Guided Meditations 4

$C \to S$ log curve

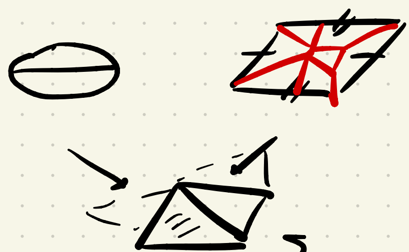
$$\left\{ \begin{array}{c} \text{Subdivisions of} \\ \mathrm{TroJac}(C/S) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Toroidal} \\ \text{Comps of} \\ \mathrm{Pic}^{[0]}(C/S) \end{array} \right\}$$

$$\Delta \longrightarrow \mathrm{LogPic}(C/S) \times_{\mathrm{TroPic}} A_\Delta := \mathrm{Pic}(\Delta)$$

$$P/\mathrm{Pic}^{[0]} \longleftrightarrow \Sigma_P \longleftarrow P$$

$$\begin{array}{c} C \\ \downarrow \\ S \end{array} \rightsquigarrow$$

Σ_C



TroJac

Σ_S

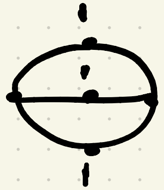
<u>Hard</u>:

- Dimension is high
- Automorphisms
- Compatible w specializations

Stability conditions solved this problem

Recall relevant ingredients

- $\Gamma' \longrightarrow \Gamma$ quasi-stable models



- Admissible divisors D

  $\rightsquigarrow$ degree $= 1$ on exceptional vertices

- Rule $\phi : V(\Gamma) \longrightarrow \mathbb{R}$

  compatible w.r.t all aut/ contractions

- ϕ-Semistable divisors

  $\forall\, S \subset \Gamma$,

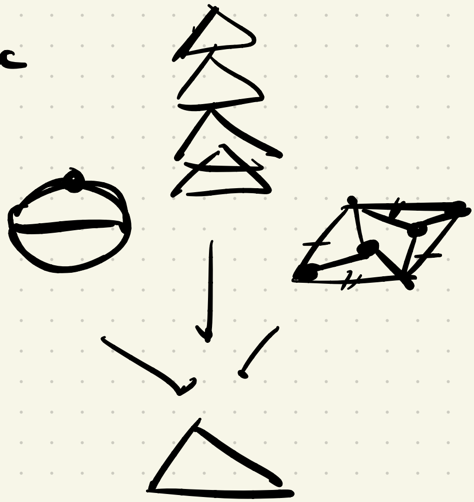
$$\phi(S) - \frac{val(S)}{2} \leq D(S) \leq \phi(S) + \frac{val(S)}{2}$$

- If ϕ is "generic" $\iff$ D is stable $\iff$ semi stable

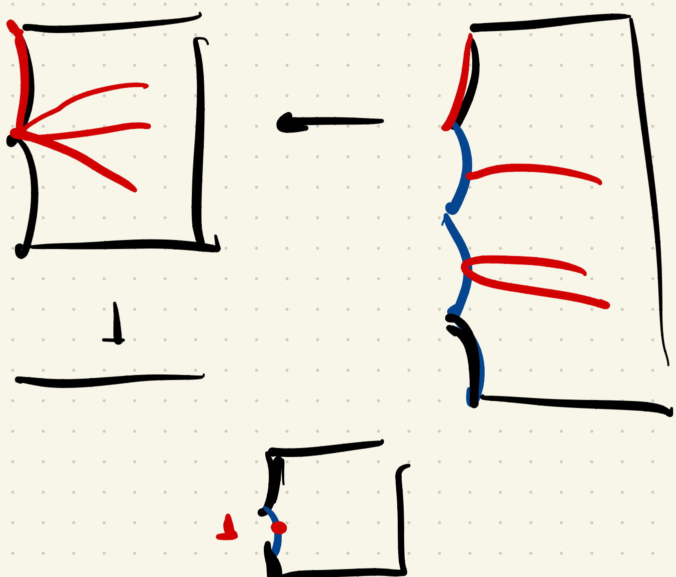
$$\mathcal{J}^{trop}(\phi) = \{\phi\text{-semistable admissible divisors on quasi-st. models of } \Sigma_C \to \Sigma_S\}$$

sub. $\downarrow$

TroJac



Tropical Reflection of "Meditation 1"



$$\varinjlim \mathrm{Pic}(C'/S')/\mathrm{PL}(C') \longrightarrow \mathrm{LogPic}(C/S)$$

