Log Geometry Lectures

Nov 25 2020

Goal for lectures is to explain some work in progress on log intersection theory.

Based on projects w/ various people

(R, W, H, P, S)

- Looking for framework that incorporates phenomena such as invariance of log invariants with respect to log blowups
- Tie in with DR, DDR... and in general LogPic, stable maps.

Apologies: Some overlap with my previous talk, and also w/ talks given in Johannes' seminar

Technical background is necessary:

- Weak semistable reduction
- Fiber products of log schemes
- Log Blowups and Strict transforms
- $CH(X)$ and piecewise polynomials on $Trop(X)$

Today: Toric geometry.

Very good toy model where the germs of all the ideas appear.

Notation

X toric variety w/ torus $T = \mathbb{G}_m^n / \mathbb{C}$

$N = \mathrm{Hom}(\mathbb{G}_m, T) =$ 1 p-subgroups

$M = \mathrm{Hom}(T, \mathbb{G}_m) = \mathrm{Hom}(N, \mathbb{Z}) =$ characters

Main thm of toric geometry: $\exists$ equivalence of categories

$$\left\{\begin{array}{c}\{\text{Toric Vars}\}\\ \text{Equivariant}\\ \text{Maps}\end{array}\right\} \longleftrightarrow \begin{array}{l}\text{Lattices } N\\ \text{Rational polyhedral}\\ \text{fans } F \subset N_{\mathbb{R}}\end{array}$$

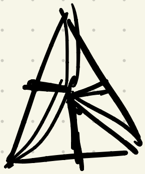
$$\begin{tikzcd}\mathcal{F} = \bigcup \sigma \arrow[d] \\ X(\mathcal{F}) = \varinjlim_{\sigma \in F} \operatorname{Spec} \mathbb{C}[S_\sigma]\end{tikzcd}$$

$$S_\sigma = \sigma^\vee \cap M = \{u \in M \mid \langle u, \sigma \rangle \geq 0\}$$

$\rightsquigarrow \mathbb{P}^2$

$$X \rightsquigarrow \mathcal{F} = \bigcup \sigma$$

w/ $\sigma^\circ = \{ v : \mathbb{G}_m \to T \mid v(0) \text{ exists}$ and is a given point $\}$



Try to reduce all geometry on X to geometry on $\mathcal{F}$.

Well known instances:

Singularities

X is non-singular iff all σ in F

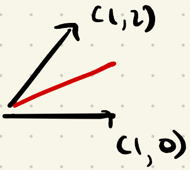
are (i) simplicial $\sigma = \underbrace{\langle v_1, \dots, v_k \rangle}_{\text{lin. indep.}}$

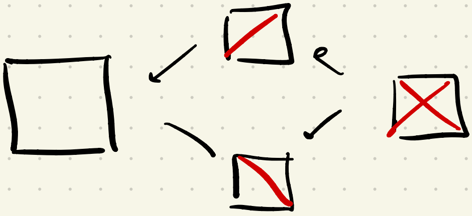
(ii) unimodular

det $\langle v_1, \dots v_k \rangle = 1$ or

$v_1, \dots v_k$ are basis for $N \cap \text{Lin}(\sigma)$.

For example, resolution of singularities is equivalent to finding subdivisions





Remark Subdivisions produce proper birational maps, but geometrically can be complicated

Star Subdivision gives a weighted blowup.

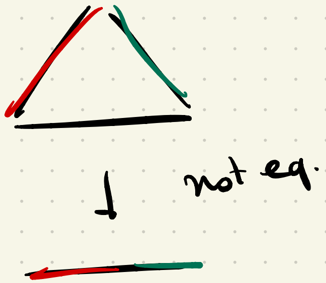
Not so well know

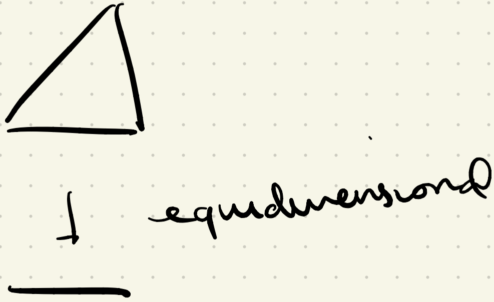
Thm (Abramovich-Karu)

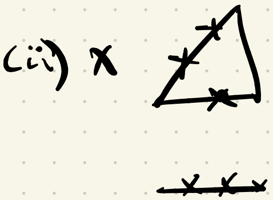
- Surjective map $X \to Y$ is

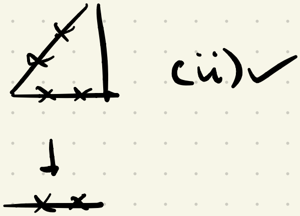
  (i) Equidimensional if maps cones onto cones

  (ii) has reduced fibers if maps integral structures onto integral structures









Definition: A map satisfying (i) and (ii) is called weakly semistable.

Remark: Should probably have been called "saturated".

<u>Thm</u> Surjective weakly s.s. morphisms are flat.

<u>Thm</u>. Every map $X \to Y$ can be turned weakly semistable after blowup*

* and root stack or finite cover

Meaning: Given $X \to Y$, $\exists$

weakly s.s.

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

This is an analogue of Raynaud-Gruson.

Idea:

Thm Can achieve X', Y' smooth.

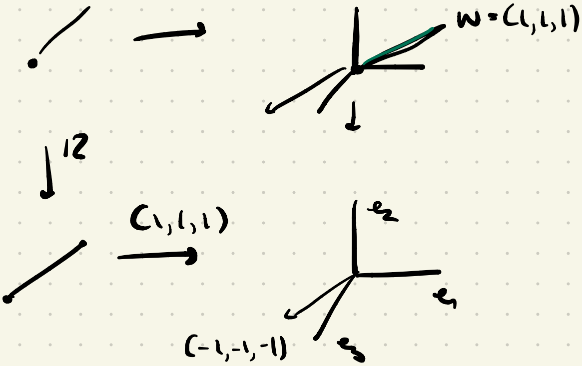
This is hard, proven by (ALT)

When $\dim Y = 1$, classical.

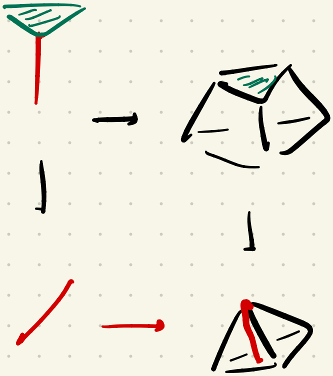
Fiber Products

Category of toric varieties has fiber products, but different than schematic ones.

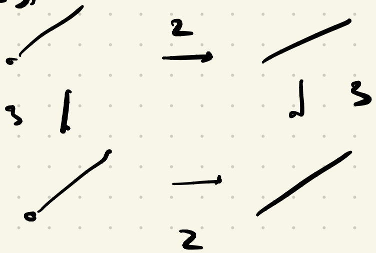
2 main pathologies



Schematically



in $\mathbb{Z}\cdot(3,2)$



$\{(y,x) \mid y^2 = x^3\}$

Thm $X \to Y$ is weakly s.s if and only if $(X \times_Y Z)^{tor} = X \times_Y Z$ for all $Z \to Y$.

Where does the descrepancy come from?

$X = X(\sigma, N)$, $Y = X(\tau, N')$, $Z = X(\kappa, N'')$

$S_\sigma = \sigma^\vee \cap M$

$$X \times_Y Z = \operatorname{Spec} \mathbb{C}[S_\sigma] \otimes_{\mathbb{C}[S_\tau]} \mathbb{C}[S_\kappa]$$

$$= \operatorname{Spec} \mathbb{C}[S_\sigma \oplus_{S_\tau} S_\kappa]$$

But a monoid P of the form S_{cone} is special:

- (P^{gp} is torsion free)
- $P \to P^{gp}$ is injective ("fine")
- $P \subset P^{gp}$ is saturated ("f.s")

$S_\sigma \oplus_{S_\tau} S_\kappa$ does not satisfy these properties

For any P, $\exists$ $P \longrightarrow P^{tor}$

Completely formally,

$$(X \times_Y Z)^{tor} = \operatorname{Spec} \mathbb{C}[(S_\sigma \oplus_{S_\tau} S_k)^{tor}]$$

P^{tor} is composition of 3 functors

- $P \longrightarrow P^{int} =$ Image of P in P^{gp}
- $P^{int} \longrightarrow P^{sat} = \{x \in P^{gp} \mid nx \in P^{int}\}$

(- $P^{sat} \longrightarrow P^{sat}/Tor \subset \bar{P}^{gp}/Tor$)

Geometrically

- $\operatorname{Spec} \mathbb{C}[P^{int}] \longrightarrow \operatorname{Spec} \mathbb{C}[P]$

  inclusion of irred comp.

