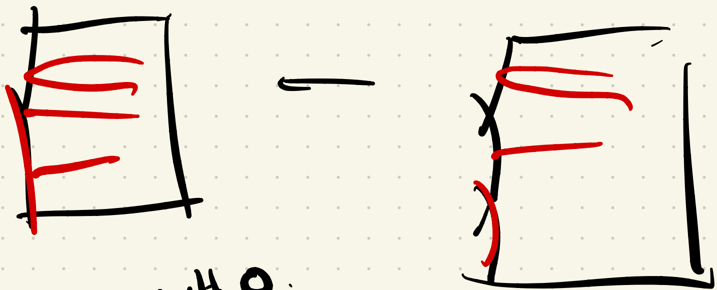
Guided Meditations 3

$X \to S$ log curve, $\Sigma_X \to \Sigma_S$

• $\mathrm{LogPic}(X/S) \text{ " = " } \varinjlim \mathrm{Pic}(Y/T) / \mathrm{PL}(Y)$



mult. 0

• $0 \to \mathrm{Pic}^{[0]}(X/S) \to \mathrm{LogPic}(X/S) \to \mathrm{TroPic}(\Sigma_X/\Sigma_S) \to 0$

$\mathrm{Pic}^0 \to \mathrm{LogPic}(X/S)$

$$
\begin{tikzcd}
BG_m \arrow[r] & BG_m^{\log} \arrow[r] \arrow[d] & BG_m^{\mathrm{trop}} \arrow[d, equal] \\
\mathrm{Div} \arrow[r] & B\mathcal{L} \arrow[r] & BP\mathcal{L}
\end{tikzcd}
$$

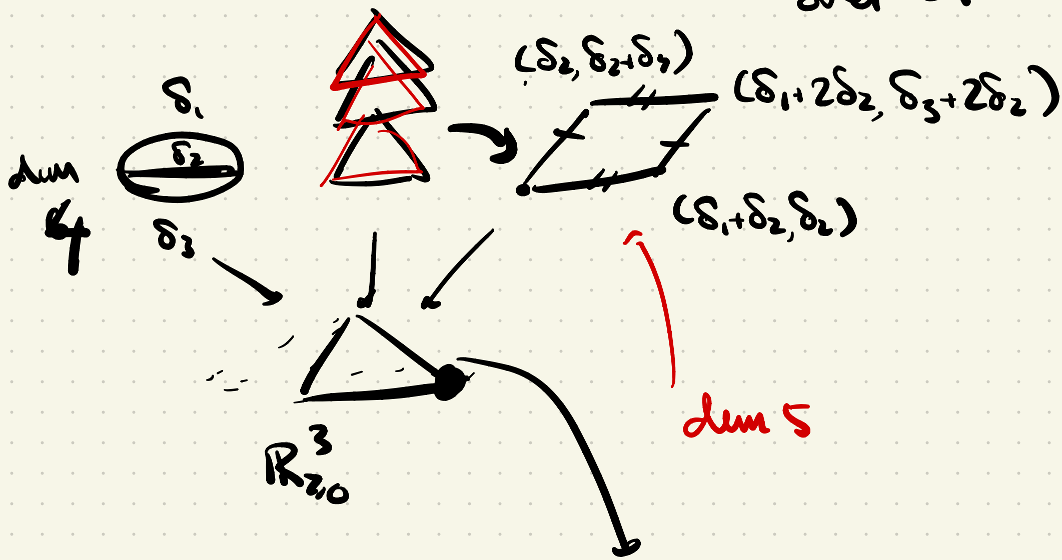
$$
\begin{tikzcd}
P \arrow[r] \arrow[d] & \overline{P} = \{ L \in BG_m, \alpha : L^{\log} \to P \} / \mathrm{iso} \\
\mathrm{trop}(P) = \{ L, \alpha : L^{\log} \xrightarrow{\sim} P : \mathrm{mult}(L) = 0 \} / \mathrm{iso} &
\end{tikzcd}
$$

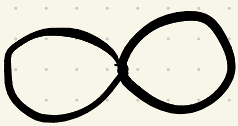
Corollary

$$\mathrm{LogPic} \times_{\mathrm{TroPic}} - \ : \ \mathrm{Cone\ Stacks}/\mathrm{TroPic} \longrightarrow \mathrm{Alg.\ Stacks}/\mathrm{LogPic}$$

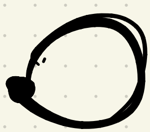
<u>Proof</u> $\Sigma \longleftrightarrow \mathcal{A}$ artin fan

$\mathrm{LogPic} \times_{\mathrm{TroPic}} \mathcal{A}$ is a $\mathrm{Pic}^{[0]}(X/S)$-torsor over $\mathcal{A}$





