $-\eta$ *is regular for* $\Gamma_{K \otimes E^*}$ *and*

$$\mathrm{SW}(X, \Gamma_{K \otimes E^*}) = (-1)^{\frac{\sigma+\chi}{4}} \mathrm{SW}(X, \Gamma_E).$$

If $b^+ = 1$ *then*

$$\mathrm{SW}^+(X, \Gamma_{K \otimes E^*}) = (-1)^{\frac{\sigma+\chi}{4}} \mathrm{SW}^-(X, \Gamma_E).$$

Proof: It is convenient to abbreviate $\widetilde{E} = K \otimes E^*$, $\widetilde{B} = -B - 2A_{\mathrm{can}}$. Consider the commutative diagram

$$\begin{array}{ccccc}
\Omega^{0,0}(X, E^*) & \overset{\wedge (i\omega)^2/2}{\longrightarrow} & \Omega^{2,2}(X, E^*) & \longrightarrow & \Omega^{0,2}(X, K \otimes E^*) \\
\partial_{-B} \downarrow & & \downarrow -\bar{\partial}_{-B}{}^* & & \downarrow -\bar{\partial}_{\widetilde{B}}{}^* \\
\Omega^{1,0}(X, E^*) & \overset{\wedge i\omega}{\longrightarrow} & \Omega^{2,1}(X, E^*) & \longrightarrow & \Omega^{0,1}(X, K \otimes E^*)
\end{array}$$

That the second square commutes is essentially the contents of Proposition 3.23. That the first square commutes can be expressed in the form

$$(\partial_{-B}\bar{\varphi}_0) \wedge i\omega = -\bar{\partial}^*_{-B}\left(\bar{\varphi}_0 \wedge (i\omega)^2/2\right)$$

for $\bar{\varphi}_0 \in C^\infty(X, E^*)$. For $E = \mathbb{C}$ and $B = 0$ this is the formula (3.12) in Lemma 3.32 with $\sigma = \bar{\varphi}_0$ and $k = 2$. In general the formula follows by taking tensor products. There is a similar diagram

$$\begin{array}{ccccc}
\Omega^{2,0}(X,E^*) & = & \Omega^{2,0}(X,E^*) & \longrightarrow & \Omega^{0,0}(X, K\otimes E^*) \\
\partial_{-B}{}^* \downarrow & & \downarrow -\bar{\partial}_{-B} & & \downarrow -\bar{\partial}_{\widetilde{B}} \\
\Omega^{1,0}(X,E^*) & \xrightarrow{\wedge i\omega} & \Omega^{2,1}(X,E^*) & \longrightarrow & \Omega^{0,1}(X, K\otimes E^*)
\end{array}$$

That the second square commutes follows again from Proposition 3.23 and the commutativity of the first square can be expressed in the form

$$(\partial_{-B}{}^*\bar{\varphi}_2) \wedge i\omega = -\bar{\partial}_{-B}\bar{\varphi}_2$$

for $\bar{\varphi}_2 \in \Omega^{2,0}(X, E^*)$. For $E = \mathbb{C}$ and $B = 0$ this is the formula (3.14) in Lemma 3.32 with $\sigma = \bar{\varphi}_2$. Taken together these equations show that

$$\bar{\partial}_B\varphi_0 + \bar{\partial}^*_B\varphi_2 = 0 \qquad \Longleftrightarrow \qquad \bar{\partial}_{\widetilde{B}}\widetilde{\varphi}_2 + \bar{\partial}^*_{\widetilde{B}}\widetilde{\varphi}_0 = 0.$$

For the second and third equations in (12.1) just note that

$$F_{\widetilde{B}} + F_{A_{\mathrm{can}}} - \eta = -\left(F_B + F_{A_{\mathrm{can}}} + \eta\right),$$

$$\langle\widetilde{\varphi}_2 \wedge \widetilde{\varphi}_0\rangle = -\langle\varphi_0 \wedge \varphi_2\rangle \in \Omega^{0,2}(X).$$

The minus sign here results from the factor $(i\omega)^2/2$ in the definition of $\widetilde{\varphi}_0 \in \Omega^{0,2}(X, \widetilde{E})$. This proves that

$$[B, \varphi_0, \varphi_2] \in \mathcal{M}(X, \Gamma_E, g, \eta) \quad \Longleftrightarrow \quad [\widetilde{B}, \widetilde{\varphi}_2, \widetilde{\varphi}_0] \in \mathcal{M}(X, \Gamma_{\widetilde{E}}, g, -\eta).$$

The relation between the signs of the Seiberg-Witten invariants for Γ_E and $\bar{\Gamma}_E = \Gamma_{K\otimes E^*}$ follows from Proposition 7.31. □

12.4 The linearized operator

It is interesting to examine the specific form of the linearized equations in the Kähler case and relate these to the Cauchy-Riemann operator. We shall only consider the case where $\varphi_2 = 0$. The linearized equations at a solution $(B, \varphi_0, 0)$ have the form

$$\begin{aligned}
d^*\alpha - i\langle i\varphi_0, \tau_0\rangle &= 0, \\
-2i(d\alpha)_\omega - \mathrm{Re}\,(\bar{\varphi}_0\tau_0) &= 0,
\end{aligned}$$

$$\begin{aligned} \bar\partial\tau_0 + \bar\partial^*\tau_2 + \alpha^{0,1}\varphi_0 &= 0, \\ 2(d\alpha)^{0,2} - \bar\varphi_0\tau_2 &= 0. \end{aligned} \tag{12.3}$$

Here the first equation asserts that the triple (α, τ_0, τ_2) is L^2-orthogonal to the orbit of $(B, \varphi_0, 0)$ under the action of the gauge group. (See Remark 8.19.) Recall from Corollary 3.28 that

$$d^*\alpha = 2i\mathrm{Im}\,(\bar\partial^*\alpha^{0,1}), \qquad -2i(d\alpha)_\omega = 2\mathrm{Re}\,(\bar\partial^*\alpha^{0,1}).$$

Moreover, note that $\langle i\varphi_0, \tau_0\rangle = \mathrm{Im}\,(\bar\varphi_0\tau_0)$. Thus the first two equations in (12.3) can be expressed in complex notation

$$2\bar\partial^*\alpha_1 - \bar\varphi_0\tau_0 = 0$$

where $\alpha_1 = \alpha^{0,1} \in \Omega^{0,1}(X)$. This shows that the linearized operator has the form

$$\mathcal{D}_{B,\varphi} : \begin{array}{c} \Omega^{0,0}(X,E) \\ \oplus \\ \Omega^{0,1}(X) \\ \oplus \\ \Omega^{0,2}(X,E) \end{array} \longrightarrow \begin{array}{c} \Omega^{0,0}(X) \\ \oplus \\ \Omega^{0,1}(X,E) \\ \oplus \\ \Omega^{0,2}(X) \end{array}$$

where

$$\mathcal{D}_{B,\varphi}\begin{pmatrix} \tau_0 \\ \alpha_1 \\ \tau_2 \end{pmatrix} = \begin{pmatrix} \bar\partial^*\alpha_1 - \bar\varphi_0\tau_0/2 \\ \bar\partial_B\tau_0 + \bar\partial_B^*\tau_2 + \alpha_1\varphi_0 \\ \bar\partial\alpha_1 - \bar\varphi_0\tau_2/2 \end{pmatrix}$$

for $\tau_0 \in \Omega^{0,0}(X,E)$, $\alpha_1 \in \Omega^{0,1}(X)$, and $\tau_2 \in \Omega^{0,2}(X,E)$. In the following it will be convenient to think of the cokernel as the quotient of the target space by the image of $\mathcal{D}_{B,\varphi}$ (rather than the orthogonal complement of the image). Let $\mathcal{O}$ denote the structure sheaf of X and $\mathcal{E}_B$ the sheaf of holomorphic sections of the bundle E with holomorphic structure $\bar\partial_B$. Thus

$$H^j(X,\mathcal{O}) = \frac{\ker\,\bar\partial}{\mathrm{im}\,\bar\partial}, \qquad H^j(X,\mathcal{E}_B) = \frac{\ker\,\bar\partial_B}{\mathrm{im}\,\bar\partial_B}.$$

Consider the map $m_\varphi : H^j(X,\mathcal{O}) \to H^j(X,\mathcal{E}_B)$ induced by multiplication with the holomorphic section φ_0. Note here that the map $\Omega^{0,j}(X) \to \Omega^{0,j}(X,E) : \alpha \mapsto \alpha\varphi_0$ intertwines the $\bar\partial$ operators, i.e.

$$\bar\partial_B(\alpha\varphi_0) = (\bar\partial\alpha)\varphi_0$$

and hence there is an induced map on cohomology. Note, however, that the map $\alpha \mapsto \alpha\varphi_0$ will not in general preserve the space of harmonic forms. The following lemma was stated by Mrowka in one of his lectures in Montréal [97].

Lemma 12.6. (Mrowka) *Let $B \in \mathcal{A}(E)$ and $0 \neq \varphi_0 \in C^\infty(X, E)$ with $F_B^{0,2} = 0$ and $\bar\partial_B \varphi_0 = 0$. Then there is an exact sequence*

$$
\begin{aligned}
0 \longrightarrow H^0(X, \mathcal{O}) &\xrightarrow{m_\varphi} H^0(X, \mathcal{E}_B) \longrightarrow \ \ker \mathcal{D}_{B,\varphi} \\
\longrightarrow H^1(X, \mathcal{O}) &\xrightarrow{m_\varphi} H^1(X, \mathcal{E}_B) \longrightarrow \operatorname{coker} \mathcal{D}_{B,\varphi} \\
\longrightarrow H^2(X, \mathcal{O}) &\xrightarrow{m_\varphi} H^2(X, \mathcal{E}_B) \longrightarrow \qquad 0.
\end{aligned}
$$

Proof: Assume without loss of generality that X is connected. Then the map $m_\varphi : H^0(X, \mathcal{O}) \to H^0(X, \mathcal{E}_B)$ is obviously injective. The first nontrivial case is the following.

Step 1: *Exactness at $H^0(X, \mathcal{E}_B)$.*

The map $H^0(X, \mathcal{E}_B) \to \ker \mathcal{D}_{B,\varphi}$ is given by

$$
s \mapsto \begin{pmatrix} s - f\varphi_0 \\ \bar\partial f \\ 0 \end{pmatrix}
$$

where the function $f : X \to \mathbb{C}$ is chosen such that

$$
d^* df + |\varphi_0|^2 f = \bar\varphi_0 s. \tag{12.4}
$$

This equation is equivalent to

$$
\bar\partial^* \bar\partial f - \bar\varphi_0 \frac{s - f\varphi_0}{2} = 0
$$

and hence the triple $(s - f\varphi_0, \bar\partial f, 0)$ belongs to the kernel of $\mathcal{D}_{B,\varphi}$. Moreover, the kernel of the map $H^0(X, \mathcal{E}_B) \to \ker \mathcal{D}_{B,\varphi}$ consists precisely of the constant multiples of φ_0.

Step 2: *Exactness at* $\ker \mathcal{D}_{B,\varphi}$.

The next map in the exact sequence is given by

$$
\ker \mathcal{D}_{B,\varphi} \to H^1(X, \mathcal{O}) : \begin{pmatrix} \tau_0 \\ \alpha_1 \\ 0 \end{pmatrix} \mapsto [\alpha_1]
$$

where $[\alpha_1] \in H^1(X, \mathcal{O})$ denotes the equivalence class of $\alpha_1 \in \ker \bar\partial$. That this map is well defined follows from the fact that $\tau_2 = 0$ for every triple $(\tau_0, \alpha_1, \tau_2) \in \ker \mathcal{D}_{B,\varphi}$. To see this apply the operator $\bar\partial_B$ to the degree-1 component of $\mathcal{D}_{B,\varphi}(\tau_0, \alpha_1, \tau_2)$ to obtain

$$
0 = \bar\partial_B \bar\partial_B^* \tau_2 + (\bar\partial \alpha_1)\varphi_0 = \bar\partial_B \bar\partial_B^* \tau_2 + \frac{1}{2}|\varphi_0|^2 \tau_2.
$$

The first equation uses the fact that $F_B^{0,2} = 0$ and $\bar{\partial}_B\varphi_0 = 0$. The second equation uses the formula $\bar{\partial}\alpha_1 = \bar{\varphi}_0\tau_2/2$. It follows that $\tau_2 = 0$ as claimed. Hence the kernel of $\mathcal{D}_{B,\varphi}$ consists of all triples $(\tau_0, \alpha_1, 0)$ which satisfy

$$\bar{\partial}_B\tau_0 + \alpha_1\varphi_0 = 0, \qquad \bar{\partial}\alpha_1 = 0, \qquad \bar{\partial}^*\alpha_1 = \bar{\varphi}_0\tau_0/2. \tag{12.5}$$

The condition $\bar{\partial}\alpha_1 = 0$ shows that the above map is well defined. Exactness at $\ker \mathcal{D}_{B,\varphi}$ is now almost obvious. $[\alpha_1] = 0$ is the zero cohomology class if and only if there exists a function $f : X \to \mathbb{C}$ with $\alpha_1 = \bar{\partial}f$. It then follows from the the first equation in (12.5) that $\bar{\partial}_B(\tau_0 + f\varphi_0) = 0$ and hence

$$s = \tau_0 + f\varphi_0 \in H^0(X, \mathcal{E}_B).$$

Moreover,

$$0 = 2\bar{\partial}^*\alpha_1 - \bar{\varphi}_0\tau_0 = d^*df + |\varphi_0|^2 f - \bar{\varphi}_0 s.$$

This shows that f satisfies (12.4) and hence $(\tau_0, \alpha_1, 0)$ satisfies (12.5) with $\alpha_1 \in \operatorname{im}\bar{\partial}$ if and only if it belongs to the image of the map $H^0(X, \mathcal{E}_B) \to \ker \mathcal{D}_{B,\varphi}$.

Step 3: *Exactness at* $H^1(X, \mathcal{O})$.

To show that the composition is zero, just note that if $(\tau_0, \alpha_1, 0) \in \ker \mathcal{D}_{B,\varphi}$ then

$$\alpha_1\varphi_0 = -\bar{\partial}_B\tau_0 \in \operatorname{im}\bar{\partial}_B.$$

Conversely, suppose that $\beta \in \Omega^{0,1}(X)$ satisfies

$$\beta \in \ker\bar{\partial}, \qquad \beta\varphi_0 \in \operatorname{im}\bar{\partial}_B.$$

We must prove that there exists a function $f : X \to \mathbb{C}$ such that $\alpha_1 = \beta + \bar{\partial}f$ is the degree-1 component of some element in the kernel of $\mathcal{D}_{B,\varphi}$. First choose any section $s \in C^\infty(X, E)$ with

$$\bar{\partial}_B s + \beta\varphi_0 = 0.$$

Then choose $f : X \to \mathbb{C}$ such that

$$d^*df + |\varphi_0|^2 \operatorname{Re} f = \bar{\varphi}_0 s - 2\bar{\partial}^*\beta$$

and note that the triple

$$\begin{pmatrix} \tau_0 \\ \alpha_1 \\ 0 \end{pmatrix} = \begin{pmatrix} s - f\varphi_0 \\ \beta + \bar{\partial}f \\ 0 \end{pmatrix}$$

satisfies (12.5) and hence belongs to the kernel of $\mathcal{D}_{B,\varphi}$.

Step 4: *Exactness at $H^1(X,\mathcal{E}_B)$.*

The map $H^1(X,\mathcal{E}_B)\to\operatorname{coker}\mathcal{D}_{B,\varphi}$ is induced by

$$\tau_1\mapsto\begin{pmatrix}0\\ \tau_1\\ 0\end{pmatrix}.$$

First we must prove that this induces a well defined map of the quotient spaces, i.e. if $\tau_1=\bar\partial_B\tau_0$ then $(0,\tau_1,0)\in\operatorname{im}\mathcal{D}_{B,\varphi}$. The key observation is that all vectors of the form $(\alpha_0,0,0)$ with $\alpha_0\in\Omega^{0,0}(X)$ are contained in the image of $\mathcal{D}_{B,\varphi}$. To see this let $f:X\to\mathbb{C}$ be the unique solution of

$$d^*df+\frac{1}{2}|\varphi_0|^2f=2\alpha_0$$

and define

$$\tau_0=-f\varphi_0,\qquad \alpha_1=\bar\partial f,\qquad \tau_2=0.$$

Then

$$\bar\partial^*\alpha_1-\bar\varphi_0\tau_0/2=\alpha_0,\qquad \bar\partial_B\tau_0+\alpha_1\varphi_0=0,\qquad \bar\partial\alpha_1=0,$$

and hence $\mathcal{D}_{B,\varphi}(\tau_0,\alpha_1,0)=(\alpha_0,0,0)$.

Now let $\tau_1\in\Omega^{0,1}(X,E)$ with $\bar\partial_B\tau_1=0$. Then the class $[\tau_1]\in H^1(X,\mathcal{E}_B)$ lies in the image of the map $m_\varphi:H^1(X,\mathcal{O})\to H^1(X,\mathcal{E}_B)$ if and only if there exist $\alpha_1\in\Omega^{0,1}(X)$ and $\tau_0\in\Omega^{0,0}(X,E)$ such that

$$\tau_1=\bar\partial_B\tau_0+\alpha_1\varphi_0,\qquad \bar\partial\alpha_1=0. \tag{12.6}$$

For such a pair τ_0,α_1 we obtain

$$\begin{pmatrix}0\\ \tau_1\\ 0\end{pmatrix}=\begin{pmatrix}\bar\partial^*\alpha_1-\bar\varphi_0\tau_0/2\\ \bar\partial_B\tau_0+\alpha_1\varphi_0\\ \bar\partial\alpha_1\end{pmatrix}+\begin{pmatrix}-\bar\partial^*\alpha_1+\bar\varphi_0\tau_0/2\\ 0\\ 0\end{pmatrix}\in\operatorname{im}\mathcal{D}_{B,\varphi}.$$

This proves both that the map $H^1(X,\mathcal{E}_B)\to\operatorname{coker}\mathcal{D}_{B,\varphi}$ is well defined and that the composition $H^1(X,\mathcal{O})\to H^1(X,\mathcal{E}_B)\to\operatorname{coker}\mathcal{D}_{B,\varphi}$ is zero. Conversely, suppose that

$$\tau_1\in\ker\bar\partial_B,\qquad (0,\tau_1,0)\in\operatorname{im}\mathcal{D}_{B,\varphi}.$$

Then there exist sections $\tau_0\in\Omega^{0,0}(X,E)$, $\alpha_1\in\Omega^{0,1}(X)$, $\tau_2\in\Omega^{0,2}(X,E)$ such that

$$\bar\partial^*\alpha_1=\bar\varphi_0\tau_0/2,\qquad \tau_1=\bar\partial_B\tau_0+\bar\partial_B^*\tau_2+\alpha_1\varphi_0,\qquad \bar\partial\alpha_1=\bar\varphi_0\tau_2/2.$$

Since $\tau_1 \in \ker \bar{\partial}_B$ we obtain

$$0 = \bar{\partial}_B \tau_1 = \bar{\partial}_B \bar{\partial}_B^* \tau_2 + (\bar{\partial}\alpha_1)\varphi_0 = \bar{\partial}_B \bar{\partial}_B^* \tau_2 + \frac{1}{2}|\varphi_0|^2 \tau_2.$$

This implies $\bar{\partial}_B^* \tau_2 = 0$ and hence α_1, τ_0, and τ_1 satisfy (12.6). Thus $[\tau_1]$ belongs to the image of the map $m_\varphi : H^1(X, \mathcal{O}) \to H^1(X, \mathcal{E}_B)$.

Step 5: *Exactness at* $\operatorname{coker} \mathcal{D}_{B,\varphi}$.

The map $\operatorname{coker} \mathcal{D}_{B,\varphi} \to H^2(X, \mathcal{O})$ is given by

$$\begin{bmatrix} \alpha_0 \\ \tau_1 \\ \alpha_2 \end{bmatrix} \mapsto \left[\alpha_2 + \frac{\bar{\varphi}_0 \tau_2}{2}\right]$$

where $\tau_2 \in \Omega^{0,2}(X, E)$ is the unique solution of

$$\bar{\partial}_B \bar{\partial}_B^* \tau_2 + \frac{1}{2}|\varphi_0|^2 \tau_2 = \bar{\partial}_B \tau_1 - \alpha_2 \varphi_0. \tag{12.7}$$

We first prove, in one stroke, that this map is well defined and that the composition is zero. Hence assume that $[\alpha_0, \tau_1, \alpha_2]$ is in the image of the map $H^1(X, \mathcal{E}_B) \to \operatorname{coker} \mathcal{D}_{B,\varphi}$. This means that it is a sum of a vector in the image of $\mathcal{D}_{B,\varphi}$ and one of the form $(0, \eta, 0)$ with $\bar{\partial}_B \eta = 0$. Thus there exist sections $\tau_0 \in \Omega^{0,0}(X, E)$, $\alpha_1 \in \Omega^{0,1}(X)$ and $\tau_2 \in \Omega^{0,2}(X, E)$ such that

$$\tau_1 - \bar{\partial}_B^* \tau_2 - \alpha_1 \varphi_0 \in \ker \bar{\partial}_B \tag{12.8}$$

and

$$\alpha_2 = \bar{\partial}\alpha_1 - \bar{\varphi}_0 \tau_2 / 2, \qquad \alpha_0 = \bar{\partial}^* \alpha_1 - \bar{\varphi}_0 \tau_0 / 2. \tag{12.9}$$

As before consider the term $\bar{\partial}_B \tau_1$ to obtain

$$\bar{\partial}_B \tau_1 = \bar{\partial}_B \bar{\partial}_B^* \tau_2 + (\bar{\partial}\alpha_1)\varphi_0 = \bar{\partial}_B \bar{\partial}_B^* \tau_2 + \frac{1}{2}|\varphi_0|^2 \tau_2 + \alpha_2 \varphi_0.$$

Hence τ_2 satisfies (12.7) and, moreover, $\alpha_2 + \bar{\varphi}_0 \tau_2 / 2 = \bar{\partial}\alpha_1$. This shows that $[\tau_0, \alpha_1, \tau_2]$ belongs to the kernel of the map $\operatorname{coker} \mathcal{D}_{B,\varphi} \to H^2(X, \mathcal{O})$. Conversely, let $\alpha_0, \tau_1, \alpha_2$ be given with

$$[\alpha_2 + \bar{\varphi}_0 \tau_2 / 2] = 0 \in H^2(X, \mathcal{O})$$

where τ_2 is defined by (12.7). Choose $\alpha_1 \in \Omega^{0,1}(X)$ with

$$\alpha_2 + \frac{1}{2}\bar{\varphi}_0 \tau_2 = \bar{\partial}\alpha_1.$$

Then it follows from (12.7) that τ_1, τ_2, and α_1 satisfy (12.8). Moreover, by definition of α_1, the first equation in (12.9) is satisfied. Let $\alpha_0' \in \Omega_0^{0,0}(X)$

be defined by the right hand side of the second equation in (12.9). Then the triple $(\alpha_0', \tau_1, \alpha_2)$ satisfies both (12.8) and (12.9). Moreover, the proof of Step 4 shows that $(\alpha_0' - \alpha_0, 0, 0) \in \operatorname{im} \mathcal{D}_{B,\varphi}$. Hence $[\alpha_0, \tau_1, \alpha_2] = [\alpha_0', \tau_1, \alpha_2]$ belongs to the image of the map $H^1(X, \mathcal{E}_B) \to \operatorname{coker} \mathcal{D}_{B,\varphi}$. This completes the proof of Step 5.

Step 6: *Exactness at* $H^2(X, \mathcal{O})$.

The composition is obviously zero, because (12.7) asserts that $\varphi_0(\alpha_2 + \bar{\varphi}_0\tau_2/2) \in \operatorname{im} \bar{\partial}_B$. Conversely, let $\alpha_2 \in \Omega^{0,2}(X)$ be given such that $\alpha_2\varphi_0 = \bar{\partial}_B\tau_1$ for some $\tau_1 \in \Omega^{0,1}(X, E)$. Then the unique solution of (12.7) is obviously $\tau_2 = 0$ and hence $[\alpha_2]$ is the image of $[0, \tau_1, \alpha_2]$ under the map $\operatorname{coker} \mathcal{D}_{B,\varphi} \to H^2(X, \mathcal{O})$. This proves exactness at $H^2(X, \mathcal{O})$. Finally, the map $m_\varphi : H^2(X, \mathcal{O}) \to H^2(X, \mathcal{E}_B)$ is obviously surjective. This proves the lemma. □

Remark 12.7 For Kähler surfaces the cohomology groups $H^1(X, i\mathbb{R})$ and $H^{2,+}(X, i\mathbb{R})$ carry natural orientations. In the case of $H^{2,+}(X, i\mathbb{R})$ the canonical orientation is determined by the isomorphism

$$H^{2,+}(X, i\mathbb{R}) \to i\mathbb{R}\omega \oplus H^{0,2}(X) : \tau \mapsto \tau_\omega\omega \oplus \tau^{0,2}$$

of Proposition 3.38. In the case of H^1 the orientation is determined by identifying it with $H^{0,1}$ via $\alpha \mapsto \alpha^{0,1}$. Equivalently, it is given by the complex structure

$$\alpha \mapsto *(\alpha \wedge \omega) = -\alpha \circ J$$

or by the canonical symplectic form

$$\Omega(\alpha, \beta) = -\int_X \alpha \wedge \beta \wedge \omega$$

for $\alpha, \beta \in H^1(X, i\mathbb{R})$. (See Exercise 3.31.) Here the minus sign is chosen because the forms are imaginary valued. These orientations give rise to an orientation of the determinant line bundle $\mathcal{D}et \to \mathcal{C}(\Gamma_E)$ defined by $\mathcal{D}et_{B,\Phi} = \det(\mathcal{D}_{B,\Phi})$. On the other hand the exact sequence in Lemma 12.6 also gives rise to a natural orientation of this determinant bundle because all the terms in the sequence (except for the kernel and cokernel) carry natural complex structures. The reader may check that both constructions give rise to the same orientation of $\mathcal{D}et$. □

Exercise 12.8 Prove that the L^2-adjoint of $\mathcal{D}_{B,\varphi}$ is the operator

$$\mathcal{D}_{B,\varphi}^* : \begin{array}{c} \Omega^{0,0}(X) \\ \oplus \\ \Omega^{0,1}(X, E) \\ \oplus \\ \Omega^{0,2}(X) \end{array} \longrightarrow \begin{array}{c} \Omega^{0,0}(X, E) \\ \oplus \\ \Omega^{0,1}(X) \\ \oplus \\ \Omega^{0,2}(X, E) \end{array}$$

given by

$$\mathcal{D}_{B,\varphi}{}^* \begin{pmatrix} \alpha_0 \\ \tau_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \bar{\partial}_B^* \tau_1 - \alpha_0 \varphi_0/2 \\ \bar{\partial}\alpha_0 + \bar{\partial}^*\alpha_2 + \bar{\varphi}_0 \tau_1 \\ \bar{\partial}_B \tau_1 - \alpha_2 \varphi_0/2 \end{pmatrix}$$

for $\alpha_0 \in \Omega^{0,0}(X)$, $\tau_1 \in \Omega^{0,1}(X, E)$, and $\alpha_2 \in \Omega^{0,2}(X)$. Prove that $\alpha_0 = 0$ for every triple $(\alpha_0, \tau_1, \alpha_2) \in \ker \mathcal{D}_{B,\varphi}{}^*$. Express the maps

$$H^1(X, \mathcal{E}_B) \to \operatorname{coker} \mathcal{D}_{B,\varphi} \to H^2(X, \mathcal{O})$$

in the exact sequence of Lemma 12.6 in terms of the kernel of $\mathcal{D}_{B,\varphi}{}^*$. □

12.5 Nontriviality of the invariants

The following theorems establish nontriviality of the Seiberg-Witten invariants for Kähler surfaces.

Theorem 12.9. (Seiberg-Witten) *Let X be a Kähler surface with $b^+ > 1$. Then X has Seiberg-Witten invariants*

$$\mathrm{SW}(X, \Gamma_{\mathrm{can}}) = 1, \qquad \mathrm{SW}(X, \Gamma_K) = (-1)^{\frac{\sigma+\chi}{4}}. \tag{12.10}$$

Moreover, if $\mathrm{SW}(X, \Gamma_E) \neq 0$ then $c_1(E)$ can be represented by a harmonic 2-form of type $(1,1)$ and

$$0 \leq c_1(E) \cdot [\omega] \leq c_1(K) \cdot [\omega]. \tag{12.11}$$

Equality can only occur if $E = 0$ or $E = K$.

Theorem 12.10. (Seiberg-Witten) *Let X be a Kähler surface with $b^+ = 1$. Then X has Seiberg-Witten invariants*

$$\mathrm{SW}^+(X, \Gamma_{\mathrm{can}}) = 1, \qquad \mathrm{SW}^-(X, \Gamma_K) = (-1)^{\frac{\sigma+\chi}{4}}. \tag{12.12}$$

Moreover,

$$\mathrm{SW}^+(X, \Gamma_E) \neq 0 \qquad \Longrightarrow \qquad c_1(E) \cdot [\omega] \geq 0$$

with equality if and only if $E = \mathbb{C}$ and

$$\mathrm{SW}^-(X, \Gamma_E) \neq 0 \qquad \Longrightarrow \qquad c_1(E) \cdot [\omega] \leq c_1(K) \cdot [\omega].$$

with equality if and only if $E = K$.

Remark 12.11. (Signs) Recall that the definition of the Seiberg-Witten invariant as an integer depends on a choice of orientation of

$$H^0(X) \oplus H^1(X) \oplus H^{2,+}(X)$$

(see page 238). In Theorem 12.9 the formula $\mathrm{SW}(X, \Gamma_{\mathrm{can}}) = 1$ is obtained by choosing the standard orientation of the cohomology of X which is induced by the Kähler structure as in Remark 12.7. Similarly in Theorem 12.10. □

Proposition 12.12. (Witten) *Every compact Kähler surface with $b^+ > 1$ has simple type.*

Remark 12.13 Theorem 12.9 shows that if X is a Kähler surface with $b^+ > 1$ and $c_1(K) \cdot [\omega] = 0$ then $c_1(K) = 0$. Moreover, in this case $c = 0$ is the only basic class. The only examples of such manifolds are the 4-torus and the $K3$-surface. (See Theorem 12.1 below.) On the other hand it follows from Theorem 12.10 that if $b^+ = 1$ and $b_1 = 0$ then $c_1(K) \neq 0$. Otherwise it would follow from Theorem 12.10 that $\mathrm{SW}^{\pm}(X, \Gamma_{\mathrm{can}})$ is equal to ± 1 in contradiction to the wall crossing formula of Theorem 9.9. Thus, for example, the canonical class of the Enriques surface $X = K3/\mathbb{Z}_2$ must be the unique nontrivial torsion class. □

Theorem 12.9 proves, for example, that the hypersurface $X_d \subset \mathbb{C}P^3$ of odd degree $d > 4$ is not diffeomorphic to the manifold

$$X_d' = \ell \mathbb{C}P^2 \# m \overline{\mathbb{C}P}^2$$

where

$$\ell = \frac{d^3 - 6d^2 + 11d - 3}{3}, \quad m = \frac{2d^3 - 6d^2 + 7d - 3}{3}$$

even though, by Freedman's theorem, these manifolds are homeomorphic. (See Proposition 3.66 for the Betti numbers of X_d.) The reason is that, by Theorem 12.9 the manifold X_d has nontrivial Seiberg-Witten invariants while those of X_d' are all zero. The latter can be proved by either applying the vanishing theorem for connected sums in Section 11.2 or by using Proposition 7.32 in conjunction with the fact that the manifolds X_d' admit metrics of positive scalar curvature. (See Theorem 2.18 and Corollary 2.19.) Note in fact that X_d does not admit a metric of positive scalar curvature. This is only the simplest example of its kind. Over the past ten years vast classes of examples of smooth 4-manifolds of given homotopy type have been found whose diffeomorphism types can be distinguished by Donaldson's invariants. (See for example the work of Fintushel-Stern [27], Gompf [40], Gompf-Mrowka [41], Friedman-Morgan [29], Kotschick [59] and the references therein.) Most of these results - in fact all of them if Witten's conjecture is true - can be proved with the Seiberg-Witten invariants.

Another consequence of the nontriviality of the Seiberg-Witten invariants is the new theorem that Kähler surfaces are indecomposable. The proof requires blowup formulae for the invariants (see Section 11.7 below). Moreover, the result extends to the symplectic category and it will be discussed in Chapter 13.

Proof of Theorem 12.9: If $\mathrm{SW}(X, \Gamma_E) \neq 0$ then the moduli space $\mathcal{M}(X, \Gamma, g, i\lambda\omega)$ is nonempty for any $\lambda \in \mathbb{R}$. Thus there exists a solution $(B, \varphi_0, \varphi_2)$ of (12.1) with $\eta = i\lambda\omega$. By Proposition 12.3 one of the components φ_0 and φ_2 is zero and the argument prceding Corollary 12.4 shows that $\varphi_2 = 0$ and $\varphi_0 \neq 0$ whenever $\lambda > 0$ is sufficiently large. Hence the bundle E with holomorphic structure $\bar{\partial}_B$ has a nonzero holomorphic section φ_0. It then follows from Proposition 3.37 that $c_1(E) \cdot [\omega] \geq 0$ with equality if and only if $E = \mathbb{C}$ is the trivial bundle. The corresponding statement for $K \otimes E^*$ follows from Proposition 12.5.

To prove that the Seiberg-Witten invariant for the trivial bundle $E = \mathbb{C}$ is 1 consider the perturbation

$$\eta = -F^+_{A_{\mathrm{can}}} + i\pi\lambda\omega, \qquad \lambda > 0.$$

By Proposition 12.3, one of the components φ_0 and φ_2 must vanish. This cannot be φ_0 because the last equation in (12.1) now takes the form

$$4i(dB)_\omega = 4\pi\lambda + |\varphi_2|^2 - |\varphi_0|^2$$

and the integral of the term on the left vanishes. Hence $\varphi_2 = 0$ and a pair $B \in \Omega^1(X, i\mathbb{R})$, $\varphi_0 \in C^\infty(X, \mathbb{C})$ satisfies (12.1) if and only if

$$\bar{\partial}_B\varphi_0 = 0, \qquad (dB)^{0,2} = 0, \qquad 4i(dB)_\omega = 4\pi\lambda - |\varphi_0|^2. \tag{12.13}$$

These equations have an obvious solution

$$B = 0, \qquad \varphi_0 \equiv \sqrt{4\pi\lambda}. \tag{12.14}$$

Here are two proofs of uniqueness up to gauge equivalence.

Argument 1: If B and φ_0 satisfy (12.13) then, by Proposition 3.25,

$$0 = 2\bar{\partial}_B^*\bar{\partial}_B\varphi_0 = {d_B}^* d_B\varphi_0 - 2i(dB)_\omega\varphi_0.$$

Take the inner product with φ_0 to obtain

$$\|d_B\varphi_0\|^2_{L_2} = \int_X 2i(dB)_\omega|\varphi_0|^2 \mathrm{dvol}$$

$$
\begin{aligned}
&= \int_X 2i(dB)_\omega \Big(|\varphi_0|^2 - 4\pi\lambda\Big)\mathrm{dvol} \\
&= -\frac{1}{2}\int_X \Big(|\varphi_0|^2 - 4\pi\lambda\Big)^2 \mathrm{dvol}.
\end{aligned}
$$

The second identity follows from the fact that the integral of the function $(dB)_\omega$ over X is zero. It follows that $d_B\varphi_0 = 0$ and $|\varphi_0|^2 \equiv 4\pi\lambda$ and this implies that $B = u^{-1}du$ and $\varphi_0(x) = u(x)^{-1}\sqrt{4\pi\lambda}$ for some function $u : X \to S^1$.

Argument 2: Suppose that B and φ_0 satisfy (12.13). Then φ_0 never vanishes because otherwise the zero set of φ_0 would be a divisor which determines a nonzero first Chern class. Hence consider the functions

$$
u = |\varphi_0|^{-1}\varphi_0 : X \to S^1, \qquad \theta = \log|\varphi_0| : X \to \mathbb{R}.
$$

Recall that $\bar\partial_{u^*B}(u^{-1}\varphi_0) = 0$. This equation can be written in the form

$$
\bar\partial(u^{-1}\varphi_0) + (u^*B)^{0,1}u^{-1}\varphi_0 = 0.
$$

With $u^{-1}\varphi_0 = |\varphi_0| = e^\theta$ it follows that $(u^*B)^{0,1} = -e^{-\theta}\bar\partial e^\theta = -\bar\partial\theta$ and, since u^*B is an imaginary valued 2-form, this implies

$$
u^*B = \partial\theta - \bar\partial\theta.
$$

Thus

$$
dB = d(u^*B) = \bar\partial\partial\theta - \partial\bar\partial\theta = -2\partial\bar\partial\theta
$$

and, by Corollary 3.29,

$$
2i(dB)_\omega = -4i(\partial\bar\partial\theta)_\omega = d^*d\theta.
$$

With $|\varphi_0| = e^\theta$ the last equation in (12.13) gives

$$
d^*d(2\theta) + e^{2\theta} = 4\pi\lambda.
$$

This is the Kazdan-Warner equation and, by Theorem D.1, the obvious solution $e^{2\theta(x)} = 4\pi\lambda$ is the only one. Hence $u^*B = 0$ and $u^{-1}\varphi_0 = \sqrt{4\pi\lambda}$ as claimed.

That the solution (12.14) is regular follows from Lemma 12.6. In the exact sequence all the maps $H^j(X,\mathcal{O}) \to H^j(X,\mathcal{E}_B)$ are isomorphisms under our assumptions and hence the kernel and cokernel of $\mathcal{D}_{B,\varphi}$ are zero. This shows that $\mathrm{SW}(X,\Gamma_{\mathrm{can}}) = \pm 1$ and it remains to consider the orientations. The moduli space is zero dimensional and so we must determine the sign

$\nu(B, \varphi_0, 0)$ associated to the canonical solution. Recall from page 238 that this sign is determined by trivializing the determinant line bundle over the path of operators $t \mapsto \mathcal{D}_{B,t\varphi}$. The above argument shows that these operators are bijective for $t > 0$ and hence there is a single crossing at $t = 0$. (See Proposition A.10.) More explicitly, the discussion in Section 12.4 shows that the operator $\mathcal{D}_{B,t\varphi}$ is given by

$$\mathcal{D}_{B,t\varphi} \begin{pmatrix} \tau_0 \\ \alpha_1 \\ \tau_2 \end{pmatrix} = \begin{pmatrix} \bar{\partial}^* \alpha_1 \\ \bar{\partial}_B \tau_0 + \bar{\partial}_B^* \tau_2 \\ \bar{\partial} \alpha_1 \end{pmatrix} + t\sqrt{4\pi\lambda} \begin{pmatrix} -\tau_0/2 \\ \alpha_1 \\ -\tau_2/2 \end{pmatrix} \tag{12.15}$$

where $\sqrt{4\pi\lambda} = \varphi_0 = \bar{\varphi}_0$. The operator $\mathcal{D}_{B,0}$ is given by the first column on the right in (12.15) and its kernel and cokernel are given by

$$\ker \mathcal{D}_{B,0} = H^{0,0}(X) \oplus H^{0,1}(X) \oplus H^{0,2}(X),$$

$$\operatorname{coker} \mathcal{D}_{B,0} = H^{0,0}(X) \oplus H^{0,1}(X) \oplus H^{0,2}(X).$$

Note here that $E = \mathbb{C}$ is the trivial bundle. The kernel and cokernel are even dimensional and their dimensions are equal. Thus the contributions from the crossing numbers is $+1$ and it remains to examine what is called $\sigma(\dot{D}_0) \in \det(D_0)$ in Proposition A.10. This number can be described as follows. The operator $\dot{\mathcal{D}}_{B,0}$ is given by the second column on the right in (12.15). Consider the restriction of this operator to the kernel of $\mathcal{D}_{B,0}$ followed by the projection onto the cokernel. This composition is an isomorphism and the sign of $\sigma(\dot{\mathcal{D}}_{B,0})$ is determined by whether or not this isomorphism is orientation preserving. Examining the last column in (12.15) we find that the operator is complex linear and hence orientation preserving and hence $\sigma(\dot{\mathcal{D}}_{B,0}) \in \det(\mathcal{D}_{B,0})$ is given by the complex orientation. Together with the $+1$ from the crossing numbers we obtain $\nu(B, \varphi_0, 0) = 1$ and hence $\mathrm{SW}(X, \Gamma_{\mathrm{can}}) = 1$ as claimed. The assertion about $\mathrm{SW}(X, \Gamma_K)$ follows from Proposition 12.5. □

Proof of Theorem 12.10: The proof of nontriviality of the invariants in Theorem 12.9 did not actually use the fact that $b^+ > 1$ and carries over word by word to the case $b^+ = 1$. Let $E \to X$ be a line bundle with $\mathrm{SW}^+(X, \Gamma_E) \neq 0$ and suppose that $c_1(E) \cdot [\omega] \leq 0$. Then

$$\varepsilon_{\Gamma_E}(g, i\lambda\omega) = \pi \left(c_1(K) - 2c_1(E)\right) \cdot [\omega] + \lambda \int_X \omega \wedge \omega > 0$$

whenever $2\lambda \mathrm{Vol}(X) + \pi c_1(K) \cdot [\omega] > 0$. Hence there exists a solution $(B, \varphi_0, \varphi_2)$ of (12.1) with $\eta = i\lambda\omega$. By Proposition 12.3, one of the components φ_0 or φ_2 must be zero. The formula preceding Corollary 12.4 shows that

$$\|\varphi_2\|^2 - \|\varphi_0\|^2 = 2\pi(2c_1(E) - c_1(K)) \cdot [\omega] - 2\lambda \int_X \omega \wedge \omega < 0$$

and hence $\varphi_2 = 0$. This shows that the bundle E with holomorphic structure $\bar{\partial}_B$ has a nonzero holomorphic section. Since $c_1(E) \cdot [\omega] \leq 0$ it follows from Proposition 3.37 that $E = \mathbb{C}$ is the trivial bundle. This proves the first assertion. The second follows from Proposition 12.5. □

Proof of Proposition 12.12: The proof relies on the generalized adjunction inequality by Kronheimer and Mrowka proved in Theorem 14.1 in Chapter 14 below. Nontriviality of the invariant $\mathrm{SW}(X, \Gamma_E)$ implies, by Proposition 12.3 with $\eta = i\pi\lambda\omega$ for some large constant $\lambda > 0$, that E carries a holomorphic structure $\bar{\partial}_B$ with a nonzero holomorphic section. Thus the cohomology class $c_1(E)$ can be represented by an effective divisor $D = \sum_i m_i V_i$ via

$$c_1(E) = \sum_i m_i \mathrm{PD}([V_i]).$$

Each V_i is the image of a holomorphic curve $u_i : \Sigma_i \to X$ where Σ_i is a connected Riemann surface of genus g_i. The proof now relies on the following three observations.

(i) If $i \neq j$ then $V_i \cdot V_j \geq 0$ with equality if and only if $V_i \cap V_j = \emptyset$.

(ii) The genus g_i of Σ_i is given by

$$2g_i - 2 = V_i \cdot V_i + c_1(K) \cdot V_i - 2d_i$$

where $d_i \geq 0$ is the number of double points of a nearby immersed pseudo-holomorphic curve. This number is zero if and only if u_i is an embedding.

(iii) If $V_i \cdot V_i \geq 0$ then it follows from the Kronheimer-Mrowka generalized adjunction inequality, proved in Chapter 14 below, that

$$0 \leq c_1(E) \cdot V_i \leq c_1(K) \cdot V_i.$$

This is because there exists a nearby embedded 2-manifold which represents the homology class $[V_i]$ and has genus $g_i + d_i$. Thus, by Theorem 14.1 below,

$$c_1(K) \cdot V_i = 2(g_i + d_i) - 2 - V_i \cdot V_i \geq |(c_1(K) - 2c_1(E)) \cdot V_i|.$$

This is equivalent to the required inequality.

Now if $V_i \cdot V_i < 0$ then it follows from (i) and (ii) that

$$c_1(E) \cdot V_i \leq m_i V_i \cdot V_i \leq -2 - V_i \cdot V_i \leq c_1(K) \cdot V_i.$$

Hence the inequality $c_1(E) \cdot V_i \leq c_1(K) \cdot V_i$ holds in all cases. Multiply this by m_i and take the sum over i to obtain

$$c_1(E) \cdot c_1(E) \le c_1(K) \cdot c_1(E).$$

This is equivalent to the inequality

$$c \cdot c \le c_1(K) \cdot c_1(K)$$

for the class $c = c_1(L_{\Gamma_E}) = 2c_1(E) - c_1(K)$. But the converse inequality must hold because the moduli space has nonnegative dimension $\dim \mathcal{M} = (c \cdot c - c_1(K) \cdot c_1(K))/4$. This proves the proposition. □

12.6 Positive scalar curvature

Suppose that X is a Kähler surface which admits a metric of positive scalar curvature. Then it must satisfy $b^+ = 1$ since otherwise all the Seiberg-Witten invariants would be zero. In [70] LeBrun proved that the only minimal Kähler surfaces which admit metrics of positive scalar curvature are $\mathbb{C}P^2$ and ruled surfaces. That minimal surfaces with a Kähler metric of positive scalar curvature are rational or ruled is an older theorem by Yau (cf [127]). The proof uses some classification theory for Kähler surfaces as discussed in Section 12.1.

Theorem 12.14. (LeBrun) *Let X be a minimal Kähler surface. Then the following are equivalent.*

(i) *X is diffeomorphic to either $\mathbb{C}P^2$ or a ruled surface.*

(ii) *X admits a Kähler metric with positive scalar curvature.*

(iii) *X admits a metric with positive scalar curvature.*

The proof relies on the following elementary but important observation. Recall that $H^{2,+}(X)$ carries a natural orientation and that, for any Riemannian metric g, ω_g denotes the unique self-dual harmonic 2-form which has norm 1 and determines the given orientation of $H^{2,+}$. Note that if g is a Kähler metric then $\omega_g = \omega$ is the corresponding Kähler form.

Lemma 12.15 *Let X be a smooth compact 4-manifold with $b^+ = 1$ and $c \in H^2(X, \mathbb{Z})$ be a nontorsion cohomology class such that*

$$c \cdot c \ge 0.$$

Then $c \cdot \omega_g \ne 0$ for any Riemannian metric g on X.

Proof: Denote by $\bar{c} \in H^2(X) = H^2(X, \mathbb{Z})/\text{torsion}$ the equivalence class of c. Then $\bar{c} = (c \cdot \omega_g)[\omega_g] + \bar{c}_0$ where $\bar{c}_0 \in H^{2,-}(X)$. If $c \cdot \omega_g = 0$ then, since c is not a torsion class, $\bar{c} = \bar{c}_0 \ne 0$ and thus $c \cdot c < 0$ in contradiction to the assumption. This proves the lemma. □

Proof of Theorem 12.14: That (i) implies (ii) is a standard construction in Kähler geometry. For the Fubini-Study metric on $\mathbb{C}P^2$ see Example 3.49.

The obvious product metric on $\Sigma \times \mathbb{C}P^1$ is Kähler and has positive scalar curvature whenever the radius of the 2-sphere is sufficiently small. The (unique) nontrivial 2-sphere bundle over Σ can be constructed as a quotient $\mathbb{H} \times \mathbb{C}P^1/\Gamma$ with the standard Kähler structure on the upper halfplane $\mathbb{H}$ and $\mathbb{C}P^1$ again with small radius. (See e.g. [85], Example 6.30.) That (ii) implies (iii) is obvious.

That (iii) implies (i) follows from Theorem 12.10. More precisely, since X has positive scalar curvature the moduli space $\mathcal{M}(X, \Gamma_{\text{can}}, g, \eta)$ is empty for small η. But Theorem 12.10 asserts that $\text{SW}^+(X, \Gamma_{\text{can}}) = 1$ and hence these empty moduli spaces correspond to the invariant $\text{SW}^-(X, \Gamma_{\text{can}}) = 0$. This means that $\varepsilon_{\Gamma_{\text{can}}}(g, 0) < 0$ and since $\varepsilon_{\Gamma_{\text{can}}}(g, 0) = \pi c_1(K) \cdot [\omega_g]$ it follows that

$$c_1(K) \cdot [\omega_g] < 0$$

for some metric g. It follows immediately that $c_1(K)$ is not a torsion class and this rules out the case of Kodaira dimension zero. There are two cases. First suppose

$$c_1(K) \cdot c_1(K) < 0.$$

Then it follows from Theorem 12.1 that $\text{Kod}(X) = -\infty$ and, moreover, that X is a ruled surface. (If $X = \mathbb{C}P^2$ then $c_1(K) \cdot c_1(K) > 0$.) Secondly assume

$$c_1(K) \cdot c_1(K) \geq 0.$$

Since $c_1(K)$ is not a torsion class it follows from Lemma 12.15 that $c_1(K) \cdot [\omega_g] < 0$ for every metric and hence, in particular,

$$c_1(K) \cdot [\omega] < 0$$

for every Kähler form ω. This implies again that X has Kodaira dimension

$$\text{Kod}(X) = -\infty$$

and it follows from Theorem 12.1 (iv) that X is diffeomorphic to $\mathbb{C}P^2$ or a ruled surface. This proves the theorem. □

Remark 12.16 In [30] Friedman and Morgan proved that every Kähler surface with positive scalar curvature is a blowup of either $\mathbb{C}P^2$ or a ruled surface. This was extended by Ono and Ohta [101] and, independently, by Liu [74] to the symplectic category. □

In [32] it was proved by Friedman and Qin that the Kodaira dimension of a Kähler surface is a diffeomorphism invariant. Shortly afterwards Kronheimer found a proof which is based on the Seiberg-Witten invariants.

Theorem 12.17. (Friedman-Qin) *If two minimal Kähler surfaces are diffeomorphic then they have the same Kodaira dimension.*

Proof: Let X and Y be two diffeomorphic minimal Kähler surfaces. Assume first that $\mathrm{Kod}(X) = -\infty$. Then X admits a metric of positive scalar curvature and so does Y. By Theorem 12.14, Y is rational or ruled and so $\mathrm{Kod}(Y) = -\infty$. (Thus there are no fake Kähler structures on ruled surfaces.) This shows that $\mathrm{Kod}(X) \geq 0$ if and only if $\mathrm{Kod}(Y) \geq 0$.

Under the assumption $\mathrm{Kod}(X) \geq 0$ the classification theorem 12.1 shows that $\mathrm{Kod}(X) = 2$ if and only if $c_1(K)^2 = 2\chi(X) + 3\sigma(X) > 0$. This is clearly a topological condition and hence $\mathrm{Kod}(X) = 2$ if and only if $\mathrm{Kod}(Y) = 2$.

Now assume $b^+ \geq 3$. Then Theorem 12.1 shows that $\mathrm{Kod}(X) = 0$ if and only if $K_X = 0$. But this means that $c = 0$ is the only basic class of X. If this is the case then 0 is the only basic class of Y and it follows from Theorem 12.9 that $K_Y = 0$ and thus $\mathrm{Kod}(Y) = 0$. This shows that $\mathrm{Kod}(X) = 0$ if and only if $\mathrm{Kod}(Y) = 0$.

Thus it remains to consider Kähler surfaces with $b^+ = 1$ and Kodaira dimension 0 or 1. These surfaces satisfy $c_1(K)^2 = 0$ and they are listed in the table in Section 12.1. In this table the surfaces with $\mathrm{Kod}(X) = 1$ are obtained from the Enriques surface or from $\mathbb{C}P^2\#9\overline{\mathbb{C}P}^2$ by logarithmic transforms and they are obviously not diffeomorphic to the finite quotients of $\mathbb{T}^4$ or $K3$ which have Kodaira dimension $\mathrm{Kod}(X) = 0$. This proves the theorem. □

We note here that the Seiberg-Witten invariants give rise to an alternative proof of the **Miyaoka-Yau inequality** (cf [128]) which was found by LeBrun (cf [71]). The proof assumes the existence of an Einstein metric.

Theorem 12.18. (Miyaoka-Yau) *Let X be a compact Kähler surface which admits an Einstein metric. Then the first and second Chern classes of TX satisfy*

$$c_1 \cdot c_1 \leq 3c_2.$$

Proof: Assume first that $b^+ > 1$. Then, by Theorem 12.9, $\mathrm{SW}(X, \Gamma_{\mathrm{can}}) \neq 0$ and $c_1(L_{\Gamma_{\mathrm{can}}}) \cdot c_1(L_{\Gamma_{\mathrm{can}}}) = 2\chi + 3\sigma$. Hence it follows from LeBrun's theorem 7.34 that

$$3\sigma \leq \chi.$$

With $c_1 \cdot c_1 = 2\chi + 3\sigma$ and $\chi = c_2$ this is equivalent to the Miyaoka-Yau inequality. Now suppose $b^+ = 1$ and $s \leq 0$. Then, by Theorem 12.10, $\mathrm{SW}^+(X, \Gamma_{\mathrm{can}}) \neq 0$ and, moreover,

$$c_1(L_{\Gamma_{\mathrm{can}}}) \cdot [\omega] = -c_1(K) \cdot [\omega] \leq 0.$$

The last inequality follows from the fact that, by (3.17),

$$-c_1(K) = c_1(TX) = \frac{1}{2\pi}[\rho_\omega] = \frac{s}{8\pi}[\omega]$$

with $s \le 0$. This shows again that X satisfies the assumptions of Theorem 7.34. This leaves the case of Einstein manifolds with positive scalar curvature. Any such manifold has a positive definite Ricci tensor (Lemma 2.7) and hence satisfies

$$b_1 = 0, \qquad b^+ = 1.$$

(See Exercise 2.31 for $b_1 = 0$ and Proposition 7.32 for $b^+ = 1$.) Under these conditions the Miyaoka-Yau inequality is obviously satisfied. □

Remark 12.19 It was proved by Tian that the only Kähler–Einstein surfaces with positive scalar curvature are $\mathbb{C}P^2$, $\mathbb{C}P^1 \times \mathbb{C}P^1$, and the k-fold blowup of $\mathbb{C}P^2$ with $3 \le k \le 8$ (cf [122]). □

Exercise 12.20 Show that $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ is a nontrivial $\mathbb{C}P^1$-bundle over $\mathbb{C}P^1$ and that $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P}^2$ is diffeomorphic to $(\mathbb{C}P^1 \times \mathbb{C}P^1) \# \overline{\mathbb{C}P}^2$. **Hint:** Identify $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P}^2$ with the submanifold

$$X = \{([z_0 : z_1 : z_2], [x_0 : x_1], [y_1 : y_2]) \,|\, x_0 z_1 = z_0 x_1,\ y_1 z_2 = z_1 y_2\}$$

of $\mathbb{C}P^2 \times \mathbb{C}P^1 \times \mathbb{C}P^1$. See also Exercise 9.4 and Example 6.34 in [85]. □

12.7 Minimal surfaces of general type

The goal of this section is to prove that for minimal Kähler surfaces of general type the only basic classes are $\pm c_1(K)$. In [126] Witten attributes this result to Tian, Yau, Kronheimer, Mrowka, Morrison, Friedman, and Morgan. We give two proofs. The first is due to Kronheimer [62] and the second was explained by Mrowka in a lecture in Montréal [97].

Theorem 12.21 *Let (X, J, ω) be a minimal Kähler surface with*

$$b^+ > 1, \qquad c_1(K) \cdot c_1(K) > 0.$$

Then the only basic classes are $c = \pm c_1(K)$.

Proof 1: The first proof relies on the observation that the first Chern class of the canonical bundle K of a minimal surface can be represented by a differential form τ of type $(1,1)$ such that

$$\tau(v, Jv) \ge 0$$

for all $v \in TX$. To prove this one uses sufficiently many holomorphic sections $s_0, s_1, \ldots, s_N$ of a sufficiently large power K^m of the canonical bundle K such that at every point $x \in X$ at least one of the sections s_j is nonzero. Consider the holomorphic map

$$X \to \mathbb{C}P^N : x \mapsto f(x) = [z_0 : \cdots : z_N]$$

where $z \in \mathbb{C}^{N+1}$ is a nonzero vector such that $z_i s_j(x) = z_j s_i(x)$ for all i, j. The pullback of the bundle $H \to \mathbb{C}P^N$ (with fiber $\mathrm{Hom}(\ell, \mathbb{C})$ over $\ell \in \mathbb{C}P^N$) is the bundle K^m. Hence the required 2-form on X is obtained by pulling back the curvature 2-form of a connection on H.

This shows that the form τ lies in the closure of the Kähler cone. More precisely, for $\varepsilon > 0$ the form $\tau + \varepsilon\omega$ is a Kähler form on X. Hence the result of Theorem 12.9 holds for this form and hence it holds for τ. This means that every basic class $c = c_1(L_\Gamma) = 2e - c_1(K)$ satisfies $0 \le [\tau]\cdot e \le [\tau]\cdot c_1(K)$ and hence

$$-[\tau] \cdot c_1(K) \le [\tau] \cdot c \le [\tau] \cdot c_1(K).$$

Since τ represents the class $c_1(K)$ it follows that

$$|c_1(K) \cdot c| \le c_1(K) \cdot c_1(K).$$

By the Hodge index theorem 3.39,

$$(c_1(K) \cdot c_1(K))(c \cdot c) \le |c_1(K) \cdot c|^2 \le (c_1(K) \cdot c_1(K))^2.$$

Since $c_1(K) \cdot c_1(K) > 0$ this implies

$$c \cdot c \le c_1(K) \cdot c_1(K).$$

However, the Seiberg-Witten invariants can only be nonzero if the virtual dimension of the moduli space is nonnegative, i.e.

$$c \cdot c \ge 2\chi(X) + 3\sigma(X) = c_1(K) \cdot c_1(K).$$

Thus $c \cdot c = c_1(K) \cdot c_1(K)$. It follows again from the Hodge index theorem that c and $c_1(K)$ are linearly dependent and hence $c = \pm c_1(K)$. This proves the theorem. □

Proof 2: The second proof assumes the existence of a Kähler-Einstein metric. By Yau's theorem a minimal Kähler surface of general type admits such a metric if and only if there exists no embedded holomorphic sphere with self-intersection number -2 (see Remark 3.65). Recall from Section 3.7 that (X, J, ω) is a Kähler-Einstein manifold if the Ricci-form ρ_ω is a constant multiple of ω. The constant is $s/4$ and hence

$$c_1(TX) = \frac{1}{2\pi}[\rho_\omega] = \frac{s}{8\pi}[\omega]$$

(see (3.17)). This implies that the canonical class $c_1(K) = -c_1(TX)$ satisfies

$$c_1(K) \cdot c_1(K) = \frac{s^2}{64\pi^2}\int_X \omega \wedge \omega = \frac{s^2 \mathrm{Vol}(X)}{32\pi^2}.$$

In particular, by the general type assumption, s must be nonzero. Now suppose that $\mathrm{SW}(X, \Gamma_E) \neq 0$. Then, by Proposition 7.33, the class $c_1(L_{\Gamma_E}) = 2c_1(E) - c_1(K)$ satisfies

$$c \cdot c \leq \frac{s^2 \mathrm{Vol}(X)}{32\pi^2} = c_1(K) \cdot c_1(K).$$

On the other hand, since the moduli space has nonnegative dimension, we must have $c \cdot c \geq c_1(K) \cdot c_1(K)$ as above. But this implies, by Proposition 7.33, that there exists a solution $(B, \varphi_0, \varphi_2)$ of the unperturbed Seiberg-Witten equations (12.1) with $\eta = 0$ such that $|\varphi_0|^2 + |\varphi_2|^2 = -s/2$. Since either φ_0 or φ_2 vanishes this is only possible if $E = \mathbb{C}$ or $E = K$. Thus the only basic classes are $c_1(L_{\Gamma_{\mathrm{can}}}) = -c_1(K)$ and $c_1(L_{\Gamma_K}) = c_1(K)$. This proves the theorem. □

Corollary 12.22 *Let X be a minimal Kähler surface with $b^+ > 1$ and $c_1(K) \cdot c_1(K) > 0$. Then the canonical class $c_1(K)$ is a diffeomorphism invariant up to a change of sign. In other words, if $f : X \to Y$ is a diffeomorphism between minimal Kähler surfaces of general type with $b^+ > 1$ then $f^* c_1(K_Y) = \pm c_1(K_X)$.*

Proof: Theorem 12.21 and Exercise 7.37. □

12.8 Monopoles and divisors

Before the Seiberg-Witten invariants were discovered it was proved by Bradlow [12] that the moduli space of vortex pairs associated to a complex line bundle E over a compact Kähler manifold X (of arbitrary dimension) can be naturally identified with the space of effective divisors representing the first Chern class of E. For Riemann surfaces this was proved independently by Garcia-Prada [36] and the result was extended to higher dimensional bundles by Okonek and Teleman [100]. Now the discussion in section 12.2 shows that the unperturbed Seiberg-Witten moduli space $\mathcal{M}(X, \Gamma_E, g)$ can be identified with the moduli space of vortex pairs of the bundle E and this gives rise to the following result.

Proposition 12.23. (Bradlow) *Let (X, ω, J, g) be a Kähler surface and $E \to X$ be a Hermitian line bundle. If*

$$0 \leq c_1(E) \cdot [\omega] < \frac{c_1(K) \cdot [\omega]}{2} + \lambda \mathrm{Vol}(X)$$

with $\lambda \in \mathbb{R}$ then there is a natural bijection

$$\mathcal{M}(X, \Gamma_E, g, i\pi\lambda\omega) \cong \mathrm{Div}^{\mathrm{eff}}(X, c_1(E)).$$

If $c_1(K) \cdot [\omega]/2 + \lambda \mathrm{Vol}(X) < c_1(E) \cdot [\omega] \leq c_1(K) \cdot [\omega]$ then there is a natural bijection $\mathcal{M}(X, \Gamma_E, g, i\pi\lambda\omega) \cong \mathrm{Div}^{\mathrm{eff}}(X, c_1(K) - c_1(E))$.

Proof: Recall that the space of effective divisors can be naturally identified with the space of isomorphism classes of pairs $(\bar{\partial}, \varphi_0)$. Here $\bar{\partial} : C^\infty(X,E) \to \Omega^{0,1}(X,E)$ is an integrable Cauchy-Riemann operator and $\varphi_0 : X \to E$ is a holomorphic section, i.e.

$$\bar{\partial} \circ \bar{\partial} = 0, \qquad \bar{\partial}\varphi_0 = 0.$$

(See Appendix F for more details.) By Proposition 3.15, every Cauchy-Riemann operator has the form $\bar{\partial} = \bar{\partial}_B$ for a unique Hermitian connection $B \in \mathcal{A}(E)$. The condition $\bar{\partial} \circ \bar{\partial} = 0$ now takes the form $F_B^{0,2} = 0$ (see Proposition 3.16). Hence $\mathrm{Div}^{\mathrm{eff}}(X, c_1(E))$ is the space of complex gauge equivalence classes of pairs (B, φ_0) where $B \in \mathcal{A}(E)$ and $\varphi_0 \in C^\infty(X,E)$ with

$$F_B^{0,2} = 0, \qquad \bar{\partial}_B \varphi_0 = 0. \tag{12.16}$$

Such a pair determines a solution of (12.1) with $\eta = i\pi\lambda\omega$ if and only if

$$4i(F_{A_{\mathrm{can}}} + F_B)_\omega = 4\pi\lambda - |\varphi_0|^2. \tag{12.17}$$

We must prove that, up to unitary gauge equivalence, there is exactly one such pair in every complex gauge equivalence class. Hence fix a pair (B, φ_0) which satisfies (12.16). Then a real gauge transformation of the form $u = e^\theta$ with $\theta : X \to \mathbb{R}$ acts on the pair (B, φ_0) by

$$u^*B - B = \bar{\partial}\theta - \partial\theta, \qquad u^*\varphi_0 = e^{-\theta}\varphi_0.$$

The reader may check that $\bar{\partial}_{u^*B}(u^*\varphi_0) = e^{-\theta}\bar{\partial}_B\varphi_0$ and hence the pair $(u^*B, u^*\varphi_0)$ still satisfies (12.16). Now $(u^*B, u^*\varphi_0)$ satisfies (12.17) if and only if

$$4i(F_{u^*B})_\omega + |u^*\varphi_0|^2 = 4\pi\lambda - 4i(F_{A_{\mathrm{can}}})_\omega.$$

Since

$$F_{u^*B} - F_B = d(u^*B - B) = d(\bar{\partial}\theta - \partial\theta) = 2\partial\bar{\partial}\theta$$

this is equivalent to

$$8i(\partial\bar{\partial}\theta)_\omega + e^{-2\theta}|\varphi_0|^2 = 4\pi\lambda - 4i(F_B + F_{A_{\mathrm{can}}})_\omega$$

Lemma 3.32 asserts that the Laplace-Beltrami operator of the Kähler metric g is given by $\Delta_g\theta = -4i(\partial\bar{\partial}\theta)_\omega$ and hence (12.17) is equivalent to

$$\Delta_g(-2\theta) + e^{-2\theta}|\varphi_0|^2 = 4\pi\lambda - 4i(F_B + F_{A_{\mathrm{can}}})_\omega. \tag{12.18}$$

This is the Kazdan-Warner equation and the integral of the term on the right is given by

$$\int_X \left(4\pi\lambda - 4i(F_B + F_{A_{\mathrm{can}}})_\omega\right) = 4\pi\left(\lambda \mathrm{Vol}(X) + \frac{c_1(K)\cdot[\omega]}{2} - c_1(E)\cdot[\omega]\right) > 0.$$

Hence it follows from Theorem D.1 that (12.18) has a unique solution $\theta : X \to \mathbb{R}$. This proves the first assertion. The second follows from the first and the proof of Proposition 12.5. □

Remark 12.24 Recall from the second proof of Theorem 12.21 that if (X, J, ω) is a Kähler-Einstein surface of general type then the unperturbed Seiberg-Witten equations (12.1) for Γ_E do not have any solutions unless $E = \mathbb{C}$ or $E = K$ or the moduli space has negative virtual dimension $c_1(E) \cdot c_1(E) - c_1(E) \cdot c_1(K) < 0$. Hence Proposition 12.23 shows that if X is a *Kähler-Einstein surfaces of general type* then a holomorphic line bundle $E \to X$ with

$$0 < c_1(E)\cdot[\omega] < \frac{c_1(K)\cdot[\omega]}{2}, \qquad c_1(E)\cdot c_1(E) \geq c_1(E)\cdot c_1(K)$$

has no holomorphic sections. Equivalently, the Poincaré dual homology class $\beta = \mathrm{PD}(c_1(E)) \in H_2(X, \mathbb{Z})$ cannot be represented by holomorphic curves. □

Proposition 12.23 can be viewed as a nonlinear version of Theorem 3.40 in Section 3.5 which relates holomorphic structures on a line bundle $E \to X$ to the moduli space $\mathcal{A}^\omega(E)/\mathcal{G}(E)$ of Hermitian Yang-Mills connections on E. As in that case there are interesting connections with symplectic geometry. The space $\mathcal{A}(E) \times C^\infty(X, E)$ carries a natural symplectic form

$$\Omega((b,\theta),(b',\theta')) = -\int_X b \wedge b' \wedge \omega + \int_X \langle i\theta, \theta'\rangle \mathrm{dvol}_X$$

for $b, b' \in \Omega^1(X, i\mathbb{R}) = T_B\mathcal{A}(E)$ and $\theta, \theta' \in C^\infty(X, E)$. This form is obviously nondegenerate and closed and it is compatible with the complex structure

$$(b, \theta) \mapsto (*(b \wedge \omega), i\theta).$$

Now consider the set $\mathcal{N} \subset \mathcal{A}(E) \times C^\infty(X, E)$ of all pairs (B, Θ) which satisfy

$$F_B^{0,2} = 0, \qquad \bar\partial_B \Theta = 0.$$

This is a complex and hence symplectic submanifold of $\mathcal{A}(E) \times C^\infty(X, E)$ whose tangent space at (B, Θ) consists of all (b, θ) which satisfy $\bar\partial b^{0,1} = 0$ and $\bar\partial_B \theta + b^{0,1}\Theta = 0$. With $*(b \wedge \omega) = -b \circ J$ it follows easily that this space is invariant under the complex structure. Now the gauge group $\mathcal{G} = \mathrm{Map}(X, S^1)$ acts on $\mathcal{N}$ by Hamiltonian symplectomorphisms and the moment map $\mu : \mathcal{N} \to \Omega^0(X, i\mathbb{R})$ is given by

$$\mu(B,\Theta) = 2(F_B + F_{A_{\mathrm{can}}})_\omega - i|\Theta|^2/2.$$

Here we identify the Lie algebra $\Omega^0(X, i\mathbb{R}) = \mathrm{Lie}(\mathcal{G})$ with its dual via the standard L^2-inner product. Hence the Seiberg-Witten moduli space can be identified with the Marsden-Weinstein quotient

$$\mathcal{M}(X, \Gamma_E, g, i\pi\lambda\omega) = \mu^{-1}(-2\pi i\lambda)/\mathcal{G} = \mathcal{N}/\!\!/\mathcal{G}.$$

On the other hand the set $\mathrm{Div}^{\mathrm{eff}}(X, c_1(E))$ of effective divisors can be naturally identified with the quotient of $\mathcal{X}$ by the action of the complexified gauge group $\mathcal{G}^c = \mathrm{Map}(X, \mathbb{C}^*)$:

$$\mathrm{Div}^{\mathrm{eff}}(X, c_1(E)) = \mathcal{N}/\mathcal{G}^c.$$

Proposition 12.23 asserts that there is a natural bijection between these two quotients, in analogy to Theorem 3.40 and to various similar problems in finite dimensional geometric invariant theory.

12.9 Elliptic surfaces

Let X be a minimal Kähler surface with $b^+ > 1$ and $b_1 = 0$. Recall from Proposition 3.63 that any such surface satisfies $c_1(K)^2 \geq 0$. If $c_1(K)^2 > 0$ then Theorem 12.21 asserts that the only basic classes are plus and minus the canonical class. Hence assume

$$c_1(K) \cdot c_1(K) = 0.$$

The computation of the Seiberg-Witten invariants for this case is based on the relation between monopoles and divisors established in Proposition 12.23. Recall that the geometric genus is defined by

$$p_g = \dim^c H^{2,0}(X) = \frac{b^+ - 1}{2}.$$

(See Proposition 3.38.) As a first model case it is interesting to consider the elliptic surface V_{-1} obtained from $\mathbb{C}P^2$ by blowing up 9 distinct points. This surface obviously satisfies $b^+ = 1$ and it is not minimal. But it is a standard model from which minimal elliptic surfaces with $b^+ > 1$ can be constructed.

Example 12.25 Consider the surface

$$V_{-1} = \mathbb{C}P^2 \# 9\overline{\mathbb{C}P}^2.$$

Obviously its Euler characteristic and signature are given by $\chi(V_{-1}) = 12$ and $\sigma(V_{-1}) = -8$. Denote by $E_1, \ldots, E_9$ the exceptional divisors and by

$S \subset V_{-1}$ a lift of the standard 2-sphere $\mathbb{C}P^1 \subset \mathbb{C}P^2$ to V_{-1} (assuming that none of the nine points lie on $\mathbb{C}P^1$). Then the homology classes $[S]$ and $[E_i]$ generate $H_2(V_{-1}, \mathbb{Z})$ with

$$S \cdot S = 1, \qquad E_i \cdot E_i = -1.$$

and all the other intersection numbers are zero. The first Chern class of the tangent bundle satisfies $c_1(TX) \cdot S = 3$ and $c_1(TX) \cdot E_i = 1$ and hence the canonical class $c_1(K_{V_{-1}}) = -c_1(TX)$ is given by

$$c_1(K_{V_{-1}}) = \mathrm{PD}\left(-3[S] - [E_1] - \cdots - [E_9]\right).$$

This class obviously satisfies $c_1(K_{V_{-1}}) \cdot c_1(K_{V_{-1}}) = 0$.