- $\operatorname{Spec} \mathbb{C}[P^{sat}] \longrightarrow \operatorname{Spec} \mathbb{C}[P^{int}]$ normalization

(- Killing torsion picks one component out of disjoint union)

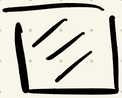
So $(X \times_Y Z)^{tor} \subset X \times_Y Z$ is "normalization of main component".

To see other components, use "extended tropicalization"

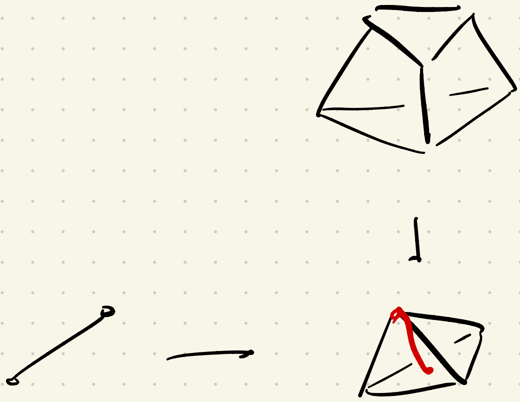
$$F = \varinjlim \mathrm{Hom}(S_\sigma, \mathbb{R}_{\geq 0})$$

$$\overline{F} = \varinjlim \mathrm{Hom}(S_\sigma, \overline{\mathbb{R}}_{\geq 0} = \mathbb{R}_{\geq 0} \cup \infty)$$

Ex: $\mathbb{A}^2$: $F =$ 

$\overline{F} =$ 

$\mathbb{P}^2$ 



~ Thm • Components of $X \times_Y Z$ are indexed by components of $\overline{F}(X) \times_{\overline{F}(Y)} \overline{F}(Z)$

- The normalization of such a component is toric, with extended fan = that component.
- If $X \to Y$ is a subdivision, no normalization is necessary.

Furthermore, $(X \times_Y Z)^{tor}$ = Strict transform of Z.

Chow groups.

Well known: Orbit - Cone Correspondence

Toric variety is stratified, strata are orbits of T.

$$\{\text{Cones of } F\} \longleftrightarrow \{\text{Strata}\}$$

$$\sigma \longrightarrow \{T\,\nu(0)\}_{\nu\in\sigma^\circ}$$

$$\sigma \text{ s.t } \nu(0)\in O(\sigma) \text{ for } \nu\in\sigma^\circ \longleftarrow O(\sigma)$$

$\dim \sigma = k$ $\dim \sigma = n-k$

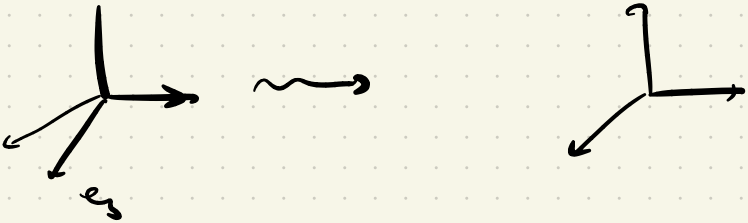
$$V(\sigma) = \overline{O(\sigma)} = \bigcup_{\tau>\sigma} O(\tau)$$

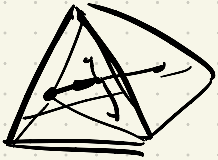
This is a toric variety. Its fan is the Star of σ.

$\mathcal{N}(\sigma) = N/_{\mathrm{Lin}(\sigma)\cap N}$ $\mathcal{M}(\sigma) = \sigma^{\perp} \cap M$

$F(V(\sigma)) = \{ \bigcup \overline{\tau} \mid \tau \geq \sigma \}$

Remark $\overline{F}(V(\sigma))$ can be seen directly in $\overline{F}(X)$, as the face perpendicular to σ.





Thm (?) $A_*(X)$ is generated by $[V(\sigma)]$

Relations come from characters

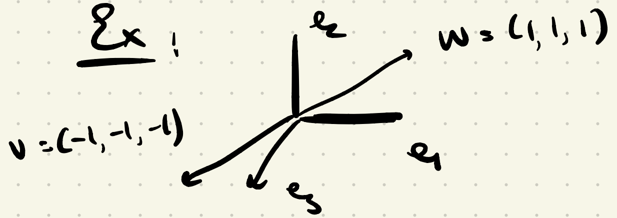
$$\mathrm{div}\,\chi^u = \sum \langle u, r_i\rangle V(\rho_i) := 0$$

More generally, for all σ, $u \in M(\sigma)$

$$\mathrm{div}\,\chi^u = \sum_{\substack{\tau > \sigma \\ \text{in codim } 1.}} \langle u, n_{\sigma,\tau}\rangle V(\tau) = 0$$

$\nwarrow$ primitive elt. on τ/σ

Ex!



$$A_2 = \langle V(e_1), V(e_2), V(e_3), V(v), V(w)\rangle \Big/$$

M

$\chi^{e_1^*}$

$$\underline{V(e_1) + V(w) = V(v)}$$

$$V(e_2) + V(w) = V(v)$$

$$V(e_3) + V(w) = V(v)$$

$$\simeq \langle V(e_1), V(w)\rangle$$

A_1 : $v_{12}, v_{13}, v_{1w}, v_{1v}, v_{23}, v_{2w}, v_{2v}, v_{3w}, v_{3v}$

$v_{12} + v_{1w} = v_{1v}$ $\quad e_2^*$,

$v_{13} + v_{1w} = v_{1v}$

$v_{12} + v_{2w} = v_{2v}$

$v_{23} + v_{2w} = v_{2v}$

$v_{13} + v_{3w} = v_{3v}$

$v_{23} + v_{3w} = v_{3v}$

$v_{1w} - v_{2w} = 0$

$v_{1w} - v_{3w} = 0$

$v_{1v} - v_{2v} = 0$

$v_{1v} - v_{3v} = 0$

v_{12}, v_{1w} basis

Chow Ring trickier.

if X is complete,

$$A^k(X) = \mathrm{Hom}(A_k(X), \mathbb{Z})$$

$$\subset \mathrm{Hom}(\{\text{Cod } k\text{-cones}\}, \mathbb{Z})$$

c : Cod k-cones $\to \mathbb{Z}$ s.t $\forall \sigma$ of cod $k+1$

$$\sum_{\tau > \sigma} c(\tau) \langle u, n_{\sigma,\tau} \rangle = 0$$

$$\iff$$

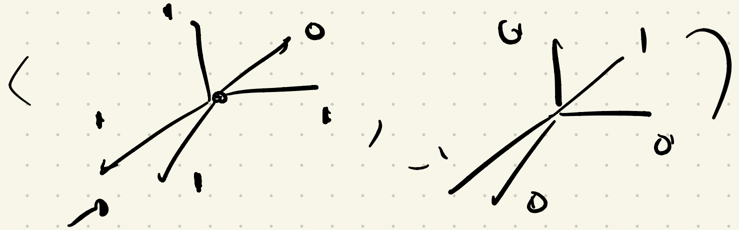
$$\sum_{\tau > \sigma} c(\tau)\, n_{\sigma,\tau} \in \mathrm{Lin}(\sigma)$$

↗

Minkowski weights

(Balanced polyhedral subcomplexes/

tropical cycles)



$A^1(Bl_0\mathbb{P}^3) = \mathbb{Z}^2$

if X not complete, more mysterious.

