Guided Meditations

Goal

- Progress on DR-type problems
- $A = (a_1, \dots, a_n)$, $\mathcal{L} \in \mathrm{Pic}(\overline{\mathcal{C}}_{g,n})$

$\rightsquigarrow$ ~~Class of the locus~~

$$\text{“}\left\{ (C, x_1, \dots, x_n) : \mathcal{O}\left(\textstyle\sum a_i x_i\right) \simeq \mathcal{L} \right\}\text{”}$$

$CH^g(\overline{\mathcal{M}}_{g,n})$

virtual

subtle over $\partial \overline{\mathcal{M}}_{g,n}$

$$\boxed{\mathcal{L} \simeq \mathcal{O}}$$

$$\mathcal{O}\left(\textstyle\sum a_i x_i\right) \simeq \mathcal{O}$$

$$\overline{M}^{rub}_{g,A}(\mathbb{P}^1)^{log/rel} \xrightarrow{\ p\ } \overline{M}_{g,n}$$

$$p_*[\overline{M}^{rub}_{g,A}(\mathbb{P}^1)^{log}]^{vir} := DR_{g,A}$$

not enough for log GW of toric varieties.

$\mathcal{L} \neq \mathcal{O}$

Other approach (D. Holmes, Marcus-Wise)

$$\begin{tikzcd}
DR_{\mathcal{L}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & M_{g,n} \arrow[d, "0"] \\
M_{g,n} \arrow[r] & Jac(C_{g,n})
\end{tikzcd}$$

$aj^*[0]$

Rigidify by $\mathbb{G}_m$

$$aj = aj_A \otimes \mathcal{L}^{-1}$$

$$aj(C, x_1, \ldots, x_n) = \mathcal{O}(\Sigma a_i x_i) \otimes \mathcal{L}^{-1}$$

$$\begin{tikzcd}
DR\mathcal{L} \arrow[r] \arrow[d] \arrow[dd, bend right=60, "p"', "\text{proper}"] \arrow[dr, phantom, "\square"] & \overline{\mathcal{M}}_{g,n} \arrow[d, "0"] \\
\mathcal{M}_{g,n} \arrow[r, "aj"] \arrow[d, "\text{not proper}"] & Jac(\overline{C}_{g,n}) \\
\overline{\mathcal{M}}_{g,n} \arrow[r, dashed, "\text{rational}"'] & Jac
\end{tikzcd}$$

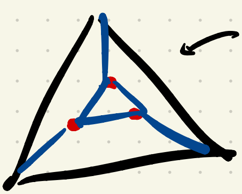
trop DR

"universal normal space containing $\mathcal{M}_{g,n}$ as an open on which aj extends"

$$p_*(aj^*([0])) \in CH^g(\overline{\mathcal{M}}_{g,n})$$

$$= DR_{g,A}(R)$$

When $R = 0$ get same class as w/ GW.



$\overline{\mathcal{M}}_{g,n}$

$$
\begin{tikzcd}
DRL \arrow[d, phantom, "\cap"] \arrow[dr, "q"] \arrow[dd, bend right=60, "p"'] & \\
\mathcal{M}^{\lozenge}_{g,n} \arrow[r, phantom, "\subset"] \arrow[d] & \overline{\mathcal{M}}^{\blacklozenge}_{g,n} \arrow[dl, "b"] \\
\mathcal{M}_{g,n} &
\end{tikzcd}
$$

choice of compactification

$b \rightsquigarrow$ log blowup

$$q_* aj^*[0] = DR^{\blacklozenge}_{g,A}(\ell) \in CH^g(\overline{\mathcal{M}}^{\blacklozenge}_{g,n})$$

refinement of $DR_{g,A}(\ell)$

$$b_* DR^{\blacklozenge}_{g,A}(\ell) = DR_{g,A}(\ell)$$

$$b^* DR_{g,A}(\ell) \neq DR^{\blacklozenge}_{g,A}(\ell)$$

$\overline{\mathcal{M}}_{g,n}$ not canonical

$\underline{DR^{\spadesuit}_{g,A}}$ as a representative of a class in $\log CH(\overline{\mathcal{M}}_{g,n})$

$$\boxed{\log DR_{g,A}(\mathcal{L})} = b^* DR_{g,A}(\mathcal{L}) + \text{correction}$$