There is a (singular) fibration $V_1 \to \mathbb{C}P^1$ with the generic fiber a 2-torus and finitely many exceptional fibers. An explicit representation of V_{-1} as an elliptic fibration over $\mathbb{C}P^1$ can be obtained by considering a pencil of cubics in $\mathbb{C}P^2$ passing through nine distinct points of intersection and blowing up each of these nine points. Then the lifts of these cubics to $V_{-1} = \mathbb{C}P^2 \# 9\overline{\mathbb{C}P}^2$ do not intersect and this gives rise to a projection $V_{-1} \to \mathbb{C}P^1$ with $\mathbb{T}^2$ as the generic fiber. Denote by $F \subset V_{-1}$ the generic fiber with its complex orientation. Then

$$c_1(K_{V_{-1}}) = \mathrm{PD}(-[F]).$$

To see this note that each of the nine exceptional divisors intersects each generic fiber in exactly one point while a generic line which does not meet those nine points intersects F in exactly three points. All these intersections are transverse with intersection number $+1$. (For more details see Griffiths and Harris [45] or [85], Example 6.26.) □

Example 12.26 An interesting fact is that the $K3$-surface can be obtained as the fiber connected sum $V_0 = V_{-1} \#_{\mathbb{T}^2} V_{-1}$. To see this just note that the Euler characteristic and signature are additive under taking fiber connected sums over a torus. Hence $\chi(V_0) = 24$ and $\sigma(V_0) = -16$ which characterizes $K3$-surfaces. □

Example 12.27 Consider the elliptic surface $V_k \to \mathbb{C}P^1$ with geometric genus, Euler characteristic, and signature given by

$$p_g(V_k) = k + 1, \qquad \chi(V_k) = 12(k+2), \qquad \sigma(V_k) = -8(k+2).$$

Thus V_0 is the $K3$-surface and the general surface V_k can be constructed as a fiber connected sum $V_k = V_{k-1} \#_{\mathbb{T}^2} V_{-1}$. Denote by $F \subset V_k$ the generic fiber with its complex orientation. Then the canonical bundle $K = K_{V_k} \to V_k$ has first Chern class

$$c_1(K_{V_k}) = \mathrm{PD}(k[F]).$$

For $k = -1$ this was proved in Example 12.25 and for the $K3$-surface V_0 see (3.24) in Proposition 3.66. For $k \geq 1$ the formula follows by examining the fiber connected sums. It is interesting to note that if $E \to V_k$ denotes the line bundle with first Chern class $c_1(E) = \mathrm{PD}([F])$ then the dimension of the space of holomorphic sections of $E^{\otimes m}$ is

$$\dim^c H^0(V_k, E^{\otimes m}) = \begin{cases} m+1 & \text{if } m \geq 0, \\ 0 & \text{if } m < 0. \end{cases}$$

(See Griffiths and Harris [45] for details.) □

Let us return to the general case where X is a Kähler surface with $b_1 = 0$. Although the main result of this section is perhaps most interesting for minimal elliptic surfaces, neither minimality nor ellipticity is required for the proof. Moreover, we also allow for the case $b^+ = 1$. Let $E \to X$ be a Hermitian line bundle and $B \in \mathcal{A}(E)$ be a connection on E with $F_B^{0,2} = 0$. Since $b_1 = 0$ it follows from Theorem 3.40 that up to complex gauge equivalence there is only one such connection. Consider the corresponding Dolbeault-de Rham complex

$$\Omega^{0,0}(X,E) \xrightarrow{\bar\partial_B} \Omega^{0,1}(X,E) \xrightarrow{\bar\partial_B} \Omega^{0,2}(X,E)$$

with cohomology groups

$$H^{0,j}(X,E) = H^j(X, \mathcal{E}_B) = \frac{\ker \bar\partial_B}{\operatorname{im} \bar\partial_B} \cong \ker \bar\partial_B \cap \ker \bar\partial_B^*.$$

Since the complex isomorphism class of the operator $\bar\partial_B$ is independent of B so are the cohomology groups $H^{0,j}(X,E)$ and we shall denote their dimensions by $h^j = h^j(E) = \dim^c H^{0,j}(X,E)$ for $j = 0,1,2$. Recall from the Hirzebruch-Riemann-Roch theorem 3.42 that the Euler characteristic of this complex is given by

$$\chi(X,E) = h^0 - h^1 + h^2 = \frac{c \cdot c - \sigma}{8}$$

where $c = 2c_1(E) - c_1(K)$. Since Kähler surfaces with $b^+ > 0$ have simple type the only interesting case is where $c \cdot c - \sigma = 2\chi + 2\sigma = 4(1 + b^+) = 8(p_g + 1)$ and hence

$$\chi(X,E) = p_g + 1.$$

On the other hand $b^+ = 1$ is equivalent to $p_g = 0$ and in this case the assumption that the moduli space has nonnegative dimension can be expressed in the form $\chi(X,E) \geq 1$. The following theorem was proved by Mrowka in [98].

Theorem 12.28. (Mrowka) *Let X be a Kähler surface with $b_1 = 0$ and $E \to X$ be a Hermitian line bundle Denote $h^j = \dim H^{0,j}(X, E)$. If $p_g > 0$ and $\chi(X, E) = p_g + 1$ then*

$$\mathrm{SW}(X, \Gamma_E) = (-1)^{h^0-1} \binom{p_g - 1}{h^0 - 1} \qquad \text{if} \qquad h^1 - h^2 < 0 < h^0,$$

and $\mathrm{SW}(X, \Gamma_E) = 0$ *otherwise. If $p_g = 0$ then*

$$\mathrm{SW}^+(X, \Gamma_E) = \begin{cases} 1, & \text{if } h^0 > 0,\ \chi(X, E) \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Exercise 12.29 Prove that the formula in Theorem 12.28 is in agreement with Theorem 12.9. What are the numbers h^0, h^1, h^2 in the case of the trivial bundle $E = \mathbb{C}$ and the canonical bundle $E = K$? □

Example 12.30 Consider the elliptic surface V_k with $p_g(V_k) = k + 1$ discussed in Example 12.27 and recall that the canonical class $c_1(K)$ is Poincaré dual to $k[F]$ where $F \subset V_k$ denotes a generic fiber. Let $E \to V_k$ be the line bundle with first Chern class

$$c_1(E) = \mathrm{PD}(q[F])$$

and denote $h^j(qF) = \dim^c H^{0,j}(X, E)$. Since $F \cdot F = 0$ it follows from the Hirzebruch-Riemann-Roch theorem that

$$\chi(X, qF) = p_g + 1 = k + 2.$$

Recall from Example 12.27 that the dimension of the space $H^0(X, E)$ of holomorphic sections of E is given by $h^0(qF) = q + 1$ for $q \geq 0$ and is zero otherwise. Since $h^2(qF) = h^0(K - qF) = h^0((k - q)F)$ it follows that

$$h^0(qF) = q + 1, \qquad h^1(qF) = 0, \qquad h^2(qF) = k - q + 1,$$

whenever $0 \leq q \leq k$. Hence Theorem 12.28 shows that

$$\mathrm{SW}(V_k, \Gamma_{qF}) = (-1)^q \binom{k}{q}, \qquad 0 \leq q \leq k. \tag{12.19}$$

All the other invariants are zero. Note, in particular, that $c = 0$ is the only basic class of the $K3$-surface V_0 in agreement with Theorem 12.9. For general k the Donaldson series is given by

$$\mathbb{D}_{V_k}\left(\left(1 + \frac{u}{2}\right) e^h\right) = e^{h \cdot h/2} \left(\frac{e^F - e^{-F}}{2}\right)^k$$

$$= 2^{2+\frac{7\chi+11\sigma}{4}} e^{h\cdot h/2} \sum_{q=0}^{k} (-1)^q \binom{k}{q} e^{qF\cdot h}.$$

The first term is an older calculation of Donaldson's invariants while the second expression comes from the computation of the Seiberg-Witten invariants. The reader may check that both formulae agree and hence confirm Witten's conjecture in this case. □

The proof of Theorem 12.28 rests on the following general principle. Suppose that $f : X \to Y$ is a smooth Fredholm map between paracompact separable Banach manifolds X and Y which satisfies the following.

(i) *Y is connected.*

(ii) *f has Fredholm index 0.*

(iii) *$f^{-1}(K)$ is compact for every compact set $K \subset Y$.*

(iv) *The determinant line bundle $\mathcal{D}et \to X$ with fiber*

$$\mathcal{D}et_x = \det(df(x)) = \Lambda^{\max} \ker\, df(x) \otimes \Lambda^{\max} \ker\, df(x)^*$$

is orientable.

Given an orientation of the bundle $\mathcal{D}et \to X$ it follows from the Sard-Smale theorem that the map f has a well-defined **degree** defined by

$$\deg(f) = \deg(f; y) = \sum_{f(x)=y} \nu_f(x)$$

for every regular value y where $\nu_f(x) = \pm 1$ is the sign obtained by comparing the obvious orientation of the line $\mathcal{D}et_x = \det(df(x)) = \mathbb{R}$ with the one induced by the bundle $\mathcal{D}et$. As in the finite dimensional case it follows from standard arguments in differential topology (e.g. Milnor [90]) that the degree of f is independent of the regular value used to define it. If Y is not connected then the number $\deg(f; y)$ depends only on the component of the regular value y. (See Appendices A and B for more details.) Now suppose that $y_0 \in Y$ is not a regular value of f but that the preimage

$$M_0 = f^{-1}(y_0)$$

is a smooth finite dimensional compact orientable manifold with tangent space $T_x M_0 = \ker\, df(x)$ for $x \in M_0$. Then the cokernels of $df(x)$ are of constant dimension for $x \in M_0$ and form a vector bundle $\operatorname{coker} df \to M_0$ with fiber $\operatorname{coker} df(x)$ over $x \in M_0$. This is called the **obstruction bundle**. Fix an orientation of M_0. Then the orientation of $\mathcal{D}et$ determines an orientation of the obstruction bundle $\operatorname{coker} df$ and we denote by $e(\operatorname{coker} df) \in H^*(M_0)$

the corresponding Euler class. The next proposition asserts that the degree of f agrees with the pairing of the Euler class of the obstruction bundle with the fundamental class of the zero set.

Proposition 12.31 *Under the above assumptions*

$$\deg(f) = \int_{M_0} e(\operatorname{coker} df).$$

Proof: For simplicity let us assume that both X and Y are Banach spaces and $y_0 = 0$. For Y this is no restriction at all (just choose a local chart near y_0) But for X the extension of the following argument to the general case requires the construction of a local exponential map near M_0 which we leave to the reader. It is useful to choose a smooth family of pseudo-inverses

$$M_0 \to \mathcal{L}(Y, X) : x \mapsto T_x$$

of the operators $D_x = df(x)$ so that

$$D_x T_x D_x = D_x, \qquad T_x D_x T_x = T_x.$$

Such operators exist locally. Then one can use a partition of unity to obtain operators which satisfy $D_x S_x D_x = D_x$. Finally define $T_x = S_x D_x S_x$. This operator family gives rise to complements

$$E_x = \operatorname{im} T_x, \qquad F_x = \ker T_x$$

of $\ker D_x$ and $\operatorname{im} D_x$:

$$X = \ker D_x \oplus E_x, \qquad Y = \operatorname{im} D_x \oplus F_x.$$

Note that $D_x : E_x \to \operatorname{im} D_x$ is bijective and that the open set

$$U_\delta = \{x + \xi \,|\, x \in M_0,\, \xi \in E_x,\, \|\xi\| < \delta\}$$

is a tubular neighbourhood of M_0 for $\delta > 0$ sufficiently small. Now choose a smooth section s of the bundle $F \to M_0$:

$$s : M_0 \to Y, \qquad s(x) \in F_x$$

which is transverse to the zero section. Let $\beta : [0, \delta] \to [0, 1]$ be a smooth cutoff function with $\beta(r) = 1$ near $r = 0$ and $\beta(r) = 0$ near $r = \delta$. Now define $\varphi : X \to Y$ by

$$\varphi(x + \xi) = s(x)\beta(\|\xi\|), \qquad x \in M_0,\ \xi \in E_x,\ \|\xi\| \leq 0,$$

and by $\varphi(x) = 0$ for $x \notin U_\delta$. If the Banach space X is uniformly convex then this function is smooth and the operators $d\varphi(x)$ all have a finite dimensional range. As in finite dimensional differential topology it is now easy to see that the degree of the function $f + t\varphi : X \to Y$ is independent of t and hence

$$\deg(f) = \deg(f + \varphi).$$

The next step is to prove that the zero set of $f + \varphi$ agrees with the zero set of the section s. Firstly, $(f + \varphi)^{-1}(0) \subset U$ and, secondly,

$$f(x + \xi) + \varphi(x + \xi) = 0 \qquad \Longleftrightarrow \qquad \xi = 0,\ \varphi(x) = 0$$

for $x \in M_0$ and $\xi \in E_x$ with $\|\xi\| < \delta$. To see this note first that the vector

$$s(x)\beta(\|\xi\|) + D_x\xi = f(x) + df(x)\xi - f(x + \xi)$$

satisfies a quadratic estimate

$$\|s(x)\beta(\|\xi\|) + D_x\xi\| \le c\|\xi\|^2.$$

Now $s(x) \in F_x$ and $D_x\xi \in \operatorname{im} D_x$ lie in complementary subspaces and the restriction of D_x to E_x is injective. Since $\xi \in E_x$ we obtain the estimate

$$\|\xi\| \le c'\|D_x\xi\| \le c''\|\xi\|^2 \le c''\delta\|\xi\|.$$

With $c''\delta < 1$ this is only possible if $\xi = 0$ and thus $s(x) = 0$. It is also easy to check that the crossing index $\nu_s(x)$ of x as a zero of s agrees with the index $\nu_{f+\varphi}(x)$ associated to x as a zero of $f + \varphi$. Hence

$$\deg(f + \varphi) = \sum_{f(x)+\varphi(x)=0} \nu_{f+\varphi}(x) = \sum_{x \in M_0,\, s(x)=0} \nu_s(x) = \int_{M_0} e(F).$$

This proves the proposition. □

Proof of Theorem 12.28: Consider the Banach manifold $\mathcal{N}^{k,p}$ introduced in Proposition 8.16. In the Kähler case this manifold consists of all triples $B \in \mathcal{A}^{k,p}(E)$, $\varphi_0 \in W^{k,p}(X, E)$, $\varphi_2 \in W^{k,p}(X, \Lambda^{0,2}T^*X \otimes E)$ which satisfy $\bar\partial_B\varphi_0 + \bar\partial_B^*\varphi_2 = 0$, $d^*(B - B_0) = 0$, and $(\varphi_0, \varphi_2) \ne (0, 0)$. Let

$$\mathcal{W}^{k-1,p} \subset W^{k-1,p}(X, \mathbb{R}) \oplus W^{k-1,p}(X, \Lambda^{0,2}T^*X)$$

denote the complement of the Γ-wall. More explicitly, the Banach space on the right can be identified with the space of imaginary valued self-dual 2-form of class $W^{k-1,p}$ via $\eta \mapsto (i\eta_\omega, \eta^{0,2})$ and the Γ-wall consists of all η for which there exists a connection $B \in \mathcal{A}^{k,p}(E)$ with $(F_B + \eta)_\omega = 0$ and $(F_B + \eta)^{0,2} = 0$. Now consider the smooth Fredholm map

$$\mathcal{F}_1 : \mathcal{N}^{k,p} \to \mathcal{W}^{k-1,p}$$

defined by

$$\mathcal{F}_1 \begin{pmatrix} B \\ \varphi_0 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} -i(F_{A_{\mathrm{can}}} + F_B)_\omega + (|\varphi_2|^2 - |\varphi_0|^2)/4 \\ -F_B^{0,2} + \bar{\varphi}_0\varphi_2/2 \end{pmatrix}.$$

This map is invariant under the obvious action of S^1 on $\mathcal{N}^{k,p}$ (by rotating the fibers of E) and hence descends to a map

$$\overline{\mathcal{F}}_1 : \mathcal{N}^{k,p}/S^1 \to \mathcal{W}^{k-1,p}.$$

Remark 8.20 asserts that this map is proper. Moreover, the proof of Theorem 7.16 shows that $\mathcal{F}_1$ is a Fredholm map (see page 294). In the case $\chi(X, E) = p_g + 1$ this map has index zero. If, moreover, $b^+ > 1$ then the Banach manifold $\mathcal{W}^{k-1,p}$ is connected and thus the map $\overline{\mathcal{F}}_1$ has a well defined degree. By definition, this agrees with the Seiberg-Witten invariant

$$\mathrm{SW}(X, \Gamma_E) = \deg(\overline{\mathcal{F}}_1).$$

In the case $b^+ = 1$ the space $\mathcal{W}^{k-1,p}$ has two components and the two corresponding degrees of $\overline{\mathcal{F}}_1$ are the two invariants $\mathrm{SW}^\pm(X, \Gamma_E)$ provided that $\chi(X, E) = 1$. In fact, for every $\eta \in \mathcal{W}^{k-1,p}$ the preimage $\overline{\mathcal{F}}_1^{\,-1}(\eta)$ agrees with the Seiberg-Witten moduli space for the perturbation η.

For the proof of Theorem 12.28 it suffices to consider the case $c_1(E) \cdot [\omega] \geq 0$ since otherwise the invariant is zero. Choose $\lambda > 0$ such that

$$0 \leq c_1(E) \cdot [\omega] < \frac{c_1(K) \cdot [\omega]}{2} + \lambda \mathrm{Vol}(X). \tag{12.20}$$

If $b^+ = 1$ then, under this assumption, the perturbation $\eta = i\pi\lambda\omega$ will determine the invariant $\mathrm{SW}^+(X, \Gamma_E)$. More generally, for any value of b^+, the condition (12.20) guarantees that every solution $(B, \varphi_0, \varphi_2)$ of the Seiberg-Witten equations with $\eta = i\pi\lambda\omega$ satisfies $\varphi_2 = 0$. By Proposition 12.23, the moduli space of these solutions can be identified with the set of effective divisors in the cohomology class $c_1(E)$. Because the bundle E has a unique holomorphic structure this is simply the projective space of the space $H^0(X, \mathcal{E})$ of holomorphic sections of E:

$$\overline{\mathcal{F}}_1^{\,-1}(i\pi\lambda\omega) = \mathrm{Div}^{\mathrm{eff}}(X, c_1(E)) = PH^0(X, \mathcal{E}) = \mathbb{P}.$$

This is evidently a manifold and we must prove that the kernel of the linearized map $d\overline{\mathcal{F}}_1(B, \varphi)$ agrees with the tangent space of $\mathbb{P}$. It suffices to prove that the kernel has the right dimension $h^0 - 1$ and we must then examine the cokernel bundle over $\mathbb{P}$.

It is useful to recall from the proof of Theorem 7.16 that the kernel and cokernel of $d\mathcal{F}_1(B,\varphi)$ are naturally isomorphic to the kernel and cokernel of the operator $\mathcal{D}_{B,\varphi}$ discussed in Section 12.4:

$$\ker\, d\mathcal{F}_1(B,\varphi_0,0) \cong \ker\, \mathcal{D}_{B,\varphi}, \qquad \operatorname{coker} d\mathcal{F}_1(B,\varphi_0,0) \cong \operatorname{coker} \mathcal{D}_{B,\varphi}$$

(see page 294). The cokernel can easily be determined from the exact sequence of Lemma 12.6. Since $b_1 = 0$ the term $H^1(X,\mathcal{O})$ vanishes and, since there is a unique holomorphic structure on E, the cohomology groups $H^j(X,\mathcal{E}_B)$ are independent of B. Hence Lemma 12.6 gives rise to the exact sequence

$$0 \stackrel{m_\varphi}{\to} H^1(X,\mathcal{E}_B) \to \operatorname{coker} \mathcal{D}_{B,\varphi} \to H^2(X,\mathcal{O}) \stackrel{m_\varphi}{\to} H^2(X,\mathcal{E}_B) \to 0.$$

Let us now consider the S^1-action and denote the cokernel bundle with fiber $\operatorname{coker} \mathcal{D}_{B,\varphi}$ over the equivalence class of a pair $[B,\varphi] \in \mathbb{P}$ by $\operatorname{coker} \mathcal{D}$. (For equivalent pairs (B,φ) and (B,φ') the corresponding cokernels of $\mathcal{D}_{B,\varphi}$ and $\mathcal{D}_{B,\varphi'}$ are to be identified via the S^1 action.) Denote by $H \to \mathbb{P}$ the anti-canonical bundle with $c_1(H) \in H^2(\mathbb{P},\mathbb{Z})$ the positive generator of the cohomology. Since S^1 rotates the fibres of E we obtain an exact sequence of vector bundles over $\mathbb{P}$:

$$0 \to H \otimes H^1(X,\mathcal{E}) \to \operatorname{coker} \mathcal{D} \to H^2(X,\mathcal{O}) \to H \otimes H^2(X,\mathcal{E}) \to 0.$$

With $p_g = \operatorname{rank} H^2(X,\mathcal{O})$ and $h^j = \operatorname{rank} H^j(X,\mathcal{E})$ it follows that the dimension of $\mathbb{P}$ and the rank of the kernel and cokernel bundles are given by*

$$\dim \mathbb{P} = \operatorname{rank}\, \ker \mathcal{D} = h^0 - 1, \qquad \operatorname{rank} \operatorname{coker} \mathcal{D} = p_g + h^1 - h^2.$$

This shows that the kernel of the linearized operator agrees with the tangent space of $\mathbb{P}$. Moreover, the total Chern class of the bundle $\operatorname{coker} \mathcal{D}$ is given by

$$c(\operatorname{coker} \mathcal{D}) = c(H)^{h^1-h^2} = (1 + c_1(H))^{h^1-h^2}.$$

If $p_g > 0$ then the assumption $\chi(X,E) = p_g + 1$ guarantees that the rank of the cokernel bundle agrees with the dimension of $\mathbb{P}$ and we must evaluate the top Chern class on $\mathbb{P}$. Note also that the condition $p_g > 0$ implies $h^1 - h^2 < h^0 - 1 = \dim \mathbb{P}$. There are three cases to consider. Firstly, if $h^0 = 0$ then $\mathbb{P} = \emptyset$ and hence the invariant is zero. Secondly, if $0 \le h^1 - h^2 < h^0 - 1$. then the top Chern class of the cokernel bundle

*To obtain the formula for the kernel of $\mathcal{D}$ use either the first part of the exact sequence in Lemma 12.6 or the formula $\operatorname{index} \mathcal{D} = \chi(X,E) - p_g - 1$.

vanishes and hence so does the invariant. Thus the only case in which the invariant can be nontrivial is

$$h^1 - h^2 < 0 \le h^0 - 1.$$

In this case the integral of the top Chern class is the coefficient of z^{h^0-1} in the power series $(1+z)^{-(h^2-h^1)}$ with $z = c_1(H)$. The exercise below shows that this coefficient is given by

$$\mathrm{SW}(X, \Gamma_E) = (-1)^{h^0-1}\binom{h^0+h^2-h^1-2}{h^0-1} = (-1)^{h^0-1}\binom{p_g-1}{h^0-1}.$$

This proves the theorem in the case $p_g > 0$. In the case $p_g = 0$ we assume $\chi(X, E) \ge 1$ and hence

$$0 \le \operatorname{rank}\operatorname{coker}\mathcal{D} = h^1 - h^2 \le h^0 - 1 = \dim \mathbb{P}.$$

If $h^0 = 0$ then the invariant is zero. Hence assume $h^0 \ge 1$. If $\chi(X, E) = 1$ then $h^1 - h^2 = h^0 - 1 \ge 0$ and in this case top Chern class is given by z^{h^0-1} giving $\mathrm{SW}^+(X, \Gamma_E) = 1$. If $\chi(X, E) > 1$ then the moduli space has positive dimension and a generic section of the bundle $\operatorname{coker}\mathcal{D} \to \mathbb{P}$ cuts out a submanifold of $\mathbb{P}$ which is cobordant to this moduli space. Thus the invariant $\mathrm{SW}^+(X, \Gamma_E)$ is given by the integral of the top Chern class of $\operatorname{coker}\mathcal{D}$ multiplied by an appropriate power of the canonical generator $z = c_1(H) \in H^2(\mathbb{P}, \mathbb{Z})$. A moment's thought shows that the resulting invariant is $\mathrm{SW}^+(X, \Gamma_E) = 1$ unless $h^0 = 0$. This proves the theorem. □

Exercise 12.32 Given $a \ge 1$ prove that

$$(1+z)^{-a} = \sum_{n=0}^{\infty}(-1)^n\binom{a+n-1}{n}z^n.$$

Corollary 12.33 *Let X be a Kähler surface with $b_1 = 0$ and suppose that there is a splitting $K = E \otimes F$ into holomorphic line bundles with*

$$h^0(E) > 0, \qquad h^0(F) > 0, \qquad \max\{h^0(E), h^0(F)\} > h^1.$$

where $h^1 = h^1(E) = h^1(F)$. Then

$$\chi(X, \mathcal{O}) \ge \chi(X, E), \qquad p_g > 0.$$

Moreover, if $\chi(X, \mathcal{O}) = \chi(X, E)$ and $h^1 > 0$ then $h^0(E) = h^0(F)$ and h^1 is even.

Proof: Abbreviate $h^0 = h^0(E)$, $h^2 = h^2(E) = h^0(F)$ and assume without loss of generality that $h^2 > h^1$. Moreover, note that $\chi(X, \mathcal{O}) = p_g + 1$.

The assumption $h^0 > 0$ guarantees, by Proposition 12.23, that the Seiberg-Witten moduli space is nonempty for a suitable perturbation $\eta = \pi i \lambda \omega$. The proof of Theorem 12.28 shows that the linearized operator satisfies

$$\dim \operatorname{coker} \mathcal{D} = p_g + h^1 - h^2, \qquad \dim \ker \mathcal{D} = h^0 - 1.$$

Hence the assumption $h^1 < h^2$ implies $p_g > 0$. Now suppose that

$$p_g < \chi(X, E) - 1.$$

or, equivalently, $c_1(E)^2 > c_1(E) \cdot c_1(K)$. Then the Seiberg-Witten moduli space has positive dimension. Moreover, the argument in the proof of Theorem 12.28 shows that this moduli space is cobordant to a submanifold of $\mathbb{P}$ cut out by a generic section of the cokernel bundle. Now the condition $h^2 > h^1$ guarantees that the top Chern class of this bundle is nonzero. More precisely, as in the proof of Theorem 12.28, this class is given by

$$c_{p_g+h^1-h^2}(\operatorname{coker} \mathcal{D}) = (-1)^{p_g+h^1-h^2} \binom{p^g - 1}{p_g + h^1 - h^2} z^{p_g+h^1-h^2}$$

where $z \in H^2(\mathbb{P}, \mathbb{Z})$ is the positive generator. Multiplying this class by a suitable power of z and integrating over $\mathbb{P}$ we obtain a nonzero Seiberg-Witten invariant in contradiction to the fact that X has simple type. Hence the bundle E must satisfy $p_g \geq \chi(X, E) - 1 > 0$. Let us now suppose that

$$p_g = \chi(X, E) - 1.$$

Then, by Theorem 12.28, the Seiberg-Witten invariant of Γ_E is given by

$$\mathrm{SW}(X, \Gamma_E) = (-1)^{h^0-1} \binom{p_g - 1}{h^0 - 1}.$$

Proposition 12.5 shows that the invariants $\mathrm{SW}(X, \Gamma_E)$ and $\mathrm{SW}(X, \Gamma_{K \otimes E^*})$ are related by a factor $(-1)^{p_g+1}$. Since $h^j(E) = h^{2-j}(K \otimes E^*)$, by Serre duality, it follows that

$$\mathrm{SW}(X, \Gamma_E) = (-1)^{h^0-h^1-1} \binom{p_g - 1}{h^0 - h^1 - 1} \qquad \text{if} \quad h^1 - h^0 < 0 < h^2.$$

Comparing the two expressions for $\mathrm{SW}(X, \Gamma_E)$ we find that they can only be equal if h^1 is even and if either $h^1 = 0$ or $h^0 = h^2$. In the latter case we have $(h^0 - 1) + (h^0 - h^1 - 1) = p_g - 1$ and so the two binomial coefficients agree. This proves the corollary. □

Remark 12.34 **(i)** If X and E satisfy the assumptions of Corollary 12.33 with

$$\chi(X,\mathcal{O}) = \chi(X,E), \qquad h^0(E) \notin \{1, p_g\}$$

then the formula of Theorem 12.28 shows that the Seiberg-Witten invariant of Γ_E is nonzero and not equal to ± 1. By Theorem 12.21 and Theorem 14.9 below, X cannot be the blowup of a surface of general type. Since $p_g > 0$ it follows from the Enriques-Kodaira classification that X must be the blowup of an elliptic surface.

(ii) Let X be a minimal Kähler surface with $b_1 = 0$ and suppose that E satisfies the assumptions of Corollary 12.33 with

$$\chi(X,\mathcal{O}) = \chi(X,E), \qquad c_1(E) \notin \{0, c_1(K)\}.$$

Then it follows again from Theorem 12.28 that the Seiberg-Witten invariant of Γ_E is nonzero. Since E is neither the trivial nor the canonical bundle, Theorem 12.21 asserts that X is not of general type and hence must be elliptic. It was pointed out to me by Stefan Bauer that this also follows from elementary arguments which do not use the Seiberg-Witten invariants. □

It is quite easy to construct minimal surfaces of general type in which there exists a nontrivial splitting $K = E \otimes F$ such that $h^0(E) > 0$ and $h^0(F) > 0$ as well as $\max\{h^0(E), h^0(F)\} > h^1$ but with $p_g + 1 > \chi(X,E)$. Examples are singular fibrations over $\mathbb{C}P^1$ whose generic fibres are surfaces of genus at least 2. Elliptic surfaces form the borderline case where such a splitting exists with $p_g + 1 = \chi(X,E)$. The splitting of the canonical bundle is also related to the factorization problem discussed in the next section.

12.10 Factorization

In [126] Witten proposed the following strategy for computing the invariants for Kähler surfaces. Choose a spin^c structure $W_E = W_{\mathrm{can}} \otimes E$ with nontrivial Seiberg-Witten invariants. By Proposition 12.3 the class $e = c_1(E)$ can be represented by a harmonic 2-form τ of type $(1,1)$. This implies that every connection $B \in \mathcal{A}(E)$ and every harmonic 2-form ζ of type $(0,2)$ satisfy

$$\int_X F_B^{2,0} \wedge \zeta = \int_X F_B \wedge \zeta = -2\pi i \int_X \tau \wedge \zeta = 0.$$

The second equality follows from the fact that ζ is closed and $F_B + 2\pi i \tau$ is exact. The first and last equality follow from the fact that the exterior product of two forms of non-complementary type vanishes pointwise. Now consider a perturbation of the form

$$\eta = -F_{A_{\mathrm{can}}}^+ + \frac{\zeta - \bar{\zeta}}{2}$$

where $\zeta \in H^{0,2}(X)$ is a harmonic 2-form of type $(0,2)$. Then the Seiberg-Witten equations take the form

$$\begin{aligned} \bar\partial_B\varphi_0 + \bar\partial_B^*\varphi_2 &= 0, \\ 2F_B^{0,2} &= \bar\varphi_0\varphi_2 - \zeta, \\ 4i(F_B)_\omega &= |\varphi_2|^2 - |\varphi_0|^2. \end{aligned} \tag{12.21}$$

As before, apply the operator $\bar\partial_B$ to the first equation in (12.21) to obtain

$$-\bar\partial_B\bar\partial_B^*\varphi_2 = \frac{1}{2}|\varphi_0|^2\varphi_2 - \frac{1}{2}\varphi_0\zeta$$

and hence

$$-\int_X \left|\bar\partial_B^*\varphi_2\right|^2 \mathrm{dvol} = \frac{1}{2}\int_X \left(|\varphi_0|^2|\varphi_2|^2 - \langle\zeta, \bar\varphi_0\varphi_2\rangle\right) \mathrm{dvol}.$$

Now use the fact that $F_B^{0,2}$ is orthogonal to ζ to obtain

$$\begin{aligned} \int_X \left|F_B^{0,2}\right|^2 \mathrm{dvol} &= \frac{1}{2}\int_X \left\langle F_B^{0,2}, \bar\varphi_0\varphi_2 - \zeta\right\rangle \mathrm{dvol} \\ &= \frac{1}{2}\int_X \left\langle F_B^{0,2}, \bar\varphi_0\varphi_2\right\rangle \mathrm{dvol} \\ &= \frac{1}{4}\int_X \langle\bar\varphi_0\varphi_2 - \zeta, \bar\varphi_0\varphi_2\rangle \\ &= -\frac{1}{2}\int_X \left|\bar\partial_B^*\varphi_2\right|^2 \mathrm{dvol}. \end{aligned}$$

Hence every solution $(B, \varphi_0, \varphi_2)$ of (12.21) must satisfy

$$F_B^{0,2} = 0, \qquad \bar\partial_B\varphi_0 = 0, \qquad \bar\partial_B^*\varphi_2 = 0.$$

This shows that the Seiberg-Witten equations can be interpreted as a factorization problem. Think of the 2-form $\bar\zeta \in \Omega^{2,0}(X)$ as a holomorphic section of the canonical bundle $K = \Lambda^{2,0}T^*X$. The equations $\bar\partial_B\varphi_0 = 0$ and $\bar\partial_B^*\varphi_2 = 0$ show that φ_0 is a holomorphic section of the bundle E and $\bar\varphi_2$ is a holomorphic section of $K \otimes E^*$. The latter follows from the fact that φ_2 is self-dual and hence $0 = \bar\partial_B^*\varphi_2 = -*\partial_B*\varphi_2 = -*\partial_B\varphi_2$. (See also the proof of Proposition 12.5.) Hence the problem of finding the solutions of the Seiberg-Witten equations is reduced to the problem of factorizing $\bar\zeta$ into

$$\bar\zeta = \varphi_0\bar\varphi_2, \qquad \varphi_0 \in H^0(X, E), \qquad \bar\varphi_2 \in H^0(X, K \otimes E^*).$$

In [126] Witten gives a formula for the sign attached to every such factorization. This gives rise to a method for computing the invariants.

Exercise 12.35 In [126] Witten uses an alternative argument to show that $F_B^{0,2} = 0$ for all solutions of (12.21). He considers the action $E(A_{\text{can}} + B, \Phi; \eta)$ of a solution and notes that it is invariant under the transformation

$$(B, \varphi_0, \varphi_2, \zeta) \mapsto (A, \varphi_0, -\varphi_2, -\zeta).$$

Moreover, the minimum value of the action remains unchanged and hence this transformation preserves the space of solutions of the equation (12.21). It follows that solutions can only exist if $F_B^{0,2} = 0$. Carry out the details of this argument. $\square$

13

SYMPLECTIC FOUR-MANIFOLDS

The purpose of this chapter is to describe some of the recent new advances in 4-dimensional symplectic topology which arose from the Seiberg-Witten invariants, mainly through the work of Taubes. Shortly after the new invariants were discovered he realized that symplectic four-manifolds have nontrivial Seiberg-Witten invariants and this immediately led to the solution of a longstanding conjecture concerning the nonexistence of symplectic structures on certain 4-manifolds. This result can also be used to prove that for every symplectic structure on the 4-torus the tangent bundle admits a symplectic trivialization. Another consequence is the extension of the Thom conjecture to the symplectic category. Taubes' deepest theorem along these lines concerns the relation between the Seiberg-Witten and the Gromov invariants and can be viewed as an existence theorem for J-holomorphic curves. This result can be combined with the work of Gromov and McDuff to derive far-reaching consequences concerning the topology of symplectic 4-manifolds. One such consequence is Kotschick's irreducibility theorem. Another is the theorem by Liu and Ohta-Ono that symplectic 4-manifolds which admit a metric of positive scalar curvature must be blow-ups of rational or ruled surfaces. Taubes himself proved that symplectic 4-manifolds with $b^+ \geq 2$ have simple type, satisfy $c_1(K) \cdot c_1(K) \geq 0$, and that they are minimal in the smooth category if and only if they are minimal in the symplectic category. He also showed that there is a unique symplectic structure on $\mathbb{C}P^2$ (with given volume and up to diffeomorphism). These and a number of other results will be proved below.

Many of these results can be viewed as symplectic versions of theorems about Kähler manifolds. For example, Taubes' existence theorem about pseudoholomorphic curves resembles Proposition 12.23 about the relation between the Seiberg-Witten equations and divisors. Also many of the results relating the topology of the manifold to properties of the canonical class (such as $c_1(K) \cdot [\omega] < 0$ implies rational or ruled) have this flavour. A notable exception is the result that for minimal Kähler surfaces of general type plus and minus the canonical class are the only basic classes. This theorem has no symplectic analogue. Thus symplectic 4-manifolds share many of the features of Kähler surfaces while in other respects they seem to have quite different properties. For example, Gompf proved that every finitely generated group is the fundamental group of a symplectic 4-manifold and Gompf and Mrowka constructed large classes of symplectic manifolds which are not homotopy equivalent to Kähler or even complex surfaces. On the other hand there are recent examples of smooth 4-manifolds with nontrivial Seiberg-Witten invariants which do not admit any symplectic structure [60]. Thus the question whether symplectic 4-manifolds are closer to Kähler surfaces or to general smooth 4-manifolds still seems to be far from understood.

This chapter begins with a discussion of the existence question for symplectic structures. Section 13.2 discusses Taubes' theorem about the nontriviality of the Seiberg-Witten invariants for symplectic 4-manifolds. His existence theorem for J-holomorphic curves is discussed in Section 13.3 along with some of its consequences. Section 13.4 is devoted to the irreducibility of symplectic 4-manifolds and Section 13.5 to some of the new results about rational and ruled surfaces by Ohta-Ono and Li-Liu. Section 13.6 gives a proof of Taubes' theorems about the nontriviality of the invariants and Section 13.7 gives an outline of Taubes' existence proof for J-holomorphic curves.

13.1 Existence of symplectic structures

A **symplectic structure** on a 4-manifold X is a closed nondegenerate 2-form ω. The nondegeneracy condition can be expressed as $\omega \wedge \omega \neq 0$ and hence the cohomology class $[\omega] \in H^2(X, \mathbb{R})$ satisfies

$$[\omega] \cup [\omega] \neq 0.$$

Recall also that every symplectic form ω is compatible with some almost complex structure J on TX (i.e. $g(v,w) = \omega(Jv, w)$ is a Riemannian metric) and that the space $\mathcal{J}(X, \omega)$ of such almost complex structures is contractible (see for example [85]). Thus any symplectic manifold carries two cohomology classes $[\omega] \in H^2(X, \mathbb{R})$ and $c = c_1(TX) \in H^2(X, \mathbb{Z})$. In 4 dimensions the class c satisfies

$$\langle c, \alpha \rangle = \alpha \cdot \alpha (\mathrm{mod}\ 2) \tag{13.1}$$

for $\alpha \in H_2(X, \mathbb{Z})$ (see Lemma 1.45) and, by the Hirzebruch signature theorem,

$$c \cdot c = 2\chi(X) + 3\sigma(X). \tag{13.2}$$

Here $\cdot$ denotes the cup-product evaluated on the fundamental class of X. Conversely, every cohomology class $c \in H^2(X, \mathbb{Z})$ which satisfies these conditions is the first Chern class of some almost complex structure on X. The next proposition shows that the isomorphism class of J is uniquely determined by c, however, there are always at least two homotopy classes of almost complex structures in each isomorphism class.

Proposition 13.1. (Wu) *Let X be a compact oriented smooth 4-manifold. There is a one-to-one correspondence between isomorphism classes of almost complex structures on X which are compatible with the orientation and integral cohomology classes $c \in H^2(X, \mathbb{Z})$ which satisfy (13.1) and (13.2).*

Proof: For every class $c \in H^2(X, \mathbb{Z})$ which satisfies (13.1) and (13.2) there exists a complex vector bundle $E \to X$ of rank 2 such that

$$c_1(E) = c, \qquad \langle c_2(E), [X]\rangle = \chi(X).$$

As a real vector bundle E is characterized, up to isomorphism, by its Euler class and the Stiefel-Whitney and Pontryagin classes. For every complex vector bundle and the tangent bundle of every orientable 4-manifold the odd Stiefel-Whitney classes vanish. For real rank-4 bundles the 4-th Stiefel-Whitney class agrees with the mod-2 reduction of the Euler class. Thus the only remaining classes are the second Stiefel-Whitney class, the Euler class, and the first Pontryagin class. These are related to the Chern classes by

$$\mathrm{w}_2(E) = c_1(E) (\mathrm{mod}\ 2), \quad e(E) = c_2(E), \quad p_1(E) = c_1(E)^2 - 2c_2(E).$$

By (13.1) the second Stiefel-Whitney class of E agrees with that of TX and, since $\langle c_2(E), [X]\rangle = \chi(X)$, the Euler classes agree. Now the Hirzebruch signature theorem for compact 4-manifolds asserts that

$$\sigma(X) = \frac{1}{3}\langle p_1(TX), [X]\rangle.$$

Hence (13.2) shows that the first Pontryagin class of E agrees with that of TX. Since all three classes agree E and TX must be isomorphic as real vector bundles. Hence TX carries an almost complex structure with the required Chern class c. Now if J and J' are two almost complex structures on TX with $c_1(TX, J) = c_1(TX, J')$ then, since the second Chern classes already agree, these bundles must be isomorphic as complex vector bundles (see Exercise 1.43). Hence there exists a real bundle isomorphism $\Phi : TX \to TX$ such that $\Phi J = J'\Phi$. This proves the proposition. □

Example 13.2 Consider the 4-manifold

$$X = \ell\mathbb{C}P^2 \# m\overline{\mathbb{C}P}^2$$

with intersection form $Q_X = \ell(1) \oplus m(-1)$ and

$$\chi(X) = 2 + \ell + m, \qquad \sigma(X) = \ell - m.$$

Denote by $\alpha_1, \ldots, \alpha_\ell, \beta_1, \ldots, \beta_m$ the obvious generators of $H_2(X, \mathbb{Z})$ and by $a_1, \ldots, a_\ell, b_1, \ldots, b_m$ their Poincaré duals. Thus $a_i \cdot a_j = -b_i \cdot b_j = \delta_{ij}$

and $a_i \cdot b_j = 0$. Any integral 2-dimensional cohomology class on X is of the form

$$c = \sum_{i=1}^{\ell} \lambda_i a_i + \sum_{j=1}^{m} \mu_j b_j.$$

This class satisfies (13.1) if and only if all the numbers λ_i and μ_j are odd. Now $2\chi(X) + 3\sigma(X) = 4 + 5\ell - m$ and hence c satisfies (13.2) if and only if

$$\sum_{i=1}^{\ell} {\lambda_i}^2 - \sum_{j=1}^{m} {\mu_j}^2 = 4 + 5\ell - m. \tag{13.3}$$

By Proposition 13.1, X admits an almost complex structure which is compatible with the given orientation if and only if there exist odd integers λ_i and μ_j which satisfy (13.3). Since the square of an odd integer is congruent to 1 mod 8 an odd solution of (13.3) can only exist if $4 + 4\ell$ is divisible by 8 and hence if ℓ is odd. On the other hand if ℓ is odd then the odd vectors

$$\lambda = (3, 1, 3, 1, \ldots, 3), \qquad \mu = (1, \ldots, 1)$$

solve (13.3). Hence the connected sum $\ell\mathbb{C}P^2 + m\overline{\mathbb{C}P}^2$ admits an almost complex structure compatible with its orientation if and only if ℓ is odd. □

Remark 13.3 The condition (13.1) asserts that the first Chern class $c = c_1(TX, J) \in H^2(X, \mathbb{Z})$ of an almost complex 4-manifold is a characteristic vector for the intersection form of X. It is a general result about unimodular quadratic forms that for any such vector the number $c \cdot c - \sigma$ is divisible by 8. In the case of almost complex 4-manifolds this follows also from the fact that the number $(c \cdot c - \sigma(X))/8$ is the complex index of the Dirac operator associated to the standard spin^c structure (see Theorem 6.22). Now the condition (13.2) shows that

$$\frac{c \cdot c - \sigma(X)}{8} = \frac{\sigma(X) + \chi(X)}{4} = \frac{1 + b^+ - b_1}{2}.$$

Hence $b^+ - b_1$ is odd for every almost complex 4-manifold. □

Exercise 13.4 Prove that every simply connected smooth 4-manifold with b^+ odd admits an almost complex structure compatible with the orientation.

In summary, there are two obvious necessary conditions for the existence of a symplectic structure on an orientable $2n$-dimensional manifold X, namely the existence of a cohomology class $a \in H^2(X, \mathbb{Z})$ such that $a \cup a \neq 0$ and the existence of an almost complex structure. In dimension 4 this leads to the following fundamental existence question for symplectic structures.

Question 1: *Let X be a compact smooth 4-manifold. Suppose that X carries two cohomology classes $a \in H^2(X, \mathbb{R})$ and $c \in H^2(X, \mathbb{Z})$ such that $a \cup a \neq 0$ and c satisfies (13.1) and (13.2). Does X carry a symplectic form ω such that $a = [\omega]$ and $c = c_1(TX, J)$ for $J \in \mathcal{J}(X, \omega)$?*

This question has recently been answered in the negative by Donaldson via his construction of symplectic submanifolds. In the spring of 1994 he proved the following theorem. (See [18], [19] and the discussion in [85], Chapter 4.)

Theorem 13.5. (Donaldson) *Let (X, ω) be a compact symplectic manifold such that the cohomology class $[\omega] \in H^2_{\mathrm{DR}}(X)$ admits an integral lift. Let $a \in H^2(X, \mathbb{Z})$ be such a lift. Then for every sufficiently large integer k there exists a codimension-2 symplectic submanifold $\Sigma_k \subset X$ which represents the Poincaré dual of the cohomology class ka.*

Donaldson actually proves that the inclusion $\Sigma_k \hookrightarrow X$ induces an isomorphism of the homotopy groups π_i for $i < n - 1$ where dim $X = 2n$. In particular the manifold Σ_k is connected whenever X is. In dimension 4 this result can be combined with the minimal genus theorem of Kronheimer and Mrowka to obtain obstructions to the existence of symplectic structures. For example, let $X_d \subset \mathbb{C}P^3$ be a complex hypersurface of $\mathbb{C}P^3$ of degree $d \geq 4$ and $\omega \in \Omega^2(X_d)$ be a symplectic form on X_d with corresponding Chern class $c_1 \in H^2(X_d, \mathbb{Z})$. Then it follows from Theorem 13.5 and Theorem 14.1 that either $c_1 = 0$ and $d = 4$ or

$$[\omega] \cdot c_1(TX) < 0.$$

In particular, c_1 cannot be a positive multiple of the class $[\omega] \in H^2(X_d, \mathbb{Z})$. The same assertion holds for the standard 4-torus $X = \mathbb{T}^4$. On these manifolds there are plenty of cohomology classes $c \in H^2(X, \mathbb{Z})$ which satisfy the conditions (13.1) and (13.2). One obvious example (in the case $d > 4$) is the class $c = -c_1$ where c_1 is the first Chern class of the standard complex structure. Donaldson's theorem asserts that this class cannot be the first Chern class of an almost complex structure which is compatible with a symplectic form on X_d in the same cohomology class as the standard symplectic structure. In view of these results one might ask the following stronger question.

Question 2: *Is there a smooth 4-manifold X which carries two cohomology classes $a \in H^2(X, \mathbb{R})$ and $c \in H^2(X, \mathbb{Z})$ satisfying (13.1), (13.2), and $a \cup a \neq 0$, but which does not carry any symplectic form at all?*

A natural first candidate for such a manifold would be

$$X = \mathbb{C}P^2 \# \mathbb{C}P^2 \# \mathbb{C}P^2.$$

By Example 13.2 this manifold carries an almost complex structure and a suitable cohomology class a. However, nobody had found a symplectic form on X and, up until very recently, nobody could prove that such a form cannot exist. In November 1994 Taubes finally settled this question using the Seiberg-Witten invariants (cf. [116] and [117]).

Remark 13.6 On any compact smooth 4-manifold there are at least two homotopy classes of almost complex structures for any cohomology class $c \in H^2(X, \mathbb{Z})$ which satisfies (13.1) and (13.2) (see Proposition 5.25). Hence one might ask the refined question which homotopy classes of almost complex structures and cohomology classes $a \in H^2(X, \mathbb{R})$ with $a \cup a \neq 0$ can be realized by symplectic forms. □

13.2 New results from the Seiberg-Witten invariants

Most of the new theorems about symplectic 4-manifolds arising from the Seiberg-Witten invariants are due to Taubes. Before describing his theorems let us first recall some preliminary facts about spin^c structures on symplectic 4-manifolds. There is a canonical spin^c structure $W_{\text{can}} = \Lambda^{0,*}T^*X$ associated to any compatible almost complex structure $J \in \mathcal{J}(X, \omega)$. Every other spin^c structure can be obtained from this one by tensoring with a line bundle $E \to X$. Denote $W_E = W_{\text{can}} \otimes E$ with the corresponding spin^c structure $\Gamma_E : TX \to \text{End}(W_E)$. The characteristic line bundle L_{Γ_E} is isomorphic to $K^* \otimes E^2$ where $K = \Lambda^{2,0}T^*X$ is the canonical bundle associated to J. The correspondence $\Gamma_E \mapsto c_1(E)$ determines a bijection from isomorphism classes of spin^c structures to $H^2(X, \mathbb{Z})$.

Remark 13.7 Recall that the symmetry $\Gamma \mapsto \bar{\Gamma}$ of complex conjugation is given by the correspondence

$$\bar{\Gamma}_E \cong \Gamma_{K \otimes E^*}.$$

This is because the bundle $W_{\text{can}} \otimes K = \Lambda^{0,*}T^*X \otimes K$ is isomorphic to $\bar{W}_{\text{can}} = \Lambda^{*,0}T^*X$. The details are as in the Kähler case. (See Section 12.3.) In particular, it follows as in Proposition 12.5 that

$$\text{SW}(X, \Gamma_{K \otimes E^*}) = (-1)^{\frac{\sigma+\chi}{4}} \text{SW}(X, \Gamma_E)$$

whenever $b^+ \geq 2$ and

$$\text{SW}^+(X, \Gamma_{K \otimes E^*}) = (-1)^{\frac{\sigma+\chi}{4}} \text{SW}^-(X, \Gamma_E).$$

whenever $b^+ = 1$. □

The following results are the analogues of Theorems 12.9 and 12.10 for the symplectic case.

Theorem 13.8. (Taubes) *Let (X, ω) be a symplectic 4-manifold with its orientation given by the volume form $\omega \wedge \omega$ and assume that $b^+ \geq 2$. Then X has Seiberg-Witten invariants*

$$\mathrm{SW}(X, \Gamma_{\mathrm{can}}) = 1, \qquad \mathrm{SW}(X, \Gamma_K) = (-1)^{\frac{\chi+\sigma}{4}}.$$

Moreover, if $E \to X$ is a complex line bundle with nonzero Seiberg-Witten invariants $\mathrm{SW}(X, \Gamma_E) \neq 0$ then the first Chern class satisfies

$$0 \leq c_1(E) \cdot [\omega] \leq c_1(K) \cdot [\omega].$$

If equality holds then either $E = \mathbb{C}$ or $E = K$.

Theorem 13.9. (Taubes) *Let (X, ω) be a symplectic 4-manifold with its orientation given by the volume form $\omega \wedge \omega$ and assume that $b^+ = 1$. Then X has Seiberg-Witten invariants*

$$\mathrm{SW}^+(X, \Gamma_{\mathrm{can}}) = 1, \qquad \mathrm{SW}^-(X, \Gamma_K) = (-1)^{\frac{\sigma+\chi}{4}}.$$

Moreover,

$$\mathrm{SW}^+(X, \Gamma_E) \neq 0 \qquad \Longrightarrow \qquad c_1(E) \cdot [\omega] \geq 0$$

with equality if and only if $E = \mathbb{C}$ and

$$\mathrm{SW}^-(X, \Gamma_E) \neq 0 \qquad \Longrightarrow \qquad c_1(E) \cdot [\omega] \leq c_1(K) \cdot [\omega].$$

with equality if and only if $E = K$.

Remark 13.10 As in the Kähler case the formula $\mathrm{SW}(X, \Gamma_{\mathrm{can}}) = 1$ is valid if the cohomology $H^0(X) \oplus H^1(X) \oplus H^{2,+}(X)$ is equipped with its canonical orientation induced by an almost complex structure $J \in \mathcal{J}(X, \omega)$ which is compatible with ω. (See Remark 13.35 below for more details about this orientation.) Sometimes it is useful to express this explicitly in the form

$$\mathrm{SW}(X, \Gamma_J, \mathrm{or}_J) = 1$$

whenever $b^+ \geq 2$ and J is compatible with some symplectic form. The other assertions can be reformulated similarly. □

Theorem 13.8 implies that symplectic 4-manifolds with $b^+ \geq 2$ do not admit metrics of positive scalar curvature (see Proposition 7.32). On the other hand the manifolds

$$X = \ell \mathbb{C}P^2 \# m \overline{\mathbb{C}P}^2$$

do admit such metrics and hence do not admit any symplectic structure unless either $\ell = 1$ or $m = 1$ (in which case a symplectic structure does exist for obvious reasons). On the other hand these manifolds always admit a cohomology class $a \in H^2(X, \mathbb{R})$ with $a \cup a \neq 0$ and, by Example 13.2, they admit an almost complex structure (compatible with some orientation) whenever either ℓ or m is odd. More generally, by Theorem 11.1, the Seiberg-Witten invariants of any connected sum $X = X_1 \# X_2$ must vanish provided that $b^+(X_j) \geq 1$ for $j = 1, 2$ and $b^+(X) - b_1(X)$ is odd. Hence no such manifold admits a symplectic structure. Kotschick extended this result and proved that, at least in the simply connected case, symplectic 4-manifolds with $b^+ \geq 2$ are irreducible (see Theorem 13.28 below). Another immediate consequence of Theorem 13.8 is the following result about symplectic 4-manifolds with vanishing first Chern class.

Corollary 13.11. (Taubes) *Let X be a compact oriented smooth 4-manifold with $b^+ \geq 2$. Assume that X admits a symplectic structure ω, compatible with its orientation, such that $c_1(TX, J) = 0$ for $J \in \mathcal{J}(X, \omega)$. Then every symplectic form ω' on X which is compatible with the given orientation satisfies $c_1(TX, J') = 0$ for $J' \in \mathcal{J}(X, \omega')$.*

Proof: By Theorem 13.8, the only basic class is zero. □

Corollary 13.12. (Taubes) *If ω is any symplectic structure on the 4-torus $\mathbb{T}^4$ then $c_1(TX, J) = 0$ for $J \in \mathcal{J}(\mathbb{T}^4, \omega)$. The same holds for symplectic structures on the K3-surface which are compatible with the standard orientation.*

Remark 13.13 In [15] Connolly, Lé Hông, and Ono obtained further restrictions on the almost complex structures which can be compatible with symplectic forms. Their results are based on the observation that two almost complex structure J_0 and J_1 on a 4-manifold X which have the same first Chern class $c_1(TX, J_0) = c_1(TX, J_1)$ may not be homotopic even though, by Proposition 13.1, they are isomorphic.* In fact, it was observed by Donaldson in [16] that there is a natural involution

$$p : \pi_0(\mathcal{J}(X)) \to \pi_0(\mathcal{J}(X))$$

which preserves the first Chern class and reverses the cohomological orientation of X as defined in Remark 13.35 below. Let us temporarily denote by Γ_J the canonical spin^c structure of the almost complex structure J and by $\mathrm{SW}(X, \Gamma_J)$ the corresponding Seiberg-Witten invariant defined with the

*Recall from Proposition 5.25 that for every spin^c structure $\Gamma : TX \to \mathrm{End}(W)$ with $c_2(W^+) = 0$ the components of the set $\mathcal{J}(TX, \Gamma)$ of almost complex structures J on TX whose canonical spin^c structure Γ_J is isomorphic to Γ are given by

$$\pi_0(\mathcal{J}(TX, \Gamma)) \cong \mathbb{Z}_2 \oplus \frac{H^3(X, \mathbb{Z})}{H^1(X, \mathbb{Z}) \cup c_1(W^+)}.$$

orientation of Remark 13.35 induced by J. Consider two almost complex structures $J, J' \in \mathcal{J}(X)$ such that

$$[J'] = p([J]).$$

In [15] Connolly, Lé Hông, and Ono show that the spin^c structure $\Gamma_{J'}$ is isomorphic to Γ_J. Since the cohomological orientation of X induced by J' is opposite to that induced by J it follows that

$$\mathrm{SW}(X, \Gamma_{J'}, \mathrm{or}_{J'}) = -\mathrm{SW}(X, \Gamma_J, \mathrm{or}_J).$$

Hence Theorem 13.8 shows that in the case $b^+ \geq 2$ the structures J and J' cannot both be compatible with symplectic forms. A similar reasoning in the case $b^+ = 1$ shows that if J and J' are compatible with symplectic forms ω and ω', respectively, then these forms determine opposite orientations of $H^{2,+}(X)$ and the wall-crossing number is $\mathrm{SW}^+(X, \Gamma_J) - \mathrm{SW}^-(X, \Gamma_J) = 2$. See [15] for further details. $\square$

Another corollary of Theorem 13.8 is obtained by combining it with the adjunction formula for symplectic submanifolds and the adjunction inequality of Kronheimer and Mrowka in Theorem 14.1. This gives rise to a proof of the generalized Thom conjecture for symplectic 4-manifolds with $b^+ \geq 2$. Moreover, by Theorem 13.9, the proof of Theorem 14.2 for the case $b^+ = 1$ generalizes immediately to the symplectic category. In [73] Li and Liu give a proof which is based on the wall-crossing formula. For $b^+ \geq 2$ the result is due to Kronheimer-Mrowka [66] and Taubes [116].

Theorem 13.14. (Thom conjecture) *Let (X, ω) be a compact symplectic 4-manifold and $\Sigma \subset X$ be a compact oriented embedded surface with*

$$\Sigma \cdot \Sigma > 0, \qquad \int_\Sigma \omega > 0.$$

Then

$$2g(\Sigma) - 2 \geq \Sigma \cdot \Sigma + c_1(K) \cdot \Sigma.$$

In particular, 2-dimensional symplectic submanifolds $C \subset X$ with $C \cdot C \geq 0$ minimize the genus in their respective homology classes.

Proof: Theorem 13.8 and Theorem 14.1 for the case $b^+ \geq 2$. Theorem 13.9 and the proof of Theorem 14.2 in Section 14.2 for the case $b^+ = 1$. $\square$

Proposition 13.15. (Kronheimer-Mrowka-Taubes) *Let X be a compact symplectic 4-manifold with $b^+ \geq 2$ and $C \subset X$ be a 2-dimensional symplectic submanifold with $C \cdot C \geq 0$. Then $c_1(K) \cdot C \geq 0$ and every line bundle $E \to X$ with $\mathrm{SW}(X, \Gamma_E) \neq 0$ satisfies*

$$0 \leq c_1(E) \cdot C \leq c_1(K) \cdot C.$$

Proof: Since $[\omega] \cdot C > 0$ the homology class of C is not torsion. Hence, by Theorem 14.1,

$$c_1(K) \cdot C = 2g - 2 - C \cdot C \geq c_1(L_{\Gamma_E}) \cdot C = 2c_1(E) \cdot C - c_1(K) \cdot C.$$

This shows that $c_1(E) \cdot C \leq c_1(K) \cdot C$. The inequality $c_1(E) \cdot C \geq 0$ follows by replacing E with $K \otimes E^*$. Consider the case $E = \mathbb{C}$ to obtain $c_1(K) \cdot C \geq 0$. □

Example 13.16 Neither of the assumptions $C \cdot C \geq 0$ and $b^+ \geq 2$ in Proposition 13.15 can be removed. If $X = \mathbb{C}P^2$ and $C = \mathbb{C}P^1$ then $b^+ = 1$ and $C \cdot C = 1$ but $c_1(K) \cdot C = -3$. If $X' = X \# \overline{\mathbb{C}P}^2$ is obtained by blowing up a point in a Kähler surface X and $C = E$ is the exceptional divisor then $C \cdot C = -1$ and $c_1(K) \cdot C = -1$. □

Consider the case where C is the symplectic submanifold of Theorem 13.5 which represents the class $[C] = \mathrm{PD}(k[\omega])$. Then Proposition 13.15 yields the second assertion of Theorem 13.8. However, the proof given by Taubes in [117] and the one given below is by a direct argument from the Seiberg-Witten equations and does not rely on Theorem 13.5.

Proposition 13.15 and the second assertion of Theorem 13.8 can be interpreted in two ways, either as a restriction on the cohomology classes $c = c_1(E)$ with nontrivial Seiberg-Witten invariants, or as a restrictions on symplectic manifolds with $c_1(K) \cdot [\omega] < 0$. Since $c_1(K) = -c_1(TX, J)$ the latter includes the so-called **monotone** symplectic manifolds which satisfy $c_1(TX, J) = \lambda[\omega]$ for $J \in \mathcal{J}(X, \omega)$ with $\lambda > 0$. Such manifolds must satisfy $b^+ = 1$. In fact, it was shown by Ohta and Ono in [101] that the only monotone symplectic 4-manifolds are the del Pezzo surfaces $S^2 \times S^2$ and $\mathbb{C}P^2$ with up to eigth points blown up. The following two remarks summarize some further consequences of Theorem 13.9 for symplectic 4-manifolds with $b^+ = 1$.

Remark 13.17 Let (X, ω) be a compact symplectic 4-manifold with $b^+ = 1$ and $b_1 = 0$ and suppose that $c_1(K)$ is a torsion class. Then the wall-crossing formula show that

$$\mathrm{SW}^+(X, \Gamma_E) + \mathrm{SW}^+(X, \Gamma_{K-E}) = 1.$$

With $E = \mathbb{C}$ this shows that $c_1(K) \neq 0$. Now if $c_1(E)$ is a torsion class then Theorem 13.9 shows that $\mathrm{SW}^+(X, \Gamma_E)$ can only be nonzero if $c_1(E) = 0$. Hence for every torsion class $c_1(E)$ it follows that either $c_1(E) = 0$ or $c_1(E) = c_1(K)$. This shows that $c_1(K)$ is the only nonzero torsion class in $H^2(X, \mathbb{Z})$ and hence, by the universal coefficient theorem, $H_1(X, \mathbb{Z}) = \mathbb{Z}_2$. Moreover, the Hirzebruch signature formula shows that $b^- = 9$ and so $Q_X = H \oplus (-E_8)$. In summary,

$$2c_1(K) = 0, \quad c_1(K) \neq 0, \quad Q_X = H \oplus (-E_8), \quad H_1(X, \mathbb{Z}) = \mathbb{Z}_2.$$

The only known example with these properties is the Enriques surface (see Example 6.28). □

Remark 13.18 Let (X, ω) be a compact symplectic 4-manifold with $b^+ = 1$ and $b_1 = 2$ and suppose that $c_1(K)$ is a torsion class. Then the wall crossing formula in Theorem 9.14 with $c = -c_1(K)$ shows that $\mathrm{SW}^-(X, \Gamma_{\mathrm{can}}) = \mathrm{SW}^+(X, \Gamma_{\mathrm{can}}) = 1$. Hence $\mathrm{SW}^+(X, \Gamma_K) = 1$ and, since $c_1(K) \cdot [\omega] = 0$ it follows from Theorem 13.9 that

$$c_1(K) = 0, \qquad Q_X = H.$$

The Kähler examples here are the hyperelliptic surfaces (see Section 12.1) and nonKähler examples were found by Fernández et al in [24, 23]. See also the survey in [86]. □

13.3 Existence of J-holomorphic curves

Much more powerful consequences for symplectic 4-manifolds can be obtained by combining the Seiberg-Witten invariants with Gromov's invariants which are obtained by counting embedded J-holomorphic curves of higher genus. In his seminal paper [47] Gromov discovered that J-holomorphic curves form a powerful tool for the study of symplectic manifolds. (An exposition of the foundations of the theory can be found in McDuff-Salamon [84].) In [107] Ruan developed these ideas further and defined the higher genus Gromov invariants for general symplectic manifolds. Recently Taubes extended the construction of Ruan to take account of disconnected J-holomorphic curves in symplectic 4-manifolds (cf. [120]). This extension is naturally related to the Seiberg-Witten invariants. In [119] and [120] Taubes proved the following remarkable theorem. The result can be viewed as the symplectic version of Proposition 12.23 about the relation between Seiberg-Witten equations and divisors in the Kähler case.

Theorem 13.19. (Taubes) *Let (X, ω) be a symplectic 4-manifold with its orientation given by the volume form $\omega \wedge \omega/2$. Assume that $b^+ \geq 2$. Let $E \to X$ be a complex line bundle with nontrivial Seiberg-Witten invariants*

$$\mathrm{SW}(X, \Gamma_E) \neq 0.$$

Then there exists an embedded symplectic submanifold $C \subset X$ which represents the class

$$[C] = \mathrm{PD}(c_1(E)).$$

Moreover, every component C_i of C satisfies

$$c_1(K) \cdot C_i \leq g(C_i) - 1 \leq C_i \cdot C_i. \tag{13.4}$$

If $E = \mathbb{C}$ is the trivial bundle then C is the empty curve. In the case $b^+ = 1$ this result continues to hold for line bundles $E \to X$ with $\mathrm{SW}^+(X, \Gamma_E) \neq 0$.

It is of some interest to examine the inequality (13.4) in more detail. Any embedded J-holomorphic curve $C \subset X$ satisfies the adjunction formula

$$2g - 2 = \alpha \cdot \alpha + c_1(K) \cdot \alpha.$$

where g is the genus of C and $\alpha = [C] \in H_2(X, \mathbb{Z})$. On the other hand, the moduli space of connected embedded J-holomorphic curves in the homology class has dimension

$$\dim \mathcal{M}^{\mathrm{Gr}}(X, J; \alpha) = \alpha \cdot \alpha - c_1(K) \cdot \alpha$$

for a generic almost complex structure J. Here the complex structure on the surface is allowed to vary. Now the symplectic submanifolds constructed in the proof of Theorem 13.19 are all J-holomorphic curves for some $J \in \mathcal{J}(X, \omega)$ which are *stable* in the sense that they persist under small perturbations of J. Hence the corresponding moduli spaces $\mathcal{M}^{\mathrm{Gr}}(\alpha_i)$ for $\alpha_i = [C_i]$ must have nonnegative dimension for each component C_i. Thus $C_i \cdot C_i \geq c_1(K) \cdot C_i$ and combining this with the adjunction formula $2g_i - 2 = C_i \cdot C_i + c_1(K) \cdot C_i$ one obtains (13.4). The proof of Theorem 13.19 goes beyond the scope of this book. We will give an outline of Taubes' arguments in Section 13.7.

Here we shall prove some of the consequences of Theorem 13.19. Recall that a compact symplectic 4-manifold (X, ω) is called **minimal** if it does not contain any symplectically embedded sphere with self-intersection number -1. If such a sphere does exist then X decomposes as a connected sum of some symplectic 4-manifold (X', ω') with $\overline{\mathbb{C}P}^2$ (see [85], Chapter 6). By induction, every compact symplectic 4-manifold is a connected sum of a minimal one with finitely many copies of $\overline{\mathbb{C}P}^2$. A first consequence of Theorem 13.19 is obtained by combining the result with positivity of intersections for J-holomorphic curves.

Corollary 13.20. (Taubes) *Let X be a minimal symplectic 4-manifold and $E, E' \to X$ be two complex line bundles. Assume that either $b^+ \geq 2$, $\mathrm{SW}(X, \Gamma_E) \neq 0$, $\mathrm{SW}(X, \Gamma_{E'}) \neq 0$, or $b^+ = 1$, $\mathrm{SW}^+(X, \Gamma_E) \neq 0$, $\mathrm{SW}^+(X, \Gamma_{E'}) \neq 0$. Then*

$$c_1(E) \cdot c_1(E') \geq 0.$$

In particular, every minimal symplectic 4-manifold with $b^+ \geq 2$ satisfies

$$c_1(K) \cdot c_1(K) \geq 0$$

or, equivalently, $3\sigma(X) + 2\chi(X) \geq 0$.

Proof: By Theorem 13.19, the Poincaré duals of both classes $c_1(E)$ and $c_1(E')$ can be represented by symplectic submanifolds

$$C = C_1 \cup \cdots \cup C_N, \qquad C' = C_1' \cup \cdots \cup C_{N'}'.$$

In [120] Taubes actually proves that the C_i and C_j' are all stable embedded J-holomorphic curves in X for some generic almost complex structure J and hence satisfy (13.4):

$$g(C_i) - 1 \leq C_i \cdot C_i, \qquad g(C_j') - 1 \leq C_j' \cdot C_j'.$$

With this convention (choosing the C_i and C_j' to be J-holomorphic) it may be necessary to allow for repeated copies of embedded J-holomorphic tori with self-intersection number zero. Since X is minimal it does not contain any embedded J-holomorphic sphere with self-intersection number -1 and hence $C_i \cdot C_i \geq 0$ and $C_j' \cdot C_j' \geq 0$ for all i and j. This implies $C_i \cdot C_j' \geq 0$ whenever $C_i = C_j'$. On the other hand, if the curves C_i and C_j' are distinct then it follows from the positivity of intersections for J-holomorphic curves that

$$C_i \cdot C_j' \geq 0.$$

This is obvious when the two curves intersect transversally. Moreover, each nontransverse intersection point is isolated (see for example Lemma 2.2.2 in [84]) and contributes a positive number (at least 2) to the intersection index (see McDuff [79] and Micallef-White [89] for details). Hence it follows that

$$c_1(E) \cdot c_1(E') = \sum_{i=1}^{N} \sum_{j=1}^{N'} C_i \cdot C_j' \geq 0.$$

Note that the assertion for $E = E'$ does not require the positivity of intersections. The assertion about $c_1(K)$ follows as the special case $E = E' = K$, by Theorems 13.8 and 13.9. □

Corollary 13.21. (Taubes) *Let X be a minimal symplectic 4-manifold with $b^+ \geq 2$ and*

$$c_1(K) \cdot c_1(K) = 0.$$

Then every homology class $\mathrm{PD}(c_1(E))$ with $\mathrm{SW}(X, \Gamma_E) \neq 0$ can be represented by a disjoint union of symplectically embedded tori with self-intersection number 0. In particular, this holds for $E = K$.

Proof: For $E = K$ this follows immediately from the proof of Corollary 13.20. For the general case suppose that $\mathrm{PD}(c_1(E))$ is represented by a symplectic submanifold $C = C_1 \cup \cdots \cup C_N$ whose components C_i satisfy (13.4). Then

$$c_1(K) \cdot C_i \leq g(C_i) - 1 \leq c_1(E) \cdot C_i.$$

Now use Proposition 13.15 to obtain equality:

$$C_i \cdot C_i = c_1(K) \cdot C_i = c_1(E) \cdot C_i = g(C_i) - 1.$$

Minimality shows that $C_i \cdot C_i \geq 0$ for all i. If $c_1(K) = 0$ then obviously $C_i \cdot C_i = 0$ and $g(C_i) = 1$ for all i. If $c_1(K) \neq 0$ let $T \subset X$ be one of the embedded symplectic tori which represent the Poincaré dual of $c_1(K)$. Then, by Proposition 13.15,

$$0 \leq c_1(E) \cdot T \leq c_1(K) \cdot T = 0.$$

Since this holds for all the tori T whose union represents $\mathrm{PD}(c_1(K))$ it follows that $c_1(E) \cdot c_1(K) = 0$ and hence $c_1(K) \cdot C_i = 0$ for all i. This proves the corollary. □

Corollary 13.22. (Taubes) *Every symplectic 4-manifold with $b^+ \geq 2$ has SW-simple type.*

Proof: Suppose $\mathrm{SW}(X, \Gamma_E) \neq 0$. Then, by duality, $SW(X, \Gamma_{K-E}) \neq 0$ and hence, by Corollary 13.20, $c_1(K-E) \cdot c_1(E) \geq 0$. On the other hand the dimension of the moduli space is $\dim \mathcal{M}(X, \Gamma_E) = c_1(E-K) \cdot c_1(E) \geq 0$ and hence this dimension is zero. □

The next corollary shows that if X and Y are diffeomorphic symplectic 4-manifolds then X is minimal if and only if Y is minimal.

Corollary 13.23. (Taubes) *Let X be a compact symplectic 4-manifold with $b^+ \geq 2$. Suppose that X contains a smoothly embedded sphere S with*

$$S \cdot S = -1.$$

Then X contains a symplectically embedded sphere $C \subset X$ with $C \cdot C = -1$ and either $[C] = [S]$ or $[C] = -[S]$. Moreover, in this case $c_1(K) \cdot [S] = \pm 1$.