$\mathrm{Hom}(\mathbb{Z}^2, \mathbb{R})/\mathbb{Z}^2$



$\mathrm{Hom}(\mathbb{Z}, \mathbb{R})/\mathbb{Z}$

$$
\begin{tikzcd}
0 \arrow[r] & M^{gp} \arrow[r] \arrow[d, equal] & \mathcal{L} \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M^{gp} \arrow[r] & P\mathcal{L} \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & 0 \\
& & \mathrm{Div} \arrow[r, equal] \arrow[d] & \mathrm{Div} \arrow[d] & \\
& & 0 & 0 &
\end{tikzcd}
$$

Constant function M^{gp}

$H^1(\Gamma, H) = \mathbb{Z}$

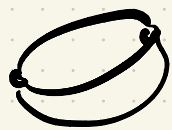
$T \to \langle s(\vec{e}), - \rangle$

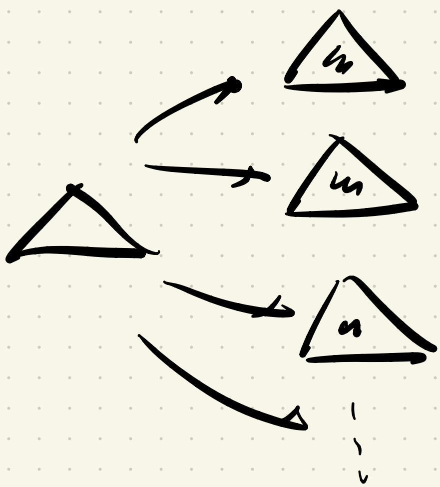
$$J = \{ (s(\vec{e})) \in \mathbb{Z} \mid s(\vec{e}) = -s(\overleftarrow{e}) \}$$

$$\mathrm{Div}^0(\Sigma_X / \Sigma_S) \longrightarrow \Sigma_S$$

$$\mathrm{Div}^0(\Gamma / \mathbb{R}_{\geq 0}^{\vee}) \longrightarrow \vee$$

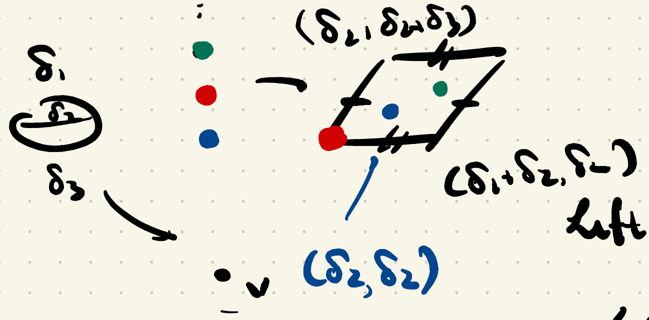
$$J = \mathbb{Z}^{E(\vec{\Gamma})}$$





$\times \mathbb{Z}$

non-separated complex



$x \rightsquigarrow \Sigma_x$

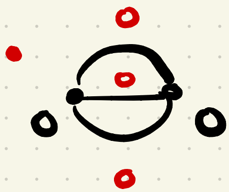
lift D to J

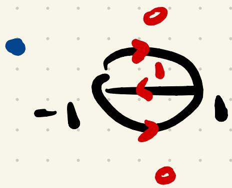
$\langle s(\vec{e}), - \rangle : H_1(\Gamma) \to N^{gp}$

Class of ϕ mod $H_1(\Gamma)$

is independent of

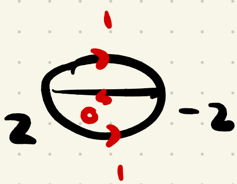
lift. $g(D) \in \dfrac{\mathrm{Hom}(H_1(\Gamma), N^{gp})^+}{H_1(\Gamma)}$





$\gamma_1 \longrightarrow \delta_2$

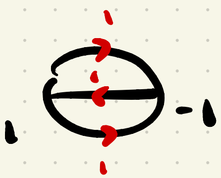
$\gamma_2 \longrightarrow \delta_2$



$\gamma_1 \longrightarrow \delta_1$

$\gamma_2 \longrightarrow \delta_3$

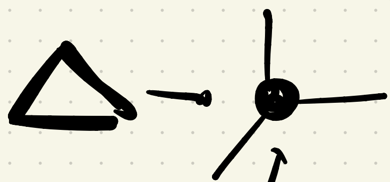
$\delta_1 = \delta_2 = \delta_3 \rightsquigarrow (\delta_2, \delta_2)$



$\gamma_1 \longrightarrow \delta_1 + \delta_2$

$\gamma_2 \longrightarrow \delta_2 + \delta_3$

Get a finite # of images at each v.