These do determine log GW theory of toric varieties

formula?

$$\log DDR_{g,A,B} = \log DR_{g,A} \log DR_{g,B}$$

$$DDR_{g,A,B} \neq DR_{g,A} \cdot DR_{g,B}$$

when $\mathcal{L} \cong \mathcal{O}$

$$p_*\left[K^{\text{rub}}_{g,A,B}(\mathbb{P}^1\times\mathbb{P}^1)^{\log}\right]^{\text{vir}}$$

Pixton's formula

Knew: When $\ell = 0$

$DDR_{g,A,B}$ is tautological

Get to formulas for $\log DR_{g,A}(\ell)$

(w/ HPPS)

Reason formulas are difficult.

$\overline{M}_{g,n}^{*}$ non explicit $\longleftrightarrow$ aj section fails

Caporaso's school

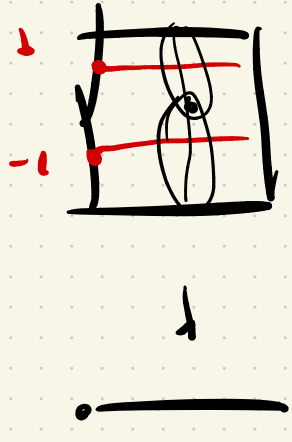
Esteves' school

Explicit $\overline{M}_{g,n}^{*}$ $\longleftrightarrow$ aj extension $\longleftrightarrow$ Compactified Jacobians

logPic

Meditation 1:

$\vartheta(\sum a_i x_i)$



$$X \to S = \operatorname{Spec} R$$

s n

multideg. 0 part of Pic

$$\mathrm{Jac}(X/S) = \mathrm{Pic}^{[0]}(X/S)$$

total deg 0

$$\mathrm{Pic}^{0}(X/S)$$

not universally closed

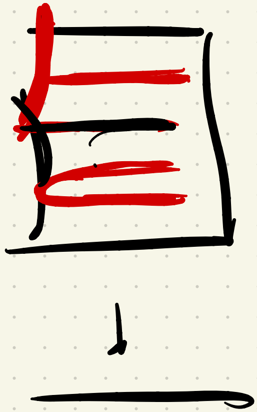
not proper

not separated.

$\mathcal{L}_n$ on X_n $\rightsquigarrow$ $g_{\alpha\beta}(n) \in \mathcal{O}^{*}_{X_n}(W_{\alpha\beta}(n))$

Over s

may have zeros/poles



$\mathcal{L}_n$ as a Cartier divisor

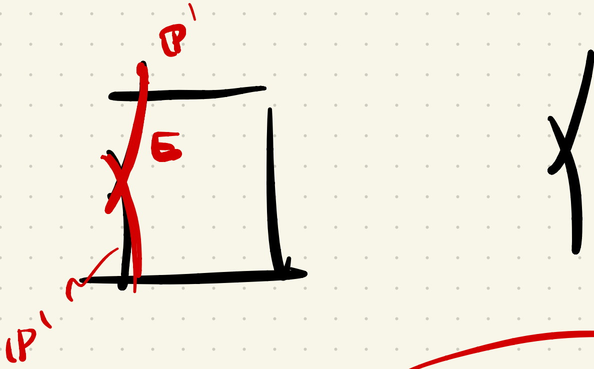
$$\mathcal{L} = \mathcal{O}(D_{vert}) \in \mathrm{Pic}^\circ$$

$$\mathcal{O}(D_n) = \mathcal{L}_n$$

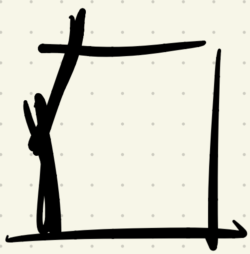
$$\overline{D}_n = D = D_{flat} + D_{vert} + D_{nodal}$$

$\{D_{vert}\}$ precisely the obstruction to Pic° being separated.