$\Gamma/\mathbb{R}_{\geq 0}$

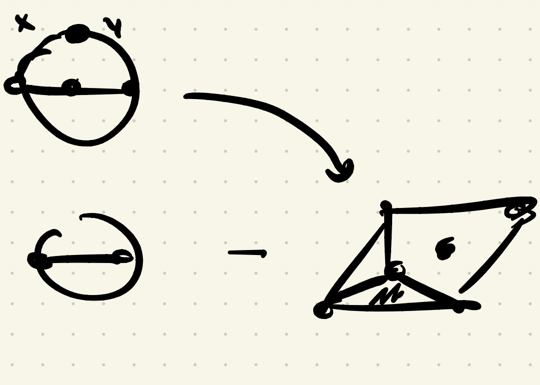
("Mod $\mathbb{Z}$")

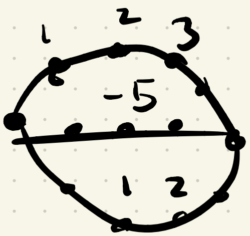
$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathbb{R} \arrow[r] \arrow[d] & \mathcal{L} \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{R} \arrow[r] & \mathrm{PL} \arrow[r] \arrow[d] & J \simeq \mathbb{Z}^{e(\Gamma)} \arrow[r] \arrow[d] & 0 \\
 & & \mathrm{Div} \arrow[r] \arrow[d] & \mathrm{Pic} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

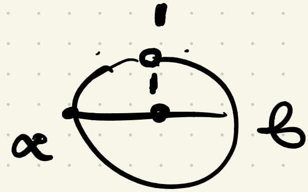
$$\mathrm{TroJac} = \mathrm{Hom}(H_1(\Gamma), \mathbb{R})/H_1(\Gamma)$$

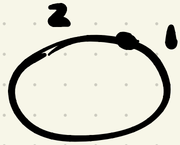
$$\Gamma' \longrightarrow \Gamma$$

$$\mathrm{Div}(\Gamma') \longrightarrow \mathrm{TroJac}(\Gamma') = \mathrm{TroJac}(\Gamma)$$







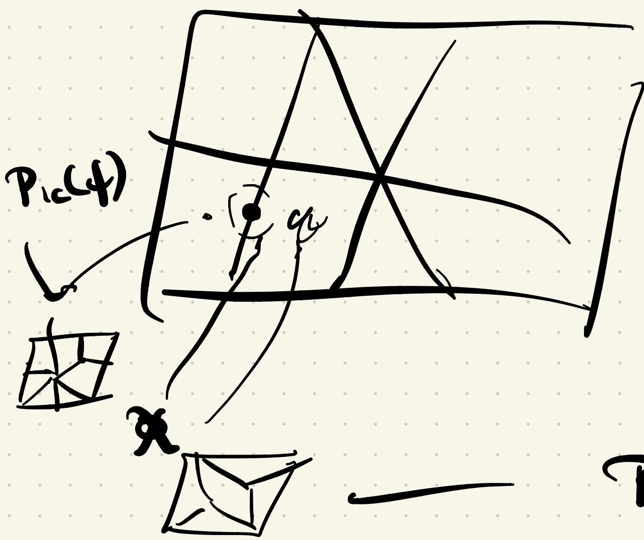


$\mathrm{Pic}(\phi) \not\to \mathrm{Pic}(\phi)$

stack

$\mathrm{Pic}(X)$

$\mathrm{Pic}(\phi) \quad \mathrm{Pic}(\phi)$



— $\mathrm{Pic}(\phi)$

non-rep. over $\overline{\mathcal{M}}_{g,n}$

S

$$\mathrm{Pic}(\phi) \subset \mathrm{Pic}(X) \supset \mathrm{Pic}(\phi)$$

$$\mathrm{Pic}^{\check{u}m}(x)$$

2

g-dim. complex

$J^{trop}(\phi)$

$J^{trop}(x)^{g+1}$

For generic ϕ, $\mathrm{Pic}(\phi)$ is special $T \to S$

$$\mathrm{Pic}(\phi)(T) = \left\{ \begin{array}{ll} C' \to C \times_S T, & \mathcal{L} \text{ on } C' \\ \text{quasi-stable}, & \phi\text{-s.stable} \end{array} \right\}$$