$\mathbb{A}^3$

$\mathrm{Hom}(H_1(\Gamma), M^{gp}) / H_1(\Gamma)$

Thm: $\mathrm{Pic}^0(X/S) \xrightarrow{\sim} \mathrm{LogPic} \times_{\mathrm{TroPic}} \mathrm{Div}(\Sigma_X/\Sigma_S)$

Proof $(P, D, u : \mathrm{trop}(P) \xrightarrow{\sim} \mathcal{O}(D))$

π_1 $\quad \mathbb{G}_m$

π_0 $\quad (P, D) \quad \mathrm{trop}(P) \simeq \mathcal{O}(D)$

$\in B\mathcal{L} \longrightarrow BP\mathcal{L}$

$\overline{P} \in B\mathbb{G}_m^{trop} = BP\mathcal{L}$ is trivial.

$\overline{P} = \{L, \alpha : L^{\log} \to P\}/\sim$ has a section,

$P = L^{\log}$

$\mathfrak{g}(\mathrm{mult}(L)) \simeq \mathfrak{g}(D)$; $D - \mathrm{mult}(L) = \mathrm{div} f$

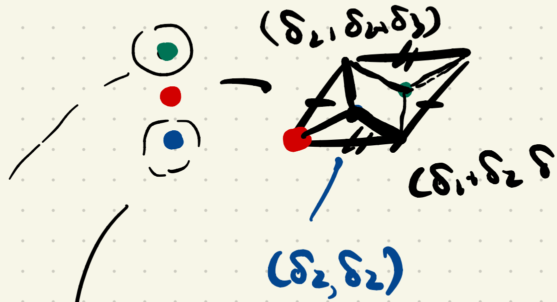
$$L(-f) \longrightarrow (L(-f)^{\log} = P, D)$$

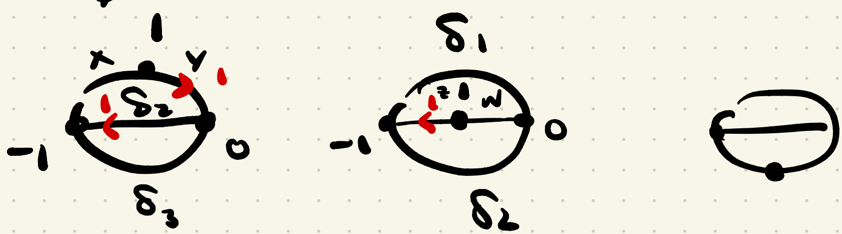
$$L, L' \rightsquigarrow (L^{\log}, \mathrm{mult}(L)) = (L'^{\log}, \mathrm{mult}(L'))$$

$$L' = L(-f), \ \mathrm{mult}(L') = \mathrm{mult}(L) \cancel{- \mathrm{div} f} \ 0$$

$\Rightarrow f$ is linear, so $L' = L$ //

Fact: $\mathrm{Pic}^\circ(X/S)$ not separated or universally closed





$\gamma_1 \to y+\delta_2$ $\quad \gamma_1 \to z$

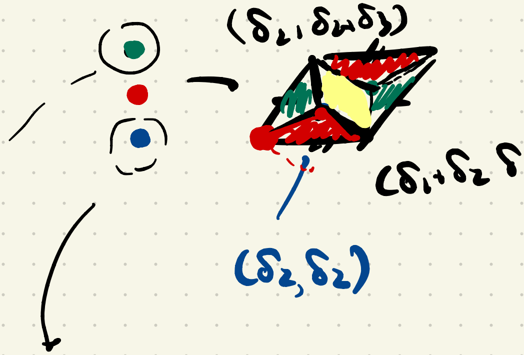
$\gamma_2 \to \delta_2$ $\quad \gamma_2 \to z$

