Guided Meditations 2

Reminder

$X \xrightarrow{p} S$ log curve

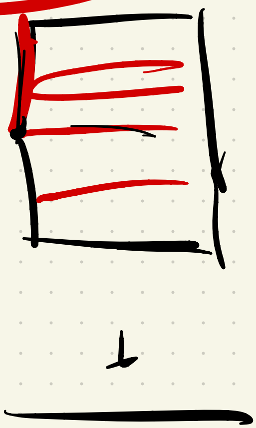
$$\underbrace{\log\mathrm{Pic}(X/S)} = p_* B\mathbb{G}_m^{\log\,+}$$

bounded monodromy

$$= \{ M_{X\times_S T}^{gp}\text{-torsors on } X\times_S T \}^+$$

- Cover
- Properness
- Tropicalization

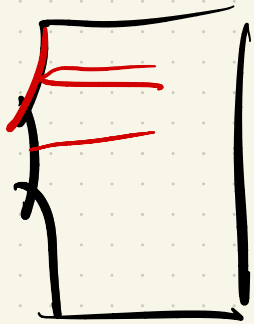
Meditation 1:



$\mathrm{Pic}^\circ(X/S)$

X toric

$\mathrm{Pic}(X) = PL(X)$



$$\mathrm{LogPic}(Y/T) = \mathrm{LogPic}(X/S)$$

Vertical Components $H^0(Y, \overline{M}_Y^{gp})$

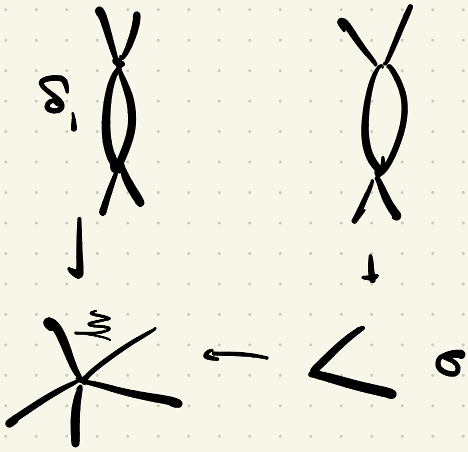
$$H^0(\overline{M}_Y^{gp}) \to \mathrm{Pic}(Y/T) \xrightarrow{\text{cover}} \mathrm{LogPic}(X/S)$$

$$\text{“}\varinjlim \mathrm{Pic}(Y/T)/H^0(\overline{M}_Y^{gp}) = \mathrm{LogPic}(X/S)\text{”}$$

$$H^0(\overline{M}_X^{gp}) \to H^1(\mathcal{O}_X^*) \to H^1(M_X^{gp}) \to H^1(\overline{M}_X^{gp})$$

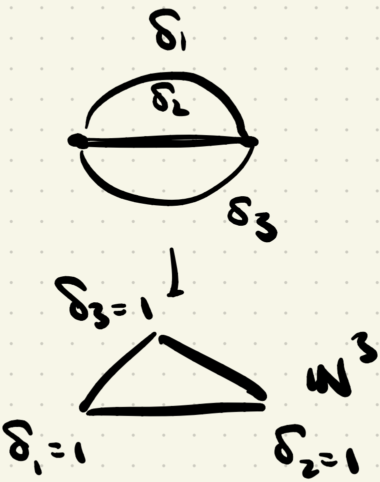
Tropicalization

$$\begin{array}{ccc} X & & \Sigma_X \\ \downarrow & \rightsquigarrow & \downarrow \\ S & & \Sigma_S \end{array}$$

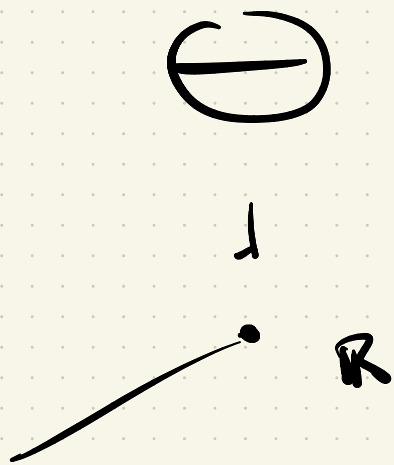


$\longleftrightarrow \quad \Gamma / M = (\text{int. points in } \sigma^\vee)$

—



$\Gamma / \mathbb{N}^3$

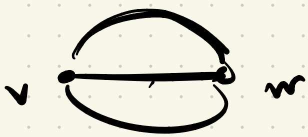


- Topology on Γ

$= \langle$ Stars of strata $\rangle$

- $PL(\Gamma) = \{ f(v) \in M^{gp} \mid \forall \vec{e}$ from v to w $f(w) - f(v) = s(\vec{e}) \cdot \delta(e) \}$

slope in $\mathbb{Z}$.