$\mathcal{O}(D_{vert})$ = limits of trivial bundles



$$\mathcal{O}(E)\big|_{X_s} = \left(\mathcal{O}(-1), \mathcal{O}(1)\right)$$



$$\mathcal{O}(X_s) \simeq \mathcal{O}$$

$$j : X_n \longrightarrow X$$

$$0 \to \mathcal{O}_X^* \to K_X^* \to \mathrm{CDiv} \to 0$$

$$\{\mathrm{Dvert}\} = \begin{array}{c} j_* \mathcal{O}_{X_n}^* / \mathcal{O}_X^* \\ \cup \\ \overline{M}_X^{gp} \end{array} \subset K_X^* / \mathcal{O}_X^*$$

Easy lemma

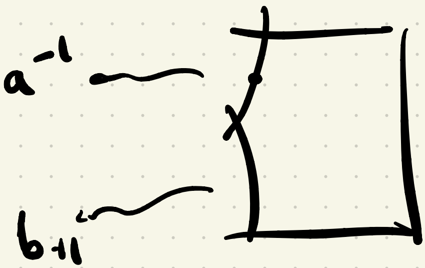
locus where M_X is trivial

if X log smooth, $U \overset{j}{\subset} X$

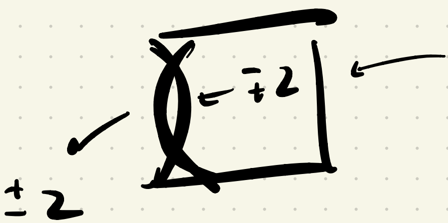
Then $j_* \mathcal{O}_U^* = M_X^{gp}$

Chip firing

Remark $H^0(X, \overline{M}_X^{gp}) = PL(\Sigma_X)$



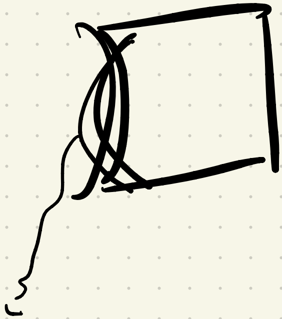
$\mathcal{O}(-1), \mathcal{O}(1)$



- Transition functions in

$$j_* \mathcal{O}_{X_U}^* = M_X^{gp}$$

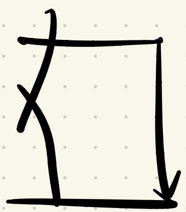
- Failure of sep. is measured by $\overline{M}_X^{gp}$



$\mathrm{Pic}^{[0]}$ <u>is</u> separated

norm.

$$\begin{tikzcd} 1 \arrow[r] & T \arrow[r] & \mathrm{Pic}^{[0]}(X_s/s) \arrow[r] & \mathrm{Pic}^{0}(X_s^{\vee}/s) \arrow[r] & 1 \\ \mathrm{Hom}(H_1(\Gamma), k^*) \arrow[ur] & & & & \end{tikzcd}$$



Look at torsors under M_X^{gp}

Illusie Kato ?

$\mathbb{G}_m^{\log} : \mathrm{LogSch} \to \mathrm{Gps}^{op}$

$\mathbb{G}_m^{\log}(S) = H^0(S, M_S^{gp})$

$\mathbb{G}_m^{\mathrm{trop}}(S) = H^0(S, \bar{M}_S^{gp})$ $\bar{M}_{gin}$

log scheme (S is log smooth)

Def $X \xrightarrow{P} S$ log curve

$\boxed{\underline{\mathrm{LogPic}(X/S)(T)}} = \{ M_{X \times_S T}\text{-torsors on } X \times_S T \}$ (bounded monodromy)

$\shortparallel$

$(P_* B\mathbb{G}_m^{\log})^{\dagger}$

$\mathbb{G}_m^{\log} \subset \mathrm{Aut}(\text{any pt})$

$\mathrm{LogPic}(X/S)(T) = (R^1(P_T)_* \mathbb{G}_m^{\log})^{\dagger}$

<u>Luca</u>: How to think about this.

Answer: Not so scary

$$0 \to \mathcal{O}_X^* \to M_X^{gp} \to \overline{M}_X^{gp} \to 0$$

extension of non-coherent by constructible

$$\rightsquigarrow H^0(\overline{M}_X^{gp}) \to \underline{\mathrm{Pic}(X)} \to H^1(M_X^{gp}) \to H^1(\overline{M}_X^{gp})$$

?