$\gamma_1 \to y+\delta_2$

$\gamma_2 \to \delta_2+\delta_3$

$2z+w+\delta_1$

$2z+w+\delta_2$

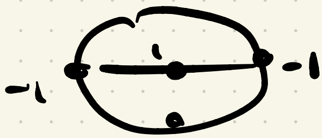


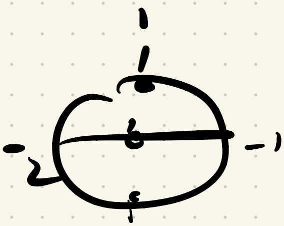
$(y+z, z)$

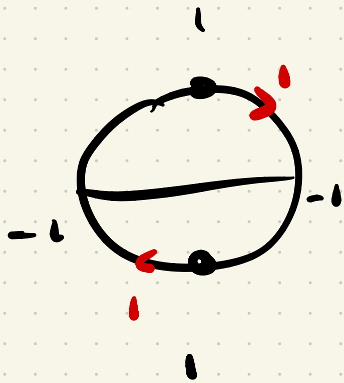
$0 \le y \le \delta_1$

$0 \le z \le \delta_2$

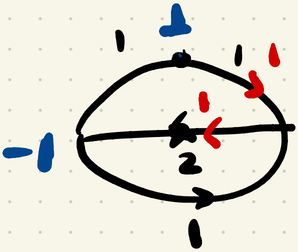
δ



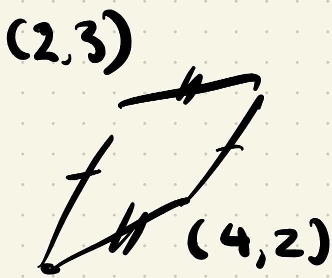




<u>Ex.</u>



$(3, 2)$



$$s_1+1+s_1 = 2s_2 = s_3$$

Observations

Could get a cover of TroPic

by using 1 subdivision of each edge

at a time

$\rightsquigarrow$ Quasi-stable model

$\cap$

Semistable

$\rightsquigarrow$ Only needed weight 1 on new components

$\rightsquigarrow$ "admissible" divisor

Tropical Moduli space Γ/σ

$$Q(\tau \to \sigma) = \left\{ \begin{array}{c} \Gamma' \\ \downarrow \leftarrow \text{q. stable} \\ \Gamma \times_\sigma \tau \end{array} \right\}$$

$$\mathcal{J}_\Gamma(\tau \to \sigma) = \left\{ \begin{array}{c} \Gamma' \\ \downarrow \\ \Gamma \times_\sigma \tau \end{array}, \ D \text{ admissible divisor} \right\}$$

Q representable by

$$K \subset \sigma \times (\mathbb{R}^2_{\geq 0})^{E(G)} / \mathbb{Z}_2$$

$$= \{(v, l_1, l_2) \mid l_1 + l_2 = \delta_v(e)\} / \mathbb{Z}_2$$

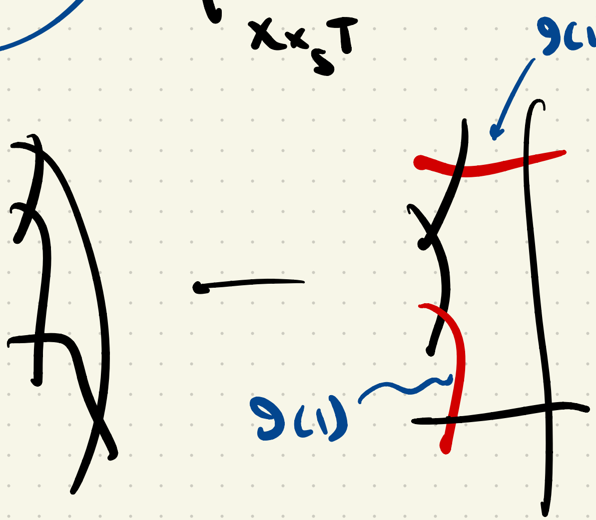
$$J_\Gamma \subset \mathrm{Div}^\circ(K)$$

Def. Let $X \to S$ be a log curve.

$$J_{X/S} = J_{\Sigma_X/\Sigma_S} \times_{\mathrm{TroPic}} \mathrm{LogPic}$$

generalized Jacobian

$$J_{X/S}(T) = \left\{ \begin{array}{c} X' \\ \downarrow \\ X \times_S T \end{array}, \ \ell \text{ admissible} \right\}$$



$\mathfrak{J}_{X/S}$ is universally closed

not separated

Observe

$\mathcal{L}(\Gamma) = M^{gp}$

$D \quad D' = D + \operatorname{div} f$

if $S \subset \Gamma$ is a minimal subgraph of Γ for f,

$$D(S) = \sum_{v \in S} D(v), \quad D'(S) = D(S) + \operatorname{div} f(S)$$

$$D' - D \geq E(S, S^c)$$

# of edges out of S

Conclusion

$\Sigma_X \to \Sigma_S$

<u>Def</u>. A stability condition on Γ

if a function $\phi : V(\Gamma) \longrightarrow \mathbb{R}$

($\phi(\Gamma) = d$ fixed degree)