$\mathrm{Pic}(\Sigma)$

"fine comp. Jacobian"

Abel-Jacobi theory

$$\begin{array}{c} C \\ \downarrow \\ S \end{array}, \quad \mathcal{L} \in \mathrm{Pic}^0(C/S)$$

$$\begin{array}{c} \overline{C}_{g,n} \\ \downarrow \\ \overline{M}_{g,n} \end{array}, \quad A = (a_1, \ldots, a_n)$$

$$\omega^{\otimes k}(-\Sigma a_i x_i) = \mathcal{L}$$

$$\begin{array}{ccc} S(\Delta) & \longrightarrow & \mathrm{Pic}(\Delta) \\ \downarrow & & \downarrow \leftarrow \text{subs} \\ S & \xrightarrow[aj]{} & \mathrm{LogJac}(C/S) \end{array}$$

$$\begin{tikzcd} \Sigma_S(\Delta) \arrow[r] \arrow[d] & \Delta \arrow[d] \\ \Sigma_S \arrow[r] & \mathrm{TropJac} \end{tikzcd}$$

most special $\Delta = J^{trop}(\phi)$

$$\Sigma_S(\phi) \quad , \qquad \begin{tikzcd} S(\phi) \arrow[r] \arrow[d] & \mathrm{Pic}(\phi) \arrow[d] \\ S \arrow[r] & \mathrm{logJac} \end{tikzcd}$$

Functor of points

$(T \to S)$ $\qquad T \to S(\phi)$

$= \{$ (i) A quasi-stable model

$C' \to C \times_S T$

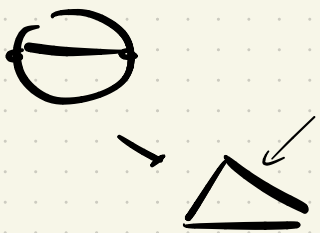
(ii) A line bundle $\mathcal{L}'$ that's ϕ-stable