Algebraization tool

- Interesting tropicalization
- Interesting Cover
- Proper

$\mathcal{O}_X^*$ not coherent

Target space

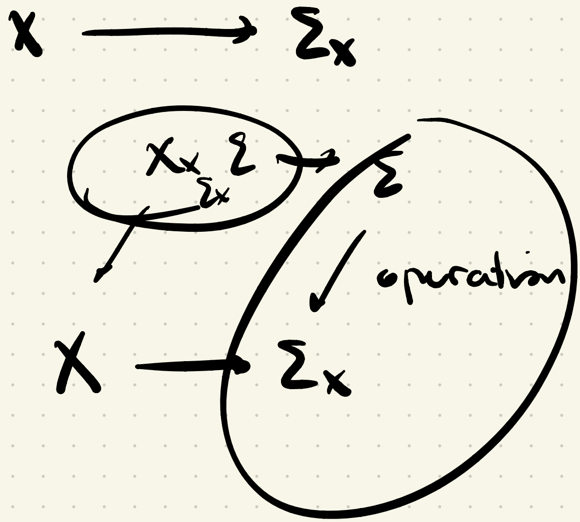
Brauer group

evil

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$

$$0 = H^2(\mathcal{O}_X) \to H^2(\mathcal{O}_X^*) \to H^3(\mathbb{Z})$$

Principle



Cover

$$0 \to \mathcal{O}_X^* \to M_X^{gp} \to \bar{M}_X^{gp} \to 0$$

$$\mathrm{Pic}(X) \to H^1(X, M_X^{gp})$$

$$\mathrm{Pic}(X/S) \longrightarrow \log\mathrm{Pic}(X/S)$$

log étale

not cover.

<u>Thm</u> (A bit difficult)

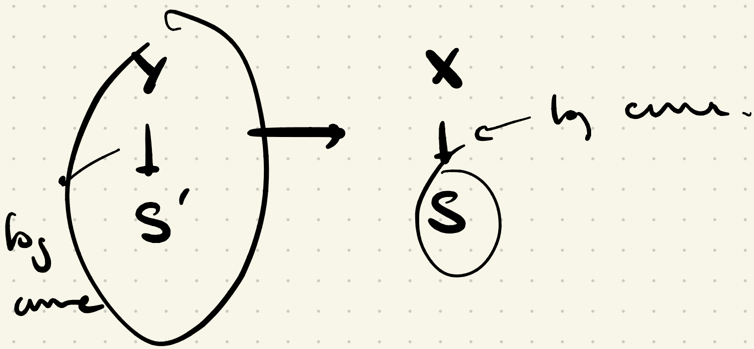
if X is log flat, if $Y \to X$ is either a log modification or a root, then

$$H^1(X, M_X^{gp}) = H^1(Y, M_Y^{gp})$$

<u>Corollary</u>

Let $X \to S$ log curve.

$$\begin{tikzcd} Y \arrow[r, "\text{mod root}"] \arrow[dr, "\text{log curve}"'] & X' \arrow[r] \arrow[d] & X \arrow[d] \\ & S' \arrow[r, "\text{log mod/root}"'] & S \end{tikzcd}
\qquad
\begin{tikzcd} Y \arrow[d] \arrow[dd, bend left] \\ X \arrow[d] \\ S \end{tikzcd}$$



$$\mathrm{LogPic}(Y/S') = \mathrm{LogPic}(X/S)$$

$$\mathrm{Pic}(Y/S') \longrightarrow \mathrm{LogPic}(X/S)$$

<u>Thm</u>

a cover

log étale cover

$$\varinjlim_{\substack{Y \to S' \\ \downarrow \quad \downarrow \\ X \to S}} \mathrm{Pic}(Y/S') \longrightarrow \mathrm{LogPic}(X/S)$$

Think of $\log\mathrm{Pic}(X/S)$ "Lawrence style"