<u>Def</u>. ϕ s.s. Divisor is a divisor D s.t

$\forall\, S \subset \Gamma$,

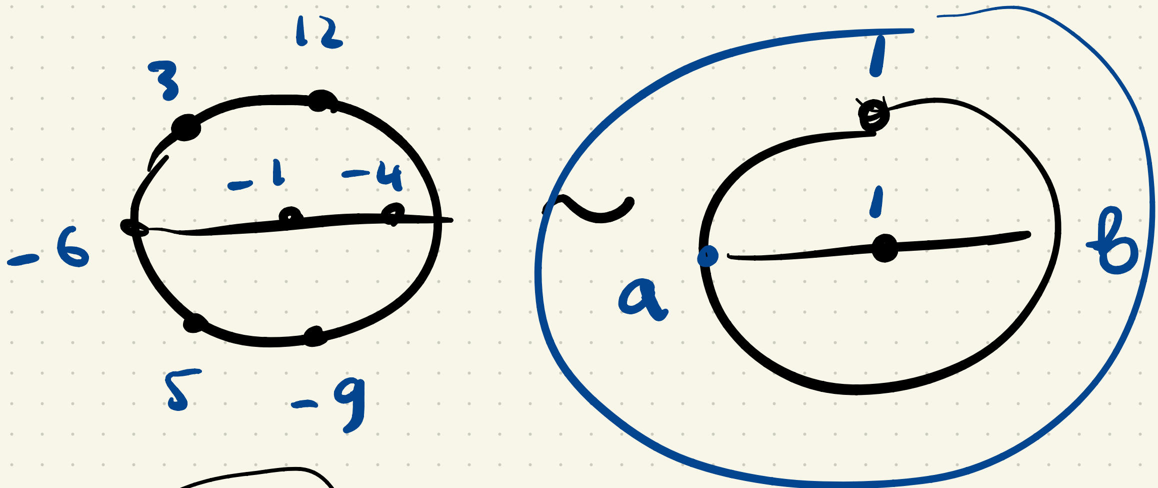
$$\phi(S) - \frac{E(S, S^c)}{2} \leq D(S) \leq \phi(S) + \frac{E(S, S^c)}{2}$$

stable means $<$ $<$

<u>Def</u>. ϕ is generic $\Leftrightarrow$ semistable $=$ stable

Thm (Cap, Abreu-Pacini)

Given D on $\Gamma'' \xrightarrow{\text{subdivision}} \Gamma$ of $\Gamma/\mathbb{R}_{\geq 0}$, and generic ϕ, $\exists!$ D' on a q-stable $\Gamma' \to \Gamma$ s.t. $D' \sim D$.



$J_{\Sigma_X/\Sigma_S}(\phi) \subset \{\phi\text{-ss. admissible divisors}\}$

$J_{\Sigma_X/\Sigma_S}(\phi) \longrightarrow \mathrm{TroJac}$

subdivision

$$\boxed{\mathrm{Pic}(\phi) = J_{\Sigma_X/\Sigma_S} \times_{\mathrm{TroPic}} \mathrm{LogPic}} \subset J_{X/S}$$

$$\downarrow \text{ subdivision}$$

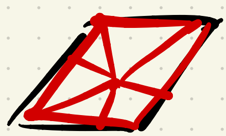
LogPic

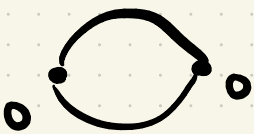
<u>**Proper**</u>

"compactified Jacobian"

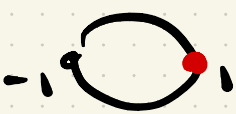
Any $\Delta \longrightarrow \mathrm{TroJac}$ pull-back