$s_1 \delta_1 = s_2 \delta_2 = s_3 \delta_3$

$f(v) \in M^{gp}$

$f + m$

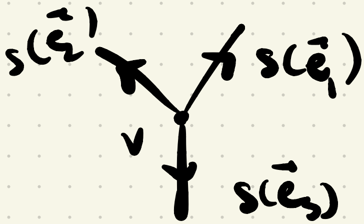
$M^{gp} \subset PL(\Gamma)$

Constant functions

$$\mathrm{Div}(\Gamma) = \left\{ \sum a_v \cdot v \right\} \simeq \mathbb{Z}^{V(\Gamma)}$$

$$\mathrm{div} : PL(\Gamma) \longrightarrow \mathrm{Div}(\Gamma)$$

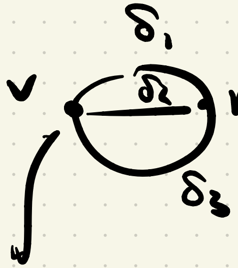
$$\mathrm{div} f = \sum_{v} \Big(\sum_{{}^{v}\vec{e} \ni v} s(\vec{e}) \Big) v$$



$$PL(\Gamma) \supset \mathrm{Ker}\,\mathrm{div} := \mathcal{L}(\Gamma)$$

$$\sum_{{}^{v}\vec{e} \ni v} s(\vec{e}) = 0$$

"balancing condition".



$/ W^3$

$/ \mathbb{R}_{\geq 0}$

value → $\mathbb{R} \times \mathbb{Z}$

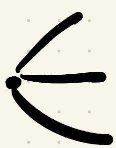
$f(v) \in \mathbb{Z}^3$

$$f(w) - f(v) = s_1 \cdot \delta_1 = s_1(1,0,0)$$

$$s_2 \cdot \delta_2 = s_2(0,1,0)$$

$$s_3 \cdot \delta_3 = s_3(0,0,1)$$

$PL(\Gamma) = \mathbb{Z}^3$

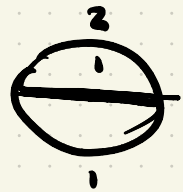


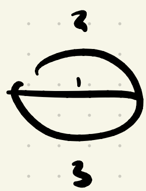
$\mathbb{Z}^3 \times \mathbb{Z}^3$

value

slope

$s_2 =$





$$PL(Y) = M^{gp} \times \mathbb{Z}^{val(v)}$$

$$\mathcal{L}(Y) = M^{gp} \times \underbrace{\mathbb{Z}_0^{val(v)}}_{\{(s(e)) : \sum s(e) = 0\}}$$

Lemma

$$\mathcal{L}(\Gamma) = M^{gp}$$ Relevant

Proof

$w > v$

if $f(w) > f(v)$ in M^{gp}

$$f(w) - f(v) = s(\vec{e}) \cdot \delta(e) > 0$$

$$\sum_{\vec{e} \ni v} s(\vec{e}) = 0$$

$\mathcal{L}$, $P\mathcal{L}$, Div

$\mathcal{T}$ = twists = tropical differentials

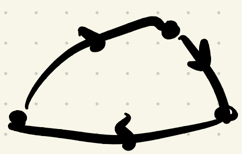
$$\mathcal{T}(\Gamma) = \{ s(\vec{e}) \in \mathbb{Z} \mid s(\vec{e}) = -s(\overleftarrow{e}) \}$$

$$\simeq \mathbb{Z}^{E(\vec{\Gamma})}$$

Pairing

$$\langle\ ,\ \rangle : \mathcal{T} \times \mathcal{T} \longrightarrow M^{gp}$$

$$\langle 1_{\vec{e}}, 1_{\vec{f}} \rangle = \begin{cases} 0 & \text{if } e \neq f \\ 1 & \text{if } \vec{e} = \vec{f} \\ -1 & \text{if } \vec{e} = \overleftarrow{f} \end{cases}$$



$$\int_P \left(\sum t(\vec{f})\, 1_{\vec{f}} \right)$$

$$P = \sum s(\vec{e})\, 1_{\vec{e}}$$

Meditation 2

— Diagram of exact seq. of sheaves

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M^{gp} \arrow[r] \arrow[d, equal] & \mathcal{L} \arrow[r] \arrow[d] & \mathcal{H} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M^{gp} \arrow[r] & P\mathcal{L} \arrow[r] \arrow[d] & J \arrow[r] \arrow[d] & 0 \\
 & & \mathrm{Div} \arrow[r, equal] \arrow[d] & \mathrm{Div} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

$\mathcal{H}$ = harmonic differentials

$J = P\mathcal{L}/M^{gp}$ sheaf iso

$$J(Y) = \mathbb{Z}^{val(v)}$$

$$P\mathcal{L}(Y) = M^{gp} \times \mathbb{Z}^{val(v)}$$

$P\mathcal{L}/M \cong T$

$$(*)$$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M^{gp} \arrow[r] \arrow[d, equal] & \mathcal{L} \arrow[r] \arrow[d] & \mathcal{A} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M^{gp} \arrow[r] & P\mathcal{L} \arrow[r] \arrow[d] & \underline{\mathcal{J}} \arrow[r] \arrow[d] & 0 \\
 & & Div \arrow[r, equal] \arrow[d] & Div \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

