Logarithmic Intersection Theory of Jacobians

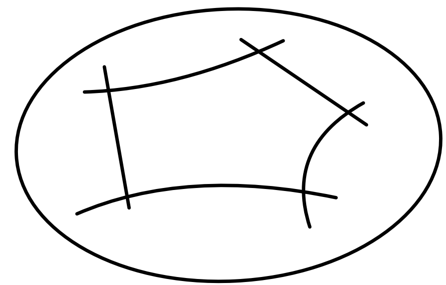
1. Log Maps

(and the problem)

smooth

normal crossings

$(X, D = D_1 \cup D_2 \cup \cdots \cup D_m)$



smooth normal crossings

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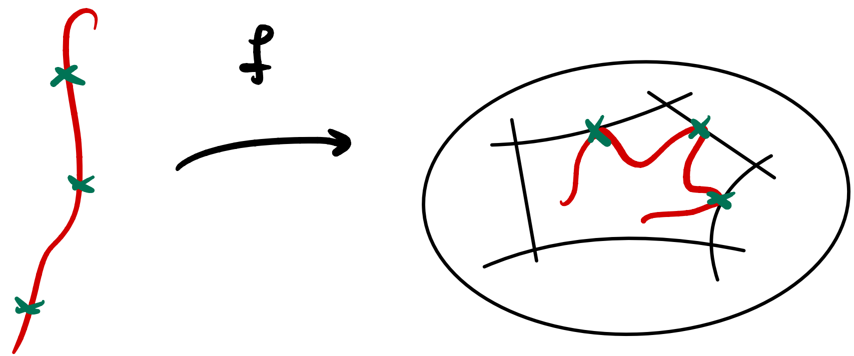


$(C, x_1, \dots, x_n)$

genus g

genus $\quad$ # of markings

$g, n \in \mathbb{N}$ fixed

$$n := \sum_{i=1}^{m} n_i$$

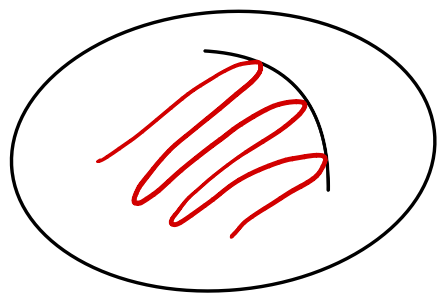
$$(x_1, \dots, x_n) := (x_{11}, x_{12}, \dots, x_{1n_1}, x_{21}, \dots \quad, x_{mn_m})$$

$$A_i = (a_{i1}, a_{i2}, \dots, a_{in_i}) \in \mathbb{N}^{n_i} \quad , \quad A = (A_1, \dots, A_m)$$

a_{ij} contact order of f at x_{ij} with D_i

$$\boxed{K_{g,A}(X)^{\circ} = \left\{ f : (C, x_1, \dots, x_n) \to X \;\middle|\; f^* D_i = \sum_j a_{ij} x_{ij} \right\}}$$

Simplest Example: D smooth



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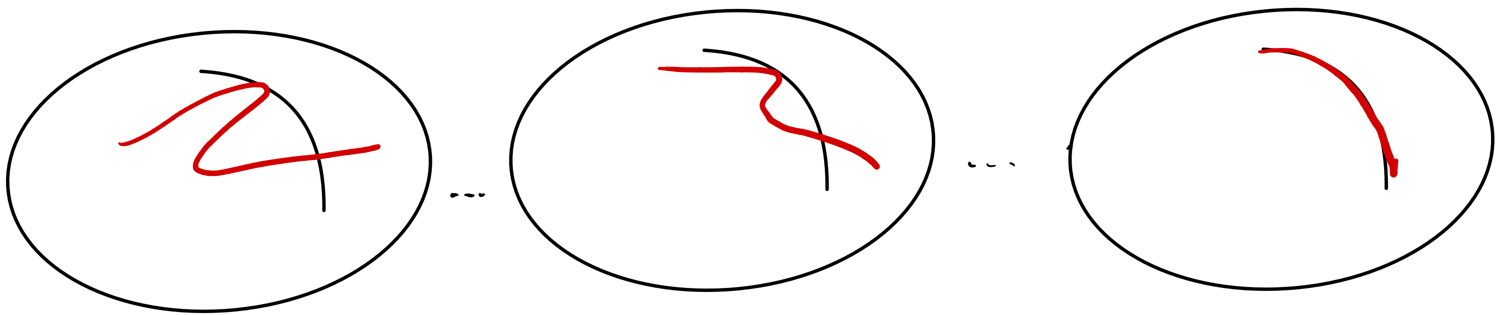
Moduli space not compact

Simplest Example: D smooth, $A = (a_1, \dots, a_n)$



Compactifications by various incarnations of

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All iterations have $[K_{g,A}(X)]^{vir}$

Compactifications by various incarnations of

$\boxed{\text{Log Maps}}$

- D smooth : Jun Li (2000) Qile Chen (2010)

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All iterations have $[K_{g,A}(X)]^{vir}$

Thm (Abr-Mar-Wise, 2012)

All theories define the same (log GW) invariants.

Variant (<u>Rubber</u>): V a toric variety, $D = V - T$

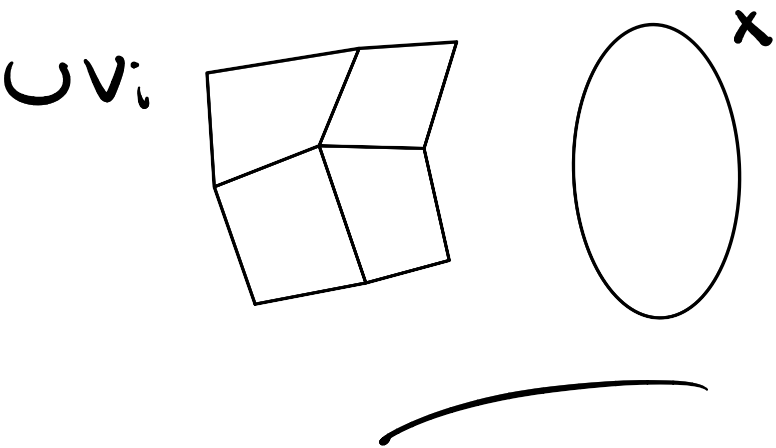
$$\mathcal{K}^{rub}_{g,A}(V)^{\circ} = \{ f: (C, x_1, \dots, x_n) \to V \mid \text{as before, but up to } T\text{-action} \}$$

+ Analogous compactifications.