Systems of line bundles $\mathcal{L}$ on

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

up to action of $\boxed{H^0(Y, \bar{M}_Y^{gp})}$ of PL functions

not Lawrence style

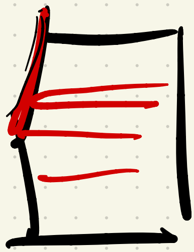
Lawrence

$$\log CH(X) = \lim_{\substack{\longrightarrow \\ Y \to X}} CH(Y)$$

small

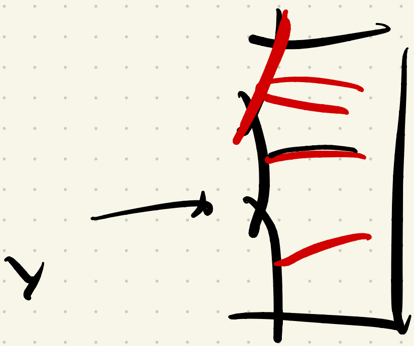
$$\begin{array}{ccc} & Y & \\ \swarrow & & \searrow \\ Y_1 & & Y_2 \end{array}$$

$$\mathcal{L} = \mathcal{L}' \otimes \mathcal{O}(-\alpha)$$



vertical components all killed in LogPic

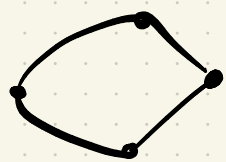
$\mathrm{Pic}^0(X/S)$



limit line bundle in a blowup.

$H^0(X, \overline{M}_X^{gp})$

$H^0(Y, \overline{M}_Y^{gp})$



$\{D_{vert}\}$ $\overline{M}_X^{gp}$

$$H^0(Y, \overline{M}_Y^{gp}) \to \boxed{\mathrm{Pic}(Y)} \to H^1(Y, M_Y^{gp}) \to H^1(Y, \overline{M}_Y^{gp})$$

$$H^1(Y, M_Y^{gp}) = H^1(X, M_X^{gp})$$

$$\alpha \mapsto \mathcal{O}(-\alpha) \mapsto \mathcal{L}$$

Tropicalization

$X \downarrow S$ $\rightsquigarrow$ System of $\Gamma_s / \overline{M}_{S,s}$ up to specialization ?

$$\begin{array}{ccc} X & & \Sigma_X \\ \downarrow & \rightsquigarrow & \downarrow \\ S & & \Sigma_S \end{array}$$

rel. dim 1

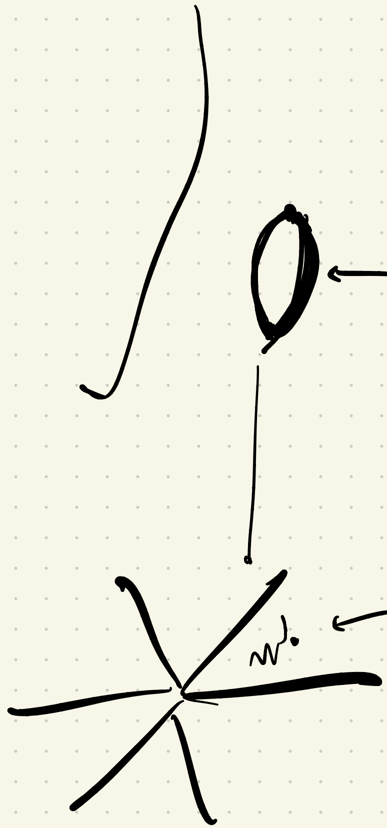
cones map onto cones

lattices onto lattices

log curve

integral

saturated



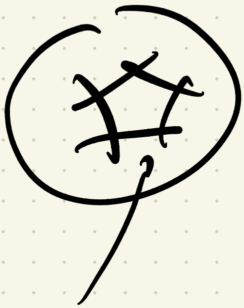
$\delta(e)(\sigma)$

$\delta(e) : \sigma \to \mathbb{R}_{\geq 0}$

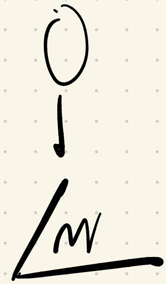
$\delta(e) \in \overline{M}_{S,s}$

σ

$\Gamma_s, \overline{M}_{s,s}$



$$\begin{array}{c} X \\ \downarrow \\ S \end{array}$$



$s \rightsquigarrow \Gamma_s$ = Dual graph of X_s

metrized by $\overline{M}_{s,s}$.

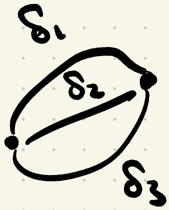
Tropicalization of LogPic.