flasque

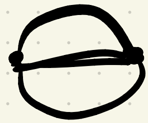
$(*) \Rightarrow$

$$\boxed{0 \to H^0(\mathcal{A}) \to \left[\mathbb{Z}^{E(\vec{\Gamma})} \to \mathbb{Z}^{V(\Gamma)}\right] \to H^1(\mathcal{A}) \to 0}$$

$$H^0(\mathcal{A}) = H_1(\Gamma) \simeq \mathbb{Z}^g$$

$$\mathcal{H}^1(\mathcal{A}) = H_0(\Gamma) = \mathbb{Z}$$

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
(**)\ 0 \arrow[r] & M^{gp} \arrow[r] \arrow[d, equal] & \mathcal{L} \arrow[r] \arrow[d] & \mathcal{A} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M^{gp} \arrow[r] & P\mathcal{L} \arrow[r] \arrow[d] & J \arrow[r] \arrow[d] & 0 \\
 & & \mathrm{Div} \arrow[r, equal] \arrow[d] & \mathrm{Div} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}
$$



$\delta_1, \delta_2, \delta_3 \in$ —

$\mathbb{R}_{\geq 0}(\delta_1, \delta_2, \delta_1)$

$$
\begin{array}{c}
H^0(\mathcal{A}) \\
\shortparallel \\
0 \longrightarrow H_1(\Gamma) \longrightarrow H^1(\Gamma, M^{gp})
\end{array}
$$

$$
= \mathrm{Hom}(H_1(\Gamma), M^{gp}) \longrightarrow H^1(\mathcal{L}) \xrightarrow{\text{deg.}} \mathbb{Z} \to 0
$$

$$
\mathbb{Z} = H^1(\mathcal{A})
$$

$$
\mathcal{H}^1(\mathcal{L})^0 = \mathcal{H}om(H_1(\Gamma), M^{gp}) \Big/ H_1(\Gamma)
$$

$$
\langle \ , \ \rangle
$$

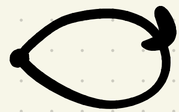
$$
\langle \ell = \textstyle\sum d_{\vec{e}}, \ \ell' = \textstyle\sum d_{\vec{f}} \rangle
$$

$$
= \delta_{\vec{e} \cap \vec{f}} = \underset{\text{edges in common}}{\textstyle\sum \delta_e}
$$

$H_1(\Gamma) \to M^{gp}$

$H_1(\Gamma) \to H^1(M^{gp})$

$= \mathrm{Hom}(H_1, M^{gp})$



$\ell = 1_e$

$$H_1(\Gamma) \to M^{gp}$$
$$\ell \mapsto \delta(e)$$
$$-\ell \mapsto -\delta(e)$$

$\langle \ell, \ell \rangle = \delta(e)$

$\langle -\ell, -\ell \rangle = \delta(e)$

$$\langle\ ,\ \rangle : T \times J \to M^{gp}$$

$H^\bullet(\mathcal{H})$

" $H_1(\Gamma) \to J$

$\ell \mapsto -$

Def

Stack

$$\mathcal{T}ro\mathcal{P}ic(\Gamma/M) = \{\mathcal{L}\text{-torsors on } \Gamma\}^{\dagger}$$

$$\text{TroPic}(\Gamma/M) = H^1(\mathcal{L})^{\dagger}$$

$$\text{TroJac}(\Gamma/M) = \text{Hom}(H_1(\Gamma), M^{gp})^{\dagger} / H_1(\Gamma)$$

$\dagger$ condition on $\mathcal{H}om(H_1(\Gamma), M^{gp})$

$\phi : H_1(\Gamma) \to M^{gp}$ has bounded monodromy

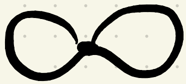
if $\forall \ell \in H_1(\Gamma)$

$$\phi(\ell) \sim \langle \ell, \ell \rangle = \delta(\ell)$$

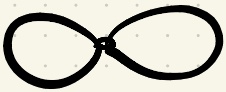
$$\boxed{\begin{array}{l} \exists\ m, n \in \mathbb{Z} \\ m\langle \ell, \ell \rangle \leq \phi(\ell) \leq n \langle \ell, \ell \rangle \end{array}}$$

Example

$\delta_1 = e_1$ $\quad$ $\delta_2 = e_2$

$\Gamma =$  , $\quad M = \mathbb{N}^2$ $\qquad (1,1)$

$(1,0) \leq (1,1)$

$H_1($$) \longrightarrow \mathbb{Z}^2$ never $(1,1) \leq m(1,0)$

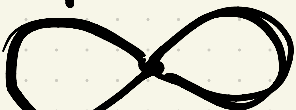
$l_1 \longrightarrow e_2$ $\qquad \cancel{\phi(l_1) = e_2}$