$$p : K^{\mathrm{rub}}_{g,A}(V) \longrightarrow \overline{\mathcal{M}}_{g,n}$$

$\boxed{\begin{array}{l}\text{\underline{The problem}: Let } (V, V-T) \text{ be a proper toric variety} \\ \quad\text{Compute} \\ \qquad DR_A(V) := p_*\left([K^{\mathrm{rub}}_{g,A}(V)]^{\mathrm{vir}}\right) \in CH^{(\dim V)\cdot g}(\overline{\mathcal{M}}_{g,n})\end{array}}$

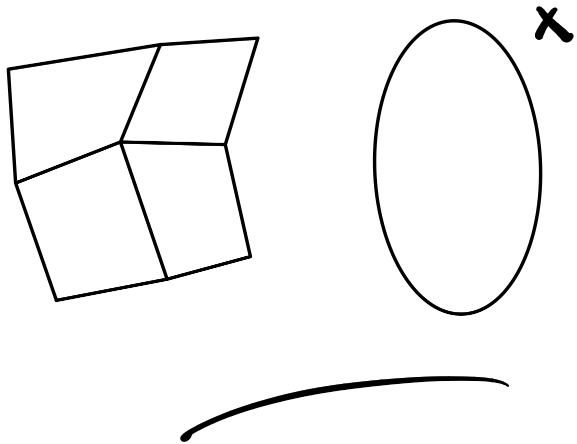
Why? (i) Ordinary GW theory



GW Theory of X
reconstructed by
log GW theory of V_i

(ACGS, Ranganathan)

Why? (i) Ordinary GW theory

$\cup V_i$ 

GW Theory of X reconstructed by log GW theory of V_i

(ACGS, Ranganathan)

Thm (Ran-Urundolil Kumaran '22)

$DR_A(V)$ determines log GW theory of V

(ii) Tautological ring $R^*(\overline{M}_{g,n}) \subset CH^*(\overline{M}_{g,n})$

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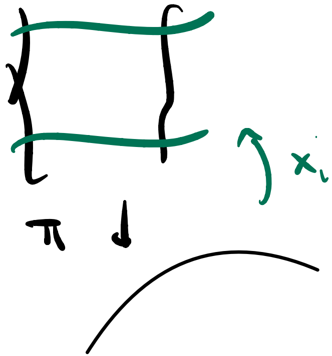
C

$\downarrow \; \pi$

S



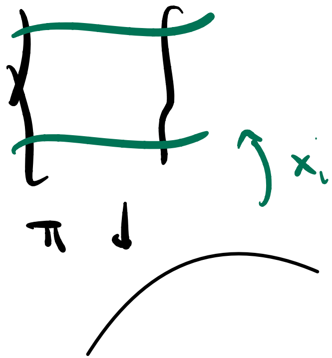
$$\pi_*\left(c_1(\omega_{C/S})^{a+1}\right) = \kappa_a$$

$$x_i^*\left(c_1(\omega_{C/S})\right) = \psi_i$$

$$c_i\left(\pi_*\omega_{C/S}\right) = \lambda_i$$

(ii) Tautological ring $R^*(\overline{M}_{g,n}) \subset CH^*(\overline{M}_{g,n})$

$$\begin{array}{c} C \\ \downarrow \\ S \end{array}$$



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$$c_i\left(\pi_* \omega_{C/S}\right) = \lambda_i$$

$$R^*(\overline{M}_{g,n}) = \langle \kappa_a, \psi_i, \lambda_i \rangle$$

+ closed under all gluing/forgetful maps

$$M_{g',n'} \longrightarrow M_{g,n}$$

Study of relations in $R^*(\overline{M}_{g,n})$ one of the biggest problems in the area.

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Study started by Mumford (1982)

Ex: λ_i are generated by the κ_d

"Mumford's formula"

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Study started by Mumford (1982)

Ex: λ_i are generated by the κ_a

"Mumford's formula"

GW theory gives most complete set of relations in $R^*(\overline{M}_{g,n})$ known to date.

Pixton - Pandharipande - Zvonkine (2013)

Pixton Relations / Conjectures.

What's known (2020)

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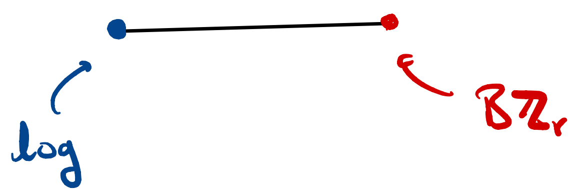
- $V = \mathbb{P}^1 \rightsquigarrow$ Pixton's formula (JPPZ, 2015)

localization on



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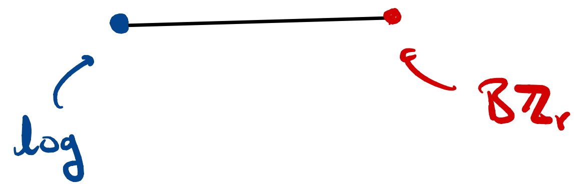


- In higher dimension, all attempts to generalize JPPZ have been unsuccessful. Cannot even show $DR_{g,A}(V) \in R^*(\overline{M}_{g,n})$