(iii) $\mathcal{L}'^{\log} \simeq \mathcal{L}^{\log}$ $\}$

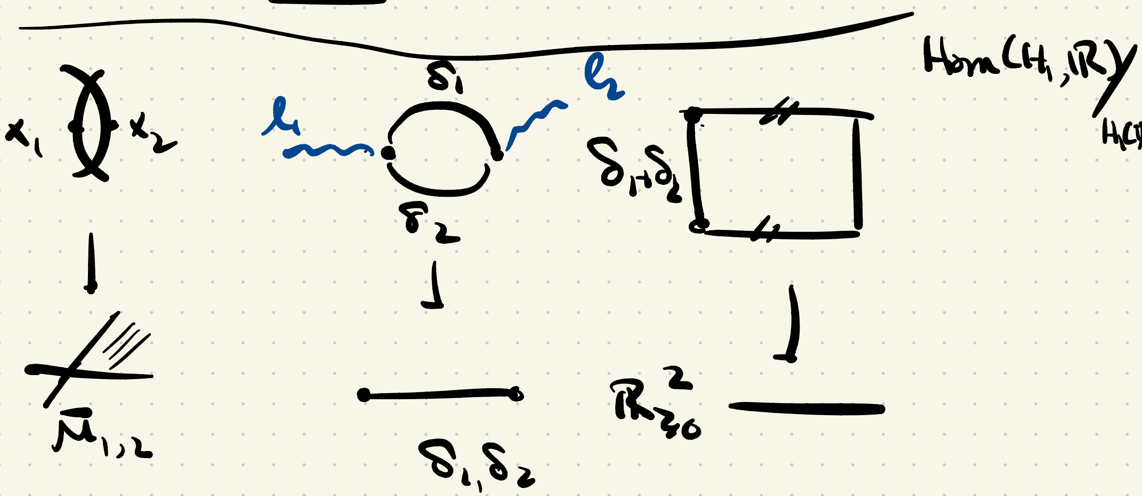
$$\iff$$

$$\{ \text{(i)}, \text{(ii)}' \text{ a PL function } \alpha \text{ on } C', \text{(iii)}' \text{ s.t. } \mathcal{L}(-\alpha) \text{ is } \phi\text{-stable} \}$$

Ex



$C \to S$



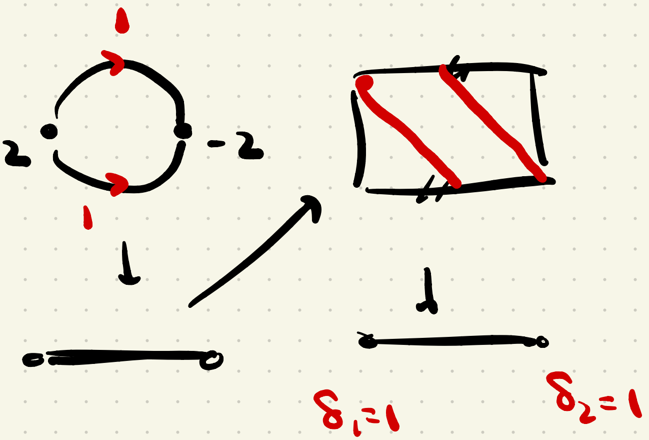
$\mathcal{L} \simeq \mathcal{O}(2x_1 - 2x_2)$ $\quad$ $A = (2,-2) = \mathrm{mult}(\mathcal{L})$

$$S \longrightarrow \mathrm{LogJac}$$

$$\Sigma_S \longrightarrow \mathrm{TroJac}$$

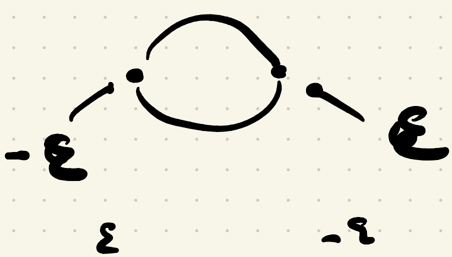
$\mathrm{Hom}(H_1(\Gamma), \mathbb{R}) / H_1$

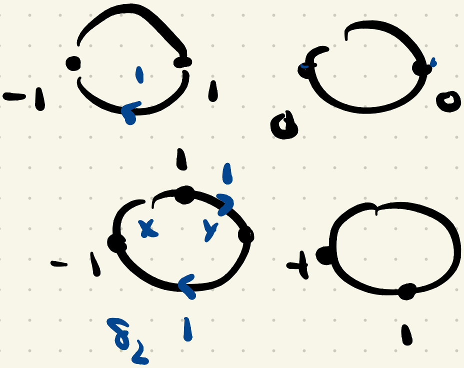
$\delta_1 - \delta_2 \pmod{\delta_1 + \delta_2}$