Proof: If there exists an embedded 2-sphere S with self-intersection number -1 then X is diffeomorphic to some connected sum

$$X \cong X' \# \overline{\mathbb{C}P}^2.$$

To see this note that the boundary of a tubular neighbourhood N of S is diffeomorphic to the 3-sphere and that N itself is diffeomorphic to the complement of a ball in $\mathbb{C}P^2$ with reversed orientation. Choose spin^c structures Γ' on X' and Γ_k on $\overline{\mathbb{C}P}^2$ by restricting the canonical spin^c structure Γ_{can} on $X = X'\#\overline{\mathbb{C}P}^2$ to the two summands. Thus

$$\Gamma_{\mathrm{can}} = \Gamma'\#\Gamma_k.$$

and the (odd) label k was chosen such that $c_1(L_{\Gamma_k}) = ke$ where $e = \mathrm{PD}([S])$. Then it follows from Theorem 13.8 and Theorem 11.2 that

$$\mathrm{SW}(X', \Gamma') = \mathrm{SW}(X, \Gamma_{\mathrm{can}}) = 1.$$

Abbreviate $\mathcal{M} = \mathcal{M}(X, \Gamma_{\mathrm{can}})$, $\mathcal{M}' = \mathcal{M}(X', \Gamma')$, $c = c_1(L_{\Gamma_{\mathrm{can}}})$, and $c' = c_1(L_{\Gamma'})$. Since $c \cdot c = c' \cdot c' - k^2$, $\chi(X) = \chi(X') - 1$, and $\sigma(X) = \sigma(X') + 1$, we have

$$\dim\, \mathcal{M} = \dim\, \mathcal{M}' - \frac{k^2 - 1}{4}.$$

If $k \neq \pm 1$ it would follow that $\tilde{c} = c \pm e$ were a basic class for X with a positive dimensional moduli space and hence X would not have simple type in contradiction to Corollary 13.22. Thus $k = \pm 1$. Assume first that $k = -1$ and thus

$$\Gamma_{\mathrm{can}} = \Gamma'\#\Gamma_{-1}$$

with $c_1(L_{\Gamma_{\mathrm{can}}}) = -c_1(K) = c' - e$. Let $E \to X$ be the complex line bundle with first Chern class $c_1(E) = e = \mathrm{PD}([S])$. Then

$$\Gamma_E = \Gamma_{\mathrm{can}} \otimes E \cong \Gamma'\#(\Gamma_{-1} \otimes E) \cong \Gamma'\#\Gamma_1$$

with $c_1(L_{\Gamma_E}) = 2c_1(E) - c_1(K) = c' + e$. Hence, by Theorem 11.2,

$$\mathrm{SW}(X, \Gamma_E) \neq 0,$$

and it follows from Theorem 13.19 that the class $[S] = \mathrm{PD}(c_1(E))$ can be represented by an embedded symplectic submanifold $C = C_1 \cup \cdots \cup C_N$. As before

$$C_i \cdot C_i = c_1(K) \cdot C_i = c_1(E) \cdot C_i = g(C_i) - 1.$$

But now the sum of the self-intersection numbers is -1 and so one of the components is a symplectically embedded sphere of self-intersection number -1. Let this be C_1 and consider the union $C' = C_2 \cup \cdots \cup C_N$. Then $C' \cdot C' = C' \cdot C = c_1(K) \cdot C' = 0$. One can show, using successive reflection diffeomorphisms localized near embedded spheres with self-intersection numbers minus one, that the homology class $A_\nu := [C] - \nu[C']$ can be represented by a smoothly embeddeded sphere for every $\nu \in \mathbb{Z}$. Hence, by the above argument, the class A_ν can be represented by an embedded symplectic submanifold for every $\nu \in \mathbb{Z}$. This implies $C' = \emptyset$ and $[C_1] = [S]$. This proves the corollary in the case $k = -1$. A similar argument in the case $k = +1$ with S replaced by $-S$ shows that in that case the class $-[S]$ can be represented by a symplectically embedded sphere with self-intersection number -1. This proves the corollary. $\square$

Corollary 13.24. (Taubes) *Up to diffeomorphism there is a unique symplectic structure on* $\mathbb{C}P^2$ *with volume* 1.

Proof: Let $e = \mathrm{PD}([\mathbb{C}P^1])$ and choose a line bundle $E \to \mathbb{C}P^2$ with $c_1(E) = e$. Then, by Example 9.11, $\mathrm{SW}^+(\mathbb{C}P^2, \Gamma_E) = 1$. Fix a symplectic form ω on $\mathbb{C}P^2$ and suppose without loss of generality that $\int_{\mathbb{C}P^1} \omega = 1$. (Otherwise replace ω by $-\omega$ and note that there is a diffeomorphism of $\mathbb{C}P^2$ inducing the reflection on H^2.) Let $J \in \mathcal{J}(X, \omega)$ be compatible with ω and denote by Γ_J the canonical spin^c structure of J. Then $\mathrm{SW}^+(\mathbb{C}P^2, \Gamma_J) = 1$ (because the two orientations of $H^{2,+}$ agree) and hence $c_1(T\mathbb{C}P^2, J) = 3e = c_1(T\mathbb{C}P^2, J_0)$. Since $\mathrm{SW}^+(X, \Gamma_E) = 1$ there exists a symplectic submanifold $C \subset \mathbb{C}P^2$ representing the class $[\mathbb{C}P^1]$. This curve must be connected (any two components would have nonzero intersection number) and the adjunction formula $2g-2 = C \cdot C - c_1(T\mathbb{C}P^2, J) \cdot C = -2$ shows that it is a sphere. Hence $[\mathbb{C}P^1]$ can be represented by a symplectically embedded sphere C. Under this condition Gromov proved in [47] that there exists a diffeomorphism $\psi : \mathbb{C}P^2 \to \mathbb{C}P^2$ such that $\psi^*\omega$ is the standard symplectic structure on $\mathbb{C}P^2$. $\square$

Remark 13.25 At the time of writing it seems to be an open question whether there is a symplectic 4-manifold (X, ω) which is homeomorphic, but not diffeomorphic, to $\mathbb{C}P^2$. If the volume is normalized to 1 then it follows from the Hirzebruch signature formula and Theorem 13.29 below that such a manifold must satisfy $c_1(TX, J) = -3[\omega]$ for $J \in \mathcal{J}(X, \omega)$. $\square$

Exercise 13.26 Use the wallcrossing formula to compute the Seiberg-Witten invariants of a *fake symplectic homotopy* $\mathbb{C}P^2$, i.e. a compact symplectic 4-manifold (X, ω) which is homeomorphic to $\mathbb{C}P^2$ and satisfies $c_1(TX, J) = -3[\omega]$ for $J \in \mathcal{J}(X, \omega)$. $\square$

13.4 Irreducibility

Definition 13.27 *A compact smooth* 4*-manifold* X *is called* **irreducible** *if in every connected sum decomposition* $X \cong X_1 \# X_2$ *one of the summands is a homotopy* 4*-sphere.*

Theorem 13.28. (Kotschick) *Every simply connected minimal symplectic* 4*-manifold with* $b^+ \geq 2$ *is irreducible.*

Proof: It follows from Theorem 11.1 that in any connected sum decomposition of X one of the summands has a negative definite intersection form. Hence assume

$$X = X' \# N$$

where Q_N is negative definite. We must prove that $b_2(N) = 0$. Suppose otherwise that $b_2(N) > 0$. Then, by Donaldson's theorem 9.6, the intersection form of N is diagonalizable. By Theorem 11.2 and Theorem 13.8, there exists a basic class $c' \in H^2(X', \mathbb{Z})$ and a characteristic vector $e \in H^2(N, \mathbb{Z})$ such that the canonical class of X is given by

$$-c_1(K) = c' + e.$$

One argues as in the proof of Corollary 13.23 that

$$e \cdot e + b_2(N) = 0.$$

(The left hand side is necessarily nonpositive, and if it were negative then there would exist a basic class $c = c' + \tilde{e}$ for X with a positive dimensional moduli space.) Now choose a basis $e_1, \ldots, e_\ell$ of $H^2(N, \mathbb{Z})$ with respect to which Q_N has diagonal form. Thus $e_i \cdot e_j = -\delta_{ij}$ and $e = \sum_i \varepsilon_i e_i$ where $\varepsilon_i = \pm 1$. Suppose without loss of generality that

$$e = -e_1 - \cdots - e_\ell$$

and consider the class $e' = e_1 - e_2 - \cdots - e_\ell$. Then, by Theorem 11.2, the class

$$2e_1 - c_1(K) = c' + e'$$

is a basic class for X and hence, by Theorem 13.19, the Poincaré dual of e_1 can be represented by an embedded symplectic submanifold and one argues as in the proof of Corollary 13.23 that this must be a sphere (or else there exists a nontorsion homology class with zero self-intersection number which is also represented by a sphere, in contradiction to Theorem 14.1.) Hence there exists a symplectically embedded sphere with self-intersection number -1 in contradiction to the assumption of minimality. This proves the theorem. □

In the non-simply connected case Kotschick proved that in every connected sum decomposition $X \cong X_1 \# X_2$ one of the summands is a homology 4-sphere whose fundamental group has no nontrivial finite quotient. In full generality, the conjecture that minimal compact symplectic 4-manifolds are irreducible seems to be still open at the time of writing.

13.5 Rational and ruled surfaces

Before the Seiberg-Witten invariants were discovered Gromov's techniques of pseudoholomorphic curves were used by McDuff in [77] to prove that every minimal symplectic 4-manifold which contains a symplectically embedded 2-sphere with nonnegative self-intersection number is a rational or ruled surface. At the time such embedded spheres were hard to come by.

Now Taubes' theorem 13.19 is a powerful existence result for J-holomorphic curves which can be combined with McDuff's theorem to round off the earlier results and give a more complete picture of rational and ruled surfaces. This concerns both the question of uniqueness of symplectic structures and that of finding topological criteria under which a minimal symplectic 4-manifold is rational or ruled. The following theorem was proved by Liu [74] and, independently, by Ohta-Ono [101]. Liu and Ohta-Ono actually prove the stronger result that every symplectic 4-manifold which admits a metric of positive scalar curvature is a blowup of a rational or ruled surface. In [101] Ohta and Ono also show that every monotone symplectic 4-manifold* is diffeomorphic to a del-Pezzo surface (i.e. to either $S^2 \times S^2$ or $\mathbb{C}P^2$ with up to eight points blown up).

Theorem 13.29. (Liu,Ohta-Ono) *Let X be a minimal symplectic 4-manifold. Then the following are equivalent.*

(i) *X admits a metric of positive scalar curvature.*

(ii) *X admits a symplectic structure ω with $c_1(K) \cdot [\omega] < 0$.*

(iii) *X is either rational or ruled.*

Theorem 13.30. (Liu) *Let X be a minimal symplectic 4-manifold with $c_1(K) \cdot c_1(K) < 0$. Then X is a ruled surface.*

This result was conjectured by Gompf in [40]. The proofs of both theorems will be given below. An immediate consequence of these results is that for a generic almost complex structure there do not exist any J-holomorphic spheres in a minimal symplectic 4-manifold which is not rational or ruled (see [86]).

Corollary 13.31 *Let X be a minimal symplectic 4-manifold which is not rational or ruled. Then there exists a set $\mathcal{J}_0(X, \omega) \subset \mathcal{J}(X, \omega)$ of compatible almost complex structures which is of the second category in the sense of Baire (a countable intersection of open and dense sets) and has the property that for every $J \in \mathcal{J}_0(X, \omega)$ X contains no J-holomorphic spheres.*

Proof: In [78] McDuff proved that if there exists an immersed J-holomorphic sphere $C \subset X$ with $c_1(TX) \cdot C = -c_1(K) \cdot C \geq 2$ then X must be rational or ruled. Since every J-holomorphic curve $u : S^2 \to X$ can be perturbed to an immersed sphere by a change in J this rules out all J-holomorphic spheres with Chern number at least 2. Moreover, J-holomorphic spheres with $c_1(TX) \cdot C \leq 0$ form moduli spaces of negative virtual dimension and hence there cannot be such spheres for a generic J. This leaves the possibility of J-holomorphic spheres C with $c_1(K) \cdot C = -1$.

*A symplectic manifold is called **monotone** if the cohomology class $[\omega]$ is a positive multiple of the first Chern class $c_1(TX, J)$ for a compatible almost complex structure $J \in \mathcal{J}(X, \omega)$.

If these are embedded then $C \cdot C = -1$ and this contradicts minimality. On the other hand, if they are not embedded, then the adjunction formula shows that $C \cdot C \geq 0$. Moreover, by Theorem 13.30, we have $c_1(K) \cdot c_1(K) \geq 0$ and the curve C satisfies $c_1(K) \cdot C < 0$ and $[\omega] \cdot C > 0$. Since $b^+ = 1$ it follows easily that $c_1(K) \cdot [\omega] < 0$. Hence Theorem 13.29 shows that X is rational or ruled, a contradiction. □

The following theorem addresses the uniqueness question for symplectic structures on ruled surfaces. We shall only state the result and refer the reader for the proof to the original papers by [73] by Li-Liu and [68] by Lalonde-McDuff.

Theorem 13.32. (Li-Liu,Lalonde-McDuff) *Let X be a 2-sphere bundle over a Riemann surface. Then for any two symplectic structures ω_0 and ω_1 which represent the same cohomology class $[\omega_0] = [\omega_1]$ there exists a diffeomorphism $\psi : X \to X$ such that $\omega_0 = \psi^*\omega_1$. More generally, any two symplectic forms on X are equivalent up to deformation and diffeomorphism.*

The second statement was established first by Li-Liu [73] and the first was then proved by Lalonde-McDuff [68]. By Moser's theorem, it suffices to prove that any two symplectic forms on a ruled surface in the same cohomology class can be connected by a path of cohomologous symplectic forms. The deformation equivalence is easier to prove once a holomorphic sphere with self-intersection number zero has been found. Under this assumption the deformation equivalence was established by McDuff in [77]. Her argument *enlarges the base* and thus changes the cohomology class of the symplectic form. The argument by Li-Liu establishes the existence of the holomorphic sphere.

The paper [73] by Li-Liu contains further results about symplectic 4-manifolds with $b^+ = 1$ which we shall not discuss in detail. For example they prove that Corollary 13.23 (about symplectically embedded spheres with self-intersection number -1) continues to hold in the case $b^+ = 1$ provided that $c_1(K) \cdot S = \pm 1$. This implies the extension of Theorem 13.32 to blowups of rational or ruled surfaces with standard canonical class. The proofs of Theorems 13.29 and 13.30 are based upon the following lemma.

Lemma 13.33. (Liu,Ohta-Ono) *If X is a minimal symplectic 4-manifold and there exists a line bundle $E \to X$ with*

$$\mathrm{SW}^+(X, \Gamma_E) \neq 0, \qquad c_1(E) \cdot c_1(E) + c_1(K) \cdot c_1(E) < 0$$

then X is rational or ruled.

Proof: By Theorem 13.19, the class $\mathrm{PD}(c_1(E))$ can be represented by a symplectic submanifold $C = C_1 \cup \cdots \cup C_N$. Since $C_i \cdot C_i = c_1(E) \cdot C_i$ the adjunction formula reads $2g(C_i)-2 = c_1(E)\cdot C_i + c_1(K)\cdot C_i$. Take the sum over all i to obtain

$$\sum_{i=1}^{N}(2g(C_i)-2) = c_1(E)\cdot c_1(E) + c_1(K)\cdot c_1(E) < 0.$$

Hence one of the components C_i is a sphere. Since X is minimal this component has nonnegative self-intersection number. Under these conditions it was proved by McDuff that X is rational or ruled (cf. [77]). □

Proof of Theorem 13.30: The argument given here was explained to me by McDuff. For any class $e \in H^2(X,\mathbb{Z})$ denote by $w(e)$ the wall-crossing number of the spin^c-structure Γ_E with $c_1(E) = e$, namely

$$w(e) = \frac{1}{k!}\int_{\mathcal{T}}\left(\frac{1}{4}\int_X \Omega\wedge\Omega\wedge(K-2e)\right)^k$$

where $2k = b_1(X)$. Here, and throughout the proof, we denote the canonical class by K (instead of $c_1(K)$). Note first that, by Corollary 13.20, we have $\mathrm{SW}^+(X,\Gamma_K) = \mathrm{SW}^-(X,\Gamma_{\mathrm{can}}) = 0$ and hence $w(0) = 1$. This shows that not all wall-crossing numbers are zero and hence the set

$$\mathcal{H} = \left\{a_1 \cup a_2 \,|\, a_i \in H^1(X,\mathbb{Z})\right\} \subset H^2(X,\mathbb{Z})$$

contains a nontorsion element whenever $b_1 \neq 0$. We now claim that there exists a class $a \in H^2(X,\mathbb{Z})$ such that

$$a\cdot a = 0, \qquad K\cdot a < 0, \tag{13.5}$$

$$q(2p-1)|K\cdot a| + (p^2-p)K\cdot K \geq 0 \quad\Longrightarrow\quad w(pK-qa)\neq 0. \tag{13.6}$$

Note here that the dimension of the moduli space for the spin^c structure Γ_{pK-qa} is given by

$$\begin{aligned}\dim \mathcal{M}(X,\Gamma_{pK-qa}) &= (pK-qa)\cdot(pK-qa) - K\cdot(pK-qa)\\ &= (p^2-p)K\cdot K + q(2p-1)|K\cdot a|.\end{aligned}$$

In the case $b_1 = 0$ condition (13.6) is automatically satisfied and the existence of a class a which satisfies (13.5) is an easy exercise. In the case $b_1 > 0$ choose a nonzero class $a \in \mathcal{H}$ and note that, by Lemma 9.15, $\Omega\wedge\Omega\wedge a = 0$ and $a\wedge a = 0$. Hence

$$\begin{aligned}w(pK-qa) &= w(pK)\\ &= \frac{1}{k!}\int_{\mathcal{T}}\left(\frac{1}{4}\int_X \Omega\wedge\Omega\wedge(1-2p)K\right)^k\end{aligned}$$

$$
\begin{aligned}
&= (1-2p)^k w(0) \\
&= (1-2p)^k.
\end{aligned}
$$

Thus we have found a class a which satisfies (13.5) and (13.6). Now for any class $e \in H^2(X,\mathbb{Z})$ the wall-crossing formula of Theorem 9.14 can be expressed in the form

$$
\mathrm{SW}^+(X,\Gamma_e) + (-1)^k \mathrm{SW}^+(X,\Gamma_{K-e}) = w(e).
$$

Hence if $w(e) \neq 0$ then one of the invariants $\mathrm{SW}^+(X,\Gamma_e)$, $\mathrm{SW}^+(X,\Gamma_{K-e})$ is nonzero.

We shall prove that for some values of p and q the class

$$
e = pK - qa
$$

satisfies the requirements of Lemma 13.33. Let us first consider the class $e = a$. For this class the moduli space has positive dimension and hence, by (13.6), $w(a) \neq 0$. Hence either $\mathrm{SW}^+(X,\Gamma_a) \neq 0$ or $\mathrm{SW}^+(X,\Gamma_{K-a}) \neq 0$. If the invariant for a is nonzero, then the class $e = a$ satisfies the requirements of Lemma 13.33 and hence X is rational or ruled. Hence assume

$$
\mathrm{SW}^+(X,\Gamma_a) = 0, \qquad \mathrm{SW}^+(X,\Gamma_{K-a}) \neq 0.
$$

If $3K\cdot a - 2K\cdot K > 0$ then the class $e = K - a$ satisfies the requirements of Lemma 13.33 and again we are done. Hence assume

$$
3K\cdot a - 2K\cdot K \leq 0.
$$

Abbreviate

$$
s = |K\cdot K|, \qquad t = |K\cdot a|, \qquad \lambda = \frac{s}{t} \leq \frac{3}{2}.
$$

We must find p and q such that $e = pK - qa$ satisfies

$$
e\cdot e + K\cdot e < 0, \qquad e\cdot e - K\cdot e \geq 0 \tag{13.7}
$$

(as well as $\mathrm{SW}^+(X,\Gamma_e) \neq 0$). If $p > 1$ then these inequalities can be expressed in the form

$$
\frac{2p+1}{p^2+p} < \frac{\lambda}{q} \leq \frac{2p-1}{p^2-p}. \tag{13.8}
$$

Equivalently $f(p) \leq \lambda/q \leq f(p-1)$ where $f(p) = (2p+1)/(p^2+p)$ is strictly decreasing with $f(1) = 3/2$. Hence for every $q \geq 1$ there exists a unique

$p \geq 2$ for which the inequality (13.8) is satisfied. For any such values of p and q the class $e = pK - qa$ satisfies (13.7). Moreover, $w(e) \neq 0$ and hence either $\mathrm{SW}^+(X, \Gamma_e) \neq 0$ or $\mathrm{SW}^+(X, \Gamma_{K-e}) \neq 0$. We must exclude the latter. To see this recall that $\mathrm{SW}^+(X, \Gamma_{K-a}) \neq 0$. Hence if $\mathrm{SW}^+(X, \Gamma_{K-e}) \neq 0$ then, by Corollary 13.20, $(K-a) \cdot (K-e) > 0$. By a simple computation

$$(K-e) \cdot (K-a) = (p-1)s - (p+q-1)t.$$

We must find p and q which satisfy (13.8) and for which the last number is negative, i.e.

$$\lambda < \frac{p+q-1}{p-1}. \tag{13.9}$$

This will be satisfied if (13.8) holds and

$$\frac{q(2p-1)}{p^2-p} < \frac{p+q-1}{p-1}.$$

But this last inequality is equivalent to $q < p$. However, for every $q \geq 1$ the unique p for which (13.8) holds is at least 2 and hence

$$\frac{5}{3} \leq \frac{2p+1}{p+1} < \lambda \frac{p}{q} \leq \frac{3}{2} \frac{p}{q}.$$

Hence $p/q > 10/9 > 1$ and thus $q < p$ as required. Having found p and q which satisfy (13.8) and (13.9) we conclude that the class $e = pK - qa$ satisfies the requirements of Lemma 13.33 and hence X is rational or ruled. □

Proof of Theorem 13.29: That (iii) implies both (i) and (ii) is obvious. We prove that (i) implies (ii). By Theorem 13.9, $\mathrm{SW}^+(X, \Gamma_{\mathrm{can}}) \neq 0$. Hence it follows as in the proof of Theorem 12.14 for the Kähler case that

$$K \cdot [\omega_g] < 0$$

for every metric g with positive scalar curvature. In particular, K is not a torsion class. If $K \cdot K < 0$ then, by Theorem 13.30, X is rational or ruled and thus satisfies (ii). Hence assume $K \cdot K \geq 0$. Under this assumption Lemma 12.15 shows that $K \cdot [\omega_g] < 0$ for every metric g and, in particular, $K \cdot [\omega] < 0$ for the symplectic form ω.

We prove that (ii) implies (iii). Again it suffices to assume $K \cdot K \geq 0$. Under this condition the identity $K \cdot K = 9 - 4b_1 - b^- \geq 0$ forces b_1 to be either 0 or 2. There are five cases to consider.

Case 1: *$b_1 = 0$ and Q_X is odd (Example: $X = \mathbb{C}P^2$).*

In this case $Q_X = (1) \oplus m(-1)$ with $m \leq 9$. Choose corresponding generators $\alpha, \beta_1, \ldots, \beta_m$ of H^2 with $\alpha \cdot \alpha = 1$, $\alpha \cdot \beta_j = 0$, and $\beta_i \cdot \beta_j = -\delta_{ij}$. Assume

$$K = \lambda\alpha + \sum_i \mu_i\beta_i, \qquad [\omega] = \alpha + \sum_i \varepsilon_i\beta_i$$

where λ and μ_i are odd integers and $\sum_i \varepsilon_i^2 < 1$. Examining $K \cdot K = 9 - m = \lambda^2 - \sum_i \mu_i^2$ we find $\lambda^2 \geq 9$. Moreover, since $[\omega] \cdot K < 0$ we must have $\lambda \leq -3$. (It is an easy consequence of the Cauchy-Schwarz inequality that if $\lambda > 0$ then $K \cdot [\omega] > 0$.) Now consider the spin^c structure $\Gamma_\alpha = \Gamma_E$ with $c_1(E) = \alpha$. This class satisfies $\alpha \cdot [\omega] = 1 > K \cdot [\omega]$ and hence it follows from Theorem 13.9 that $\mathrm{SW}^-(X, \Gamma_\alpha) = 0$. Since the moduli space has positive dimension $\alpha \cdot \alpha - K \cdot \alpha > 0$ the wall-crossing formula of Theorem 9.9 asserts that $\mathrm{SW}^+(X, \Gamma_\alpha) = 1$. Since $\alpha \cdot \alpha + K \cdot \alpha = 1 + \lambda < 0$ it follows from Lemma 13.33 that X is rational or ruled.

Case 2: *$b_1 = 0$ and $Q_X = H$ (Example: $X = S^2 \times S^2$).*

Choose classes $\alpha, \beta \in H^2(X, \mathbb{Z})$ such that $\alpha \cdot \alpha = \beta \cdot \beta = 0$ and $\alpha \cdot \beta = 1$. Since $K \cdot [\omega]$ suppose without loss of generality that

$$K = \lambda\alpha + \mu\beta, \qquad [\omega] = \alpha + \varepsilon\beta$$

where $\lambda < 0$ and $\mu < 0$ are even integers and $\varepsilon > 0$. Now argue as in Case 1 that $\alpha \cdot [\omega] = \varepsilon > K \cdot [\omega]$ and hence $\mathrm{SW}^-(X, \Gamma_\alpha) = 0$. Since $\alpha \cdot \alpha - K \cdot \alpha = -\mu > 0$ it follows again from the wall-crossing formula that $\mathrm{SW}^+(X, \Gamma_\alpha) = 1$ and finally, since $\alpha \cdot \alpha + K \cdot \alpha = \mu < 0$, Lemma 13.33 asserts that X is rational or ruled.

Case 3: *$b_1 = 0$ and $Q_X = (-E_8) \oplus H$ (Example: Enriques surface, but not with positive scalar curvature).*

In this case choose additional generators $\gamma_1, \dots, \gamma_8$ of H^2 corresponding to $-E_8$. Thus $\gamma_i \cdot \gamma_i = -2$ and $\gamma_i \cdot \gamma_j = 0, 1$ according to the Dynkin diagram. Now argue as above with

$$K = \lambda\alpha + \mu\beta + \sum_{i=1}^{8} \nu_i\gamma_i, \qquad [\omega] = \alpha + \varepsilon\beta + \sum_{i=1}^{8} \delta_i\gamma_i.$$

where λ, μ, ν_i are even integers and $\varepsilon > 0$. Since $K \cdot K = 0$ and K is not torsion we have $\lambda\mu > 0$. It follows easily from the Cauchy-Schwarz inequality (for the γ-part of K and $[\omega]$) that

$$\left| \sum_{i,j=1}^{8} \delta_i \nu_j \gamma_i \cdot \gamma_j \right| \leq 2\sqrt{\lambda\mu\varepsilon} \leq \varepsilon|\lambda| + |\mu|.$$

Hence the condition $K \cdot [\omega] < 0$ implies that $\lambda < 0$ and $\mu < 0$. (The H-part of the product $K \cdot [\omega]$ dominates the E_8-part and hence determines the sign.) Now consider the spin^c structure Γ_α with characteristic class $K-2\alpha$. This class again satisfies

$$\alpha \cdot [\omega] = \varepsilon > K \cdot [\omega], \qquad \alpha \cdot \alpha - K \cdot \alpha = -\mu > 0$$

and it follows as in Step 2 that $\mathrm{SW}^-(X, \Gamma_\alpha) = 0$ and $\mathrm{SW}^+(X, \Gamma_\alpha) = 1$. Since $\alpha \cdot \alpha + K \cdot \alpha = \mu < 0$ Lemma 13.33 implies that X is rational or ruled. On the other hand, no rational or ruled surface has intersection form $(-E_8) \oplus H$ and hence there is no manifold with positive scalar curvature which has this intersection form.

Case 4: *$b_1 = 2$ and Q_X is odd (Example: nontrivial S^2-bundle over $\mathbb{T}^2$).*

In this case we must have $K \cdot K = 0$ and $b^- = 1$. To see this note that, since $K \cdot [\omega] < 0$, we have $\mathrm{SW}^-(X, \Gamma_{\mathrm{can}}) = SW^+(X, \Gamma_K) = 0$ and hence not all the wall crossing numbers are zero. Hence not all the cup-products of two classes in $H^1(X, \mathbb{Z})$ can vanish. Any such nonzero cup-product is a class in H^2 with square zero and hence $b^- \geq 1$. Since $0 \leq K \cdot K = 9 - 4b_1 - b^- = 1 - b^-$ it follows that $b^- = 1$ and $K \cdot K = 0$ as claimed. Thus $Q_X = (1) \oplus (-1)$. Choose α, β with $\alpha \cdot \alpha = 1$, $\beta \cdot \beta = -1$, $\alpha \cdot \beta = 0$. Then, by reversing the sign of α and β if necessary, we have

$$K = \lambda\alpha + \lambda\beta, \qquad [\omega] = \alpha + \varepsilon\beta$$

where $\lambda < 0$ and $-1 < \varepsilon < 1$. (As in Case 2 the α-coefficients of K and ω must have opposite sign.) Now consider the spin^c structure Γ_e corresponding to the class

$$e = \alpha - \beta.$$

Since $e \cdot [\omega] = 1 + \varepsilon > 0 > K \cdot [\omega]$ it follows again that $\mathrm{SW}^-(X, \Gamma_e) = 0$. Moreover,

$$e \cdot e = 0, \qquad K \cdot e = 2\lambda < 0.$$

and hence the moduli space has positive dimension. Now, since the wall crossing number of K is nonzero the class $\alpha + \beta$ cannot be in $\mathcal{H}$ and hence $e = \alpha - \beta \in \mathcal{H}$. Thus it follows as in the proof of Theorem 13.30 that the wall-crossing number of e agrees with that of K and hence is 1. Thus $\mathrm{SW}^+(X, \Gamma_e) = 1$. Moreover, $e \cdot e + K \cdot e = 2\lambda < 0$ and hence Lemma 13.33 shows that X is rational or ruled.

Case 5: *$b_1 = 2$ and Q_X is even (Example: $X = \mathbb{T}^2 \times S^2$).*

In this case $b^- = 1$, $K \cdot K = 0$, and $Q_X = H$. Choose α, β as in Case 2 with $\alpha \cdot \alpha = \beta \cdot \beta = 0$ and $\alpha \cdot \beta = 1$. Then K is a nonzero multiple of either α or β. Suppose, without loss of generality

$$K = \lambda\alpha, \qquad [\omega] = \alpha + \varepsilon\beta$$

where $\lambda < 0$ and $\varepsilon > 0$. Consider the spin^c structure Γ_β. First note that $\beta \cdot \beta = 0 > K \cdot \beta$ and hence the moduli space has positive dimension. Moreover, as in Case 4, $\beta \in \mathcal{H}$ and hence the wall-crossing number of Γ_β is $w(K-2\beta) = w(K) = 1$. Since $\beta \cdot [\omega] = 1 > K \cdot [\omega]$ we have $\mathrm{SW}^-(X, \Gamma_\beta) = 0$ and $\mathrm{SW}^+(X, \Gamma_\beta) = 1$. With $\beta \cdot \beta + K \cdot \beta = \lambda < 0$ it follows again from Lemma 13.33 that X is rational or ruled. This completes the proof of the theorem. □

Proof of Theorem 13.32: The proof of Theorem 13.29 shows that for every symplectic structure ω on a ruled surface X there exists a symplectically embedded 2-sphere with nonnegative self-intersection number. Under these conditions it was proved by McDuff in [77] that ω is deformation equivalent to a standard symplectic structure. □

Note that the proofs in this section reflect the divison into positive curvature (sphere), zero curvature (torus), and negative curvature (higher genus) for both complex curves and Kähler surfaces. Even though the rational and ruled surfaces belong to the *positive scalar curvature group* there is a natural further subdivision into

the positive case $K \cdot K > 0$**:** $\mathbb{C}P^2$, $S^2 \times S^2$ (Cases 1 and 2 in the proof of Theorem 13.29),

the elliptic case $K \cdot K = 0$**:** Sphere bundles over $\mathbb{T}^2$ (Cases 4 and 5 in the proof of Theorem 13.29),

general type $K \cdot K < 0$**:** Sphere bundles over surfaces of higher genus (Theorem 13.30).

13.6 Proofs of Taubes' theorems

Let us begin by examining the Seiberg-Witten equations in the symplectic case. Given a symplectic 4-manifold (X, ω) with $J \in \mathcal{J}(X, \omega)$ consider the canonical spin^c structure $\Gamma_{\mathrm{can}} : TX \to \mathrm{End}(W_{\mathrm{can}})$ with $W_{\mathrm{can}} = \Lambda^{0,*}T^*X$ and $L_{\Gamma_{\mathrm{can}}} = K^*$. Consider the spin^c connection ∇_{can} on W_{can} introduced in (6.8). Recall from Lemma 6.16 that the induced connection on $K^* = L_{\Gamma_{\mathrm{can}}}$ is given by $\widetilde{\nabla}$ and, as in the Kähler case, denote the corresponding *virtual connection* on ${L_{\Gamma_{\mathrm{can}}}}^{1/2} = K^{-1/2}$ by A_{can}. Its curvature is the 2-form $F_{A_{\mathrm{can}}} = \frac{1}{2}\mathrm{trace}^c(\widetilde{R}) \in \Omega^2(X, i\mathbb{R})$. where $\widetilde{R} \in \Omega^2(X, \mathrm{End}(TX))$ denotes the curvature tensor of the connection $\widetilde{\nabla}$ on TX. By Lemma 3.21,

$$F_{A_{\mathrm{can}}}(u, v) = -\frac{i}{4}\mathrm{trace}(JR(u, v)) + \frac{i}{16}\mathrm{trace}(J[\nabla_u J, \nabla_v J]). \tag{13.10}$$

Recall that the second term on the right is of type $(1,1)$.

Now take the tensor product with a Hermitian line bundle $E \to X$ with a connection $B \in \mathcal{A}(E)$. Thus consider the bundles

$$W_E^+ = (\Lambda^{0,0} \oplus \Lambda^{0,2}) \otimes E, \quad W_E^- = \Lambda^{0,1} \otimes E, \quad L_{\Gamma_E} = K^* \otimes E^2.$$

where $\Lambda^{p,q} = \Lambda^{p,q}T^*X$. The corresponding spin^c connection on W_E is given by $\nabla_A = \nabla_{\mathrm{can}} + B$ and the induced connection on the *virtual bundle* ${L_{\Gamma_E}}^{1/2} = K^{-1/2} \otimes E$ is given by $A = A_{\mathrm{can}} + B$ where $A_{\mathrm{can}} \in \mathcal{A}(K^{-1/2})$ is the connection associated to the standard spin^c structure. The curvature 2-form of the connection $A = A_{\mathrm{can}} + B$ is given by $F_A = F_{A_{\mathrm{can}}} + F_B$. The Seiberg-Witten equations for this connection take the following form.

Proposition 13.34 *In the symplectic case the Seiberg-Witten equations for the pair* $(A_{\mathrm{can}} + B, \Phi)$ *and the perturbation* $\eta \in \Omega^{2,+}(X, g)$ *take the form*

$$\begin{aligned} \bar{\partial}_A \varphi_0 + \bar{\partial}_A^* \varphi_2 &= 0, \\ 2(F_{A_{\mathrm{can}}} + F_B + \eta)^{0,2} &= \bar{\varphi}_0 \varphi_2, \\ 4i(F_{A_{\mathrm{can}}} + F_B + \eta)_\omega &= |\varphi_2|^2 - |\varphi_0|^2. \end{aligned} \tag{13.11}$$

where $\Phi = (\varphi_0, \varphi_2) \in \Omega^{0,0}(X, E) \times \Omega^{0,2}(X, E)$.

As in the Kähler case this result follows immediately from Theorem 6.17 and Lemma 4.62. The only difference is that in the Kähler case the curvature $F_{A_{\mathrm{can}}}$ is of type $(1,1)$ and so the corresponding term does not appear in the second equation of (12.1). In order to specify the sign in the definition of the Seiberg-Witten invariants it is necessary to specify an orientation of $H^1(X, i\mathbb{R}) \oplus H^{2,+}(X, i\mathbb{R})$. In the Kähler case this is quite obvious, since H^1 for example carries a natural complex structure $\alpha \mapsto *(\alpha \wedge \omega) = -\alpha \circ J$. In the symplectic case, however, the situation is slightly more complicated. For example, H^1 need not be even dimensional.

Remark 13.35. (Orientation) If (X, ω) is a compact connected symplectic 4-manifold then the cohomology group

$$H^0(X, i\mathbb{R}) \oplus H^1(X, i\mathbb{R}) \oplus H^{2,+}(X, i\mathbb{R})$$

admits a canonical orientation. To see this fix an almost complex structure $J \in \mathcal{J}(X, \omega)$ and consider the operator family

$$D_t : \Omega^{0,1}(X) \to \Omega^{0,0}(X) \oplus \Omega^{0,2}(X)$$

defined by

$$D_t \tau_1 = (\bar{\partial}^* \tau_1, \bar{\partial}\tau_1 - t\bar{\tau}_1 \circ N_J/4)$$

where $N_J : TX \otimes TX \to TX$ denotes the Nijenhuis tensor of J. For $t = 0$ this operator is complex linear and for $t = 1$ it is isomorphic to the self-duality operator $D^+ = d^* \oplus d^+$ (see the proof of Corollary 3.43 and the discussion thereafter). More precisely, there is a commutative diagram

$$\begin{array}{ccc}
\Omega^{0,1}(X) & \xrightarrow{D_1} & \Omega^{0,0}(X) \oplus \Omega^{0,2}(X) \\
\downarrow & & \downarrow \\
\Omega^1(X, i\mathbb{R}) & \xrightarrow{D^+} & \Omega^0(X, i\mathbb{R}) \oplus \Omega^{2,+}(X, i\mathbb{R})
\end{array}$$

where the vertical isomorphisms are given by $\tau_1 \mapsto \tau_1 - \bar{\tau}_1$ and $(\tau_0, \tau_2) \mapsto (2i\mathrm{Im}\,\tau_0, i(\mathrm{Re}\,\tau_0)\omega + \tau_2 - \bar{\tau}_2)$. The commutativity of the diagram can be expressed in the explicit form

$$d^*\beta = 2i\mathrm{Im}\,\bar{\partial}^*\beta^{0,1}, \quad (d\beta)_\omega = i\mathrm{Re}\,(\bar{\partial}^*\beta^{0,1}), \quad (d\beta)^{0,2} = \bar{\partial}\beta^{0,1} + \frac{1}{4}\beta^{1,0} \circ N_J.$$

for $\beta \in \Omega^1(X, i\mathbb{R})$ with $\tau_1 = \beta^{0,1}$. (See Corollary 3.28 and Proposition 3.16.) Now the determinant line of the complex linear operator D_0 carries a natural orientation and trivializing the determinant line bundle along the path $t \mapsto D_t$ gives an orientation of $\det(D_1)$. Since $\ker D_1 = H^1(X, i\mathbb{R})$ and $\mathrm{coker}\, D_1 = H^0(X, i\mathbb{R}) \oplus H^{2,+}(X, i\mathbb{R})$ this gives the required orientation of $H^0 \oplus H^1 \oplus H^{2,+}$. A simple homotopy argument shows that this orientation is independent of the choice of the almost complex structure $J \in \mathcal{J}(X, \omega)$ used to define it. □

The proofs of Theorems 13.8 and 13.9 are much the same as in the Kähler case. Taubes' original papers [116] and [117] contain a more complicated argument. The simplification in the proof below was indicated by Taubes in [118].

Proof of Theorems 13.8 and 13.9: Let $E \to X$ be a line bundle and consider the twisted spin^c structure $W_E = \Lambda^{0,*}T^*X \otimes E$. As in the Kähler case, consider the perturbation

$$\eta = i\widetilde{F}^+ + \pi\lambda\omega$$

with $\lambda > 0$. In the Kähler case λ was any positive number, but in the symplectic case λ will be chosen large. The Seiberg-Witten equations take the form

$$\begin{aligned}
\bar{\partial}_B\varphi_0 + \bar{\partial}_B^*\varphi_2 &= 0, \\
2F_B^{0,2} &= \bar{\varphi}_0\varphi_2, \\
4i(F_B)_\omega &= 4\pi\lambda + |\varphi_2|^2 - |\varphi_0|^2.
\end{aligned} \tag{13.12}$$

The difference from the Kähler case lies in the formula

$$\bar{\partial}_B\bar{\partial}_B\varphi_0 = F_B^{0,2}\varphi_0 - \frac{1}{4}(\partial_B\varphi_0) \circ N_J$$

of Proposition 3.16. In order to deal with the additional term involving the Nijenhuis tensor it will be necessary to choose λ large. It is convenient to introduce the rescaled sections

$$\psi_0 = \frac{1}{\sqrt{\lambda}}\varphi_0, \qquad \psi_2 = \frac{1}{\sqrt{\lambda}}\varphi_2.$$

as in Taubes [118]. Then (13.12) takes the form

$$\begin{aligned} \bar\partial_B\psi_0 + \bar\partial_B^*\psi_2 &= 0, \\ \frac{1}{\lambda}2F_B^{0,2} &= \bar\psi_0\psi_2, \\ \frac{1}{\lambda}4i(F_B)_\omega &= 4\pi + |\psi_2|^2 - |\psi_0|^2. \end{aligned} \tag{13.13}$$

In the symplectic case a solution (B, ψ_0, ψ_2) of (13.13) satisfies

$$\begin{aligned} 0 &= \left\|\bar\partial_B\psi_0 + \bar\partial_B^*\psi_2\right\|^2 \\ &= \left\|\bar\partial_B\psi_0\right\|^2 + \left\|\bar\partial_B^*\psi_2\right\|^2 + 2\left\langle\bar\partial_B\bar\partial_B\psi_0, \psi_2\right\rangle \\ &= 2\left\|\bar\partial_B\psi_0\right\|^2 + 2\left\langle F_B^{0,2}\psi_0, \psi_2\right\rangle - \frac{1}{2}\left\langle(\partial_B\psi_0)\circ N_J, \psi_2\right\rangle \\ &= 2\left\|\bar\partial_B\psi_0\right\|^2 + \lambda\left\|\bar\psi_0\psi_2\right\|^2 - \frac{1}{2}\left\langle(\partial_B\psi_0)\circ N_J, \psi_2\right\rangle \end{aligned}$$

and hence

$$2\left\|\bar\partial_B\psi_0\right\|^2 + \lambda\left\|\bar\psi_0\psi_2\right\|^2 = \frac{1}{2}\left\langle(\partial_B\psi_0)\circ N_J, \psi_2\right\rangle. \tag{13.14}$$

Here all norms and inner products are L^2-norms and L^2-inner products on X. Now recall the formula $2\bar\partial_B^*\bar\partial_B\psi_0 = d_B{}^*d_B\psi_0 - 2i(F_B)_\omega\psi_0$, from Proposition 3.25. Take the inner product with ψ_0 and use the formula

$$\int_X 2i(F_B)_\omega \mathrm{dvol} = \int_X i(F_B)_\omega\omega\wedge\omega = \int_X \omega\wedge iF_B = 2\pi[\omega]\cdot c_1(E)$$

to obtain

$$\begin{aligned} 2\left\|\bar\partial_B\psi_0\right\|^2 &= \left\|d_B\psi_0\right\|^2 - \int_X |\psi_0|^2 2i(F_B)_\omega \mathrm{dvol} \\ &= \left\|d_B\psi_0\right\|^2 - 8\pi^2[\omega]\cdot c_1(E) + \int_X \left(4\pi - |\psi_0|^2\right)2i(F_B)_\omega \\ &= \left\|d_B\psi_0\right\|^2 - 8\pi^2\lambda[\omega]\cdot c_1(E) \\ &\quad + \frac{\lambda}{2}\int_X \left(4\pi - |\psi_0|^2\right)\left(4\pi + |\psi_2|^2 - |\psi_0|^2\right) \\ &= \left\|d_B\psi_0\right\|^2 + 2\pi\lambda\left\|\psi_2\right\|^2 + \frac{\lambda}{2}\int_X \left(4\pi - |\psi_0|^2\right)^2 \\ &\quad - \frac{\lambda}{2}\left\|\bar\psi_0\psi_2\right\|^2 - 8\pi^2[\omega]\cdot c_1(E). \end{aligned}$$

Inserting this formula into (13.14) gives

$$\begin{aligned}\|d_B\psi_0\|^2 + \frac{\lambda}{2}\left\|\bar{\psi}_0\psi_2\right\|^2 + 2\pi\lambda\|\psi_2\|^2 + \frac{\lambda}{2}\int_X\left(4\pi - |\psi_0|^2\right)^2 \\ = 8\pi^2[\omega]\cdot c_1(E) + \frac{1}{2}\langle(\partial_B\psi_0)\circ N_J, \psi_2\rangle.\end{aligned}$$

The last term on the right is the *bad* one. It vanishes in the Kähler case but in the symplectic case it is not possible to control its sign. However, if λ is sufficiently large, this term can be estimated by the positive terms on the left. The inequality $ab \le \frac{\delta}{2}a^2 + \frac{1}{2\delta}b^2$ gives

$$\frac{1}{2}\langle(\partial_B\psi_0)\circ N_J, \psi_2\rangle \le \delta\|d_B\psi_0\|^2 + \frac{c}{\delta}\|\psi_2\|^2$$

for any $\delta > 0$. In the case $\pi\lambda \ge c/\delta$ this leads to

$$\begin{aligned}(1-\delta)\|d_B\psi_0\|^2 + \frac{\lambda}{2}\left\|\bar{\psi}_0\psi_2\right\|^2 + \left(2\pi\lambda - \frac{c}{\delta}\right)\|\psi_2\|^2 + \frac{\lambda}{2}\int_X\left(4\pi - |\psi_0|^2\right)^2 \\ \le 8\pi^2[\omega]\cdot c_1(E). \qquad (13.15)\end{aligned}$$

It follows that the Seiberg-Witten equations (13.13) cannot have any solutions with the perturbation $2\pi\lambda > c/\delta$ unless $[\omega]\cdot c_1(E) \ge 0$. Hence, in the case $b^+ > 1$ the first Chern class of every complex line bundle $E \to X$ with nonzero Seiberg-Witten invariants $\mathrm{SW}(X, \Gamma_E) \ne 0$ must satisfy

$$[\omega]\cdot c_1(E) \ge 0.$$

The same holds in the case $b^+ = 1$ if $SW^+(X, \Gamma_E) \ne 0$. Moreover, if $[\omega]\cdot c_1(E) = 0$ then any solution must satisfy

$$d_B\psi_0 = 0, \qquad |\psi_0| = \sqrt{4\pi}, \qquad \psi_2 = 0.$$

In particular, this shows that the bundle E has a nonvanishing section ψ_0 and hence admits a trivialization. By duality, if either $b^+ > 1$ and $SW(X, \Gamma_E) \ne 0$ or $b^+ = 1$ and $SW^-(X, \Gamma_E) \ne 0$ then $[\omega]\cdot c_1(E) \le [\omega]\cdot c_1(K)$ and equality implies that E is isomorphic to K. This proves the second assertion of Theorems 13.8 and 13.9. For the trivial bundle $E = X \times \mathbb{C}$ the conclusion is that every solution of (13.13) has the form

$$u^*B = 0, \qquad u^{-1}\psi_0 = \sqrt{4\pi}, \qquad \psi_2 = 0 \qquad (13.16)$$

where $u = |\psi_0|^{-1}\psi_0 : X \to S^1$. Hence the moduli space $\mathcal{M}(X, \Gamma, g, \eta)$ for the standard spin^c structure consists of a single point. To complete the proof of Theorems 13.8 and 13.9 it remains to show that this point is regular and to examine the orientations.

The linearized operator

Consider the standard solution $B = 0$, $\psi_0 = \sqrt{4\pi}$, $\psi_2 = 0$. Linearizing (13.13) we obtain

$$\begin{aligned} d^*\beta - \lambda i\langle i\psi_0, \tau_0\rangle &= 0, \\ 2i(d\beta)_\omega + \lambda \mathrm{Re}\,(\bar\psi_0\tau_0) &= 0, \\ \bar\partial\tau_0 + \bar\partial^*\tau_2 + \beta^{0,1}\psi_0 &= 0, \\ 2(d\beta)^{0,2} - \lambda\bar\psi_0\tau_2 &= 0, \end{aligned} \tag{13.17}$$

for $\beta \in \Omega^1(X, i\mathbb{R})$, $\tau \in \Omega^{0,0}(X)$, $\tau_2 \in \Omega^{0,2}(X)$. The first equation expresses the condition that the triple (β, τ_0, τ_2) is orthogonal to the tangent space of the orbit of $(B, \varphi_0, 0)$ under the action of the gauge group with respect to the inner product of the norm $\|\beta\|^2 + \lambda\|\tau_0\|^2 + \lambda\|\tau_2\|^2$. As in the Kähler case it follows from Corollary 3.28 that the first two equations are equivalent to

$$\bar\partial^*\beta - \frac{\lambda}{2}\bar\psi_0\tau_0 = 0.$$

Moreover, in the symplectic case we have, by Proposition 3.16,

$$(d\beta)^{0,2} = (d\beta^{0,1})^{0,2} + (d\beta^{1,0})^{0,2} = \bar\partial\beta^{0,1} + \frac{1}{4}\beta^{1,0} \circ N_J$$

Denote $\tau_1 = \beta^{0,1}$ so that $\bar\tau_1 = \beta^{1,0}$ and recall that $\psi_0 = \sqrt{4\pi}$ and $B = 0$. Then the linearized operator has the form

$$\mathcal{D} = \mathcal{D}_1 : \begin{matrix} \Omega^{0,0}(X) \\ \oplus \\ \Omega^{0,1}(X) \\ \oplus \\ \Omega^{0,2}(X) \end{matrix} \longrightarrow \begin{matrix} \Omega^{0,0}(X) \\ \oplus \\ \Omega^{0,1}(X) \\ \oplus \\ \Omega^{0,2}(X) \end{matrix}$$

where

$$\mathcal{D}_t \begin{pmatrix} \tau_0 \\ \tau_1 \\ \tau_2 \end{pmatrix} = \begin{pmatrix} \bar\partial^*\tau_1 \\ \bar\partial\tau_0 + \bar\partial^*\tau_2 \\ \bar\partial\tau_1 \end{pmatrix} + t \begin{pmatrix} -\sqrt{\pi}\lambda\tau_0 \\ \sqrt{4\pi}\tau_1 \\ -\bar\tau_1 \circ N_J/4 - \sqrt{\pi}\lambda\tau_2 \end{pmatrix}$$

for $\tau_i \in \Omega^{0,i}(X, E)$ and $0 \le t \le 1$. We prove that the operator $\mathcal{D}_t$ is bijective for $0 < t \le 1$ provided that λ is sufficiently large. To see this suppose that $\tau = (\tau_0, \tau_1, \tau_2)$ is in the kernel of $\mathcal{D}_t$. Then

$$t\sqrt{\pi}\lambda\tau_0 = \bar\partial^*\tau_1, \qquad t\sqrt{\pi}\lambda\tau_2 = \bar\partial\tau_1 - t\bar\tau_1 \circ N_J/4,$$

and

$$t\sqrt{\pi}\lambda(\bar{\partial}\tau_0 + \bar{\partial}^*\tau_2) + 2\pi\lambda t^2\tau_1 = 0.$$

Inserting the first two equations into the third we find

$$\bar{\partial}\bar{\partial}^*\tau_1 + \bar{\partial}^*\bar{\partial}\tau_1 + 2\pi\lambda t^2\tau_1 = t\bar{\partial}^*(\bar{\tau}_1 \circ N_J/4).$$

Take the L^2-inner product with τ_1 to obtain

$$\begin{aligned}\left\|\bar{\partial}^*\tau_1\right\|^2 + \left\|\bar{\partial}\tau_1\right\|^2 + 2\pi\lambda\left\|t\tau_1\right\|^2 &= \langle\bar{\partial}\tau_1, t\bar{\tau}_1 \circ N_J/4\rangle \\ &\le c\left\|\bar{\partial}\tau_1\right\|\left\|t\tau_1\right\| \\ &\le \frac{1}{2}\left\|\bar{\partial}\tau_1\right\|^2 + \frac{c^2}{2}\left\|t\tau_1\right\|^2\end{aligned}$$

If $2\pi\lambda > c^2/2$ then this implies that $\tau_1 = 0$ and hence $\tau_0 = 0$ and $\tau_2 = 0$. This shows that the unique solution of the Seiberg-Witten equations (13.13) is regular for $\lambda > 0$ sufficiently large and hence $\mathrm{SW}(X, \Gamma_{\mathrm{can}}) = \pm 1$ (respectively $\mathrm{SW}^+(X, \Gamma_{\mathrm{can}}) = \pm 1$ in the case $b^+ = 1$). In view of Remark 13.35 the discussion of orientations is similar to the Kähler case and will be omitted. □

13.7 Relation with the Gromov invariants

The goal of this section is to explain the main ideas of Taubes' proof of Theorem 13.19 about the existence of J-holomorphic curves from the Seiberg-Witten invariants. We begin with the following result about the behaviour of the solutions of (13.13) for $\lambda \to \infty$. The proof is based on the estimate (13.15) in the case where $[\omega] \cdot c_1(E) > 0$.

Proposition 13.36 *There exist constants $c > 0$ and $\lambda_0 > 0$ such that every solution (B, ψ_0, ψ_2) of (13.13) with $\lambda \ge \lambda_0$ satisfies the inequalities*

$$\left\|\bar{\partial}_B\psi_0\right\|^2 + \left\|\widetilde{\nabla}_B\psi_2\right\|^2 \le \frac{c}{\lambda}, \qquad \left\|\psi_2\right\|^2 + \left\|\bar{\psi}_0\psi_2\right\|^2 \le \frac{c}{\lambda^2}, \tag{13.18}$$

$$\begin{aligned}8\pi^2[\omega] \cdot c_1(E) - \frac{c}{\lambda} &\le \left\|\partial_B\psi_0\right\|^2 + \frac{\lambda}{2}\int_X \left(4\pi - |\psi_0|^2\right)^2 \\ &\le 8\pi^2[\omega] \cdot c_1(E) + \frac{c}{\lambda}.\end{aligned} \tag{13.19}$$

Proof: Using the identity $2\bar{\partial}_B\bar{\partial}_B^*\psi_2 = \widetilde{\nabla}_B{}^*\widetilde{\nabla}_B\psi_2 + 2i(F_B + 2F_{A_{\mathrm{can}}})_\omega\psi_2$ from Proposition 3.25 one obtains

$$2\left\|\bar{\partial}_B^*\psi_2\right\|^2 = \left\|\widetilde{\nabla}_B\psi_2\right\|^2 + \frac{\lambda}{2}\int_X |\psi_2|^4 + 2\pi\lambda\left\|\psi_2\right\|^2$$

$$- \frac{\lambda}{2} \left\| \bar{\psi}_0 \psi_2 \right\|^2 + \int_X 4i(F_{A_{\mathrm{can}}})_\omega |\psi_2|^2.$$

Inserting this and the above formula

$$\begin{aligned} \left\| \bar{\partial}_B \psi_0 \right\|^2 &= \left\| \partial_B \psi_0 \right\|^2 + 2\pi\lambda \left\| \psi_2 \right\|^2 + \frac{\lambda}{2} \int_X \left(4\pi - |\psi_0|^2 \right)^2 \\ &\quad - \frac{\lambda}{2} \left\| \bar{\psi}_0 \psi_2 \right\|^2 - 8\pi^2 [\omega] \cdot c_1(E) \end{aligned} \tag{13.20}$$

into

$$2 \left\| \bar{\partial}_B \psi_0 \right\|^2 + 2 \left\| \bar{\partial}_B^* \psi_2 \right\|^2 + 2\lambda \left\| \bar{\psi}_0 \psi_2 \right\|^2 = \langle (\partial_B \psi_0) \circ N_J, \psi_2 \rangle$$

(see (13.14)) one obtains the inequality

$$\begin{aligned} &(1-\delta) \left\| \partial_B \psi_0 \right\|^2 + \left\| \bar{\partial}_B \psi_0 \right\|^2 + \left\| \widetilde{\nabla}_B \psi_2 \right\|^2 + \lambda \left\| \bar{\psi}_0 \psi_2 \right\|^2 \\ &+ \frac{\lambda}{2} \int_X \left(\left(4\pi - |\psi_0|^2 \right)^2 + |\psi_2|^4 \right) + \left(4\pi\lambda - \frac{c}{\delta} \right) \left\| \psi_2 \right\|^2 \\ &\quad \leq 8\pi^2 [\omega] \cdot c_1(E) \end{aligned} \tag{13.21}$$

for $0 < \delta \leq 1$ which slightly strengthens (13.15). Now write the formula (13.20) in the form

$$\begin{aligned} &\left\| \partial_B \psi_0 \right\|^2 - \left\| \bar{\partial}_B \psi_0 \right\|^2 - \frac{\lambda}{2} \left\| \bar{\psi}_0 \psi_2 \right\|^2 + 2\pi\lambda \left\| \psi_2 \right\|^2 \\ &\quad + \frac{\lambda}{2} \int_X \left(4\pi - |\psi_0|^2 \right)^2 \mathrm{dvol} = 8\pi^2 [\omega] \cdot c_1(E), \end{aligned}$$

multiply this equation by $(\delta - 1)$, and add it to (13.21) to obtain

$$\begin{aligned} &\left\| \bar{\partial}_B \psi_0 \right\|^2 + \left\| \widetilde{\nabla}_B \psi_2 \right\|^2 + \lambda \left\| \bar{\psi}_0 \psi_2 \right\|^2 \\ &+ \frac{\lambda}{2} \int_X \left(|\psi_2|^4 + \delta (4\pi - |\psi_0|^2)^2 \right) + \left(2\pi\lambda - \frac{c}{\delta} \right) \left\| \psi_2 \right\|^2 \\ &\quad \leq \delta 8\pi^2 [\omega] \cdot c_1(E) \end{aligned}$$

for $0 < \delta \leq 1$. With $\delta = c/\pi\lambda$ this implies (13.18). Finally, (13.19) follows from (13.18) and (13.20). This proves the proposition. □

If $\mathrm{SW}(X, \Gamma_E) \neq 0$ then there must be a solution (B, ψ_0, ψ_2) for every λ and it is interesting to consider the limit $\lambda \to \infty$. As Taubes points out in [118], the section ψ_0 of E would like to be equal to $\sqrt{4\pi}$ everywhere as λ gets large while the section ψ_2 of $K^* \otimes E$ would like to be zero. But

ψ_0 cannot be nonzero everywhere unless E is trivial. Thus the geometric picture is that ψ_0 will converge to $\sqrt{4\pi}$ almost everywhere in X but will be zero somewhere. In [118] Taubes investigates the behaviour of the zero sets

$$C_\lambda = {\psi_0}^{-1}(0) \subset X$$

as $\lambda \to \infty$. Heuristically, the first inequality in (13.18) suggests that ψ_0 will look more and more like a holomorphic section as $\lambda \to \infty$. Recall that in the Kähler case either ψ_0 or ψ_2 must vanish, and for large λ this can only be ψ_2. Examining (13.20) we find that in the Kähler case the constant c in (13.19) can be chosen to be zero provided that λ is sufficiently large. If there were a pointwise estimate of the form

$$|\bar{\partial}_B \psi_0| < |\partial_B \psi_0|$$

on the zero set C_λ of ψ_0 then it would follow immediately that C_λ is a symplectic submanifold representing the Poincaré dual of $e = c_1(E)$. Any such submanifold is a J-holomorphic curve for some $J \in \mathcal{J}(X, \omega)$. Note, however, that if the almost complex structure J is fixed arbitrarily then there may not be any embedded J-holomorphic curves representing the class $\mathrm{PD}(e)$. (Think of a Kähler situation where every divisor representing e is singular.) On the other hand, for a generic almost complex structure all J-holomorphic curves representing $\mathrm{PD}(e)$ will be embedded. In his proof of Theorem 13.19 in [119] Taubes proceeds as follows.

Step 1: Pointwise estimates

The first step is to give pointwise estimates for the solutions of (13.13). Some of these estimate can be interpreted as a pointwise version of Proposition 13.36.

Proposition 13.37. (Taubes) *There exist constants $c > 0$ and $\lambda_0 > 0$ such that every solution (B, ψ_0, ψ_2) of (13.13) with $\lambda \geq \lambda_0$ satisfies the following pointwise inequalities**

$$|\psi_0|^2 \leq 4\pi + \frac{c}{\lambda^2}, \qquad |\psi_2|^2 \leq \frac{c}{\lambda}\left(4\pi - |\psi_0|^2 + \frac{c}{\lambda^2}\right), \tag{13.22}$$

$$|F_B^+| \leq \frac{\lambda}{\sqrt{8}}\left(4\pi - |\psi_0|^2\right) + c, \tag{13.23}$$

*Taubes denotes the connection on E by a, the section of E by α, and the $(0,2)$-form with values in E by β. Moreover, he calls the scaling parameter r and rescales such that $\alpha \to 1$ almost everywhere. Thus

$$r = 4\pi\lambda, \qquad \alpha = \psi_0/\sqrt{4\pi}, \qquad \beta = \psi_2/\sqrt{4\pi}, \qquad a = B$$

in Taubes' notation and his generic constants are called z instead of c.

$$|F_B^-| \le \frac{\lambda}{\sqrt{8}}\left(1+\frac{c}{\sqrt{\lambda}}\right)\left(4\pi - |\psi_0|^2\right) + c, \tag{13.24}$$

$$|d_B\psi_0|^2 + \lambda|\widetilde{\nabla}_B\psi_2|^2 \le c\lambda\left(4\pi - |\psi_0|^2 + \frac{c}{\lambda^2}\right). \tag{13.25}$$

The proof of (13.22) is quite similar to the proof of Proposition 13.36. The key idea is to use (13.13) and the identities

$$2\bar{\partial}_B^*\bar{\partial}_B\psi_0 = d_B{}^*d_B\psi_0 - 2i(F_B)_\omega\psi_0,$$

$$\bar{\partial}_B\bar{\partial}_B^*\psi_2 = \frac{1}{2}\widetilde{\nabla}_B{}^*\widetilde{\nabla}_B\psi_2 + i(F_B + 2F_{A_{\mathrm{can}}})_\omega\psi_2$$

from Proposition 3.25 to prove that the function $u_\delta = 4\pi - |\psi_0|^2 - |\psi_2|^2/\delta + c\delta/\lambda$ satisfies the pointwise inequality

$$d^*du_\delta + \lambda|\psi_0|^2u_\delta \ge 0$$

(for λ sufficiently large and $\delta = c'/\lambda$) with equality only possible at points where $u_\delta > 0$. It follows that u_δ cannot have a negative minimum and hence $u_\delta \ge 0$ which proves (13.22). The inequality for F_B^+ follows easily from (13.22) and the pointwise identity

$$|F_B^+|^2 = \frac{\lambda^2}{8}\left((4\pi - |\psi_0|^2)^2 + 2(4\pi + |\psi_0|^2)|\psi_2|^2 + |\psi_2|^4\right).$$

To prove this identity use (13.13), the formula $F_B^+ = (F_B^+)_\omega\omega + F_B^{0,2} + F_B^{2,0}$ and the fact that $|\omega|^2 = 2$ and $|F_B^{0,2}| = |F_B^{2,0}|$. The estimate for F_B^- is considerably harder to establish. Both (13.24) and (13.25) are based on examining the expression $d^*du + |\psi_0|^2u$ for $u = |F_B^-|^2$, respectively $u = |d_B\psi_0|^2 + \lambda|\widetilde{\nabla}_B\psi_2|^2$. These proofs involve the following energy inequality.

Lemma 13.38. (Taubes) *There exist constants $c > 0$ and $\lambda_0 > 0$ such that every solution (B, ψ_0, ψ_2) of (13.13) with $\lambda \ge \lambda_0$ satisfies the inequality*

$$2\pi[\omega]\cdot c_1(E) - \frac{c}{\lambda} \le \frac{\lambda}{2}\int_X \left|4\pi - |\psi_0|^2\right| \le 2\pi[\omega]\cdot c_1(E) + \frac{c}{\lambda}. \tag{13.26}$$

Proof: Use the identity $2\pi[\omega]\cdot c_1(E) = \int_X (2iF_B)_\omega \mathrm{dvol}$, the third equation in (13.13), and the inequality $\|\psi_2\|^2 \le c/\lambda$ of Proposition 13.36 to prove (13.26) with the absolute value signs removed. Then use (13.22). □

Step 2: Monotonicity

The second and crucial step of the proof is a so-called **monotonicity formula** for the local energy

$$\mathcal{E}_U(\psi_0) = \frac{\lambda}{2}\int_U \left|4\pi - |\psi_0|^2\right|.$$

This formula gives estimates for the local energy over small geodesic balls $B_r(x) \subset X$ of radius r centered at x.

Proposition 13.39. (Taubes) *There exist constants $c > 0$, $\lambda_0 > 0$, and $\delta > 0$ such that every solution (B, ψ_0, ψ_2) of (13.13) with $\lambda \geq \lambda_0$ satisfies the following.*

(i) *If $1/4\lambda \leq r^2 \leq \delta$ then $\mathcal{E}_{B_r(x)}(\psi_0) \leq cr^2$.*

(ii) *If $1/4\lambda \leq r^2 \leq \delta$ and $|\psi_0(x)| \leq \sqrt{\pi}$ then $\mathcal{E}_{B_r(x)}(\psi_0) \geq r^2/c$.*

Sketch of proof: The proof is based on an inequality of the form

$$\mathcal{E}_{B_r} \leq \frac{r}{2}(1+cr)\left(1+\frac{c}{\sqrt{\lambda}}\right)\frac{d}{dr}\mathcal{E}_{B_r} + cr^4. \tag{13.27}$$

This inequality should be read essentially in the form $\varepsilon(r) = \mathcal{E}_{B_r}$ is *approximately smaller than* $r\varepsilon'(r)/2$. The proof involves the curvature estimates in Proposition 13.37. The key identity is

$$\frac{\lambda}{2}\int_U (4\pi - |\psi_0|^2 + |\psi_2|^2)\mathrm{dvol} = \int_{\partial U} \theta \wedge iF_B$$

where $d\theta = \omega$ in U. By (13.22), this leads to the inequality

$$\frac{\lambda}{2}\int_{B_r(x)} \left|4\pi - |\psi_0|^2\right| \mathrm{dvol} \leq \int_{\partial B_r(x)} \theta \wedge iF_B + \frac{cr^4}{\lambda}.$$

It is here where the precise estimate of the curvature in (13.23) and (13.24) with the factor $\lambda/\sqrt{8}$ is needed in order to obtain the factor $r/2$ in (13.27). More precisely, the inequalities (13.23) and (13.24) imply

$$|F_B| \leq \frac{\lambda}{2}\left(1+\frac{c}{\sqrt{\lambda}}\right)\left|4\pi - |\psi_0|^2\right| + \sqrt{2}c$$

and the 1-form θ can be chosen such that

$$|\theta| \leq \frac{r}{2}(1+cr).$$

Inserting these two inequalities in the previous one gives (13.27).

To integrate (13.27) consider the function

$$f(r) = \frac{2}{1+c/\sqrt{\lambda}}\log\left(c+\frac{1}{r}\right)$$

and note that (13.27) implies

$$\frac{d}{dr}\left(e^{f(r)}\varepsilon(r)\right) \geq -ce^{f(r)}r^3$$

where $\varepsilon(r) = \mathcal{E}_{B_r}$ and the constant c is adjusted appropriately. Now integrate the last inequality from $r_0 = \lambda^{-1/2}$ to r and use the estimate $1/cr^2 \leq e^{f(r)} \leq c/r^2$ for $\lambda^{-1/2} \leq r \leq \delta$ to obtain

$$\varepsilon(r) \geq \frac{r^2}{c}\frac{\varepsilon(r_0)}{{r_0}^2} - r^4.$$

Then (i) follows from the energy inequality of Lemma 13.38. The proof of the lower bound in (ii) involves the estimate (13.25) on the derivatives (if $|\psi_0|$ stays away from $\sqrt{4\pi}$ at x then it does so in a uniform neighbourhood of radius approximately $\lambda^{-1/2}$). For further details see [119]. □

The monotonicity estimate is the crucial step in the proof. It implies, roughly speaking, that the zero set of ψ_0 can be covered by approximately $N_\rho \leq c/\rho^2$ balls of radius ρ whenever $1/\sqrt{\lambda} \leq \rho \leq \rho_0$ and it follows that these zero sets are of Hausdorff dimension 2 and satisfy a uniform bound on their 2-dimensional Hausdorff measure.

Corollary 13.40. (Taubes) *There exist constants $c > 0$, $\lambda_0 > 0$, $\rho_0 > 0$ such that every solution (B, ψ_0, ψ_2) of (13.13) with $\lambda \geq \lambda_0$ satisfies the following. For every $\rho > 0$ with*

$$\frac{1}{\sqrt{\lambda}} < \rho < \rho_0$$

the set

$$Z(\psi_0) = \left\{x \in X \mid |\psi_0(x)|^2 < \pi\right\}$$

can be covered by $N_\rho \leq c/\rho^2$ geodesic balls of radius ρ.

Proof: Let N be the maximal number of disjoint balls of radius $\rho/2$ centred at points in ${\psi_0}^{-1}(0)$. Let $B_i = B_{\rho/2}(x_i)$ be N such balls. Then it follows from Proposition 13.39 that $\mathcal{E}_{B_i}(\psi_0) \geq \rho^2/4c$ and hence

$$\frac{N\rho^2}{4c} \leq \sum_{i=1}^{N} \mathcal{E}_{B_i}(\psi_0) \leq \mathcal{E}_X(\psi_0) \leq 2\pi[\omega] \cdot c_1(E) + 1.$$

The last inequality follows from Lemma 13.38 with λ sufficiently large. This shows that $N \leq c'/\rho^2$. Moreover, since N was chosen maximal, it follows that the balls $B_\rho(x_i)$ cover the set ${\psi_0}^{-1}(0)$. □

In particular, Corollary 13.40 can be used to prove a refined estimate of the curvature, namely that F_B^- satisfies the same estimate as F_B^+ in (13.23).

Proposition 13.41. (Taubes) *There exist constants $c > 0$ and $\lambda_0 > 0$ such that every solution (B, ψ_0, ψ_2) of (13.13) with $\lambda \geq \lambda_0$ satisfies the following pointwise inequality*

$$|F_B^{\pm}| \leq \frac{\lambda}{\sqrt{8}} \left(4\pi - |\psi_0|^2\right) + c.$$

Step 3: The zero set of ψ_0

In this step Taubes proves an exponential decay estimate for the solutions of (13.13) with large λ away from the zero set of ψ_0. This estimate has the following form.

Proposition 13.42. (Taubes) *There exist constants $c > 0$ and $\lambda_0 > 0$ such that for every solution (B, ψ_0, ψ_2) of (13.13) with $\lambda \geq \lambda_0$ the function $u : X \to \mathbb{R}$ defined by*

$$u = \lambda^2|\psi_2|^2 + \lambda|\widetilde{\nabla}_B\psi_2|^2 + |d_B\psi_0|^2 + \lambda(4\pi - |\psi_0|^2) + |F_B|$$

satisfies the inequality

$$u(x) \leq c\lambda \exp\left(-\frac{\sqrt{\lambda}}{c} d(x, \psi_0^{-1}(0))\right)$$

for all $x \in X$.

Note first that for $d(x, \psi_0^{-1}(0)) \leq 1/\sqrt{\lambda}$ the result follows immediately from the pointwise estimates in Proposition 13.37. For $d(x, \psi_0^{-1}(0)) \geq 1/\sqrt{\lambda}$ Taubes proves that the function

$$v = \lambda^2|\psi_2|^2 + c\lambda|\widetilde{\nabla}_B\psi_2|^2 + c'|d_B\psi_0|^2$$

with suitable constants $c' > c > 1$ satisfies an estimate

$$d^*dv + \frac{\lambda}{16}v \leq 0. \tag{13.28}$$

The key tool for proving this inequality is a local analysis of the perturbed Seiberg-Witten equations over $\mathbb{R}^4 = \mathbb{C}^2$. With the standard complex and symplectic structures on $\mathbb{C}^2$ these are the vortex equations and they have *model solutions* whose zero sets are the zero sets of complex polynomials in two variables. The result is proved by comparing the solutions on X with these model solutions. The main conclusion is that for every $\delta > 0$ there exists a constant $c_\delta > 0$ such that

$$|\psi_0(x)|^2 < 4\pi - \delta \qquad \Longrightarrow \qquad d(x, {\psi_0}^{-1}(0)) < \frac{c_\delta}{\sqrt{\lambda}}.$$

This eventually leads to a proof of the estimate (13.28). With (13.28) established one can use a comparison argument to obtain the exponential decay for v. The estimate for the last two terms in u can then be reduced to that of v.