$$\mathrm{Pic}(\Delta) = \Delta \times_{\mathrm{TroJac}} \mathrm{LogJac}$$



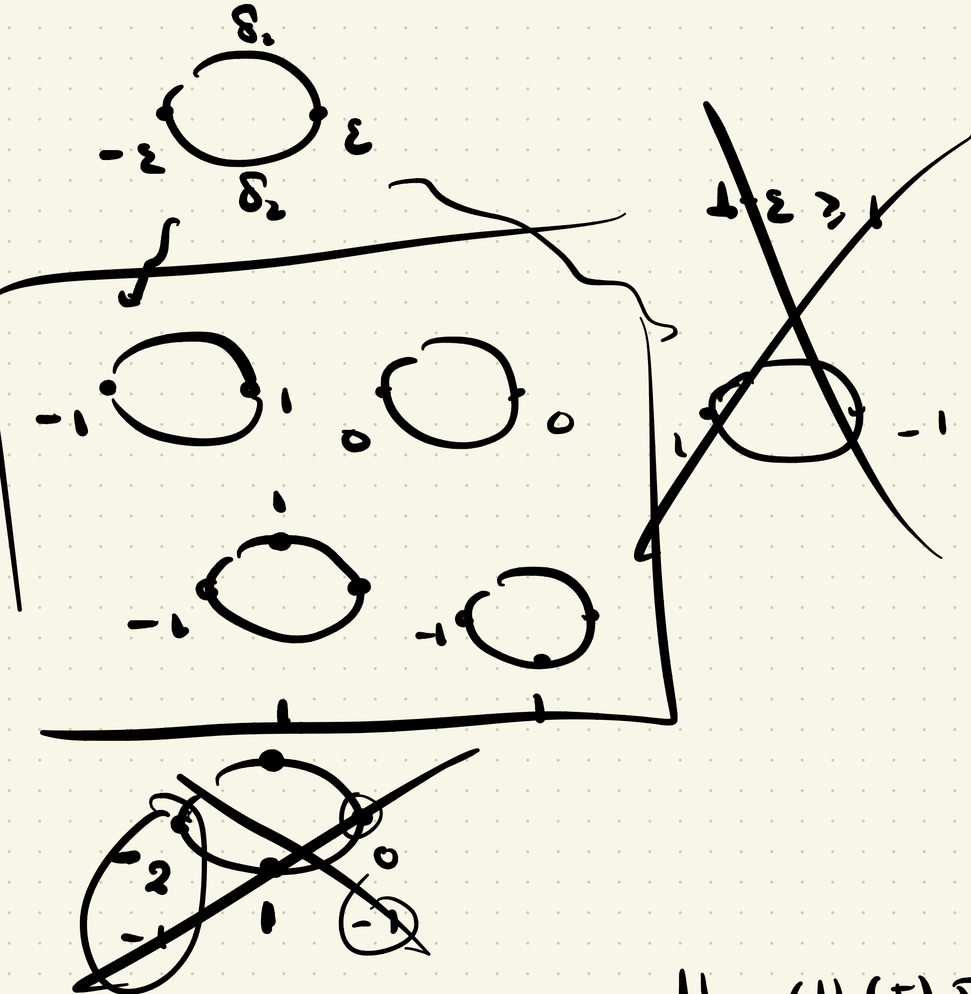


$\rightsquigarrow$ not generic

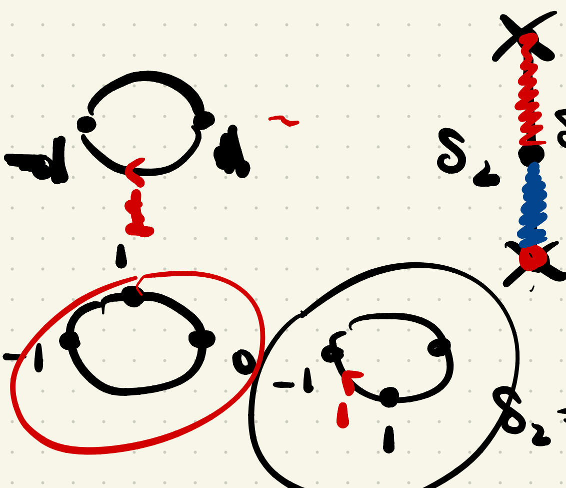


$\rightsquigarrow$ semistable but not stable

$$0 - \frac{2}{2} \le 1 \le 0 + \frac{2}{2}$$

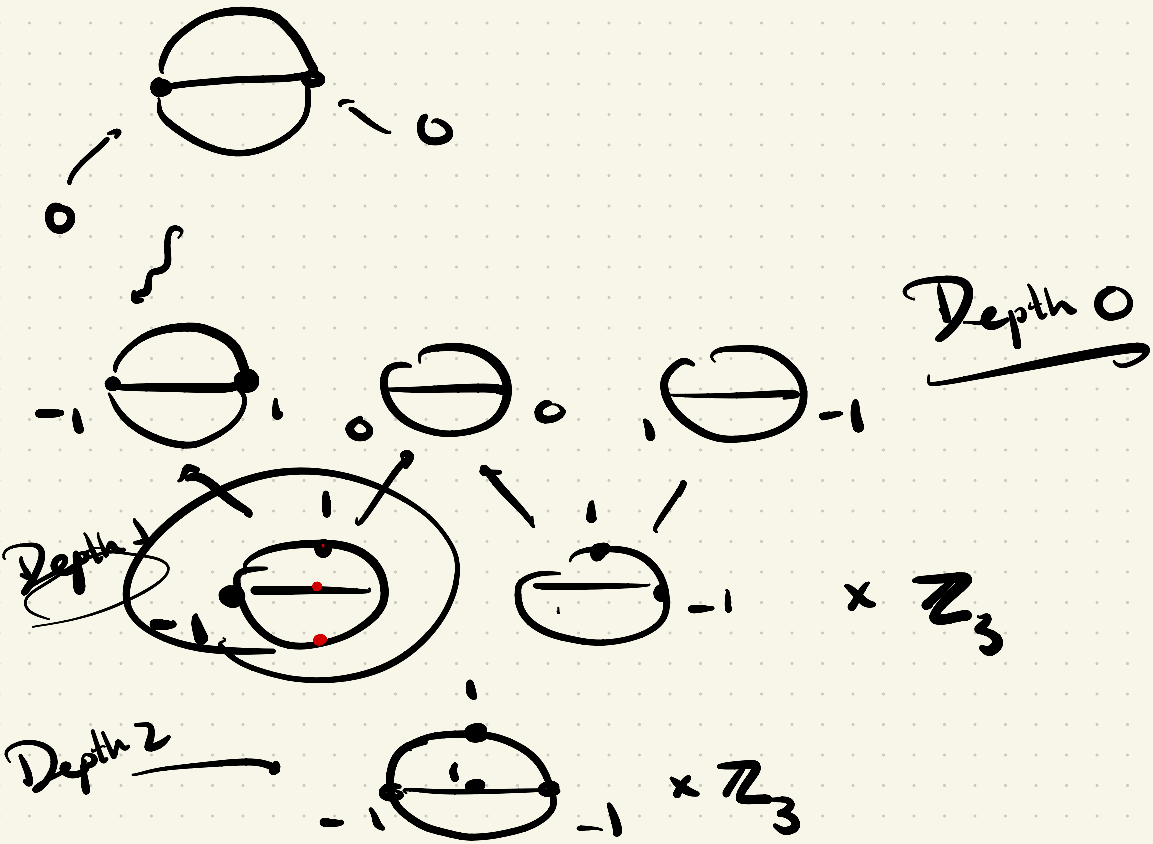


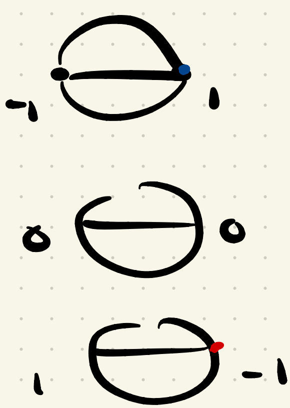