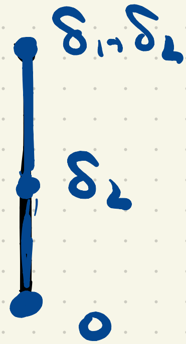


Choose a stab. condition

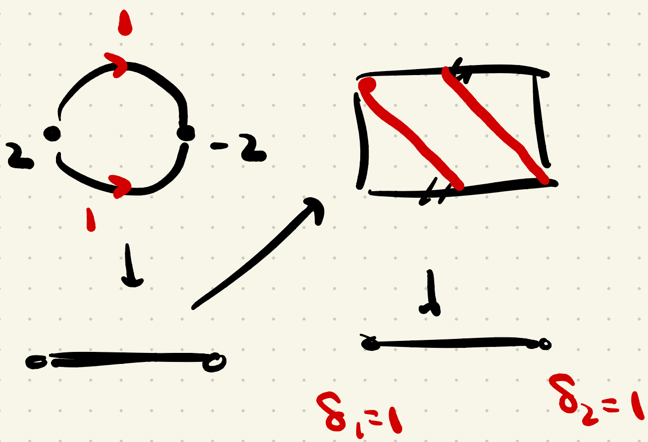
$\phi = 0$







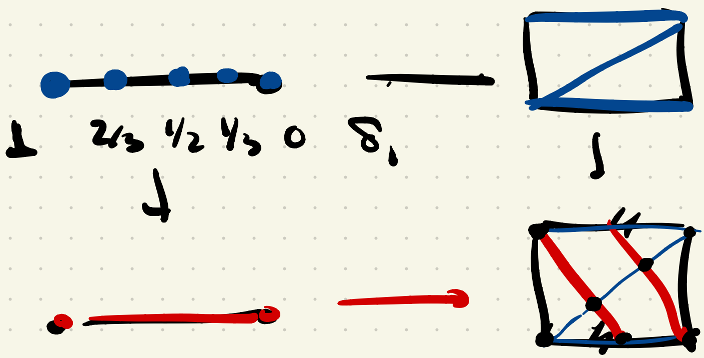
$\delta_2 + y \,,\ 0 \le y \le \delta_1$



$\delta_1+\delta_2=1$

$(2/3, 1/3)$

$(2,1)$



$\delta_2 = \delta_1 - \delta_2$

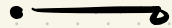
$(\mathrm{mod}\ \delta_1, \delta_2)$

$0 = \delta_1 - \delta_2$

(mod)

How do I compute $S(\phi)$, given ϕ: ex. $(-2,2)$

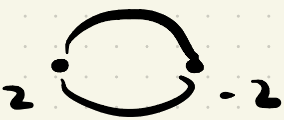
- Cone by cone



- List ϕ-stable divisors on C

<u>Solve</u> tropical alg. problem

$$A - D = \operatorname{div} f$$



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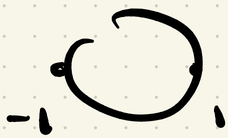
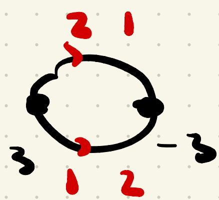
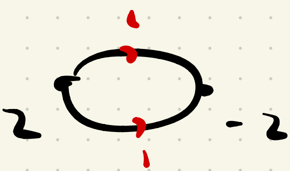