$l_2 \longrightarrow e_2$ $\qquad \langle l_1, l_1 \rangle = e_1$

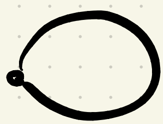
$e_1 \longrightarrow 0$

$\mathbb{N}^2 \longrightarrow \mathbb{N}$

$l_1 \longmapsto e_2$

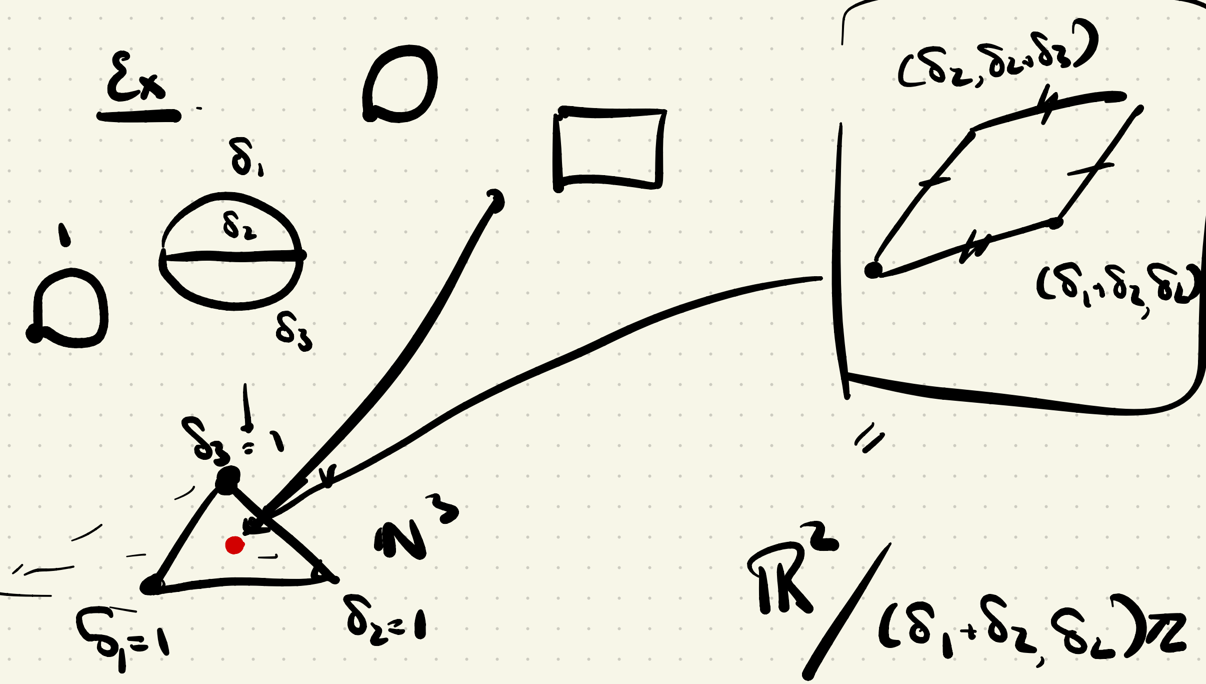
$H_1($$) \longrightarrow \mathbb{Z}^2$

Contract l_1

$H_1($$) \longrightarrow \mathbb{Z}$ $\qquad e_2$

$1 \quad 0$

Ex

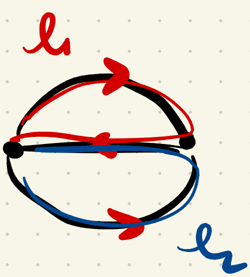


$$= \mathbb{R}^2 \Big/ (\delta_1+\delta_2, \delta_2)\mathbb{Z} \oplus (\delta_2, \delta_2+\delta_3)\mathbb{Z}$$

$\mathbb{R}_{\geq 0} V$

$$\mathrm{Hom}(H_1(\Gamma), N^3)^{+} \Big/ H_1(\Gamma)$$

$$\mathrm{TropJac} = \mathrm{Hom}(H_1(\Gamma), \mathbb{R}) \Big/ H_1(\Gamma)$$

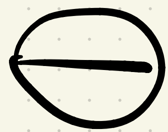


l_1 : $l_1 \longrightarrow \delta_1 + \delta_2$
$l_2 \longrightarrow \delta_2$

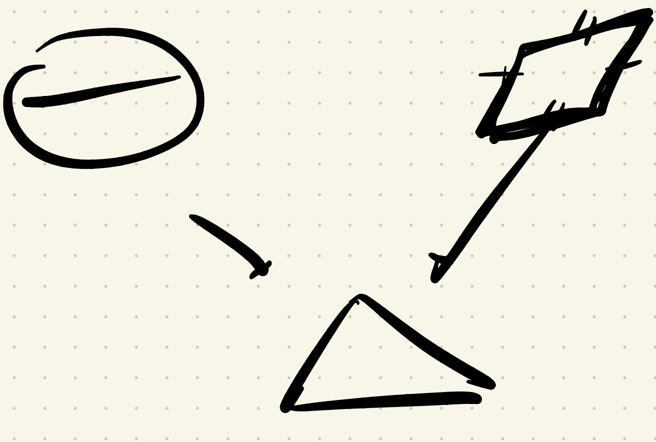
l_2 : $l_1 \longrightarrow \delta_2$
$l_2 \longrightarrow \delta_2 + \delta_3$

$$\begin{array}{ccccc} \$ & X & \rightsquigarrow & \Sigma_X \\ & \downarrow & & \downarrow \\ \downarrow & S & & \Sigma_S \\ \bullet & & & \end{array}$$