However, Pic can be described differently

$$Pic(X) = PL(F(X))/M$$

$$= \{\text{Piecewise linear}\}/\{\text{linear}\}$$

Thm (Brion)

if X is smooth

$$A^*(X) = PP(F(X))/M$$

The isomorphism sends function

that takes 1 on ray r, 0 on other rays to $q(\vartheta(v(r)))$.

Can compute $PP^*(x)$ via Stanley-Reisner ring:

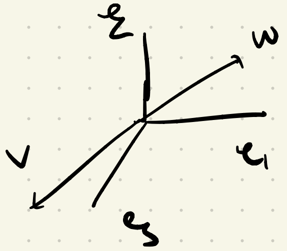
$$SR = k[\{x_r\}] \Big/ \cdot\ x_{r_1} \cdots x_{r_k} = 0 \text{ if } \langle r_1, \dots, r_k \rangle$$

do not form a cone

Quotient also by $\langle \sum \langle u, r_i \rangle x_{r_i} \rangle_{u \in M}$

to get

$PP^\circ(X) /$



$$\underline{\Bbbk[x_1, x_2, x_3, x_w, x_v]}$$

$$x_1 x_2 x_3 = 0$$

$$x_v x_w = 0$$

$$x_1 + x_w = x_v$$

$$x_2 + x_w = x_v$$

$$x_3 + x_w = x_v$$

$$\simeq \Bbbk[x_1, x_w] / x_1^3 = 0$$

$$x_1 x_w + x_w^2 = 0$$

<u>deg 1</u>: x_1, x_w

deg 2: $x_1^2, x_1 x_w$

deg 3; $x_1^2 x_w$

deg 4 and up $= 0$

In full generality

<u>Thm</u> (Payne) $PP^*(X) = A^*([X/T])$

for any toric variety.

Lecture 2.

Foundations

etale sheaf
of monoids

Log Scheme : pair $(X, \mathcal{M}_X)$

$$\mathcal{M}_X \xrightarrow{\alpha} \mathcal{O}_X$$

$$\alpha^{-1}(\mathcal{O}_X^*) = \mathcal{O}_X^*$$

Maps $(X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$

$$X \to Y$$

$+$

$$\begin{tikzcd} f^{-1}\mathcal{M}_Y \arrow[r] \arrow[d] & \mathcal{M}_X \arrow[d] \\ f^{-1}\mathcal{O}_Y \arrow[r] & \mathcal{O}_X \end{tikzcd}$$

Examples

$X = X(F)$ toric variety

$\mathcal{M}_X = \{ f \in \mathcal{O}_X \mid f$ is a unit on torus$\}$

$\mathcal{M}_X(X) =$ monomials $k^* M$

- A bit more generally, if P is a monoid,

  $\operatorname{Spec} k[P]$ has log structure

  $M_{k[P]}$ generated by $\{x^p\}_{p \in P}$

- $(X, \mathcal{O}_X^*)$ ← crucial

- $(X, \mathcal{O}_X)$ ← rarely used

Embedding

$$\begin{tikzcd} \mathrm{Sch} \arrow[r] & \mathrm{LogSch} \arrow[r, "\mathrm{Forget}"] & \mathrm{Sch} \\ X \arrow[r] & (X, \mathcal{O}_X^*) \arrow[r] & X \end{tikzcd}$$

As a category, LogSch is nice

- Closed under finite inverse limits

E.g has fiber products

(fiber product of schemes, pushout of log str.)

Ex:

$$\begin{tikzcd}
(Y, f^*M_Y) \arrow[r, "\text{“strict map”}"] \arrow[d] & (X, M_X) \arrow[d] \\
(Y, \mathcal{O}_Y^*) \arrow[r] & (X, \mathcal{O}_X^*)
\end{tikzcd}$$

But LogSch too large for geometry.

Put finiteness condition on log structure

__Def__. (X, M_X) is <u>coherent</u> if $\forall x \in X$, $\exists$ etale neighborhood (U, u) of (X, x), a monoid P, and

$$\text{map } U \xrightarrow{\text{strict}} \operatorname{Spec} k[P]$$

(i.e. $M_U = f^* M_P$)

"Chart"

Usually demand more:

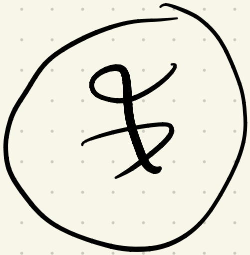
Charts can be chosen

- Integral ($P \subset P^{gp}$)
- Integral + finitely gen. ("fine")
- Fine and saturated ("f.s")

For the most part you can live your log life here f.s log schemes are essentially category generated by toric varieties (w/ interesting log str.) schemes (w/ trivial), and étale localization.

3 main examples:

(i)



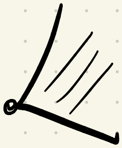
all toroidal embeddings

(X, D)

$\mathcal{M}_X = \{ f \in \mathcal{O}_X \mid f \text{ unit away from } D \}$

(ii) Log Point

P monoid with no units

$\bullet \xrightarrow{i} \operatorname{Spec} k[P] =$ 

$i^* \mathcal{M}_P$ a non-trivial log str.

Explicitly : $(\operatorname{Spec} k, P \oplus k^*)$

with $P \oplus k^* \longrightarrow k$

$p \longrightarrow 0$

(iii) Log Curves.

$$\begin{tikzcd} X = \text{nodal curve} \arrow[d, dash] \\ S = (\operatorname{Spec} k, M_S) \text{ log point} \end{tikzcd}$$

Log structure looks as follows

- $\overline{M}_X = \overline{M}_S$ at generic point of irred component
- $\overline{M}_{X,q} = \overline{M}_S \oplus_{\mathbb{N}} \mathbb{N}^2$ at node q

  $$\left.\begin{array}{ccc} \mathbb{N} & \longrightarrow & \overline{M}_S \\ 1 & \longmapsto & \delta_q \end{array}\right\} \text{this is part of the data}$$

- Potentially there are marked points w/ log structure, and then

$$\overline{M}_{X,p} = \overline{M}_S \oplus \mathbb{N}$$

What's more interesting are the morphisms

Many ways to put log structure on a scheme, but once you choose, morphisms are restricted.

Ex: $(\mathrm{Spec}\, k, P \oplus k^*) \longrightarrow (\mathrm{Spec}\, k, k^*)$ ok

$(\mathrm{Spec}\, k, k^*) \longrightarrow (\mathrm{Spec}\, k, P \oplus k^*)$ not ok.

$$\begin{tikzcd}
P \arrow[r] \arrow[ddd, bend right] & u(p) \arrow[dd, bend left] \\
P \oplus k^* \arrow[r] \arrow[d] & k^* \arrow[d] \\
k \arrow[r] & k \\
0 \arrow[r] & 0 \neq u(p)
\end{tikzcd}$$

(ii) E.g. $\bar{S} \subset X$ ← toric variety

↗

closed st-al

2 log structures:

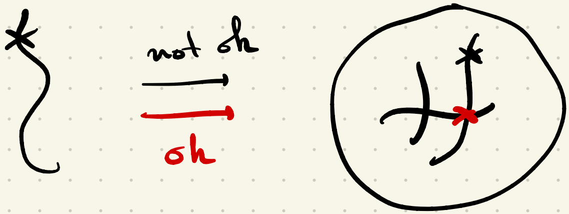
$$(\bar{S}, i^*\mathcal{M}_X) \longrightarrow (X, \mathcal{M}_X) \quad \text{ok}$$

$$(\bar{S}, \mathcal{M}_{\bar{S}}) \not\longrightarrow (X, \mathcal{M}_X) \quad \text{no map}$$

↖

standard toric log structure

(iii)