$=$



$= \operatorname{div} f$

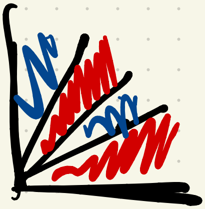


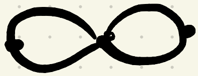
$= \operatorname{div} f$

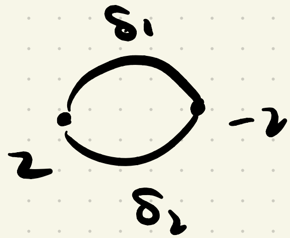
$\delta_1 = \delta_2$

$2\delta_2 = \delta_1$

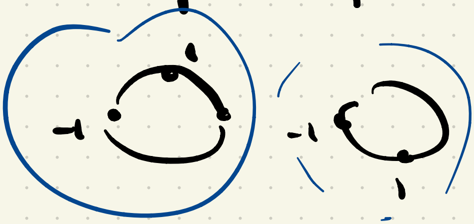
$\delta_1 = 2\delta_2$







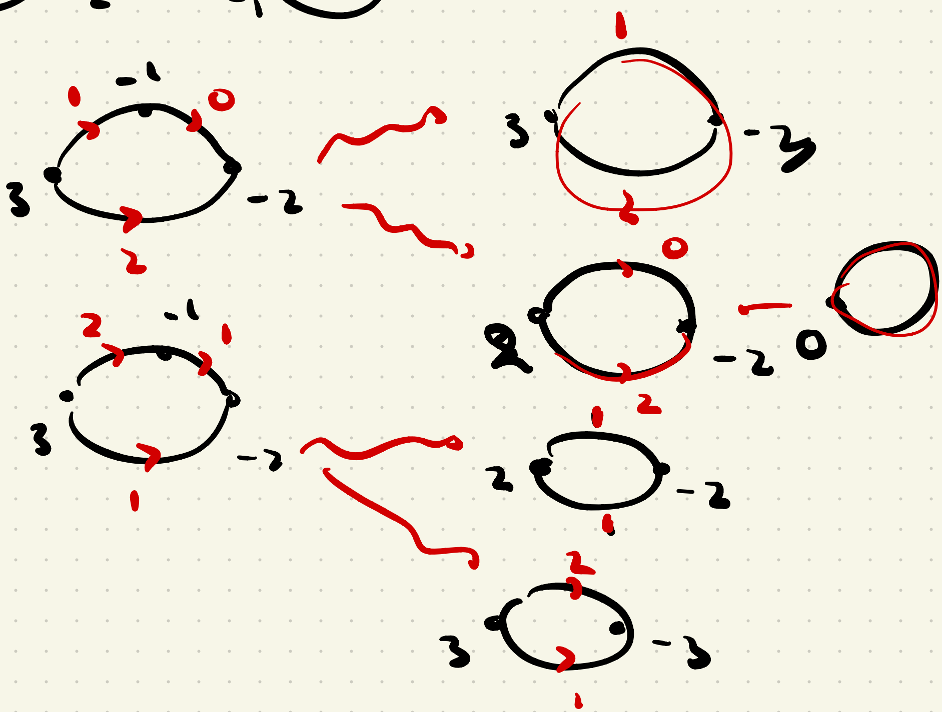
- Higher depth

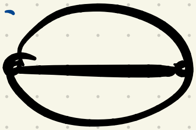


acyclic twist

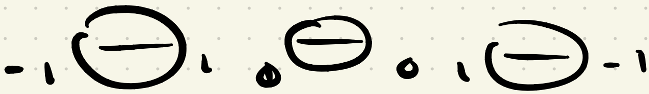
$= d_w(t)$

=





$A = (3,-3)$, $\phi = 0$

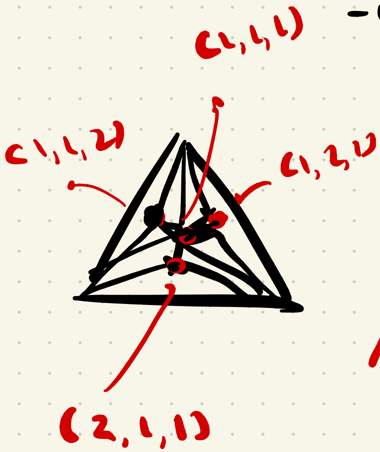


6 

× Symetries 

× Sym

× Sym. 3



(surprisingly easy)

$A = 3$

log DR cycle

log smooth

$\mathcal{U}$ open dense

Holmes $\quad C \to S, \mathcal{L}$

$$\mathcal{U} \longrightarrow \mathrm{Pic}^{[0]}(C/S)$$
$$\cup$$
$$S \dashrightarrow$$

not proper

$$\begin{tikzcd}
\mathrm{DRR} \arrow[r] \arrow[d, "j"'] \arrow[ddr, "p"] & S \arrow[d, "o"] \\
\mathcal{U} \subset S^{\diamond} \arrow[r, "aj^{\diamond}"] & \mathrm{Pic}^{[0]}(C/S) \\
\subset S^{\bullet} &
\end{tikzcd}$$

$$P_*\left(aj^{\diamond *}([0])\right) = CH^{g}(S^{\bullet})$$

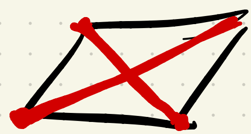
$$(\log CH(S))$$

$[0] \in \mathrm{Log}CH(\mathrm{LogPic})$ ← not alg.