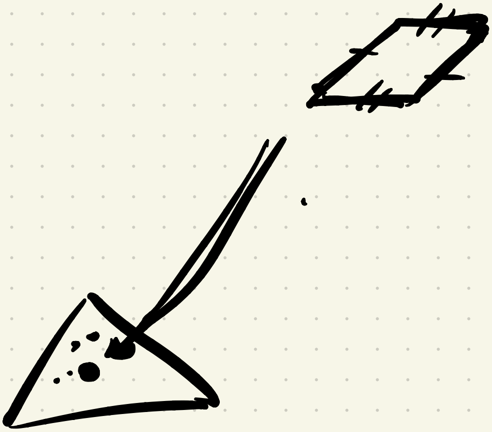
$(\mathbb{N}^3 \oplus k^*)$

Γ = dual graph of X_s = 

$$\mathrm{Hom}(H_1(\Gamma), M^{sp})^{+} / H_1(\Gamma)$$



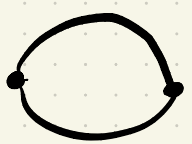
$$\begin{tikzcd}
\mathrm{TropJac}(\Sigma_X/\Sigma_S) \arrow[r] & \Sigma_S \arrow[d, dashed] \\
& \mathrm{Hom}(H_1(\Sigma_X|_v), \mathbb{R}) / H_1(\Sigma_X|_v)
\end{tikzcd}$$



N

$\mathbb{Z}$, minimal

$$\mathrm{Sections}(\sigma, \mathrm{TropJac}) = \mathrm{Hom}(H_1(\Gamma), M^{gp})^{+} / H_1(\Gamma)$$



$$\mathbb{R}/(\delta_1+\delta_2)\mathbb{Z}$$

$$\mathbb{Z}/(\delta_1+\delta_2)\mathbb{Z}$$

$$L \rightsquigarrow L^{\log}$$

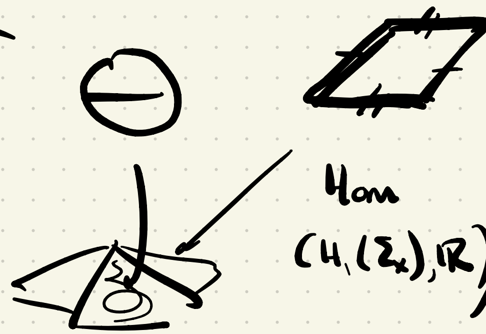
$$\begin{tikzcd} \mathrm{Pic} \arrow[r] \arrow[dr, bend right] & \mathrm{LogPic} \arrow[d] \\ & \mathrm{TroPic} \end{tikzcd}$$

On compact type

$$\mathrm{LogPic}^0 \cong \mathrm{Pic}^{[0]}$$

$\boxed{\text{Tropicalization map.}}$

$$\begin{tikzcd} X \arrow[d] & & \Sigma_X \arrow[d] \\ S & \rightsquigarrow & \Sigma_S \end{tikzcd}$$



$\mathrm{Hom}(H_1(\Sigma_x), \mathbb{R}) / H_1(\Sigma_x)$

Tropicalization LogPic $\to$ TroPic

$$\begin{cases} 0 \to \mathbb{G}_m \to \mathbb{G}_m^{\log} \to \mathbb{G}_m^{\text{trop}} \to 0 & \text{on } X \\ 0 \to \mathcal{L} \to P\mathcal{L} \to \mathrm{Div} \to 0 & \text{on } \Sigma_X \end{cases}$$

$$\boxed{X \xrightarrow{\text{trop}} \Sigma_X}$$

$$\boxed{R(\text{trop})_* \overline{M}_X^{gp} = P\mathcal{L}}$$

Sequence of sheaves of ab. groups

$$0 \to K \to G \xrightarrow{\pi} H \to 0$$

LES of group stacks

$$H \to BK \to BG \to BH \to \cdots$$

$$G \to G'$$

$BG \to BG'$ (extension of structure group)

$$P \longrightarrow P \times^G G' = P \times G' \big/ (p, g\cdot g') = (gp, g') \text{ for } g \in G$$

$$= \{ L \in BK,\ \alpha : L \times^K G \to G \}$$

$$\begin{array}{ccc} BK \times_{BG} O & \longrightarrow & O \\ \downarrow & & \downarrow \\ BK & \longrightarrow & BG \end{array}$$