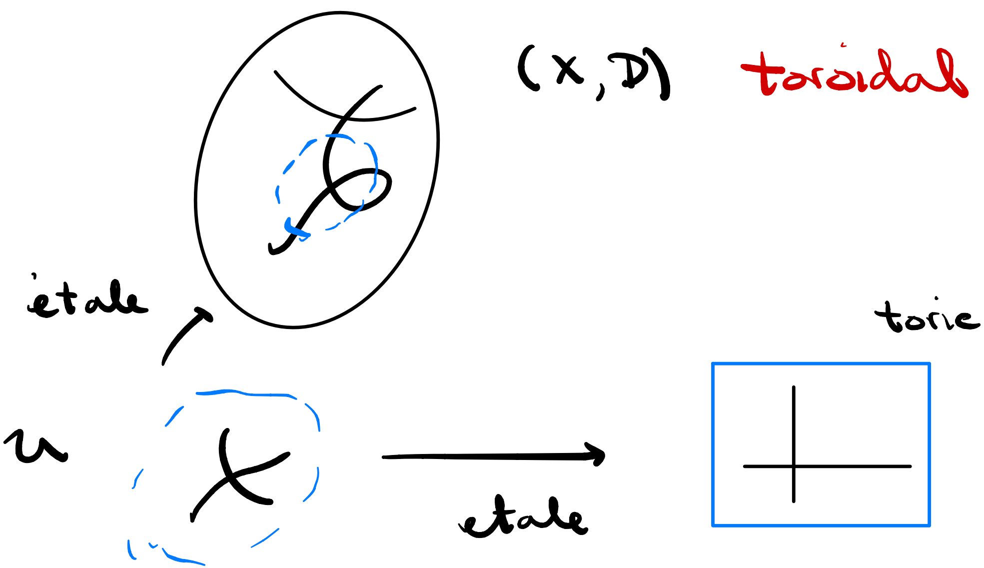
Some new ideas are needed

2. Log Intersection Theory

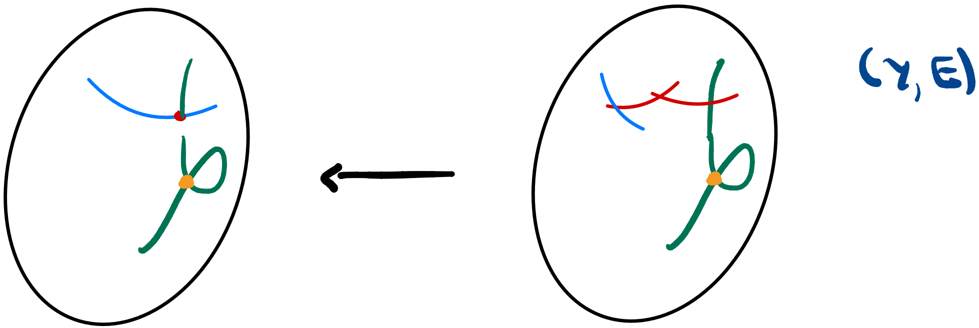
What is it?

What is log

(X, D) toroidal embedding



- (X, D) is **stratified**.



Def: A simple log blowup is an iterated blowup along boundary strata.

What is intersection theory?

Study of $CH^*(X)$

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Study of $CH^*(X)$

What is log intersection theory?

Study of $\log CH^*(X,D) := \varinjlim_{(Y,E)\to(X,D)} CH^*(Y)$

Perspective: $X - D = \mathcal{U}$ is the object of interest.

Truly interesting classes arise from processes that produce a class $\beta_Y \in CH(Y)$ for all Y

s.t β_Y eventually stabilizes:

$$\beta_Z = \beta_Y|_Z \quad \text{for } Z \to Y$$

Truly interesting classes arise from processes that produce a class $b_Y \in CH(Y)$ for all Y

s.t b_Y eventually stabilizes:

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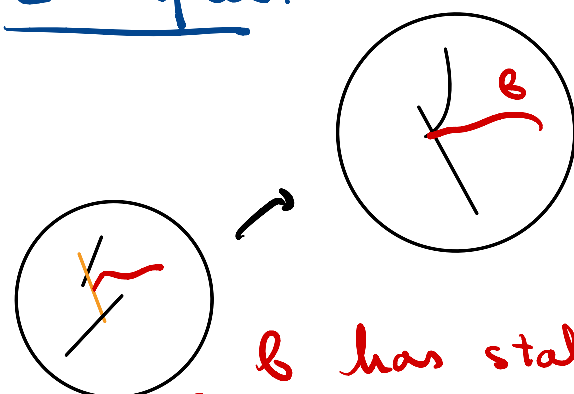
Examples:



b^0 cycle on U

$b_Y = \overline{b^0}$ in $CH_*(Y)$

b has stabilized here

(ii)

$$\begin{tikzcd} K^{\mathrm{rub}}_{g,A}(V) \arrow[r, "p"] \arrow[dr, dashed, "\sim\exists q"'] & \overline{\mathcal{M}}_{g,n} \\ & \overline{\mathcal{M}}'_{g,n} \arrow[u, "f"'] \end{tikzcd}$$

$$q_*[K^{\mathrm{rub}}_{g,A}(V)]^{\mathrm{vir}} \in CH_*(\overline{\mathcal{M}}'_{g,n})$$

$\rightsquigarrow$ The process stabilizes to produce a class

$$\log DR_{g,A}(V) \in \log CH^*(\overline{\mathcal{M}}_{g,n})$$

Conceptual advantages

(i) $$\begin{array}{ccc} \log CH^*(\overline{\mathcal{M}}_{g,n}) & \longrightarrow & CH^*(\overline{\mathcal{M}}_{g,n}) \\ \log DR_{g,A}(V) & \longrightarrow & DR_{g,A}(V) \end{array}$$

Conceptual advantages

(i) $\log CH^*(\overline{\mathcal{M}}_{g,n}) \longrightarrow CH^*(\overline{\mathcal{M}}_{g,n})$

$\log DR_{g,A}(V) \longrightarrow DR_{g,A}(V)$

(ii) log DR satisfies birational invariance

+ product formula (Ran, Herr '18)

$$\rightsquigarrow \log DR_{g,(A_1,\dots,A_m)}(V) = \prod_{i=1}^{m} \log DR_{g,A_i}(\mathbb{P}^1)$$

Recall:

The problem: Let $(V, V-T)$ be a proper toric variety

Compute

$$DR_A(V) := p_*\left([K^{\text{rub}}_{g,A}(V)]^{\text{vir}}\right) \in CH^{(\dim V)\cdot g}(\overline{M}_{g,n})$$

Thus:

Suffices to determine $\log DR_{g,A}(\mathbb{P}^1) \in \log CH(\overline{M}_{g,n})$

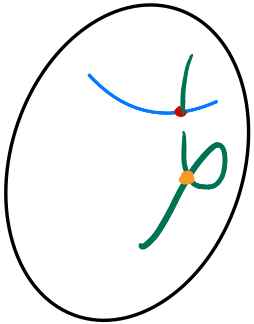
As it stands, $\log CH^*(X, D)$ is "poor".

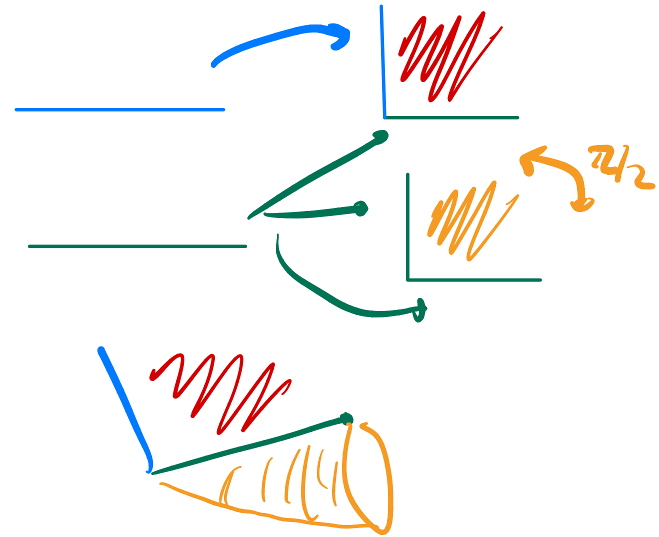
~ It has allowed me to use res. of sing. methods in Chow

~ But I have traded away explicitness.

Recover Explicitness by combinatorics.