Γ/M — tropical curve metrized by M.

Several structures

- Γ has a topology

Stars of strata



$\delta(e) \in M$

$M = \mathbb{N}^3 \hookleftarrow \overline{M}_{g,n}$

$\delta_i = e_i$

$\$$

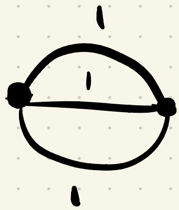
Does not depend on M

2. $PL(\Gamma) = \{ f(v) \in M^{gp} \quad \forall v \in V(\Gamma) :$

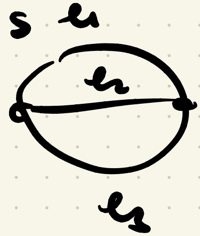
M^{gp} $\quad f(w) - f(v) = s(\vec{e})\,\delta(e) \leftarrow M$

$\in \mathbb{Z}$

$\forall \vec{e} : v \longrightarrow w$

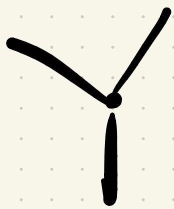


$M = \mathbb{R}_{\geq 0}$ $\quad PL(\Gamma) = \mathbb{R} \times \mathbb{Z}$



$M = \mathbb{N}^3$ $\quad PL(\Gamma) = \mathbb{Z}^3$

$$f(w) = f(v) + s \cdot e_1$$



$M^{gp} \times \mathbb{Z}$

$$PL(U) = M^{gp} \times \mathbb{Z}^{val(v)}$$

$$\mathrm{Dir}(\Gamma) = \left\{ \sum a_v \cdot v \;\middle|\; a_v \in \mathbb{Z},\ v \in V(\Gamma) \right\}$$

Do ag theory on Γ

$$PL(\Gamma) \xrightarrow{\mathrm{div}} \mathrm{Dir}(\Gamma)$$

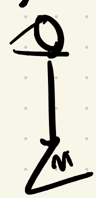
$$\mathrm{div}\, f = \sum_v \Big(\sum_{v \vec{e} \ni v} s(\vec{e}) \Big) v$$

<u>Def</u>. $\mathcal{L} = \mathrm{Ker}\,\mathrm{div}$

$\mathcal{L}$-torsors on Γ (+ b.m.)

How to compute $H^1(\Gamma, \mathcal{L})$

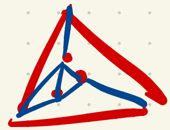
$$\begin{tikzcd} X \arrow[d] \arrow[r, phantom, "\rightsquigarrow"] & \Sigma_X \arrow[d] \\ S & \Sigma_S \end{tikzcd}$$



M_X-torsors $\longleftarrow$ $\mathcal{L}$-torsors

$$\begin{array}{c} X \\ \downarrow \\ S \end{array}$$

$\mathcal{M}_{g,n}^{\diamond} \subset \overline{\mathcal{M}}_{g,n}^{\diamond}$



$\downarrow$

$$\begin{tikzcd} & \mathrm{Pic}(\phi) \arrow[d] \arrow[dl, hook'] \\ \mathrm{Pic}^{[0]}(X/S) \arrow[r, hook] & \mathrm{LogPic}(X/S) \end{tikzcd}$$

not alg. stack

remarkable

$$\begin{tikzcd} \mathcal{M}_{g,n}^{\diamond} \subset \overline{\mathcal{M}}_{g,n}(\phi) \arrow[r, "aj_\phi"] \arrow[d] & \mathrm{Pic}(\phi) \arrow[r] \arrow[d, "\text{blowups}"] & \bullet \arrow[d] \\ \overline{\mathcal{M}}_{g,n} \arrow[r, "aj"'] \arrow[rr, bend right] & \mathrm{LogPic}^{0}(\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n}) \arrow[r] & \mathrm{TroPic} \end{tikzcd}$$

$DR = aj^*([0])$ $\quad [0] \in CH^g(\mathrm{LogPic})$

$aj_\phi^*([0]) \in \overline{\mathcal{M}}_{g,n}(\phi)$ $\quad \mathrm{LogCH}^g(\mathrm{LogPic})$