G

$$\boxed{\boxed{H \longrightarrow BK} \longrightarrow BG \longrightarrow BH}$$

<u>Fact</u>: $G \longrightarrow G'$

$$\{\text{Trivializations } P \times^G G' \to G'\}$$

$$\updownarrow$$

$$\{G\text{-maps } P \to G'\}$$

$\alpha(p, g') = g' \cdot \beta(p)$

$\alpha : P \times^G G' \to G'$

$\beta(p) = \alpha(p, 1)$

β

Image of H in BK = K-torsors w/ map to G

$H \longrightarrow BK$

$h \rightsquigarrow \partial(-h) = \{g \mid \pi(g) = h\} \subset G$

$L \xrightarrow{\alpha} G \simeq \alpha(L) \subset G \simeq \partial(-\pi(\alpha(l))) \subset G$

$$BK \longrightarrow BG$$

$$L \longmapsto L \times^K G \longrightarrow H$$

$$(\ell, g) \longmapsto \pi(g)$$

$$BG \longrightarrow BH$$

$$P \longmapsto P \times^G H = P/K = \{L \in BK, \ \alpha : L \to P\}/\text{iso}$$

$$= \{L \in BK, \ \alpha : L \times^K G \to P\}/\text{iso}$$

Specialize to $\mathbb{G}_m \to \mathbb{G}_m^{\log} \to \mathbb{G}_m^{\text{trop}}$

$$B\mathbb{G}_m \longrightarrow B\mathbb{G}_m^{\log} \longrightarrow B\mathbb{G}_m^{\text{trop}}$$

$$L \longmapsto L^{\log}$$

$$P \longmapsto P/\mathbb{G}_m = \{L, \ \alpha : L^{\log} \to P\}/\text{iso}$$

$$0 \to \mathcal{L} \to P\mathcal{L} \to \mathrm{Div} \to 0$$

on X

$$\begin{tikzcd}
 & BG_m^{\log} \arrow[r] & BG_m^{\mathrm{trop}} \arrow[d, phantom, "\cong"] \\
P \arrow[rru, squiggly] & & \\
 & B\mathcal{L} \arrow[r] & BP\mathcal{L}
\end{tikzcd}$$

on Σ_X

$$P \rightsquigarrow P/G_m \longrightarrow \mathrm{Div}$$

x fiber in P $\quad \mathcal{O}(-x)$ is a line bundle

$\mathrm{mult}(\mathcal{O}(-x))$

1

$$\begin{tikzcd}
\mathbb{G}_m^{trop} \arrow[r] \arrow[d, "\wr"] & B\mathbb{G}_m \arrow[r] \arrow[d] & B\mathbb{G}_m^{log} \arrow[r] \arrow[d, "trop"] & B\mathbb{G}_m^{trop} \arrow[d, "\wr"] \\
P\mathcal{L} \arrow[r] & Div \arrow[r] & B\mathcal{L} \arrow[r] & BP\mathcal{L} \arrow[r] & BDiv
\end{tikzcd}$$

tropicalization

$$trop(P) = \left\{ L \in B\mathbb{G}_m,\ \alpha : L^{log} \to P \right\} / iso$$

s.t $mult(L) = 0$

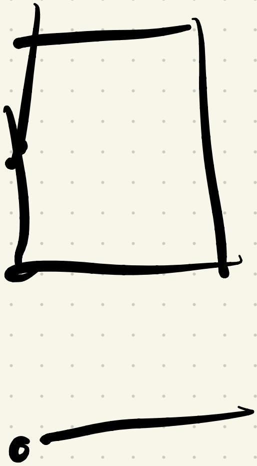
e

$$\to \mathcal{L}og Pic^d(X/S$$

)

$$\to TroPi$$

$$c^d(X/S) \to 0$$



$$\begin{array}{c} X \\ \big| \\ S \end{array}$$

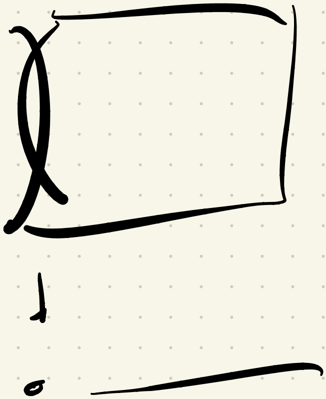
$\mathcal{L}_n \longmapsto P$ on X/S $\mathbb{G}_m^{\log}$-torsor

$$P \longmapsto \bar{P}$$

$$\mathrm{LogPic}(X/S) \to \mathrm{TroPic}$$

P may or may not be $\cong L^{\log}$

$$PL \to \mathrm{Div} \to \mathrm{TroJac}$$



mult.

$$\dot{\mathrm{Dir}} \to \dot{\mathrm{TroJac}}$$

If S is durr.

$$\lim_{\to} \mathrm{Dir}(\Gamma') / \mathrm{PL}(\Gamma') \simeq \mathrm{TroJac}$$

L

Corollary.