(a) Cone complexes





(X, D)

$$= \varinjlim U$$

$$\Sigma_{(X,D)} = \varinjlim \sigma_U$$

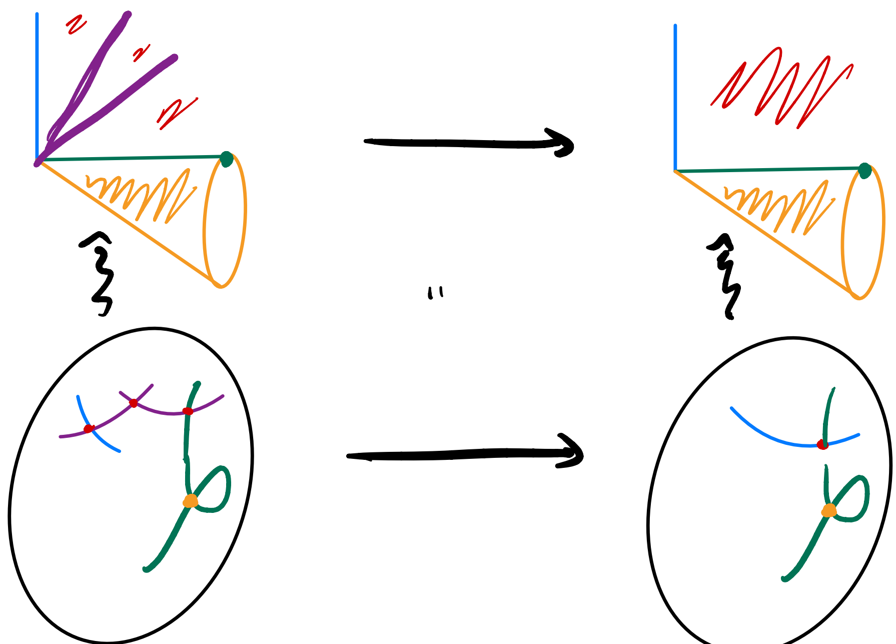
$$\mathcal{A}_{(X,D)} = \varinjlim \left[V(\sigma_U) / T_{V(\sigma_U)} \right]$$

[KKMSD 73, ACP '12, CCUW '17]

[AW]

- $\exists$ an equivalence of categories

$$\left\{ \begin{array}{c} \text{Log Blowups} \\ \text{of } (X, D) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Subdivisions} \\ \Sigma_X' \to \Sigma_X \end{array} \right\}$$



Toric V	Toroidal (X,D)
Fan Σ_V	Cone complex $\Sigma_{(X,D)}$
$[V/T]$	Artin fan $\mathcal{A}_X$
Toric blowup $\parallel$ $\Sigma_V' \longrightarrow \Sigma_V$	Log Blowup $\parallel$ $\Sigma'_{(X,D)} \longrightarrow \Sigma_{(X,D)}$

(b)

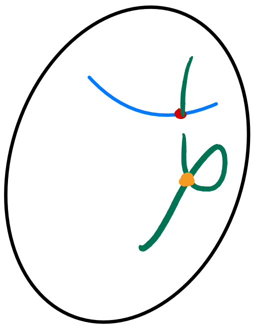
$$\begin{tikzcd} PP(\Sigma_X) \arrow[r, "\Phi"] \arrow[d, phantom, "\cong" rotate=90] & CH^*(X) \\ CH^*(A_X) \arrow[ur] & \end{tikzcd}$$

[MPS, NR]

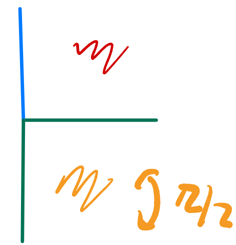
Roughly: PL function with slope 1 along ray ρ corresponding to divisor D_ρ goes to $c_1(\mathcal{O}(D_\rho))$. Extend by excess intersection formula + linearity

Example

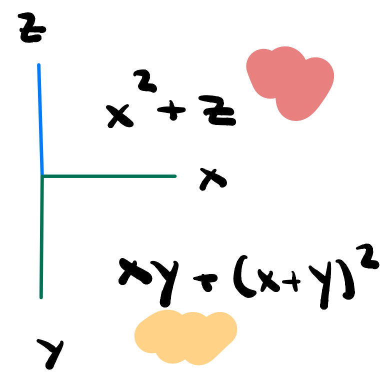
(X, D)



$\Sigma_{(X,D)}$



$[\smile] + [\,\cdot\,] + j_*[c_1(j : \cdot \to X)] \xleftarrow{\phi}$



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Toric V	Toroidal (X, D)
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Toric blowup $=$ $\Sigma_V' \longrightarrow \Sigma_V$	Log Blowup $=$ $\Sigma_{(X,D)}' \longrightarrow \Sigma_{(X,D)}$
$PP(\Sigma_V) \simeq CH_T(V)$ $\phi \searrow \quad \swarrow$ $CH^*(V)$ (Payne '06)	$PP(\Sigma_{(X,D)}) \simeq CH^*(\mathcal{A}_X)$ $\phi \searrow \quad \swarrow$ $CH^*(X)$ analogue of non-equivariant limit

Paradigm: How to write a formula in $\log CH^*$

$$b \in \log CH^*(X, D)$$

- Find $\Sigma_{(Y,E)} \longrightarrow \Sigma_{(X,D)}$ s.t

  (i) $b \in CH^*(Y)$

  (ii) Understand $\Sigma_{(Y,E)}$ very well.

- Write b in terms of $CH^*(X)$, $PP(\Sigma_{(Y,E)})$

Some new ideas are needed

Some new ideas are needed

but also:

Some old ideas are needed

3. Jacobians

Fix $A = (a_1, \dots, a_n)$, $\sum a_i = 0$, $(C, x_1, \dots, x_n)$

smooth

$$\exists \; C \xrightarrow{f} \mathbb{P}^1 \text{ s.t } \iff \mathcal{O}_C\left(\sum a_i x_i\right) \simeq \mathcal{O}_C$$

$$f^{-1}(0) - f^{-1}(\infty) = \sum a_i x_i$$

Map Condition

Line Bundle Condition

So let's use the Jacobian

Dan Abramovic, embedding a curve of genus 3 in its Jacobian.