Practically $\forall \Delta \to$ TropJac

$[0] \in CH(\mathrm{Pic}(\Delta))$ ← fund. object

$$\begin{tikzcd}
\mathrm{DRL} \arrow[r] \arrow[d] & S \arrow[d, "0"] \\
S^0 \arrow[r] \arrow[d] & \mathrm{Pic}^{[0]}(C/S) \arrow[d, phantom, "\cap"] \\
S(\Delta) \arrow[r, "aj_\Delta"'] \arrow[d] & \mathrm{Pic}(\Delta) \arrow[d] \\
S \arrow[r, "aj"'] & \mathrm{LogPic}
\end{tikzcd}$$



if I choose Δ to contain 0 as a vertex

$aj^*([0])$

$$\{aj_\Delta^*([0])\}$$

$$\begin{tikzcd}
\Sigma_S^{\diamond} \arrow[r] \arrow[d] & \Sigma_S \arrow[d, "\circ"] \\
\Sigma_S(\Delta) \arrow[r] \arrow[d] & \Delta \arrow[d] \\
\Sigma_S \arrow[r] & \mathrm{Trodae}
\end{tikzcd}$$

<u>Thm</u> $p_* aj^{\diamond *}([0]) = aj_\Delta^*([0])$.

When Δ comes from ϕ generic, close to 0

$$S(\phi) \rightsquigarrow \cdot\, C(\phi) \to C \times_S S(\phi)$$

q-stable

$$\cdot\ \mathcal{L}(\phi) = \mathcal{L}(-\alpha)$$

ϕ-st. bundle

$$\mathrm{uniDR}(C(\phi)/S(\phi), \mathcal{L}(\phi)) \in CH^g_{op}(S(\phi))$$

Bae - Holmes - Pandharipande - Schmitt - Swartz?

BHPSS ← 130

Thm $aj_\phi^*([0]) = \mathrm{uniDR}(C(\phi)/S(\phi), \mathcal{L}(\phi))$

Pixton's formula

$\bar{C}_{g,n}/\bar{\mathcal{M}}_{g,n}$ $\mathcal{O}(\sum a_i x_i)$

$\rightsquigarrow$

DR

$\lim_{k\to} P^k$

r

$S(\phi)$

$\downarrow$

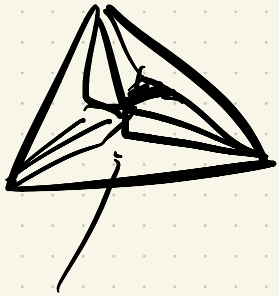
S

rep. of logDR on $S(\phi)$

Piecewise polynomial on $\Sigma(\phi)$

$$P_{g,A}(\phi) = \left(\prod a_i^2 \psi_i\right)\left(\sum_{(\Gamma', \text{twist})} f(\Gamma', \text{twist})\right)$$

q-stable twist



r-dependence

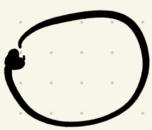
$$p_*\left(DR_{g,A}(\phi)\cdot DR_{g,B}(\phi)\right)$$

$$= \dots$$

$$\prod a_i^2 \psi_i \left(\sum_{\Gamma} \underline{\quad\quad} \right)$$

$$e \subset \left(\mathbb{R}_{\geq 0} \times \mathbb{R}^2_{\geq 0}\right)^{E(\Gamma)} \Big/ \mathbb{Z}_2$$

$$\left\{ (x, (\ell(e), x_1(e), x_2(e)) \mid x_1(e)+x_2(e)=\ell(e) \right\}$$

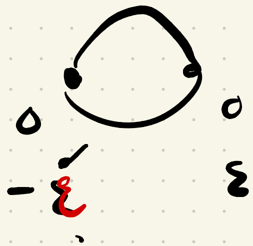


$\circlearrowleft \mathbb{Z}/2$

$\circlearrowleft \mathbb{Z}_2$

$\mathbb{Z}/2$

$\mathbb{R}_{\geq 0}$ ~~$\times \mathbb{R}$~~



$$\phi(s) - \frac{val(s)}{2} \le D(s) \le \phi(s) + \frac{val(s)}{2}$$

$-1 \le -1 \le 1$

$-1 \le \cancel{1} \le 1$