Common to break study of log scheme in 2 pieces

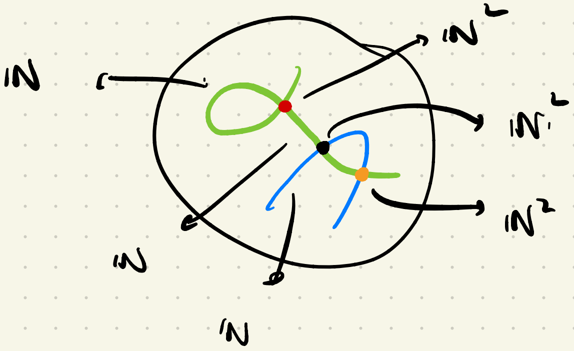
$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathcal{M}_X^{gp} \longrightarrow \bar{\mathcal{M}}_X^{gp} \longrightarrow 0$$

↗ algebraic

↖ combinatorial

Combinatorial aspects

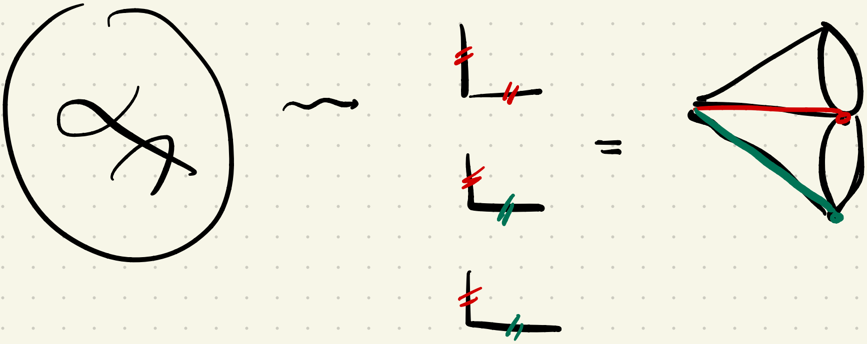
$\overline{\mathcal{M}}_x^{gr}$ gives stratification



Tend to organize into "cone complex"/fan

When $x \rightsquigarrow y$ generization, $\overline{\mathcal{M}}^\vee_{x,y} \xhookrightarrow[\text{face}]{} \overline{\mathcal{M}}^\vee_{x,x}$

$$\varinjlim_{x \in X} \overline{\mathcal{M}}^\vee_{x,x} := C(X)$$



- If X is toric w/ toric log structure:

$$C(X) = F$$, but cocharacter lattice N is lost

However, ~~orbit~~ strata-cone correspondence holds verbatim

- Map $X \longrightarrow Y$ induces

$$C(X) \longrightarrow C(Y)$$

Not induced by global map of lattices, but still *piecewise linear*.

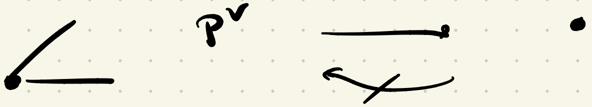
Still have that if stratum S of X maps to stratum T of Y,

$$\mathrm{int}(\sigma_S) \subset \mathrm{int}(\sigma_T)$$

Can see previous examples in this language

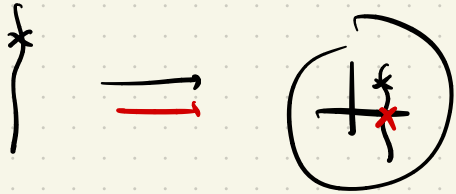
(ib)

$$\begin{tikzcd}(\operatorname{Spec} k, P \oplus k^*) \arrow[r] \arrow[d, phantom, "\wr"] & (\operatorname{Spec} k, k^*) \arrow[l, bend left, "/" marking] \\ {} & {}\end{tikzcd}$$

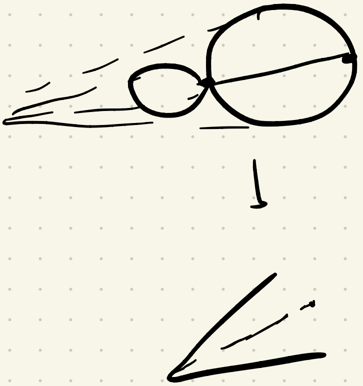


(interior cannot go to interior!)

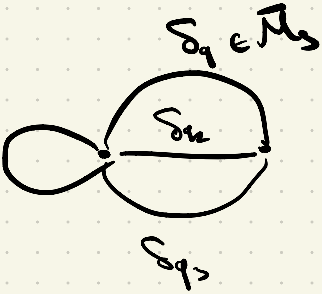
ii)



$$\begin{tikzcd} C \arrow[d, dash] & \text{log curve} \\ S = (\operatorname{Spec} k, \bar{M}_S) & \end{tikzcd}$$



Proxy:



$C(X)$ captures a very simple operation, which is crucial.

–

A *subdivision* $C' \longrightarrow C(X)$ determines a proper map $X' \longrightarrow X$ with $C' = C(X')$

Locally, if $\sigma = \mathcal{M}_{X,x}^{\vee}$, you have chart

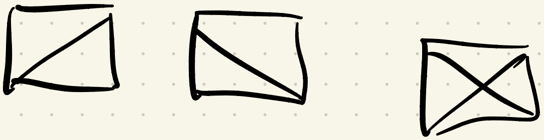
$$\begin{tikzcd}
X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X(\Sigma) \arrow[d] & \text{subdivision} \\
X \arrow[r] & \operatorname{Spec} k[\bar{\mathcal{M}}_{X,x}] & \\
 & & \text{toric,}
\end{tikzcd}$$

X' is called a log modification.

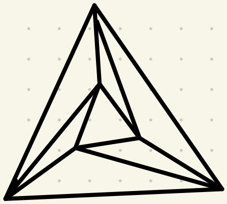
One way to create log modifications is to start with sheaf of ideals $\mathcal{I} \subset \bar{\mathcal{M}}_X^{gp}$.

Corresponding modification is then called a *log blowup*

Lots of possible subdivisions

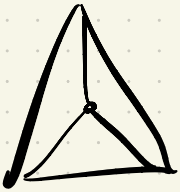


log blowups



log modification,
not log blowup.

Most important ones - star subdivisions

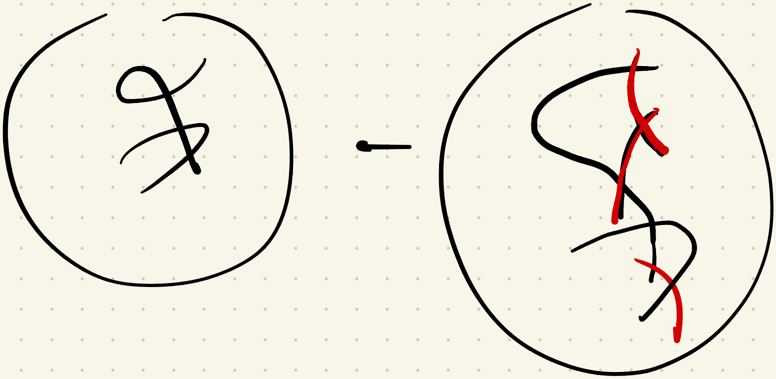


log blowup

If X is smooth and toroidal, star subdivision at barycenter = blowup at smooth stratum

Star subdivisions are cofinal

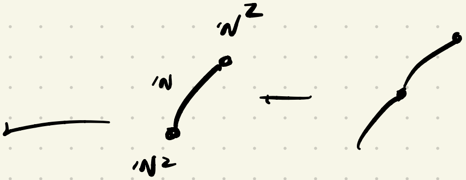
Intuitively, there are far fewer log blowups than blowups



True for the toroidal case.

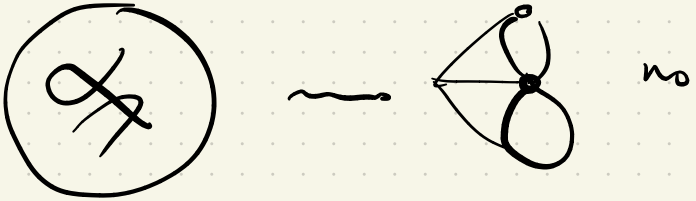
Other extreme:

$(\bullet, \mathbb{N}^2)$

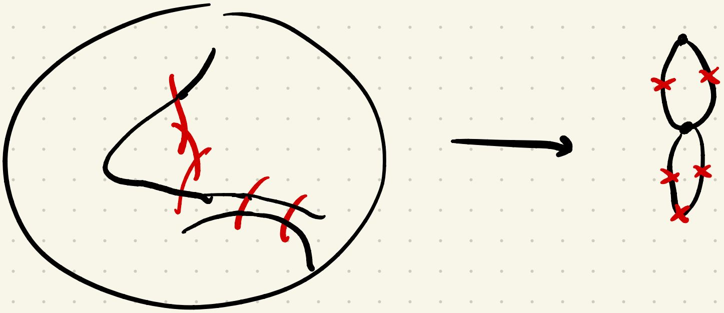


etc.