Step 4: Convergence

Choose a sequence $\lambda_n \to \infty$ and let $(B_n, \psi_{0,n}, \psi_{2,n})$ be a corresponding sequence of solutions of (13.13). In [119] Taubes considers the sequence of currents (linear functionals) $\mathcal{F}_n : \Omega^2(X) \to \mathbb{R}$ defined by

$$\mathcal{F}_n(\tau) = \frac{i}{2\pi} \int_X F_{B_n} \wedge \tau$$

for $\tau \in \Omega^2(X)$. Consider $\mathcal{F}_n$ as a linear functional on the space of continuous differential forms with the L^∞ norm. The corresponding norm of $\mathcal{F}_n$ is given by

$$\|\mathcal{F}_n\| = \sup_{0 \neq \tau \in \Omega^2(X)} \frac{\mathcal{F}_n(\tau)}{\|\tau\|_{L^\infty}}.$$

It follows easily from Proposition 13.41 and Lemma 13.38 that the sequence $\mathcal{F}_n$ is bounded in this norm, namely

$$\begin{aligned}
\|\mathcal{F}_n\| &\leq \frac{1}{2\pi} \|F_{B_n}\|_{L^1} \\
&\leq \frac{\lambda}{4\pi} \int_X \left|4\pi - |\psi_{0,n}|^2\right| \mathrm{dvol} + \frac{c}{\lambda} \\
&\leq [\omega] \cdot c_1(E) + \frac{2c}{\lambda}.
\end{aligned}$$

Here the second inequality uses the exponential decay estimate of Proposition 13.42 with $u = |F_B|$. By Alaoglu's theorem, $\mathcal{F}_n$ has a subsequence (still denoted by $\mathcal{F}_n$) which converges in the weak-$*$-topology. Thus there exists a current $\mathcal{F} : \Omega^2(X) \to \mathbb{R}$ (continuous with respect to the L^∞-norm) such that

$$\mathcal{F}(\tau) = \lim_{n \to \infty} \frac{i}{2\pi} \int_X F_{B_n} \wedge \tau$$

for every $\tau \in \Omega^2(X)$. It follows quite easily from Corollary 13.40 and Proposition 13.42 that this current is supported in a set of Hausdorff dimension 2 and that $\mathcal{F}$ has type $(1,1)$, i.e. that $\mathcal{F}(\tau) = 0$ for all forms τ of type $(0,2)$ or $(2,0)$.* Taubes then proves that $\mathcal{F}$ has positive intersection number with

*One is tempted to try to prove this by using the identity $2{F_B}^{0,2} = \lambda\bar{\psi}_0\psi_2$ and the L^2-estimate (13.18) in Proposition 13.36. However, this will only give a uniform bound on $\mathcal{F}_n(\tau)$ for $\tau \in \Omega^{0,2}(X)$ rather than convergence to zero. To obtain the required convergence to zero one needs the more subtle exponential decay estimates of Proposition 13.42.

every pseudoholomorphic disc whose boundary lies outside the support of $\mathcal{F}$. The intersection number can be defined by evaluating $\mathcal{F}$ on a differential form which is supported near the disc and restricts to a volume form with volume 1 on every fiber of the normal bundle. The proof that this number is positive is highly nontrivial and requires the full power of the earlier estimates.

Step 5: Existence of a J-holomorphic curve

The final task is to show that under these conditions the support of $\mathcal{F}$ is a pseudoholomorphic curve. This means that there exists a Riemann surface Σ (not necessarily connected) and a J-holomorphic curve $u : \Sigma \to X$ such that

$$\mathcal{F}(\tau) = \int_\Sigma u^*\tau$$

for every $\tau \in \Omega^2(X)$. This result is true for any current $\mathcal{F}$ which is supported in a closed set C of finite 2-dimensional Hausdorff measure and has positive intersection numbers with pseudoholomorphic discs. This is essentially a regularity theorem for C. One first shows that C is the image of a continuous map $u : \Sigma \to X$ (locally), secondly that this map can be chosen Lipschitz continuous, thirdly that C has an open and dense set of differentiable points, fourthly that T_xC is a complex subspace of (T_xX, J) at every regular point, and finally one has to analyse the structure of C near the singular set. We shall not discuss the details of these arguments which are all carefully explained in [119].

The Gromov invariants

The higher genus Gromov invariants for general (semi-positive) symplectic manifolds were first defined by Ruan in [107]. In [120] Taubes extended Ruan's construction to include disconnected curves and take proper account of multiply covered tori of self-intersection number zero. A detailed exposition of the genus-zero invariants can also be found in McDuff-Salamon [84]. Here is a very sketchy outline of the definition of these invariants.

Fix a Riemann surface Σ of genus g and consider the moduli space

$$\mathcal{M}^{\mathrm{Gr}}(X, J; \alpha, g)$$

of all equivalence classes of pairs $[u, j]$ where $j \in \mathcal{J}(\Sigma)$ is a complex structure on Σ and $u : \Sigma \to X$ is a (j, J)-holomorphic map which represents the class α. The equivalence relation is given by the obvious action of the diffeomorphism group $\mathrm{Diff}(\Sigma)$ on $\mathrm{Map}(\Sigma, X) \times \mathcal{J}(\Sigma)$. If

$$2g - 2 = c_1(K) \cdot \alpha + \alpha \cdot \alpha \tag{13.29}$$

the map $u : \Sigma \to X$ is an embedding for every pair $[u, j] \in \mathcal{M}^{\mathrm{Gr}}(X, J; \alpha, g)$. Standard Fredholm theory as in [84] asserts that, for a generic almost complex structure $J \in \mathcal{J}(X,\omega)$, the space of parametrized J-holomorphic curves $u : \Sigma \to X$ (with a fixed complex structure $j \in \mathcal{J}(\Sigma)$) is a smooth manifold of dimension $4-4g-2c_1(K)\cdot\alpha$. Varying j increases the dimension by $6g-6$, i.e. the dimension of Teichmüller space. Hence the moduli space has dimension

$$\dim \mathcal{M}^{\mathrm{Gr}}(X,J;\alpha,g) = 2g-2-2c_1(K)\cdot\alpha.$$

As was observed by Gromov [47] this formula continues to hold in the cases $g=1$ and $g=0$. If g and α satisfy (13.29) then the dimension formula can be expressed as

$$\dim \mathcal{M}^{\mathrm{Gr}}(X,J;\alpha,g) = \alpha\cdot\alpha - c_1(K)\cdot\alpha = 2d(\alpha). \tag{13.30}$$

Note that this agrees with the dimension of the Seiberg-Witten moduli space $\mathcal{M}^{\mathrm{SW}}(X,\Gamma_E)$ whenever $\alpha = \mathrm{PD}(e)$. However, in general the zero set of the section ψ_0 might not be connected. Suppose that it consists of N components $C_1,\ldots,C_N$. It is then necessary to consider J-holomorphic curves defined on a disconnected Riemann surface $\Sigma = \Sigma_1\cup\cdots\cup\Sigma_N$. Suppose that the components Σ_i have genera g_i and that these satisfy the condition

$$\sum_{i=1}^{N}(2g_i-2) = c_1(K)\cdot\alpha + \alpha\cdot\alpha \tag{13.31}$$

The corresponding moduli space

$$\mathcal{M}^{\mathrm{Gr}}(X,J;\alpha,g_1,\ldots,g_N)$$

is defined as before as the moduli space of unparametrized embedded J-holomorphic curves $\Sigma\to X$. Note, in particular, that the diffeomorphism type of Σ may interchange components. If α_i denotes the homology class represented by $u(\Sigma_i)$ then $\alpha_i\cdot\alpha_j = 0$ for $i\neq j$ and the dimension of $\mathcal{M}^{\mathrm{Gr}}(X,J;\alpha,g_1,\ldots,g_N)$ is the sum of the dimensions of the $\mathcal{M}^{\mathrm{Gr}}(X,J;\alpha_i,g_i)$. The individual dimensions are $\alpha_i\cdot\alpha_i - c_1(K)\cdot\alpha_i$ and their sum is $\alpha\cdot\alpha - c_1(K)\cdot\alpha$. Now consider the space

$$\mathcal{M}^{\mathrm{Gr}}(X,J;\alpha) = \bigcup_{g_i}\mathcal{M}^{\mathrm{Gr}}(X,J;\alpha,g_1,\ldots,g_N)$$

where the union runs over all N and all N-tuples $(g_1,\ldots,g_N)$ which satisfy (13.31). Since only finitely many homology classes can be represented by J-holomorphic curves it follows that only finitely many of these spaces are nonempty. They all have the same dimension and thus

$$\dim \mathcal{M}^{\mathrm{Gr}}(X,J;\alpha) = \alpha\cdot\alpha - c_1(K)\cdot\alpha.$$

The crucial compactness theorem asserts that if the moduli space is zero dimensional then it is a finite set. The proof is based on Gromov's compactness theorem. Even if the complex structure j on Σ is fixed holomorphic spheres in X can bubble off and thus a sequence of curves may converge to a so-called cusp-curve. (A bouquet of J-holomorphic curves with J-holomorphic spheres attached to a base curve.) However, such cusp-curves form again moduli spaces of strictly lower dimension, and if the original moduli space was zero-dimensional then the spaces of cusp-curves have negative dimension and hence must be empty (see [84] and [107] for details). Now in the case at hand the complex structure on Σ is allowed to vary and thus more complicated degenerations can occur. One example is a curve of genus g degenerating into a curve of genus $g-1$ with a self-intersection. However, one can show again that all the resulting moduli spaces of such generalized cusp-curves have smaller dimension and must again be empty. The remaining difficulty is to consider moduli spaces of J-holomorphic tori with self-intersection number zero. This can only occur when

$$\alpha \cdot \alpha = c_1(K) \cdot \alpha = 0$$

and then the moduli space $\mathcal{M}^{\mathrm{Gr}}(X, J; k\alpha)$ is zero-dimensional for all k. In this case it is important to take proper account of multiply covered tori and in [120] Taubes explains in detail how to do this. His *counting principle* for multiply covered tori involves interesting new ideas. He considers four different operators, labelled by the elements of $H^1(C, \mathbb{Z}_2)$ and defined by twisting the Cauchy-Riemann operator with a real line bundle over C whose first Stiefel-Whitney class is the given element of $H^1(C, \mathbb{Z}_2)$. Associated to these operators there is a map

$$\delta_C : H^1(C, \mathbb{Z}_2) \to \mathbb{Z}_2$$

defined by a certain mod(2)-spectral flow. Taubes then defines an integer $\nu(C, m)$ depending on this map δ_C and the multiplicity $m \geq 1$. This integer represents the contribution of the m-fold cover of C to the Gromov invariant. In the case $m = 1$ this integer $\nu(C, 1) = \pm 1$ is the standard sign associated to C. Now consider a J-holomorphic curve

$$C = \sum_{i=1}^{N} m_i C_i$$

with $m_i = 1$ unless C_i is a torus with $C_i \cdot C_i = 0$ and define

$$\nu(C) = \sum_{i=1}^{N} \nu(C_i, m_i).$$

For every $\alpha \in H_2(X, \mathbb{Z})$ with $d(\alpha) = 0$ the set of such curves representing the class α is finite and the Gromov invariant is defined by

$$\mathrm{Gr}(X, \alpha) = \sum_{i=1}^{N} \nu(C_i, m_i).$$

In [120] Taubes proves that, with his definition of $\nu(C_i, m_i)$, this number is independent of the choice of the generic almost complex structure $J \in \mathcal{J}(X, \omega)$ used to define it and that it depends only on the isotopy class of the symplectic form ω. In the case $d(\alpha) > 0$ Taubes proves a similar theorem for J-holomorphic curves passing through $d(\alpha)$ given points in X.

The Gromov invariant can be regarded as an integer valued function

$$\mathrm{Gr} : H_2(X, \mathbb{Z}) \to \mathbb{Z}$$

which vanishes for classes with $d(\alpha) < 0$. Similarly, the Seiberg-Witten invariant can be regarded as a map

$$\mathrm{SW} : H^2(X, \mathbb{Z}) \to \mathbb{Z}$$

which assigns an integer $\mathrm{SW}(X, \Gamma_E)$ to each cohomology class $e = \mathrm{PD}(E)$. In [119, 120, 121] Taubes proved that

$$\mathrm{SW} = \mathrm{Gr} \circ \mathrm{PD}.$$

The first striking observation is that the moduli spaces $\mathcal{M}^{\mathrm{SW}}(X, \Gamma_E)$ and $\mathcal{M}^{\mathrm{Gr}}(X, J; \mathrm{PD}(c_1(E)))$ have the same dimension. Secondly, Theorem 13.19 shows how the solutions of the Seiberg-Witten equations generate J-holomorphic curves C in the homology class $[C] = \mathrm{PD}(c_1(E))$. In [121] Taubes proves the converse and shows how embedded J-holomorphic curves C can be used to construct solutions of the Seiberg-Witten (13.13) for large λ. His idea is, roughly, to use the vortex equations on $\mathbb{C}$ (with a single zero) and *glue them in* on the normal bundle of the J-holomorphic curve C. He thus obtains the following beautiful theorem.

Theorem 13.43. (Taubes) *For every compact symplectic 4-manifold X and every nontrivial line bundle $E \to X$ the Seiberg-Witten invariant of the spin^c structure Γ_E agrees with the Gromov invariant of the Poincaré dual of the first Chern class of E:*

$$\mathrm{SW}(X, \Gamma_E) = \mathrm{Gr}(X, \mathrm{PD}(c_1(E))).$$

14

EMBEDDED SURFACES

This chapter gives a proof of the generalized adjunction inequality by Kronheimer and Mrowka. An important ingredient in the proof is the blowup formula for the Seiberg-Witten invariants proved in Chapter 11. The proof is based on the study of Seiberg-Witten monopoles on tubes $\mathbb{R} \times Y$ where Y is a compact 3-manifold (see Section 10.1). The first section gives an overview over the main theorems and Section 14.2 contains the proofs.

14.1 The generalized adjunction formula

It has been a longstanding conjecture in Kähler geometry that complex curves C in compact Kähler surfaces X should minimize the genus in their respective homology classes. The genus of such a curve is given by the adjunction formula

$$2g(C) - 2 = C \cdot C + c_1(K) \cdot C$$

where $K = \Lambda^{2,0}T^*X$ denotes the canonical bundle with $c_1(K) = -c_1(TX)$. Hence the conjecture can be restated in the form

$$2g(\Sigma) - 2 \geq \Sigma \cdot \Sigma + c_1(K) \cdot \Sigma$$

for every embedded surface $\Sigma \subset X$, complex or not, which is homologous to a complex curve. This conjecture is attributed to Thom for the case $X = \mathbb{C}P^2$. In 1993 it was confirmed by Kronheimer and Mrowka for a large class of Kähler manifolds X with $b^+ \geq 2$ but not, at the time, for $\mathbb{C}P^2$ (cf. [63], [64], [65]). With the advent of the Seiberg-Witten invariants Kronheimer and Mrowka quickly realized that these give rise to much simpler proofs which can be extended to the case $b^+ = 1$ (cf. [66]). Later on Taubes observed that these results extend to the symplectic category (cf. [116], [117], [118]). All these results require the assumption $\Sigma \cdot \Sigma \geq 0$ which so far nobody has been able to remove.

The next theorem is the generalized adjunction inequality. It is the Seiberg-Witten version of the earlier theorem by Kronheimer and Mrowka relating the genus of embedded surfaces to the D-basic classes (see Theorem 7.40). The following theorem was proved by Kronheimer and Mrowka in November 1994 (cf [66]) and independently by Morgan, Szabó, and Taubes (cf [96]). A version of this result for immersed spheres (which does not require the assumption of nonnegative self-intersection number) was proved by Fintushel and Stern (cf [27]).

Theorem 14.1. (Kronheimer-Mrowka) *Let X be a smooth compact oriented 4-manifold with $b^+ - b_1$ odd and $b^+ \geq 2$. Moreover, let $\Gamma : TX \to \mathrm{End}(S)$ be a spin^c structure with nonzero Seiberg-Witten invariants*

$$\mathrm{SW}(X, \Gamma) \neq 0$$

and $\Sigma \subset X$ be a compact oriented embedded surface with self-intersection number

$$\Sigma \cdot \Sigma \geq 0.$$

Moreover, if Σ is a 2-sphere suppose that the homology class $[\Sigma] \in H_2(X, \mathbb{Z})$ is not a torsion class. Then

$$2g(\Sigma) - 2 \geq \Sigma \cdot \Sigma + |c_1(L_\Gamma) \cdot \Sigma|. \tag{14.1}$$

Note the obvious example of an embedded 2-sphere which is homologous to zero. Any such sphere has self-intersection number zero and satisfies $c_1(L_\Gamma) \cdot \Sigma = 0$ for every spin^c structure Γ. It does not satisfy the inequality (14.1). Note also that the genus inequality in Theorem 14.1 can also be interpreted as a vanishing theorem for the Seiberg-Witten invariants. Namely, if a cohomology class $c = c_1(L_\Gamma)$ violates the inequality (14.1) for some embedded surface Σ then the corresponding Seiberg-Witten invariant must be zero.

The proof of Theorem 14.1 will be given in Section 14.2. It is an immediate consequence that embedded complex curves in Kähler surfaces minimize the genus in their respective homology classes. Before stating this result let us consider the case $b^+ = 1$. In [66] Kronheimer and Mrowka proved the Thom conjecture for the projective plane. Their proof can be combined with some elementary surgery arguments to obtain the following result. A proof, in this generality, will appear in a forthcoming paper by Morgan, Szabó, and Taubes [96]. The proof of Theorem 14.2 given below is based on the techniques of Kronheimer and Mrowka and generalizes (word by word) to the symplectic category.

Theorem 14.2. (Morgan-Szabo-Taubes) *Let X be a compact Kähler surface with $b^+ = 1$ and $\Sigma \subset X$ be a compact oriented embedded surface satisfying*

$$\Sigma \cdot \Sigma \geq 0, \qquad \int_\Sigma \omega > 0.$$

Then

$$2g(\Sigma) - 2 \geq \Sigma \cdot \Sigma + c_1(K) \cdot \Sigma. \tag{14.2}$$

Note that the condition $[\omega] \cdot \Sigma > 0$ cannot be removed. If $[\omega] \cdot \Sigma < 0$ then one gets the inequality $2g(\Sigma) - 2 \geq \Sigma \cdot \Sigma - c_1(K) \cdot \Sigma$ and there are examples where $c_1(K) \cdot \Sigma > 0$ and equality holds. (Consider for example the sphere in $\mathbb{C}P^2$ representing the class $-H$.) It is an immediate consequence of Theorems 14.1 and 14.2 that holomorphic curves in Kähler surfaces with nonnegative self-intersection number minimize the genus in their respective homology classes. This is the generalized Thom conjecture and various different proofs were given by Kronheimer-Mrowka [66], Morgan-Szabó-Taubes [96], and Mrowka-Ozsváth-Yu [99].

Corollary 14.3. (Generalized Thom conjecture) *Let X be a compact Kähler surface and $C \subset X$ be an embedded (nonconstant) complex curve with*

$$C \cdot C \geq 0$$

Then every embedded surface $\Sigma \subset X$ which represents the same homology class as C has genus

$$g(\Sigma) \geq g(C).$$

Proof: Any embedded complex curve satisfies

$$2g(C) - 2 = C \cdot C + c_1(K) \cdot C, \qquad \int_C \omega > 0.$$

In particular, the homology class of C is never a torsion class. In the case $b^+ = 1$ the result now follows immediately from (14.2) in Theorem 14.2. If $b^+ > 1$ then, by Theorem 12.9, $\mathrm{SW}(X, \Gamma_{\mathrm{can}}) = 1$ and $c_1(L_{\Gamma_{\mathrm{can}}}) = -c_1(K)$. Hence in this case the inequality (14.1) in Theorem 14.1 implies (14.2) and this proves the corollary. □

Corollary 14.4. (Kronheimer–Mrowka) *Let $\Sigma \subset \mathbb{C}P^2$ be an embedded surface representing the homology class $[\Sigma] = d[\mathbb{C}P^1] \in H_2(\mathbb{C}P^2, \mathbb{Z})$ with $d > 0$. Then*

$$g(\Sigma) \geq \frac{(d-1)(d-2)}{2}.$$

Remark 14.5 Assume $b^+ > 1$. Then, as a byproduct of the proof of Corollary 14.3, one obtains the inequality

$$0 \leq c_1(E) \cdot C \leq c_1(K) \cdot C$$

for every holomorphic curve $C \subset X$ with $C \cdot C \geq 0$ and every line bundle $E \to X$ with $\mathrm{SW}(X, \Gamma_E) \neq 0$. To see this just note that

$$c_1(K) \cdot C = 2g - 2 - C \cdot C \geq |c_1(L_{\Gamma_E}) \cdot C|.$$

With $c_1(L_{\Gamma_E}) = 2c_1(E) - c_1(K)$ the required inequality follows. □

No counterexample is known to the assertion that complex curves minimize the genus in their homology class. This should hold in full generality without any restriction on the self-intersection number.

The minimal genus as an invariant

In [97] Mrowka suggested the definition of a function $\mathrm{km} : H_2(X, \mathbb{Z}) \to \mathbb{Z}$ by

$$\mathrm{km}(\alpha) = \min_{[\Sigma]=\alpha} 2g(\Sigma) - 2 - \Sigma \cdot \Sigma$$

where the minimum is over all compact connected oriented embedded surfaces representing the class α. This function has the following property.

Lemma 14.6. (Mrowka) *If $\alpha \cdot \alpha \geq 0$ then*

$$\mathrm{km}(n\alpha) \leq n\, \mathrm{km}(\alpha).$$

for every positive integer n.

Proof: Let $\Sigma \subset X$ be a compact connected oriented embedded surface representing the class $[\Sigma] = \alpha$. Suppose that $\mathrm{km}(\alpha) = 2g(\Sigma) - 2 - \Sigma \cdot \Sigma$. Then the normal bundle $\nu_\Sigma \to \Sigma$ has degree

$$\deg(\nu_\Sigma) = \alpha \cdot \alpha = \Sigma \cdot \Sigma.$$

Now any oriented real rank-2 bundle of degree N over a Riemann surface admits a section which intersects the zero section in precisely N points such that all intersection points are transverse with positive intersection number. Moreover, the intersection points can be prescribed. A simple induction argument then shows that for any positive integer n there exist n different sections $s_1, \ldots, s_n : \Sigma \to \nu_\Sigma$ intersecting pairwise in precisely N points with positive intersection number. These give rise to n different surfaces $S_1, \ldots, S_n$, all representing the class α, such that S_i and S_j intersect in precisely $N = \alpha \cdot \alpha$ points with intersection number 1 whenever $i \neq j$. Moreover each surface S_j has the same genus as Σ. Consider the embedded surface Σ_n obtained from the surfaces S_i by removing all the $n(n-1)N/2$ intersection points. This surface has genus

$$g(\Sigma_n) = ng(\Sigma) + \frac{n(n-1)}{2}\Sigma \cdot \Sigma + 1 - n.$$

Moreover, Σ_n represents the class $n\alpha$ and hence has self-intersection number

$$\Sigma_n \cdot \Sigma_n = n^2\, \Sigma \cdot \Sigma.$$

These two identities give

$$2g(\Sigma_n) - 2 - \Sigma_n \cdot \Sigma_n = n(2g(\Sigma) - 2 - \Sigma \cdot \Sigma) = n\, \mathrm{km}(\alpha). \qquad \Box$$

Suppose now that $b^+ > 1$ and $b^+ - b_1$ is odd and that X has nontrivial Seiberg-Witten invariants. Then Theorem 14.1 shows that $\mathrm{km}(\alpha) \ge 0$ for every $\alpha \in H_2(X, \mathbb{Z})$ with $\alpha \cdot \alpha \ge 0$. In [97] Mrowka posed the question whether the function km is a norm (at least on the set of classes α with nonnegative self-intersection number). Alternatively, he suggested to consider the function

$$\mathrm{KM}(\alpha) = \liminf_{n \to \infty} \frac{\mathrm{km}(n\alpha)}{n}. \tag{14.3}$$

Theorem 14.1 shows that there are finitely many cohomology classes

$$K_1, \ldots, K_s \in H^2(X, \mathbb{Z}),$$

namely the SW-basic classes, such that

$$\mathrm{KM}(\alpha) \ge \max_i |K_i \cdot \alpha|. \tag{14.4}$$

Is it possible that this estimate is sharp? For example, if X is a Kähler surface with $b^+ > 1$ then the estimate is sharp for all classes α which can be represented by embedded holomorphic curves. Moreover, in this case the maximum is given by $K \cdot \alpha$ where K denotes the canonical class.

Exercise 14.7 Suppose that the classes α_1 and α_2 can be represented by connected oriented embedded surfaces Σ_1 and Σ_2, respectively, such that

$$\mathrm{km}(\alpha_i) = |\chi(\Sigma_i)| - \alpha_i \cdot \alpha_i$$

and Σ_1 and Σ_2 intersect transversally with each intersection point contributing intersection number 1. Prove that under these conditions

$$\mathrm{km}(\alpha_1 + \alpha_2) \le \mathrm{km}(\alpha_1) + \mathrm{km}(\alpha_2).$$

Prove also that

$$\mathrm{KM}(\lambda\alpha) = |\lambda| \, \mathrm{KM}(\alpha)$$

for every $\lambda \in \mathbb{Z}$ and every $\alpha \in H_2(X, \mathbb{Z})$. □

Remark 14.8 The above definition is reminiscent of **Thurston's norm** $\mathrm{Th} : H_2(Y, \mathbb{Z}) \to \mathbb{Z}$ on the homology of a 3-manifold Y defined by

$$\mathrm{Th}(\alpha) = \min_{[\Sigma]=\alpha} |\chi(\Sigma)|.$$

As above the minimum is over all compact oriented embedded surfaces $\Sigma \subset Y$ which represent the homology class α, are possibly disconnected, and do not contain any component diffeomorphic to the 2-sphere. Thurston proved that the function Th is a norm, that is

$$\mathrm{Th}(\alpha) \geq 0, \qquad \mathrm{Th}(\alpha+\beta) \leq \mathrm{Th}(\alpha) + \mathrm{Th}(\beta), \qquad \mathrm{Th}(\lambda\alpha) = |\lambda|\,\mathrm{Th}(\alpha)$$

for $\alpha, \beta \in H_2(X, \mathbb{Z})$ and $\lambda \in \mathbb{Z}$. Moreover, Gabai proved that there exist cohomology classes $\tau_1, \ldots, \tau_s \in H^2(X, \mathbb{Z})$ such that

$$\mathrm{Th}(\alpha) = \max_i |\tau_i \cdot \alpha|.$$

These classes τ_i are in fact the Euler classes of taut foliations of Y. □

Blowup formula

The following blowup formula for the Seiberg-Witten invariants is due to Morgan-Szabó-Taubes and is a special case of Theorem 11.2 proved in chapter 11. This result can be viewed as a generalization of the fact that the symplectic or Kähler structures of a 4-manifold are preserved when taking connected sums with $\overline{\mathbb{C}P}^2$.

Theorem 14.9. (Morgan-Szabó-Taubes) *Let X be a compact oriented smooth 4-manifold with $b^+(X) \geq 2$ and consider the connected sum*

$$X' = X \# \overline{\mathbb{C}P}^2.$$

Denote by $e = \mathrm{PD}([S]) \in H^2(X', \mathbb{Z})$ the Poincaré dual of the fundamental class of the standard 2-sphere $S \subset \overline{\mathbb{C}P}^2$. If $c \in H^2(X, \mathbb{Z})$ is a basic class for X and $k \in \mathbb{Z}$ is an odd integer with

$$d(c) + \frac{1-k^2}{8} \geq 0$$

where $8d(c) = c \cdot c - 2\chi(X) - 3\sigma(X)$ then

$$c' = c + ke \in H^2(X', \mathbb{Z})$$

is a basic class for X'. Conversely, every basic class in $H^2(X', \mathbb{Z})$ is of this form.

The proof of the inequality $g(\Sigma) \geq g(C)$ in Corollary 14.4 can be carried out without relying on the blowup formula of Theorem 14.9. Instead one can use the blowup construction in the Kähler category (see for example [85], Chapter 6) and examine the canonical class K' of the blown up manifold X'. This approach will be used in the proof of Theorem 14.2.

14.2 Proof of the Thom conjecture

The proofs of Theorems 14.1 and 14.2 are based on the following proposition which is concerned with the Seiberg-Witten monopole equations on X for a sequence of metrics with *long necks*. More precisely, let $\Sigma \subset X$ be a compact oriented embedded smooth 2-manifold with trivial normal bundle, i.e.

$$\Sigma \cdot \Sigma = 0.$$

Write

$$X = X_1 \cup_Y X_2$$

where X_1 is a closed tubular neighbourhood of Σ and X_2 is the closure of $X - X_1$. Since $\Sigma \cdot \Sigma = 0$ it follows that the boundary $Y = \partial X_1$ is diffeomorphic to the product $\Sigma \times S^1$. For any $T > 0$ choose a metric g on X which in a neighbourhood of Y is a product metric on $Y \times (-\varepsilon, \varepsilon)$. Suppose further that the corresponding metric on $Y = S^1 \times \Sigma$ is a product metric with a constant curvature metric on Σ. Now stretch the neck, i.e. consider a family of metrics g_T on X with respect to which X is isometric to the manifold

$$X_T = X_1 \cup [0, T] \times Y \cup X_2$$

with the obvious product metric on the tube $[0, T] \times Y$. The following proposition is due to Kronheimer and Mrowka as are all the results in this section. However, the proof given below differs slightly from the one in [66].

Proposition 14.10 *Suppose that the moduli space $\mathcal{M}(X_T, \Gamma_T, g_T)$ of unperturbed Seiberg-Witten monopoles is nonempty for every sufficiently large $T > 0$. Then either*

$$2g(\Sigma) - 2 \geq |c_1(L_\Gamma) \cdot \Sigma|$$

or $c_1(L_\Gamma) \cdot \Sigma = 0$.

Proof: Let (A_T, Φ_T) be a Seiberg-Witten monopole on X_T with the metric g_T and perturbation $\eta = 0$. It satisfies the a priori estimate

$$\sup_{X_T} |\Phi_T|^2 \leq -\frac{1}{2} \inf_{X_T} s$$

of Lemma 7.13. Note in particular that the infimum of the scalar curvature s on X_T is indpendent of T. Moreover, it follows from Proposition 7.3 that the Seiberg-Witten action of the pair (A_T, Φ_T) is given by

$$\begin{aligned} E(A_T, \Phi_T) &= \int_{X_T} \left(|\nabla_{A_T} \Phi_T|^2 + \frac{s}{4} |\Phi_T|^2 + \frac{1}{4} |\Phi_T|^4 + |F_{A_T}|^2 \right) \mathrm{dvol} \\ &= -\pi^2 c_1(L_\Gamma)^2. \end{aligned}$$

Here $c_1(L_\Gamma)^2$ denotes the cup-product of $c_1(L_\Gamma)$ with itself, evaluated on the fundamental class of X. Now consider the restriction of the solution

(A_T, Φ_T) to the tube $[0,T]\times Y$. Assume without loss of generality that this solution is in temporal gauge. Then

$$\dot{\Phi}_T = D_{A_T}\Phi_T, \qquad \gamma(\dot{A}_T + *F_{A_T}) = (\Phi_T\Phi_T^*)_0\,.$$

By Proposition 10.9 (with $\eta = 0$) the energy of (A_T, Φ_T) on $[0,T]\times Y$ agrees with the Seiberg-Witten action E^{SW} on this domain and hence

$$\begin{aligned}
E(A_T, \Phi_T; [0,T]\times Y) &= E(A_T, \Phi_T; X_T) - E(A_T, \Phi_T; X_1 \cup X_2)\\
&= -\pi^2 c_1(L_\Gamma)^2 - E(A_T, \Phi_T; X_1 \cup X_2)\\
&\le -\pi^2 c_1(L_\Gamma)^2 - \frac{1}{4}\int\limits_{X_1\cup X_2}\left(s|\Phi_T|^2 + |\Phi_T|^4\right)\\
&= -\pi^2 c_1(L_\Gamma)^2 + \frac{1}{16}\int\limits_{X_1\cup X_2}\left(s^2 - (s+|\Phi_T|^2)^2\right)\\
&\le -\pi^2 c_1(L_\Gamma)^2 + \frac{1}{16}\int\limits_X s^2 \mathrm{dvol}.
\end{aligned}$$

Hence there is a uniform upper bound for the energy

$$E(A_T, \Phi_T; [0,T]\times Y) = 2\int\limits_0^T\int\limits_Y\left(\left|\dot{\Phi}_T\right|^2 + \left|\dot{A}_T\right|^2\right)\mathrm{dvol}\,dt \le c.$$

Now there is a refinement of the compactness theorem 7.12 which asserts that every sequence of solutions of the Seiberg-Witten equations (A_ν, Φ_ν) on a noncompact manifold X, with the property that the functions Φ_ν satisfy a uniform L^∞-estimate, has a subsequence which up to gauge equivalence converges uniformly with all derivatives on every compact subset of X. Apply this theorem to the manifold $(0,1)\times Y$ and a family of solutions $A_T(s_T+t), \Phi_T(s_T+t)$, $0\le t\le 1$, where s_T is chosen such that

$$E(A_T, \Phi_T; [s_T, s_T+1]\times Y) \le \frac{c}{T}.$$

Such an s_T exists for obvious reasons whenever T is an integer. The refined compactness theorem now asserts that there exists a sequence $T_\nu \to \infty$ and a sequence of gauge transformations $u_\nu : (0,1)\times Y \to S^1$ such that $({u_\nu}^* A_\nu, {u_\nu}^{-1}\dot{u}_\nu, {u_\nu}^{-1}\Phi_\nu)$ converges uniformly with all derivatives on every compact subset of $(0,1)\times Y$. Here $A_\nu(t) = A_{T_\nu}(s_{T_\nu}+t)$ and $\Phi_\nu(t) = \Phi_{T_\nu}(s_{T_\nu}+t)$. The limit

$$A = \lim_{\nu\to\infty} {u_\nu}^* A_\nu, \qquad \Psi = \lim_{\nu\to\infty} {u_\nu}^{-1}\dot{u}_\nu, \qquad \Phi = {u_\nu}^{-1}\Phi_\nu$$

is a solution of (10.5) with $\eta = 0$ on the domain $(0,1) \times Y$ which has zero energy. Hence

$$\dot{A} = d\Psi, \qquad \nabla_t \Phi = 0, \qquad D_A\Phi = 0, \qquad \gamma(*F_A) = (\Phi\Phi^*)_0.$$

This shows that for each t the pair $(A(t), \Phi(t))$ is a critical point of the Chern-Simons functional. Here the metric on $Y = \Sigma \times S^1$ was chosen arbitrarily. The argument is valid for any metric.

It is important to note that the bundle $W \to Y$ was obtained by restricting the bundle $W^+ \to X$ to the boundary $Y = \Sigma \times S^1$ of a tubular neighbourhood of Σ. Hence evaluating the first Chern class of W on a slice $\Sigma = \Sigma \times \{e^{i\theta}\}$ gives the number

$$c_1(W) \cdot \Sigma = c_1(L_\Gamma) \cdot \Sigma.$$

Now recall that the metric was chosen to be of constant curvature on Σ. By rescaling if necessary, assume without loss of generality that the area of Σ is 1. In other words the metric on Σ is chosen to be a Kähler-Einstein metric with Ricci form $\rho_\omega = (s/2)\omega$ where ω is the volume form on Σ with $\int_\Sigma \omega = 1$. For the factor $s/2$ see Lemma 2.7. Recall that the form $(2\pi)^{-1}\rho_\omega$ represents the first Chern class of $T\Sigma$ and hence

$$2 - 2g = \int_\Sigma c_1(T\Sigma) = \frac{1}{2\pi}\int_\Sigma \rho_\omega = \frac{s}{4\pi}\int_\Sigma \omega = \frac{s}{4\pi}.$$

This shows that the scalar curvature of Σ, and hence of $Y = \Sigma \times S^1$, is

$$s = 4\pi(2 - 2g).$$

By Lemma 10.4 the solution (A, Φ) of (10.2) satisfies either $\Phi = 0$ or

$$\sup_Y |F_A| = \frac{1}{2}\sup_Y |\Phi|^2 \le -\frac{s}{4} = \pi(2g - 2).$$

Assume first that $\Phi \ne 0$. Since the first Chern class of the bundle W is represented by the 2-form iF_A/π, it follows that

$$|c_1(W) \cdot \Sigma| = \left|\int_\Sigma \frac{iF_A}{\pi}\right| \le \frac{\sup_Y |F_A|}{\pi} = 2g - 2.$$

This is the required inequality. If $\Phi = 0$ then $F_A = 0$ and hence

$$c_1(L_\Gamma) \cdot \Sigma = c_1(W) \cdot \Sigma = 0.$$

This proves the proposition. □

Proof of Theorem 14.1: The proof consists of three steps.

Step 1: *The theorem holds when* $\Sigma \cdot \Sigma = 0$ *and* $c_1(L_\Gamma) \cdot \Sigma \neq 0$.

Since $\mathrm{SW}(X, \Gamma) \neq 0$ the moduli space $\mathcal{M}(X_T, \Gamma_T, g_T)$ is nonempty for every $T > 0$. Since $c_1(L_\Gamma) \cdot \Sigma \neq 0$ it follows from Proposition 14.10 that $2g(\Sigma) - 2 \geq |c_1(L_\Gamma) \cdot \Sigma|$. This proves Step 1.

Step 2: *The theorem holds when* $\Sigma \cdot \Sigma > 0$.

Assume without loss of generality that

$$c_1(L_\Gamma) \cdot \Sigma \geq 0.$$

Otherwise replace Γ by $\bar{\Gamma}$. Now consider the connected sum

$$X' = X \# \ell \overline{\mathbb{C}P}^2$$

with $\ell = \Sigma \cdot \Sigma$. Denote by $S_1, \dots, S_\ell \subset X'$ the embedded spheres representing the generators of the second homology groups of the ℓ copies of $\overline{\mathbb{C}P}^2$. Then

$$S_i \cdot S_i = -1$$

for every i and $S_i \cdot S_j = 0$ for $i \neq j$. Consider the connected sum

$$\Sigma' = \Sigma \# S_1 \# \cdots \# S_\ell.$$

This is an embedded surface of the same genus as Σ and it has self-intersection number

$$\Sigma' \cdot \Sigma' = \Sigma \cdot \Sigma - \ell = 0.$$

Over X' consider the spin^c structure $\Gamma' : TX' \to \mathrm{End}(W')$ which over X agrees with Γ and satisfies

$$c_1(L_{\Gamma'}) \cdot S_i = 1.$$

Then it follows from Theorem 14.9 that $\mathrm{SW}(X', \Gamma') \neq 0$. Moreover,

$$c_1(L_{\Gamma'}) \cdot \Sigma' = \ell + c_1(L_\Gamma) \cdot \Sigma = \Sigma \cdot \Sigma + c_1(L_\Gamma) \cdot \Sigma.$$

By assumption this number is positive. Hence Step 1 shows that

$$\begin{aligned} 2g(\Sigma) - 2 &= 2g(\Sigma') - 2 \\ &\geq c_1(L_{\Gamma'}) \cdot \Sigma' \\ &= \Sigma \cdot \Sigma + c_1(L_\Gamma) \cdot \Sigma \\ &= \Sigma \cdot \Sigma + |c_1(L_\Gamma) \cdot \Sigma|. \end{aligned}$$

This proves Step 2.

Step 3: *The theorem holds when* $\Sigma \cdot \Sigma = 0$ *and* $c_1(L_\Gamma) \cdot \Sigma = 0$.

It remains to prove that under these hypotheses Σ cannot be a sphere. Assume, by contradiction, that Σ is a sphere with

$$\Sigma \cdot \Sigma = 0, \qquad c_1(L_\Gamma) \cdot \Sigma = 0.$$

Then, by assumption, the homology class $[\Sigma]$ is not a torsion class. The contradiction is obtained by the following beautiful trick due to Kronheimer and Mrowka [65]. First note that there exists a connected embedded Riemann surface T such that

$$T \cdot \Sigma > 0, \qquad T \cdot T > 0.$$

The corresponding homology class exists for algebraic reasons and every 2-dimensional homology class can be represented by an embedded surface. (See the footnote on page 28.) Assume that T intersects Σ transversally. Any intersection point with negative intersection index can be removed by the following procedure. Given any pair $x^\pm \in T \cap \Sigma$ of intersection points with opposite intersection indices, choose a path $\gamma : [-1, 1] \to \Sigma$ which runs from x^- to x^+ and meets T only at the endpoints. A neighbourhood of this path can be embedded into $\mathbb{R}^4$ such that Σ corresponds to the (x_0, x_1)-plane, the path γ corresponds to an interval on the x_1-axis, and T corresponds to two planes parallel to the (x_2, x_3)-plane through the points $x_1 = -1$ and $x_1 = +1$ with $x_0 = 0$. These two planes are equipped with opposite orientations. Connect them by a tube around the x_1-axis in the (x_1, x_2, x_3)-subspace and cut out the two discs in T centered at $x_1 = \pm 1$, $x_2 = x_3 = 0$. This procedure does not change the homology class of T, removes the two intersection points, and increases the genus of T by 1. After this surgery the submanifold T intersects Σ in $T \cdot \Sigma$ points and each intersection index is $+1$. Now construct a submanifold Σ_m in the homology class

$$[\Sigma_m] = [T] + m[\Sigma]$$

as follows. By assumption the normal bundle of Σ is trivial. Choose m disjoint copies of Σ which differ by small parallel translations in the direction of a nonzero normal vector field and then smooth out the intersection points of these copies with T. This gives rise to an embedded surface Σ_m with

$$\Sigma_m \cdot \Sigma_m = T \cdot T + 2m\, T \cdot \Sigma.$$

Moreover, joining T with a copy of Σ increases the genus by $\Sigma \cdot T - 1$. Hence the genus of Σ_m is $g(\Sigma_m) = g(T) + m\, T \cdot \Sigma - m$ and thus

$$2g(\Sigma_m) - \Sigma_m \cdot \Sigma_m = 2g(T) - T \cdot T - 2m.$$

On the other hand $c_1(L_\Gamma) \cdot \Sigma_m = c_1(L_\Gamma) \cdot T$ and with m sufficiently large it follows that

$$2g(\Sigma_m) - 2 < \Sigma_m \cdot \Sigma_m + c_1(L_\Gamma) \cdot \Sigma_m.$$

Since $\Sigma_m \cdot \Sigma_m > 0$, this contradicts the second part of the proof. Hence the original assumption that Σ be a sphere must have been false. This completes the proof of Theorem 14.1. □

Proof of Theorem 14.2: The proof consists of four steps.

Step 1: *The theorem holds when*

$$\Sigma \cdot \Sigma = 0, \qquad c_1(K) \cdot \Sigma > 0,$$

and

$$(c_1(K) + \mathrm{PD}([\Sigma])^2 > 0, \qquad (c_1(K) + \mathrm{PD}([\Sigma]) \cdot [\omega] > 0.$$

Consider the metric g_T on the manifold $X_T = X_1 \cup [0, T] \times Y \cup X_2$ as in Step 1 in the proof of Theorem 14.1. This need not be a Kähler metric. Denote by ω_T the unique 2-form on X_T which is self-dual and harmonic with respect to g_T, has L^2-norm 1 (also with respect to g_T), and represents the orientation of $H^{2,+}(X)$ which is determined by the Kähler structure. Since the class $c_1(K) + \mathrm{PD}([\Sigma])$ has positive square it follows from the Hodge index theorem that

$$1 \le (c_1(K) + \mathrm{PD}([\Sigma])) \cdot (c_1(K) + \mathrm{PD}([\Sigma])) \le ((c_1(K) + \mathrm{PD}([\Sigma])) \cdot [\omega_T])^2$$

for every $T > 0$. In particular, the product $(c_1(K) + \mathrm{PD}([\Sigma])) \cdot [\omega_T]$ cannot change sign and hence

$$(c_1(K) + \mathrm{PD}([\Sigma])) \cdot [\omega_T] \ge 1$$

for every T. Now ω_T is a self-dual harmonic 2-form with respect to g_T and hence satisfies

$$\Delta_T \omega_T = 0, \qquad \|\omega_T\|_{L^2(X_T, g_T)} = 1.$$

Here Δ_T denotes the Hodge Laplacian on X_T associated to the metric g_T. Now a standard elliptic bootstrapping argument (using cutoff functions on the cylindrical part $[0, T] \times Y$) shows that for every integer ℓ there exists a constant $c_\ell > 0$ such that

$$\|\omega_T\|_{W^{\ell,2}([s,s+1]\times Y)} \le c_\ell$$

for every T and every $s \in [0, T - 1]$. This shows that for every sequence $T_\nu \to \infty$ there is a subsequence $T_{\nu'} \to \infty$ and a sequence $s_{\nu'} \in [0, T_{\nu'} - 1]$

such that $\omega_{T'_\nu}$ converges to zero on the domain $[s_{\nu'}, s_{\nu'}+1]\times Y$ in, say, the C^1-norm. Hence

$$\lim_{T\to\infty}\int_\Sigma \omega_T = 0$$

and it follows that $c_1(K)\cdot[\omega_T] > 0$ for T sufficiently large. With $c_1(\Gamma_{\mathrm{can}}) = -c_1(K)$ this gives

$$\varepsilon_{\Gamma_{\mathrm{can}}}(g_T, 0) = \pi c_1(K)\cdot[\omega_T] > 0.$$

By Theorem 12.10, $\mathrm{SW}^+(X, \Gamma_{\mathrm{can}}) = 1$ and hence $\mathcal{M}(X_T, \Gamma_{\mathrm{can}}, g_T)$ is non-empty for large T. By Proposition 14.10, either $2g-2 \geq |c_1(K)\cdot\Sigma|$ or $c_1(K)\cdot\Sigma = 0$. The latter is ruled out by assumption and hence the former must hold. This proves Step 1.

Step 2: *The theorem holds when*

$$\Sigma\cdot\Sigma > 0, \qquad \Sigma\cdot\Sigma + c_1(K)\cdot\Sigma > 0,$$

and

$$(c_1(K) + \mathrm{PD}([\Sigma]))^2 > 0, \qquad \int_\Sigma \omega + c_1(K)\cdot[\omega] > 0.$$

The proof relies on the blowup construction in the Kähler category and is similar to the proof of Step 2 in Theorem 14.1. Let $\ell = \Sigma\cdot\Sigma > 0$ and consider the manifold

$$X' = X\#\ell\overline{\mathbb{C}P}^2, \qquad \Sigma' = \Sigma\#\bar{S}_1\#\cdots\#\bar{S}_\ell,$$

where the S_i denote the exceptional divisors (with their orientations as holomorphic curves) and $\bar{S}_i$ indicates the reversed orientation. More explicitly, one can think of X' as the manifold X with ℓ balls of (sufficiently small) radius $r > 0$ removed and the resulting boundary 3-spheres identified, via the Hopf map, with 2-spheres of area πr^2. Thus

$$\int_{S_i} \omega' = \pi r^2$$

for every i where ω' denotes the Kähler form on X' (see [85], Chapter 6, for more details). Denote by $c_1(K) \in H^2(X', \mathbb{Z})$ the lift of the canonical class of X to X'. Then the canonical class of X' is given by

$$c_1(K') = c_1(K) - \sum_{i=1}^{\ell} e_i$$

where $e_i = -\mathrm{PD}([S_i]) \in H^2(X', \mathbb{Z})$ satisfies $e_i \cdot S_i = 1$ and $e_i \cdot S_j = 0$ for $i \neq j$. Note that

$$\mathrm{PD}([\Sigma']) = \mathrm{PD}([\Sigma]) + \sum_{i=1}^{\ell} e_i$$

where we again identify the class $\mathrm{PD}([\Sigma]) \in H^2(X, \mathbb{Z})$ with its lift to X'. Thus $c_1(K') + \mathrm{PD}([\Sigma']) = c_1(K) + \mathrm{PD}([\Sigma])$ and this shows that

$$\Sigma' \cdot \Sigma' = 0, \qquad c_1(K') \cdot \Sigma' = \Sigma \cdot \Sigma + c_1(K) \cdot \Sigma > 0,$$

and

$$(c_1(K') + \mathrm{PD}([\Sigma']))^2 = (c_1(K) + \mathrm{PD}([\Sigma]))^2 > 0,$$
$$(c_1(K') + \mathrm{PD}([\Sigma'])) \cdot \omega' = (c_1(K) + \mathrm{PD}([\Sigma])) \cdot \omega > 0.$$

Hence it follow from Step 1 that

$$2g(\Sigma) - 2 = 2g(\Sigma') - 2 \geq c_1(K') \cdot \Sigma' = \Sigma \cdot \Sigma + c_1(K) \cdot \Sigma.$$

This proves Step 2.

Step 3: *The theorem holds when*

$$\Sigma \cdot \Sigma > 0, \qquad \int_\Sigma \omega > 0.$$

Recall from the proof of Lemma 14.6 that for every integer $n \geq 1$ there exists an oriented embedded surface Σ_n with

$$[\Sigma_n] = n[\Sigma]$$

and

$$2g(\Sigma_n) - 2 - \Sigma_n \cdot \Sigma_n = n(2g(\Sigma) - 2 - \Sigma \cdot \Sigma)).$$

For n sufficiently large the surface Σ_n satisfies the requirements of Step 2 and hence

$$2g(\Sigma_n) - 2 - \Sigma_n \cdot \Sigma_n \geq c_1(K) \cdot \Sigma_n = n\, c_1(K) \cdot \Sigma.$$

This proves Step 3.

Step 4: *The theorem holds when*

$$\Sigma \cdot \Sigma = 0, \qquad \int_\Sigma \omega > 0.$$

The proof is similar to that of Step 3 in Theorem 14.1. As in that case choose an embedded surface T with

$$T \cdot T > 0, \qquad T \cdot \Sigma > 0.$$

and such that T intersects Σ transversally in precisely $N = T \cdot \Sigma$ distinct points, each contributing $+1$ to the intersection number. As before consider the surface $\Sigma_m = T \# m\Sigma$ with self-intersection number

$$\Sigma_m \cdot \Sigma_m = T \cdot T + 2mT \cdot \Sigma$$

and genus

$$g(\Sigma_m) = g(T) + mg(\Sigma) + mT \cdot \Sigma - m.$$

At this point the proofs diverge because the surface Σ in the proof of Theorem 14.1 had genus 0. Now the surface Σ_m satisfies the requirements of Step 3 for $m > 0$ sufficiently large and hence

$$\begin{aligned} 0 &\leq 2g(\Sigma_m) - 2 - \Sigma_m \cdot \Sigma_m - c_1(K) \cdot \Sigma_m \\ &= m\Big(2g(\Sigma) - 2 - c_1(K) \cdot \Sigma\Big) \\ &\quad + 2g(T) - 2 - T \cdot T - c_1(K) \cdot T. \end{aligned}$$

This proves Step 4 and the theorem. $\square$

15

VORTEX EQUATIONS OVER RIEMANN SURFACES

15.1 Vortex pairs

Let Σ be a compact oriented Riemann surface and $\gamma : T\Sigma \to \mathrm{End}(W)$ be a spin^c structure. Thus W is a rank-2 Hermitian vector bundle and γ satisfies the usual equations (4.18). Denote by $\mathcal{A}(\Sigma) = \mathcal{A}(\Sigma, \gamma)$ the space of spin^c connections on W which are compatible with the Levi-Civita connection on Σ. Recall from Section 6.1 that the tangent space of $\mathcal{A}(\Sigma)$ is and affine space whose parallel vector space is the space $\Omega^1(\Sigma, i\mathbb{R})$ of imaginary valued 1-forms on Σ. Throughout I shall use $A \in \mathcal{A}(\Sigma)$ as a label for the corresponding spin^c connection $\nabla_A : C^\infty(\Sigma, W) \to \Omega^1(\Sigma, W)$.

Recall from Section 4.4 that W carries a natural complex structure $I \in C^\infty(\Sigma, \mathrm{End}(W))$, different from the standard structure i, which in a local positively oriented orthonormal frame e_1, e_2 of $T\Sigma$ is given by

$$I = \gamma(e_2)\gamma(e_1).$$

This structure is independent of the choice of the orthonormal frame and agrees with the automorphism $\Gamma(\varepsilon)$ in the discussion defore Lemma 4.40. Thus the subbundles $W^\pm \subset W$ of positive and negative spinors are the $\pm i$ eigenspaces of I. In other words the endomorphism I has the form

$$I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

with respect to the splitting $W = W^+ \oplus W^-$. Note that changing the metric on Σ within the same conformal class does not affect the complex structure I.

Consider the space $\mathcal{A}(\Sigma) \times C^\infty(\Sigma, W)$ of pairs (A, Θ) where $A \in \mathcal{A}(\Sigma)$ is a spin^c connection on W and $\Theta : \Sigma \to W$ is a section. This manifold carries a natural symplectic structure Ω given by

$$\Omega((\alpha, \theta), (\beta, \tau)) = -\int_\Sigma \alpha \wedge \beta + \int_\Sigma \langle I\theta, \tau \rangle \,\mathrm{dvol}_\Sigma$$

for $\alpha, \beta \in \Omega^1(\Sigma, i\mathbb{R}) = T_A\mathcal{A}(\Sigma)$ and $\theta, \tau \in C^\infty(\Sigma, W)$. It also carries a natural complex structure

$$(\alpha, \theta) \mapsto (*\alpha, I\theta)$$

which is compatible with this symplectic form in the sense that the pairing $\Omega((\alpha,\theta),(*\beta,I\tau))$ defines the standard L^2 inner product on $\Omega^1(\Sigma,i\mathbb{R})\times C^\infty(\Sigma,W)$.

A complex submanifold

Denote by $\Theta_0\in W^+$ and $\Theta_1\in W^-$ the two components of $\Theta\in W$. Consider the set

$$\mathcal{B}(\Sigma)\subset\mathcal{A}(\Sigma)\times C^\infty(\Sigma,W)$$

of all pairs (A,Θ) which satisfy

$$D_A\Theta=0,\qquad \Theta_0\neq 0,\qquad \Theta_1=0 \tag{15.1}$$

where $D_A: C^\infty(X,W)\to C^\infty(X,W)$ denotes the Dirac operator associated to the spin^c connection ∇_A.

Proposition 15.1 *The space $\mathcal{B}(\Sigma)$ is a complex, and hence symplectic, submanifold of $\mathcal{A}(\Sigma)\times C^\infty(\Sigma,W)$.*

Proof: The formal tangent space $T_{(A,\Theta)}\mathcal{B}(\Sigma)$ consists of all pairs $(\alpha,\theta)\in\Omega^1(\Sigma,i\mathbb{R})\times C^\infty(\Sigma,W)$ which satisfy

$$D_A\theta+\gamma(\alpha)\Theta=0,\qquad \theta_1=0. \tag{15.2}$$

This space is invariant under the complex structure $(\alpha,\theta)\mapsto(*\alpha,I\theta)$ because

$$D_AI+ID_A=0,\qquad \gamma(*\alpha)=\gamma(v)I=-I\gamma(v).$$

Moreover, the space $\mathcal{B}(\Sigma)$ is indeed a submanifold. To see this consider the operator $\mathcal{D}:\Omega^1(\Sigma,i\mathbb{R})\times C^\infty(\Sigma,W^+)\to C^\infty(\Sigma,W^-)$ given by

$$\mathcal{D}(\alpha,\theta_0)=D_A\theta_0+\gamma(\alpha)\Theta_0.$$

Its L^2-adjoint operator $\mathcal{D}^*: C^\infty(\Sigma,W^-)\to\Omega^1(\Sigma,i\mathbb{R})\times C^\infty(\Sigma,W^+)$ has the form

$$\mathcal{D}^*\tau_1=\begin{pmatrix} i\mathrm{Re}\,\langle i\gamma(.)\Theta_0,\tau_1\rangle \\ D_A^*\tau_1\end{pmatrix}.$$

It is a simple exercise to check that

$$\mathcal{D}\mathcal{D}^*\tau_1=D_AD_A^*\tau_1+|\Theta_0|^2\tau_1.$$

This operator is invertible whenever $\Theta_0\neq 0$. Hence, in an appropriate Sobolev space setting, it follows from the implicit function theorem that $\mathcal{B}(\Sigma)$ is a (Banach or Fr'echet) submanifold of $\mathcal{A}(\Sigma)\times C^\infty(\Sigma,W)$ whose tangent space at (A,Θ_0) consists of the solutions of (15.2). The analytical details are standard and can be safely left to the reader. □

Symplectic quotients

It is interesting to examine the symplectic geometry of the space $\mathcal{A}(\Sigma) \times C^\infty(\Sigma, W)$ more closely. The gauge group $\mathcal{G}(\Sigma) = \mathrm{Map}(\Sigma, S^1)$ acts on this space by linear symplectomorphisms

$$(A, \Theta) \mapsto (u^*A, u^{-1}\Theta)$$

for $A \in \mathcal{A}(\Sigma)$ and $\Theta \in C^\infty(\Sigma, W)$. The formula

$$D_{u^*A}(u^{-1}\Theta) = u^{-1}D_A\Theta$$

shows that the complex submanifold $\mathcal{B}(\Sigma)$ is invariant under this action. The infinitesimal action is generated by the vector fields $\mathcal{X}_\xi$ on $\mathcal{A}(L) \times C^\infty(\Sigma, W)$ defined by

$$\mathcal{X}_\xi(A, \Theta) = (-d\xi, \xi\Theta)$$

for $\xi \in \Omega^0(\Sigma, i\mathbb{R}) = \mathrm{Lie}(\mathcal{G}(\Sigma))$. The next lemma shows that the vector fields $\mathcal{X}_\xi$ are all Hamiltonian.

Lemma 15.2 *The vector field $\mathcal{X}_\xi$ is generated by the Hamiltonian function $H_\xi : \mathcal{A}(\Sigma) \times C^\infty(\Sigma, W) \to \mathbb{R}$ given by*

$$H_\xi(A, \Theta) = -\int_\Sigma \xi \left(F_A + \frac{|\Theta_1|^2 - |\Theta_0|^2}{2} i\omega \right).$$

Here $\omega \in \Omega^2(\Sigma)$ denotes the volume form of the given Riemannian metric.

Proof: For $A \in \mathcal{A}(\Sigma)$, $\alpha \in \Omega^1(\Sigma, i\mathbb{R})$, and $\Theta, \theta \in C^\infty(\Sigma, W)$

$$\begin{aligned}
\Omega(\mathcal{X}_\xi(A, \Theta), (\alpha, \theta)) &= \int_\Sigma d\xi \wedge \alpha + \int_\Sigma \langle \xi I\Theta, \theta \rangle \omega \\
&= -\int_\Sigma \xi d\alpha + \int_\Sigma \mathrm{Im}\, \xi \langle iI\Theta, \theta \rangle \omega \\
&= -\int_\Sigma \xi d\alpha + \int_\Sigma \mathrm{Im}\, \xi \left(\langle \Theta_1, \theta_1 \rangle - \langle \Theta_0, \theta_0 \rangle \right) \omega \\
&= -\int_\Sigma \xi \Big(d\alpha + \langle \Theta_1, \theta_1 \rangle i\omega - \langle \Theta_0, \theta_0 \rangle i\omega \Big) \\
&= dH_\xi(A, \Theta)(\alpha, \theta).
\end{aligned}$$

This proves the lemma. □

Identify the dual of the Lie algebra $\mathrm{Lie}(\mathcal{G}(\Sigma)) = \Omega^0(\Sigma, i\mathbb{R})$ with the space of 2-forms $\mathrm{Lie}(\mathcal{G}(\Sigma))^* = \Omega^2(\Sigma, i\mathbb{R})$ via the obvious pairing. Then

Lemma 15.2 shows that the moment map of the action of $\mathcal{G}(\Sigma)$ on $\mathcal{A}(\Sigma) \times C^\infty(\Sigma, W)$ is the map

$$\mu : \mathcal{A}(\Sigma) \times C^\infty(\Sigma, W) \to \Omega^2(\Sigma, i\mathbb{R})$$

given by

$$\mu(A, \Theta) = F_A + \frac{|\Theta_1|^2 - |\Theta_0|^2}{2} i\omega.$$

Note that $\mu(u^*A, u^{-1}\Theta) = \mu(A, \Theta)$. The moment map of the action on $\mathcal{B}(\Sigma)$ is simply the restriction of μ. It is interesting to consider the Marsden-Weinstein quotient

$$\mathcal{M}_d(\Sigma) = \mathcal{B}(\Sigma) /\!\!/ \mathcal{G}(\Sigma) = \mu^{-1}(0)/\mathcal{G}(\Sigma).$$

Here $d = c_1(W^+) \cdot [\Sigma]$ denotes the degree of the line bundle $W^+ \to \Sigma$. The set $\mu^{-1}(0)$ consists of the pairs $(A, \Theta_0) \in \mathcal{A}(\Sigma) \times C^\infty(\Sigma, W^+)$ which satisfy

$$D_A \Theta_0 = 0, \qquad *iF_A = -\frac{|\Theta_0|^2}{2}, \qquad \Theta_0 \neq 0. \tag{15.3}$$

The quotient space $\mathcal{M}_d(\Sigma)$ is called the moduli space of **vortex pairs**. It turns out that this space is a compact smooth finite dimensional manifold of real dimension $2d$. It carries a natural symplectic and complex structure. However, the space itself as well as its symplectic and complex structures depend on the metric on Σ. It will be shown below that, as the metric on Σ varies without changing the volume form, the different symplectic structures on $\mathcal{M}_d(\Sigma)$ can be naturally identified. In other words, there is a natural symplectic connection on the bundle over the space of metrics on Σ with fixed volume form whose fibers are the spaces $\mathcal{M}_d(\Sigma)$. Another important fact is that, via the zero-sets of the sections Θ_0, the space $\mathcal{M}_d(\Sigma)$ can be naturally identified with the d-fold symmetric product of the surface Σ provided that $1 \leq d \leq g - 2$ where g is the genus of Σ. This will also be proved below.

Relation with Cauchy-Riemann operators

A Riemannian metric on Σ determines a volume form ω and hence a complex structure J on Σ (see Example 3.1). Hence $T\Sigma$ is a complex vector bundle over Σ. Its dual bundle

$$K = T^*\Sigma = \Lambda^{1,0}T^*\Sigma$$

is the **canonical bundle** with Chern number $c_1(K) = 2g - 2 = -c_1(T\Sigma)$ where $g = g(\Sigma)$ denotes the genus.

Lemma 15.3 *There is a natural isomorphism $W^- \to K^* \otimes W^+$.*

Proof: First note that

$$\gamma(Jv)\theta = \gamma(v)I\theta$$

for $v \in T_z\Sigma$ and $\theta \in W_z$. To see this assume without loss of generality that $|v| = 1$. Then the vectors v, Jv form a positively oriented orthonormal basis of $T_z\Sigma$ and hence $\gamma(v)\gamma(Jv)\theta = -I\theta$ for $\theta \in W_z$. Since $\gamma(v)^* = \gamma(v)^{-1} = -\gamma(v)$ the equation follows. Now consider the isomorphism $W^- \to K^* \otimes W^+ : \theta_1 \mapsto \lambda_{\theta_1}$ given by

$$\lambda_{\theta_1}(v) = -\frac{1}{\sqrt{2}}\gamma(v)\theta_1$$

for $\theta_1 \in W_z^-$ and $v \in T_z\Sigma$. The formula $\gamma(Jv)\theta_1 = -i\gamma(v)\theta_1$ for $\theta_1 \in W_z^-$ shows that the 1-form $\lambda_{\theta_1} : T_z\Sigma \to W_z^+$ is complex anti-linear as required. The inverse isomorphism $K^* \otimes W^+ \to W^-$ is given by the formula

$$\theta_1 = \frac{\sqrt{2}}{|v|^2}\gamma(v)\lambda(v)$$

for $v \in T_z\Sigma - \{0\}$ and $\lambda \in K_z^* \otimes W_z^+$. Here the vector $|v|^{-2}\gamma(v)\lambda(v) \in W_z^-$ is independent of the choice of v. The reader may check that the isomorphism $\theta_1 \mapsto \lambda_{\theta_1}$ is unitary. This proves the Lemma. $\square$

It is sometimes useful to denote

$$W^+ = E, \qquad W^- = K^* \otimes E.$$

Thus the sections of $W \to \Sigma$ correspond to pairs (θ, λ) consisting of a section of E and a $(0,1)$-form on Σ with values in E:

$$C^\infty(\Sigma, W) = C^\infty(\Sigma, E) \oplus \Omega^{0,1}(\Sigma, E).$$

This reproves the fact that, up to isomorphism, any spin^c structure on Σ can be obtained from the standard spin^c structure $\mathbb{C} \oplus K^*$ by tensoring with a line bundle $E \to \Sigma$ (see Theorem 5.8). The Levi-Civita connection on Σ determines a connection on K and hence, with the trivial connection on the trivial bundle $\mathbb{C}$, a spin^c connection ∇_{A_0} on the standard spin^c structure $W_0 = \mathbb{C} \oplus K^*$ Think of the label A_0 as a connection on the line bundle $\det(W_0)^{1/2} = K^{-1/2}$. The space $\mathcal{A}(\Sigma)$ of spin^c connections on $W = W_0 \otimes E$, compatible with the Levi-Civita connection on Σ, can be identified with the space of connections on $\det(W)^{1/2} = K^{-1/2} \otimes E$. The elements of $\mathcal{A}(\Sigma) = \mathcal{A}(K^{-1/2} \otimes E)$ will be denoted by $A_0 + A$ where A_0 is the standard connection on $K^{-1/2}$ and $A \in \mathcal{A}(E)$. Theorem 6.17 shows that the Dirac operator D_{A+A_0} is given by

$$\frac{1}{\sqrt{2}}D_{A_0+A} = \begin{pmatrix} 0 & \bar\partial_A^* \\ \bar\partial_A & 0 \end{pmatrix}.$$

In the 2-dimensional case this can be checked easily by direct calculation.

Remark 15.4 In terms of the splitting $W = E \oplus K^* \otimes E$ the symplectic form on the space $\mathcal{A}(E) \oplus C^\infty(\Sigma, E) \oplus \Omega^{0,1}(\Sigma, E)$ can now be expressed in the form

$$\Omega((\alpha, \theta, \lambda), (\alpha', \theta', \lambda')) = -\int_\Sigma \alpha \wedge \alpha' + \int_\Sigma \Big(\langle i\theta, \theta' \rangle - \langle i\lambda, \lambda' \rangle \Big) \mathrm{dvol}_\Sigma$$

for $\theta, \theta' \in C^\infty(\Sigma, E)$ and $\lambda, \lambda' \in \Omega^{0,1}(\Sigma, E)$. □

INSERT:

0. EXAMPLES OF SOLUTIONS

1. VORTEX EQUATIONS ARE EQUIVALENT TO SYMMETRIC PRODUCTS OF Σ

2. RATIONAL COHOMOLOGY OF SYMMETRIC PRODUCTS AND THE ACTION OF THE MAPPING CLASS GROUP

3. SYMPLECTIC CONNECTION ON THE BUNDLE OVER THE MODULI SPACE OF COMPLEX STRUCTURES ON Σ WHOSE FIBER IS THE SYMMETRIC PRODUCT

4. RELATION BETWEEN HOLOMORPHIC CURVES IN SYMMETRIC PRODUCT AND SEIBERG-WITTEN MONOPOLES ON $\Sigma \times \mathbb{C}$.

15.2 Symmetric products

15.3 A symplectic connection

15.4 Holomorphic curves and Seiberg-Witten monopoles

15.5 Boundary value problems

Let Y be a compact Riemannian 3-manifold with boundary $\partial Y = \Sigma$ equipped with a spin^c structure $\gamma : TY \to \mathrm{End}(W)$. With the spin^c structure fixed denote by $\mathcal{A}(Y) = \mathcal{A}(Y, \gamma)$ the space of spin^c connections on W which are compatible with the Levi-Civita connection on TY. Recall from Section 10.1 that there is a natural 1-form $\mathcal{F}$ on the space $\mathcal{A}(Y) \times C^\infty(Y, W)$ defined by

$$\mathcal{F}(A, \Theta; \alpha, \theta) = -\int_Y F_A \wedge \alpha - \frac{1}{2} \int_Y \langle \Theta, \gamma(\alpha)\Theta \rangle \,\mathrm{dvol}_Y - \int_Y \langle D_A \Theta, \theta \rangle \,\mathrm{dvol}_Y$$

for $\alpha \in \Omega^1(Y, i\mathbb{R})$ and $\theta \in C^\infty(Y, W)$. If the boundary is nonempty then this 1-form is not closed. Its differential is given by the formula in Lemma 15.5 below. Denote $\Sigma = \partial Y$ and consider the restricted bundle $W|_\Sigma$ and the restriction $\gamma : T\Sigma \to \mathrm{End}(W)$. This defines a spin^c structure on Σ. For $y \in \Sigma$ denote by $\nu(y) \in T_y Y$ the outward unit normal and consider the complex structure on the bundle $W \to \Sigma$ defined by

$$I = -\gamma(\nu) \in C^\infty(\Sigma, \mathrm{End}(W)).$$

Then $I = \gamma(e_2)\gamma(e_1)$ for every positively oriented orthonormal basis e_1, e_2 of $T\Sigma$. Hence I depends only on the surface Σ and not on the ambient manifold Y. Moreover, I agrees with the complex structure considered in Section 15.1. Let $\iota : \Sigma \to Y$ denote the inclusion of the boundary and consider the restriction map

$$r : \mathcal{A}(Y) \times C^\infty(Y, W) \to \mathcal{A}(\Sigma) \times C^\infty(\Sigma, W)$$

defined by

$$r(A, \Theta) = (\iota^* A, \Theta \circ \iota)$$

for $A \in \mathcal{A}(Y)$ and $\Theta \in C^\infty(Y, W)$.

Lemma 15.5 *The differential of the* 1*-form* $\mathcal{F}$ *on* $\mathcal{A}(Y) \times C^\infty(Y, W)$ *is given by*

$$d\mathcal{F} = r^*\Omega.$$

Proof: Linearizing the 1-form $\mathcal{F}$ gives rise to the formally self-adjoint operator $D_{A,\Phi} : \Omega^1(Y, i\mathbb{R}) \times C^\infty(Y, W) \to \Omega^1(Y, i\mathbb{R}) \times C^\infty(Y, W)$ defined by

$$\begin{aligned}
\langle D_{A,\Theta}(\alpha,\theta),(\beta,\tau)\rangle &= \frac{d}{dt}\bigg|_{t=0} \mathcal{F}(A + t\alpha, \Theta + t\theta; \beta, \tau) \\
&= \frac{d}{dt}\bigg|_{t=0} - \int_Y F_{A+t\alpha} \wedge \beta \\
&\quad - \frac{d}{dt}\bigg|_{t=0} \int_Y \langle D_{A+t\alpha}(\Theta + t\theta), \tau\rangle \,\mathrm{dvol}_Y \\
&\quad - \frac{1}{2}\frac{d}{dt}\bigg|_{t=0} \int_Y \langle \Theta + t\theta, \gamma(\beta)(\Theta + t\theta)\rangle \\
&= -\int_Y d\alpha \wedge \beta - \int_Y \langle \theta, \gamma(\beta)\Theta\rangle \\
&\quad - \int_Y \langle D_A\theta + \gamma(\alpha)\Theta, \tau\rangle \,\mathrm{dvol}_Y \\
&= \int_Y \big\langle {*d\alpha} - \gamma^{-1}((\Theta\theta^* + \theta\Theta^*)_0), \beta\big\rangle \,\mathrm{dvol}_Y \\
&\quad - \int_Y \langle D_A\theta + \gamma(\alpha)\Theta, \tau\rangle \,\mathrm{dvol}_Y.
\end{aligned}$$

Here $\langle \cdot, \cdot \rangle$ denotes either the inner product on T^*Y or the real inner product on W. Thus the operator $D_{A,\Theta}$ is given by

$$D_{A,\Theta}(\alpha, \theta) = \begin{pmatrix} *d\alpha - \gamma^{-1}((\Theta\theta^* + \theta\Theta^*)_0) \\ -D_A\theta - \gamma(\alpha)\Theta \end{pmatrix}.$$

(Compare this with the formula (10.4) for the gradient of the Chern-Simons functional.) Here γ^{-1} is to be understood as a fiberwise linear isomorphism $\mathrm{End}_0(W) \to T^*Y \otimes \mathbb{C}$ which assigns to every traceless endomorphism of W_y a complex valued real linear functional $T_yY \to \mathbb{C}$. If the endomorphism is Hermitian then the linear functional is imaginary valued. For two sections $\theta, \tau \in C^\infty(Y, W)$ there is a natural vector field $v = v(\theta, \tau) \in \mathrm{Vect}(Y)$ defined by

$$v(\theta, \tau) = \sum_{j=1}^{3} \langle \gamma(e_j)\theta, \tau \rangle \, e_j$$

in a local orthonormal frame e_1, e_2, e_3 of TY. Note that any other vector field $w \in \mathrm{Vect}(Y)$ satisfies the pointwise identity

$$\langle v(\theta, \tau), w \rangle = \langle \gamma(w)\theta, \tau \rangle \, .$$

Moreover, the covariant divergence of $v(\varphi, \psi)$ is given by

$$\mathrm{div}(v(\theta, \tau)) = \langle D_A\theta, \tau \rangle - \langle \theta, D_A\tau \rangle \, .$$

To see this just note that the divergence of any vector field $v = \sum_i v_i e_i$ is given by $\mathrm{div}(v) = \sum_i \partial_i v_i + \mathrm{div}(e_i) v_i$ and calculate the right hand side. The details of this are left to the reader. Now the differential of $\mathcal{F}$ is given by

$$\begin{aligned}
& d\mathcal{F}(A, \Theta)((\alpha, \theta), (\beta, \tau)) \\
& \quad = \langle D_{A,\Theta}(\alpha, \theta), (\beta, \tau) \rangle - \langle (\alpha, \theta), D_{A,\Theta}(\beta, \tau) \rangle \\
& \quad = -\int_Y d\alpha \wedge \beta + \int_Y \alpha \wedge d\beta \\
& \qquad - \int_Y \Big(\langle D_A\theta, \tau \rangle - \langle \theta, D_A\tau \rangle \Big) \mathrm{dvol}_Y \\
& \quad = -\int_Y d(\alpha \wedge \beta) - \int_Y \mathrm{div}(v(\theta, \tau)) \mathrm{dvol}_Y \\
& \quad = -\int_\Sigma \alpha \wedge \beta - \int_\Sigma \langle v(\theta, \tau), \nu \rangle \mathrm{dvol}_\Sigma \\
& \quad = -\int_\Sigma \alpha \wedge \beta - \int_\Sigma \langle \gamma(\nu)\theta, \tau \rangle \mathrm{dvol}_\Sigma \\
& \quad = -\int_\Sigma \alpha \wedge \beta + \int_\Sigma \langle I\theta, \tau \rangle \mathrm{dvol}_\Sigma .
\end{aligned}$$

Here the fourth equality is Stokes' theorem (see the hint in Exercise 2.36). □

Part V

APPENDIX

APPENDIX A

FREDHOLM THEORY

This appendix gives an introduction to linear Fredholm theory. The first section discusses the basic stability properties of Fredholm operators. It includes a brief exposition of the topological index as a K-theory class. Section A.2 is devoted to the determinant line bundle over the space of Fredholm operators. This material plays a crucial role in proving that moduli spaces are orientable and for finding a canonical orientation. The final section derives an explicit formula for the trivialization of the determinant bundle along a path of Fredholm operators in terms of a crossing number (in the case of index zero).