$$\mathrm{Hom}(H_1(\Gamma),\mathbb{R}) \big/ H_1(\Gamma)$$

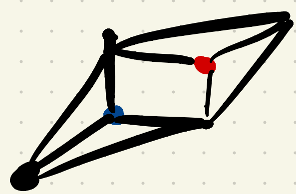


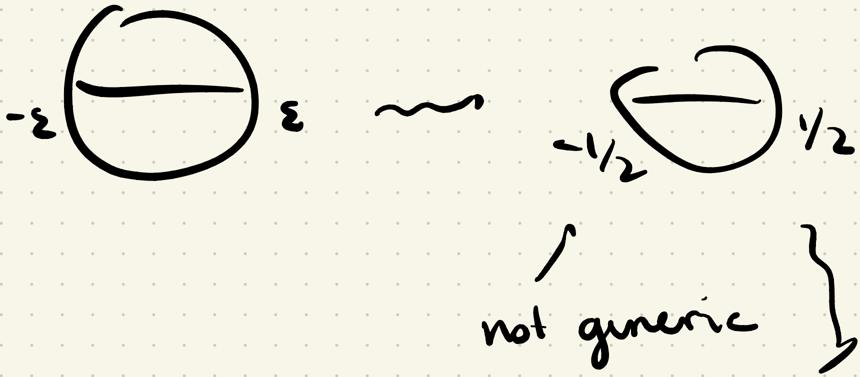
$$\delta_1+\delta_2 \simeq \mathbb{R}\big/(\delta_1+\delta_2)\mathbb{Z}$$

$\delta_2 = x+y \quad 0 \leq x \leq \delta_2$

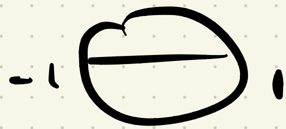

0
0
Depth 0
-1
1
0
0
1
-1
Depth 1
1
-1
1
-1
× $\mathbb{Z}_3$
Depth 2
1
1
-1
-1
× $\mathbb{Z}_3$