$\exists$ exact sequence

$$0 \to \mathrm{Pic}^{[0]}(X/S) \to \mathcal{L}og\mathrm{Pic}^{d}(X/S) \to \mathrm{TroPic}^{d}(X/S) \to 0$$

Corollary:

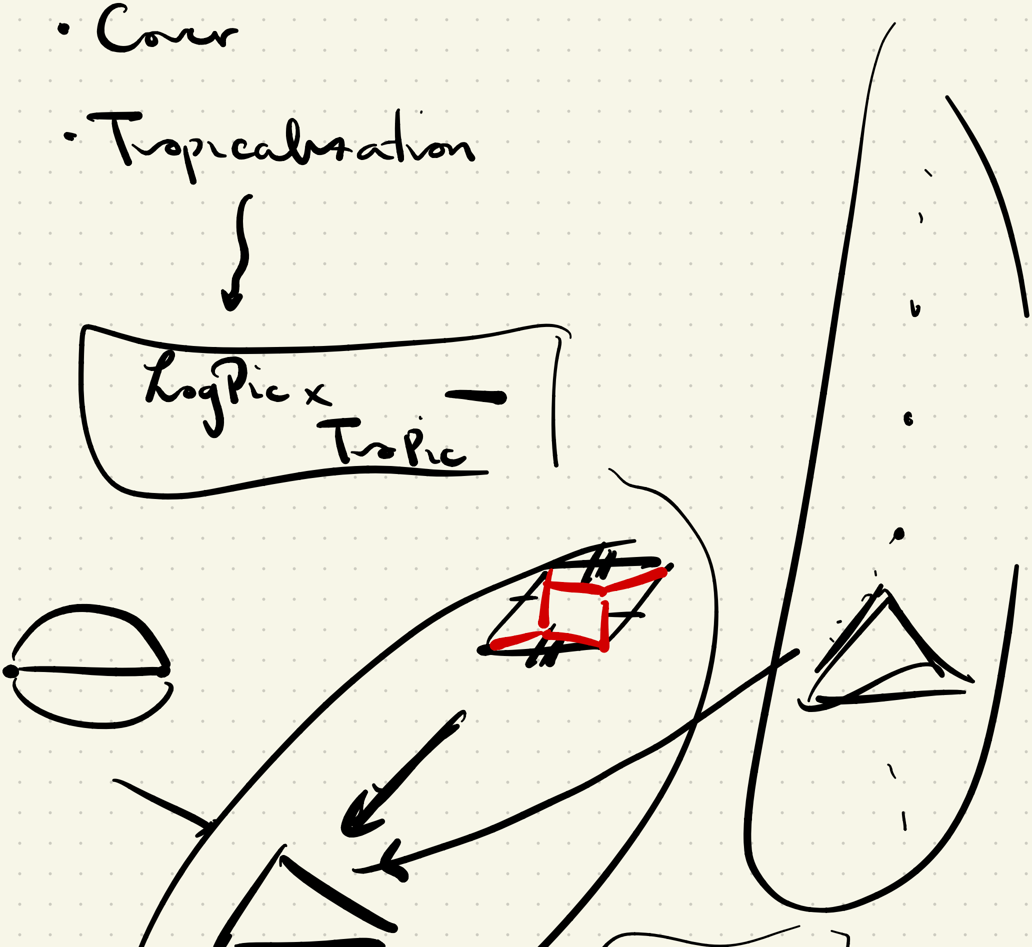
$\mathrm{LogPic} \times_{\mathrm{TroPic}} : \mathrm{Tropical\ Spaces}/_{\mathrm{TroPic}} \longrightarrow \mathrm{Alg\ Stacks}$

<u>Proof</u>: $\Sigma \to \mathrm{TroPic} \longleftrightarrow A \to \mathrm{TroPic}$

$\mathrm{LogPic} \times_{\mathrm{TroPic}} A$ is a $\mathrm{Pic}^{[0]}(X/S)$ over A.

- Proper
- Cover
- Tropicalization

$\mathrm{LogPic} \times_{\mathrm{TroPic}} -$



Dir

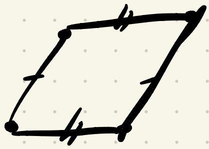
$\mathrm{Pic}(X) = \mathrm{LogPic} \times_{\mathrm{TroPic}} \cdot$

$\mathrm{Div} \longrightarrow \mathrm{TroPic}$

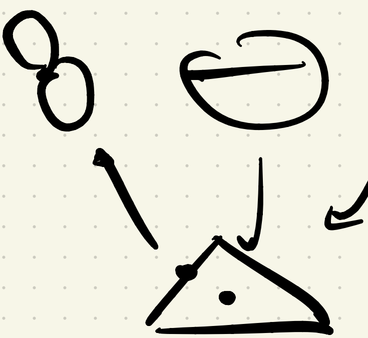
$H^1(M^{gp})$

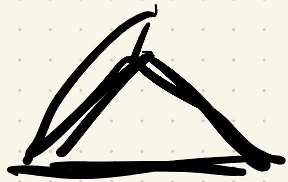
Div^0

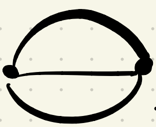
$(\delta_2, \delta_2+\delta_1)$



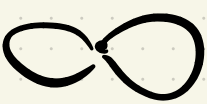
$(\delta_1+\delta_2, \delta_2)$





a  $-a$ $\quad \mathrm{Div}^0 \cong \mathbb{Z}$

$\$$

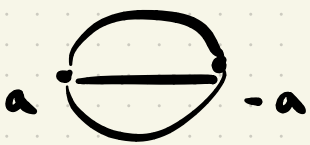
 $\mathrm{Div} = 0$

$$\mathrm{Div} \longrightarrow \mathrm{TroPic}$$

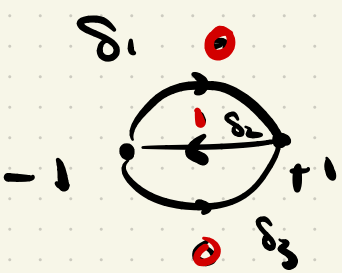
$$PL^{C} \quad D \longrightarrow [\bar{D}].$$