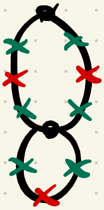
Ex: For (X, D) smooth toroidal, D is snc iff $C(X)$ can be linearly embedded in a vector space.

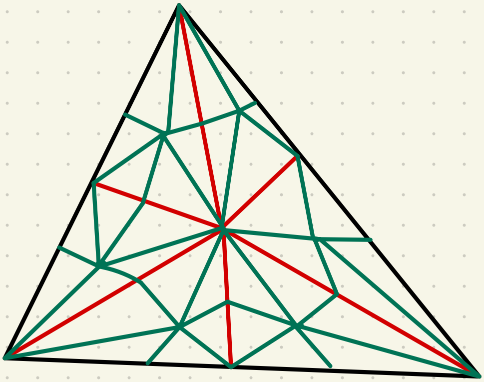


Can always achieve:



Algorithmically: Second barycentric subdivision





Parallel approach to combinatorics:

To a log scheme X, we can assign

a stack $\mathcal{L}og_X$

$$\mathcal{L}og_X \longrightarrow Sch/\underline{X}$$

$$(Y, M_Y) \to X \quad \rightsquigarrow \quad Y \xrightarrow{f} \underline{X}$$

w/ strict maps

$$X \subset \mathcal{L}og_X$$

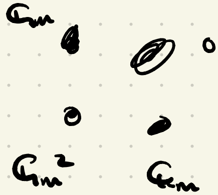
$$Y \to \underline{X} \rightsquigarrow (Y, f^* M_X) \to X$$

$\mathcal{L}og_X$ is very large, but essentially combinatorial.

For monoid P, let

$$\mathcal{Z}_P := \left[\operatorname{Spec} k[P] \big/ \operatorname{Spec} k[P^{gp}] \right]$$

$\mathcal{Z}_{\mathbb{N}^2} =$



For example

$$\mathcal{H}om_{LogSch}(X, \mathcal{Z}_P) = \mathrm{Hom}_{Mon}(P, \overline{M}_X(X))$$

if $U \to X$ is etale, + $U \to \operatorname{Spec} k[P]$

is global chart,

$$\coprod_{P \to Q} U \times_{\mathcal{Z}_P} \mathcal{Z}_Q \xrightarrow{\text{etale cover}} \mathcal{L}og_U$$

In fact,

$$\mathcal{L}og_U = \varinjlim_{\mathrm{Mon} \backslash P} U \times_{\mathcal{Z}_P} \mathcal{Z}_Q$$

and $\mathcal{L}og_U$ cover $\mathcal{L}og_X$.

Special case

$$\mathcal{L}og = \mathcal{L}og_{(\operatorname{Spec} k, k^*)} = \varinjlim_{P} \mathcal{Z}_P$$

For any log scheme X,

$$\begin{tikzcd}
X \arrow[r] \arrow[dr] & \mathcal{L}og \text{ factors through} \\
& \varinjlim_{x \in X} \mathcal{Z}_{\overline{M}_{X,x}} := \mathcal{A}_X \arrow[u, "\leftarrow \text{étale}"'] \leftarrow \text{Artin fan of } X
\end{tikzcd}$$

Same information as $C(X)$

$$C(X) = \varinjlim_{x \in X} \overline{M}_{X,x}^{\vee} \quad \text{versus} \quad \varinjlim_{x \in X} \mathcal{Z}_{\overline{M}_{X,x}} = \mathcal{A}_X$$

In fact

Thm (CCUW)

$$\{\text{Artin fans}\} \longleftrightarrow \{\text{Cone Complexes}\}$$

In this context

$$\mathcal{A}_{C'} \to \mathcal{A}_X \quad \rightsquigarrow \quad \begin{array}{l}\text{Subdivision} \\ C' \to C(X)\end{array}$$

$$\begin{tikzcd} X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{A}_{C'} \arrow[d] \\ X \arrow[r] & \mathcal{A}_X \end{tikzcd}$$

So combinatorally equivalent.

Advantage:

Deformation theory of log maps is governed by Log or Artin fan.

(i) $X \longrightarrow Y$ is log smooth/flat/etale

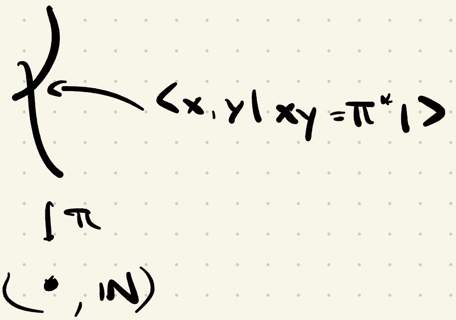
if $\mathcal{L}og_X \longrightarrow \mathcal{L}og_Y$ is smooth/flat/etale

More intuitively (locally on X)

$$\begin{tikzcd} X \arrow[r, "\text{sm/flat/etale}"] & Y \times_{\mathcal{A}_Y} \mathcal{A}_X \arrow[r] \arrow[d] & \mathcal{A}_X \arrow[d] \\ & Y \arrow[r] & \mathcal{A}_Y \end{tikzcd}$$

<u>E.g</u>

- Toroidal embedding — log smooth over (Spec k, k^*)
- log point (Spec k, P) — never log smooth over (Spec k, k^*) unless P = 0.
- Nodal curves are log smooth for appropriate log structures



(ii) ∃ log cotangent complex $\mathcal{L}^{\log}_{X/Y}$ for maps

$$X \longrightarrow Y$$

It is simply $\mathcal{L}_{X/\mathcal{L}og_Y}$

Has (mostly) expected formal properties.

key to defining obstruction theory for log stable maps.

Connection with previous lecture.

Main points:

- For a toric variety, schematic and toric fiber product are different
- Toric fiber product is a normalization of a main component, basically strict transform in case of interest

- There is a class of maps where the two notions agree. These are the semistable maps (cones onto cones, lattices onto lattices). Every map can be turned semistable.

- For non-semistable maps, the excess component are governed by fiber product of extended tropicalization.

The same holds essentially verbatim for f.s. log schemes

- f.s. fiber product does not have the same underlying scheme as fiber product.

The issue is exactly the same:

if $X \to \operatorname{Spec} k[P]$, $Y \to \operatorname{Spec} k[Q]$, $Z \to \operatorname{Spec} k[R]$

are local charts,

fiber product in all log schemes wants to have chart $\operatorname{Spec} k[\underbrace{P \oplus_Q R}_{\text{not f.s.}}]$

f.s. fiber product wants to have chart

$$\operatorname{Spec} k[(P \oplus_Q R)^{sat}]$$

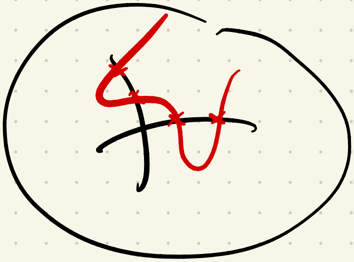
The process of going from $M \to M^{sat}$ keeps precisely the normalization of the main component.

- For semistable maps, the two notions coincide.

Semistable maps between log smooth objects look like

(i) flat, reduced fiber families

(ii) Inclusions transversal to the boundary



- The excess components are parametrized by fiber product $\overline{C}(X) \times_{\overline{C}(Y)} \overline{C}(Z)$

When $X \to Y$ is a log blowup, and Z has generically trivial log structure

- $(X \times_Y Z)^{f.s} \subset X \times_Y Z$

strict transform.