A.1 Linear Fredholm operators

A bounded linear operator $D : X \to Y$ between Banach spaces is called a **Fredholm operator** if it has finite dimensional kernel, closed range, and finite dimensional cokernel $Y/\mathrm{im}\, D$. The **index** of a Fredholm operator D is defined by

$$\mathrm{index}\, D = \dim \ker\, D - \dim \mathrm{coker}\, D.$$

Here the kernel and cokernel are to be understood as real vector spaces. If D is a complex linear Fredholm operator between complex Banach spaces then it is important to distinguish between the real and the complex Fredholm index. Obviously, the real Fredholm index is twice the complex Fredholm index. The following lemma plays an important role in establishing the Fredholm property for a given linear operator D.

Lemma A.1 *Let X, Y, Z be Banach spaces. Assume that $D : X \to Y$ is a bounded linear operator and $K : X \to Z$ is a compact operator. Assume that there is a constant $c > 0$ such that*

$$\|x\|_X \le c \left(\|Dx\|_Y + \|Kx\|_Z \right) \tag{A.1}$$

for $x \in X$. Then D has closed range and finite dimensional kernel.

Proof: To prove that the kernel of D is finite dimensional it suffices to show that the unit ball in $\ker D$ is compact. To see this choose a bounded sequence $x_\nu \in X$ such that

$$\|x_\nu\| \le 1, \qquad Dx_\nu = 0.$$

Since x_ν is bounded there exists a subsequence (still denoted by x_ν) such that Kx_ν converges. Since $x_\nu \in \ker D$ it follows from the estimate (A.1) that x_ν is a Cauchy sequence. Since X is complete the subsequence converges.

Now assume without loss of generality that D is injective. Otherwise replace X by a complement of $\ker D$. Such a complement exists by the Hahn-Banach theorem: Pick a basis $x_1, \dots, x_N$ of $\ker D$ and choose $x_k^* \in X^*$ such that $\langle x_k^*, x_j \rangle = \delta_{jk}$. Then $X_0 = \{x \in X \mid \langle x_1^*, x\rangle = \cdots = \langle x_N^*, x\rangle = 0\}$ is the required complement.

Let $y \in \mathrm{cl}(\mathrm{im}\, D)$. Then there exists a sequence $x_\nu \in X$ such that

$$y = \lim_{\nu\to\infty} Dx_\nu.$$

We prove first that x_ν is bounded. Otherwise, passing to a subsequence, we may assume that $\|x_\nu\|$ converges to ∞. Then the sequence

$$\xi_\nu = \|x_\nu\|^{-1} x_\nu$$

is of norm 1 and $D\xi_\nu$ converges to 0. Passing to a further subsequence we may assume that $K\xi_\nu$ converges. In view of (A.1) ξ_ν is a Cauchy sequence. The limit $\xi = \lim_{\nu\to\infty} \xi_\nu$ is of norm 1 and $D\xi = 0$, a contradiction. Hence the sequence x_ν is bounded. It follows again from the compactness of K, estimate (A.1), and the completeness of X that x_ν has a converging subsequence. Let x be its limit. Then $y = Dx$. □

Corollary A.2 *Let X and Y be Banach spaces and $D : X \to Y$ be a bounded linear operator with closed range and finite dimensional kernel.*

(i) *For every compact operator $K : X \to Y$ the operator $D + K$ also has closed range and finite dimensional kernel.*

(ii) *There exists a constant $\varepsilon > 0$ such that if $P : X \to Y$ is a bounded linear operator with $\|P\| < \varepsilon$ then $D + P$ has closed range and finite dimensional kernel.*

Proof: Suppose $\dim \ker D = n$ and choose a bounded linear operator $K_0 : X \to \mathbb{R}^n$ such that the restriction of K_0 to $\ker D$ is a vector space isomorphism. Then the operator $X \to Y \oplus \mathbb{R}^n : x \mapsto (Dx, K_0 x)$ is injective and has closed range. Hence, by the open mapping theorem, there exists a constant $c > 0$ such that

$$\|x\|_X \le c\left(\|Dx\|_Y + \|K_0 x\|_{\mathbb{R}^n}\right).$$

Hence for any compact operator $K : X \to Y$ we have

$$\|x\|_X \le c\left(\|(D+K)x\|_Y + \|Kx\|_Y + \|K_0 x\|_{\mathbb{R}^n}\right).$$

Similarly, if $\|P\| < 1/c$, then

$$\|x\|_X \leq \frac{c}{1 - c\|P\|} \left(\|(D + P)x\|_Y + \|K_0 x\|_{\mathbb{R}^n} \right).$$

Hence the assertions follows from Lemma A.1. $\square$

Corollary A.3 *Let X and Y be Banach spaces and $D : X \to Y$ be a bounded linear operator with closed range and finite dimensional cokernel.*

(i) *For every compact operator $K : X \to Y$ the operator $D + K$ also has closed range and finite dimensional cokernel.*

(ii) *There exists a constant $\varepsilon > 0$ such that if $P : X \to Y$ is a bounded linear operator with $\|P\| < \varepsilon$ then $D + P$ has closed range and finite dimensional cokernel.*

Proof: D has closed range if and only if D^* has closed range and it has finite dimensional cokernel if and only if D^* has finite dimensional kernel. Hence the result follows from Corollary A.2. $\square$

A bounded linear operator $D : X \to Y$ is Fredholm if and only if it is invertible modulo a compact operator. This means that there exists a bounded linear operator $T : Y \to X$ such that both $DT - \mathbb{1}$ and $TD - \mathbb{1}$ are compact operators.* If both $D : X \to Y$ and $T : Y \to Z$ are Fredholm operators then so is $TD : X \to Z$ and

$$\operatorname{index} TD = \operatorname{index} D + \operatorname{index} T. \tag{A.2}$$

A bounded linear operator $D : X \to Y$ is Fredholm if and only if its dual operator $D^* : Y^* \to X^*$ is. Their indices are related by

$$\operatorname{index} D^* = -\operatorname{index} D.$$

The most important properties of Fredholm operators are related to their stability under perturbations. The assertions about the Fredholm property follow immediately from Corollaries A.2 and A.3. The assertions about the index are easy exercises.

Theorem A.4 *Let $D : X \to Y$ be a Fredholm operator*

(i) *If $K : X \to Y$ is a compact operator then $D + K$ is a Fredholm operator and* $\operatorname{index}(D + K) = \operatorname{index} D$.

(ii) *There exists a constant $\varepsilon > 0$ such that if $P : X \to Y$ is a bounded linear operator with $\|P\| < \varepsilon$ then $D + P$ is a Fredholm operator and* $\operatorname{index}(D + P) = \operatorname{index} D$.

*To prove the existence of T use the construction of a pseudo-inverse in Proposition B.7. To prove the converse use Lemma A.1.

The last statement asserts that the set of Fredholm operators is open with respect to the uniform operator topology and the index is constant on each component.

The topological index

Here is a brief discussion of the **index bundle** or the **topological index** of a family of Fredholm operators as a K-theory class. For a full exposition the reader is referred to the original work of Atiyah in [3, 4]. Let X and Y be complex Banach spaces and M be a finite dimensional compact manifold. Suppose that

$$D : M \to \mathcal{L}(X, Y)$$

is a smooth map such that $D(p)$ is a complex linear Fredholm operator for every $p \in M$. If $D(p)$ is surjective for all p then the kernels form a vector bundle $\ker D \to M$. Local trivializations can be found by choosing a pseudo-inverse $T_0 : Y \to X$ of $D_0 = D(p_0)$ and considering the projection $\ker D(p) \to \ker D(p_0) : x \mapsto x - T_0 D_0 x$. (See Proposition B.7.) In general, when D is neither injective nor surjectve, it is interesting to consider the formal difference

$$\mathcal{IND}(D) = \ker D \ominus \operatorname{coker} D \in K(M)$$

That this is a well defined element in the K-theory of M can be seen as follows. It is easy to construct a smooth map $\Phi : M \to \mathcal{L}(\mathbb{C}^N, Y)$ for N sufficiently large such that the operator

$$D(p) \oplus \Phi(p) : X \oplus \mathbb{C}^N \to Y$$

is surjective for every p. Just do this locally in the neighbourhood of a point and use cutoff functions. Hence there is a vector bundle $\ker(D \oplus \Phi) \to M$ and the topological index of D can be defined as the K-theory class

$$\mathcal{IND}(D) = \ker(D \oplus \Phi) \ominus \mathbb{C}^N \in K(M).$$

That the right hand side is independent of Φ is an easy exercise. When D is onto just note that the bundle $\ker(D) \oplus \mathbb{C}^N$ is naturally isomorphic to $\ker(D \oplus \Phi)$. In the general case consider $D \oplus \Phi \oplus \Psi$ where Ψ is another such map. The reader may check that the topological index of the adjoint operator is given by

$$\mathcal{IND}(D^*) = -\mathcal{IND}(D).$$

The notation $\mathcal{IND}(D) = \ker D \ominus \operatorname{coker} D$ can now be justified as follows. If the kernel and cokernel of D are of constant dimension and form actual vector bundles over M choose a bundle $E \to M$ such that $\operatorname{coker} D \oplus E = \mathbb{C}^N$ is the trivial bundle. Then there exists a map $\Phi : \mathbb{C}^N \to Y$ such that

$\ker\,\Phi = E$ and $D \oplus \Phi$ is always onto. Hence $\ker\,(D \oplus \Phi) = \ker\,D \oplus E$ and thus

$$\mathcal{IND}(D) = \ker\,D \oplus E \ominus \mathbb{C}^N = \ker\,D \ominus \operatorname{coker} D$$

as claimed. Another important observation is the fact that the K-theory class $\mathcal{IND}(D) \in K(M)$ is invariant under homotopy and hence under compact perturbations. Thus

$$\mathcal{IND}(D + K) = \mathcal{IND}(D)$$

for every map $K : M \to \mathcal{K}(X, Y)$ into the space of compact operators. These facts follow directly from the definition of the topological index via stabilization. Finally, note that there is an obvious generalization to Banach space bundles $E \to M$ and $F \to M$ with $D(p) : E_p \to F_p$ a Fredholm section of the bundle of linear operators. If the corresponding definition of the topological index is applied to the finite dimensional case then

$$\mathcal{IND}(D) = E \ominus F.$$

This can be seen either by noting that the zero map is a compact perturbation, or by stabilizing with $\mathbb{C}^N = F \oplus F'$ and $D' : E \oplus F' \oplus F \to F$ given by $D'(x, y, y') = y - Dx$. Then $\ker D' \cong E \oplus F'$.

A.2 Determinant line bundles

Let X and Y be Banach spaces and denote by $\mathcal{F}(X, Y)$ the space of all Fredholm operators $D : X \to Y$. The **determinant** of a Fredholm operator $D \in \mathcal{F}(X, Y)$ is defined as the 1-dimensional real vector space

$$\det(D) = \Lambda^{\max}(\ker\,D) \otimes \Lambda^{\max}(\ker\,D^*).$$

Our goal is to show that as D varies the vector spaces $\det(D)$ fit together to form a locally trivial line bundle. Hence it is important to keep track of the isomorphisms which identify different 1-dimensional vector spaces.

Think of the real line $\mathbb{R}$ as a 1-dimensional real vector space. For any two 1-dimensional real vector spaces V and W we shall use the notation $V = W$ to mean that the spaces are *naturally* isomorphic. This means that there is an obvious choice of isomorphism between them. For example, if V is 1-dimensional there is a natural isomorphism

$$V \otimes V^* \to \mathbb{R} : v \otimes v^* \mapsto v^*(v).$$

This notion of natural isomorphism can be more precisely expressed in the language of category theory. Denote by $\mathcal{V}$ the category of 1-dimensional real vector spaces and isomorphisms. Two functors $\mathcal{F}_0, \mathcal{F}_1 : \mathcal{C} \to \mathcal{V}$ are called

naturally isomorphic if there exists a functor $\mathcal{T} : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{V})$ which assigns to every object $A \in \mathrm{Ob}(\mathcal{C})$ a vector space isomorphism $\mathcal{T}(A) : \mathcal{F}_0(A) \to \mathcal{F}_1(A)$ such that $\mathcal{T}(B) \circ \mathcal{F}_0(T) = \mathcal{F}_1(T) \circ \mathcal{T}(A)$ for $A, B \in \mathrm{Ob}(\mathcal{C})$ and $T \in \mathrm{Mor}(A, B)$. In all our examples the isomorphism $\mathcal{T}(A)$ is obvious and we shall simply use the notation $\mathcal{F}_0(A) = \mathcal{F}_1(A)$. The reader may check that the notation $V \otimes V^* = \mathbb{R}$ is an example of this convention.

The **highest exterior power** $\Lambda^{\max} V$ of a finite dimensional vector space V is the space of equivalence classes $v_1 \wedge \ldots \wedge v_n$ of ordered n-tuples in V, where $\dim V = n$. Two such n-tuples $v_1 \wedge \ldots \wedge v_n$ and $w_1 \wedge \ldots \wedge w_n$ are equivalent iff either both form a basis and the induced isomorphism of V has determinant 1, or both n-tuples consist of linearly dependent vectors. Hence a nonzero vector in $\Lambda^{\max} V$ determines an orientation of V. When V has dimension 0 we define $\Lambda^{\max} V = \mathbb{R}$.

The **tensor product** $V \otimes W$ of two 1-dimensional real vector spaces V and W is the space of equivalence classes $v \otimes w$ of ordered pairs $(v, w) \in V \times W$ where $\lambda v \otimes w = v \otimes \lambda w$ for all $\lambda \in \mathbb{R}$. Thus $v_1 \otimes w_1 = v_2 \otimes w_2$ iff there exists a number $\lambda \in \mathbb{R}$ such that either $(v_1, w_2) = \lambda(v_2, w_1)$ or $(v_2, w_1) = \lambda(v_1, w_2)$. It follows that there is a natural isomorphism

$$V \otimes W = \Lambda^2(V \oplus W)$$

and hence

$$\Lambda^{\max} V \otimes \Lambda^{\max} W = \Lambda^{\max}(V \oplus W)$$

for any two finite dimensional real vector spaces V and W.

For every finite dimensional real vector space V there is a natural isomorphism

$$\Lambda^{\max} V \otimes \Lambda^{\max}(V^*) = \mathbb{R}$$

is given by $(v_1 \wedge \ldots \wedge v_n) \otimes (v_1^* \wedge \ldots \wedge v_n^*) \mapsto \det(v_k^*(v_j))$. This is a nondegenerate pairing and hence $\Lambda^{\max}(V^*) = (\Lambda^{\max} V)^*$. Since the positioning of the parentheses makes no difference we shall use the notation $\Lambda^{\max} V^*$. This is the space of volume forms on V. Conversely, every isomorphism $\Lambda^{\max} V \otimes \Lambda^{\max} W \to \mathbb{R}$ induces an isomorphism $\Lambda^{\max} V \to \Lambda^{\max} W^*$.

Lemma A.5 *Let V be a finite dimensional vector space and $W \subset V$ be a linear subspace. Then there is a natural isomorphism*

$$\Lambda^{\max} V = \Lambda^{\max} W \otimes \Lambda^{\max}(V/W).$$

Proof: Let $N = \dim V$ and $n = \dim W$. Denote by $\mathcal{B}(V, W)$ the set of all bases $v_1, \ldots, v_N$ of V whose first n elements span W. For $v \in V$ denote $[v] = v + W$. The map $\mathcal{B}(V, W) \to \Lambda^{\max} W \otimes \Lambda^{\max}(V/W)$ defined by

$$(v_1, \ldots, v_N) \mapsto (v_1 \wedge \ldots \wedge v_n) \otimes ([v_{n+1}] \wedge \ldots \wedge [v_N])$$

induces the required isomorphism. □

Theorem A.6 *The space*

$$\det(X,Y) = \{(D,\sigma)\,|\,D \in \mathcal{F}(X,Y), \sigma \in \det(D)\}$$

is a line bundle over $\mathcal{F}(X,Y)$.

Proof: We must prove that the space $\det(X,Y)$ admits a local trivialization in a neighborhood of every Fredholm operator $D \in \mathcal{F}(X,Y)$. Assume first that D is onto. Then there is an estimate

$$\|y^*\|_{Y^*} \le c\,\|D^*y^*\|_{X^*}\,.$$

Hence the operator $D+P : X \to Y$ is onto whenever P is sufficiently small. By Theorem A.4 both operators D and $D+P$ have the same Fredholm index and hence their kernels are of the same dimension. Now choose a right inverse $T : Y \to X$ of D so that $DT = 1\!\!1_Y$. Then there is an isomorphism

$$\ker\,(D+P) \to \ker\,D : x \mapsto x + TPx$$

whenever P is sufficiently small. Hence the kernels of $D+P$ form a locally trivial vector bundle over $\mathcal{F}(X,Y)$ in a neighborhood of a surjective Fredholm operator.

We now reduce the general case to the surjective case. First observe that given any Fredholm operator $D_0 : X \to Y$ there exists a positive integer N and an injective linear map $\Phi : \mathbb{R}^N \to Y$ such that the operator $D_0 \oplus \Phi : X \oplus \mathbb{R}^N \to Y$ defined by

$$D_0 \oplus \Phi(x,\zeta) = D_0x + \Phi\zeta$$

is surjective. To see this let $N = \dim\operatorname{coker} D_0$ and choose $y_1,\ldots,y_N \in Y$ which span a complement of the range of D_0 in Y. Then the linear map $\Phi\zeta = \sum_j \zeta_j y_j$ is as required.

Let $D : X \to Y$ be a Fredholm operator such that $D \oplus \Phi$ is onto and consider the exact sequence

$$0 \longrightarrow \ker\,D \longrightarrow \ker\,(D\oplus\Phi) \longrightarrow \operatorname{im} D \cap \operatorname{im}\Phi \longrightarrow 0$$

where the second map is $\ker\,D \to \ker\,(D\oplus\Phi) : x \mapsto (x,0)$ and the third map is $\ker\,(D\oplus\Phi) \to \operatorname{im} D\cap\operatorname{im}\Phi : (x,\xi) \mapsto \Phi\xi = -Dx$. This sequence shows that there is a natural isomorphism

$$\operatorname{im} D \cap \operatorname{im}\Phi = \frac{\ker\,(D\oplus\Phi)}{\ker\,D}$$

and hence, by Lemma A.5,

$$\Lambda^{\max} \ker (D \oplus \Phi) = \Lambda^{\max}(\ker\ D) \otimes \Lambda^{\max}(\operatorname{im} D \cap \operatorname{im} \Phi). \tag{A.3}$$

We claim that $\Lambda^{\max}(\operatorname{im} D \cap \operatorname{im} \Phi)$ is naturally isomorphic to $\Lambda^{\max}(\ker\ D^*)$. To see this note first that, since $\Phi : \mathbb{R}^N \to Y$ is injective, $\Lambda^{\max}\operatorname{im} \Phi = \mathbb{R}$. Now use Lemma A.5 for the inclusion $\operatorname{im} D \cap \operatorname{im} \Phi \subset \operatorname{im} \Phi$ to obtain

$$\begin{aligned}
\Lambda^{\max}(\operatorname{im} D \cap \operatorname{im} \Phi) &= \Lambda^{\max} \left(\frac{\operatorname{im} \Phi}{\operatorname{im} D \cap \operatorname{im} \Phi} \right)^* \\
&= \Lambda^{\max} \left(\frac{Y}{\operatorname{im} D} \right)^* \\
&= \Lambda^{\max}(\ker\ D^*).
\end{aligned}$$

Here the second isomorphism uses the fact that $\operatorname{im} D + \operatorname{im} \Phi = Y$. The last isomorphism uses the fact that the dual space of a quotient Y/Y_1 agrees with the annihilator of Y_1 in Y^*. With $Y_1 = \operatorname{im} D$ this annihilator is given by the kernel of D^*. Thus we have proved that $\Lambda^{\max}(\operatorname{im} D \cap \operatorname{im} \Phi) \cong \Lambda^{\max}(\ker\ D^*)$. Combining this isomorphism with (A.3) we obtain

$$\det(D) = \Lambda^{\max}(\ker\ D) \otimes \Lambda^{\max}(\ker\ D^*) = \Lambda^{\max} \ker (D \oplus \Phi) = \det(D \oplus \Phi)$$

as required. □

Exercise A.7 Let $D : X \to Y$ be a Fredholm opertor and $\Phi : \mathbb{R}^N \to Y$ be a linear map (not necessarily injective) such that $D \oplus \Phi$ is onto.

(i) Prove that $\dim \ker D^* + \dim (\operatorname{im} D \cap \operatorname{im} \Phi) = N$.

(ii) Given a basis $x_1, \ldots, x_k$ of $\ker\ D$ and a basis $y_1^*, \ldots, y_\ell^*$ of $\ker D^*$ prove that there exists a basis $\zeta_1, \ldots, \zeta_N$ of $\mathbb{R}^N$ and vectors $\xi_{\ell+1}, \ldots, \xi_N \in X$ such that

$$\begin{aligned}
\langle y_i^*, \Phi\zeta_j \rangle &= \delta_{ij}, \qquad && i, j = 1, \ldots, \ell, \\
D\xi_j + \Phi\zeta_j &= 0, && j = \ell + 1, \ldots, N, \\
\det(\zeta_1 \cdots \zeta_N) &= 1.
\end{aligned}$$

Prove that the map $\Lambda^{\max}(\ker\ D) \otimes \Lambda^{\max}(\ker\ D^*) \to \Lambda^{\max} \ker (D \oplus \Phi)$ given by

$$\begin{aligned}
&(x_1 \wedge \ldots \wedge x_k) \otimes (y_1^* \wedge \ldots \wedge y_\ell^*) \\
&\qquad \mapsto (x_1, 0) \wedge \ldots \wedge (x_k, 0) \wedge (\xi_{\ell+1}, \zeta_{\ell+1}) \wedge \ldots \wedge (\xi_N, \zeta_N)
\end{aligned}$$

is a well defined linear isomorphism.

(iii) Prove that the isomorphism $\det(D) \to \det(D \oplus \Phi)$ of (ii) agrees with the one constructed in the proof of Theorem A.6. □

Exercise A.8 Assume $N = 1$ so that $\Phi : \mathbb{R} \to Y$ is given by $\Phi t = ty$ for some vector $y \in Y$. Assume that both $D : X \to Y$ and $D \oplus \Phi : X \oplus \mathbb{R} \to Y$ are onto. Choose a vector $\xi \in X$ such that

$$D\xi + y = 0$$

Prove that the map $\Lambda^{\max} \ker D \to \Lambda^{\max} \ker (D \oplus \Phi)$ given by

$$x_1 \wedge \ldots \wedge x_k \mapsto (x_1, 0) \wedge \ldots \wedge (x_k, 0) \wedge (\xi, 1)$$

is the isomorphism of Exercise A.7. □

A.3 Crossing numbers

To gain a better understanding of the line bundle $\det(X, Y) \to \mathcal{F}(X, Y)$ we shall interprete trivializations of this line bundle along a path $D : [0, 1] \to \mathcal{F}(X, Y)$ as a crossing number in the case of Fredholm index zero. Denote

$$\mathcal{F}^0(X, Y) = \{D \in \mathcal{F}(X, Y) \,|\, \operatorname{index} D = 0\}$$

and for each integer $k \geq 0$ consider the subset

$$\mathcal{F}^0_k(X, Y) = \{D \in \mathcal{F}(X, Y) \,|\, \operatorname{index} D = 0, \dim \ker D = k\}$$

of Fredholm operators of index 0 with k-dimensional kernel. This is a submanifold of codimension k^2. The tangent space at the operator $D \in \mathcal{F}^0_k(X, Y)$ is given by

$$T_D\mathcal{F}^0_k(X, Y) = \{P \in \mathcal{L}(X, Y) \,|\, Px \in \operatorname{im} D \text{ for all } x \in \ker D\}.$$

A complement of this space is the space of linear operators from the kernel of D to a complement of the image of D. The union

$$\overline{\mathcal{F}}^0_1(X, Y) = \bigcup_{k \geq 1} \mathcal{F}^0_k(X, Y)$$

is a kind of stratified subvariety of codimension 1 whose complement (in the space of Fredholm operators of index zero) is the space of invertible operators.

Now consider a path $[0, 1] \to \mathcal{F}^0(X, Y) : t \mapsto D_t$ with invertible endpoints D_0 and D_1. Assume that the path is continuously differentiable (in the strong operator topology) and define the operator $\dot{D}_t : X \to Y$ by

$$\dot{D}_t x = \frac{d}{dt} D_t x$$

for $x \in X$. Call a point $t \in [0,1]$ a **crossing** if $\ker D_t > 0$. A crossing is called **regular** if

$$x \in \ker D_t, \quad \dot{D}_t x \in \operatorname{im} D_t \qquad \Longrightarrow \qquad x = 0.$$

This means that $\dot{D}_t$ maps the kernel of D_t bijectively onto a complement of $\operatorname{im} D_t$. A **simple crossing** is a regular crossing t with $D_t \in \mathcal{F}^0_1(X,Y)$. Note that the operator D_{t+s} is invertible for small s whenever t is a regular crossing. Hence every regular crossing is isolated. If $t \mapsto D_t$ is a path with only regular crossings we define the **crossing index** to be the number

$$\mu(\{D_t\}) = \sum_t \dim\ker\, D_t. \tag{A.4}$$

We shall prove that the mod 2 reduction of this number is a homotopy invariant and determines the sign of the map $\det(D_0) \to \det(D_1)$ arising from a trivialization of the determinant line bundle $\det(X,Y)$ along the path $t \mapsto D_t$. More precisely, consider the line bundle

$$L = \{(t,\sigma) \,|\, t \in [0,1],\, \sigma \in \det(D_t)\}$$

over the unit interval. A trivialization of this line bundle gives rise to an isomorphism $\det(D_0) \to \det(D_1)$. Since the 1-dimensional vector space $\det(D_t)$ inherits a norm from X and Y, this isomorphism can be chosen uniquely as an isometry. Since $\det(D_0) = \det(D_1) = \mathbb{R}$ this isomorphism is given by multiplication with a real number of modulus 1 which we denote by

$$\nu(\{D_t\}) \in \{\pm 1\}.$$

Proposition A.9 *Let $[0,1] \to \mathcal{F}^0(X,Y) : t \mapsto D_t$ be a continuously differentiable path with invertible endpoints D_0 and D_1 and only regular crossings. Then any trivialization of the determinant line bundle over this path gives rise to an isomorphism $\det(D_0) = \mathbb{R} \to \det(D_1) = \mathbb{R}$ of sign*

$$\nu(\{D_t\}) = (-1)^{\mu(\{D_t\})} = \prod_t (-1)^{\dim\ker\, D_t}. \tag{A.5}$$

Here the product runs over all crossings t. In particular, the crossing index mod 2 is invariant under homotopies with fixed endpoints.

Proof: Let us consider a path $t \mapsto D_t$ with a single crossing at $t = 0$. By choosing a suitable splitting of X and Y we may assume without loss of generality that $X = Y = X_0 \oplus \mathbb{R}^k$ and

$$D_0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}, \qquad \dot{D}_0 = \begin{pmatrix} A & 0 \\ B & \mathbb{1} \end{pmatrix}.$$

Consider the linear map $\Phi : \mathbb{R}^k \to Y = X_0 \oplus \mathbb{R}^k$ given by $\Phi z = (0, z)$. Then we have

$$D_t \oplus \Phi = \begin{pmatrix} \mathbb{1} + tA & 0 & 0 \\ tB & t\mathbb{1} & \mathbb{1} \end{pmatrix} + O(t^2).$$

Hence a trivialization of the kernels of the operators $D_t \oplus \Phi$ is given by embeddings $\iota_t : \mathbb{R}^k \to X \oplus \mathbb{R}^k = (X_0 \oplus \mathbb{R}^k) \oplus \mathbb{R}^k$ of the form

$$\iota_t = \begin{pmatrix} 0 \\ \mathbb{1} \\ -t\mathbb{1} \end{pmatrix} + O(t^2).$$

Here the two upper blocks represent the X-component of $\ker(D_t \oplus \Phi) \subset X \oplus \mathbb{R}^k$ while the third block represents the $\mathbb{R}^k$-component. Thus the induced map

$$\mathbb{R}^k \to \ker(D_{-\varepsilon} \oplus \Phi) \to \ker(D_\varepsilon \oplus \Phi) \to \mathbb{R}^k$$

is given by $-\mathbb{1} + O(\varepsilon)$. This map is orientation reversing if k is odd and orientation presrving if k is even. This proves the formula (A.5) in the case of a single regular crossing. The general case is an obvious consequence. Moreover, it is an obvious consequence of the crossing formula (A.5) that the crossing index mod 2 is a homotopy invariant. This proves the proposition. □

It is also interesting to consider paths $[0,1] \to \mathcal{F}^0(X,Y) : t \mapsto D_t$ where the operators D_0 and D_1 are not invertible. Suppose that the crossings at $t = 0$ and $t = 1$ are regular. This means that the linear operator $\dot{D}_0$ maps the kernel of D_0 bijectively onto a complement of the image of D_0

$$\mathrm{im}\, D_0 \oplus \left\{ \dot{D}_0 \xi \,|\, \xi \in \ker\, D_0 \right\} = Y,$$

and similarly for D_1. It follows that the operators $\dot{D}_0$ and $\dot{D}_1$ determine nonzero elements

$$\sigma(\dot{D}_0) \in \det(D_0), \qquad \sigma(\dot{D}_1) \in \det(D_1)$$

which can be defined as follows. Choose bases $\xi_1, \ldots, \xi_k$ of $\ker D_0$ and $\eta_1, \ldots, \eta_k$ of $\ker {D_0}^*$ such that

$$\langle \eta_i, \dot{D}_0 \xi_j \rangle = \delta_{ij}$$

and define

$$\sigma(\dot{D}_0) = (\xi_1 \wedge \ldots \wedge \xi_k) \otimes (\eta_1 \wedge \ldots \wedge \eta_k) \in \det(D_0).$$

It is easy to see that this vector is independent of the choice of the bases. If D_0 is invertible we use the convention $\sigma(\dot{D}_0) = 1 \in \det(D_0) = \mathbb{R}$ and similarly for D_1.

Proposition A.10 *Let* $[0,1] \to \mathcal{F}^0(X,Y) : t \mapsto D_t$ *be a continuously differentiable path with only regular crossings. Then a trivialization of the determinant bundle gives rise to an isomorphism*

$$\det(D_0) \to \det(D_1) : \sigma(\dot{D}_0) \mapsto \nu(\{D_t\})\sigma(\dot{D}_1)$$

where

$$\nu(\{D_t\}) = \prod_{0\le t<1} (-1)^{\dim\ker\, D_t}.$$

Here the product runs over all crossings t *including the one at* $t = 0$ *but excluding the one at* $t = 1$.

Proof: Let us first consider paths with a single crossing at $t = 0$. Given any linear map $\Phi : \mathbb{R}^k \to Y$ such that $D_0 \oplus \Phi$ is onto we must examine the composition

$$\det(D_0) \to \Lambda^{\max}\ker(D_0) \to \Lambda^{\max}\ker(D_\varepsilon \oplus \Phi) \to \Lambda^{\max}\mathbb{R}^k = \mathbb{R}$$

where the second map arises from a trivialization of the vector bundle $L_t = \ker(D_t \oplus \Phi)$ and the last map is induced by the obvious isomorphism $\ker(D_\varepsilon \oplus \Phi) \to \mathbb{R}^k : (\xi, z) \mapsto z$. The first map arises from the fact that $\operatorname{im} D_0 \oplus \operatorname{im}\Phi = Y$ and can be explicitly described as follows. If $\eta_1, \dots, \eta_k$ is a basis of $\ker\, D^*$, choose a dual basis $y_1, \dots, y_k \in \operatorname{im}\Phi$ and define

$$\Lambda^{\max}\ker\, {D_0}^* \to \mathbb{R} : \eta_1 \wedge \ldots \wedge \eta_k \mapsto \frac{1}{\det(\Phi^{-1}y_1, \dots, \Phi^{-1}y_k)}.$$

This map is easily seen to be linear. If $\xi_1, \dots, \xi_k \in \ker D_0$ and $\eta_1, \dots, \eta_k \in \ker\, {D_0}^*$ are chosen as above with $\langle \eta_i, \dot{D}_0\xi_j\rangle = \delta_{ij}$, and $\Phi : \mathbb{R}^k \to Y$ is defined by

$$\Phi z = \sum_{j=1}^k z_j \dot{D}_0 \xi_j$$

then the map $\Lambda^{\max}\ker\, {D_0}^* \to \mathbb{R}$ sends $\eta_1 \wedge \ldots \wedge \eta_k$ to 1. Hence in this case the map $\det(D_0) \to \Lambda^{\max}\ker\, D_0$ sends $\sigma(\dot{D}_0)$ to $\xi_1 \wedge \ldots \wedge \xi_k$.

Now assume that $D_t = D_0 + t\dot{D}_0$. Then

$$\ker(D_t \oplus \Phi) = \Big\{(\xi, z) \in X \times \mathbb{R}^k \,|\, \xi \in \ker\, D_0,\ z_j = -t\langle \eta_j, \dot{D}_0\xi\rangle\Big\}$$

A trivialization of these kernels on the interval $[0,1]$ gives rise to the map

$$\ker\, D_0 \to \mathbb{R}^k : \xi \mapsto -\begin{pmatrix} \langle \eta_1, \dot{D}_0\xi\rangle \\ \vdots \\ \langle \eta_k, \dot{D}_0\xi\rangle \end{pmatrix}.$$

Hence the map $\ker D_0 \to \mathbb{R}^k$ sends the basis $\xi_1, \dots, \xi_k$ to minus the standard basis of $\mathbb{R}^k$. Hence the resulting map $\Lambda^{\max} \to \mathbb{R}$ sends $\xi_1 \wedge \dots \wedge \xi_k$ to $(-1)^k$. Since the first map $\det(D_0) \to \Lambda^{\max} \ker D_0$ sends $\sigma(\dot{D}_0)$ to $\xi_1 \wedge \dots \wedge \xi_k$ the proposition is proved in the case of an affine path $D_t = D_0 + t\dot{D}_0$. The general case is an easy consequence since $D_t - D_0 - t\dot{D}_0 = O(t^2)$. Thus we have proved that the crossing at $t = 0$ contributes the factor $(-1)^{\dim \ker D_0}$ to the map $\det(D_0) \to \det(D_1)$. That the crossing at $t = 1$ does not contribute follows from the same argument with t replaced by $-t$. The contributions of the intermediate crossings $0 < t < 1$ are given by Proposition A.9. This completes the proof. □

Exercise A.11 Prove that the formula of Proposition A.10 is consistent with reversing time, i.e. compare the paths $t \mapsto D_t$ and $t \mapsto D_{1-t}$. □

Remark A.12 If $X = Y = H$ is an infinite dimensional Hilbert space then it is easy to construct a loop of Fredholm operators of index zero with crossing number 1. Choose an orthonormal basis $e_0, e_1, e_2, \dots$. For $0 \le t \le 1/2$ define $A_t \in \mathcal{L}(H)$ to be a rotation by angle $2\pi t$ in the (e_{2j}, e_{2j+1})-planes for $j \ge 0$ so that $A_0 = \mathrm{id}$ and $A_{1/2} = -\mathrm{id}$. Then for $1/2 \le t \le 1$ define $A_t \in \mathcal{L}(H)$ to be a rotation by angle $2\pi t$ in the (e_{2j-1}, e_{2j})-planes for $j \ge 1$ so that

$$A_1 x = -\langle e_0, x\rangle e_0 + \sum_{j=1}^{\infty} \langle e_j, x\rangle e_j.$$

Now connect A_1 to the identity by a straight line. The resulting loop $t \mapsto A_t$ has a single crossing with crossing index one. Hence the determinant bundle over the space of Fredholm operators on H (of index zero) does not admit a trivialization. □

Exercise A.13 Construct a loop of surjective Fredholm operators of index 1 whose kernels form a Moebius band. The existence of such a loop shows that the above correspondence between trivializations of determinant bundles and crossing numbers does not have an obvious generalization to operators of nonzero Fredholm index. □

Exercise A.14 Suppose that X and Y are complex Banach spaces and $[0,1] \to \mathcal{F}(X,Y) : t \mapsto D_t$ is a path of Fredholm operators which are all complex linear. Then the one dimensional real vector space $\det(D_t)$ inherits a natural orientation from the complex structures of $\ker D_t$ and $\ker {D_t}^*$. Prove that any trivialization of the real line bundle

$$\bigcup_t \det(D_t)$$

gives rise to an orientation preserving isomorphism

$$\det(D_0) \to \det(D_1).$$

Hints: Consider the complex line bundle

$$\det^c(X,Y) \longrightarrow \mathcal{F}^c(X,Y)$$

whose fiber over a complex linear Fredholm operator $D \in \mathcal{F}^c(X,Y)$ is the 1-dimensional complex vector space

$$\det^c(D) = \Lambda^{\max}_{\mathbb{C}} \ker D \otimes_{\mathbb{C}} \Lambda^{\max}_{\mathbb{C}} \ker D^*.$$

Here $D^* : Y^* \to X^*$ denotes the real adjoint operator and if $J : X \to X$ is multiplication by i then the complex structure on X^* is given by the dual operator $J^* : X^* \to X^*$. Show that the proof of Theorem A.6 carries over to the complex category and hence $\det^c(X,Y)$ is a locally trivial complex line bundle over $\mathcal{F}^c(X,Y)$. Now for every finite dimensional complex vector space V there is a natural quadratic map

$$\Lambda^{\max}_{\mathbb{C}} V \to \Lambda^{\max}_{\mathbb{R}} V : \tau \mapsto i^n (-1)^{\frac{n(n-1)}{2}} \tau \wedge \bar{\tau}.$$

where $\bar{\tau} \in \Lambda^{\max}_{\mathbb{C}} \bar{V}$. Here $\bar{V}$ denotes the vector space with the opposite complex structure and both $\Lambda^{\max}_{\mathbb{C}} V$ and $\Lambda^{\max}_{\mathbb{C}} \bar{V}$ are natural linear subspaces of $\Lambda^{\mathrm{mid}}_{\mathbb{R}} V \otimes \mathbb{C}$. With these conventions the above map sends $v_1 \wedge \ldots \wedge v_n$ to $v_1 \wedge (Jv_1) \wedge \ldots \wedge v_n \wedge (Jv_n)$. □

APPENDIX B

TRANSVERSALITY

This appendix provides the necessary analytical background material for the proof that moduli spaces form smooth finite dimensional manifolds. The first section is devoted to the inverse and implicit function theorems on infinite dimensional Banach spaces. The *Kuranishi model* can be viewed as a kind of extension of the implicit function theorem which reduces the local analysis of the zero set of a Fredholm map near a singular point to a finite dimensional model. This is discussed in Section B.2 along with Furuta's technique for obtaining a global Kuranishi model. As in the finite dimensional case the theorem of Sard plays a crucial role in proving the existence of a regular value. Smale's extension of this result to the infinite dimensional setting is proved in Section B.3. The final section deals with applications to transversality problems.

B.1 Implicit function theorem

Let X and Y be Banach spaces and $f : X \to Y$ be a smooth map. For every $x \in X$ denote by $df(x) : X \to Y$ the differential of f at x. If this operator is bijective then its inverse $df(x)^{-1} : Y \to X$ is a bounded linear operator by the open mapping theorem. The inverse function theorem asserts that f has a local inverse near every point x at which $df(x)$ is invertible. Denote by $B_r^X(x)$ the open ball of radius r centered at x in the Banach space X. Abbreviate $B_r^X = B_r^X(0)$.

Theorem B.1. (Inverse function theorem) *Let $f : X \to Y$ be continuously differentiable. Suppose that the linearized operator $D = df(x_0) : X \to Y$ has a bounded inverse. Choose constants $c > 0$ and $\delta > 0$ such that*

$$\left\|D^{-1}\right\| \le c, \qquad \|df(x) - D\| \le \frac{1}{2c}$$

for $\|x - x_0\|_X < \delta$. Then the following holds.

(i) *The restriction of f to $U = B_\delta^X(x_0)$ is injective and $f(U) = V$ is an open set in Y containing the ball $B^Y_{\delta/2c}(f(x_0))$.*

(ii) *The inverse map $f^{-1} : V \to U$ is continuously differentiable with $df^{-1}(y) = df(f^{-1}(y))^{-1}$ for $y \in V$. Moreover, if f is of class C^ℓ for some integer ℓ then so is f^{-1}.*

(iii) *If $x_1, x_2 \in B_\delta^X(x_0)$ then $\|x_1 - x_2\| \le 2c \|f(x_1) - f(x_2)\|$.*

Lemma B.2 *Let X be a Banach spaces and $\psi : X \to X$ be a continuously differentiable map such that $\psi(0) = 0$ and*

$$\|\mathbb{1} - d\psi(x)\| \le \gamma$$

for all $x \in X$ with $\|x\| < R$ for some constant $\gamma < 1$. Then the restriction of ψ to B_R is injective, $\psi(B_R)$ is an open set, and $\psi^{-1} : \psi(B_R) \to B_R$ is continuously differentiable with $d\psi^{-1}(y) = d\psi(\psi^{-1}(y))^{-1}$. Moreover,

$$B_{R(1-\gamma)} \subset \psi\left(B_R\right) \subset B_{R(1+\gamma)}.$$

Proof: Consider the map $\varphi = \mathrm{id} - \psi : X \to X$. Then $\|d\varphi(x)\| \le \gamma$ for all $x \in X$ with $\|x\| < R$ and hence φ is a contraction:

$$\|\varphi(x_1) - \varphi(x_2)\| \le \gamma \|x_1 - x_2\|.$$

This implies

$$\|\psi(x_1) - \psi(x_2)\| \le \|x_1 - x_2\| + \|\varphi(x_1) - \varphi(x_2)\| \le (1 + \gamma) \|x_1 - x_2\|,$$

$$\|\psi(x_1) - \psi(x_2)\| \ge \|x_1 - x_2\| - \|\varphi(x_1) - \varphi(x_2)\| \ge (1 - \gamma) \|x_1 - x_2\|$$

for $x_1, x_2 \in B_R$. The first inequality shows that $\psi(B_R) \subset B_{R(1+\gamma)}$ and the second inequality shows that ψ is injective on B_R. Now let $y \in B_{R(1-\gamma)}$ and consider the map $x \mapsto \varphi(x) + y$. This is a contraction of the closed ball of radius $R - \varepsilon$ where $\|y\| = (1 - \gamma)(R - \varepsilon)$. Hence it has a unique fixed point x with $\|x\| \le R - \varepsilon$. But the equation $\varphi(x) + y = x$ is equivalent to $\psi(x) = y$. This shows that $B_{R(1-\gamma)} \subset \psi(B_R)$.

Now let $y = \psi(x)$ with $x \in B_R$. Choose $\varepsilon > 0$ such that $B_\varepsilon(x) \subset B_R$. Then

$$B_{\varepsilon(1-\gamma)}(\psi(x)) \subset \psi(B_\varepsilon(x)) \subset \psi(B_R)$$

and hence the set $\psi(B_R)$ is open. The same argument with ε arbitrarily small shows that ψ^{-1} is continuous. It remain to prove that ψ^{-1} is continuously differentiable. To see this fix a point $x_0 \in B_R$ and denote $y_0 = \psi(x_0)$, $D = d\psi(x_0)$. Then $\|\mathbb{1} - D\| \le \gamma$ and hence D has an inverse

$$D^{-1} = \sum_{k=0}^{\infty} (\mathbb{1} - D)^k, \qquad \left\|D^{-1}\right\| \le \frac{1}{1-\gamma}.$$

Since ψ is continuously differentiable there exists, for every $\varepsilon > 0$, a $\delta > 0$ such that if $x \in B_R$ with $\|x - x_0\| < \delta/(1-\gamma)$ then

$$\|\psi(x) - \psi(x_0) - D(x - x_0)\| \le \varepsilon(1-\gamma)^2 \|x - x_0\|.$$

Choose δ so small that $B_\delta(y_0) \subset \psi(B_R)$. Then $\|y - y_0\| < \delta$ implies $x = \psi^{-1}(y) \in B_R$ and $\|x - x_0\| < \delta/(1-\gamma)$. Hence

$$\begin{aligned}\left\|\psi^{-1}(y) - \psi^{-1}(y_0) - D^{-1}(y - y_0)\right\| &= \left\|D^{-1}\left(y - y_0 - D(x - x_0)\right)\right\| \\ &\le \frac{1}{1-\gamma}\left\|\psi(x) - \psi(x_0) - D(x - x_0)\right\| \\ &\le \varepsilon(1-\gamma)\left\|x - x_0\right\| \\ &\le \varepsilon\left\|y - y_0\right\|.\end{aligned}$$

This shows that ψ^{-1} is differentiable at y_0 with $d\psi^{-1}(y_0) = D^{-1} = d\psi(\psi^{-1}(y_0))^{-1}$. □

Proof of Theorem B.1: Assume without loss of generality that $x_0 = 0$ and $f(0) = 0$. Consider the map $\psi : U = B_\delta^X \to X$ given by

$$\psi(x) = D^{-1}f(x).$$

Its differential satisfies $\mathbb{1} - d\psi(x) = \mathbb{1} - D^{-1}df(x) = D^{-1}(D - df(x))$ and hence

$$\|\mathbb{1} - d\psi(x)\| \le c\|D - df(x)\| \le \frac{1}{2}$$

for $x \in B_\delta^X$. Hence it follows from Lemma B.2 with $R = \delta$ and $\gamma = 1/2$ that ψ has a continuously differentiable inverse on B_δ^X with $\psi(B_\delta^X)$ an open set containing $B_{\delta/2}^X$. Thus

$$f(B_\delta^X) = D\psi(B_\delta^X) \supset DB_{\delta/2}^X \supset B_{\delta/2c}^Y$$

and the required inverse of f is given by $f^{-1}(y) = \psi^{-1}(D^{-1}y)$. It is continuously differentiable and the formula $df^{-1}(y) = df(f^{-1}(y))^{-1}$ follows easily from the chain rule. Since df is continuous so is df^{-1}. This proves (i) and (ii) with $\ell = 1$. The last assertion in (ii) follows by induction. To prove (iii) note that

$$\|x_1 - x_2\| \le 2\left\|\psi(x_1) - \psi(x_2)\right\| \le 2c\left\|f(x_1) - f(x_2)\right\|.$$

Here the first inequality is taken from the proof of Lemma B.2 and the second follows from the fact that $f = D \circ \psi$ and $\|D^{-1}\| \le c$. □

A smooth map $f : X \to Y$ between Banach spaces is called **Fredholm** if the linearized operator $df(x) : X \to Y$ is Fredholm for every $x \in X$. Since the Fredholm index is invariant under small perturbations the index of $df(x)$ is independent of the choice of x. It will be denoted by $\operatorname{index} f$. For any smooth map $f : X \to Y$, Fredholm or not, a vector $y \in Y$ is called a **regular value** of f if $df(x) : X \to Y$ is onto and has a right inverse for every $x \in f^{-1}(y)$. The implicit function theorem asserts that $f^{-1}(y)$ is a smooth manifold for every regular value of f. Moreover, if f is a Fredholm map then the dimension of $f^{-1}(y)$ is finite and agrees with the Fredholm index of f.

Theorem B.3. (Implicit function theorem) *If $f : X \to Y$ is of class C^ℓ and y is a regular value of f then*

$$\mathcal{M} = f^{-1}(y) \subset X$$

is a C^ℓ Banach manifold. If, moreover, f is a Fredholm map then $\mathcal{M}$ is finite dimensional and

$$\dim \mathcal{M} = \operatorname{index} f.$$

If $x_0 \in \mathcal{M}$ then, by assumption, the operator $D = df(x_0) : X \to Y$ is surjective and has a right inverse $T : Y \to X$ such that $DT = \mathrm{id}_Y$. The existence of such an inverse is equivalent to the existence of a splitting

$$X = \ker D \oplus \operatorname{im} T.$$

Here $\ker D$ consists of the solutions of the linearized equation $df(x_0)\xi = 0$ and we expect the space of solutions of the full nonlinear equations to *look like* the kernel of D locally near x_0. More precisely, the implicit function theorem asserts that there exists a smooth map $\varphi : \ker D \to Y$ with $d\varphi(0) = 0$ such that

$$f(x) = 0 \qquad \Longleftrightarrow \qquad x = x_0 + \xi + T\varphi(\xi), \quad D\xi = 0$$

for x near x_0. (See Figure B.1.) The implicit function theorem also asserts that if x_0 is an approximate solution of $f(x_0) \simeq 0$ and the inverse of f is uniformly bounded near x_0 then there exists a *true solution* of $f(x) = 0$ near x_0. More precisely, we have the following result.

FIG. B.1. Implicit function theorem

Proposition B.4 *Let $f : X \to Y$ be a continuously differentiable map between Banach spaces. Suppose that $D = df(x_0) : X \to Y$ is onto with a right inverse $T : Y \to X$ such that*

$$\|T\| \le c, \qquad \|df(x) - D\| \le \frac{1}{2c} \tag{B.1}$$

whenever $\|x - x_0\| \le \delta$. *Suppose further that* $x_1 \in X$ *satisfies*

$$\|f(x_1)\| < \frac{\delta}{4c}, \qquad \|x_1 - x_0\| < \frac{\delta}{8}. \tag{B.2}$$

Then there exists a unique $x \in X$ *such that* $f(x) = 0$, $x - x_1 \in \operatorname{im} T$, *and* $\|x - x_0\| \le \delta$. *Moreover,* $\|x - x_1\| \le 2c \, \|f(x_1)\|$.

Proof: Consider the map $\psi : X \to X$ defined by

$$\psi(x) = x + T(f(x) - Dx).$$

Then $\mathbb{1} - d\psi(x) = -T(D - df(x))$ and hence, by (B.1),

$$\|\mathbb{1} - d\psi(x)\| \le c\|df(x) - D\| \le 1/2$$

for $\|x - x_0\| < \delta$. By Lemma B.2, ψ maps $B_\delta(x_0)$ bijectively onto some open set in X with $B_{\delta/2}(\psi(x_0)) \subset \psi(B_\delta(x_0))$. Now, by (B.2),

$$\begin{aligned} \|x_1 - TDx_1 - \psi(x_0)\| &= \|\psi(x_1) - \psi(x_0) - Tf(x_1)\| \\ &\le 2\|x_1 - x_0\| + c\|f(x_1\| \\ &\le \delta/2. \end{aligned}$$

Hence there exists a unique $x \in B_\delta(x_0)$ with $\psi(x) = x_1 - TDx_1$. The latter is equivalent to $f(x) = 0$ and $x - x_1 \in \ker(\mathbb{1} - TD) = \operatorname{im} T$. Moreover, the proof of Lemma B.2 shows that

$$\|x - x_1\| \le 2\|\psi(x) - \psi(x_1)\| = 2\|Tf(x_1)\| \le 2c\|f(x_1)\|.$$

This proves the proposition. □

Proof of Theorem B.3: Suppose that T is a right inverse of $D = df(x_0)$ where $f(x_0) = 0$. Then (B.1) is satisfied for $c \ge 1$ sufficiently large and $\delta > 0$ sufficiently small. Moreover, if $\xi \in \ker D$ with $\|\xi\| \le \delta/8$ then $x_1 = x_0 + \xi$ satisfies (B.2). Hence for any such ξ there exists a unique $x \in B_\delta(x_0)$ with

$$f(x) = 0, \qquad x - x_0 - \xi \in \operatorname{im} T.$$

Since T is injective there exists a unique $\varphi(\xi) \in Y$ such that

$$x = x_0 + \xi + T\varphi(\xi).$$

Let ψ be as in the proof of Proposition B.4. Then

$$\varphi(\xi) = D\psi^{-1}(x_0 - TDx_0 + \xi) - Dx_0. \tag{B.3}$$

To see this note that $\psi(x) = x_0 - TDx_0 + \xi$ and hence $x_0 + \xi + T\varphi(\xi) = \psi^{-1}(x_0 - TDx_0 + \xi)$. Now apply D to both sides of the equation and use $DT = \mathrm{id}_Y$. By Theorem B.1, φ is of class C^ℓ. The formula $d\varphi(0) = 0$ is a simple exercise. Moreover, if $\|x - x_0\| < \varepsilon = \delta/8(1 + c\|D\|)$ and $f(x) = 0$, write $x - x_0 = \xi + T\eta$ with $\xi \in \ker D$ and $\eta \in Y$. Then $\xi = (\mathbb{1} - TD)(x - x_0)$ and hence $\|\xi\| < \delta/8$ and thus $\eta = \varphi(\xi)$. This shows that $f^{-1}(0) \cap B_\varepsilon(x_0)$ is the image of the C^ℓ-chart $\xi \mapsto x_0 + \xi + T\varphi(\xi)$ defined on some open subset of $\ker D$. This proves Theorem B.3. □

In the Banach space setting the existence of a right inverse does not follow from the fact that D is onto. Such a right inverse exists if and only if the kernel of D has a complement in X. In particular, every surjective Fredholm operator has a right inverse. This generalizes to operators of the form $D \oplus L : X \oplus Z \to Y$ defined by

$$D \oplus L(x, z) = Dx + Lz$$

where $D : X \to Y$ is Fredholm.

Lemma B.5 *Assume $D : X \to Y$ is a Fredholm operator and $L : Z \to Y$ is a bounded linear operator such that $D \oplus L : X \oplus Z \to Y$ is onto. Then $D \oplus L$ has a right inverse. Moreover, the projection $\Pi : \ker(D \oplus L) \to Z$ is a Fredholm operator with* $\ker \Pi \cong \ker D$ *and* $\operatorname{coker} \Pi \cong \operatorname{coker} D$ *and hence*

$$\operatorname{index} \Pi = \operatorname{index} D.$$

Proof: Choose a complement X_1 of $\ker D$ in X and finitely many vectors $z_1, \ldots, z_N \in Z$ such that $Lz_1, \ldots, Lz_N$ span a complement of $\operatorname{im} D$ in Y. Then a right inverse of $D \oplus L$ is the operator

$$Y \to X \oplus Z : y \mapsto \left(x, \sum_{\nu=1}^N \lambda_\nu z_\nu \right)$$

where x and $\lambda_1, \ldots, \lambda_N$ are chosen such that

$$x \in X_1, \qquad y = Dx + \sum_{\nu=1}^N \lambda_\nu L z_\nu.$$

Now $\ker \Pi = \ker D \oplus 0$ and $\operatorname{im} \Pi = L^{-1}(\operatorname{im} D)$ and hence

$$\frac{Z}{\operatorname{im} \Pi} = \frac{Z}{L^{-1}(\operatorname{im} D)} = \frac{\operatorname{im} L}{\operatorname{im} D \cap \operatorname{im} L} = \frac{Y}{\operatorname{im} D}.$$

The second isomorphism is induced by L and the last equality follows from the fact that $\operatorname{im} D + \operatorname{im} L = Y$. □

Exercise B.6 Let X be an infinite dimensional Banach space. In contrast to the finite dimensional case a C^∞ smooth function $\varphi : X \to \mathbb{R}$ which vanishes outside the unit ball need not be bounded (even though every point in X has a neighbourhood in which the function is bounded). Construct an example of an unbounded smooth function with support in the unit ball. **Hint:** There is an infinite sequence of pairwise disjoint balls of radius 1/4 which are all contained in the unit ball of radius 1. □

B.2 The Kuranishi model

The Kuranishi model gives a local description for the zero set of a smooth map $f : X \to Y$ between two Banach spaces. Assume that $f(0) = 0$ and denote $D = df(0) : X \to Y$. By assumption, D is a bounded linear operator. A **pseudo-inverse** of D is a bounded linear operator $T : Y \to X$ which satisfies

$$TDT = T, \qquad DTD = D.$$

The next proposition gives a necessary and sufficient criterion for the existence of a pseudo-inverse. It shows, in particular, that every Fredholm operator admits a pseudo-inverse.

Proposition B.7 *A bounded linear operator $D : X \to Y$ admits a pseudo-inverse if and only if D satisfies the following*

(i) *D has closed range,*

(ii) *The kernel of D has a complement in X.*

(iii) *The image of D has a complement in Y.*

Proof: Assume first that D satisfies (i), (ii) and (iii), denote

$$X_0 = \ker D, \qquad Y_1 = \operatorname{im} D,$$

and choose complements X_1 and Y_0 so that

$$X = X_0 \oplus X_1, \qquad Y = Y_0 \oplus Y_1.$$

Then X_1 and Y_1 are Banach spaces and the restriction of D to X_1 determines a bijective bounded linear operator $D_1 : X_1 \to Y_1$. The reader may check that the operator $T : Y \to X$ defined by

$$T(y_0 + y_1) = {D_1}^{-1} y_1$$

for $y_0 \in Y_0$ and $y_1 \in Y_1$ is a pseudo-inverse. Conversely, if $T : Y \to X$ is a pseudo-inverse of D then the required complements are given by

$$X_1 = \operatorname{im} T, \qquad Y_0 = \ker T.$$

To see this just note that

$$P = TD : X \to X, \qquad Q = DT : Y \to Y$$

are projection operators with $\operatorname{im} P = \operatorname{im} T$, $\ker P = \ker D$ and $\operatorname{im} Q = \operatorname{im} D$, $\ker Q = \ker T$. This proves the proposition. $\square$

Remark B.8 Let G be a compact Lie group acting on the Banach spaces X and Y by strongly continuous maps $\mathrm{G} \to \mathcal{L}(X) : g \mapsto \Phi_g$ and $\mathrm{G} \to \mathcal{L}(Y) : g \mapsto \Psi_g$. Suppose that $D : X \to Y$ is an equivariant bounded linear operator. If D admits a pseudo-inverse then it admits an equivariant pseudo-inverse. To see this note that if T is any pseudo-inverse of D then the operator

$$T_g = \Phi_g T {\Psi_g}^{-1}$$

is also a pseudo-inverse. Hence the average

$$S = \int_{\mathrm{G}} T_g \, d\mu(g),$$

with respect to the Haar measure $d\mu$ on G with $\mathrm{Vol}(\mathrm{G}) = 1$, is equivariant and satisfies $DSD = D$. It follows that the operator

$$R = SDS = \int_{\mathrm{G}} \int_{\mathrm{G}} T_g D T_h \, d\mu(g) d\mu(h)$$

is an equivariant pseudo-inverse of D. $\square$

Theorem B.9. (Kuranishi) *Let X and Y be Banach spaces and $f : X \to Y$ be a smooth map such that $f(0) = 0$. Suppose that the operator $D = df(0) : X \to Y$ has a pseudo-inverse $T : Y \to X$ and denote*

$$Y_0 = \ker T.$$

Then there exist an open neighbourhood U of 0 in X, a local diffeomorphism $g : U \to g(U) \subset X$, and a smooth map $f_0 : U \to Y_0$ such that

$$f \circ g(x) = Dx + f_0(x)$$

for $x \in U$ and

$$g(0) = 0, \quad dg(0) = 1\!\!1, \quad f_0(0) = 0, \quad df_0(0) = 0.$$

Moreover, if f is equivariant with respect to the action of some compact Lie group G *on X and Y then the maps g and f_0 can be chosen equivariant.*

Proof: Consider the smooth map $\psi : X \to X$ defined by

$$\psi(x) = x + T(f(x) - Dx).$$

This map satisfies

$$\psi(0) = 0, \qquad d\psi(0) = 0.$$

Hence it follows from the inverse function theorem B.1 that ψ has a local inverse defined on some open neighbourhood U of 0. Define $g : U \to X$ and $f_0 : U \to Y_0$ by

$$g = \psi^{-1}, \qquad f_0 = (\mathbb{1} - DT) \circ f \circ \psi^{-1}.$$

Then the formula

$$D\psi(x) = Dx + DT(f(x) - Dx) = DTf(x)$$

shows that

$$D = DT \circ f \circ \psi^{-1}$$

and hence

$$f \circ g = f \circ \psi^{-1} = D + (\mathbb{1} - DT) \circ f \circ \psi^{-1} = D + f_0.$$

If f is equivariant choose an equivariant pseudo-inverse T of D (see Remark B.8). Then the above formulae show that g and f_0 are also equivariant. This proves the theorem. □

Remark B.10 Let g and f_0 be as in Theorem B.9. Then the local zero set of f near $x = 0$ can be identified with the zero set of $f_0 : U \cap X_0 \to Y_0$ where $X_0 = \ker D$:

$$f^{-1}(0) \cap g(U) = \{g(x) \,|\, x \in U,\, Dx = 0,\, f_0(x) = 0\}.$$

To see this just note that $f_0(x) \in Y_0$ where Y_0 is a complement of the image of D and hence

$$Dx + f_0(x) = 0 \qquad \Longleftrightarrow \qquad Dx = 0, \quad f_0(x) = 0.$$

This observation is particularly interesting when D is a Fredholm operator. In this case the kernel and cokernel of D are finite dimensional and hence $f_0 : X_0 \to Y_0$ is a smooth map between finite dimensional vector spaces. □

Remark B.11 If $f = D + \hat{f}$ where $\hat{f} : X \to Y$ is a quadratic map then the proof of Theorem B.9 shows that

$$g^{-1} - \mathbb{1} = T \circ \hat{f}, \qquad f_0 \circ g^{-1} = (\mathbb{1} - DT) \circ \hat{f}$$

are quadratic maps. □

Global Kuranishi model

It is sometimes interesting to find a global Kuranishi model. Such models were used by Furuta in his proof of the 10/8-conjecture. (See Chapter 9.) The following discussion explains Furuta's construction in a more abstract setting. Suppose that

$$P_n : X \to X, \qquad Q_n : Y \to Y$$

are sequences of projection operators such that

$$DP_n = Q_n D, \quad TQ_n = P_n T, \quad \operatorname{im} Q_n \subset \operatorname{im} D, \quad \ker D \subset \ker P_n,$$

and

$$\lim_{n\to\infty} P_n x = 0, \qquad \lim_{n\to\infty} Q_n y = 0$$

for all $x \in X$ and $y \in Y$. Then one can define the functions ψ_n, g_n, f_n as in the proof of Theorem B.9 with $P = TD$ replaced by P_n and $Q = DT$ replaced by Q_n. Thus

$$\psi_n(x) = x + TQ_n(f(x) - Dx).$$

and

$$g_n = {\psi_n}^{-1}, \qquad f_n = (\mathbb{1} - Q_n) \circ \left(D + (f - D) \circ {\psi_n}^{-1}\right).$$

One checks easily as in the proof of Theorem B.9 that

$$f \circ g_n = Q_n D + f_n$$

and that the zero set of f on $g_n(U_n)$ is the image of the zero set of the restriction $f_n : X_n \to Y_n$ where $X_n = \ker P_n$ and $Y_n = \ker Q_n$. Thus

$$f^{-1}(0) \cap g_n(U_n) = \{g_n(x) \,|\, x \in X_n \cap U_n,\ f_n(x) = 0\}.$$

Here the set $U_n \subset X$ is the domain of the inverse of ψ_n. The key point is that if the operators $df(x) - D$ are uniformly compact then these domains will in the limit fill out the whole Banach space X. To see this note that

$$d\psi_n(x) = \mathbb{1} - TQ_n(df(x) - D).$$

If f is of class C^1 then the map $X \to \mathcal{L}(X, Y) : x \mapsto df(x) - D$ is continuous in the uniform operator topology. Moreover the sequence Q_n converges to zero in the strong operator topology and since the operators $df(x) - D$ are uniformly compact it follows that the composition $Q_n(df(x) - D)$ converges to zero in the norm topology.

Lemma B.12 *Let X, Y, Z be Banach spaces and $Q_n : Y \to Z$ be a sequence of bounded linear operators such that*

$$\lim_{n\to\infty} Q_n y = 0$$

for all $y \in Y$. Moreover, let $\{K_\alpha\}_{\alpha\in A}$ be a collection of bounded linear operators $K_\alpha : X \to Y$, indexed by a set A, such that the set

$$B = \{K_\alpha x \,|\, \alpha \in A,\, x \in X,\, \|x\| \le 1\} \subset Y$$

has compact closure. Then

$$\lim_{n\to\infty} \sup_\alpha \|Q_n K_\alpha\|_{\mathcal{L}(X,Z)} = 0.$$

Proof: By the uniform boundedness principle, there exists a constant $c > 0$ such that

$$\|Q_n\|_{\mathcal{L}(Y,Z)} \le c$$

for every n. Given $\varepsilon > 0$ cover the set B by finitely many balls of radius $\varepsilon/2c$ centered at $y_1, \ldots, y_N$. Now choose $n_0 \in \mathbb{N}$ such that

$$\|Q_n y_j\|_Z \le \frac{\varepsilon}{2}$$

for $j = 1, \ldots, N$ and $n \ge n_0$. Given $\alpha \in A$ and $x \in X$ with $\|x\| \le 1$ choose a j with $\|y_j - K_\alpha x\| \le \varepsilon/2c$. Then

$$\|Q_n K_\alpha x\| \le \|Q_n\| \, \|K_\alpha x - y_j\| + \|Q_n y_j\| \le c\, \|K_\alpha x - y_j\| + \frac{\varepsilon}{2} \le \varepsilon.$$

Hence $\|Q_n K_\alpha\| \le \varepsilon$ for $\alpha \in A$ and $n \ge n_0$. This proves the lemma. □

Now suppose that the operators $df(x) - D$ are uniformly compact in the ball of radius R. Then the previous lemma shows that there exists an integer n such that

$$\|x\| \le R \qquad \Longrightarrow \qquad \|\mathbb{1} - d\psi_n(x)\| \le 1/2.$$

Lemma B.2 now shows that ψ_n has an inverse on the entire ball of radius R with

$$\psi_n(B_{R(1-\gamma)/(1+\gamma)}) \subset B_{R(1-\gamma)} \subset \psi_n\left(B_R\right).$$

This will be of particular interest if the zero set of f is contained in the ball of radius $R(1-\gamma)(1+\gamma)^{-1}$.

B.3 Sard-Smale theorem

The following infinite dimensional version of Sard's theorem is due to Smale [113].

Theorem B.13. (Sard-Smale) *Let X and Y be separable Banach spaces and $U \subset X$ be an open set. Suppose that $f : U \to Y$ is a C^∞ smooth Fredholm map. Then the set*

$$Y_{\mathrm{reg}} = \{y \in Y \mid \operatorname{im} df(x) = Y \text{ for all } x \in U \text{ with } f(x) = y\}$$

of regular values of f is of the second category in the sense of Baire (a countable intersection of open and dense sets).

The separability condition is essential. A Banach space X is called **separable** if it admits a dense sequence. Since every metric space is paracompact so is every Banach space. This means that every open cover of X admits a locally finite refinement. In the separable case every locally finite cover is countable.* Since the existence of a countable refinement implies the existence of a countable subcover this proves the following.

Proposition B.14 *Let X be a separable Banach space. Then every open cover of X admits a countable subcover.*

Proof of Theorem B.13: By Proposition B.14 it suffices to prove that every point $x \in U$ admits a closed neighbourhood V such that the set of regular values of the restriction $f|_V$ is open and dense in Y. Assume without loss of generality that $0 \in U$ and consider a local Kuranishi model

$$f \circ g = D + f_0$$

near $x = 0$ where $D = df(0)$, $T : Y \to X$ is a pseudo-inverse of D, $g : W \to X$ is a local diffeomorphism defined in a bounded closed neighbourhood W of 0 and $f_0 : W \to \ker T$ is a smooth map. Recall that there are splittings

$$X = X_0 \oplus X_1, \qquad Y = Y_0 \oplus Y_1$$

with

$$X_0 = \ker D, \quad X_1 = \operatorname{im} T, \qquad Y_0 = \ker T, \quad Y_1 = \operatorname{im} D.$$

Think of $g : W \to X$ as a coordinate chart on X and write the equation $f(g(x)) = y$ in the form

$$y_0 = f_0(x_0, x_1), \qquad y_1 = D_1 x_1,$$

for $x = x_0 + x_1 \in W$ with $x_i \in X_i$. Here $D_1 : X_1 \to Y_1$ denotes the restriction of D to X_1. It follows from this description that $y = y_0 + y_1$ is a regular value of $f|_{g(W)}$ if and only if y_0 is a regular value of the map

*If the cover $\{U_\alpha\}_\alpha$ is locally finite and $\{x_i\}_i$ is a dense sequence then the set of pairs (α, i) with $x_i \in U_\alpha$ is countable. Since every U_α contains some point x_i the map $(\alpha, i) \mapsto \alpha$ is surjective.

$$x_0 \mapsto f_0(x_0, {D_1}^{-1}y_1)$$

defined on the closed set of all $x_0 \in X_0$ with $x_0 + {D_1}^{-1}y_1 \in W$. Hence it follows from the finite dimensional version of Sard's theorem that the set of regular values of $f|_{g(W)}$ is dense in Y. We prove that this set is open. Let $y_\nu = y_{\nu,0} + y_{\nu,1} \in Y$ be a sequence of singular values of $f|_{g(W)}$ which converges to y. Choose $x_\nu \in W$ with $f(g(x_\nu)) = y_\nu$ such that $df(g(x_\nu))$ is not surjective. Then the sequence $x_{\nu,1} = TDx_\nu = Ty_\nu$ converges and, since W is bounded, the sequence $x_{\nu,0} = x_\nu - TDx_\nu \in \ker D$ is bounded. Passing to a subsequence we may assume that $x_{\nu,0}$ converges as well, and hence, so does $x_\nu = x_{\nu,0} + x_{\nu,1}$. Since W is closed, the limit point $x = \lim_{\nu\to\infty} x_\nu$ lies again in W and $f(g(x)) = y$. Moreover, $df(g(x))$ is the limit of operators with a nontrivial cokernel and hence cannot be surjective. Hence y is a singular value of $f|_{g(U_0)}$. This shows that the set of regular values of the restriction $f : g(W) \to Y$ is open and dense in Y. Thus the theorem is proved. □

Recall that in Sard's theorem for finite dimensional manifolds the degree of smoothness required depends on the difference of the dimensions of the source manifold X and the target manifold Y. In the above proof this theorem is applied to the function $f_0 : \ker D \to \operatorname{coker} D$. Hence Theorem B.13 continues to hold for functions of class C^ℓ for some finite, but sufficiently large, number ℓ which depends on the Fredholm index of f_0. More precisely, ℓ must nmust be at least 1 and $\ell \geq \mathrm{index}(f_0) + 2$.