Θ-divisor, Mumford's formula
abelian variety,
Fourier-Mukai...
Jac X

$$\begin{tikzcd} \mathrm{Pic}^\circ(C_{g,n}) \arrow[d] \\ \mathcal{M}_{g,n} \arrow[u, bend right, "a_{J_A}"'] \end{tikzcd}$$

$$a_{J_A}(C, x_1, \dots, x_n) \simeq \mathcal{O}_C\left(\sum a_i x_i\right)$$

$$\boxed{\mathrm{DR}_{g,A}\big|_{\mathcal{M}_{g,n}} = a_{J_A}^*([0])}$$

$$\begin{tikzcd} \mathrm{Pic}^0(C_{g,n}) \arrow[d] \\ \mathcal{M}_{g,n} \arrow[u, bend right, "{a_{J_A}(C,x_1,\dots,x_n) \simeq \mathcal{O}_C(\sum a_i x_i)}"'] \end{tikzcd}$$

$$\boxed{DR_{g,A}\big|_{\mathcal{M}_{g,n}} = a_{J_A}^*([0])}$$

Diagram can be extended to compact type

$$\begin{tikzcd} \mathrm{Pic}^{[0]}(C_{g,n}^{ct}) \arrow[d] \\ \overline{\mathcal{M}}_{g,n}^{ct} \arrow[u, bend right, "a_{J_A}"'] \end{tikzcd}$$

Thm (Cav-Mar-Wise '13): $aj_A^*([0]) = DR_{g,A}$ on $\overline{M}_{g,n}^{ct}$

In fact, first calculations were perfomed using Jacobians

Hain (2003) using $\Theta^g/g! = [0]$

Grushevsky-Zakharov (2013)

Extending to $\overline{M}_{g,n}$ is subtle.

Reason: Abelian Varieties don't degenerate well

(A_g not compact)

Extending to $\overline{M}_{g,n}$ is subtle.

Reason: Abelian Varieties don't degenerate well

($\mathcal{A}_g$ not compact)

Mumford's philosophy: Good (toroidal) compactifications exist. But

[A, F-C, O]

- not canonical
- depend on combinatorial choices

i) start with a set of periods $Y \subset \tilde{G}(K)$ where $\tilde{G} \cong G_m^r \times S$, satisfying suitable conditions;

ii) construct a kind of compactification:

$$\begin{tikzcd} \tilde{G} \arrow[rr, phantom, "\underset{\text{open}}{\subset}"] \arrow[dr] & & \tilde{P} \arrow[dl] \\ & S & \end{tikzcd}$$

such that the action of Y by translation extends to $\tilde{P}$, and Y acts freely and discontinuously (in the *Zariski* topology) on $\tilde{P}_0$. Unlike the case of curves, $\tilde{P}$ is neither unique nor canonical!

this has not yet been done. Instead I conclude the paper with many examples. For me, one of the most enjoyable features of this research was the beauty of the examples which one works out without a great deal of extra effort. In fact, the non-uniqueness of $\tilde{P}$ gives one freedom to seek for the most elegant solutions in any particular case.

(Mumford '72)

$$\begin{tikzcd}
\mathrm{Pic}^\circ \arrow[r] \arrow[d] & \mathrm{Pic}(\Delta) \arrow[d] \\
\mathcal{M}_{g,n} \arrow[r, phantom, "\subset"] & \overline{\mathcal{M}}_{g,n}
\end{tikzcd}$$

toroidal comp.
w/ comb. data
Δ.

But it doesn't quite help.

$\exists$ no $\mathrm{Pic}(\Delta)$ with $\sigma_A : \overline{\mathcal{M}}_{g,n} \to \mathrm{Pic}(\Delta)$


JacX

Marcus-Wise '17, Holmes '17: Succeeded in extending

$a_{J_A}^*([0])|_{\overline{M}_{g,n}^{ct}}$ along those lines

+ proved extension $= DR_{g,A}$

But: Extension also insufficient for calculation

Roughly

$$\begin{tikzcd}
\mathrm{DRL} \arrow[r, "a_{J_A}"] \arrow[d] \arrow[dd, bend left, "P_*"] & \overline{\mathcal{M}}_{g,n} \arrow[d, "o"] \\
\left(\begin{matrix}\text{Non sep.} \\ \text{space}\end{matrix}\right) \arrow[r, "a_{J_A}"] \arrow[d] & \mathcal{P}ic \\
\overline{\mathcal{M}}_{g,n} &
\end{tikzcd}$$

non-separated

Extension $P_*(a_{J_A}^{!}[0])$. Must push from DRL.

But no real understanding of space DRL

The problem of compactified Jacobians can be

Perfectly organized

The problem of compactified Jacobians can be

perfectly organized

Thm (MW , 2022-∞ - 2022)[1]

$$\begin{tikzcd} \mathrm{Pic}^0(\overline{C}_{g,n}) \arrow[r, phantom, "\subset"] \arrow[d] & \mathrm{LogPic}^0(\overline{C}_{g,n}) \arrow[d] \\ \mathcal{M}_{g,n} \arrow[r, phantom, "\subset"] & \overline{\mathcal{M}}_{g,n} \end{tikzcd}$$

(i) • proper , group , smooth

---

1. Personal Recollection

(ii) Every $\mathrm{Pic}(\Delta)$ is a log blowup of LogPic

Means: LogPic has a fan Σ_{LogPic}

Mumford's comb. data (Δ) is exactly

a subdivision $\Delta \longrightarrow \Sigma_{\mathrm{LogPic}}$

cone complex

family of real tori over $\Sigma_{\overline{\mathcal{M}}_{g,n}}$

not cone complex

(iii) Not representable by toroidal alg. stack

algebraic log space, not log algebraic space

(iv)

$$\exists \ \alpha_{J_A} : \overline{\mathcal{M}}_{g,n} \longrightarrow \mathcal{L}og\mathrm{Pic}$$

Dan Abramovic, embedding a curve of genus 3 in its log Jacobian^2.

2. Personal Aspiration

(iv)

$$\exists \ a_{J_A} : \overline{\mathcal{M}}_{g,n} \longrightarrow \mathcal{L}og\mathrm{Pic}$$

Great: Let's take $a_{J_A}^*([0])$, for

$$[0] \in CH^*(\mathcal{L}og\mathrm{Pic})$$

(iv)

$$\exists \ a_{J_A} : \overline{\mathcal{M}}_{g,n} \longrightarrow \mathcal{L}og\mathcal{P}ic$$

Great: Let's take $a_{J_A}^*([0])$, for

$$[0] \in \mathrm{log}CH^*(\mathcal{L}og\mathcal{P}ic) = \varinjlim_{\Delta} CH^*(\mathcal{P}ic(\Delta))$$

i.e $a_{J_A}^* : \mathrm{log}CH(\mathcal{L}og\mathcal{P}ic) \longrightarrow \mathrm{log}CH(\overline{\mathcal{M}}_{g,n})$.