- Other components are projective bundles over strata of Z.

Intersection theory.

Look again at $0 \to \mathcal{O}_X^* \to M_X^{gp} \to \overline{M}_X^{gp} \to 0$

Easy: $H^0(\overline{M}_X^{gp})$ = Piecewise linear functions $C(X) \to \mathbb{Z}$.

Long exact sequence

$$H^0(M_X^{gp}) \longrightarrow PL(C(X)) \longrightarrow Pic(X)$$

$$\alpha \longrightarrow \mathcal{O}(-\alpha)$$

$\mathcal{O}(-\alpha)$ = torsor of lifts of α

When X is smooth, log smooth

$$PL(C(X)) = \mathbb{Z}^{\#\text{rays of } C(X)}$$

$$e_\rho \longrightarrow \mathcal{O}(D)$$

More generally

$$X \longrightarrow \mathcal{A}_X$$

induces a map

$$\begin{tikzcd} CH^*(\mathcal{A}_X) \arrow[r] \arrow[d, equal] & CH^*(X) \\ PP(C(X)) & \end{tikzcd}$$

Payne's theorem

Essentially extension of $H^0(\bar{\mathcal{M}}_X^{gp}) \to Pic(X)$

by linearity.

Lecture 3

Want to study geometry of $\overline{\mathcal{M}}_{g,n}$.

In log geometry, there are two approaches

(i) Log Stable maps	(ii) Log line bundles
$K_\Gamma(V)$ into some toroidal V	on $\overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$
• Honest, hard-working space	• Mysterious, underdeveloped
Well studied by now	but gives intuition

next lecture ⟶ DR ⟵ today

Warm up.

$\mathrm{Spec}\,k[P]$ with its canonical log str.

$$\mathrm{Hom}_{\mathrm{LogSch}}(X, \mathrm{Spec}\,k[P])$$

$$= \mathrm{Hom}_{\mathrm{Mon}}(P, \mathcal{M}_X(X))$$

$$= \{ \phi \in \mathrm{Hom}(P^{gp}, \mathcal{M}_X(X)^{gp}) \text{ s.t } \phi(P) \subset \mathcal{M}_X(X) \}$$

Proof

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & \mathcal{M}_X(X) \arrow[d] \\
k[P] \arrow[r] & \mathcal{O}_X(X)
\end{tikzcd}$$

Cone complex $C(\mathrm{Spec}\,k[P]) = P^\vee$

$\mathrm{Hom}(C, P^\vee)$ is determined

by $\mathrm{Hom}(\sigma, P^\vee)$ and equals

$\mathrm{Hom}(P, \sigma^\vee)$

Can formally "lift" $C(\mathrm{Spec}\,k[P])$ to LogSch by

$$\mathrm{Hom}(X, C(\mathrm{Spec}\,k[P])) := \mathrm{Hom}(P, \overline{M_X}(X))$$

This functor is representable by

$$\mathcal{A}_P = \left[\mathrm{Spec}\,k[P] \big/ \mathrm{Spec}\,k[P^{gp}] \right]$$

(Different way to phrase equivalence between cone complexes and Artin fans)

A bit more generally:

$X = X(\mathcal{F}, N)$ a toric variety, $M = N^\vee$

$$\mathrm{Hom}_{\mathrm{LogSch}}(Y, X) = \left\{ \phi \in \mathrm{Hom}(M, M(Y)^{gp}) : \begin{array}{l} \text{locally on } Y,\ \phi \text{ takes} \\ \sigma^\vee \cap M \text{ into } M(Y) \text{ for some } \sigma \in F \end{array} \right\}$$

Suppose $\tilde{X} \to X$ is a subdivision of X.

Last time we saw $\tilde{X} \to X$ is log étale

(In fact: Log étale maps are generated by subdivisions, $\underbrace{\text{orbifold structure}}_{\text{didn't discuss}}$, classical étale maps)

Can also see from functor of points that they are monomorphisms of log schemes.

Evgeny asked: What about category of log schemes "up to log ~~étale~~" subdivision

I'll interpret the question a certain way and give an answer.

$\mathcal{C}$ new category

$$\operatorname{Ob}\mathcal{C} = \operatorname{Ob}\mathrm{LogSch}$$

$$\operatorname{Mor}_{\mathcal{C}}(Y, X) = \varinjlim_{\substack{\tilde{Y} \to Y \\ \text{subdivision}}} \operatorname{Hom}(\tilde{Y}, X).$$

Features:

(i) In $\mathcal{C}$, $\tilde{Y} \simeq Y$ for a log blowup

(same co-functor (?) of points)

(ii) Let $X = T = \operatorname{Spec} k[M]$ torus.

Then $\operatorname{Hom}(Y, T) = \operatorname{Hom}(M, \mathcal{O}(Y)^{*})$

So $\operatorname{Mor}_{\mathcal{C}}(Y, T) = \varinjlim \operatorname{Hom}(\tilde{Y}, T)$

$$= \operatorname{Hom}(Y, T)$$

nothing changes.

(iii) Let X = toric variety compactifying T.

Then $\mathrm{Mor}(Y, X) = \varinjlim \mathrm{Hom}(\tilde{Y}, X)$

$$= \varinjlim \{ \phi : M \to M(\tilde{Y})^{gp} = M(Y)^{gp}$$

s.t locally on $\tilde{Y}$,

$$\phi(\sigma^\vee \cap M) \subset M(\tilde{Y})$$

$$\Updownarrow$$

$$M(\tilde{Y})^\vee \subset \sigma$$

The $M(\tilde{Y})$ get smaller and smaller

$$= \mathrm{Hom}(M, M(Y)^{gp})$$

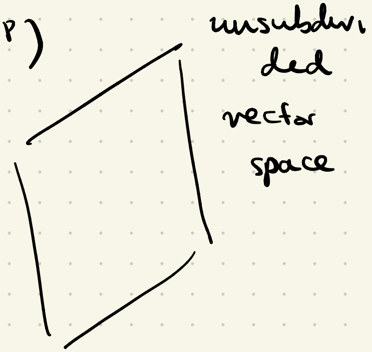
So X becomes a group object!

Instead of working with $\mathcal{C}$, I'll take a slightly different approach

Def (Kato) Functor on LogSch.

$$T^{\log}(Y) = \mathrm{Hom}(M, M(Y)^{gp})$$

$$T^{trop}(Y) = \mathrm{Hom}(M, \overline{M(Y)}^{gp})$$



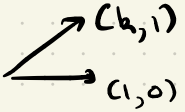
(i) $\exists$ exact sequence

$$0 \to T \to T^{\log} \to T^{trop} \to 0$$

(ii) In $\mathcal{C}$, $T, T^{\log}, T^{trop}$ are representable by T, $X(F)$, $[X(F)/T]$ respectively.

(iii) $T^{\log}$ and T^{trop} are not representable in LogSch.

Pf Suffices to do $\mathbb{G}_m^{trop}$. Suppose X represents.

Let $S_k = (\bullet,$  $)$

$$\varinjlim S_k = S_\infty = (\bullet, \underset{(1,0)}{\longrightarrow})$$

$$\mathbb{Z} = \mathrm{Hom}(\varinjlim S_k, \overline{\mathbb{G}}_m^{trop}) \neq \varprojlim \mathrm{Hom}(S_k, \overline{\mathbb{G}}_m^{trop}) = \mathbb{Z}^2$$

This is a very good toy model for what follows.