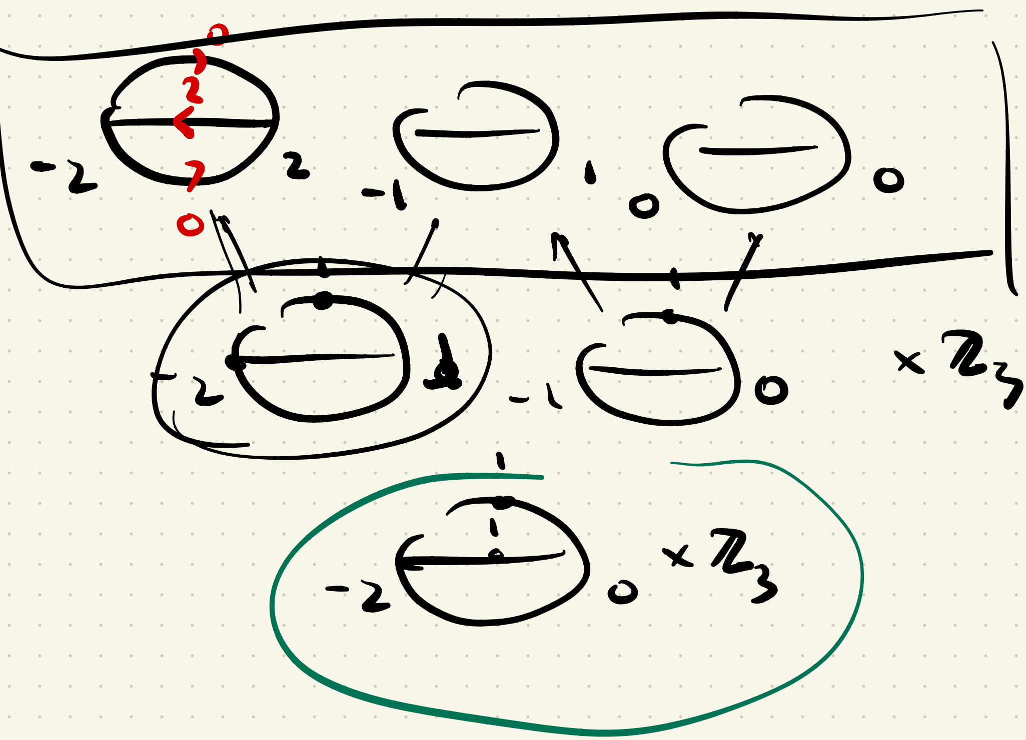

-1
1
0
0
1
-1

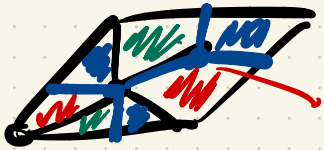




$$\phi(S) \pm \frac{e(S, S^c)}{2}$$





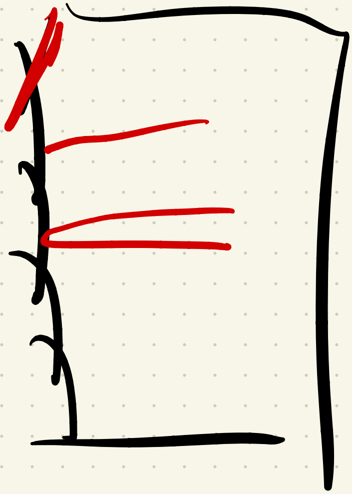


Francesca: Let $X \to S$ a log curve over DVR, $\mathcal{L}_\eta$ is a line bundle on X_η. When is there a limit on X?

Algorithm

- Write $\mathcal{L}_\eta = \mathcal{O}(D_\eta)$
- $D = \overline{D_\eta}$

- Blowup X to make D^{strict} Cartier divisor.



$\mathcal{L} = \mathcal{O}(D^{\text{struct}})$

Compute its image in LogPic TroPic

$\overset{\circ}{H^1}(X', \overline{M}_X'^{gp})$

$H^1(X', M_X'^{gp})$

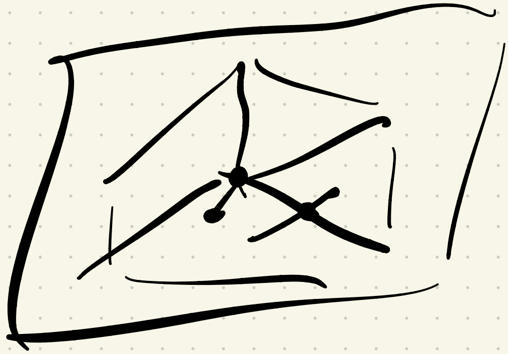
$H^1(X, M_X^{gp})$

• Push to $H^1(X, \overline{M}_X^{gp})$

$H^1(\Gamma, P\mathcal{L})$

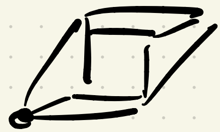
Compute ϕ-stable rep for any stab. condition (generic)

$$\phi : \overset{d}{V(\Gamma)} \longrightarrow \mathbb{R}$$

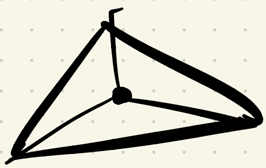


0

$$\overline{\mathcal{M}}_{g,n} \dashrightarrow \mathrm{Pic}^{[0]}$$



$2g_v - 2 + n$



$$\mathrm{tDR} = \begin{tikzcd} \mathcal{M}_{g,n}^{\Diamond} \arrow[r] \arrow[d] & \mathrm{Pic}^{[0]} \arrow[r] \arrow[d] & \bullet \arrow[d] & \phi \\ \mathcal{M}_{g,n}(\Delta) \arrow[r] \arrow[d, "aj_\Delta"] & \mathrm{Pic}(\Delta) \arrow[r] \arrow[d] & \Delta \arrow[d] & \\ \overline{\mathcal{M}}_{g,n} \arrow[r] & \mathrm{LogPic} \arrow[r] & \mathrm{TroPic} & \end{tikzcd}$$

$\overline{M}_{g,n}(\phi)$