$\mathcal{L}$-worth of ?$\mathcal{L}$

$$\mathrm{Div}^{\circ} \longrightarrow \mathrm{Hom}(H_1(\Gamma), M^{gp})^{\dagger} / H_1(\Gamma)$$



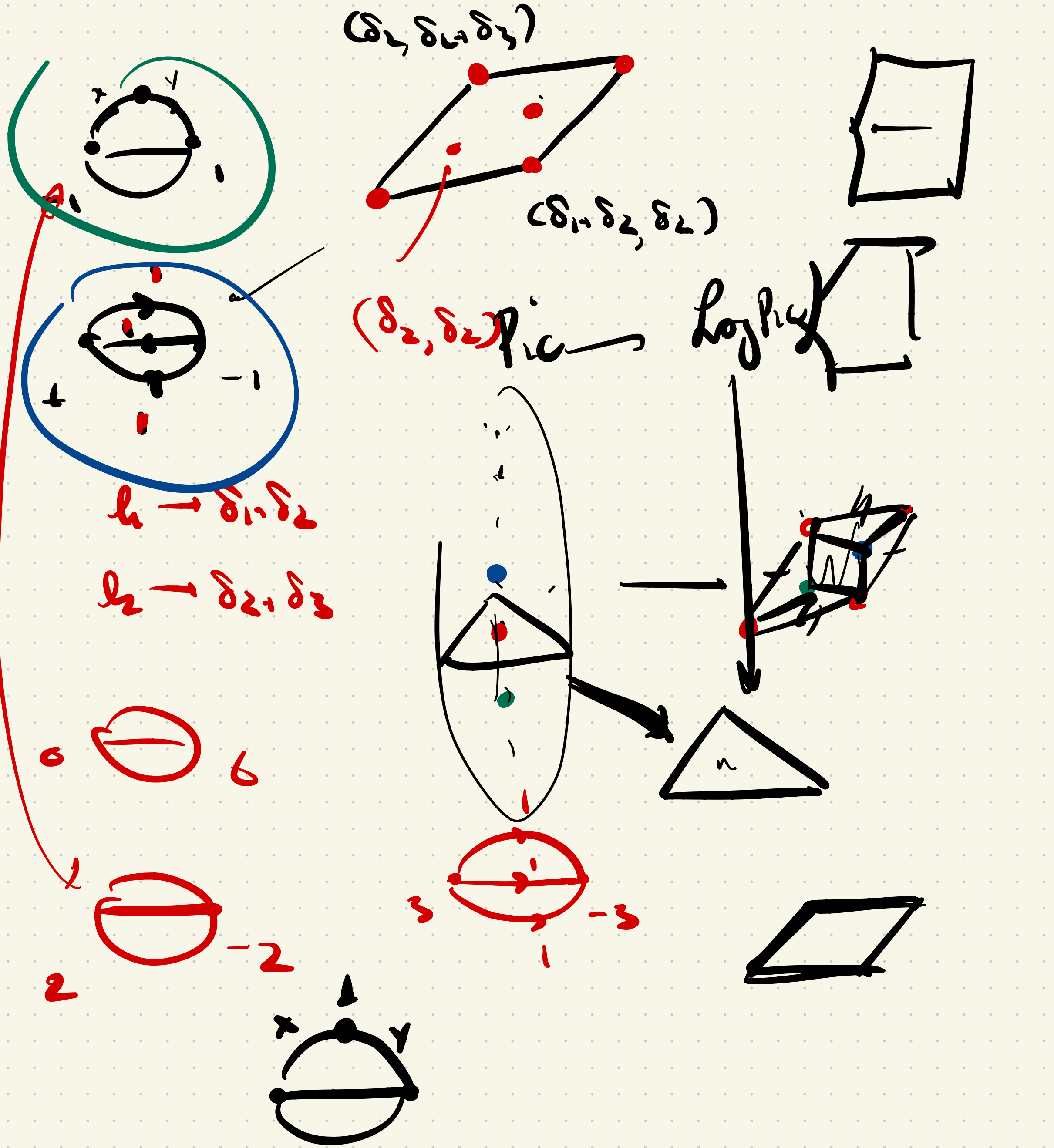
Choose an elt. of J representing.



$$H_1(\Gamma) \longrightarrow \mathbb{R}$$

$$\ell_1 \longrightarrow \delta_2$$

$$\ell_2 \longrightarrow \delta_2$$



(δ2, δ1+δ3)
(δ1+δ2, δ2)
(δ2, δ2)
Pic
LogPic
l1 → δ1+δ2
l2 → δ2+δ3
x
y
-1
0
6
2
-2
3
-3
1

Div $\longrightarrow$ Trolae

Tropical alg. map.