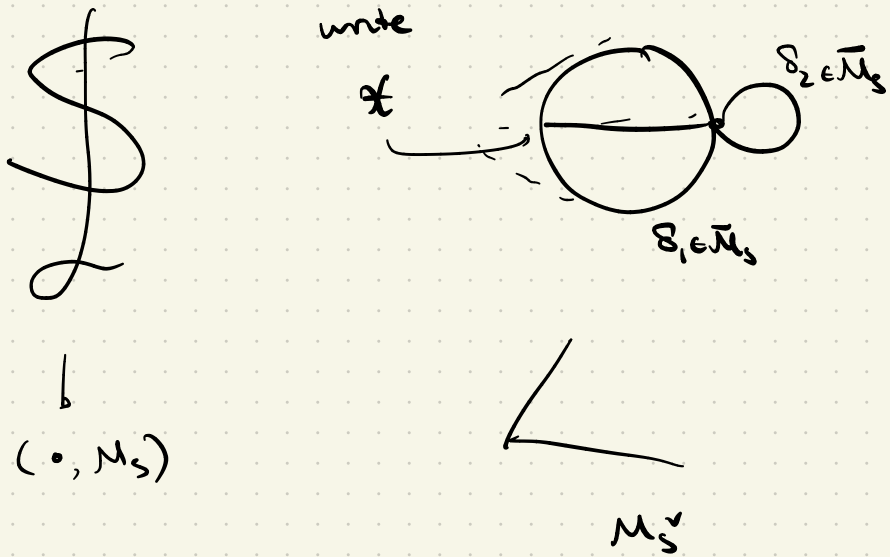
LogPic

$$\begin{tikzcd} X \arrow[d] \\ S \arrow[d, phantom, "\shortparallel"] \\ (\bullet, M_S) \end{tikzcd}$$

log curve: log smooth, weakly semistable, proper.

No log structures on markings.

for expository purposes



Consider

$$0 \to \mathcal{O}_X^* \to M_X^{gp} \to \overline{M}_X^{gp} \to 0$$

Illusie: Study M_X^{gp}-torsors on X.

Intuition

$$\begin{tikzcd}
X_U \arrow[r] \arrow[d, "\text{smooth}"'] & X \arrow[d, "\text{nodal}"] \\
U \arrow[r, phantom, "\subset"] & S \\
\text{dense open} & \text{smooth}
\end{tikzcd}$$

$\mathrm{Pic}^0(X)$ not separated

$\mathrm{Pic}^{[0]}(X)$ not universally closed.

L_u line bundle on X_u has

transition functions $g_{ab} \in \mathcal{O}(X_u)^*$

but over $S = \overline{U}$ g_{ab} may not go to a unit.

However,

$$(\text{Easy}) \quad (i_{X_u})_* \mathcal{O}_{X_u}^* = M_X^{gp}$$

So expect an M_X^{gp} torsor as limit.

<u>Def</u>

$$\underline{\text{LogPic}} \longrightarrow \text{LogSch}/S$$

$$\left\{ \begin{matrix} M_{X \times_S T}^{gp}\text{-torsors} \\ \text{on } X \times_S T \end{matrix} \right\}^{+} \rightsquigarrow T \to S$$

$+ \longrightarrow$ ignore in this talk.

LogPic = Sheaf of iso classes

Properties: $\mathrm{LogPic}(X/S)$ is

- log smooth
- proper
- group.

But what does it look like? I'll give 3 descriptions.

Description 1. Look at $0 \to \mathcal{O}_X^* \to M_X^{gp} \to \overline{M}_X^{gp} \to 0$

LES $\quad H^0(\overline{M}_X^{gp}) \to \mathrm{Pic}(X) \to H^1(M_X^{gp}) \to H^1(\overline{M}_X^{gp})$

S-points of LogPic

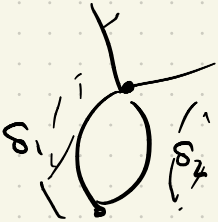
For any log scheme

$$H^0(\overline{M}_X^{gp}) = P\mathcal{L}(C(X))$$

(Pf: $H^0(\overline{M}_X^{gp}) = \mathrm{Hom}(X, \mathcal{A}_{\mathbb{N}}) = \mathrm{Hom}(\mathcal{A}_X, \mathcal{A}_{\mathbb{N}})$

$= \mathrm{Hom}(C(X), \mathbb{N})$)

Explicitly here



$$PL(X) = \{ f \in (\overline{M}_s)^{V(X)} : \forall e \text{ with}$$

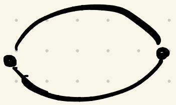
slope

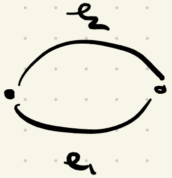
$$end(e) = w, in(e) = v,$$

$$\exists\, s(e) \in \mathbb{Z} \text{ s.t}$$

$$f(w) - f(v) = s(e)\delta_e \}$$

X





$\overline{M}_s = \mathbb{R}_{\geq 0}$

$\mathbb{R}_{\geq 0} \times \mathbb{Z} = PL(X)$

$\overline{M}_s = \mathbb{N}^2$

$PL(X) = \overline{M}_s^{sp}$

Remark: In toric case, $H^0(\overline{M}_X^{gp}) \to \mathrm{Pic}(X)$

is the homomorphisms $PP^1(F) \longrightarrow A^1(X)$

of lecture 1

<u>**Miracle**</u>: For any log blowup $\tilde{X} \to X$

$$H^1(\tilde{X}, M_{\tilde{X}}^{gp}) = H^1(X, M_X^{gp})$$

Thm (MW) $\quad \mathrm{Pic}(\tilde{X}) \to \mathrm{LogPic}(\tilde{X}) = \mathrm{LogPic}(X)$

cover $\mathrm{LogPic}(X)$ as $\tilde{X} \to X$ ranges over

all log blowups of X.

"Formula" : $\mathrm{LogPic}(X) \text{ "="} \varinjlim_{\tilde{X} \to X} \mathrm{Pic}(\tilde{X}) \big/ P\ell(\tilde{X})$

Log line bundles = equivalence classes

of line bundles on s.s.

models up to action of

$P\ell(\tilde{X})$.

Description 2: Try to find $C(\mathrm{LogPic}(X/S))$

$$:= \mathrm{TroPic}(\mathcal{X})$$

But how? LogPic is only given as a functor. So I can try to give

TroPic as a functor on cone complexes

↓

affine cones

↓

monoids

(think: CFG over Sch ⟷ CoFG over Rings)

Idea: Write corresponding moduli problem on $\mathcal{X}$.

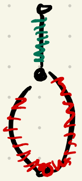
Pathway $\bar{M}_X^{gp} \longleftrightarrow P\mathcal{L}(\mathcal{X})$

$M_X^{gp} \longleftrightarrow \mathcal{L}(\mathcal{X})$

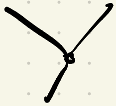
I'll define in a minute

Geometry on $\mathfrak{X}$ ("Tropical Geometry")

(i) $\mathfrak{X}$ has a topology generated by stars of strata



(ii) $PL(X)$ is a sheaf on $\mathfrak{X}$



$$PL = \bar{M}_S^{gp} \times \mathbb{Z}^e$$

$$PL = \bar{M}_S^{gp} \times \mathbb{Z}$$

(iii) There is a sheaf of divisors Div

$$Div(U) = \bigoplus_{v \in U} \mathbb{Z}$$

and a map

$$ord_v : PL \longrightarrow \mathbb{Z}$$

$$ord_v(f) = \sum_{v \in e} s(e)$$

(iv) $\exists \ \mathrm{div} : PL \longrightarrow \mathrm{Div}$

$$\mathrm{div}(f) = \sum (\mathrm{ord}_v f)\, v$$

Def: $\mathcal{L} = \ker \mathrm{div}$

$$= \{ f \in PL(\mathfrak{X}) : \sum_{v \in e} s(e) = 0 \ \forall v \}$$

("balanced functions")

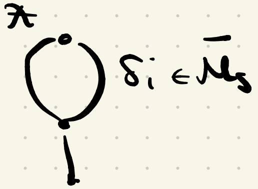
LogPic : Given $T \to S$, form $X \times_S T$, take $H^1(M_{X \times_S T}^{\,gp})$

$$T \longrightarrow S \rightsquigarrow M_T^\vee \longrightarrow M_S^\vee$$

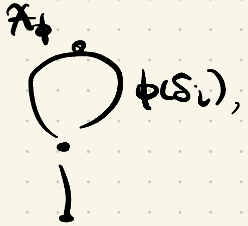
$$\longleftrightarrow$$

$$M_S \longrightarrow M_T$$

Given $\phi : M_S \longrightarrow P$,



$\rightsquigarrow$



$\mathcal{L}_\phi$ = P-valued linear functions on $\mathfrak{X}_\phi$

Def $\mathrm{TroPic}(\phi : M_S \to P) = H^1(\mathfrak{X}_\phi, \mathcal{L}_\phi)^\dagger$