In practice: $\forall\ \mathrm{Pic}(\Delta)$

$$
\begin{tikzcd}
\overline{\mathcal{M}}_{g,A}(\Delta) \arrow[r, "a_{J,A}(\Delta)"] \arrow[d, "\text{explicit subdivision}"'] & \mathrm{Pic}(\Delta) \arrow[d] \\
\overline{\mathcal{M}}_{g,n} \arrow[r, "a_{J,A}"'] & \mathrm{LogPic}
\end{tikzcd}
$$

Class $\quad a_{J,A}(\Delta)^*([0]) \in CH^*(\mathcal{M}_{g,A}(\Delta))$

Thm (HMPPS): For any $\mathrm{Pic}(\Delta)$,

$$a_{J,A}(\Delta)^*([0]) = \log DR_{g,A} \in \log CH^*(\overline{\mathcal{M}}_{g,n})$$

4. Synthesis

Recall paradigm:

- Find $\Sigma_{(Y,E)} \longrightarrow \Sigma_{(X,D)}$ s.t

  (i) $\mathcal{B} \in CH^*(Y)$

  (ii) Understand $\Sigma_{(Y,E)}$ very well.

- Write $\mathcal{B}$ in terms of $CH^*(X)$, $PP(\Sigma_{(Y,E)})$

Recall paradigm:

✓ • Find $\Sigma_{(Y,E)} \longrightarrow \Sigma_{(X,D)}$ s.t $\Sigma M_{g,A}(\Delta)$

✓ (i) $b \in CH^*(Y)$ $a_{J_A}^*(\Delta)[o]$

(ii) Understand $\Sigma_{(Y,E)}$ very well.

• Write b in terms of $CH^*(X)$, $PP(\Sigma_{(Y,E)})$

Doing much better, but not done.

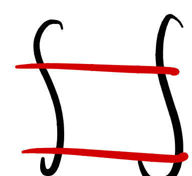
We take the toroidal compactifications of

Oda-Seshadri '79, Caporaso '94, Esteves '97, Melo '15, Kass-Pagani '17

Numerical data of a "stability condition" θ $\rightsquigarrow$ really good models $\mathrm{Pic}(\Delta)$

$$\mathrm{Pic}(\theta) = \{ C', L \mid \mathrm{multdeg}\, L \text{ is close to } \theta \}$$

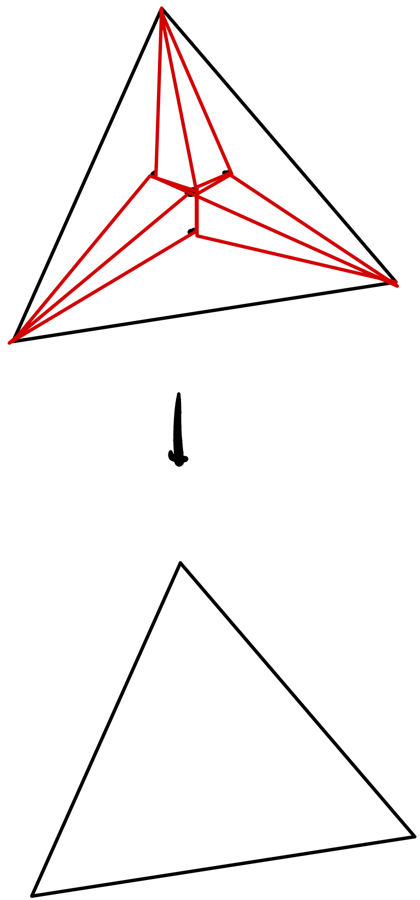
quasi-stable



$$\begin{tikzcd}
\overline{\mathcal{M}}_{g,A}(\theta) \arrow[r, "a_{J_A}(\theta)"] \arrow[d] & \mathrm{Pic}(\theta) \arrow[d] \\
\overline{\mathcal{M}}_{g,n} \arrow[r] & \mathrm{LogPic}
\end{tikzcd}$$

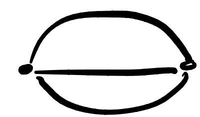
$$\{(C', \alpha \in PL(\Sigma_{C'})) \mid a_{J_A} \otimes \mathcal{O}(\Phi(\alpha)) = \theta\text{-stable}\}$$

$$\Sigma_{\overline{\mathcal{M}}_{g,A}(\theta)} = \{\Gamma', \alpha \mid \operatorname{multdeg} a_{J_A} + \operatorname{div} \alpha = \theta\text{-st}\}$$



$$\Sigma_{M_{g,A}}(\theta)$$

$$\Sigma_{M_{g,n}}^{trop}$$



$\theta = 0$

Thm (HMPPS)

implemented on computer

$\exists$ an explicit formula for $\log DR_{g,A}$

in terms of

$$R^*(\overline{\mathcal{M}}_{g,n}) + PP(\Sigma_{\mathcal{M}_{g,A}(\vartheta)})$$

$$\cap$$

$$CH^*(\overline{\mathcal{M}}_{g,n})$$

To write the formula, we must first define two special strict piecewise power series $\mathfrak{P}, \mathfrak{L}$ on $\widetilde{\Sigma}^\theta$. The functions $\mathfrak{P}, \mathfrak{L}$ are uniquely determined by their restriction to the interior cones of $\widetilde{\Sigma}^\theta$. As discussed in Section 1.7.2, the interior cones correspond to a stable graph $\Gamma \in \mathcal{G}_{g,n}$ together with a tuple $(\widehat{\Gamma}, D, I)$. We will define the functions $\mathfrak{P}, \mathfrak{L}$ on the associated cone $\sigma_{\widehat{\Gamma},I} \in \widetilde{\Sigma}^\theta_\Gamma$.

- The definition of $\mathfrak{P}$ requires a sum over weightings: for a positive integer r, an *admissible weighting mod* r on $\widehat{\Gamma}$ is a flow w with values in $\mathbb{Z}/r\mathbb{Z}$ such that

$$\mathrm{div}(w) = D \in (\mathbb{Z}/r\mathbb{Z})^{V(\widehat{\Gamma})}\,.$$

We define

$$\mathrm{Cont}^r_{(\widehat{\Gamma},D,I)} = \sum_w r^{-h^1(\widehat{\Gamma})} \prod_{e \in E(\widehat{\Gamma})} \exp\left(\overbrace{\frac{\overline{w}(\vec{e}) \cdot \overline{w}(\overleftarrow{e})}{2}}^{\in \mathbb{Z}/r\mathbb{Z}} \widehat{\ell}_e\right) \in \mathbb{Q}[[\widehat{\ell}_e : e \in E(\widehat{\Gamma})]]\,, \quad \text{PL}$$

where the sum runs over admissible weightings w mod r. Inside the exponential, $\overline{w}(\vec{e})$ and $\overline{w}(\overleftarrow{e})$ denote the unique representative of $w(\vec{e}) \in \mathbb{Z}/r\mathbb{Z}$ and $w(\overleftarrow{e}) \in \mathbb{Z}/r\mathbb{Z}$ in $\{0, \ldots, r-1\}$.