B.4 Thom-Smale transversality

Let X be an n-dimensional smooth manifold and

$$\pi : E \to X$$

be an m-dimensional real vector bundle. A smooth section $f : X \to E$ is said to be **transversal to the zero section** if the linear map

$$Df(x) = \Pi(x) \circ df(x) : T_xX \to E_x$$

is onto for every zero $x \in X$ of f. Here $df(x) : T_xX \to T_{(x,f(x))}E$ denotes the differential of f and $\Pi(x) : T_{(x,0)}E \to E_x$ denotes the projection onto the vertical subspace $E_x \subset T_{(x,0)}E$. This projection is well defined since the tangent space $T_{(x,0)}E$ at a point $(x, 0)$ in the zero section splits naturally as $T_{(x,0)}E = T_xX \oplus E_x$.

Proposition B.15 *If f is transversal to the zero section then the set*

$$\mathcal{M}(f) = \{x \in X \mid f(x) = 0\}$$

is a submanifold of X of dimension $n - m$. The tangent space to $\mathcal{M}(f)$ at x is given by $T_x\mathcal{M}(f) = \ker Df(x)$.

The proof is an application of the implicit function theorem and reduces to a simple exercise in local coordinates which is left to the reader. Now let Z be an N-dimensional manifold (the *parameter space*) and

$$\pi : E \to X \times Z$$

be a real vector bundle of rank m. Let $F : X \times Z \to E$ be a section of this bundle and denote by $\iota_z : X \to X \times Z$ the inclusion $\iota_z(x) = (x, z)$. Think of $E_z = {\iota_z}^* E$ as the restriction of E to $X \times \{z\}$ and consider the section $f_z = {\iota_z}^* F : X \to E_z$ defined by

$$f_z(x) = F(x, z).$$

If F is transversal to the zero section then the space

$$\mathcal{M}(F) = \{(x, z) \in X \times Z \,|\, F(x, z) = 0\}$$

is a manifold of dimension $n - m + N$. It intersects $X \times \{z\}$ in the set $\mathcal{M}(f_z)$.

Proposition B.16 *Assume that $F : X \times Z \to E$ is transversal to the zero section. Then $z \in Z$ is a regular value of the projection $\pi : \mathcal{M}(F) \to Z$ if and only if f_z is transversal to the zero section.*

Proof: The differential of F at a zero $(x, z) \in \mathcal{M}(F)$ can be written as a sum

$$DF(x, z)(\xi, \zeta) = Df_z(x)\xi + D_z F(x, z)\zeta$$

for $\xi \in T_x X$ and $\zeta \in T_z Z$. Since F is transversal to the zero section this map is onto for every $(x, z) \in \mathcal{M}(F)$. By Proposition B.16 the tangent space to $\mathcal{M}(F)$ at (x, z) is the set of all pairs $(\xi, \zeta) \in T_x X \times T_z Z$ such that

$$Df_z(x)\xi + D_z F(x, z)\zeta = 0. \tag{B.4}$$

Now the differential $d\pi(x, z) : T_{(x,z)}\mathcal{M}(F) \to T_z Z$ is just the map $(\xi, \zeta) \mapsto \zeta$. Hence $d\pi(x, z)$ is onto if and only if for every $\eta \in T_z Z$ there exists a $\xi \in T_x X$ such that (B.4) is satisfied. But this means that

$$\operatorname{im} D_z F(x, z) \subset \operatorname{im} Df_z(x).$$

By assumption $\operatorname{im} D_z F(x, z) + \operatorname{im} Df_z(x) = E_x$. Hence $d\pi(x, z)$ is onto if and only if $Df_z(x)$ is onto. This proves the proposition. □

If $\mathcal{M}(F)$ satisfies the second axiom of countability (can be covered by an atlas consisting of countably many charts) then it follows from Sard's theorem that the set Z_{reg} of regular values of π is of the second category in the sense of Baire (a countable intersection of open and dense sets).

In particular the set is dense. By Propositions B.15 and B.16 $\mathcal{M}(f_z)$ is a manifold for every $z \in Z_{\mathrm{reg}}$. Moreover, the following proposition shows that, if the parameter manifold Z is connected, then for any two regular values z_0 and z_1 the manifolds $\mathcal{M}_{z_0}$ and $\mathcal{M}_{z_1}$ are cobordant.

Proposition B.17 *Assume that f_{z_0} and f_{z_1} are transversal to the zero section. Denote by $\mathcal{Z}$ the set of all smooth paths $z : [0,1] \to Z$ with $z(0) = z_0$ and $z(1) = z_1$. There exists a set $\mathcal{Z}_{\mathrm{reg}} \subset \mathcal{Z}$ of the second category in the sense of Baire such that for every $z = \{z_t\} \in \mathcal{Z}$ the space*

$$\mathcal{M}(\{z_t\}) = \{(t, x) \,|\, t \in [0,1],\, F(x, z_t) = 0\}$$

is a smooth manifold of dimension $n - m + 1$ with boundary

$$\partial\mathcal{M}(\{z_t\}) = \mathcal{M}(z_1) - \mathcal{M}(z_0).$$

Here the minus sign indicates the reversal of orientation.

Proof: Denote by $\mathcal{Z}^\ell$ the space of C^ℓ-paths $z : [0,1] \to Z$ with $z(0) = z_0$ and $z(1) = z_1$. Then there is a vector bundle

$$\mathcal{E}^\ell \to [0,1] \times X \times \mathcal{Z}^\ell$$

whose fiber over $(t, x, \{z_t\})$ is the space E_{x,z_t}. This bundle has a section

$$\mathcal{F} : [0,1] \times X \times \mathcal{Z}^\ell \to \mathcal{E}^\ell$$

given by $\mathcal{F}(t, x, \{z_t\}) = F(x, z_t)$. We prove that this map is transverse to the zero section: if $\mathcal{F}(t, x, \{z_t\}) = 0$ then the differential $D\mathcal{F}(t, x, \{z_t\}) : \mathbb{R} \times T_x X \times T_{\{z_t\}}\mathcal{Z} \to E_{x,z_t}$ is given by

$$D\mathcal{F}(t, x, \{z_t\})(\tau, \xi, \{\zeta_t\}) = DF(x, z_t)(\xi, \zeta_t + \tau \dot{z}_t).$$

For $t = 0$ and $t = 1$ this operator is surjective as a function of ξ alone, and for other values of t it is surjective as a function of ξ and ζ because ζ_t can be choosen arbitrarily. Hence $\mathcal{F}$ is transverse to the zero section and its zero set is therefore an infinite dimensional Banach manifold

$$\mathbb{M}^\ell = \left\{(t, x, \{z_t\}) \in [0,1] \times X \times \mathcal{Z}^\ell \,|\, F(x, z_t) = 0\right\}.$$

This is a kind of *universal manifold* which incorporates all paths in $\mathcal{Z}^\ell$. The obvious projection

$$\pi : \mathbb{M}^\ell \to \mathcal{Z}^\ell$$

is a Fredholm map between separable Banach manifolds with Fredholm index

$$\operatorname{index}\pi = n - m + 1.$$

In fact, Lemma B.5 shows that the kernel of $d\pi$ has dimension at most $n+1$ and the cokernel has dimension at most m. Hence it follows from the Sard-Smale theorem B.13 that, for ℓ sufficiently large, the set

$$\mathcal{Z}^\ell_{\mathrm{reg}}$$

of regular values of π is of the second category in the sense of Baire. It follows from the same argument as in Proposition B.16 that for every $\{z_t\} \in \mathcal{Z}^\ell_{\mathrm{reg}}$ the section $[0,1]\times X \to \mathcal{E} : (t,x) \mapsto F(x,z_t)$ is transverse to the zero section. Its zero set is the required manifold $\mathcal{M}(\{z_t\})$. This proves the proposition in the C^ℓ category.

The C^∞ statement can easily be reduced to the C^ℓ statement by an argument which is due to Taubes. Denote by

$$\mathcal{Z}_{\mathrm{reg}}$$

the set of all paths $\{z_t\} \in \mathcal{Z}$ for which the section $[0,1]\times X \to E : (t,x) \mapsto F(x,z_t)$ is transverse to the zero section. Similarly, for every compact set $K \subset X$ denote by

$$\mathcal{Z}_{K,\mathrm{reg}}$$

the set of those paths $\{z_t\} \in \mathcal{Z}$ for which the section $[0,1]\times K \to E : (t,x) \mapsto F(x,z_t)$ is transverse to the zero section. Then

$$\mathcal{Z}_{\mathrm{reg}} = \bigcap_K \mathcal{Z}_{K,\mathrm{reg}}$$

and this is a countable intersection since X can be exhausted by a sequence of compact sets. We claim that each set $\mathcal{Z}_{K,\mathrm{reg}}$ is open and dense. Opennes is an obvious consequence of the compactness of K. To prove that $\mathcal{Z}_{K,\mathrm{reg}}$ is dense in $\mathcal{Z}$ consider first the set $\mathcal{Z}^\ell_{K,\mathrm{reg}}$ defined in a similar way in the C^ℓ-category. This set is open in $\mathcal{Z}^\ell$ with respect to the C^ℓ topology and, since $\mathcal{Z}^\ell_{\mathrm{reg}} \subset \mathcal{Z}^\ell_{K,\mathrm{reg}}$ it is also dense. This implies that the set $\mathcal{Z}_{K,\mathrm{reg}}$ is dense in $\mathcal{Z}$. To see this approximate a given smooth path $z = \{z_t\} \in \mathcal{Z}$ by a sequence of paths $z_\nu \in \mathcal{Z}^\ell_{K,\mathrm{reg}}$ and then approximate z_ν by a C^∞ smooth path z'_ν. Then

$$z'_\nu \in \mathcal{Z}^\ell_{K,\mathrm{reg}} \cap \mathcal{Z} = \mathcal{Z}_{K,\mathrm{reg}}$$

converges to z. Thus we have proved that the sets $\mathcal{Z}_{K,\mathrm{reg}}$ are all open and dense in $\mathcal{Z}$. Hence $\mathcal{Z}_{\mathrm{reg}} \subset \mathcal{Z}$ is a countable intersection of open and dense sets as required. □

There are important generalizations of these ideas to situations where the manifolds X and Z as well as the fibers of the vector bundle E are infinite dimensional. In many cases the operator $Df_z(x) : T_xX \to E_{(x,z)}$ (differentiation with respect to x) is a Fredholm operator and the resulting moduli spaces $\mathcal{M}(f_z)$ are finite dimensional manifolds whose dimension is the Fredholm index of the operator $Df_z(x)$. In contrast the operator $D_zF(x,z) : T_zZ \to E_{(x,z)}$ (differentiation with respect to the parameter) will in general not be Fredholm but have a dense range. In other words one must prove that the parameter manifold Z is *sufficiently rich* to guarantee that the operator $DF(x,z)$ is always onto. It then follows from Lemma B.5 that the operator $DF(x,z)$ has a right inverse and by Theorem B.3 the space $\mathcal{M}(F)$ is an infinite dimensional Banach manifold. The proof of Proposition B.16 generalizes word by word to the infinite dimensional situation and shows that f_z is transversal to the zero section for every regular value z of the projection $\pi : \mathcal{M}(F) \to Z$. Finally, the Sard-Smale theorem B.13 is required to prove that the set of regular values of π is of the second category in the sense of Baire.

APPENDIX C

ELLIPTIC REGULARITY

This appendix gives an introduction to the theory of second order elliptic partial differential equations on Euclidean space. The first section explains the necessary background material about Sobolev spaces, embedding theorems, and interpolation and product estimates. The second section gives a sketch of the proof of regularity for the weak L^2 solutions of an elliptic equation with Dirichlet boundary conditions. Section C.3 gives a proof of the Calderón-Zygmund inequality and Section C.4 shows how this can be used to establish the L^p-theory for general second order elliptic operators. The regularity theorems play a central role in establishing the Fredholm properties of elliptic operators between suitable Sobolev spaces. Excellent references for the material of this appendix are Gilbarg-Trudinger [39] and Simon [111].

C.1 Sobolev spaces

Let $\Omega \subset \mathbb{R}^n$ be an open set. Throughout $C^\infty(\bar{\Omega})$ denotes the space of restrictions of smooth functions on $\mathbb{R}^n$ to $\bar{\Omega}$ and $C_0^\infty(\Omega)$ the space of smooth compactly supported functions on Ω. We begin by mentioning two fundamental inequalities for smooth compactly supported functions $u, v : \mathbb{R}^n \to \mathbb{R}$, namely, **Hölder's inequality**

$$\|uv\|_{L^1} \le \|u\|_{L^p} \|v\|_{L^q}$$

for $1/p + 1/q = 1$ and **Young's inequality**

$$\|u * v\|_{L^p} \le \|u\|_{L^1} \|v\|_{L^p} .$$

Here

$$u * v(x) = \int_{\mathbb{R}^n} u(x-y)v(y)\,dy$$

denotes the convolution and

$$\|u\|_{L^p} = \left(\int_{\mathbb{R}^n} |u|^p \right)^{1/p}$$

denotes the L^p-norm for $1 \le p < \infty$.

Weak derivatives

Let $u : \Omega \to \mathbb{R}$ be locally (Lebesgue) integrable and fix a multi-index $\nu = (\nu_1, \ldots, \nu_n)$. A locally integrable function $u_\nu : \Omega \to \mathbb{R}^n$ is called the **weak derivative** of u corrsponding to ν if

$$\int_\Omega u(x)\partial^\nu \varphi(x)\,dx = (-1)^{|\nu|}\int_\Omega u_\nu(x)\varphi(x)\,dx$$

for all $\varphi \in C_0^\infty(\Omega)$. The weak derivative, if it exists, is (almost everywhere) uniquely determined by u and we write

$$\partial^\nu u := u_\nu.$$

The divergence theorem shows that every C^k-function $u : \Omega \to \mathbb{R}$ has weak derivatives up to order k and these agree with the strong derivatives.

Fix an integer $k \geq 1$ and a number $1 \leq p \leq \infty$. The **Sobolev space** $W^{k,p}_{\mathrm{loc}}(\Omega)$ is defined as the set of function $u \in L^p_{\mathrm{loc}}(\Omega)$ for which all the weak partial derivatives $\partial^\nu u$ of order $|\nu| = \nu_1 + \cdots + \nu_n \leq k$ exist and are locally p-integrable (respectively locally bounded in the case $p = \infty$). Denote by $W^{k,p}(\Omega)$ the space of all functions $u \in W^{k,p}_{\mathrm{loc}}(\Omega)$ with $\partial^\nu u \in L^p(\Omega)$ for $|\nu| \leq k$. The $W^{k,p}$**-Sobolev-norm** of $u \in W^{k,p}(\Omega)$ is defined by

$$\|u\|_{k,p} = \left(\int_\Omega \sum_{|\nu|\leq k} |\partial^\nu u(x)|^p\,dx\right)^{1/p}.$$

For $k = 0$ let $W^{0,p}(\Omega) = L^p(\Omega)$ denote the standard L^p-space. The symbol $W^{k,p}_0(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$.

Exercise C.1 Prove that $W^{k,p}(\Omega)$ is a Banach space, and is reflexive for $1 < p < \infty$. Prove that $W^{k,p}_0(\Omega)$ is separable for $1 \leq p < \infty$. **Hint:** Think of $W^{k,p}(\Omega)$ as a closed subspace of $L^p(\Omega, \mathbb{R}^N)$ for a suitable integer N. Every closed subspace of a Banach space is complete. Every closed subspace of a reflexive Banach space is reflexive. For separability use the fact that every smooth function $u : \Omega \to \mathbb{R}$ can be approximated by a sequence of polynomials with rational coefficients, where the convergence is uniform for each derivative on every compact set. □

Mollifiers

Let $\rho : \mathbb{R}^n \to \mathbb{R}$ be a smooth nonnegative function such that

$$\operatorname{supp}\rho \subset B_1, \qquad \int_{\mathbb{R}^n} \rho = 1.$$

For $\delta > 0$ denote

$$\rho_\delta(x) = \delta^{-n}\rho(\delta^{-1}x).$$

Given a locally integrable function $u : \Omega \to \mathbb{R}$ define

$$u_\delta(x) = \rho_\delta * u(x) = \int_{B_\delta(x)} \rho_\delta(x-y)u(y)\,dy \tag{C.1}$$

for $x \in \Omega_\delta$. Here $B_\delta(x)$ denotes the open ball of radius δ about x and

$$\Omega_\delta = \left\{x \in \Omega \,|\, \bar{B}_\delta(x) \subset \Omega\right\}.$$

The function $u_\delta : \Omega_\delta \to \mathbb{R}$ is smooth and the smoothing operator $u \mapsto u_\delta$ is called a **mollifier**. If $u \in W^{k,p}_{\mathrm{loc}}(\Omega)$ then it is a simple consequence of the definition of weak derivatives that the (strong) derivatives of u_δ are given by the mollified (weak) derivatives of u:

$$\partial^\alpha(\rho_\delta * u) = \rho_\delta * \partial^\alpha u. \tag{C.2}$$

Hence Young's inequality asserts that

$$\|u_\delta\|_{W^{k,p}(\Omega_\delta)} \le \|u\|_{W^{k,p}(\Omega)} \tag{C.3}$$

Now one checks easily that for every continuous function $f : \Omega \to \mathbb{R}$ the function $\rho_\delta * f$ converges to f uniformly on compact sets as $\delta \to 0$. Since the continuous functions form a dense subset of $L^p_{\mathrm{loc}}(\Omega)$ for $1 \le p < \infty$ it follows from the uniform estimate (C.3) that f_δ converges to f in the L^p-norm over compact subsets of Ω whenever $f \in L^p_{\mathrm{loc}}(\Omega)$. Combining this with (C.2) one finds that

$$\lim_{\delta\to 0} \|u - u_\delta\|_{W^{k,p}(K)} = 0 \tag{C.4}$$

for every $u \in W^{k,p}_{\mathrm{loc}}(\Omega)$, every compact subset $K \subset \Omega$, and every $p \in [1,\infty)$.

Approximation by smooth functions

Our next goal is to prove that for a large class of domains $\Omega \subset \mathbb{R}^n$ the Sobolev space $W^{k,p}(\Omega)$ can be identified with the completion of $C^\infty(\bar{\Omega})$ with respect to the $W^{k,p}$-norm. An open set $\Omega \subset \mathbb{R}^n$ is called a **Lipschitz domain** if the boundary can locally be represented as the graph of a Lipschitz function. Explicitly, this means that for every $x \in \partial\Omega$ there exist a neighbourhood U of x, a unit vector $\xi \in S^{n-1}$, and a Lipschitz continuous function $f : \xi^\perp \to \mathbb{R}$ such that $f(0) = 0$ and

$$\partial\Omega \cap U = \{x + \eta + f(\eta)\xi \,|\, \eta \in \xi^\perp,\ |\eta| < \delta\}$$

for some $\delta > 0$. The motivation for this definition lies in the following observation.

Exercise C.2 Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Show that for every function $u \in W^{k,p}(\Omega)$ there exists a sequence of open sets $\Omega_j \subset \mathbb{R}^n$ and a sequence of functions $u_j \in W^{k,p}(\Omega_j)$ such that

$$\bar{\Omega} \subset \Omega_j, \qquad \lim_{j\to\infty} \|u_j - u\|_{W^{k,p}(\Omega)} = 0.$$

Hint: Use the fact that

$$\lim_{t\to 0} \int_\Omega |f(x - t\xi) - f(x)|^p \, dx = 0$$

for $f \in L^p(\mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$. This holds obviously for continuous functions and for L^p-functions since continuous functions are dense in L^p. □

Proposition C.3 *If $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain then $C^\infty(\bar{\Omega})$ is dense in $W^{k,p}(\Omega)$.*

Proof: If $u \in W^{k,p}(\Omega)$ and $\varepsilon > 0$ then, by Exercise C.2, there exists a function $v \in W^{k,p}(\Omega')$ with $\bar{\Omega} \subset \Omega'$ such that $\|u - v\|_{W^{k,p}(\Omega)} < \varepsilon/2$. By (C.4), there exists a $\delta > 0$ such that

$$\|\rho_\delta * v - v\|_{W^{k,p}(\Omega)} \le \varepsilon/2.$$

Hence $\|\rho_\delta * v - u\|_{W^{k,p}(\Omega)} \le \varepsilon$ and this proves the proposition. □

The previous proposition shows that for Lipschitz domains the Sobolev space $W^{k,p}(\Omega)$ can also be defined as the completion of the space $C^\infty(\bar{\Omega})$ with respect to the $W^{k,p}$-norm.

Poincaré's inequality

It is somewhat less than obvious that a function $u \in W^{1,p}(\Omega)$ whose derivatives all vanish must be constant on every component of Ω. The proof requires the following fundamental estimate.

Lemma C.4. (Poincaré's inequality) *Let $1 < p < \infty$ and $\Omega \subset \mathbb{R}^n$ be a bounded open domain. Then for $u \in W_0^{1,p}(\Omega)$*

$$\|u\|_{L^p(\Omega)} \le \operatorname{diam}(\Omega) \, \|\nabla u\|_{L^p(\Omega)} .$$

If $\Omega = Q^n = (0,1)^n$ is the unit square then every $u \in W^{1,p}(\Omega)$ with $\int_{Q^n} u = 0$ satisfies

$$\|u\|_{L^p(Q^n)} \le n \, \|\nabla u\|_{L^p(Q^n)} .$$

Proof: It suffices to prove the first statement for $u \in C_0^\infty(\Omega)$. Suppose without loss of generality that $\Omega \subset \{x_n > 0\}$ and $0 \in \partial\Omega$. Then

$$u(x) = \int_0^{x_n} \partial_n u(x_1, \dots, x_{n-1}, t)\, dt.$$

Since $|x_n| \le \mathrm{diam}(\Omega)$ it follows from Hölder's inequality that

$$|u(x)|^p \le \mathrm{diam}(\Omega)^{p-1} \int_0^\infty |\partial_n u(x_1, \dots, x_n)|^p\, dt.$$

Now integrate both sides over $\mathbb{R}^{n-1} \times [0, \mathrm{diam}(\Omega)]$ to obtain

$$\int_\Omega |u|^p \le \mathrm{diam}(\Omega)^p \int_\Omega |\partial_n u|^p.$$

This proves the first assertion. The second assertion is proved by induction over n. For $n = 1$ the estimate is an easy exercise. Hence assume that the estimate is proved for $n \ge 1$ and let $u \in C^\infty(Q^{n+1})$ be of mean value zero. Define

$$v(t) = \int_{Q^n} u(x_1, \dots, x_n, t)\, dx_1 \cdots dx_n.$$

Since $v \in C^\infty(Q^1)$ has mean value zero

$$\int_0^1 |v(t)|^p\, dt \le \int_0^1 |\dot{v}(t)|^p\, dt \le \int_{Q^{n+1}} |\partial_{n+1} u|^p\ dx.$$

The last step follows from Hölder's inequality. By induction hypothesis, we have

$$\int_{Q^n} |u(x,t) - v(t)|^p\, dx \le n^p \int_{Q^n} |\nabla u(x,t)|^p\, dx$$

for every t. Integrate over t to obtain $\|u - v\|_{L^p} \le n \|\nabla u\|_{L^p}$. Combining this with the previous estimate gives $\|u\|_{L^p(Q^{n+1})} \le (n+1) \|\nabla u\|_{L^p(Q^{n+1})}$. This proves the second statement for smooth functions. In the general case it follows from Proposition C.3. □

Corollary C.5 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain and $u \in W^{1,p}(\Omega)$ with weak derivatives $\partial u/\partial x_j \equiv 0$ for $j = 1, \dots, n$. Then u is constant on each connected component of Ω (after redefining u on a set of measure zero if necessary). If, moreover, $u \in W_0^{1,p}(\Omega)$ then $u = 0$ almost everywhere.*

Proof: By Lemma C.4, u is locally constant, in the sense that each point $x \in \Omega$ has a neighbourhood U_x in which u is almost everywhere equal to its mean value $c_x = (\mathrm{Vol}(U_x))^{-1} \int_{U_x} u$. Now on each component of Ω the local mean value c_x is independent of x. □

The previous corollary can also be obtained as a consequence of the next exercise which shows that for any open set $\Omega \subset \mathbb{R}^n$ the Sobolev space $W^{1,\infty}_{\mathrm{loc}}(\Omega)$ can be naturally identified with the space $C^{0,1}_{\mathrm{loc}}(\Omega)$ of locally Lipschitz continuous functions on Ω.

Exercise C.6 **(i)** If $u \in L^1_{\mathrm{loc}}(\Omega)$ prove that $u_\delta(x)$ converges to $u(x)$ for almost every $x \in \Omega$.

(ii) Show that $C^{0,1}_{\mathrm{loc}}(\Omega) \subset W^{1,\infty}(\Omega)$. **Hint:** Let $u : \Omega \to \mathbb{R}$ be locally Lipschitz continuous and fix a vector $\xi \in \mathbb{R}^n$. Prove that the sequence $u_j(x) = j(u(x - \xi/j) - u(x))$ has a subsequence which converges weakly in $L^2(K)$ for every compact subset $K \subset \Omega$. Prove that the limit function $u^\xi : \Omega \to \mathbb{R}$ is the weak derivative of u in the direction ξ, i.e. $\int_\Omega u^\xi \varphi = -\int_\Omega u \langle \nabla \varphi, \xi \rangle$ for all $\varphi \in C_0^\infty(\Omega)$. Prove that u^ξ is locally bounded.

(iii) Show that $W^{1,\infty}(\Omega) \subset C^{0,1}_{\mathrm{loc}}(\Omega)$. **Hint:** If $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ prove that for $\delta > 0$ the functions $u_\delta = \rho_\delta * u$ are locally Lipschitz continuous with the Lipschitz constant $c = \sup_{B_{r+\delta}(x)} |\nabla u|$ over $B_r(x)$. Now use (i) to prove that u is locally Lipschitz continuous (possibly after redefining it on a set of measure zero). □

Extension

Define the **Hölder norm**

$$\|u\|_{C^\mu} = \sup_{x,y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^\mu} + \sup_{x \in \Omega} |u(x)|$$

for $0 < \varepsilon \le 1$ and

$$\|u\|_{C^{k,\mu}} = \sum_{|\nu| \le k} \|\partial^\nu u\|_{C^\varepsilon} .$$

Denote by $C^{k,\mu}(\Omega)$ the space of all C^k-functions $u : \Omega \to \mathbb{R}$ with finite Hölder norm $\|u\|_{C^{k,\mu}}$. A **$C^{k,\mu}$-diffeomorphism** is a bijective map $\psi : U \to V$ (between open sets in $\mathbb{R}^n$) such that both ψ and ψ^{-1} are of class $C^{k,\mu}$. An open set $\Omega \subset \mathbb{R}^n$ is called a **$C^{k,\mu}$-domain** if every point $x \in \partial\Omega$ has a neighbourhood $U \subset \mathbb{R}^n$ which is $C^{k,\mu}$-diffeomorphic to some open set $V \subset \mathbb{R}^n$ in such a way that $U \cap \partial\Omega$ is identified with $V \cap (\mathbb{R}^{n-1} \times \{0\})$. Note that every Lipschitz domain is a $C^{0,1}$-domain but not vice versa.

Proposition C.7 *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{k-1,1}$-domain and $\Omega' \subset \mathbb{R}^n$ be an open set with $\bar{\Omega} \subset \Omega'$. Then there exists a bounded linear operator $E : W^{k,p}(\Omega) \to W_0^{k,p}(\Omega')$ such that $Eu|_\Omega = u$ for every $u \in W^{k,p}(\Omega)$.*

Exercise C.8 This exercise shows that the Sobolev space $W^{k,p}$ is preserved by composition (on the right) with $C^{k-1,1}$-diffeomorphisms.

(i) Show that $u \in W^{k+1,p}(\Omega)$ if and only if $u \in W^{1,p}(\Omega)$ and the weak derivatives $\partial_i u = \partial u / \partial x_i$ lie in $W^{k,p}(\Omega)$ for $i = 1, \ldots, n$.

(ii) If $u \in W^{k,p}(\Omega)$ and $v \in W^{k,\infty}(\Omega)$ show that $uv \in W^{k,p}(\Omega)$ and

$$\|uv\|_{k,p} \le c \|u\|_{k,p} \|v\|_{k,\infty}$$

where the constant c depends only on k and n.

(iii) Let $\Omega, \Omega' \subset \mathbb{R}^n$ be bounded open domains and $\psi : \bar{\Omega}' \to \bar{\Omega}$ be a $C^{k-1,1}$-diffeomorphism (that is ψ is the restriction of a $C^{k-1,1}$-diffeomorphism between suitable open neighbourhoods of the closures). Show that if $u \in W^{k,p}(\Omega)$ then $u \circ \psi \in W^{k,p}(\Omega)$ and

$$\|u \circ \psi\|_{W^{k,p}(\Omega')} \leq c \|u\|_{W^{k,p}(\Omega)}$$

where the constant c is independent of u. **Hint:** Use (i), (ii), and Exercise C.2. Prove this by induction over k. $\Box$

Proof of Proposition C.7: The proof is taken from [39]. First consider the case $\Omega = \mathbb{H}^n = \{x_n > 0\}$ and define the extension operator $E_0 : C^{k-1,1}(\mathbb{H}^n) \to C^{k-1,1}(\mathbb{R}^n)$ by

$$E_0 u(x_1, \ldots, x_{n-1}, x_n) = \sum_{i=1}^{k} c_i u(x_1, \ldots, x_{n-1}, -x_n/i)$$

for $x_n \leq 0$ where the constants $c_1, \ldots, c_k$ are chosen such that

$$\sum_{i=1}^{k} c_i \left(-\frac{1}{i}\right)^m = 1, \qquad m = 0, \ldots, k-1.$$

One checks easily that the derivatives up to order $k-1$ match on the boundary, that if $u(x) = 0$ for $|x| \geq R$ then $E_0 u(x) = 0$ for $|x| \geq kR$, and that for compactly supported functions there is an estimate

$$\|E_0 u\|_{W^{k,p}(\mathbb{R}^n)} \leq c_0 \|u\|_{W^{k,p}(\mathbb{H}^n)}.$$

Now for any bounded $C^{k-1,1}$-domain Ω choose an open cover

$$\bar{\Omega} \subset U_0 \cup \ldots \cup U_N$$

with $\bar{U}_0 \subset \Omega$ and open sets $U_1', \ldots, U_N'$ with

$$\bar{U}_j \subset U_j' \subset \Omega'$$

such that there exist $C^{k-1,1}$-diffeomorphisms $\psi_j : U_j' \to B_k(0)$ with

$$\psi_j(U_j) = B_1(0), \qquad \psi_j(U_j' \cap \Omega) = B_k(0) \cap \mathbb{H}^n.$$

Then choose a partition of unity $\beta_j : \mathbb{R}^n \to [0,1]$ such that

$$\operatorname{supp} \beta_j \subset U_j, \qquad \sum_{j=1}^{n} \beta_j(x) = 1 \ \text{ for } x \in \Omega.$$

Define $E : C^{k-1,1}(\Omega) \to C_0^{k-1,1}(\Omega')$ by

$$Eu = \beta_0 u + \sum_{j=1}^{N} [E_0(\beta_j u \circ {\psi_j}^{-1})] \circ \psi_j.$$

It follows easily from Exercise C.8 that E extends to a bounded linear operator from $W^{k,p}(\Omega) \to W_0^{k,p}(\Omega')$. □

Sobolev embedding theorems

A function with weak derivatives need not be continuous. Consider for example the function

$$u(x) = |x|^{-\alpha}$$

with $\alpha \in \mathbb{R}$ in the domain $\Omega = B_1 = \{x \in \mathbb{R}^n \,|\, |x| < 1\}$. Then $\partial_j u = -\alpha x_j |x|^{-\alpha-2}$. (This holds pointwise for $x \neq 0$ and in the sense of weak derivatives whenever $\alpha < n-1$.) By induction,

$$|\partial^\nu u(x)| \leq c_\nu |x|^{-\alpha-|\nu|}.$$

Now the function $x \mapsto |x|^{-\beta}$ is integrable on B_1 if and only if $\beta < n$. Hence the derivatives of u up to order k will be p-integrable whenever $\alpha p + kp < n$. If $kp < n$ choose $0 < \alpha < n/p - k$ to obtain a function which is in $W^{k,p}(B_1)$ but not continuous at 0. For $kp > n$ this construction fails and, in fact, in this case every $W^{k,p}$-function is continuous.

Theorem C.9 *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and suppose that $kp > n$ and $0 < \mu = k - n/p < 1$. Then there exists a constant $c > 0$ such that*

$$\|u\|_{C^{0,\mu}} \leq c\,\|u\|_{W^{k,p}}$$

for $u \in C^\infty(\bar{\Omega})$. The inclusion $W^{k,p}(\Omega) \hookrightarrow C^0(\bar{\Omega})$ is compact.

Theorem C.10 *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and suppose that $kp < n$. Then there exists a constant $c > 0$ such that*

$$\|u\|_{L^{np/(n-kp)}} \leq c\,\|u\|_{W^{k,p}}$$

for $u \in C^\infty(\bar{\Omega})$. If $q < np/(n-kp)$ then the inclusion $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact.

These are the *Sobolev embedding theorems.* The compactness statement in Theorem C.10 is known as *Rellich's theorem.* Proofs can be found in Gilbarg-Trudinger [39] for example. The main ideas will be indicated below.

In particular, Theorem C.9 shows that if $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain then

$$C^\infty(\bar{\Omega}) = \bigcap_{k=1}^{\infty} W^{k,p}(\Omega)$$

for $1 \le p \le \infty$.

Exercise C.11 Show that the inclusion $W^{1,2}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$ is not compact. □

Exercise C.12 The case $kp = n$ is the socalled **Sobolev borderline case**. If $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain then there is a continuous inclusion $W^{k,p}(\Omega) \hookrightarrow C^0(\bar{\Omega})$ for $kp > n$ but not for $kp \le n$. Construct a sequence of function $u_j \in W^{1,n}(\mathbb{R}^n)$ on the unit disc $B^n = \{x \in \mathbb{R}^2 \,|\, |x|^2 < 1\}$ such that

$$u_j(0) = 1, \qquad \lim_{j\to\infty} \|u_j\|_{W^{1,n}} = 0.$$

Deduce that $W^{1,n}(\Omega) \not\subset C^0(\Omega)$ for any open set $\Omega \subset \mathbb{R}^n$. **Hint:** Consider the function $u(x) = \log|x|/\log\delta$, $\delta \le |x| \le 1$, with $u(x) = 1$ for $|x| \le \delta$. □

Exercise C.13 This exercise shows that the assumption of a Lipschitz domain in Theorem C.10 cannot be removed. Consider the bounded open set $\Omega \subset \mathbb{R}^2$ defined by

$$\begin{aligned}\Omega = &\left\{(x,y) \in \mathbb{R}^2 \,|\, 0 < x < 1,\, 0 < y < \frac{1}{2}\right\} \\ &\cup \bigcup_{m=0}^{\infty} \left\{(x,y) \in \mathbb{R}^2 \,|\, \frac{1}{2^{2m+1}} < x < \frac{1}{2^{2m}},\, \frac{1}{2} \le y < 1\right\}.\end{aligned}$$

Show that the embedding $W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ is not compact. Find a smooth function $u \in W^{1,2}(\Omega)$ such that $u \notin L^q(\Omega)$ for any $q > 2$. □

The assertions of Theorems C.9 and C.10 for $k \ge 2$ follow easily from the case $k = 1$. Moreover, in view of Proposition C.7, it suffices to prove these results for $W_0^{1,p}(\Omega)$.

Lemma C.14 *Every $u \in C_0^\infty(\mathbb{R}^n)$ satisfies the estimates*

$$\sup_{x\ne y} \frac{|u(x) - u(y)|}{|x-y|^\mu} \le c\,\|\nabla u\|_{L^p}, \qquad \sup|u| \le c\left(\|u\|_{L^p} + \|\nabla u\|_{L^p}\right)$$

for $p > n$ and $\mu = 1 - n/p$, where $c = 2^{n+1}{\omega_n}^{-1/p}\left((p-1)/(p-n)\right)^{1-1/p}$. Here ω_n denotes the area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$.

Proof: First suppose that $B \subset \mathbb{R}^n$ is a bounded convex set with nonempty interior and $u : B \to \mathbb{R}$ is a smooth function with mean value zero. Then u satisfies the inequality

$$|u(x)| \le \frac{d^n \omega_n}{nV{\omega_n}^{1/p}} \left(\frac{p-1}{p-n}\right)^{1-1/p} d^{1-n/p} \left\|u\right\|_{L^p(B)}, \tag{C.5}$$

where $d = \mathrm{diam}(B)$ and $V = \mathrm{Vol}(B)$. To see this note first that, since $\int_B u = 0$,

$$u(x) = \frac{1}{V}\int_B \int_0^1 \langle \nabla u(x + t(y-x)), x - y\rangle \, dtdy.$$

Hence

$$\begin{aligned}
V|u(x)| &\le \int_{|y|\le d}\int_0^1 |\nabla u(x+ty)|\,|y| \; dtdy \\
&= \int_0^d r^{n-1}\left(\int_{|\eta|=1}\int_0^1 |\nabla u(x+tr\eta)| \; rdtdS(\eta)\right) dr \\
&= \int_0^d r^{n-1}\left(\int_{|y|\le r} |y|^{1-n}\,|\nabla u(x+y)| \; dy\right) dr \\
&\le \frac{d^n}{n}\int_B |y-x|^{1-n}\,|\nabla u(y)| \; dy \\
&\le \frac{d^n}{n}\left(\int_{|y|\le d} |y|^{q-nq}\,dy\right)^{1/q} \left\|\nabla u\right\|_{L^p(B)}.
\end{aligned}$$

The last step follows from the Hölder inequality with $1/p + 1/q = 1$. The integral can be easily computed and one obtains (C.5). Now apply (C.5) to the case $B = B_r(x_0)$ with $x_0 = \frac{1}{2}(x+y)$ and $r = \frac{1}{2}|x-y|$. Then $d = |x-y|$ and $d^n\omega_n/nV = 2^n$. Hence, with

$$u_B = \frac{1}{V}\int_B u,$$

one obtains

$$\begin{aligned}
|u(x) - y(y)| &\le |u(x) - u_B| + |u_B - u(y)| \\
&\le \frac{2^{n+1}}{{\omega_n}^{1/p}}\left(\frac{p-1}{p-n}\right)^{1-1/p} |x-y|^{1-n/p} \left\|\nabla u\right\|_{L^p}
\end{aligned}$$

for every smooth function $u : \mathbb{R}^n \to \mathbb{R}$. This proves the first assertion of the lemma. The second inequality is left as an exercise. □

Lemma C.15 *Assume $p < n$. Then every $u \in C_0^\infty(\mathbb{R}^n)$ satisfies the estimates*

$$\|u\|_{L^{np/(n-p)}} \le \frac{np-p}{\sqrt{n}(n-p)} \|\nabla u\|_{L^p}.$$

Proof (due to Nirenberg): The identity

$$u(x) = \int_{-\infty}^{x_i} \partial_i u(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)\, dt$$

shows that

$$|u(x)|^{n/(n-1)} \le \prod_{i=1}^n \left(\int_{-\infty}^{\infty} |\partial_i u(x)| dx_i \right)^{1/(n-1)}.$$

Integrating over $x_1, \ldots, x_n$ and in each step using the generalized Hölder inequality

$$\|v_1 \cdots v_m\|_{L^1} \le \|v_1\|_{L^m} \cdots \|v_m\|_{L^m}$$

with $m = n-1$ one finds

$$\|u\|_{L^{n/(n-1)}} \le \prod_{i=1}^n \left(\int |\partial_i u| \right)^{1/n} \le \frac{1}{n} \sum_{i=1}^n \int |\partial_i u| \le \frac{1}{\sqrt{n}} \int |\nabla u|.$$

This proves the lemma for $p = 1$. To prove it in general consider the $L^{n/(n-1)}$-norm of the function

$$v = |u|^\alpha, \qquad \alpha = \frac{np-p}{n-p}.$$

Since

$$|\nabla v| = \alpha |u|^{\alpha-1} |\nabla u|, \qquad \frac{\alpha n}{n-1} = \frac{np}{n-p},$$

one obtains

$$\begin{aligned} \left(\int |u|^{np/(n-p)} \right)^{1-1/n} &\le \frac{\alpha}{\sqrt{n}} \int |u|^{\alpha-1} |\nabla u| \\ &\le \frac{\alpha}{\sqrt{n}} \left(\int |u|^{\alpha q - q} \right)^{1/q} \left(\int |\nabla u|^p \right)^{1/p} \\ &\le \frac{\alpha}{\sqrt{n}} \left(\int |u|^{np/(n-p)} \right)^{1-1/p} \left(\int |\nabla u|^p \right)^{1/p}. \end{aligned}$$

The second estimate is Hölder's inequality with $1/p + 1/q = 1$ and the last estimate uses the identity $\alpha q - q = np/(n-p)$. This proves the lemma in the general case. □

Proof of Theorems C.9 and C.10: By Lemma C.14, there is an inclusion $W_0^{1,p}(\Omega) \hookrightarrow C^{0,1-n/p}(\Omega)$ for $p > n$ and hence, by the Arzela-Ascoli theorem, the inclusion $W_0^{1,p}(\Omega) \hookrightarrow C^0(\bar{\Omega})$ is compact whenever Ω is bounded. Similarly, by Lemma C.15, there is an inclusion $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $p < n$ and $q = np/(n-p)$. That this inclusion is compact for bounded domains Ω and $q < np/(n-p)$ requires a separate argument. The inequality

$$\|u\|_{L^q} \le \|u\|_{L^1}^{\lambda} \|u\|_{L^{np/(n-p)}}^{1-\lambda}, \qquad \frac{1}{q} = \lambda + \frac{1-\lambda}{np/(n-p)},$$

for $q < np/(n-p)$ shows that it suffices to prove that the inclusion

$$\iota : W_0^{1,p}(\Omega) \hookrightarrow L^1(\Omega)$$

is compact for bounded domains. To see this denote by

$$S_\delta : L^1(\Omega) \to L^1(\Omega)$$

the smoothing operator

$$S_\delta f = \rho_\delta * f.$$

By the Arzela-Ascoli theorem, S_δ is compact. Namely, if u_i is a bounded sequence in $L^1(\Omega)$ then the sequence $S_\delta u_i \in C^0(\bar{\Omega})$ is bounded and equicontinuous and so has a subsequence which converges in $C^0(\bar{\Omega})$ and hence in $L^1(\Omega)$. It follows that the composition

$$S_\delta \circ \iota : W_0^{1,p}(\Omega) \to L^1(\Omega)$$

is compact. Moreover, integrating the inequality

$$\begin{aligned} |u(x) - u_\delta(x)| &= \left| \int_{|y|\le 1} \rho(y) \int_0^\delta \langle \nabla u(x-ty), y \rangle \, dtdy \right| \\ &\le \int_{|y|\le 1} \rho(y) \int_0^\delta |\nabla u(x-ty)| \, dtdy \end{aligned}$$

one finds

$$\|u - S_\delta u\|_{L^1} \le \delta \|u\|_{L^1} \le \delta \mathrm{Vol}(\Omega)^{1-1/p} \|\nabla u\|_{L^p}$$

for $u \in W_0^{1,p}(\Omega)$. This shows that the operators $S_\delta \circ \iota : W_0^{1,p}(\Omega) \to L^1(\Omega)$ converge to ι in the uniform operator topology as $\delta \to 0$ and hence the limit operator ι is compact. This proves Theorems C.9 and C.10 with $W^{k,p}(\Omega)$ replaced by $W_0^{k,p}(\Omega)$, but without any condition on the domain Ω. To prove the results in the stated form one simply combines the corresponding embedding theorems for $W_0^{k,p}$ with the extension theorem (Proposition C.7). This last step is left to the reader. □

Interpolation

To gain an intuitive understanding of Sobolev spaces it is often useful to think of a $W^{k,q}$-function as having $k-n/q$ continuous derivatives. Then the Sobolev embedding theorem C.10 can be phrased in the form that there is a continuous inclusion $W^{k,q} \hookrightarrow W^{j,p}$ whenever $W^{k,q}$-functions have more derivatives than $W^{j,p}$-functions, i.e. $j \le k$ and $j - n/p \le k - n/q$. Care must be taken in the borderline case $k - n/q = 0$. A proof of the following interpolation inequality can be found, for example, in [28].

Proposition C.16. (Gagliardo-Nirenberg) *Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with C^k boundary. Suppose that $j, k \ge 0$ are integers with $j < k$ and $1 \le p, q, r \le \infty$ with $k - n/q + n/r \ge 0$ and*

$$j - \frac{n}{p} = \lambda\left(k - \frac{n}{q}\right) + (1-\lambda)\left(-\frac{n}{r}\right), \qquad \frac{j}{k} \le \lambda \le 1.$$

If $(k-j)q = n$ assume also that $\lambda \ne 1$. Then there exists a constant $c > 0$ such that

$$\|u\|_{W^{j,p}} \le c\,\|u\|_{W^{k,q}}^{\lambda}\,\|u\|_{L^r}^{1-\lambda}$$

for $u \in W^{k,q}(\Omega)$.

Product estimates

The case $kp > n$ should be viewed as the *good case* where *everything works*; for example, composition with a smooth function and products.

Proposition C.17 *Assume $kp > n$. Then there exists a constant $c = c(k,p) > 0$ such that*

$$\|uv\|_{W^{k,p}} \le c\left(\|u\|_{W^{k,p}}\|v\|_{L^\infty} + \|u\|_{L^\infty}\|v\|_{W^{k,p}}\right)$$

$$\|f \circ u\|_{W^{k,p}} \le c\left(\|f\|_{C^k} + 1\right)\|u\|_{W^{k,p}}$$

for $u, v \in C^\infty(\bar{\Omega})$ and $f \in \mathbb{C}^k(\mathbb{R})$.

The proof of Proposition C.17 is a straighforward exercise. It makes use of Hölder's inequality and of the general interpolation inequality of Gagliardo-Nirenberg in Proposition C.16.

Proposition C.18 Assume $kp > n$ and $f \in C^\infty(\mathbb{R})$. Then the map

$$W^{k,p}(\Omega) \to W^{k,p}(\Omega) : u \mapsto f \circ u$$

is a C^∞-map of Banach spaces.

The following proposition contains some more refined product estimates.

Proposition C.19 *Fix two constants $p, q \geq 1$ and integers $j, k, n \geq 1$. Assume throughout that $j \leq k$ and $j - n/p \leq k - n/q$. Then the following holds.*

(i) *If $j - n/p < k - n/q < 0$ then there exists a constant $c > 0$ such that*

$$\|fg\|_{W^{j,p}} \leq c \|f\|_{W^{k,q}} \|g\|_{W^{j,r}}$$

for $f, g \in C_0^\infty(\mathbb{R}^n)$ where $r > p$ is defined by

$$j - \frac{n}{p} = k - \frac{n}{q} + j - \frac{n}{r}.$$

(ii) *If $j - n/p = k - n/q < 0$ then there exists a constant $c > 0$ such that*

$$\|fg\|_{W^{j,p}} \leq c \|f\|_{W^{k,q}} \left(\|g\|_{W^{j,n/j}} + \|g\|_{L^\infty}\right)$$

for $f, g \in C_0^\infty(\mathbb{R}^n)$.

(iii) *If $k - n/q = 0$ then for every $\varepsilon > 0$ there exists a constant $c = c(\varepsilon) > 0$ such that*

$$\|fg\|_{W^{j,p}} \leq c \|f\|_{W^{k,q}} \|g\|_{W^{j,p+\varepsilon}}$$

for $f, g \in C_0^\infty(\mathbb{R}^n)$.

(iv) *If $k - n/q > 0$ then there exists a constant $c > 0$ such that*

$$\|fg\|_{W^{j,p}} \leq c \|f\|_{W^{k,q}} \|g\|_{W^{j,p}}$$

for $f, g \in C_0^\infty(\mathbb{R}^n)$.

Proof: Let α and β be multi-indices with $|\alpha| = i$ and $|\beta| = j - i$. Let $s \geq 1$ and $t \geq 1$ be defined by

$$i - \frac{n}{s} = k - \frac{n}{q}, \qquad \frac{1}{s} + \frac{1}{t} = \frac{1}{p}. \tag{C.6}$$

Since $i - n/s = k - n/q < 0$ it follows that $s < \infty$ and since $j - n/p < k - n/q = i - n/s \leq j - n/s$ it follows that $s > p$. Hence $p < t < \infty$. Now t satisfies

$$j - i - \frac{n}{t} = j - \frac{n}{r}.$$

Hence there are Sobolev embeddings $W^{k,q} \hookrightarrow W^{i,s}$ and $W^{j,r} \hookrightarrow W^{j-i,t}$ and, by Hölder's inequality,

$$\left\|(\partial^\alpha f)(\partial^\beta g)\right\|_{L^p} \leq \|f\|_{W^{i,s}} \|g\|_{W^{j-i,t}} \leq c \|f\|_{W^{k,q}} \|g\|_{W^{j,r}}.$$

Take the sum over all multi-indices α and β with $|\alpha| + |\beta| = k$ to obtain the required estimate (i).

To prove (ii) assume that $kq < n$ and $j - n/p = k - n/q < 0$. If s, t are defined by (C.6) as before and r is as in (i) then $r = n/j$ and $p \le s < \infty$. However, care must be taken when $i = j$. This is the only case where $s = p$ and hence $t = \infty$. In this case $W^{j,r} = W^{j,n/j}$ does not embed into $W^{j-i,t} = L^\infty$ and one obtains

$$\|(\partial^\alpha f)g\|_{L^p} \le \|f\|_{W^{j,p}} \|g\|_{L^\infty} \le c\,\|f\|_{W^{k,q}} \|g\|_{L^\infty}$$

for $|\alpha| = j$. This proves (ii).

Now assume $kq = n$ and proceed as before with $|\alpha| = i$, $|\beta| = j - i$, and $s = n/i$, $t = np/(n - ip)$. Then $p \le s \le \infty$ and the case $s = \infty$ occurs with $i = 0$. In this case

$$\left\|f(\partial^\beta g)\right\|_{L^p} \le \|f\|_{L^{p(p+\varepsilon)/\varepsilon}} \|g\|_{W^{j,p+\varepsilon}} \le c\,\|f\|_{W^{k,q}} \|g\|_{W^{j,p+\varepsilon}}$$

for $|\alpha| = j$. The other cases can be treated as in (i), (ii). This proves (iii).

For $kq > n$ the argument is as above with $s = n/i$ and $t = np/(n - ip)$. Then $1/s + 1/t = 1/p$ as before and

$$i - \frac{s}{n} = 0 < k - \frac{n}{q}, \qquad j - i - \frac{n}{t} = j - \frac{n}{p}.$$

Hence there is a Sobolev embedding $W^{k,q} \hookrightarrow W^{i,s}$ but there only is an inclusion $W^{j,p} \hookrightarrow W^{j-i,t}$ as long as $t \ne \infty$. The latter case may occur if $ip = n$ and $s = p$. However, this can be resolved by choosing $s > p$ and $t = sp/(s - p)$ but with s so close to p that there is still an inclusion $W^{k,q} \hookrightarrow W^{i,s}$ The rest of the argument is as before and is left to the reader. This proves the lemma. □

Trace theorems

It is somewhat nontrivial to understand the restriction of functions with weak derivatives to lower-dimensional submanifolds. For example the obvious fact that the restriction of a C^k-function to a hyperplane is also of class C^k has no analogue in the realm of Sobolev spaces. A $W^{k,p}$-function *loses derivatives* when restricted to the boundary.

Proposition C.20 *For $1 < p < \infty$ and $\Omega \subset \mathbb{R}^n$ with smooth boundary there exists a constant $c = c(p, \Omega) > 0$ such that*

$$\|u\|_{L^p(\partial\Omega)} \le c\,\|u\|_{L^p(\Omega)}^{1-1/p} \|u\|_{W^{1,p}(\Omega)}^{1/p}.$$

for every $u \in C^\infty(\bar{\Omega})$.

Proof: For $x \in \partial\Omega$ let $\nu(x)$ denote the outward unit normal vector. Choose a smooth function $f : \Omega \to \mathbb{R}^n$ such that $f(x) = \nu(x)$ for $x \in \partial\Omega$. Then

$$
\begin{aligned}
\int_{\partial\Omega} |u|^p &= \int_\Omega \mathrm{div}(f|u|^p) \\
&= \int_\Omega \big((\mathrm{div} f)|u|^p + p\langle f, \nabla u\rangle u|u|^{p-2}\big) \\
&\le c_1 \int_\Omega |u|^{p-1}\left(|u| + |\nabla u|\right) \\
&\le c_1 \left(\int_\Omega |u|^p\right)^{(p-1)/p} \left(\int_\Sigma (|u| + |\nabla u|)^p\right)^{1/p} \\
&\le c_2 \left\|u\right\|_{L^p}^{p-1} \left\|u\right\|_{W^{1,p}}.
\end{aligned}
$$

The last but one estimate follows from Hölder's inequality. □

Proposition C.21 *Assume $\partial\Omega$ is a smooth manifold and let $u \in W^{k,p}(\Omega)$. Then $u \in W_0^{k,p}(\Omega)$ if and only if $\partial^\nu u$ vanishes on $\partial\Omega$ for $|\nu| \le k-1$.*

Proof: If $u \in W_0^{k,p}(\Omega)$ then Proposition C.20 shows that $\partial^\nu u$ vanishes on $\partial\Omega$ for $|\nu| \le k-1$. The proof of the converse is an exercise with hints. It is enough to consider the case $k = 1$. Assume that $u \in W^{1,p}(\Omega)$ vanishes on $\partial\Omega$. Choose a family of smooth cutoff functions $\beta_\delta : \Omega \to [0,1]$ such that $\beta_\delta(x) = 1$ for $d(x,\partial\Omega) > \delta$ and $\beta_\delta(x) = 0$ for $d(x,\partial\Omega) < \delta/2$. Now prove that $\beta_\delta u$ converges to u in the $W^{1,p}$-norm as δ tends to zero. The tricky part is the estimate

$$
\left\|(\nabla\beta_\delta)u\right\|_{L^p(\Omega)} \le c\left\|\nabla u\right\|_{L^p(\Omega\setminus\Omega_\delta)}
$$

where $\Omega_\delta = \{x \in \Omega \,|\, B_\delta(x) \subset \Omega\}$ and c is independent of u and δ. Finally use the convolution with $\rho_{\delta/4}$ as in (C.1) to approximate u by smooth functions with compact support. □

C.2 Elliptic regularity: L^2-theory

Consider the linear second order differential operator

$$
L = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) \tag{C.7}
$$

where the $a_{ij} : \mathbb{R}^n \to \mathbb{R}$ are smooth functions with $a_{ij} = a_{ji}$. This operator is called **uniformly elliptic** if there exists a constant $\mu > 0$ such that

$$
\mu|\xi|^2 \le \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \le \frac{1}{\mu}|\xi|^2 \tag{C.8}
$$

for all $x, \xi \in \mathbb{R}^n$. On a compact set Q this holds if and only if the matrix with entries $a_{ij}(x)$ is positive definite for all $x \in Q$. It is interesting to consider the corresponding bilinear form $B : W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$ defined by

$$B(u,v) = \int_\Omega \sum_{i,j} \frac{\partial u}{\partial x_i} a_{ij} \frac{\partial v}{\partial x_j} \tag{C.9}$$

for $u, v \in W_0^{1,2}(\Omega)$. Integration by parts shows that $B(u,\varphi) = \langle u, L\varphi\rangle$ for $u, \varphi \in C_0^\infty(\Omega)$. Since both sides of the equation depend continuously on u with respect to the $W^{1,2}$-norm the identity continues to hold for all $u \in W_0^{1,1}(\Omega)$. Hence $u \in W_0^{1,2}(\Omega)$ is a weak solution of $Lu = f$ if and only if

$$B(u,\varphi) = \langle f, \varphi\rangle \tag{C.10}$$

for all $\varphi \in C_0^\infty(\Omega)$. Now the ellipticity condition (C.8) shows that $B(u,u) \geq \mu \left\|\nabla u\right\|_{L^2}^2$ and hence, by Poincaré's inequality in Lemma C.4

$$B(u,u) \geq \delta \left\|u\right\|_{W^{1,2}}^2 \tag{C.11}$$

for every $u \in W_0^{1,2}(\Omega)$, provided that $\delta > 0$ is chosen sufficiently small. This is the **Gårding inequality**. It shows, for example, that every weak solution $u \in W_0^{1,2}(\Omega)$ of $Lu = 0$ must vanish. Thus, for every $f \in L^2(\Omega)$ the equation $Lu = f$ has at most one weak solution $u \in W_0^{1,2}(\Omega)$. On the other hand, it follows from the Riesz representation theorem (for the functional $\varphi \mapsto \langle f, \varphi\rangle$ on the Hilbert space $W_0^{1,2}(\Omega)$ with inner product B) that such a weak solution exists. The central problem is to prove that every weak solution is regular (i.e. is of class $W^{k+2,2}$ whenever f is of class $W^{k,2}$).

Theorem C.22. (Interior regularity) *Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with smooth boundary and L be an elliptic operator on Ω with C^{k+1}-coefficients satisfying (C.8). Suppose that $u \in W_{\mathrm{loc}}^{1,2}(\Omega)$ is a weak solution of $Lu = f$ with $f \in W_{\mathrm{loc}}^{k,2}(\Omega)$, i.e.*

$$B(u,\varphi) = \langle f, \varphi\rangle$$

for all $\varphi \in C_0^\infty(\Omega)$. Then $u \in W_{\mathrm{loc}}^{k+2,2}(\Omega)$ and $Lu = f$. Moreover, for every compact subset $K \subset \Omega$ and every integer $k \geq 0$, there exists a constant $c = c(K, \Omega, L, k) > 0$ such that

$$\left\|u\right\|_{W^{k+2,2}(K)} \leq c \left(\left\|Lu\right\|_{W^{k,2}(\Omega)} + \left\|u\right\|_{L^2(\Omega)}\right)$$

for $u \in W_{\mathrm{loc}}^{k+2,2}(\Omega)$.

Lemma C.23 *Let Ω and L be as in Theorem C.22 (with $k = 0$) and let $B : W_{\mathrm{loc}}^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \to \mathbb{R}$ be defined by (C.9). Then there exists a family* *of constants $c_{K,\Omega} > 0$, one for every compact subset $K \subset \Omega$, such that the following holds. If $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ satisfies the inequality*

$$B(u, \varphi) \leq A \|\varphi\|_{L^2}$$

for all $\varphi \in W^{1,2}_0(\Omega)$ and some constant $A > 0$, then $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$ and

$$\|u\|_{W^{2,2}(K)} \leq c_{K,\Omega} \left(A + \|u\|_{L^2(\Omega)} \right)$$

for every compact subset $K \subset \Omega$.

Proof: The proof consists of four steps.

Step 1: *We can assume without loss of generality that u has compact support.*

Let $\zeta : \Omega \to [0, 1]$ be a cutoff function which is equal to 1 on some given compact subset $K \subset \Omega$ and vanishes near $\partial\Omega$. Then there is a constant $c > 0$, depending only on ζ and L, such that

$$|B(\zeta u, \varphi) - B(u, \zeta\varphi)| \leq c \|u\|_{W^{1,2}} \|\varphi\|_{L^2}$$

for all $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and all $\varphi \in C^\infty_0(\Omega)$. Step 1 follows immediately from this inequality inequality. The proof of the inequality is left to the reader.

Step 2: *Suppose that $u : \Omega \to \mathbb{R}$ has compact support, extend u to R^n by $u(x) = 0$ for $x \notin \Omega$, and define the difference quotient*

$$u^h(x) = \frac{u(x + he_\ell) - u(x)}{h}$$

where e_ℓ denotes the standard basis vector in $\mathbb{R}^n$. This difference quotient has the following properties:

(i) $(u + v)^h = u^h + v^h$.

(ii) $(uv)^h = u^h \tilde{v} + uv^h$ *where* $\tilde{v}(x) = v(x + he_\ell)$.

(iii) $\partial^\nu(u^h) = (\partial^\nu u)^h$.

(iv) $\int_\Omega uv^h = -\int_\Omega u^{-h} v$ *for h sufficiently small.*

(v) *If $u \in W^{k,p}(\Omega)$ has compact support then*

$$\left\|u^h\right\|_{W^{k-1,p}(\Omega)} \leq \|u\|_{W^{k,p}(\Omega)}$$

for h sufficiently small.

(vi) *If there exist constants $\delta > 0$ and $c > 0$ such that $\left\|u^h\right\|_{W^{k,p}(\Omega)} \leq c$ for $|h| < \delta$ then $\partial_\ell u \in W^{k,p}(\Omega)$ and $\|\partial_\ell u\|_{W^{k,p}(\Omega)} \leq c$.*

These are standard observations and the proofs are straight forward. The last assertion follows from Alaoglu's theorem: Every bounded sequence in a reflexive Banach space has a weakly convergent subsequence. Applying this to the sequence u^{h_i} with $h_i \to 0$ we obtain a subsequence converging weakly to the weak derivative $\partial_\ell u$.

Step 3: *For $u, \varphi \in W_0^{1,2}(\Omega)$ and h sufficiently small we have*

$$\tilde{B}(u^h, \varphi) + B(\mathfrak{u}, \varphi^{-h}) = -\int_\Omega \sum_{i,j} (\partial_i \varphi)(a_{ij})^h (\partial_j u),$$

where $\tilde{B}(u,v) = \int_\Omega (\partial_i u)\tilde{a}_{ij}(\partial_j u)$ with $\tilde{a}_{ij}(x) = a_{ij}(x + he_\ell)$.

The proof uses the rules of Step 2 and is left as an exercise.

Step 4: *There exist constants $c > 0$ and $\varepsilon > 0$ such that*

$$\left\| u^h \right\|_{W^{1,2}} \le c\,(A + \|u\|_{W^{1,2}})$$

for $|h| < \varepsilon$.

This is the crucial step of the proof. From Step 3 we have an inequality

$$\tilde{B}(\varphi, u^h) \le A \left\| \varphi^{-h} \right\|_{L^2} + c_1 \|\varphi\|_{W^{1,2}} \|u\|_{W^{1,2}} \le \|\varphi\|_{W^{1,2}} (A + c_1 \|u\|_{W^{1,2}})$$

for all $\varphi \in C_0^\infty(\Omega)$ and all sufficiently small constants $h \in \mathbb{R}$. Here the constant $c_1 > 0$ depends on the C^1-norm of the coefficients a_{ij}. Now fix h such that u^h is supported in Ω and choose a sequence $\varphi_\nu \in C_0^\infty(\Omega)$ converging to u^h in the $W^{1,2}$-norm. Then in the limit we obtain the inequality

$$\tilde{B}(u^h, u^h) \le \left\| u^h \right\|_{W^{1,2}} (A + c_1 \|u\|_{W^{1,2}}).$$

On the other hand it follows from the Gårding inequality (C.11) that

$$\delta \left\| u^h \right\|_{W^{1,2}}^2 \le B(u^h, u^h).$$

Combining these last two inequalities we find

$$\left\| u^h \right\|_{W^{1,2}} \le \delta^{-1} (A + c_1 \|u\|_{W^{1,2}}).$$

This proves Step 4. The result now follows immediately from Step 4 and Step 2 (vi). □

Proof of Theorem C.22: For $k = 0$ the assertion follows immediately from Lemma C.23 with $A = \|Lu\|_{L^2(\Omega)} = \|f\|_{L^2(\Omega)}$. Now suppose, by induction, that the result has been proved for some $k \ge 0$. Then we know that $u \in W^{k+2,2}_{\mathrm{loc}}(\Omega)$. In particular, $\partial_\ell u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ satisfies the following crucial identity

$$B(\partial_\ell u, \varphi) + B(u, \partial_\ell \varphi) = -\int_\Omega \sum_{i,j} (\partial_i u)(\partial_\ell a_{ij})(\partial_j \varphi).$$

Since $f \in W^{K+1,2}_{\mathrm{loc}}(\Omega)$, we can integrate by parts and obtain

$$B(\partial_\ell u, \varphi) = \int_\Omega \varphi \Big(\partial_\ell f + \sum_{i,j} \partial_j ((\partial_i u)(\partial_\ell a_{ij})) \Big) = \int_\Omega \varphi f',$$

for all $\varphi \in C_0^\infty(\Omega)$, where $f' \in W^{k,2}_{\mathrm{loc}}(\Omega)$. Hence the induction hypothesis shows that $\partial_\ell u \in W^{k+2,2}_{\mathrm{loc}}(\Omega)$. Moreover, applying the induction hypothesis to the compact subset $K \subset \Omega'$ where Ω' is an open set with $\overline{\Omega'} \subset \Omega$, we obtain the estimate

$$\begin{aligned}
\|\partial_\ell u\|_{W^{k+2,2}(K)} &\le c_1 \left(\|f'\|_{W^{k,2}(\Omega')} + \|\partial_\ell u\|_{L^2(\Omega')} \right) \\
&\le c_2 \left(\|f\|_{W^{k+1,2}(\Omega')} + \|u\|_{W^{k+2,2}(\Omega')} \right) \\
&\le c_3 \left(\|f\|_{W^{k+1,2}(\Omega)} + \|u\|_{L^2(\Omega)} \right).
\end{aligned}$$

This proves the theorem. □

Theorem C.24. (Boundary regularity) *Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with smooth boundary and L be an elliptic operator on Ω with C^{k+1}-coefficients satisfying (C.8). Suppose that $u \in W^{1,2}_0(\Omega)$ is a weak solution of $Lu = f$ with $f \in W^{k,2}(\Omega)$, i.e.*

$$B(u, \varphi) = \langle f, \varphi \rangle$$

for all $\varphi \in C_0^\infty(\Omega)$. Then $u \in W^{k+2,2}(\Omega)$ and $Lu = f$. Moreover, for every integer $k \ge 2$, there exists a constant $c = c(\Omega, L, k) > 0$ such that

$$\|u\|_{W^{k+2,2}(K)} \le c \left(\|Lu\|_{W^{k,2}(\Omega)} + \|u\|_{L^2(\Omega)} \right)$$

for $u \in W^{k+2,2}(\Omega) \cap W^{1,2}_0(\Omega)$.

We will not give a proof of this result. It can easily be established with the same techniques as Theorem C.22. The difficult part of the proof is again the case $k = 1$. The rest follows by induction as above. The main idea is to localize the argument and change coordinates near a boundary point such that $\partial\Omega \cong \mathbb{R}^{n-1}$ in the new coordinates. In these new coordinates the operator is still elliptic and one can again employ the difference quotient technique to establish the existence of the $n-1$ additional tangential derivatives, i.e. $\partial_i \partial_j u \in L^2$ for all j and all $i \le n-1$. The only missing derivative is the second derivative $\partial_n \partial_n u$ in the direction normal to the boundary, and it follows from the equation $Lu = f$ that $\partial_n \partial_n u \in L^2$. The details of these arguments will not be carried out here.

Corollary C.25 *The operator*

$$L : W^{k+2,2}(\Omega) \cap W_0^{1,2}(\Omega) \to W^{k,2}(\Omega)$$

*is bijective. Moreover, for every smooth function $f : \bar{\Omega} \to \mathbb{R}$ there exists a unique smooth solution $u : \bar{\Omega} \to \mathbb{R}$ of the equation $Lu = f$ which vanishes on the boundary (***Dirichlet boundary condition***).*

Proof: Given $f \in L^2(\Omega)$ choose $u \in W_0^{1,2}(\Omega)$ such that $B(u, \varphi) = \langle f, \varphi \rangle$ for all $\varphi \in W_0^{1,2}(\Omega)$. Then, by Theorem C.22, if $f \in W^{k,2}(\Omega)$ then $u \in W^{k+2,2}(\Omega)$. This shows that L is onto. Injectivity is obvious. Moreover, if f is smooth then $u \in W^{k,2}(\Omega)$ for all k. Hence, by Proposition C.21, u is smooth and vanishes on the boundary. □

C.3 The Calderón-Zygmund inequality

Denote by

$$\Delta = -\frac{\partial^2}{\partial {x_1}^2} - \frac{\partial^2}{\partial {x_2}^2} \cdots - \frac{\partial^2}{\partial {x_n}^2}$$

the Laplace-operator on $\mathbb{R}^n$. A C^2-function $u : \Omega \to \mathbb{R}$ on an open set $\Omega \subset \mathbb{R}^n$ is called **harmonic** if $\Delta u = 0$. Harmonic functions are real analytic. (If $n = 2$ then a function is harmonic iff it is locally the real part of a holomorphic function.) Harmonic functions are characterized by the **mean value property**

$$u(x) = \frac{n}{\omega_n r^2} \int_{B_r(x)} u(\xi)\, d\xi, \qquad B_r(x) \subset \Omega.$$

Here $\omega_n = 2\pi^{n/2}\Gamma(n/2)^{-1}$ is the volume of the unit sphere in $\mathbb{R}^n$. In particular, $\omega_2 = 2\pi$.

The **fundamental solution** of Laplace's equation is

$$K(x) = \begin{cases} (2\pi)^{-1} \log |x|, & n = 2, \\ (2-n)^{-1} \omega_n^{-1} |x|^{2-n}, & n \ge 3. \end{cases}$$

Its first and second derivatives are given by

$$K_j(x) = \frac{x_j}{\omega_n |x|^n}, \qquad K_{jk}(x) = \frac{n x_j x_k}{\omega_n |x|^{n+2}}, \qquad K_{jj}(x) = \frac{n x_j^2 - |x|^2}{\omega_n |x|^{n+2}}$$

where $K_j = \partial K / \partial x_j$ and $K_{jk} = \partial^2 K / \partial x_j \partial x_k$. In particular, $\Delta K = 0$. The function K and its first derivatives K_j are integrable on compact sets while the second derivatives are not. Hence $\partial_j(K * f) = K_j * f$ for compactly supported functions $f : \mathbb{R}^n \to \mathbb{R}$ but there is no such formula for the second derivatives. Moreover, since neither K nor its derivatives are integrable on $\mathbb{R}^n$, care must be taken for functions f which do not have compact support.

Every compactly supported C^2-function $u : \mathbb{R}^n \to \mathbb{R}$ satisfies

$$u = K * \Delta u$$

and $\partial_j u = K_j * \Delta u$, where $*$ denotes convolution. Conversely,

$$\Delta(K * f) = f, \qquad \Delta(K_j * f) = \partial_j f$$

for $f \in C_0^\infty(\mathbb{R}^n)$ (see [53]). This is **Poisson's identity**. In general $K * f$ will not have compact support. Let $f \in L^1_{\mathrm{loc}}(\Omega)$ and call $u \in L^1_{\mathrm{loc}}(\Omega)$ a **weak solution** of $\Delta u = f$ if

$$\int_\Omega u\Delta\varphi = \int_\Omega f\varphi$$

for $\varphi \in C_0^\infty(\Omega)$. Similarly call $u \in L^1_{\mathrm{loc}}(\Omega)$ a **weak solution** of $\Delta u = \partial_j f$ with $f \in L^1_{\mathrm{loc}}$ if

$$\int_\Omega u\Delta\varphi = -\int_\Omega f\partial_j\varphi$$

for $\varphi \in C_0^\infty(\Omega)$.

Lemma C.26 *Let $u, f \in L^1(\mathbb{R}^n)$ with compact support.*

(i) *u is a weak solution of $\Delta u = f$ if and only if $u = K * f$.*

(ii) *u is a weak solution of $\Delta u = \partial_j f$ if and only if $u = K_j * f$.*

Proof: If $u = K * f$ then

$$\int u\Delta\varphi = \int (K * f)\Delta\varphi = \int f(K * \Delta\varphi) = \int f\varphi$$

for $\varphi \in C_0^\infty(\mathbb{R}^n)$. Conversely, suppose that u is a weak solution of $\Delta u = f$. Choose $\rho_\delta : \mathbb{R}^n \to \mathbb{R}$ as in (C.1). Then

$$\int (\Delta\rho_\delta * u)\varphi = \int u(\rho_\delta * \Delta\varphi) = \int f(\rho_\delta * \varphi) = \int (\rho_\delta * f)\varphi$$

for every $\varphi \in C_0^\infty(\mathbb{R}^n)$. Hence $\Delta\rho_\delta * u = \rho_\delta * f$. Since $\rho_\delta * u \in C_0^\infty(\mathbb{R}^n)$ $\rho_\delta * u = K * \rho_\delta * f$. Take the limit $\delta \to 0$ to obtain $u = K * f$. This proves (i). The proof of (ii) is similar and is left to the reader. □

Theorem C.27. (Calderón-Zygmund inequality) *For* $1 < p < \infty$ *there exists a constant* $c = c(n,p) > 0$ *such that*

$$\|\nabla(K_j * f)\|_{L^p} \le c\,\|f\|_{L^p} \tag{C.12}$$

for $f \in C_0^\infty(\mathbb{R}^n)$ *and* $j = 1,\dots,n$.

This theorem is the fundamental estimate for the L^p-theory of elliptic operators. We include here a proof following Gilbarg and Trudinger [39]. The proof requires the following three lemmata. The first is the case $p = 2$.