As in [36, Appendix], for each fixed degree in the variables $\widehat{\ell}_e$, the element $\mathrm{Cont}^r_{(\widehat{\Gamma},D,I)}$ is polynomial in r for sufficiently large r. We denote by $\mathrm{Cont}_{(\widehat{\Gamma},D,I)}$ the polynomial in the variables $\widehat{\ell}_e$ obtained by substituting $r = 0$ into the polynomial expression for $\mathrm{Cont}^r_{(\widehat{\Gamma},D,I)}$. We define

$$\text{PP} \quad \Rightarrow \quad \mathfrak{P}\big|_{\sigma_{\widehat{\Gamma},I}} = \mathrm{Cont}_{(\widehat{\Gamma},D,I)}\big|_{\widehat{\ell} = \widehat{\ell}(\ell)} \in \mathbb{Q}[[\ell_e : e \in E(\Gamma)]]\,,$$

where we use the variable substitution $\widehat{\ell} = \widehat{\ell}(\ell)$ associated to $\sigma_{\widehat{\Gamma},I}$ from Claim 2. We claim that these functions fit together to give a well-defined strict piecewise power series $\mathfrak{P}$ on $\widetilde{\Sigma}^\theta$.

- To define $\mathfrak{L}$ on $\widetilde{\Sigma}^\theta$, we fix a vertex $v_0 \in V(\widehat{\Gamma})$. For every length assignment $\widehat{\ell}$ in the cone $\tau_{\widehat{\Gamma},I}$ and any vertex $v \in V(\widehat{\Gamma})$, let $\gamma_{v_0 \to v}$ be a path from v_0 to v in $\widehat{\Gamma}$. We define

$$\alpha(v) = \sum_{\vec{e} \in \gamma_{v_0 \to v}} I(\vec{e}) \cdot \widehat{\ell}_e\,, \tag{16}$$

where the sum is over the oriented edges $\vec{e}$ constituting the path $\gamma_{v_0 \to v}$. The defining equations of $\tau_{\widehat{\Gamma},I}$ imply that for $\widehat{\ell} \in \tau_{\widehat{\Gamma},I}$ the expression (16) is independent of the chosen path $\gamma_{v_0 \to v}$. We define

$$\mathfrak{L} = \sum_{v \in V(\widehat{\Gamma})} \underbrace{(D + \underline{\deg}_{k,A})(v)}_{\in \mathbb{Z}} \cdot \underbrace{\alpha(v)\big|_{\widehat{\ell} = \widehat{\ell}(\ell)}}_{\text{PL}} \in \mathbb{Q}[\ell_e : e \in E(\Gamma)]\,. \tag{17}$$

The substitution of variables $\widehat{\ell} = \widehat{\ell}(\ell)$, which give the inverse of the isomorphism $\tau_{\widehat{\Gamma},I} \to \sigma_{\widehat{\Gamma},I}$ and thus have image in $\tau_{\widehat{\Gamma},I}$, ensure that the expression is independent of the choice of the paths $\gamma_{v_0 \to v}$. The expression is also independent of the base vertex v_0, which follows from the fact that the divisor $D + \underline{\deg}_{k,A}$ has total degree 0 on $\widehat{\Gamma}$.

For the $\mathsf{logDR}_{g,A}$ formula, in addition to $\mathfrak{P}$ and $\mathfrak{L}$, we will also require the tautological class

$$\eta = k^2\kappa_1 - \sum_{i=1}^{n} a_i^2 \psi_i \in \underline{\mathsf{R}^*(\overline{\mathcal{M}}_{g,n})}\,. \tag{18}$$

$\Phi : PP \to CH$

Define the mixed degree logarithmic class

$$\mathbf{P}_{g,A}^{\theta} = \exp\left(-\frac{1}{2}(\eta + \underbrace{\Phi(\mathfrak{L})}_{PL})\right) \cdot \underbrace{\Phi(\mathfrak{P})}_{PP} \in \mathsf{logR}^*(\overline{\mathcal{M}}_{g,n})\,, \tag{19}$$

where Φ is the extension of the map (11) to piecewise power series as described at the end of Section 1.7.1.

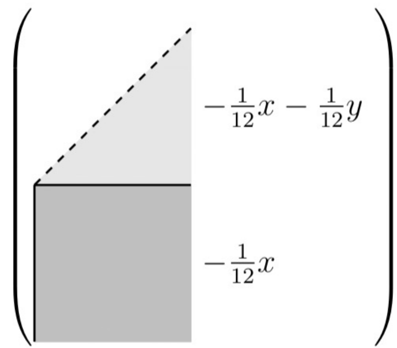
Theorem B. *Let θ be a small nondegenerate stability condition. The log double ramification cycle is the degree g part of $\mathbf{P}_{g,A}^{\theta}$,*

$$\mathsf{logDR}_{g,A} = \mathbf{P}_{g,A}^{g,\theta} \in \mathsf{logR}^g(\overline{\mathcal{M}}_{g,n})\,.$$

Example

JPPZ

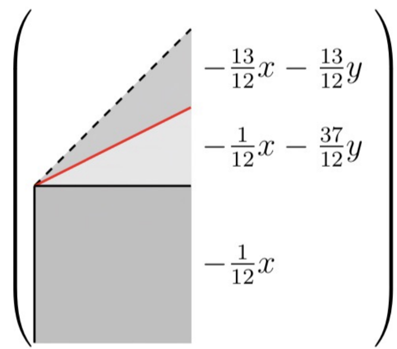
$$\mathsf{DR}_{1,(3,-3)} = \frac{9}{2}(\psi_1 + \psi_2) + \Phi$$



$$\in \mathsf{R}^1(\overline{\mathcal{M}}_{1,2}),$$

g A

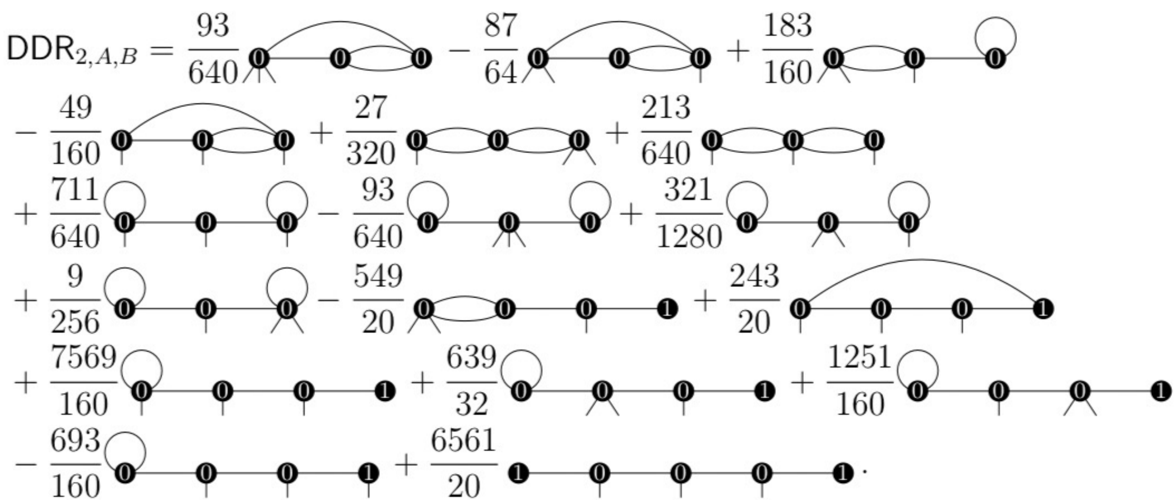
$$\log\mathsf{DR}_{1,(3,-3)} = \frac{9}{2}(\psi_1 + \psi_2) + \Phi$$



$$\in \log\mathsf{R}^1(\overline{\mathcal{M}}_{1,2}).$$

Corollary: All $DR_{g,A}(V)$ are determined

Example $g=2$, $A=(3,-3,0)$, $B=(0,3,-3)$



5. A novel approach

~~5. A novel approach~~

A novel take on an old approach

The geometry of $\mathcal{M}_{g,A}(\partial)$ is in fact extremely favorable

$$\begin{array}{c} \mathcal{C}_{g,A}(\partial), \quad \mathcal{L} = a_{J_A} \otimes \mathcal{H}(\Phi(\alpha)) \\ \pi \downarrow \\ \mathcal{M}_{g,A}(\partial) \end{array}$$

<u>Thm</u> (AMP)

$$\log DR_{g,A} = c_g(-R\pi_* \mathcal{L})$$

So $\log DR_{g,A}$ can be calculated by GRR

<u>Thm</u> (AMP): Let $C \xrightarrow{f} S$ be a family of curves.

The GRR formula lifts to $\log CH$. The result is an explicit formula in terms of

$$CH^*(S), \quad \varinjlim_{\Sigma_{S'} \to \Sigma_S} PP(\Sigma_{S'})$$

The approach leads to **new** formulas

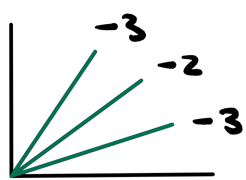
Mumford

$$\lambda_1 = \frac{\kappa_1 + \delta}{12}$$

AMP

$$\log DR_{1,(3,-3)} =$$

$$\frac{9}{2}\psi_1 + \frac{9}{2}\psi_2 + \frac{3}{2}\psi_1 - \frac{3}{2}\psi_2$$

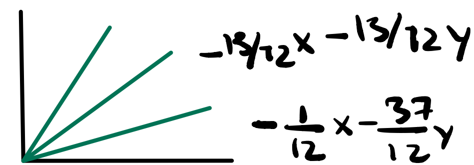
$+$  $- \lambda_1$

Pixton

$$\lambda_1 = \delta_{irr}/12 \qquad (g=1)$$

HMPPS

$$\log DR_{1,(3,-3)} =$$

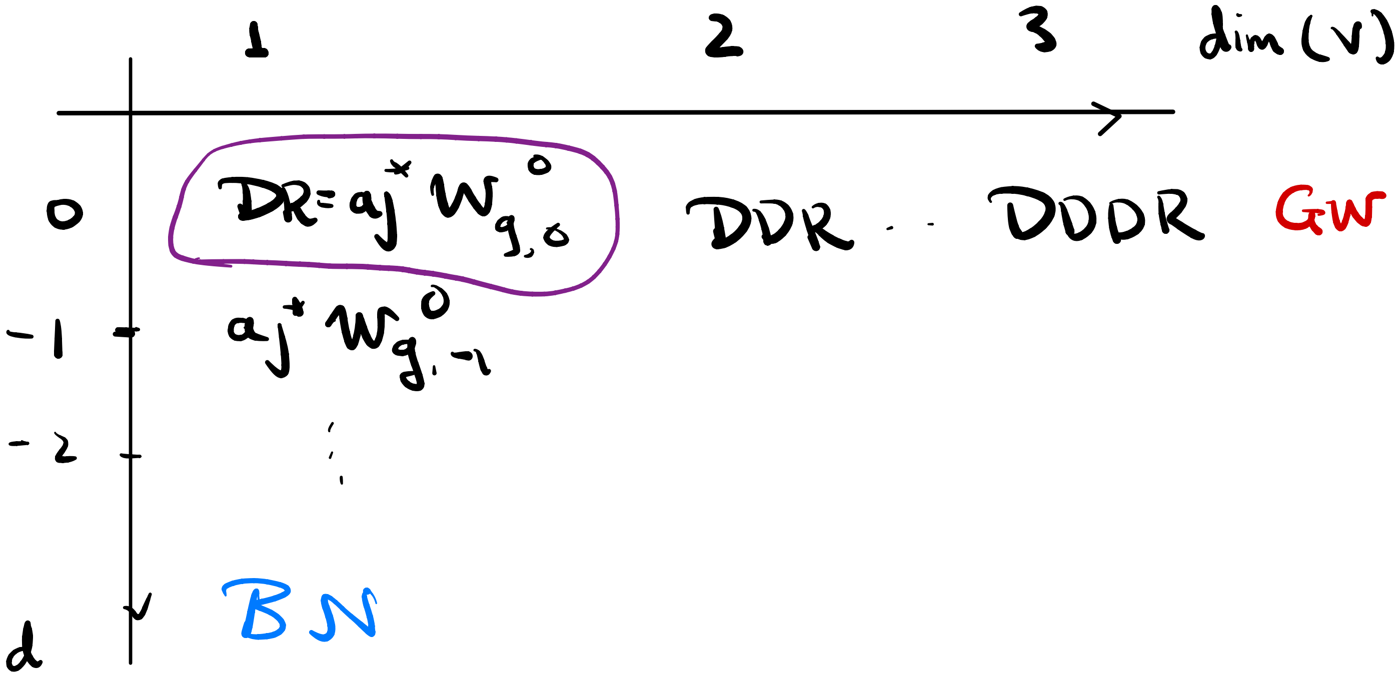
$\frac{9}{2}(\psi_1+\psi_2) +$ 

Relations: $\psi_1 = \psi_2$, $\lambda_1 = \delta_{irr}/12$ on $\overline{M}_{1,2}$

In fact, the method calculates the (virtual) classes of any universal $W^r_{g,d}$ pulled back to $\overline{M}_{g,n}$

$$W^0_{g,0} = [0] \rightsquigarrow \log DR$$

$$W^0_{g,d<0} = 0 \rightsquigarrow \text{Relations on } \overline{M}_{g,n}$$



1
2
3
dim(V)
0
DR = $a_j^* W^0_{g,0}$
DDR . . .
DDDR
GW
-1
$a_j^* W^0_{g,-1}$
-2
BN
d