Lemma C.28 *The estimate (C.12) holds for* $p = 2$ *with* $c = 1$.

Proof: Since $u(x) = K_j * f(x) \to 0$ as $|x| \to \infty$ we have

$$\|\nabla u\|_{L^2}^2 = \langle u, \Delta u\rangle = \langle u, \partial_j f\rangle = -\langle \partial_j u, f\rangle \le \|\nabla u\|_{L^2}\,\|f\|_{L^2}\,.$$

Divide both sides by $\|\nabla u\|_{L^2}$ to obtain the required estimate. □

For any measurable function $f : \mathbb{R}^n \to \mathbb{R}$ define

$$\mu(t,f) = \left|\{x \in \mathbb{R}^2 \,|\, |f(x)| > t\}\right|$$

for $t > 0$ where $|A|$ denotes the Lebesgue measure of the set A.

Lemma C.29 *For* $1 \le p < \infty$ *and* $f \in L^p(\mathbb{R}^n)$

$$t^p \mu(t,f) \le \int |f(x)|^p\,dx = p\int_0^\infty s^{p-1}\mu(s,f)\,ds.$$

Moreover, $\mu(t, f+g) \le \mu(t/2, f) + \mu(t/2, g)$.

Proof: Integrate the function $F : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by $F(x,t) = pt^{p-1}$ for $0 \le t \le |f(x)|$ and $F(x,t) = 0$ otherwise. □

Apply the previous Lemma to the function $\partial_k(K_j * f)$. By Lemma C.28,

$$\|\partial_k(K_j * f)\|_{L^2} \le \|f\|_{L^2}$$

and hence

$$\mu(t, \partial_k(K_j * f)) \le \frac{1}{t^2}\int |f(x)|^2\,dx \tag{C.13}$$

The next lemma establishes a similar inequality with the L^2-norm on the right replaced by the L^1-norm. Theorem C.27 is then proved by intepolation for $1 < p < 2$.

Lemma C.30 *There exists a constant* $c = c(n) > 0$ *such that every function* $f \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ *satisfies the following estimate for* $j,k = 1,\dots,n$:

$$\mu(t, \partial_k(K_j * f)) \le \frac{c}{t}\int |f(x)|\,dx.$$

Proof: The proof has three steps. We abbreviate $Tf := \partial_k(\partial_j K * f)$.

STEP 1. *There exists a constant $c = c(n) > 0$ such that the following holds. Let B be a countable union of closed cubes $Q_i \subset \mathbb{R}^n$ with disjoint interiors. Suppose that $h \in L^1(\mathbb{R}^n)$ has support in B and satisfies*

$$\int_{Q_i} h = 0$$

for every i. Then

$$\mu(t, Th) \le c\left(\mathrm{Vol}(B) + \frac{1}{t}\left\|h\right\|_{L^1}\right).$$

Denote by $h_i \in L^1(\mathbb{R}^n)$ the function which is equal to h on Q_i and equal to zero on $\mathbb{R}^n \setminus Q_i$. Let q_i be the center of Q_i and suppose that Q_i has sidelength $2r_i$. Then the maximal distance of any point in Q_i to q_i is $\sqrt{n}r_i$. Hence, for $x \notin Q_i$, we have

$$\begin{aligned}
|Th_i(x)| &= \left|\int_{Q_i} \left(\partial_k K_j(x-y) - \partial_k K_j(x-q_i)\right) h_i(y)\,dy\right| \\
&\le \max_{y\in Q_i} \left|\partial_k K_j(x-y) - \partial_k K_j(x-q_i)\right| \left\|h\right\|_{L^1(Q_i)} \\
&\le \sqrt{n}r_i \max_{y\in Q_i} \left|\nabla\partial_k K_j(x-y)\right| \left\|h\right\|_{L^1(Q_i)} \\
&\le c_1 r_i \max_{y\in Q_i} \frac{1}{|x-y|^{n+1}} \left\|h\right\|_{L^1(Q_i)} \\
&\le \frac{c_1 r_i}{d(x,Q_i)^{n+1}} \left\|h\right\|_{L^1(Q_i)}.
\end{aligned}$$

(We denote by c_1, c_2, c_3 constants which depend only on n.) Let

$$P_i := \left\{x \in \mathbb{R}^n \,|\, |x-q_i| < 2\sqrt{n}r_i\right\} \supset Q_i.$$

Then $d(x, Q_i) \ge |x-q_i| - \sqrt{n}r_i$ for $x \in \mathbb{R}^n \setminus P_i$. Hence

$$\begin{aligned}
\int_{\mathbb{R}^n\setminus P_i} |Th_i|\,dx &\le c_1 r_i \int_{|x|>2\sqrt{n}r_i} \frac{dx}{\left(|x| - \sqrt{n}r_i\right)^{n+1}} \left\|h\right\|_{L^1(Q_i)} \\
&= c_1 r_i \int_{2\sqrt{n}r_i}^{\infty} \frac{\omega_n \rho^{n-1}\,d\rho}{(\rho - \sqrt{n}r_i)^{n+1}} \left\|h\right\|_{L^1(Q_i)} \\
&\le c_1 \omega_n 2^{n-1} r_i \int_{\sqrt{n}r_i}^{\infty} \frac{d\rho}{\rho^2} \left\|h\right\|_{L^1(Q_i)} \\
&= c_2 \left\|h\right\|_{L^1(Q_i)}.
\end{aligned}$$

Hence, with $A := \bigcup_i P_i$ we obtain

$$\int_{\mathbb{R}^n \setminus A} |Th| \, dx \le \sum_i \int_{\mathbb{R}^n \setminus P_i} |Th_i| \, dx \le c_2 \sum_i \|h\|_{L^1(Q_i)} = c_2 \|h\|_{L^1} .$$

Since $\mathrm{Vol}(A) \le \sum_i \mathrm{Vol}(P_i) = c_3 \sum_i \mathrm{Vol}(Q_i) = c_3 \mathrm{Vol}(B)$, it follows that

$$\begin{aligned} t\mu(t, Th) &\le t\mathrm{Vol}(A) + t\,|\{x \in \mathbb{R}^n \setminus A \,|\, |Th(x)| > t\}| \\ &\le t\mathrm{Vol}(A) + \int_{\mathbb{R}^n \setminus A} |Th(x)| \, dx \\ &\le c_4 \left(t\mathrm{Vol}(B) + \|h\|_{L^1} \right), \end{aligned}$$

where $c_4 := \max\{c_2, c_3\}$. This proves Step 1.

Step 2. *Let $f \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and $t > 0$. Then there exists a countable collection of closed cubes $Q_i \subset \mathbb{R}^n$ with disjoint interiors satisfying the following.*

(i) $t\mathrm{Vol}(Q_i) < \|f\|_{L^1(Q_i)} \le 2^n t\mathrm{Vol}(Q_i)$ *for every i.*

(ii) $|f(x)| \le t$ *for almost every $x \in \mathbb{R}^n \setminus B$, where $B := \bigcup_i Q_i$.*

For $k \in \mathbb{Z}^n$ and $\ell \in \mathbb{Z}$ denote

$$Q(k, \ell) := \left\{ x \in \mathbb{R}^n \,|\, 2^{-\ell} k_i \le x_i \le 2^{-\ell}(k_i + 1),\, i = 1, \dots, n \right\}.$$

Let

$$\mathcal{Q} := \{Q(k, \ell) \,|\, k \in \mathbb{Z}^n,\, \ell \in \mathbb{Z}\}$$

and $\mathcal{Q}_0 \subset \mathcal{Q}$ be the set of all $Q \in \mathcal{Q}$ satisfying

$$t\mathrm{Vol}(Q) < \|f\|_{L^1(Q)}$$

and

$$Q \subsetneq Q' \in \mathcal{Q} \qquad \Longrightarrow \qquad \|f\|_{L^1(Q')} \le t\mathrm{Vol}(Q').$$

Then every decreasing sequence of cubes in $\mathcal{Q}$ contains at most one element of $\mathcal{Q}_0$. Hence every $Q \in \mathcal{Q}_0$ satisfies assertion (i) and any two cubes in $\mathcal{Q}_0$ have disjoint interiors. Now let

$$B := \bigcup_{Q \in \mathcal{Q}_0} Q.$$

Then

$$x \in \mathbb{R}^n \setminus B, \quad x \in Q \in \mathcal{Q} \qquad \Longrightarrow \qquad \frac{1}{\mathrm{Vol}(Q)} \|f\|_{L^1(Q)} \le t.$$

(Otherwise take a maximal cube $Q \in \mathcal{Q}$ that satisfies $t\mathrm{Vol}(Q) < \|f\|_{L^1(Q)}$ and contains x. This cube would belong to $\mathcal{Q}_0$ and so $x \in B$.) Thus we have proved that, for every $x \in \mathbb{R}^n \setminus B$, there is a sequence of decreasing cubes $Q_\ell \in \mathcal{Q}$ containing x such that $\mathrm{Vol}(Q_\ell)^{-1}\|f\|_{L^1(Q_\ell)} \le t$. Hence it follows from Lebesgue's differentiation theorem that $|f(x)| \le t$ for almost every $x \in \mathbb{R}^n \setminus B$. This proves Step 2.

Step 3. *We prove the lemma.*

Fix a constant $t > 0$, let the Q_i be as in Step 2, and denote $B := \bigcup_i Q_i$. Then, by Step 2, $t\mathrm{Vol}(B) \le \|f\|_{L^1}$. Define $g, h : \mathbb{R}^n \to \mathbb{R}$ by

$$g(x) := \begin{cases} f(x), & \text{for } x \notin B, \\ \mathrm{Vol}(Q_i)^{-1}\int_{Q_i} f, & \text{for } x \in Q_i, \end{cases} \qquad h := f - g.$$

Then $\|g\|_{L^1} \le \|f\|_{L^1}$ and $\|h\|_{L^1} \le 2\|f\|_{L^1}$. Moreover, h vanishes in $\mathbb{R}^n \setminus B$ and has mean value zero in each cube Q_i. Hence h satisfies the requirements of Step 1. Hence there exists a constant c, depending only on n, such that

$$\mu(t, Th) \le c\left(\mathrm{Vol}(B) + \frac{1}{t}\|h\|_{L^1}\right) \le \frac{3c}{t}\|f\|_{L^1}.$$

Moreover, it follows from Step 2 that $|g(x)| \le 2^n t$ for almost every $x \in \mathbb{R}^n$. Hence, by Lemma C.29,

$$\mu(t, Tg) \le \frac{\|g\|_{L^2}^2}{t^2} \le \frac{2^n\|g\|_{L^1}}{t} \le \frac{2^n\|f\|_{L^1}}{t}.$$

Combining these inequalities we obtain from Lemma C.29 that

$$\mu(2t, Tf) \le \mu(t, Tg) + \mu(t, Th) \le \frac{2^{n+1} + 6c}{2t}\|f\|_{L^1}.$$

This proves Lemma C.30. □

Proof of Theorem C.27: First assume $1 < p < 2$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function with compact support. Then by Lemma C.29

$$\begin{aligned}\int_{\mathbb{R}^n} |\partial_k(K_j * f)(x)|^p\,dx &= p\int_0^\infty t^{p-1}\mu(t, \partial_k(K_j * f))\,dt \\ &\le \left(pc\int_0^1 t^{p-2}\,dt + p\int_1^\infty t^{p-3}\,dt\right)\|f\|_{L^1} \\ &= \left(\frac{pc}{p-1} + \frac{p}{2-p}\right)\|f\|_{L^p}.\end{aligned}$$

Here we have used Lemma C.30 for $t < 1$ and (C.13) for $t > 1$. This proves the estimate for $1 < p < 2$. For $2 < p < \infty$ we use duality. Let $1 < q < 2$ such that $1/p + 1/q = 1$. Then

$$\int g(x)\partial_k(K_j * f)(x)\,dx = \int \partial_k(K_j * g)(x)f(x) \le c\,\|f\|_{L^p}\,\|g\|_{L^q}$$

and hence $\|\partial_k(K_j * f)\|_{L^p} \le \|f\|_{L^p}$. □

C.4 Elliptic regularity: L^p-theory

Theorem C.31. (Elliptic estimate) *Let $1 < p < \infty$, $k \ge 0$ be an integer, and $\Omega \subset \mathbb{R}^n$ be an open domain. Let L be a uniformly elliptic differential operator of the form (C.7) on Ω. Then for every compact subset $Q \subset \Omega$ there exists a constant $c > 0$ such that*

$$\|u\|_{W^{k+2,p}(Q)} \le c\left(\|Lu\|_{W^{k,p}(\Omega)} + \|u\|_{L^p(\Omega)}\right)$$

for $u \in C^\infty(\bar{\Omega})$. Moreover, the inequality continues to hold for $Q = \Omega$ if u vanishes on the boundary.

Proof: We prove the inequality for $k = 0$. The general case then follows easily by induction. Assume first that L has constant coefficients. Then, by change of variables, we may assume that $L = \Delta$ is the standard Laplacian. Choose an open neighborhood U of Q such that $\mathrm{cl}(U) \subset \Omega$. Let $\beta \in C_0^\infty(\Omega)$ be a smooth cutoff function such that $\beta(x) = 1$ for $x \in U$. Then, by Theorem C.27, the function $v = K * \beta\Delta u$ satisfies an estimate

$$\|v\|_{W^{2,p}(U)} \le c_1\,\|\beta\Delta u\|_{L^p(\Omega)} \le c_2\,\|\Delta u\|_{L^p(\Omega)}\,.$$

The function $v - u$ is harmonic in U. By the mean value property for harmonic functions there exists a constant $c_3 > 0$ such that

$$\|v-u\|_{W^{2,p}(Q)} \le c_3\,\|v-u\|_{L^p(U)} \le c_3\left(\|v\|_{L^p(U)} + \|u\|_{L^p(U)}\right).$$

Hence

$$\begin{aligned}\|u\|_{W^{2,p}(Q)} &\le \|v\|_{W^{2,p}(Q)} + \|v-u\|_{W^{2,p}(Q)}\\ &\le c_4\left(\|v\|_{W^{2,p}(U)} + \|u\|_{L^p(U)}\right)\\ &\le c_5\left(\|\Delta u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}\right).\end{aligned}$$

This proves the result (for $k = 0$) in the case of constant coefficients. Moreover, the constant c_5 depends continuously on the coefficients and hence can be chosen independent of them as long as (C.8) is satisfied. Next we prove that there exist constants $c > 0$ and $r > 0$ such that for every point $x_0 \in Q$ we have

$$\mathrm{supp}(u) \subset B_r(x_0) \qquad \Longrightarrow \qquad \|u\|_{W^{2,p}} \le c\left(\|Lu\|_{L^p} + \|u\|_{L^p}\right). \quad \text{(C.14)}$$

We have already proved that this holds (with a uniform constant c) if L is replaced by the operator L_0 with constant coefficients $a_{0ij} \equiv a_{ij}(x_0)$. In general, the inequality (C.14) follows from the fact that

$$\|L_0 u - Lu\|_{L^p} \leq \varepsilon \|u\|_{W^{2,p}} + c' \|u\|_{L^p}$$

whenever $r > 0$ is sufficiently small. Note here that the first order terms can be estimated by the interpolation inequality

$$\|u\|_{W^{1,p}} \leq c \|u\|_{W^{2,p}}^{1/2} \|u\|_{L^p}^{1/2}$$

of Proposition C.16 (with $p = q = r$, $j = 1$, $k = 2$). This proves (C.14) and the result for $k = 0$ now follows by a standard partition of unity argument which is left to the reader.

To prove the result with $Q = \Omega$ and u vanishing on $\partial\Omega$ cover the boundary by finitely many coordinate charts which map $\partial\Omega$ to a hyperplane. Then use a reflection argument to prove the estimate for operators which in these coordinates have constant coefficients. More precisely, u is a smooth function on $x_n \geq 0$ which vanishes on $x_n = 0$ then the function

$$u(x_1, \ldots, x_{n-1}, -x_n) = -u(x_1, \ldots, x_n)$$

is twice continuously differentiable and hence satisfies the required estimate for operators with constant coefficients. The general case can then be reduced to that of constant coefficients by the same arguments as above. This proves the inequality for $k = 0$. For general k it follows by a standard induction argument involving an estimate for $LDu - DLu$ where D is a first order differential operator given by differentiation in the direction of a vector field which is tangent to $\partial\Omega$. □

Theorem C.32. (Elliptic regularity) *Fix constants $1 < p, q < \infty$ with $1/p + 1/q = 1$. Let Ω and L be as in Theorem C.31. If $u \in W_0^{1,p}(\Omega)$ is a weak solution of $Lu = f$ with $f \in W^{k,p}(\Omega)$, i.e. $\langle u, L\varphi\rangle = \langle f, \varphi\rangle$ for all $\varphi \in C_0^\infty(\Omega)$, then $u \in W^{k+2,p}(\Omega)$.*

Proof: We only prove the result for $p \geq 2$. The case $p \leq 2$ can be reduced to this by duality. This argument will be omitted. Given $f \in W^{k,p}(\Omega)$ choose a sequence f_i of smooth functions converging to f in the $W^{k,p}$-norm. Then, by Corollary C.25, there exists a unique smooth function $u_i : \bar{\Omega} \to \mathbb{R}$ which satisfies $Lu_i = f_i$ and vanishes on the boundary. Moreover, Theorem C.31 shows that there is an estimate

$$\|u_i\|_{W^{k+2,p}(\Omega)} \leq c \left(\|f_i\|_{W^{k,p}(\Omega)} + \|u_i\|_{L^p(\Omega)} \right)$$

Hence u_i has a subsequence converging weakly in $W^{k+2,p}(\Omega)$. Call the limit u_0. Then $u_0 \in W^{k+2,p}(\Omega)$ and $Lu_0 = f$. Hence $B(u - u_0, \varphi) =$

$\langle u-u_0, L\varphi\rangle = 0$ for all $\varphi \in C_0^\infty(\Omega)$ and, by choosing a sequence $\varphi_i \to u-u_0$ in $W_0^{1,2}(\Omega)$, we obtain $u = u_0$. □

Remark C.33 In view of Lemma A.1 and Rellich's theorem the estimate of Theorem C.31 shows that the operator

$$L : W^{k+2,p}(\Omega) \cap W_0^{1,p}(\Omega) \to W^{k,p}(\Omega)$$

has finite dimensional kernel and closed range. The elliptic regularity theorem is equivalent to the assertion that this operator has finite dimensional cokernel. In fact, it follows that the operator is bijective. □

Remark C.34 Elliptic regularity for systems of PDEs can easily be reduced to Theorem C.31 whenever the highest order terms have diagonal form. □

Corollary C.35 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary. Assume $u \in W^{k,p}(\Omega)$ with $k \geq 1$. The following are equivalent.*

(i) *$u \in W_0^{1,p}(\Omega)$.*

(ii) *u vanishes on $\partial\Omega$.*

(iii) *There exists a sequence $u_\nu \in C^\infty(\bar{\Omega})$ such that*

$$\lim_{\nu\to\infty} \|u_\nu - u\|_{W^{k,p}} = 0, \qquad u_\nu|\partial\Omega = 0.$$

In particular, the intersection $W^{k,p}(\Omega) \cap W_0^{1,p}(\Omega)$ is a closed linear subspace of $W^{k,p}(\Omega)$.

Proof: Proposition C.21 and the proof of Theorem C.32. □

APPENDIX D

THE KAZDAN-WARNER EQUATION

Let X be a compact connected oriented Riemannian n-manifold and denote by $\Delta = d^*d$ the Laplace-Beltrami operator. The **Kazdan-Warner** equation has the form

$$\Delta u + e^u h = f. \tag{D.1}$$

Here f and h are given real valued functions on X and the goal is to find a solution $u : X \to \mathbb{R}$.

Theorem D.1. (Kazdan-Warner) *Fix a constant $p > n/2$. Let $h, f \in L^p(X)$ such that*

$$\inf_X \varphi \geq 0 \qquad \Longrightarrow \qquad \int_X h\varphi \geq 0$$

for every test function $\varphi \in C^\infty(X)$ and

$$\int_X f > 0, \qquad \int_X h > 0.$$

Then (D.1) has a unique solution $u \in W^{2,p}(X, \mathbb{R})$. Moreover, if h is smooth then so is u.

Note that the condition $\int_X f > 0$ is necessary for the existence of a solution u because the function Δu always has mean value zero.

The kernel of the Laplace operator $\Delta : W^{2,p}(X) \to L^p(X)$ consists of the constant functions and hence is 1-dimensional. Moreover its range consists of all functions with mean value zero. Hence the restriction of Δ to the space of functions of mean value zero is invertible. In conjunction with the Sobolev embedding theorem $W^{2,p}(X) \hookrightarrow C^0(X)$ for $p > n/2$ this proves the following.

Lemma D.2 *For every $p > n/2$ there exists a constant $c_0 = c_0(X, p) > 0$ such that the following holds. For every $f_0 \in L^p(X)$ with $\int_X f_0 = 0$ there exists a unique solution $u_0 \in W^{2,p}(X)$ of*

$$\Delta u_0 = f_0, \qquad \int_X u_0 = 0.$$

This solution satisfies $\|u_0\|_{L^\infty} \leq c_0 \|f_0\|_{L^p}$.

The assertion of Theorem D.1 can easily be reduced to the case where f is constant. To see this let

$$A = \frac{1}{\mathrm{Vol}(X)} \int_X f$$

and denote by $u_0 \in W^{2,p}(X)$ the unique solution of

$$\Delta u_0 = f - A, \qquad \int_X u_0 = 0.$$

Then a function $v \in W^{2,p}(X)$ satisfies $\Delta v + he^{u_0}e^v = A$ if and only $u = u_0 + v$ satisfies (D.1). Moreover, the function h is nonnegative with positive mean value if and only if the function he^{u_0} has the same property. Hence it remains to prove Theorem D.1 for the equation

$$\Delta u + e^u h^2 = A \tag{D.2}$$

where A is a positive constant. Here the square is introduced for convenience of the exposition.

Consider the open set

$$\mathcal{H} = \left\{ h \in L^{p/2}(X) \,\Big|\, \int_X h^2 > 0 \right\} \subset L^{p/2}(X)$$

and denote by

$$\mathcal{U} = \left\{ (h,u) \in \mathcal{H} \times W^{2,p}(X) \,|\, \Delta u + e^u h^2 = A \right\}$$

the space of solutions of (D.2). The next lemma shows that this space is a Banach manifold.

Lemma D.3 *The space $\mathcal{U} \subset L^{p/2}(X) \times W^{2,p}(X)$ is a smooth Banach manifold and the projection*

$$\pi : \mathcal{U} \to \mathcal{H}$$

defined by $\pi(h,u) = h$ is a local diffeomorphism near every point in $\mathcal{U}$.

Proof: Consider the smooth map $\mathcal{F} : L^{p/2}(X) \times W^{2,p}(X) \to L^p(X)$ defined by

$$\mathcal{F}(h,u) = \Delta u + e^u h^2 - A.$$

Its differential is given by

$$d\mathcal{F}(h,u)(\hat{h},\hat{u}) = \Delta\hat{u} + e^u h^2 \hat{u} + 2e^u h\hat{h}.$$

Since the function $e^u h^2$ is not identically zero it follows that the operator

$$\Delta + e^u h^2$$

is invertible. To see this note first that this operator is Fredholm and has index zero. (In the L^2 setting it is self-adjoint.) Hence it suffices to prove injectivity. Assume that $\hat{u} \in W^{2,p}(X)$ satisfies

$$\Delta\hat{u} + e^u h^2 \hat{u} = 0.$$

Take the inner product with $\hat{u}$ to obtain

$$\int_X \left(|d\hat{u}|^2 + e^u \, |h\hat{u}|^2 \right) = 0.$$

It follows that $\hat{u}$ vanishes on a set of positive measure, namely where h is nonzero. Moreover, $d\hat{u} \equiv 0$ and hence $\hat{u}$ is constant. By the previous remark this constant must be zero. Hence the operator $\Delta + e^u h^2$ is invertible for every pair $(h, u) \in \mathcal{H} \times W^{2,p}(X)$. Moreover, it is easy to see that $d\mathcal{F}(h, u)$ has a right inverse whenever $h \not\equiv 0$.

This shows that 0 is a regular value of $\mathcal{F}$ and so, by Theorem B.3, the space

$$\mathcal{U} = \mathcal{F}^{-1}(0) \subset L^{p/2}(X) \times W^{2,p}(X)$$

of solutions of (D.2) is a Banach manifold. Its tangent space at (h, u) is the kernel of $d\mathcal{F}(h, u)$:

$$T_{(h,u)}\mathcal{U} = \left\{ (\hat{h}, \hat{u}) \,\middle|\, \Delta\hat{u} + e^u h^2 \hat{u} + 2e^u h\hat{h} = 0 \right\}.$$

Now the linearized projection operator

$$d\pi(h, u) : T_{(h,u)}\mathcal{U} \to L^p(X)$$

is given by $(\hat{h}, \hat{u}) \mapsto \hat{h}$. This operator is bijective if and only if for every $\hat{h} \in L^{p/2}(X)$ there exists a unique $\hat{u} \in W^{2,p}(X)$ such that $(\hat{h}, \hat{u}) \in T_{(h,u)}\mathcal{U}$. This follows again from the invertibility of the operator $\Delta + e^u h^2$. Now the inverse function theorem B.1 shows that every pair $(h, u) \in \mathcal{U}$ has a neighbourhood $\mathcal{V}$ such that the restriction of π to $\mathcal{V}$ is a diffeomorphism $\mathcal{V} \to \pi(\mathcal{V}) \subset \mathcal{H}$. □

The next lemma is the key to the proof of Theorem D.1. It gives an a priori estimate for the solutions of (D.2).

Lemma D.4 *There exists a function $\varphi : [0, \infty)^2 \to (0, \infty)$, depending only on X and p, such that*

$$\|u\|_{L^\infty} \le \varphi\left(\left\|h^2\right\|_{L^p}, \frac{1}{\mathrm{Vol}(X)}\int_X h^2\right)$$

for every solution (h,u) *of (D.2).*

Remark D.5 The proof of Lemma D.4 shows that

$$\varphi(t,B) = \left|\log\left(\frac{A}{B}\right)\right| + 2c_0A\mathrm{Vol}(X)^{1/p} + 2\frac{c_0At}{B}\exp\left(4\frac{c_0At}{B}\right)$$

where

$$t = \left\|h^2\right\|_{L^p}, \qquad B = \frac{1}{\mathrm{Vol}(X)}\int_X h^2$$

and $c_0 > 0$ is as in Lemma D.2. Note that

$$B\mathrm{Vol}(X)^{1/p} \le t.$$

This can be restated in the form that the L^p norm of the mean value is bounded above by the L^p norm of the original function. The proof uses the Hölder inequality. □

Proof of Lemma D.4: The lemma is proved in four steps.

Step 1 *If* $u, h \in C^\infty(X)$ *satisfy (D.2) then*

$$u(x) \le 4\frac{c_0At}{B} + \log\left(\frac{A}{B}\right)$$

for every $x \in X$. *Here* t *and* B *are as in Remark D.5.*

The function $h_0 = h^2 - B$ has mean value zero and hence there is a unique function $v_0 \in C^\infty(X)$ such that

$$\Delta v_0 = -h_0 = B - h^2, \qquad \int_X v_0 = 0.$$

By Lemma D.2, this function satisfies

$$\|v_0\|_{L^\infty} \le c_0\|h_0\|_{L^p} \le c_0\left(\left\|h^2\right\|_{L^p} + B\mathrm{Vol}(X)^{1/p}\right) \le 2c_0t.$$

I claim that

$$u(x) \le \log\left(\frac{A}{B}\right) + \frac{A}{B}\left(v_0(x) + 2c_0t\right) \tag{D.3}$$

for all $x \in X$. The assertion of Step 1 is an immediate consequence of (D.3). To prove (D.3) consider the function

$$w_\varepsilon(x) = \log\left(\frac{A+\varepsilon}{B}\right) + \frac{A+\varepsilon}{B}\left(v_0(x) + 2c_0 t\right) - u(x).$$

Choose a point $x_\varepsilon \in X$ at which this function attains its minimum:

$$w_\varepsilon(x_\varepsilon) = \inf_X w_\varepsilon, \qquad \Delta w_\varepsilon(x_\varepsilon) \le 0.$$

The last inequality holds because the positive definite Laplace-Beltrami operator agrees at a minimum with minus the ordinary Laplace operator (i.e. minus the sum of the second derivatives in an orthonormal frame.) Now

$$\begin{aligned}
0 &\ge \Delta w_\varepsilon(x_\varepsilon) \\
&= \frac{A+\varepsilon}{B}\Delta v_0(x_\varepsilon) - \Delta u(x_\varepsilon) \\
&= \frac{A+\varepsilon}{B}\left(B - h(x_\varepsilon)^2\right) + e^{u(x_\varepsilon)} h(x_\varepsilon)^2 - A \\
&= \varepsilon + h(x_\varepsilon)^2\left(e^{u(x_\varepsilon)} - \frac{A+\varepsilon}{B}\right).
\end{aligned}$$

This implies that $h(x_\varepsilon) \ne 0$ and

$$u(x_\varepsilon) < \log\left(\frac{A+\varepsilon}{B}\right).$$

Hence $w_\varepsilon(x_\varepsilon) > 0$ and so $w_\varepsilon(x) > 0$ for all $x \in X$. The limit $\varepsilon \to 0$ gives the required inequality (D.3).

Step 2 *If $u, h \in C^\infty(X)$ satisfy (D.2) then the function $A - e^u h^2$ has mean value zero. The unique solution $u_0 \in C^\infty(X)$ of*

$$\Delta u_0 = A - e^u h^2, \qquad \int_X u_0 = 0$$

satisfies

$$\|u_0\|_{L^\infty} \le c_0 A \mathrm{Vol}(X)^{1/p} + \frac{c_0 A t}{B} \exp\left(4\frac{c_0 A t}{B}\right).$$

By Lemma D.2

$$\|u_0\|_{L^\infty} \le c_0 \left\|A - e^u h^2\right\|_{L^p} \le c_0 A \mathrm{Vol}(X)^{1/p} + c_0 t e^{\sup u}.$$

Hence the assertion follows from Step 1.

Step 3 *If h, u, and u_0 are as in Step 2 then*

$$u = u_0 - \log\left(\frac{1}{A\mathrm{Vol}(X)}\int_X e^{u_0}h^2\right)$$

and

$$\|u\|_{L^\infty} \le 2\,\|u_0\|_{L^\infty} + \left|\log\left(\frac{A}{B}\right)\right|.$$

Since $\Delta(u-u_0)=0$ it follows that $u=u_0+c$ for some constant c. The value of the constant is determined by the fact that the function $A-e^{u_0+c}h^2$ has mean value zero. Hence

$$c = -\log\left(\frac{1}{A\mathrm{Vol}(X)}\int_X e^{u_0}h^2\right)$$

as claimed. Now observe that

$$\exp\left(-\|u_0\|_{L^\infty}\right)h^2 \le e^{u_0}h^2 \le \exp\left(\|u_0\|_{L^\infty}\right)h^2.$$

Integrating this over X gives rise to the inequality

$$\exp\left(-\|u_0\|_{L^\infty}\right)\frac{B}{A} \le \frac{1}{A\mathrm{Vol}(X)}\int_X e^{u_0}h^2 \le \exp\left(\|u_0\|_{L^\infty}\right)\frac{B}{A}.$$

Taking logarithms one finds

$$|c| \le \|u_0\|_{L^\infty} + \left|\log\left(\frac{A}{B}\right)\right|.$$

Since $u=u_0+c$ this proves Step 3.

Step 4 *Proof of the lemma.*

It follows from Step 2 and Step 3 that every smooth solution (h,u) of (D.2) with $h\not\equiv 0$ satisfies

$$\begin{aligned}
\|u\|_{L^\infty} &\le \left|\log\left(\frac{A}{B}\right)\right| + 2\,\|u_0\|_{L^\infty} \\
&\le \left|\log\left(\frac{A}{B}\right)\right| + 2c_0A\mathrm{Vol}(X)^{1/p} + 2\frac{c_0At}{B}\exp\left(4\frac{c_0At}{B}\right) \\
&= \varphi(t,B).
\end{aligned}$$

This proves the lemma for smooth solutions (h,u).

Now suppose, by contradiction, that there is a pair $(h_0, u_0) \in \mathcal{U}$ with

$$\|u_0\|_{L^\infty} > \varphi\left(\left\|h_0^2\right\|_{L^p}, \frac{1}{\mathrm{Vol}(X)}\int_X h_0^2\right).$$

Consider the projection $\pi : \mathcal{U} \to \mathcal{H}$ and let $\mathcal{H} \to W^{2,p}(X) : h \mapsto u_h$ be a local inverse which assigns to every $h \in L^{p/2}(X)$ near h_0 the unique solution $u = u_h$ of (D.2) near u_0. This map is continuous and hence

$$\|u_h\|_{L^\infty} > \varphi\left(\left\|h^2\right\|_{L^p}, \frac{1}{\mathrm{Vol}(X)}\int_X h^2\right)$$

for h sufficiently close to h_0. Choose a smooth function h near h_0. Then, by elliptic regularity, the function u is also smooth and hence the inequality contradicts the first part of the proof. □

Proof of Theorem D.1: Consider the projection

$$\pi : \mathcal{U} \to \mathcal{H}$$

of Lemma D.3. Lemma D.4 shows that the set $\pi^{-1}(h)$ of solutions $u \in W^{2,p}(X)$ of (D.2) for a given function h is compact. To see this just note that every sequence $u_\nu \in \pi^{-1}(h)$ satisfies a uniform estimate

$$\sup_\nu \|u_\nu\|_{L^\infty} \le c_1.$$

Since $\Delta u_\nu = A - e^{u_\nu} h^2$ it follows from the Calderón-Zygmund inequality for the Laplace-Beltrami operator that

$$\sup_\nu \|u_\nu\|_{W^{2,p}} \le c_2.$$

By the Arzela-Ascoli theorem, the inclusion $W^{2,p}(X) \hookrightarrow C^0(X)$ is a compact operator. Hence there exists a subsequence, still denoted by u_ν, which converges in the sup-norm. Since u_ν is a solution of (D.2) it follows again from the Calderón-Zygmund inequality that u_ν converges in the $W^{2,p}$-norm. This proves compactness. On the other hand, by Lemma D.3 the points in $\pi^{-1}(h)$ are isolated with respect to the $W^{2,p}$-norm. This shows that $\pi^{-1}(h)$ is a finite set for every h.

Now for any smooth path

$$[0, 1] \to \mathcal{H} : t \mapsto h_t$$

consider the space

$$M = \{(t, u) \,|\, 0 \le t \le 1,\ (h_t, u) \in \mathcal{U}\}.$$

It follows by the same arguments as above that M is compact. Moreover, by Lemma D.3, M is a smooth 1-manifold and the obvious projection $M \to [0,1]$ is a covering fibration. Hence

$$\#\pi^{-1}(h_0) = \#\pi^{-1}(h_1).$$

This shows that the number of points in $\pi^{-1}(h)$ is independent of $h \in \mathcal{H}$. To show that this number is 1 just consider the constant function $h(x) \equiv 1$. Then the equation

$$\Delta u + e^u = A$$

has an obvious solution

$$u(x) \equiv \log A.$$

We must prove that this is the only solution. To see this consider, for any solution u, a point $x_0 \in X$ where u attains its maximum. At this point

$$0 \le \Delta u(x_0) = A - e^{u(x_0)}.$$

Hence $e^{u(x_0)} \le A$ and so

$$e^{u(x)} \le A$$

for all $x \in X$. On the other hand the formula $\Delta u = A - e^u$ shows that

$$\int_X (A - e^u) = 0$$

and this implies $e^{u(x)} \equiv A$ as claimed. $\square$

APPENDIX E

UNIQUE CONTINUATION

The goal of this appendix is to prove that every section in the kernel of the Dirac operator which vanishes on some open set must vanish everywhere (provided that the manifold is connected). This result can be proved with Aronszajn's theorem [2]. However, the techniques of Agmon and Nirenberg in [1] give rise to a much simpler proof which uses the first order nature of the equation. Apparently, this proof does not carry over to second order operators such as the Laplacian. The methods described in this appendix were also used by Donaldson and Kronheimer in [21], pp 150–152, to prove a unique continuation theorem for anti-self-dual instantons.

E.1 The Agmon-Nirenberg theorem

Let H be a Hilbert space and $A(t)$ be a family of (unbounded) symmetric operators on H with domains $\mathrm{dom}(A(t)) \subset H$. The operators $A(t)$ are not required to be self-adjoint although in the main applications they will be and, moreover, their domains will be independent of t. However, in some interesting cases these operators are symmetric with respect to time-dependent inner products. This case will be dealt with in Section E.2 while applications to the Dirac operator are discussed in Section E.3. The following theorem is a special case of a result by Agmon and Nirenberg [1].

Theorem E.1. (Agmon-Nirenberg) *Let H be a real Hilbert space and $A(t) : \mathrm{dom}(A(t)) \to H$ be a family of symmetric linear operators. Assume that $x : [0, T] \to H$ is continuously differentiable in the weak topology such that $x(t) \in \mathrm{dom}(A(t))$ and*

$$\|\dot{x}(t) - A(t)x(t)\| \leq c_1 \|x(t)\| \tag{E.1}$$

for every $t \in [0, T]$, where $\dot{x}(t) \in H$ denotes the time derivative of x. Assume further that the function $t \mapsto \langle x(t), A(t)x(t)\rangle$ is also continuously differentiable and satisfies

$$\frac{d}{dt}\langle x, Ax\rangle - 2\langle \dot{x}, Ax\rangle \geq -c_2 \|Ax\| \|x\| - c_3 \|x\|^2 . \tag{E.2}$$

Then the following holds.

(i) *If $x(0) = 0$ then $x(t) = 0$ for all $t \in [0, T]$.*

(ii) *If $x(0) \neq 0$ then $x(t) \neq 0$ for all $t \in [0,T]$ and, moreover,*

$$\log \|x(t)\|^2 \geq \log \|x(0)\|^2 - \left(2\frac{\langle x(0), A(0)x(0)\rangle}{\|x(0)\|^2} + \frac{b}{a}\right)\frac{e^{at}-1}{a} - 2c_1 t$$

where $a = 2c_2$ and $b = 2{c_1}^2 + {c_2}^2/2 + 2c_3$.

Remark E.2 In the applications discussed here the operators $A(t)$ are self-adjoint differential operators on some compact manifold. The inequality (E.1) allows for *lower order perturbations* which are not self-adjoint. The general version of Theorem E.1 in [1] allows for highest order perturbations which are not self-adjoint, but which are, in a precise quantitative way, dominated by the self-adjoint part. This more general version of the theorem is not needed for the applications to the Dirac operator. □

Remark E.3 Assume, for example, that the operators $A(t)$ are all self-adjoint with time-independent domain $V = \mathrm{dom}(A(t)) \subset H$. Then V is a Hilbert space in its own right and the operators $A(t)$ are all bounded linear operators from V to H. Assume that the map $[0,T] \to A(t)$ is continuously differentiable in the weak operator topology. Then a function

$$x \in C^1([0,T], H]) \cap C^0([0,T], V)$$

satisfies the requirements of Theorem E.1 if and only if

$$\langle \dot{A}x, x\rangle \geq -c_2 \|Ax\| \|x\| - c_3 \|x\|^2$$

and (E.1) holds. The reader may check that under these assumptions the function $t \mapsto \langle A(t)x(t), x(t)\rangle$ is continuously differentiable. □

The idea of the proof of Theorem E.1 is to use the convexity of the function $t \mapsto \log \|x(t)\|^2$. The key step is the following lemma.

Lemma E.4 *Let $A(t)$ and $x(t)$ be as in Theorem E.1 and define*

$$\varphi(t) = \log \|x(t)\|^2 - \int_0^t \frac{2\langle x(s), \dot{x}(s) - A(s)x(s)\rangle}{\|x(s)\|^2}\, ds$$

for $0 \leq t \leq T$ wherever $x(t) \neq 0$. Then φ is twice continuously differentiable and

$$\ddot{\varphi} + a\,|\dot{\varphi}| + b \geq 0$$

where $a = 2c_2$ and $b = 2{c_1}^2 + {c_2}^2/2 + 2c_3$.

Proof: Define $f(t) = \dot{x}(t) - A(t)x(t)$ and note that the derivative of the function φ is given by

$$\dot{\varphi} = \frac{2\langle x, \dot{x}\rangle}{\|x\|^2} - \frac{2\langle x, f\rangle}{\|x\|^2} = \frac{2\langle x, Ax\rangle}{\|x\|^2}.$$

Hence

$$\begin{aligned}
\ddot{\varphi} &= \frac{2\frac{d}{dt}\langle Ax, x\rangle}{\|x\|^2} - \frac{4\langle Ax, x\rangle\langle \dot{x}, x\rangle}{\|x\|^4} \\
&\geq \frac{4\langle Ax, Ax+f\rangle - 2c_2\|Ax\|\|x\| - 2c_3\|x\|^2}{\|x\|^2} - \frac{4\langle Ax, x\rangle\langle Ax+f, x\rangle}{\|x\|^4}.
\end{aligned}$$

Here the second step follows from the inequality (E.2) and the identity $\dot{x} = Ax + f$. The terms on the right hand side can now be organized as follows

$$\begin{aligned}
\ddot{\varphi} &\geq \frac{4}{\|x\|^2}\left(\|Ax\|^2 - \frac{\langle Ax, x\rangle^2}{\|x\|^2}\right) + \frac{4}{\|x\|^2}\left\langle Ax - \frac{\langle Ax, x\rangle}{\|x\|^2}x, f\right\rangle \\
&\quad - 2c_2\frac{\|Ax\|}{\|x\|} - 2c_3. \\
&= \frac{4}{\|x\|^2}\left\|Ax - \frac{\langle Ax, x\rangle}{\|x\|^2}x\right\|^2 + \frac{4}{\|x\|^2}\left\langle Ax - \frac{\langle Ax, x\rangle}{\|x\|^2}x, f\right\rangle \\
&\quad - 2c_2\frac{\|Ax\|}{\|x\|} - 2c_3.
\end{aligned}$$

Now abbreviate

$$\xi = \frac{x}{\|x\|}, \qquad \eta = \frac{Ax}{\|x\|}.$$

Then $\dot{\varphi} = 2\langle \xi, \eta\rangle$ and the previous inequality can be written in the form

$$\begin{aligned}
\ddot{\varphi} &\geq 4\|\eta - \langle \eta, \xi\rangle\xi\|^2 + 4\left\langle \eta - \langle \eta, \xi\rangle\xi, \frac{f}{\|x\|}\right\rangle - 2c_2\|\eta\| - 2c_3 \\
&\geq 4\|\eta - \langle \eta, \xi\rangle\xi\|^2 - 2\|\eta - \langle \eta, \xi\rangle\xi\|\frac{2\|f\|}{\|x\|} - 2c_2\|\eta\| - 2c_3 \\
&\geq 2\|\eta - \langle \eta, \xi\rangle\xi\|^2 - \frac{2\|f\|^2}{\|x\|^2} - 2c_2\|\eta\| - 2c_3 \\
&\geq 2\|\eta - \langle \eta, \xi\rangle\xi\|^2 - 2{c_1}^2 - 2c_2\|\eta\| - 2c_3.
\end{aligned}$$

The last but one inequality uses the fact that $\alpha\beta \leq \alpha^2/2 + \beta^2/2$ and the last inequality uses $\|f\| \leq c_1\|x\|$.

To obtain the desired inequality

$$\ddot{\varphi} + a|\dot{\varphi}| + b \geq 0$$

it remains to prove that

$$2\left\|\eta-\langle\eta,\xi\rangle\xi\right\|^2-2{c_1}^2-2c_2\left\|\eta\right\|-2c_3\geq -a\left|\dot\varphi\right|-b.$$

Since $\dot\varphi=2\langle\xi,\eta\rangle$ this is equivalent to

$$c_2\left\|\eta\right\|\leq\left\|\eta-\langle\eta,\xi\rangle\xi\right\|^2+\frac{a}{2}\left|\langle\eta,\xi\rangle\right|+\left(\frac{b}{2}-{c_1}^2-c_3\right).$$

Now the norm squared of η can be expressed in the form

$$\left\|\eta\right\|^2=u^2+v^2,\qquad u=\left\|\eta-\langle\eta,\xi\rangle\xi\right\|,\qquad v=\left|\langle\eta,\xi\rangle\right|.$$

Hence the desired inequality has the form

$$c_2\sqrt{u^2+v^2}\leq u^2+\frac{a}{2}v+\left(\frac{b}{2}-{c_1}^2-c_3\right).$$

But since $c_2\sqrt{u^2+v^2}\leq c_2u+c_2v$ and $c_2u\leq u^2+{c_2}^2/4$ this is satisfied with $a/2=c_2$ and $b/2-{c_1}^2-c_3={c_2}^2/4$. This proves the lemma. □

Lemma E.5 *Let $\varphi,\psi:[0,T]\to\mathbb{R}$ be twice continouously differentiable functions which satisfy*

$$\ddot\varphi+a|\dot\varphi|+b\geq 0,\qquad \ddot\psi+a|\dot\psi|+b=0$$

for two constants $a,b>0$. If

$$\psi(0)\leq\varphi(0),\qquad \dot\psi(0)\leq\dot\varphi(0)$$

then $\psi(t)\leq\varphi(t)$ for all $t\in[0,T]$.

Proof: Consider the function $\rho(t)=\varphi(t)-\psi(t)$. This function is twice continuously differentiable on $[0,T]$ and satisfies

$$\begin{aligned}\ddot\rho+a|\dot\rho| &= \ddot\varphi-\ddot\psi+a|\dot\varphi-\dot\psi|\\ &\geq \ddot\varphi-\ddot\psi+a|\dot\varphi|-a|\dot\psi|\\ &\geq 0\end{aligned}$$

and $\rho(0)\geq 0$, $\dot\rho(0)\geq 0$. This implies $\dot\rho(t)\geq 0$ for all $t\in[0,T]$. Suppose otherwise that there exists a time $0<t_1\leq T$ such that $\dot\rho(t_1)<0$. Let $t_0\geq 0$ be the largest time less than t_1 such that $\dot\rho(t_0)=0$. Then $\dot\rho(t)<0$ for $t_0<t\leq t_1$ and hence

$$\frac{d}{dt}\left(e^{-at}\dot\rho(t)\right)=e^{-at}\left(\ddot\rho(t)-a\dot\rho(t)\right)=e^{-at}\left(\ddot\rho(t)+a\left|\dot\rho(t)\right|\right)\geq 0$$

for $t_0 \le t \le t_1$. This shows that $e^{-at}\dot\rho(t) \ge \dot\rho(t_0) \ge 0$ for all $t \in [t_0, t_1]$ in contradiction to the assumption. Hence $\dot\rho(t) \ge 0$ for all $t \in [0, T]$ and, since $\rho(0) \ge 0$, it follows that $\rho(t) \ge 0$ for all t. This proves the lemma. □

Proof of Theorem E.1: Let $x(t)$ and $A(t)$ be as in Theorem E.1 and $\varphi(t)$, a, b as in Lemma E.4. Assume first that $x(0) \ne 0$ and let $t_1 > 0$ such that $x(t) \ne 0$ for all $t \in [0, t_1]$. Then the function $\varphi(t)$ is defined on the interval $[0, t_1]$. Consider the function

$$\psi(t) = \varphi(0) - \left(|\dot\varphi(0)| + \frac{b}{a}\right)\frac{e^{at}-1}{a} + \frac{bt}{a}.$$

This function satisfies

$$\dot\psi(t) = -\left(|\dot\varphi(0)| + \frac{b}{a}\right)e^{at} + \frac{b}{a}, \qquad \ddot\psi(t) = -a\left(|\dot\varphi(0)| + \frac{b}{a}\right)e^{at}.$$

Hence $\dot\psi(t) \le 0$ for all $t \in [0, t_1]$ and $\ddot\psi + a|\dot\psi| + b = \ddot\psi - a\dot\psi + b = 0$. Moreover, $\psi(0) = \varphi(0)$ and $\dot\psi(0) = -|\dot\varphi(0)| \le \dot\varphi(0)$. Hence it follows from Lemma E.5 that $\psi(t) \le \varphi(t)$ for all $t \in [0, t_1]$. Now the formula

$$\log\|x(t)\|^2 = \varphi(t) + \int_0^t \frac{2\langle x(s), \dot x(s) - A(s)x(s)\rangle}{\|x(s)\|^2}\,ds \ge \psi(t) - 2c_1 t$$

shows that

$$\|x(t)\|^2 \ge e^{-c_1 t + \psi(t)} \tag{E.3}$$

for every $t \in [0, t_1]$. This implies $x(t) \ne 0$ for all $t \in [0, T]$. Suppose otherwise that there is a $t_0 > 0$ such that $x(t) \ne 0$ for $0 \le t < t_0$ and $x(t_0) = 0$. Then the function $x(t)$ must satisfy the estimate (E.3) for $0 \le t < t_0$ and hence $x(t)$ cannot converge to zero as $t \to t_0$. This contradiction shows that $x(t) \ne 0$ for all $t \in [0, T]$ and (E.3) holds on the entire interval. This proves (ii). Statement (i) follows by reversing time. More precisely, note that the functions

$$y(t) = x(T - t), \qquad B(t) = -A(T - t)$$

satisfy all the requirements of Theorem E.1. Hence if $y(0) \ne 0$ then $y(T) \ne 0$ or, conversely, $y(T) = 0$ implies $y(0) = 0$. This means that $x(0) = 0$ implies $x(T) = 0$. Since this holds for any interval $[0, T]$ the theorem is proved. □

E.2 Time-dependent inner products

There are interesting applications to operator families $A(t)$ on a Hilbert space which are self-adjoint with respect to a time-dependent family of inner products which are all compatible with the standard inner product on H. Any such family of inner products can be expressed in the form

$$\langle x, y\rangle_t = \langle Q(t)x, Q(t)y\rangle \tag{E.4}$$

for some invertible bounded linear operators $Q(t) : H \to H$. Without loss of generality one can consider operators $Q(t)$ which are self-adjoint. Assume throughout that these operators satisfy the following conditions.

(H1) *The operator $Q(t)$ is self-adjoint for every t and there exists a constant $\delta > 0$ such that*

$$\delta \|x\| \le \|Q(t)x\| \le \delta^{-1} \|x\|$$

for all $x \in H$ and $t \in [0, T]$. Moreover, the map $[0, T] \to \mathcal{L}(H) : t \mapsto Q(t)$ is continuously differentiable in the weak operator topology and there exists a constant $c_0 > 0$ such that

$$\left\|\dot{Q}(t)\right\|_{\mathcal{L}(H)} \le c_0$$

for all $t \in [0, T]$.

Example E.6 Let X be a compact oriented smooth manifold equipped with a time-dependent family of Riemannian metrics g_t. Then the Hilbert space

$$H = L^2(X)$$

of L^2 functions on X is independent of the choice of the metric. However, the inner product on H does depend on g_t. Declare the inner product induced by g_0 to be the standard inner product on H. For every t and every $x \in X$ let $P_t(x) : T_xX \to T_xX$ be the unique endomorphism which is symmetric and positive definite with respect to g_0 and satisfies

$$g_t(v, w) = g_0(P_t v, P_t w)$$

for $v, w \in T_xX$. Then the volume forms of g_0 and g_t are related by

$$\mathrm{dvol}_t = \det(P_t)\mathrm{dvol}_0.$$

Hence the two inner products on $H = L^2(X)$ are related by the pointwise multiplication operator $Q_t : L^2(X) \to L^2(X)$ given by

$$f \mapsto Q_t f = \sqrt{\det(P_t)} f.$$

Now let $A(t) = \Delta_t$ be the Laplace-Beltrami operator of the metric g_t. This operator is self-adjoint with respect to the inner product induced by the volume form dvol_t of the metric g_t. The domain of Δ_t is the Sobolev space $W^{2,2}(X)$. This space is independent of the metric and is preserved by the operators Q_t (whenever the metrics are of class C^2 say). Note, however, that only the L^2-inner products are related by the operator Q_t but not the $W^{k,2}$ inner products for $k \geq 1$. The reader may also note that this example easily generalizes to the Hilbert space of L^2-sections of a bundle $E \to X$ where the inner products on both E and TX vary with t. □

Theorem E.7 *Let H be a real Hilbert space and $Q(t) \in \mathcal{L}(H)$ be a family of (bounded) self-adjoint operators on H which satisfy $(H1)$. Let $A(t)$: $\mathrm{dom}(A(t)) \to H$ be a family of (unbounded) linear operators such that $A(t)$ is symmetric with respect to the inner product (E.4). Assume that $x : [0,T] \to H$ is continuously differentiable in the weak topology such that $x(t) \in \mathrm{dom}(A(t))$ and*

$$\|\dot{x}(t) - A(t)x(t)\|_t \leq c_1 \|x(t)\|_t \tag{E.5}$$

for every $t \in [0,T]$. Assume further that the function $t \mapsto \langle x(t), A(t)x(t)\rangle_t$ is also continuously differentiable and satisfies

$$\begin{aligned} \frac{d}{dt}\langle x(t), A(t)x(t)\rangle_t - 2\langle \dot{x}(t), A(t)x(t)\rangle_t & \\ \geq -c_2 \|A(t)x(t)\|_t \|x\|_t - c_3 \|x(t)\|_t^2 & \end{aligned} \tag{E.6}$$

for every $t \in [0,T]$. Then the following holds.

(i) *If $x(0) = 0$ then $x(t) = 0$ for all $t \in [0,T]$.*

(ii) *If $x(0) \neq 0$ then $x(t) \neq 0$ for all $t \in [0,T]$ and, moreover,*

$$\log \|x(t)\|_t^2 \geq \log \|x(0)\|_t^2 - \left(2\frac{\langle x(0), A(0)x(0)\rangle_t}{\|x(0)\|_t^2} + \frac{b}{a}\right)\frac{e^{at}-1}{a} - 2\tilde{c}_1 t$$

where $a = 2\tilde{c}_2$ and $b = 2\tilde{c}_1^2 + \tilde{c}_2^2/2 + 2\tilde{c}_3$ with

$$\tilde{c}_1 = c_1 + \frac{c_0}{\delta}, \qquad \tilde{c}_2 = c_2 + \frac{2c_0}{\delta}, \qquad \tilde{c}_3 = c_3.$$

Proof: The result can easily be reduced to Theorem E.1. Define

$$\tilde{A} = QAQ^{-1}, \qquad \tilde{x} = Qx, \qquad \tilde{f} = \dot{Q}x + Qf$$

with $\mathrm{dom}(\tilde{A}(t)) = Q(t)\mathrm{dom}(A(t))$ where $f = \dot{x} - Ax$ as before. Then the operator $A(t)$ is symmetric with respect to the inner product (E.4) if and only if $\tilde{A}(t)$ is symmetric with respect to the standard inner product. (Moreover, one can easily check that $A(t)$ is self-adjoint with respect to (E.4) if and only if $\tilde{A}(t)$ is self-adjoint with respect to the standard inner product. However, this is not needed for the proof.) It also easy to see that

$$\dot{x} = Ax + f \qquad \Longleftrightarrow \qquad \dot{\tilde{x}} + \tilde{A}\tilde{x} = \tilde{f}.$$

It remains to show that under the assumptions of Theorem E.7 the triple $\tilde{A}$, $\tilde{x}$, $\tilde{f}$ satisfies the requirements of Theorem E.1 with suitably modified constants $\tilde{c}_i$. Firstly, note that

$$\|\tilde{f}\| = \|\dot{Q}x + Qf\| \le c_0 \|x\| + \|f\|_t \le c_0\delta^{-1} \|x\|_t + c_1 \|x\|_t$$

and hence $\tilde{x}$ satisfies (E.1) with c_1 replaced by $\tilde{c}_1 = c_1 + c_0/\delta$. Secondly, the function

$$t \mapsto \langle \tilde{x}(t), \tilde{A}(t)\tilde{x}(t)\rangle = \langle x(t), A(t)x(t)\rangle_t$$

is continuously differentiable and a simple calculation shows that

$$\frac{d}{dt}\langle \tilde{x}, \tilde{A}\tilde{x}\rangle - 2\langle \dot{\tilde{x}}, \tilde{A}\tilde{x}\rangle = \frac{d}{dt}\langle x, Ax\rangle_t - 2\langle \dot{x}, Ax\rangle_t + 2\langle \dot{Q}x, QAx\rangle.$$

Hence

$$\begin{aligned} \frac{d}{dt}\langle \tilde{x}, \tilde{A}\tilde{x}\rangle - 2\langle \dot{\tilde{x}}, \tilde{A}\tilde{x}\rangle &\ge -c_2 \|Ax\|_t \|x\|_t - c_3 \|x\|_t^2 - 2\|\dot{Q}x\|\|\tilde{A}\tilde{x}\| \\ &\ge -c_2 \|\tilde{x}\| \|\tilde{A}\tilde{x}\| - c_3 \|\tilde{x}\|^2 - 2c_0\delta^{-1} \|\tilde{x}\| \|\tilde{A}\tilde{x}\|. \end{aligned}$$

This shows that $\tilde{x}$ satisfies (E.2) with c_2 and c_3 replaced by $\tilde{c}_2 = c_2 + 2c_0/\delta$ and $\tilde{c}_3 = c_3$. Hence $\tilde{x}$ and $\tilde{A}$ satisfy the requirements of Theorem E.1 and this proves Theorem E.7. □

E.3 Application to Dirac operators

The goal of this section is to use Theorem E.7 in order to derive a unique continuation theorem for the solutions of the Dirac equation. Let X be a connected Riemannian manifold of real dimension m which is equipped with a spin^c structure

$$\Gamma : TX \to \mathrm{End}(W).$$

It is not necessary here to assume that Γ is an irreducible representation of the bundle of Clifford algebras. Hence the rank of the Hermitian vector bundle W may be bigger than 2^n when $m = 2n$ or $m = 2n + 1$. In particular, this includes bundles of the form $W = S \otimes E$ where $S \to X$ is a minimal spin or spin^c representation and $E \to X$ is any Hermitian vector bundle. We shall consider spin^c connections on W which are compatible with any connection on TX, not necessarily with the Levi-Civita connection. Moreover, it is convenient to work with connections on W of class $W^{1,p}$ with

$$p > m = \dim X.$$

More precisely, fix a reference connection A_0 on W which is compatible with the Levi-Civita connection and denote by

$$\mathcal{A}^{1,p}(\Gamma)$$

the space of all connections A on W which have the form $A = A_0 + a$ where $a \in W^{1,p}(X, \mathrm{End}(W))$ and there exists a $b \in W^{1,p}(X, \mathrm{End}(TX))$ such that

$$[a(u), \Gamma(v)] = \Gamma(b(u)v)$$

for all $u, v \in T_xX$. For every $A \in \mathcal{A}^{1,p}(\Gamma)$ denote by $D_A : W^{1,p}(X, W) \to L^p(X, W)$ the corresponding Dirac operator. Recall that this operator is defined by

$$D_A\Phi = \sum_{i=0}^{m} \Gamma(e_i)\nabla_{A,e_i}\Phi$$

for any local orthonormal frame $e_1, \ldots, e_m$. Recall from Lemma C.23 that any $\Phi \in W^{1,p}(X, W)$ in the kernel of the Dirac operator is necessarily of class $W^{2,p}$. The proof carries over to manifolds of arbitrary dimension and general spin^c representations.

The restriction $p > m$ is chosen for technical reasons. The unique continuation theorem should remain valid for connections of class $W^{1,p}$ with any $p > m/2$, however, for $p \le m$ the connection potential is no longer continuous and this leads to complications in the proof. Namely, the treatment of the general case would require an extension of Theorem E.7 which allows for solutions of the inequality $\|\dot{x}(t) - A(t)x(t)\| \le V(t)\,\|x(t)\|$ where the function V is not bounded but lies in some space $L^p([0, T], H)$. However, for the purposes of this book the following theorem suffices.

Theorem E.8 *Let X be a connected Riemannian manifold of real dimension m equipped with a spin^c structure $\Gamma : TX \to \mathrm{End}(W)$. Assume that the metric and the spin^c structure are of class C^3. Let $A \in \mathcal{A}^{1,p}(X)$ for some $p > m$ and suppose that $\Phi \in W^{2,p}(X, W)$ is a solution of the Dirac equation*

$$D_A\Phi = 0.$$

If Φ vanishes on some open set then $\Phi(x) = 0$ for all $x \in X$.

Proof: Assume first that $m = 2n$ is even. We shall prove that for every compact set $K \subset X$ there exists a number $\varepsilon > 0$ such that if Φ vanishes in a neighbourhood of a point $x_0 \in K$ then Φ vanishes in $B_\varepsilon(x_0)$. This implies immediately that the set of points $x \in X$ such that Φ vanishes in a neighbourhood of x is closed. This set is obviously open and so it must either be empty or coincide with X.

The constant ε will simply be any number smaller than the minimal injectivity radius on the set K. Near any point $x_0 \in K$ choose local geodesic polar coordinates. This coordinate chart identifies the ball of radius ε in $\mathbb{R}^{2n}$, centered at the origin, with the geodesic ball $B_\varepsilon(x_0)$. The chart is of class C^2 whenever the metric is of class C^3. It determines a metric $g_{ij}(x)$ on $\mathbb{R}^{2n}$ (of class C^2) with respect to which the straight lines $t \mapsto tx$ are geodesics for all $x \in \mathbb{R}^{2n}$. In such coordinates the ordinary sphere of radius r centered at zero agrees with the geodesic sphere. Denote by g_r the metric on S^{2n-1} induced by the embedding $x \mapsto rx$ and rescaled to standard size by the factor $1/r^2$. Thus the inner product of two tangent vectors $\xi, \eta \in T_x S^{2n-1} = x^\perp$ with respect to this metric is given by

$$\langle \xi, \eta \rangle_r = \xi^T g(rx) \eta.$$

The metric on $\mathbb{R}^{2n}$ in polar coordinates $(0, \infty) \times S^{2n-1} \to \mathbb{R}^{2n} : (r, x) \mapsto rx$ is then given by

$$g = dr \otimes dr + \frac{1}{r^2} g_r.$$

In particular the radial vector field $\partial/\partial r$ has unit length.

Now choose a local Hermitian trivialization of the bundle W^+. In this trivialization the fibers of W^+ are simply identified with $\mathbb{C}^N$ for some integer $N > 0$ and Φ is just map $\mathbb{R}^{2n} \to \mathbb{C}^N$. Consider the maps

$$\varphi_r : S^{2n-1} \to \mathbb{C}^N$$

defined by

$$\varphi_r(x) = \Phi(rx), \qquad |x| = 1.$$

Use the unit radial vector field $-\partial/\partial r$ to identify the bundle W^- with W^+, i.e. $-\Gamma(\partial/\partial r)$ maps the given frame of W^+ to a reference frame of W^-. Thus the fibres of both W^+ and W^- are identified with $\mathbb{C}^N$. In this trivialization the spin^c structure is a map $\mathbb{R}^{2n} \to \mathrm{Hom}(\mathbb{R}^{2n}, \mathbb{C}^{N \times N})$ (of class C^2) denoted by $\Gamma(x; \xi) \in \mathbb{C}^{N \times N}$ for $x, \xi \in \mathbb{R}^{2n}$. Here $\xi \in \mathbb{R}^{2n}$ is to be understood as the tangent vector at x and the map $\xi \mapsto \Gamma(x; \xi)$ is linear and satisfies the usual condition (4.18). Note that the radial vector field is just given by $x \mapsto x$ and thus our choice of frame for W^- means that

$$\Gamma(x; x) = -\mathbb{1}.$$

Consider the spin^c structure $\Gamma_r : TS^{2n-1} \to \mathbb{C}^{N \times N}$ on S^{2n-1} defined by

$$\Gamma_r(x; \xi) = \Gamma(rx; \xi)$$

for $\xi \in T_x S^{2n-1} = x^\perp$.

Let $\nabla = \nabla_{A_0}$ be the fixed reference connection on W^+ (in the given local coordinates and local trivialization) which is compatible with the Levi-Civita connection. Assume without loss of generality that this connection is in radial gauge. (Otherwise change the local frame of W^+.) The corresponding Dirac operator on the sphere of radius r is denoted by D_r and is given by

$$D_r \varphi_r(x) = \sum_{i=1}^{2n-1} \Gamma(rx; e_i) \nabla_{e_i} \varphi_r(x)$$

for any local orthonormal frame $e_1, \ldots, e_{2n-1}$ of $T_x S^{2n-1} = x^\perp$ with respect to the inner product $g_r(x) = g(rx)$. This operator is self-adjoint with respect to the metric g_r. With these conventions the Dirac equation on $\mathbb{R}^{2n}$ takes the form

$$\frac{\partial}{\partial r} \varphi_r = \frac{1}{r} D_r \varphi_r. \tag{E.7}$$

The factor $1/r$ arises from the fact that all the derivatives of $\varphi_r(x) = \Phi(rx)$ with respect to x carry a factor r which has to be cancelled in order to obtain the original Dirac equation.

Let $a = \sum_j a(x) dx_j$ be the spin^c connection with

$$a_j \in W^{1,p}(\mathbb{R}^{2n}, \mathbb{C}^{N \times N}).$$

The induced connection on the sphere of radius r is the endomorphism valued 1-form

$$a_r = \sum_{j=1}^{2n} a_j(rx) dx_j$$

on S^{2n-1} with corresponding zeroth order perturbation term

$$\Gamma_r(a_r) := \sum_{j=1}^{2n} \Gamma(rx; e_j) a(rx; e_j).$$

Also denote by $b_r : S^{2n-1} \to \mathbb{C}^{N \times N}$ the radial part of the 1-form a given by

$$b_r(x) = \sum_{j=1}^{2n} a_j(rx) x_j$$

Then the perturbed Dirac equation takes the form

$$\frac{\partial}{\partial r} \varphi_r + b_r \varphi_r = \frac{1}{r} D_r \varphi_r + \Gamma_r(a_r) \varphi_r. \tag{E.8}$$

It is convenient to rewrite this equation with the new time coordinate $t = \log(r) \in (-\infty, \log\varepsilon)$. Thus $r = e^t$ and the equation (E.8) takes the form

$$\frac{\partial}{\partial t}\varphi(t) = D(t)\varphi(t) + B(t)\varphi(t) \tag{E.9}$$

where

$$D(t) = D_{\exp(t)}, \qquad \varphi(t) = \varphi_{\exp(t)}$$

and $B(t) : S^{2n-1} \to \mathbb{C}^{N\times N}$ is given by

$$B(t) = \exp(t)\left(\Gamma_{\exp(t)}(a_{\exp(t)}) - b_{\exp(t)}\right).$$

Recall that the operator $D(t) : C^\infty(S^{2n-1}, \mathbb{C}^N) \to C^\infty(S^{2n-1}, \mathbb{C}^N)$ is self-adjoint with respect to the metric $g_{\exp(t)}$ and this metric is continuously differentiable as a function of t. The reasoning of Example E.6 shows in fact that the corresponding multiplication operators $Q(t)$ satisfy the hypothesis $(H1)$ of Section E.2.

The assumption $p > m$ guarantees that the functions $a_j : \mathbb{R}^{2n} \to \mathbb{C}^{N\times N}$ are uniformly bounded (in the ball of radius ε) and hence there exists a constant $c_1 > 0$ such that

$$\sup_{t<\log(\varepsilon)} \|B(t)\|_{L^\infty(S^{2n-1})} \le c_1. \tag{E.10}$$

Now consider the function

$$(-\infty, \log\varepsilon) \to L^2(S^{2n-1}, \mathbb{C}^N) : t \mapsto \varphi(t).$$

By assumption, the original function Φ is of class $W^{2,p}$ on $\mathbb{R}^{2n}$ and since $p > 2n$ this implies that Φ is continuously differentiable as a function on $\mathbb{R}^{2n}$. Hence the function $t \mapsto \varphi(t)$ with values in L^2 is also continuously differentiable. Moreover, by (E.9) and (E.10),

$$\|\dot\varphi(t) - D(t)\varphi(t)\|_{L^2} = \|B(t)\varphi(t)\|_{L^2} \le c_1 \|\varphi(t)\|_{L^2}.$$

This shows that φ satisfies the condition (E.5) of Theorem E.7.

To check the condition (E.6) consider the function

$$t \mapsto \langle\varphi(t), D(t)\varphi(t)\rangle_{L^2} = \int_{S^{2n-1}} \langle\varphi(t), D(t)\varphi(t)\rangle \mathrm{dvol}_t$$

where dvol_t denotes the volume form of the metric $g_{\exp(t)}$ on S^{2n-1}. That this function is continuously differentiable follows from the fact that the map $t \mapsto D(t)\varphi(t)$ (with values in $L^2(S^{2n-1}, \mathbb{C}^N)$) is continuous in the norm topology and that, for every fixed section $\theta \in L^2(S^{2n-1}, \mathbb{C}^N)$, the map $t \mapsto D(t)\theta$ is continuously differentiable in the norm topology with derivative $\dot{D}(t)\theta$. Consider the formula

$$\frac{d}{dt}\langle\varphi, D\varphi\rangle_{L^2} - 2\langle\dot{\varphi}, D\varphi\rangle_{L^2} = \langle\varphi, \dot{D}\varphi\rangle_{L^2} + \langle\varphi, \dot{Q}D\varphi\rangle_{L^2,g_0}. \tag{E.11}$$

This equation is to be understood pointwise for every t. The last term simply involves the derivatives of the metric and hence does not cause any problems. The first term on the right concerns the variation of the operator $D(t)$ with time. To give a formula for this variation note that the local frame $e_j(t,x)$ as well as the spin^c structure $\Gamma(e^t x;\xi)$ and the spin^c connection ∇_t depend on t. Their time derivatives will be denoted by

$$\dot{\Gamma}_{e^t}(x;\xi) = \frac{d}{dt}\Gamma(e^t x;\xi), \quad \dot{e}_j(t,x) = \frac{d}{dt}e_j(t,x), \quad \dot{\nabla}_\xi = \frac{\partial}{\partial t}\nabla_\xi - \nabla_\xi\frac{\partial}{\partial t}$$

for a fixed $x \in S^{2n-1}$ and a fixed tangent vector $\xi \in T_x S^{2n-1}$. With these conventions the operator $\dot{D}(t)$ is given by

$$\dot{D}(t) = \sum_{i=1}^{2n-1}\left(\dot{\Gamma}_{e^t}(e_i)\nabla_{e_i} + \Gamma_{e^t}(\dot{e}_i)\nabla_{e_i} + \Gamma_{e^t}(e_i)\dot{\nabla}_{e_i} + \Gamma_{e^t}(e_i)\nabla_{\dot{e}_i}\right).$$

Here the dependence on the point $x \in S^{2n-1}$ is dropped in the notation. This formula shows that

$$\left\|\dot{D}(t)\theta\right\|_{L^2} \le c\,\|\theta\|_{W^{1,2}} \le c_2\,\|D(t)\theta\|_{L^2} + c_3\,\|\theta\|_{L^2}. \tag{E.12}$$

for every $\theta : S^{2n-1} \to \mathbb{C}^N$. The equations (E.11) and (E.12) together show that condition (E.6) of Theorem E.7 is satisfied with $H = L^2(S^{2n-1},\mathbb{C}^N)$, $A(t)$ replaced by $D(t)$, $x(t)$ replaced by $\varphi(t)$, and $Q(t)$ determined by the metric $g_{\exp(t)}$ on S^{2n-1}.

Since the original section $\Phi : \mathbb{R}^{2n} \to \mathbb{C}^N$ vanishes in some neighbourhood of the origin it follows that

$$\varphi(t) = 0, \qquad t < -T,$$

for T sufficiently large. Hence, by Theorem E.7, $\varphi(t) = 0$ for all $t \in (-\infty, \log\varepsilon]$ and this shows that Φ vanishes in the ε-ball about zero. This proves the theorem in the case where X has even dimension $2n$. The odd case can easily be reduced to the even case by considering the manifold $X \times \mathbb{R}$. Namely, every solution of the Dirac equation on X defines a solution of the corresponding Dirac equation on $X \times \mathbb{R}$ which is translation invariant in the $\mathbb{R}$-direction. This proves Theorem E.8. □

Remark E.9 In [21], pp 150–152, Donaldson and Kronheimer used a similar technique to prove a unique continuation theorem for anti-self-dual instantons on a connected 4-manifold X. For example, let A be a connection (of class $W^{1,p}$ with $p > 4$) on a principal G-bundle $P \to X$ where G is a compact Lie group. Let $\omega \in \Omega^{2,+}(X, \mathfrak{g}_P)$ be a self-dual Lie algebra valued 2-form (of class $W^{2,p}$) which satisfies

$$d_A^* \omega + d_A \xi = 0$$

for some section $\xi \in \Omega^0(X, \mathfrak{g}_P)$ (also of class $W^{2,p}$). If ω and ξ vanish on some open set then they must vanish everywhere. This can be reduced to Theorem E.7 with the same techniques as in the proof of Theorem E.8 (see [21] for details). In particular, when $\mathrm{G} = S^1$, this means that an anti-self-dual harmonic 2-form which vanishes on some open set must vanish everywhere. □

The result of the previous remark as well as Theorem E.8 can also be proved with Aronszajn's theorem [2]. Here is a statement of that theorem which applies to general second order elliptic differential equations.

Theorem E.10. (Aronszajn) *Let $\Omega \subset \mathbb{R}^n$ be a connected open set and L be an elliptic operators on Ω of the form*

$$L = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)$$

with coefficients $a_{ij} \in C^{2,1}(\Omega)$ satisfying

$$\delta|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \delta^{-1}|\xi|^2$$

for $x \in \Omega$, $\xi \in \mathbb{R}^n$, and some constant $\delta > 0$. Let $u = (u_1, \ldots, u_N) \in W^{2,2}(\Omega, \mathbb{R}^N)$ and suppose that there exists a constant $M > 0$ such that

$$|Lu_\nu(x)| \leq M \sum_{\mu=1}^{N} (|u_\mu(x)| + |\nabla u_\mu(x)|) \tag{E.13}$$

for $\nu = 1, \ldots, N$ and almost every $x \in \Omega$. If u has a zero of infinite order then $u \equiv 0$ in Ω.

A **zero of infinite order** is a point $x_0 \in \Omega$ such that for every integer $k > 0$ there exists a constant c_k with

$$\int_{B_r(x_0)} |u| \leq c_k r^k$$

for $r \leq 1$. The technique of proof of Theorem E.10 goes back to Carleman [14]. It is based on the generalized Carleman inequality

$$\int_{\mathbb{R}^n} \left(\frac{|u(x)|^2}{|x|^{2\alpha+4}} + \frac{|\nabla u(x)|^2}{|x|^{2\alpha+2}} \right) dx \leq c \int_{\mathbb{R}^n} \frac{|Lu(x)|^2}{|x|^{2\alpha}} \, dx \tag{E.14}$$

for smooth functions $u : \mathbb{R}^n - \{0\} \to \mathbb{R}$ with compact support where the constant c is independent of u and α. A self-contained proof of Theorem E.10 can be found in Kazdan [56].

In the case of anti-self-dual harmonic 2-forms Theorem E.10 can be used for a single scalar equation where L is the Laplace-Beltrami operator on X. In the case of the Dirac equation use the Weitzenböck formula and apply Theorem E.10 to the Bochner Laplacian $L = \nabla^* \nabla$ in a suitable local frame in which the highest order term is of diagonal form. Note that the lower order terms involve the curvature of the connection A and hence the resulting unique continuation theorem requires connections with bounded curvature. Thus the result obtained is weaker than Theorem E.8 which holds for connections with curvature in L^p where $p > m = \dim X$.

Remark E.11 In special cases there are stronger results for second order equations. For example in [52] Jerison and Kenig proved a unique continuation theorem for equations of the form

$$\Delta u + V u = 0$$

where

$$\Delta = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

is the standard Laplacian in $\mathbb{R}^n$ and $V \in L^{n/2}$. (In contrast, Aronszajn's theorem requires V to be bounded.) This is the strongest possible result of its kind because for $p < n/2$ it is easy to find potential functions of class L^p which do not have the unique continuation property. (**Hint:** Try the function $u(x) = \prod_i e^{-|x_i|^{-\varepsilon}}$.) For general second order elliptic operators the corresponding statement seems to be an open question. The papers [38] by Garofalo and Lin and [56] by Kazdan treat general second order elliptic operators but allow only for isolated singularities of the potential V. $\square$

APPENDIX F

LINE BUNDLES AND DIVISORS

The goal of this appendix is to explain some backround material from algebraic geometry necessary for the understanding of divisors on complex manifolds. The first section discusses some elementary facts about holomorphic functions of several complex variables. Section F.2 deals with algebraic properties of the ring of convergent power series, and Section F.3 proves Hilbert's Nullstellensatz. Section F.4 deals with analytic hypersurfaces. Section F.5 discusses basic properties of the multiplicity function m_f which assigns to each point in the domain of a holomorphic function f the order of the point as a zero of f. Section F.6 is devoted to divisors and Section F.7 to line bundles. Excellent references are Atiyah-MacDonald [5], Bochner-Martin [10], and Griffiths-Harris [45].

F.1 Several complex variables

Consider the ring $\mathcal{O} = {}_n\mathcal{O}$ of convergent power series in n complex variables. A **power series** has the form

$$f(z) = \sum_\nu a_\nu z^\nu$$

where the sum runs over all multi-indices $\nu = (\nu_1, \ldots, \nu_n)$ and **convergent** means that there exists a real number $r > 0$ such that the series converges uniformly and absolutely in the domain $|z| \le r$. It follows from elementary analysis that $\mathcal{O}$ is a ring and that $f \in \mathcal{O}$ is invertible if and only if $f(0) \ne 0$. Invertible elements are called **units**. For every $f \in {}_n\mathcal{O}$ we shall denote by $\operatorname{dom} f \subset \mathbb{C}^n$ the domain of (absolute) convergence.

Using Cauchy's integral formula and Abel's lemma one can prove that every holomorphic function $f : U \to \mathbb{C}$ on some open set $U \subset \mathbb{C}^n$ can locally be represented as a power series

$$f(z+\zeta) = f_z(\zeta) = \sum_\nu a_\nu \zeta^\nu, \qquad a_\nu = \frac{\partial^\nu f(z)}{\nu!}.$$

This implies that if f vanishes to infinite order at some point $z \in U$ then f vanishes in some neighbourhood of z. Hence the set of all points $z \in U$ at which f vanishes to infinite order is open and closed. This implies the following.

Theorem F.1. (Identitätssatz) *Let $U \subset \mathbb{C}^n$ be a connected open set and $f : U \to \mathbb{C}$ be a holomorphic function which vanishes to infinite order at a point $z_0 \in U$. Then $f(z) = 0$ for all $z \in U$.*

Consider the map

$$m : {}_n\mathcal{O} \to \mathbb{N}$$

which assigns to every convergent power series $f \in {}_n\mathcal{O}$ the order of $\zeta = 0$ as a zero of f. Thus $m(f) = 0$ for every unit f and otherwise $m = m(f)$ is the unique positive integer such that the coefficients a_ν of f vanish for $|\nu| \le m-1$ and there is at least one nonzero coefficient with $|\nu| = m$. Thus

$$m(f) = \min\{m \in \mathbb{Z} \,|\, \exists \nu \ni \ |\nu| = m,\, a_\nu \neq 0\}. \tag{F.1}$$

The next lemma shows that m is a homomorphism from the multiplicative semigroup of power series to the additive semigroup of nonnegative integers. Note that convergence is not required for this result.

Lemma F.2 *If $f, g \in {}_n\mathcal{O}$ then*

$$m(fg) = m(f) + m(g).$$

Proof: For every positive integer m denote by $\mathcal{I}_m \subset \mathbb{N}^n$ the set of multi-indices ν with $|\nu| = m$. Consider the convolution pairing

$$\mathbb{C}^{\mathcal{I}_\ell} \times \mathbb{C}^{\mathcal{I}_m} \to \mathbb{C}^{\mathcal{I}_{\ell+m}} : (a, b) \mapsto a * b$$

defined by

$$(a * b)_\nu = \sum_{|\lambda|=\ell} a_\lambda b_{\nu-\lambda}$$

for $|\nu| = \ell + m$. We must prove that this pairing is nondegenerate, that is, $a \neq 0$ and $b \neq 0$ imply $a * b \neq 0$. To see this consider the polynomials

$$\varphi(z) = \sum_{|\lambda|=\ell} a_\lambda \zeta^\lambda, \qquad \psi(z) = \sum_{|\mu|=m} b_\mu \zeta^\mu.$$

Then the condition $a * b = 0$ is equivalent to $\varphi(\zeta)\psi(\zeta) \equiv 0$. Suppose $a \neq 0$ and choose a point $\zeta \in \mathbb{C}^n$ with $\varphi(\zeta) \neq 0$. Then ψ vanishes in a neighbourhood of ζ and hence, by Theorem F.1, must vanish everywhere. Hence $b = 0$. This proves nondegeneracy and the assertion of the lemma is an immediate consequence. □

Exercise F.3 Prove that for every $f \in {}_n\mathcal{O}$ there exist constants $c > 0$ and $\delta > 0$ such that $|f(\zeta)| \le c|\zeta|^{m(f)}$ for every $\zeta \in \mathbb{C}^n$ with $|\zeta| \le \delta$. □

F.2 Unique factorization

A convergent power series $p \in {}_n\mathcal{O}$ is called **irreducible** if $p = fg$ implies that either f or g is a unit. It is called a **prime** if $p|fg$ implies that $p|f$ or $p|g$. Obviously, every prime is irreducible. The converse is a nontrivial fact which is equivalent to the following fundamental theorem.

Theorem F.4 *Every nonunit $f \in {}_n\mathcal{O}$ is a product of finitely many primes.*

It is obvious from the definition of *prime* that the factorization into primes is unique up to order and multiplication by units. Likewise, it is obvious from the definition of *irreducible* that every nonunit f factors into finitely many irreducibles.* Moreover, the existence of factorizations into primes is equivalent to the uniqueness for factorizations into irreducibles. Rings with this property are called **unique factorization domains** or briefly **factorial**. Every principal ideal domain has this property, however, for $n > 1$ the ring ${}_n\mathcal{O}$ is no longer a principal ideal domain. Unique factorization domains have the following properties.

Lemma F.5. (Gauss) *If $\mathcal{R}$ is factorial then so is $\mathcal{R}[x]$.*

Lemma F.6 *If $\mathcal{R}$ is factorial and $u, v \in \mathcal{R}[x]$ are relatively prime then there exist relatively prime polynomials $\alpha, \beta \in \mathcal{R}[x]$ and a nonzero element $0 \neq \gamma \in \mathcal{R}$ such that*

$$\alpha u + \beta v = \gamma.$$

The proof of Theorem F.4 is by induction over n. It is based on the notion of a **Weierstrass polynomial of order m in z_0**, that is, a convergent power series $\omega \in {}_{n+1}\mathcal{O}$ of the form

$$\omega(z_0, z) = z_0{}^m + \sum_{j=1}^{m} a_j z_0{}^{m-j}$$

where the $a_j = a_j(z_1, \ldots, z_n)$ are nonunits, i.e. $a_j(0) = 0$. A convergent power series $f \in {}_{n+1}\mathcal{O}$ is called **distinguished in z_0 of order m** if it satisfies

$$f(z_0, 0, \ldots, 0) = z_0{}^m \cdot f_0(z_0)$$

where $f_0(0) \neq 0$. It is easy to see that any nonzero convergent power series can be brought into this form by a linear change of coordinates in $\mathbb{C}^{n+1}$. In fact one can find such coordinates simultaneaously for any countably subset of ${}_{n+1}\mathcal{O}$.†

*If f is not a unit and is not irreducible then there exist nonunits f_1 and f_2 such that $f = f_1 f_2$. Now continue by induction. Since $m(f_1) + m(f_2) = m(f)$ and $m(f_i) > 0$ this process must terminate after finitely many steps.

†For every $f \in {}_n\mathcal{O}$ the set of unit vectors $\zeta \in S^{2n-1} \subset \mathbb{C}^n$, for which the function $t \mapsto f(t\zeta)$ (in one complex variable) is nonzero, is open and dense in S^{2n-1}.

Theorem F.7. (Weierstrass preparation theorem) *Suppose that $f \in {}_{n+1}\mathcal{O}$ is distinguished of order m in z_0. Then f decomposes uniquely in the form $f = \omega \cdot e$ where ω is a Weierstrass polynomial of order m and e is a unit.*

Theorem F.8. (Weierstrass division theorem) *If ω is a Weierstrass polynomial of degree m then for every $f \in {}_{n+1}\mathcal{O}$ there exists a $g \in {}_{n+1}\mathcal{O}$ and a polynomial $r \in {}_n\mathcal{O}[z_0]$ such that*

$$f = g\omega + r, \qquad \deg r < m.$$

Exercise F.9 Let $f \in {}_{n+1}\mathcal{O}$ be a Weierstrass polynomial in z_0. Prove that f is irreducable in ${}_{n+1}\mathcal{O}$ if and only if it is irreducible in ${}_n\mathcal{O}[z_0]$. Prove that two Weierstrass polynomials $f, g \in {}_{n+1}\mathcal{O}$ are coprime in ${}_{n+1}\mathcal{O}$ if and only if they are coprime in ${}_n\mathcal{O}[z_0]$. **Hint:** If $f \in {}_{n+1}\mathcal{O}$ is distinguished in the variable z_0 then so are the factors $f_i \in {}_{n+1}\mathcal{O}$ in any decomposition $f = f_1 \cdots f_k$. Thus each factor in a decomposition of a Weierstrass polynomial is a product of a Weierstrass polynomial and a unit. □

Proof of Theorem F.4: Suppose, by induction, that ${}_n\mathcal{O}$ is factorial and let $\omega \in {}_{n+1}\mathcal{O}$. By Theorems F.7 and F.8 we may assume without loss of generality that ω is a Weierstrass polynomial. Thus ω is a polynomial in z_0 with coefficients in $R = {}_n\mathcal{O}$. Moreover, it is a prime in this ring if and only if it is prime in ${}_{n+1}\mathcal{O}$. Hence the result follows from Gauss's lemma F.5. For more details see [45], p 10, or [55], p 81. □

Lemma F.10 *Suppose that $f, g \in {}_n\mathcal{O}$ are relatively prime. Then f_z and g_z are relatively prime for z sufficiently small.*

Proof: Replace n by $n+1$ and assume without loss of generality that f and g are both Weierstrass polynomials in z_0. Then f and g are relatively prime in the polynomial ring ${}_n\mathcal{O}[z_0]$ and, by Lemma F.6, there exist $\alpha, \beta \in {}_n\mathcal{O}[z_0]$ such that

$$\alpha f + \beta g = \gamma, \qquad 0 \neq \gamma \in {}_n\mathcal{O}.$$

Now suppose that $f(z_0, z) = 0$ and $g(z_0, z) = 0$ and that $f_{z_0,z}$ and $g_{z_0,z}$ have a common factor h which is not a unit. Then h divides γ_z and this implies $h \in {}_n\mathcal{O}$. Since h divides $f_{z_0,z}$ and vanishes at 0 it follows that $f(z_0 + \zeta_0, z) = 0$ for ζ_0 sufficiently small. But for z sufficiently small this is impossible because f is a Weierstrass polynomial in z_0. This proves the lemma. □

A ring $\mathcal{R}$ is called *Noetherian* if every nonempty set of ideals in $\mathcal{R}$ has a maximal element. An equivalent condition is that every ideal $\mathcal{I} \subset \mathcal{R}$ is finitely generated (as a module over $\mathcal{R}$). This means that there exist finitely many elements $a_1, \ldots, a_k \in \mathcal{I}$ such that every $a \in \mathcal{I}$ can be expressed in the form $a = \sum_i f_i a_i$ with $f_i \in \mathcal{R}$.

Theorem F.11 *The ring ${}_n\mathcal{O}$ is Noetherian.*

Proof: The proof is by induction over n. For $n = 0$ the ring ${}_0\mathcal{O} = \mathbb{C}$ is obviously Noetherian. Hence assume that ${}_n\mathcal{O}$ is Noetherian for some $n \geq 0$ and let $\mathcal{I} \subset {}_{n+1}\mathcal{O}$ be an ideal. By a generic linear change of coordinates, we may assume without loss of generality that $\mathcal{I}$ contains some element f which is distinguished of order m in z_0. For any such element it is a consequence of the Weierstrass division theorem that the map

$$({}_n\mathcal{O})^m \to \frac{{}_{n+1}\mathcal{O}}{f \cdot {}_n\mathcal{O}} : (a_0, \ldots, a_{m-1}) \mapsto \sum_{j=1}^{m} a_j {z_0}^{m-j}$$

is an isomorphism of free ${}_n\mathcal{O}$-modules. By the induction hypothesis the ideal $\bar{\mathcal{I}} = \mathcal{I}/f \cdot {}_n\mathcal{O} \subset {}_{n+1}\mathcal{O}/f \cdot {}_n\mathcal{O}$ is finitely generated. Let $\bar{f}_1, \ldots, \bar{f}_k$ be generators and denote $f_0 = f$. Then $f_0, f_1, \ldots, f_k$ generate $\mathcal{I}$. □

F.3 Nullstellensatz

Given an ideal $\mathcal{I} \subset {}_n\mathcal{O}$ denote by

$$\mathcal{V}(\mathcal{I}) = \{z \in \mathbb{C}^n \,|\, f(z) = 0 \;\forall\, f \in \mathcal{I} \text{ with } z \in \operatorname{dom} f\}$$

the common zero set. For any (Zariski) closed subset $\mathcal{V} \subset \mathbb{C}^n$ denote by

$$\mathcal{I}(\mathcal{V}) = \{f \in {}_n\mathcal{O} \,|\, \exists \varepsilon > 0 \ni f|_{\mathcal{V}\cap B_\varepsilon} = 0\}$$

the ideal of all power series which vanish on $\mathcal{V}$ (intersected with some nighbourhood of zero). The **radical** of an ideal $\mathcal{I} \subset {}_n\mathcal{O}$ is defined by

$$\sqrt{\mathcal{I}} = \left\{f \in {}_n\mathcal{O} \,|\, \exists\, k \geq 1 \ni f^k \in \mathcal{I}\right\}.$$

The next theorem is *Hilbert's Nullstellensatz*. It asserts that the radical of $\mathcal{I}$ agrees with the ideal of power series which vanish on $\mathcal{V}(\mathcal{I})$.

Theorem F.12. (Nullstellensatz) *For every ideal $\mathcal{I} \subset {}_n\mathcal{O}$*

$$\mathcal{I}(\mathcal{V}(\mathcal{I})) = \sqrt{\mathcal{I}}.$$

In particular, if $p \in {}_n\mathcal{O}$ is irreducible then $p|f$ if and only if f vanishes on the zero set of p (intersected with some neighbourhood of zero).

A nontrivial ideal $\mathcal{I} \subset \mathcal{R}$ is called **irreducible** if it cannot be expressed as the intersection of two ideals which are both not equal to $\mathcal{I}$. It is called **prime** if $fg \in \mathcal{I}$ implies that either $f \in \mathcal{I}$ or $g \in \mathcal{I}$. It is called **primary** if $fg \in \mathcal{I}$ implies that either $f \in \mathcal{I}$ or $g \in \sqrt{\mathcal{I}}$. It follows easily from the definitions that if $\mathcal{I}$ is a primary ideal then $\sqrt{\mathcal{I}}$ is a prime ideal and that every prime ideal is irreducible. In [5], Chapter 7, it is proved that in a Noetherian ring every ideal is a finite intersection of irreducible ideals and every irreducible ideal is primary. Thus in a Noetherian ring $\mathcal{R}$ the radical of an ideal $\mathcal{I}$ is the intersection of all prime ideals $\mathcal{J} \subset \mathcal{R}$ with $\mathcal{I} \subset \mathcal{J}$:

$$\sqrt{\mathcal{I}} = \bigcap_{\substack{\mathcal{I} \subset \mathcal{J} \\ \mathcal{J}\ prime}} \mathcal{J}. \tag{F.2}$$

This shows that it suffices to prove Hilbert's Nullstellensatz for prime ideals $\mathcal{I} \subset {}_n\mathcal{O}$. Any such ideal is generated by finitely many primes $p_1, \dots, p_k \in {}_n\mathcal{O}$.* Geometrically, one can think of $\mathcal{V}(\mathcal{I})$ as the common zero set of the p_i and the Nullstellensatz asserts that every $f \in {}_n\mathcal{O}$ which vanishes on this common zero set can be expressed in the form $f = \sum_{i=1}^k f_i p_i$.

An ideal $\mathcal{I} \subset {}_n\mathcal{O}$ is called **regular** if there exists an integer k with $0 \le k \le n-1$ and power series $f_{k+1}, \dots, f_n \in \mathcal{I}$ such that

$$0 \ne f \in \mathcal{I} \qquad \Longrightarrow \qquad \frac{\partial f}{\partial z_j} \not\equiv 0 \text{ for some } j > k \tag{F.3}$$

and, for each j, the power series f_j is distinguished of order j in the variable z_j and is independent of the variables $z_{j+1}, \dots, z_n$. By the Weierstrass preparation theorem, we may assume without loss of generality that each f_j is a Weierstrass polynomial in z_j with coefficients in ${}_{j-1}\mathcal{O}$. (In the case $j = 1$ we use the convention ${}_0\mathcal{O} = \mathbb{C}$.) It follows easily by induction that for every ideal $\mathcal{I} \subset {}_n\mathcal{O}$, which is not zero and not equal to ${}_n\mathcal{O}$, there exists a linear transformation $\Psi \in \mathrm{GL}(n, \mathbb{C})$ such that the ideal $\mathcal{J} = \{f \circ \Psi \mid f \in \mathcal{I}\}$ is regular.†

Proof of Theorem F.12: In view of (F.2) and the above remarks it suffices to prove the result for regular prime ideals. Hence assume that there exist an integer k with $0 \le k \le n-1$ and power series $f_{k+1}, \dots, f_n \in \mathcal{I}$ such that (F.3) holds and each f_j is a Weierstrass polynomial in z_j with coefficients in ${}_{j-1}\mathcal{O}$, i.e.

$$f_j(z) = {z_j}^{m_j} + \sum_{\nu=0}^{m_j-1} a_{j\nu}(z_1, \dots, z_{j-1}){z_j}^{\nu}, \qquad j = k+1, \dots, n, \tag{F.4}$$

*Given any finite set of generators, use the prime factorization property to replace these generators by suitable prime factors. *Exercise:* Show that the prime generators of $\mathcal{I}$ are determined uniquely up to multiplication by units.

†First choose any nonzero element $f_n \in \mathcal{I}$ and apply a transformation such that f_n is distinguished in z_n. If all nonzero elements of $\mathcal{I}$ satisfy $\partial_n f \not\equiv 0$ we are done. Otherwise choose a nonzero element $f_{n-1} \in \mathcal{I}$ with $\partial_n f_{n-1} \equiv 0$ and apply a linear transformation of the variables $z_1, \dots, z_{n-1}$ after which f_{n-1} is distinguished in z_{n-1}. Now proceed by induction.

where $m_j > 0$ and $a_j \in {}_{j-1}\mathcal{O}$ vanishes at the origin. Assume without loss of generality that the f_j are irreducible. Otherwise replace f_j by one of its prime factors which, by Exercise F.9, can be chosen to be a Weierstrass polynomial in z_j with coefficients in ${}_{j-1}\mathcal{O}$. We claim first that

$$\mathcal{I} = \langle f_{k+1}, \dots, f_n \rangle.$$

Suppose, by contradiction, that there exists an element $f \in \mathcal{I}$ which is coprime to $f_{k+1}, \dots, f_n$. Then an easy induction argument, based on Lemma F.6 and the Weierstrass division theorem, shows that there exist power series $\alpha_{k+1}, \dots, \alpha_n, \beta \in {}_n\mathcal{O}$ such that

$$\sum_{j=k+1}^{n} \alpha_j f_j + \beta f = \gamma, \qquad 0 \neq \gamma \in {}_k\mathcal{O}. \tag{F.5}$$

Since γ is a nonzero element of $\mathcal{I}$ this contradicts (F.3). Thus we have proved that $\mathcal{I}$ is generated by the f_j.

Now suppose, by contradiction, that $\mathcal{I} \neq \mathcal{I}(\mathcal{V}(\mathcal{I}))$ and choose a powerseries $f \in \mathcal{I}(\mathcal{V}(\mathcal{I})) - \mathcal{I}$. Then f is coprime to f_j for all j and hence, as above, there exist power series $\alpha_j, \beta \in {}_n\mathcal{O}$ such that (F.5) holds. Thus $\gamma \in {}_k\mathcal{O} \subset {}_n\mathcal{O}$ is a power series in the variables $z_1, \dots, z_k$ which vanishes on $\mathcal{V}(\mathcal{I}) \cap B_\varepsilon(0)$ for some $\varepsilon > 0$. A moment's thought shows that for each sufficiently small vector $x = (z_1, \dots, z_k) \in \mathbb{C}^k$ there exists a point $y = (z_{k+1}, \dots, z_n) \in \mathbb{C}^{n-k}$ such that $|y| < \varepsilon/2$ and $z = (x, y) \in \mathcal{V}(\mathcal{I})$. Namely, given $z_1, \dots, z_j$ with $k \leq j < n$ choose z_{j+1} such that $f_{j+1}(z_1, \dots, z_{j+1}) = 0$. Hence $\gamma(x) = 0$ for every sufficiently small vector $x \in \mathbb{C}^k$ and thus $\gamma \equiv 0$, condradicting (F.5). This proves the theorem. □

F.4 Analytic varieties

The local case

A subset $\mathcal{V} \subset \mathbb{C}^n$ is called a **local analytic variety in $\mathbb{C}^n$ at zero** if there exist an open neighbourhood $U \subset \mathbb{C}^n$ of zero and finitely many power series $f_1, \dots, f_k \in {}_n\mathcal{O}$ such that $U \subset \operatorname{dom} f_j$ for all j and $\mathcal{V} = \{z \in U \mid f_1(z) = \cdots = f_k(z) = 0\}$. If $\mathcal{I} = \langle f_1, \dots, f_k \rangle$ denotes the ideal generated by the f_j then $\mathcal{V} \cap U = \mathcal{V}(\mathcal{I}) \cap U$ and, conversely, the Nullstellensatz asserts that the ideal $\sqrt{\mathcal{I}}$ can be recovered from $\mathcal{V}$ as the set of all power series $f \in {}_n\mathcal{O}$ which vanish on $\mathcal{V} \cap U$ for some neighbourhood U of zero, namely $\sqrt{\mathcal{I}} = \mathcal{I}(\mathcal{V})$. Two local analytic varieties $\mathcal{V}_1$ and $\mathcal{V}_2$ at zero are called **equivalent** if there exists an open neighbourhood $U \subset \mathbb{C}^n$ of zero such that $\mathcal{V}_1 \cap U = \mathcal{V}_2 \cap U$. Obviously, equivalent varieties give rise to the same ideal $\mathcal{I}(\mathcal{V}_1) = \mathcal{I}(\mathcal{V}_2)$. Thus there is a one-to-one correspondence between equivalence classes of local analytic varieties in $\mathbb{C}^n$ at zero and ideals $\mathcal{I} \subset {}_n\mathcal{O}$ which agree with their radical $\sqrt{\mathcal{I}} = \mathcal{I}$. The empty variety corresponds to $\mathcal{I} = {}_n\mathcal{O}$ and the total space $\mathcal{V} = \mathbb{C}^n$ to $\mathcal{I} = \{0\}$. Denote by $[\mathcal{V}]$ the equivalence class of a local analytic variety at zero. Note that these equivalence classes form a lattice with lattice operations

$$[\mathcal{V}_1] \cap [\mathcal{V}_2] = [\mathcal{V}_1 \cap \mathcal{V}_2], \qquad [\mathcal{V}_1] \cup [\mathcal{V}_2] = [(\mathcal{V}_1 \cup \mathcal{V}_2) \cap U]$$

for a sufficiently small neighbourhood U of zero. One can think of the set of radical ideals as the dual lattice with operations $(\mathcal{I}_1, \mathcal{I}_2) \mapsto \sqrt{\mathcal{I}_1 + \mathcal{I}_2}$ and $(\mathcal{I}_1, \mathcal{I}_2) \mapsto \mathcal{I}_1 \cap \mathcal{I}_2$:

$$\mathcal{I}(\mathcal{V}_1 \cap \mathcal{V}_2) = \sqrt{\mathcal{I}(\mathcal{V}_1) + \mathcal{I}(\mathcal{V}_2)}, \qquad \mathcal{I}(\mathcal{V}_1 \cup \mathcal{V}_2) = \mathcal{I}(\mathcal{V}_1) \cap \mathcal{I}(\mathcal{V}_2).$$

A local analytic variety $\mathcal{V}$ is called **irreducible** if it does not decompose as a union $[\mathcal{V}] = [\mathcal{V}_1] \cup [\mathcal{V}_2]$ of two local analytic varieties which are both not equivalent to $\mathcal{V}$. Equivalently, the corresponding ideal $\mathcal{I} = \mathcal{I}(\mathcal{V})$ cannot be expressed as the intersection $\mathcal{I} = \mathcal{I}_1 \cap \mathcal{I}_2$ of two ideals which are both not equal to $\mathcal{I}$. But this means that $\mathcal{I}(\mathcal{V})$ is irreducible. Since in a Noetherian ring every irreduducible ideal $\mathcal{I} = \sqrt{\mathcal{I}}$ is prime, we have proved the following.

Proposition F.13 *Let $\mathcal{V} \subset \mathbb{C}^n$ be a local analytic variety at zero. Then $\mathcal{V}$ is irreducible if and only if $\mathcal{I}(\mathcal{V})$ is a prime ideal.*

Proposition F.14 *Every local analytic variety $\mathcal{V} \subset \mathbb{C}^n$ at zero decomposes uniquely into a finite union of irreducible subvarieties.*

Proof: Proposition F.13 and (F.2). □

Exercise F.15 A local analytic variety $\mathcal{V} \subset \mathbb{C}^n$ at zero is called **nonsingular** (at zero) if there exist generators $f_1, \dots f_k$ of $\mathcal{I}(V)$ such that the linear functionals $df_1(0), \dots, df_k(0)$ on $\mathbb{C}^n$ are linearly independent. Prove that if this holds then $\mathcal{I}$ is a prime ideal and the f_i are irreducible. Show that a nonempty local analytic hypersurface $\mathcal{V} \subset \mathbb{C}^n$ at zero with $\mathcal{I}(\mathcal{V}) = \langle f \rangle$ is nonsingular iff $m(f) = 1$ and singular iff $m(f) > 1$, where $m(f)$ is defined by (F.1). □

Let $\mathcal{V} \subset \mathbb{C}^n$ be an irreducible local analytic variety at zero. The **codimension** of $\mathcal{V}$ is defined as the minimal number of generators of the ideal $\mathcal{I} = \mathcal{I}(\mathcal{V})$. A local analytic variety of codimension 1 is called an **analytic hypersurface**. In this case $\mathcal{I} = \mathcal{I}(\mathcal{V}) = \langle f \rangle$ is a principal ideal. The next lemma characterizes generators of radical ideals.

Lemma F.16 Let $\mathcal{I} = \langle f \rangle \subset {}_n\mathcal{O}$ be a principal ideal. Then the following are equivalent.

(i) $\mathcal{I} = \sqrt{\mathcal{I}}$.

(ii) $f = p_1 \cdots p_k$ is a product of irreducibles which are pairwise relatively prime.

(iii) There exists a vector $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{C}^n$ such that f and $\sum_j \omega_j \partial_j f$ are relatively prime.

Proof: We prove that (i) is equivalent to (ii). Let $f = p_1^{a_1} \cdots p_k^{a_k}$ be the prime decomposition of f. If $a_j > 1$ for some j then $p_1 \cdots p_k \in \sqrt{\mathcal{I}} - \mathcal{I}$. Conversely, suppose that $a_j = 1$ for all j and let $g \in \sqrt{\mathcal{I}}$. Then g^m is divisible by f for some $m \in \mathbb{N}$ and hence g^m is divisible by p_j for all j. Since each p_j is irreeducible, g is divisible by p_j for all j. Since the p_j are pairwise relatively prime, g is divisible by $p_1 \cdots p_k = f$. Hence $g \in \mathcal{I}$.

We prove that (ii) implies (iii). Abbreviate $D = \sum_{j=1}^n \omega_j \partial_j$ and choose $\omega \in \mathbb{C}^n$ such that $Dp_i \not\equiv 0$ for all $i \in \{1, \ldots, k\}$. Then p_i does not divide Dp_i for all i. This implies that p_i does not divide

$$Df = \sum_{i=1}^k (Dp_i) \prod_{i' \neq i} p_{i'}$$

for all i. Hence f and Df are relatively prime.

We prove that (iii) implies (ii). If (ii) does not hold and $f = p_1^{a_1} \cdots p_k^{a_k}$ is the prime decomposition of f then $a_i > 1$ for some i. Hence p_i divides $Df = \sum_j \omega_j \partial_j f$ for all $\omega \in \mathbb{C}^n$ and thus (iii) does not hold. This proves the lemma. □

For $f \in {}_n\mathcal{O}$ and $U \subset \operatorname{dom} f$ we denote

$$\mathcal{V}(f, U) = \{z \in U \mid f(z) = 0\},$$

$$\mathcal{V}_s(f, U) = \{z \in U \mid f(z) = 0,\, df(z) = 0\}.$$

Proposition F.17 *Let $\mathcal{V} \subset \mathbb{C}^n$ be a local analytic hypersurface and $f \in {}_n\mathcal{O}$ be a generator of $\mathcal{I}(\mathcal{V})$. Then the following holds.*

(i) *The set $\mathcal{V}(f, U) - \mathcal{V}_s(f, U)$ is dense in $\mathcal{V}(f, U)$ for every sufficiently small neighbourhood $U \subset \operatorname{dom} f$ of zero.*

(ii) *The set $U - \mathcal{V}(f, U)$ is connected for every sufficiently small connected neighbourhood U of zero.*

(iii) *If f is irreducible and relatively prime to $g \in {}_n\mathcal{O}$ then for every $\varepsilon > 0$ there exists a neighbourhood $U \subset B_\varepsilon(0)$ of zero such that $\mathcal{V}(f, U) - \mathcal{V}_s(fg, U)$ is connected.*

Proof: By assumption the ideal $\mathcal{I} = \mathcal{I}(\mathcal{V}) = \langle f \rangle$ agrees with it radical $\sqrt{\mathcal{I}}$. By Lemma F.16, there exists a vector $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{C}^n$ such that f and $Df = \sum_{j=1}^n \omega_j \partial_j f$ are relatively prime. By Lemma F.10, f_z and Df_z are relatively prime for z sufficiently small. Hence, by the Nullstellensatz, there is a sequence $z_\nu \to z$ with $f(z_\nu) = 0$ and $Df(z_\nu) \neq 0$. This proves (i).

To prove (ii) choose a connected neighbourhood $U \subset \mathrm{dom}\, f$ of zero and, for every multi-index ν, define the set

$$\mathcal{V}_\nu(f,U) = \{z \in U \,|\, \partial^\nu f(z) = 0,\, \exists\, j \ni \partial_j \partial^\nu f(z) \neq 0\}\,.$$

These are a complex codimension-1 submanifolds and their union contains $\mathcal{V}(f,U)$. Thus $\mathcal{V}(f,U)$ is contained in the union of at most countably many complex codimension-1 submanifolds of U. Since U is connected it follows by a standard transversality argument that $U - \mathcal{V}(f,U)$ is connected.

To prove (iii) let us assume that $\mathcal{I}(\mathcal{V})$ is a principal ideal with an irreducible generator f and let $g \in {}_n\mathcal{O}$ be relatively prime to f. We must prove that there exists a neighbourhood $U \subset \mathbb{C}^n$ of zero such that the set $\mathcal{V}(f,U) - \mathcal{V}_s(fg,U)$ is connected. To see this let us assume, without loss of generality, that f and g are Weierstrass polynomial in z_n of positive degree. Then f is relatively prime to both g and $\partial_n f$, hence to their product, and hence to $\partial_n(fg)$. Thus, by Lemma F.6, there exist polynomials $\alpha, \beta \in {}_{n-1}\mathcal{O}[z_n]$ such that

$$\alpha f + \beta \partial_n(fg) = \gamma, \qquad 0 \neq \gamma \in {}_{n-1}\mathcal{O}. \tag{F.6}$$

Denote $z' = (z_1, \ldots, z_{n-1})$. Choose a small connected neighbourhood $U' \subset \mathbb{C}^{n-1}$ of zero which is contained in the domain of convergence of all the power series appearing in (F.6) and let $U \subset \mathbb{C}^n$ be a corresponding product neighbourhood. Note that if $z = (z', z_n) \in \mathcal{V} \cap U$ with $\gamma(z') \neq 0$ then $f(z) = 0$ and $g(z)\partial_n f(z) = \partial_n(fg)(z) \neq 0$ and thus, in particular, $z \in \mathcal{V}(f,U) - \mathcal{V}_s(fg,U)$. We claim that $\mathcal{V}(f,U) - \gamma^{-1}(0)$ is connected. To see this let

$$\mathcal{V}' \subset \mathcal{V}(f,U) - \gamma^{-1}(0)$$

be one of the components and consider the projection*

$$\mathcal{V}' \to U' - \gamma^{-1}(0) : z = (z_1, \ldots, z_n) \mapsto z' = (z_1, \ldots, z_{n-1}).$$

This is a covering fibration and, by (i), the base is connected. Hence the number of points in the fiber is independent of the choice of the base point. Let this number be ℓ and consider the function

$$f'(z) = \prod_{j=1}^{\ell} (z_n - w_j(z'))$$

where for each $z' \in U'$ with $\gamma(z') \neq 0$ the points $w_1(z'), \ldots, w_\ell(z') \in \mathbb{C}$ are the unique z_n-coordinates with $(z', w_j(z')) \in \mathcal{V}'$. If $z = (z', z_n) \in U$

*With slight abuse of notation, we denote by $\gamma^{-1}(0)$ both the set of all $z' \in U'$ with $\gamma(z') = 0$ and the set of all $z = (z', z_n) \in U$ with $\gamma(z') = 0$.

with $\gamma(z') \neq 0$ then the construction shows that $f'(z) = 0$ if and only if $z \in \mathcal{V}'$. Moreover f' is holomorphic wherever defined and bounded. Using the Weierstrass preparation theorem for γ and the Cauchy integral formula one can show that under these hypotheses f' extends to a holomorphic function on U'. The zero locus of this extension is precisely the closure of $\mathcal{V}'$. In particular, $\mathcal{V}(f', U) \subset \mathcal{V}(f, U)$ and hence, by Hilbert's Nullstellensatz, some power of f is divisible by f'. Since f is irreducible, this implies that f' is equal to some power of f up to multiplication by a unit. (The power is 1 but this is immaterial.) Hence $\mathcal{V}(f', U) = \mathcal{V}(f, U)$ and so

$$\mathcal{V}(f, U) - \gamma^{-1}(0) = \mathcal{V}'$$

is connected. It follows that $\mathcal{V}(f, U) - \mathcal{V}_s(fg, U)$ is also connected provided that U is chosen sufficiently small. To see this note first that f does not vanish on the line $z = (0, z_n)$ and hence does not divide γ. Thus f and γ are relatively prime and, by Lemma F.10, f_z and $\gamma_{z'}$ are relatively prime for $z = (z', z_n)$ sufficiently small. Hence no power of $\gamma_{z'}$ is divisible by f_z. By the Nullstellensatz, this shows that every point $z \in \mathcal{V}$ which is sufficiently close to zero can be approximated by a sequence $z_\nu = (z'_\nu, z_{n,\nu})$ with $\gamma(z'_\nu) \neq 0$ and $f(z_\nu) = 0$. It follows that every point $z \in \mathcal{V}(f, U) - \mathcal{V}_s(fg, U)$ can be connected by a short path in $\mathcal{V}(f, U) - \mathcal{V}_s(fg, U)$ to a point in $\mathcal{V}(f, U) - \gamma^{-1}(0)$. Hence $\mathcal{V}(f, U) - \mathcal{V}_s(fg, U)$ is connected, as claimed. □

The global case

Let X be a complex manifold. For $x \in X$ denote by ${}_X\mathcal{O}_x = \mathcal{O}_x$ the ring of (equivalence classes of) local holomorphic functions on X defined in some neighbourhood of x. Two such holomorphic functions are equivalent if they agree in some neighbourhood of x. Given any holomorphic chart near x this ring can be naturally identified with the ring ${}_n\mathcal{O}$ of convergent power series. In particular, $\mathcal{O}_x$ is factorial and there is a semigroup homomorphism

$$m_x : \mathcal{O}_x \to \mathbb{N}$$

defined as in (F.1). A closed subset $V \subset X$ is called an **analytic hypersurface** if for every point $x \in V$ there exists a neighbourhood U of x and a nonzero local holomorphic function $f : U \to \mathbb{C}$ such that $V \cap U = f^{-1}(0)$. As an element of $\mathcal{O}_x$ the local holomorphic function f decomposes as $f = {p_1}^{a_1} \cdots {p_k}^{a_k}$ for irreducibles $p_1, \ldots, p_n \in \mathcal{O}_x$. Obviously, the zero locus of f agrees with that of $f_0 = p_1 \cdots p_k$. Moreover, by Hilbert's Nullstellensatz, f_0 is minimal in the sense that every $f \in \mathcal{O}_x$ which vanishes on V near x is divisible by f_0 in $\mathcal{O}_x$. Such a function $f_0 : U_0 \to \mathbb{C}$ is called a **defining function for** V **at** x. In other words the variety V determines a principal ideal

$$\mathcal{I}_x = \mathcal{I}_x(V) \subset \mathcal{O}_x$$

at every point $x \in X$ which is defined as the set of all equivalence classes of holomorphic maps $f : U \to \mathbb{C}$ which are defined in some neighbourhood $U \subset X$ of x and vanish on $U \cap V$. Note that $\mathcal{I}_x \subset \mathcal{O}_x$ is always nonzero and is a proper ideal if and only if $x \in V$. Any defining function f of V at x is a generator of $\mathcal{I}_x$ and the number $m_x(f)$ is independent of the choice of the generator. This determines a map $m_V : X \to \mathbb{N}$ defined by

$$m_V(x) = \inf_{f \in \mathcal{I}_x(V)} m_x(f). \tag{F.7}$$

Call x a **smooth point** of V if $m_V(x) = 1$ and a **singular point** if $m_V(x) > 1$. Denote the singular part by

$$V_s = \{x \in V \mid m_V(x) > 1\}.$$

Thus x is a smooth point of V iff $df(x) \neq 0$ for some (and hence every) defining function $f \in \mathcal{O}_x$. It follows that the smooth points form a complex codimension-1 submanifold $V - V_s \subset X$. By Proposition F.17 (i), the set $V - V_s$ is dense in V.* The following theorem introduces the crucial notion of irreducible analytic hypersurfaces. The proof follows [45], p 21/22.

Theorem F.18 *Let X be a complex manifold and $V \subset X$ be an analytic hypersurface. Then the following are equivalent.*

(i) *$V - V_s$ is connected.*

(ii) *If $V_1, V_2 \subset X$ are analytic hypersurfaces with $V_1 \cup V_2 = V$ then either $V_1 = V$ or $V_2 = V$.*

If these conditions are satisfied then V is called **irreducible**. *Moreover, every analytic hypersurface of a compact complex manifold decomposes uniquely into finitely many irreducible ones.*

Proof: We prove that (i) implies (ii). If (ii) does not hold then $V = V_1 \cup V_2$ is the union of two analytic hypersurfaces V_1 and V_2 neither of which is equal to V. Then $V - V_s$ can be expressed as the disjoint union of the three subsets

$$V - V_s = (V_1 - (V_s \cup V_2)) \cup (V_2 - (V_s \cup V_1)) \cup (V_1 \cap V_2 - V_s).$$

The first two subsets are obviously open in $V - V_s$. Moreover, $V_1 - V_2$ and $V_2 - V_1$ are nonempty and, since $V - V_s$ is dense in V and V_1 and V_2 are closed, it follows that $V_1 - (V_s \cup V_2)$ and $V_2 - (V_s \cup V_1)$ are nonempty. We claim that the third subset $V_1 \cap V_2 - V_s$ is open in $V - V_s$. To see this let

*Let $f \in \mathcal{O}_x$ be a defining function for V at x. By Proposition F.17 (iii), the set $\mathcal{V}(f, U) - \mathcal{V}_s(f, U)$ is dense in $\mathcal{V}(f, U)$ for every sufficiently small neighbourhood U of x. Since $V_s \cap U \subset \mathcal{V}_s(f, U)$ there exists a sequence $x_\nu \in V - V_s$ converging to x.

$x \in V_1 \cap V_2 - V_s$ and let $f_1, f_2, f \in \mathcal{O}_x$ be defining functions of V_1, V_2, V, respectively. Then $f_1 f_2 \in \mathcal{I}_x(V)$ and hence f divides $f_1 f_2$. Moreover, since $m(f) = 1$, f is a prime and hence divides either f_1 or f_2. Suppose f divides f_1. Then there exists a neighbourhood U of x such that $V \cap U \subset V_1 \cap U$ and hence $V \cap U = V_1 \cap U$. This implies that $\mathcal{I}_x(V_1) = \mathcal{I}_x(V) \subset \mathcal{I}_x(V_2)$, hence f_2 divides f, and since $0 < m(f_2) \leq m(f) = 1$ this implies that f_2 and f differ by a unit. Thus $\mathcal{I}_x(V_1) = \mathcal{I}_x(V) = \mathcal{I}_x(V_2)$ and so there exists a neighbourhood U of x such that $V_1 \cap U = V_2 \cap U$. This shows that the set $V_1 \cap V_2 - V_s$ is open in $V - V_s$. Thus we have decomposed the set $V - V_s$ into three open subsets, two of which are nonempty. Hence $V - V_s$ is disconnected.

We prove that (ii) implies (i). We argue indirectly, and assume that $V - V_s$ is disconnected. We claim that the closure of each component is again an analytic hypersurface. More precisely, fix a component V' of $V - V_s$ and consider its closure

$$\mathrm{cl}(V') \subset V' \cup V_s.$$

We must prove that this set admits a defining function at each point $x \in \mathrm{cl}(V) \cap V_s$. To construct such a function let us choose a local defining function $f : U \to \mathbb{C}$ for V in some neighbourhood of x. Let $f = f_1 \cdots f_k$ be a prime factorization of f and suppose that the factors f_j all converge in U. By Proposition F.17 (iii), with f replaced by f_i and g replaced by the product of the other factors, the set $\mathcal{V}(f_i, U) - \mathcal{V}_s(f, U)$ is connected (for a suitably chosen neighbourhood U). Moreover, Proposition F.17 (ii) shows that the zero locus of f_i is the closure of $\mathcal{V}(f_i, U) - \mathcal{V}_s(f, U)$. Thus a local defining function for $\mathrm{cl}(V')$ is the product f' of those f_i whose zero locus $\mathcal{V}(f_i, U) - \mathcal{V}_s(f, U)$ is contained in V'. This shows that $\mathrm{cl}(V')$ is an analytic hypersurface. Since $V - V_s$ is disconnected it follows that V is the union of two distinct analytic hypersurfaces V_1 and V_2.

The same argument shows that if X is compact then $V - V_s$ has only finitely many components. Let V' be the closure of such a component. Then V' is an analytic hypersurface and $V'_s \subset V_s$. By assumption $V' - V_s$ is connected. Moreover, the proof of Proposition F.17 shows that every point in $V' - V'_s$ can be connected by a short path to a point in $V' - V_s$. Hence $V' - V'_s$ is connected and thus V' is irreducible. This proves the theorem. $\square$

Proposition F.19 *Let X be a compact complex surface. Then $V \subset X$ is an irreducible analytic hypersurface if and only if there exists a compact connected Riemann surface Σ and a holomorphic map $u : \Sigma \to X$ such that $V = u(\Sigma)$. Moreover, V_s is a finite set for every analytic hypersurface V.*

Proof: It suffices to prove this locally. Hence let $f \in {}_2\mathcal{O}$ be a convergent power series in the variables x and y and assume without loss of generality that f is a Weierstrass polynomial in y. Choose polynomials $\alpha, \beta \in {}_1\mathcal{O}[y]$ such that (F.6) is satisfied for some nonzero power series $\gamma = \gamma(x)$:

$$\alpha(x,y)f(x,y) + \beta(x,y)\frac{\partial f}{\partial y}(x,y) = \gamma(x).$$

Then $x = 0$ is an isolated zero of γ and hence $(x,y) = (0,0)$ is an isolated singularity of $V = f^{-1}(0)$. The intersection of the complement $V - \{(0,0)\}$ with a sufficiently small polydisc D^2 decomposes into finitely many components corresponding to the irreducible factors of f. Suppose first that $V - \{(0,0)\}$ is connected and consider the covering projection

$$\pi : V - \{(0,0)\} \to \mathbb{C} - \{0\}, \qquad \pi(x,y) = x.$$

Suppose that this is an m-fold cover. Then there exists a holomorphic map $\varphi : D \to \mathbb{C}^2$ defined on a sufficiently small disc $D \subset \mathbb{C}$ centered at zero such that $\varphi(D - \{0\}) = V - \{(0,0)\} \cap D^2$ and

$$\pi \circ \varphi(z) = z^m.$$

To see this just note that the projection π has a local holomorphic inverse near every point and hence any such map φ is uniquely determined its value at a single point. A simple covering argument shows that φ exists globally and that the restriction to $D - \{0\}$ is a bijection onto $(V - \{(0,0)\}) \cap D^2$. This defines a local complex manifold structure on $(V - V_s) \cup \{\mathrm{pt}\}$. In general, if $V - \{(0,0)\}$ has several components, one such chart is required for each component, i.e. for each irreducible factor in a local defining function.

Conversely, we must prove that the image of any local holomorphic map $u : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ is the zero locus of some holomorphic function $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$. Consider the equations $x = u_1(z)$ and $y = u_2(z)$. Suppose that $z = 0$ is a zero of u_i with order m_i where $m = m_1 \leq m_2$. Thus $u_1(z) = (zv_1(z))^m$ where $v_1(0) \neq 0$. Replacing z by $zv_1(z)$ we may assume without loss of generality that

$$x = u_1(z) = z^m, \qquad y = u_2(z).$$

Then the function

$$f(x,y) = \prod_{z^m = x} (y - u_2(z))$$

is continuous and is holomorphic where $x \neq 0$. Hence it extends to a holomorphic map on a neighbourhood of $(0,0)$ and its zero locus is obviously the image of u. The proof of irreducibility is left as an exercise. □

F.5 Multiplicity

Let $U \subset \mathbb{C}^n$ be an open set. Denote by $\mathcal{O}(U)^*$ the set of all holomorphic functions $f : U \to \mathbb{C}$ which do not vanish on any open subset of U. There is a natural semigroup homomorphism $\mathcal{O}(U)^* \to \mathrm{Map}(U,\mathbb{N}) : f \mapsto m_f$ which assigns to $f \in \mathcal{O}(U)^*$ the **multiplicity map**

$$m_f(z) = m(f_z) = \min\{|\nu| \,|\, \partial^\nu f(z) \neq 0\}.$$

Note that the points with $m_f(z) = 1$ form a codimension-1 complex submanifold of U and that the set of points with $m_f(z) = 0$ is open.

Proposition F.20 *Let $U \subset \mathbb{C}^n$ be a connected open set and $f : U \to \mathbb{C}$ and $g : U \to \mathbb{C}$ be holomorphic maps. Then the following are equivalent.*

(i) *$m_f(z) \leq m_g(z)$ for all $z \in U$.*

(ii) *For every compact subset $K \subset U$ there exists a constant $c > 0$ such that $|g(z)| \leq c|f(z)|$ for $z \in K$.*

(iii) *There exists a holomorphic function $u : U \to \mathbb{C}$ such that $g = uf$.*

Proof: Obviously (iii) implis (ii). We prove that (ii) implies (i). Suppose, by contradiction, that there exists a point $z \in U$ with $m_g(z) < m_f(z)$. Denote by ψ the homogeneous polynomial of order $m = m_g(z)$ determined by the partial derivatives of g of order m at z. By assumption, ψ is nonzero. Choose $\zeta \in \mathbb{C}^n$ with $\psi(\zeta) \neq 0$ and note that $|\psi(t\zeta)| = |\psi(\zeta)|\,|t|^m$ for $t \in \mathbb{C}$. By Exercise F.3, there exist constants $c > 0$ and $\varepsilon > 0$ such that

$$|f(z+t\zeta)| \leq c|t|^{m+1}, \qquad |g(z+t\zeta) - \psi(t\zeta)| \leq c|t|^{m+1}$$

for $|t| \leq \varepsilon$. With ε sufficiently small it follows that $|g(z+t\zeta)| \geq \delta|t|^m$ and hence

$$\frac{|g(z+t\zeta)|}{|f(z+t\zeta)|} \geq \frac{\delta}{c|t|}$$

for $|t| \leq \varepsilon$. This contradicts (ii).

Next we prove that (i) implies (iii). By Theorem F.1, it suffices to show that u exists locally. Near a point $z \in U$ consider the power series $f_z, g_z \in {}_n\mathcal{O}$. If $m(f_z) = 0$ then $f_z(0) \neq 0$ and so (iii) is satisfied near zero with $u_z = g_z/f_z$. Hence suppose that $m(f_z) > 0$ and thus neither f_z nor g_z are units. By Theorem F.4, choose a prime factorization

$$f_z = {p_1}^{a_1} \cdots {p_k}^{a_k}.$$

Then $p_j(\zeta) = 0$ implies $f_z(\zeta) = 0$, hence $m_g(z+\zeta) \geq m_f(z+\zeta) > 0$, and hence $g_z(\zeta) = 0$. By Hilbert's Nullstellensatz F.12, g_z is divisible by p_j. Hence the prime decomposition of g_z has the form

$$g_z = {p_1}^{b_1} \cdots {p_k}^{b_k} h$$

for some $h \in {}_n\mathcal{O}$ which is either a unit or is not divisible by p_j for any j. Now it follows again from Hilbert's Nullstellensatz that, for each j, there exists a point $\zeta \in \mathbb{C}^n$ with $p_j(\zeta) = 0$ and $p_i(\zeta) \neq 0$ for $i \neq j$ and $h(\zeta) \neq 0$. At this point ζ we have, by Lemma F.2,

$$m_f(z+\zeta) = a_j m_{p_j}(\zeta), \qquad m_g(z+\zeta) = b_j m_{p_j}(\zeta).$$

Since $m_{p_j}(\zeta) > 0$ this implies $b_j \geq a_j$. This proves the result locally. The global statement follows immediately from unique continuation. □

Meromorphic functions

A **meromorphic function** on X is a function $v : \mathrm{dom}(v) \to \mathbb{C}$, defined on an open and dense subset $\mathrm{dom}(v) \subset X$, such that for every $x \in X$ there exists a neighbourhood U of x and two holomorphic functions $f, g : U \to \mathbb{C}$ such that

$$v(x) = f(x)/g(x)$$

for $x \in U \cap \mathrm{dom}(v)$. The choice of the domain is immaterial. The intersection of two domains is is still open and dense and if two meromorphic functions v_1 and v_2 agree on the intersection of their domains then they agree on any point to which they can be continuously extended. Hence from now on, with slight abuse of notation, we shall write $v : X \to \mathbb{C}$ even if v is not defined on all of X. Note that every meromorphic function $v : X \to \mathbb{C}$ determines a multiplicity map $m_v : X \to \mathbb{Z}$ defined by

$$m_v(x) = m_f(x) - m_g(x)$$

for $x \in X$ where $f, g : U \to \mathbb{C}$ are as above. By Lemma F.2, the number $m_v(x)$ is independent of the choice of the holomorphic functions f and g. Note that the multiplicity map m_v is defined at every point of X regardless of whether or not v is defined at this point. As a matter of fact, the value of $m_v(x)$ is of particular interest at those points where v is undefined. It follows from Proposition F.20 that v extends continuously at a point x_0 if and only if $m_v(x) \geq 0$ in some neighbourhood of x_0.* Call x a **pole of order** m if $m_v(x) = -m$ and a **zero of order** m if $m_v(x) = m$. Note that $v : X \to \mathbb{C}$ is everywhere defined, and hence holomorphic, if and only if $m_v(x) \geq 0$ for all $x \in X$. However, this is not a very interesting case because on a compact complex manifold all global holomorphic functions are constant (by the maximum principle).

F.6 Divisors

Definition F.21. (Weil divisor) *A* **divisor** *on a complex manifold X is formal sum*

*$m_v(x) \geq 0$ does not imply that x belongs to the maximal domain of v. Consider the basic example $v(z_1, z_2) = z_1/z_2$.

$$D = \sum_{j=1}^{\ell} m_j V_j$$

where $0 \neq m_j \in \mathbb{Z}$ *and the* V_j *are distinct irreducible hypersurfaces.*

Definition F.22 *A* **divisor** *on a complex manifold* X *is a map* $m : X \to \mathbb{Z}$ *such that for every point* $x \in X$ *there exists a neighbourhood* U *of* x *and two holomorphic functions* $f, g : U \to \mathbb{C}$ *such that*

$$m(x) = m_f(x) - m_g(x)$$

for $x \in U$.

Associated to every Weil divisor $D = \sum_j m_j V_j$ is a function

$$m = \sum_{j=1}^{\ell} m_j m_{V_j} : X \to \mathbb{Z}$$

where m_{V_j} is defined by (F.7). This function m evidently satisfies the requirements of Definition F.22. Namely given any point $x \in X$ choose a neighbourhood U with defining functions $f_j : U \to \mathbb{C}$ for the varieties V_j and define

$$f = \sum_{m_j > 0} m_j f_j, \qquad g = \sum_{m_j < 0} m_j f_j.$$

Conversely, suppose that $m : X \to \mathbb{Z}$ is a divisor in the sense of Definition F.22 and consider the set

$$V = \mathrm{cl}\left(\{x \in X \mid m(x) \neq 0\}\right).$$

This set is a complex hypersurface. To see this fix a point $x \in X$ and choose functions $f, g : U \to \mathbb{C}$ near x with $m = m_f - m_g$. Choose local prime decompositions

$$f = {f_1}^{a_1} \cdots {f_s}^{a_s}, \qquad g = {g_1}^{b_1} \cdots {g_t}^{b_t} \tag{F.8}$$

in $\mathcal{O}_x$ Assume without loss of generality that the f_i and g_j are pairwise relatively prime at x and hence, by Lemma F.10, at each point in U if U is chosen sufficiently small. We shall prove that $V \cap U$ agrees with the zero locus $\mathcal{V}(fg, U) = \{x \in U \mid f(x)g(x) =\}$. Firstly, if $y \in V \cap U$ then $m(y) = \sum_i a_i m_{f_i}(y) - \sum_j b_j m_{g_j}(y) \neq 0$, hence one of the functions f_i or g_j vanishes at y, and hence $y \in \mathcal{V}(fg, U)$. Conversely, it follows from the coprime assumption on the f_i and g_j that $(fg)_y$ is a defining function for $\mathcal{V}(fg, U)$ at y for every $y \in U$. Hence, if $y \in \mathcal{V}(fg, U) - \mathcal{V}_s(fg, U)$ then y is a regular zero of one of the functions f_i or g_j and all the others do not vanish at y. This implies $m(y) \neq 0$ and hence $y \in V$. By Proposition F.17 (ii), this shows that $\mathcal{V}(fg, U) = \mathrm{cl}_U(\mathcal{V}(fg, U) - \mathcal{V}_s(fg, U)) \subset V$. Thus we have proved that $V \cap U = \mathcal{V}(fg, U)$ and hence V is a hypersurface. It also follows that either $m_f(x) \neq 0$ or $m_g(x) \neq 0$ (or both) at every point of V.

By Theorem F.18, V decomposes uniquely as a union of finitely many distinct irreducible hypersurfaces

$$V = \bigcup_{j=1}^{\ell} V_j.$$

The V_j are the closures of the components of $V - V_s$. We claim that m is constant on $V_j - V_s$. First note that, by definition of V_j, this set is connected and hence it suffices to prove that m is locally constant. Let $x \in V_j - V_s$ and choose any defining function f_j for V_j at x. Moreover, choose f and g near x such that they are relatively prime and satisfy $m = m_f - m_g$. Recall that either $m_f(x) \neq 0$ or $m_g(x) \neq 0$. If they are both nonzero and relatively prime then $x \in V_s$ in contradiction to our assumption. Hence assume without loss of generality that $m_f(x) \neq 0$ and $m_g = 0$ near x. If $f_j(y) = 0$ for y near x then $x \in V$, hence $m(y) = m_f(y) \neq 0$, and hence $f(y) = 0$. Since f_j is a defining function for V_j this shows that f_j divides f. Write $f = {f_j}^{m_j} u$ for some positive integer m_j where u is not divisible by f_j. Then u must be a unit because otherwise x would not be a smooth point of V. Hence $m_f(y) = m_j m_{f_j}(y) = m_j$ for all y near x. Thus we have proved that m is constant on $V_j - V_s$ for every j. Denote this constant by m_j. Then m_j is a nonzero integer and

$$m = \sum_{j=1}^{\ell} m_j m_{V_j}.$$

This equation is satisfied on $V - V_s$, by definition of m_j. In a neighbourhood of a singularity this formula can be proved again by choosing prime decompositions (F.8) of f and g. Then, if $f_r(y) = 0$ for some $y \in V_j - V_s$ near x and some r, we must have $a_r = m_j > 0$ and all the other factors of f and g are nonzero at this point y. Similarly, if $g_r(y) = 0$ for some $y \in V_j - V_s$ near x then $b_r = -m_j > 0$ and again all the other factors of g and f are nonzero at this point. Thus each V_j with $m_j > 0$ is the common zero set of finitely many f_r with $a_r = m_j$ and thus m_{V_j} is the sum of the corresponding functions m_{f_r}. Similarly, each V_j with $m_j > 0$ is the common zero set of finitely many g_r with $b_r = -m_j$ and thus m_{V_j} is the sum of the corresponding functions m_{g_r}. This proves that the formula $m = \sum_j m_j m_{V_j}$ continues to hold on the singular set. Thus we have proved that the two definitions F.21 and F.22 are equivalent. Moreover, Proposition F.20 shows that Definition F.22 is equivalent to the following.

Definition F.23. (Cartier divisor) *A* **divisor** *on a complex manifold X is a system $\{U_\alpha, f_\alpha/g_\alpha\}_\alpha$ where $\{U_\alpha\}_\alpha$ is an open cover of X, $f_\alpha : U_\alpha \to \mathbb{C}$ and $g_\alpha : U_\alpha \to \mathbb{C}$ are holomorphic functions which do not vanish on any open subset of U_α, and there exist nowhere vanishing holomorphic functions $u_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathbb{C}$ such that*

$$f_\beta/g_\beta = u_{\beta\alpha} f_\alpha/g_\alpha \tag{F.9}$$

on $U_\alpha \cap U_\beta$.

Theorem F.1 shows that quotient f_α/g_α in the above definition is defined on a dense open subset of U_α. Hence the functions $u_{\beta\alpha}$ in (iii) are uniquely determined by the divisor $\{U_\alpha, f_\alpha/g_\alpha\}_\alpha$. Moreover these functions form a **cocycle** in the sense that

$$u_{\gamma\beta} u_{\beta\alpha} = u_{\gamma\alpha}, \qquad u_{\alpha\alpha} = 1.$$

Here the first equation holds on the domain $U_\alpha \cap U_\beta \cap U_\gamma$. Two Cartier divisors $\{U_\alpha, f_\alpha/g_\alpha\}_\alpha$ and $\{V_\beta, \varphi_\beta/\psi_\beta\}_\beta$ are called **equivalent** if their union is a divisor, i.e. if for every pair α, β there exists a nonvanishing holomorphic function $v_{\beta\alpha} : U_\alpha \cap V_\beta \to \mathbb{C}^*$ such that $\varphi_\beta g_\alpha = v_{\beta\alpha} f_\alpha \psi_\beta$. The equivalence of the two definition is given by the correspondence

$$m(x) = m_{f_\alpha}(x) - m_{g_\alpha}(x) \tag{F.10}$$

for $x \in U_\alpha$. It follows from Lemma F.2 that $m(x)$ is independent of α and hence determines a well-defined divisor in the sense of definition F.22. Moreover, two equivalent Cartier divisors obviously determine the same function m. The converse follows from Proposition F.20. Namely, given a divisor $m : X \to \mathbb{Z}$ choose an open cover $\{U_\alpha\}_\alpha$ of X such that on each U_α there exist nonzero holomorphic functions $f_\alpha, g_\alpha : U_\alpha \to \mathbb{Z}$ which satisfy (F.10). Then, by Lemma F.2,

$$m_{f_\alpha g_\beta} = m_{f_\alpha} + m_{g_\beta} = m_{f_\beta} + m_{g_\alpha} = m_{f_\beta g_\alpha}$$

on $U_\alpha \cap U_\beta$ and hence the existence of the functions $u_{\beta\alpha}$ follows from Proposition F.20. Throughout denote by

$$\mathrm{Div}(X) = \{\text{Weil divisors}\} \cong \frac{\{\text{Cartier divisors}\}}{\text{equivalence}}$$

the set of divisors. The reader may check that divisors form a group. In the case of Weil divisors this is obvious because these are simply defined as the elements of the free abelian group generated by algebraic hypersurfaces. In the case of multiplicity functions $m : X \to \mathbb{Z}$ the group operation is addition and the neutral element is $m(x) \equiv 0$. In the case of Cartier divisors the group operation is given by choosing a common refinement of the two open covers and multiplying the functions. The neutral element is the divisor with open cover $U_\alpha = X$ consisting of a single set with $f_\alpha(x) = 1$ and $g_\alpha(x) = 1$. Note that this diviser is equivalent to any other Cartier divisor for which f_α and g_α never vanish. The inverse of a divisor is given by interchanging f_α and g_α. The group of divisors carries a natural partial order. The nonnegative cone consists of the *effective* divisors.

Definition F.24 *A divisor $m : X \to \mathbb{Z}$ is called* **effective** *if $m(x) \geq 0$ for all $x \in X$. The corresponding Weil divisor $D = \sum_j m_j V_j$ is* **effective** *iff either $m_j > 0$ for all j or $D = \emptyset$. A Cartier divisor $\{U_\alpha, f_\alpha/g_\alpha\}_\alpha$ is called* **effective** *if $g_\alpha = 1$ for all α. The semigroup of effective divisors is denoted by* $\mathrm{Div}^{\mathrm{eff}}(X)$.

That the two definitions of effective are equivalent under the above correspondence follows again from Proposition F.20. Obviously, if $\{U_\alpha, f_\alpha\}_\alpha$ is an effective Cartier divisor, then the corresponding multiplicity function m, defined by (F.10), is nonnegative. Conversely, if $m : X \to \mathbb{Z}$ is a nonnegative divisor and $g, h : U \to \mathbb{C}$ are two holomorphic functions with $m|_U = m_g - m_h$, then $m_g \geq m_h$ and hence, by Proposition F.20, there exists another holomorphic function $f : U \to \mathbb{C}$ such that $g = fh$. By Lemma F.2, $m|_U = m_g - m_h = m_f$. This shows that there is an effective Cartier divisor $\{U_\alpha, f_\alpha\}_\alpha$ with $m|_{U_\alpha} = m_{f_\alpha}$ for all α.

F.7 Line bundles

Associated to every Cartier divisor $\{U_\alpha, f_\alpha/g_\alpha\}_\alpha$ there is a natural holomorphic line bundle

$$E \to X.$$

Explicitly this line bundle can be defined as the set of equivalence classes $[x, z, \alpha]$ with $x \in U_\alpha$ and $z \in \mathbb{C}$ where

$$[x, z, \alpha] \equiv [x, u_{\beta\alpha}(x)z, \beta]$$

whenever $x \in U_\alpha \cap U_\beta$ with $u_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathbb{C}^*$ defined by (F.9). A holomorphic section of E can be described as a collection of holomorphic maps $v_\alpha : U_\alpha \to \mathbb{C}$ which satisfy

$$v_\beta = u_{\beta\alpha} v_\alpha. \tag{F.11}$$

The section $s : X \to E$ is then given by $s(x) = [x, v_\alpha(x), \alpha]$ for $x \in U_\alpha$. Comparing (F.9) and (F.11) one finds that holomorphic sections of E determine global meromorphic functions $v : X \to \mathbb{C}$ via

$$v(x) = v_\alpha(x) g_\alpha(x)/f_\alpha(x). \tag{F.12}$$

for $x \in U_\alpha$. Note that the right hand side is only defined where $f_\alpha(x)$ is nonzero and that it is independent of α, i.e. if $x \in U_\alpha \cap U_\beta$ with $f_\alpha(x) \neq 0$ and $f_\beta(x) \neq 0$ then the two expressions for $v(x)$ agree. It follows immediately from the definitions that the multiplicity function of v satisfies

$$m_v(x) + m(x) \geq 0 \tag{F.13}$$

for all $x \in X$. Conversely, if $v : X \to \mathbb{C}$ is a meromorphic function whose multiplicity function satisfies (F.13) then it follows from Proposition F.20 that the meromorphic function $v_\alpha = v f_\alpha / g_\alpha$ extends to a holomorphic function on U_α for every α and hence v is of the form (F.12). Thus there is a one-to-one correspondence of holomorphic sections of E and meromorphic functions on X which satisfy (F.13). The space of holomorphic sections of E is commonly denoted by $H^0(X, E)$ and thus we have

$$H^0(X, E) \cong \{v : X \to \mathbb{C} \,|\, v \text{ is meromorphic, } m_v + m \geq 0\}.$$

An interesting case is when the constant function $v(x) = 1$ determines a holomorphic section of E. Since $m_1(x) = 0$ this is the case if and only if $m(x) \geq 0$ for all x, i.e. when m is an effective divisor. Thus effective divisors are in one-to-one correspondence to isomorphism classes of pairs (E, s) where $E \to X$ is a holomorphic line bundle and $s : X \to E$ is a nonzero holomorphic section:

$$\mathrm{Div}^{\mathrm{eff}}(X) \cong \frac{\big\{(E, s) \,|\, E \to X \text{ hol. line bundle, } 0 \neq s \in H^0(X, E)\big\}}{\text{isomorphisms}}.$$

The section s associated to an effective divisor $\{U_\alpha, f_\alpha\}_\alpha$ is determined by the functions $v_\alpha = f_\alpha$.

Sometimes it is convenient to denote the line bundle associated to the divisor $m : X \to \mathbb{Z}$ by E_m. This is slightly inaccurate since m does not actually determine a line bundle but only an isomorphism class (unless one wants to use a maximal open covering). Note that the correspondence

$$\mathrm{Div}(X) \to \mathrm{Pic}(X) : m \mapsto E_m$$

is a surjective group homomorphism from divisors to the **Picard group** $\mathrm{Pic}(X)$ of isomorphism classes of holomorphic line bundles. The group operation for line bundles is given by the tensor product and the neutral element is the trivial bundle $\mathbb{C}$.

There is an induced group homomorphism

$$\mathrm{Div}(X) \to H^2(X, \mathbb{Z}) : m \mapsto c_1(E_m).$$

Consider for example the case where $m(x)$ is either 0 or 1 at each point $x \in X$. Then m is the indicator function of a complex submanifold $V_m \subset X$ of codimension 1 and the cohomology class $c_1(E_m)$ is the Poincaré dual of the fundamental class of V_m:

$$c_1(E_m) = \mathrm{PD}([V_m]).$$

The general situation is more complicated but it is still possible to make sense of this formula. For example, if X is a complex surface then, by Proposition F.19, every irreducible hypersurface V_i is the image of a holomorphic map $u_i : \Sigma_i \to X$ and hence carries a fundamental cycle. In this case

$$c_1(E_m) = \sum_i m_i \mathrm{PD}([V_i]), \qquad m = \sum_i m_i m_{V_i}. \tag{F.14}$$

Given a cohomology class $e \in H^2(X, \mathbb{Z})$ denote the set of divisors with first Chern class e by

$$\mathrm{Div}(X, e) = \{\text{divisors } m \text{ with } c_1(E_m) = e\}.$$

Similarly, $\mathrm{Div}^{\mathrm{eff}}(X, e)$ denotes the set of effective divisors $m : X \to \mathbb{Z}$ with $c_1(E_m) = e$. One can think of E as a fixed complex vector bundle with first Chern class e and then $\mathrm{Div}^{\mathrm{eff}}(X, e)$ can be identified with the set of isomorphism classes of pairs $(\bar\partial, s)$ where $\bar\partial : \Omega^k(X, E) \to \Omega^{k+1}(X, E)$ is a Cauchy-Riemann operator with $\bar\partial \circ \bar\partial = 0$ and $s : X \to E$ is a nonzero section in the kernel of $\bar\partial$:

$$\mathrm{Div}^{\mathrm{eff}}(X, e) \cong \frac{\left\{(\bar\partial, s) \,|\, \bar\partial \circ \bar\partial = 0,\ \bar\partial s = 0,\ s \neq 0\right\}}{\text{gauge equivalence}}.$$

Here a smooth complex gauge transformation $u : X \to \mathbb{C}^*$ acts on a pair $(\bar\partial, s)$ by $u^*(\bar\partial, s) = (u^{-1} \circ \bar\partial \circ u, u^{-1} s)$.

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