Fun calculation (ϕ-points) ("harmonic" differentials)

$$0 \to P^{gp} \to \mathcal{L}_\phi^{gp} \to \mathcal{H} \to 0$$

$$\begin{tikzcd}
\rightsquigarrow 0 \arrow[r] & \mathcal{H}^0(\mathcal{H}) = H_1(\mathcal{X}) \arrow[r] & \underbrace{H^1(\mathcal{X}, P^{gp})}_{\mathrm{Hom}(H_1(\mathcal{X}), P^{gp})} \arrow[r] & \mathrm{TroPic} \arrow[r, "\text{"deg"}"] & H^1(\mathcal{H}) \arrow[d, phantom, "\wr"] \\
& & & & H_0(\mathcal{X}) \arrow[d, phantom, "\wr"] \\
& & & & \mathbb{Z}
\end{tikzcd}$$

$\mathcal{H}_1(\mathcal{X}) \longrightarrow \mathrm{Hom}(H_1(\mathcal{X}), P^{gp})$ intersection pairing

induced by $\mathbb{Z}^{E(X)} \times \mathbb{Z}^{E(X)} \longrightarrow \bar{M}_S^{gp}$

$$\langle e, e' \rangle = \delta_{ene'}$$

Degree 0 piece

$$\mathrm{TroJac}(\mathcal{X}) = \mathrm{Hom}(H_1(\mathcal{X}), P^{gp}) / \mathcal{H}_1(\mathcal{X})$$

$$= T^{trop} / H_1(\mathcal{X})$$

$$T = \mathrm{Spec}\, k[H_1(\mathcal{X})]$$

Thm (MW)

There is a tropicalization map $\mathrm{LogPic}(X/S) \to \mathrm{TroPic}(X)$

giving an exact sequence of sheaves

$$0 \to \mathrm{Pic}^{[0]}(X/S) \to \mathrm{LogPic}(X/S) \to \mathrm{TroPic}(X/S) \to 0$$

Caution: LogPic is not representable

e.g its cone complex in deg 0 in TroJac,

looks like an unsubdivided real torus

$\longrightarrow \mathrm{TroJac}(\mathbb{R}_{\geq 0})$

$\overline{M}_S = \mathbb{R}_{\geq 0}$

$= \mathbb{R}^2/\mathbb{Z}^2$

$\simeq$

However :

Thm (KKN) Subdivisions of TroJac

$\longleftrightarrow$ $H_1(X)$-invariant subdivisions of T^{trop}

$\longleftrightarrow$ Subdivisions of $\mathcal{L}og\mathrm{Pic}^0(X/S)$

correspond precisely to toroidal compactifications of $\mathrm{Pic}^{[0]}(X/S)$.

Very much like toy model:

- $\mathcal{L}og\mathrm{Pic}$ a group
- not representable
- In $\mathcal{C}$, representable by any toroidal compactification of $\mathrm{Pic}^{[0]}$.

Symbolically

$$\mathcal{L}og\mathrm{Pic} = \varprojlim (\text{Toroidal compactifications})$$

Perpendicular statement For X/S $\swarrow$ log smooth

(HMOP) The subgroups of $\mathrm{TroJac}(X)(\bar{M}_S)$ correspond to group models of $\mathrm{Pic}^{[0]}(X/S)$.

The finite subgroups correspond to the separated group models.

In particular, (separated) Néron model exits $\iff$ $\mathrm{TroJac}(X)(\bar{M}_S)$ is finite.

Relation w/ DR. Fix $\alpha = (\alpha_1, \dots, \alpha_n)$, $\Sigma a_i = 0$

On $M_{g,n}$, $\exists$ $aj : M_{g,n} \longrightarrow \mathrm{Pic}^0(C_{g,n})$

$$(C, x_1, \dots, x_n) \longrightarrow \mathcal{O}_C(\Sigma a_i x_i)$$

$$DR|_{M_{g,n}} = aj^{-1}(0).$$

On $\overline{M}_{g,n}$, aj does not extend.

Can define DR by GW theory (next talk)

Blowing up $\overline{M}_{g,n}$ to resolve aj (Holmes, Marcus-Wise)

aj extends to a map $\overline{M}_{g,n} \longrightarrow \mathrm{LogPic}^0(\overline{C}_{g,n}/\overline{M}_{g,n})$

$$\begin{tikzcd}
DR \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{g,n} \arrow[d, "0"] \\
\overline{M}_{g,n} \arrow[r, "aj"'] & \mathrm{LogPic}^0(\overline{C}_{g,n}/\overline{M}_{g,n})
\end{tikzcd}$$

But this only gives scheme structure.

Cycle structure — I don't know

There is a problem: What A^*

can we use, since LogPic is not representable?

There are candidates.

For a log scheme X, can define

$$A_*^{\log}(X) = \varprojlim_{\tilde{X} \to X} A_*(\tilde{X})$$

Chow homology theory, natural from point of view of log Gromov-Witten theory.

(Not a ring)

Chow cohomology

$$\text{(Barrott)} \quad A^*_{\log}(X) = \varinjlim_{\tilde{X} \to X} A^*(\tilde{X})$$

(this is what Rahul called LogCh in his talk)

e.g for $\mathrm{LogPic}(X/S)$, this makes sense

as $\varinjlim A^*$ (Toroidal models of $\mathrm{Pic}^{[0]}(X)$)

and is the Chow theory of a group object.

Hope that identities proved by formal methods for $D|_{M_{g,n}}$ would extend in appropriate LogCh.

this is a bit subtle. I'll discuss next time.

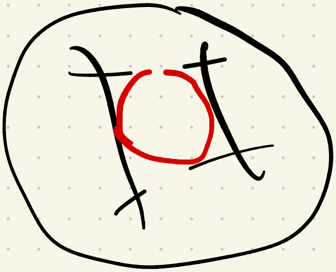
Lecture 4 (Based on ongoing work with D. Ranganathan)

Log Stable maps.

Fix V log smooth scheme, i.e. toroidal pair (V,D).

Fix discrete data $\Gamma = (g, n, \underset{H_2(V,\mathbb{Z})}{\underset{\ni}{\beta}}, \overset{\text{contact orders}}{\vec{\alpha}})$

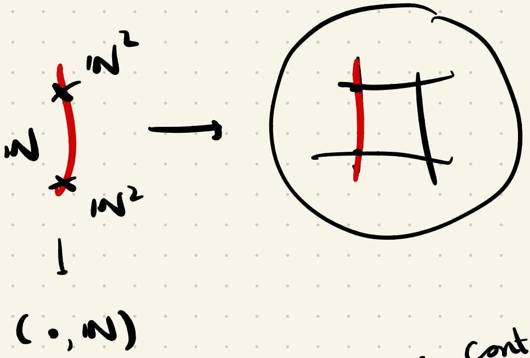
Problem: Compactify space of maps into V with discrete data Γ.

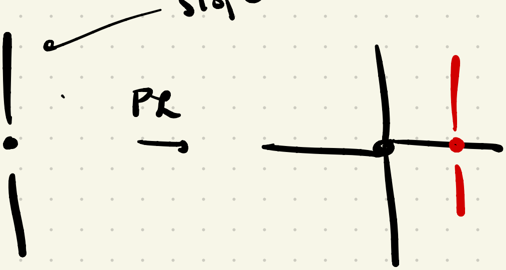


In the limit, you may lose contact order completely if a component of C falls into D, or the contact order may want to "jump" to a different component of D.

Idea: A log map can regain the contact order information even in the degenerate situations, essentially via the cone complexes

Example





slope keeps contact orders

Def (ACGS)

$$K_\Gamma(V)(S) = \left\{ \begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\pi"'] & V \times S \arrow[dl] \\ S & \end{tikzcd} \;\middle|\; \begin{array}{l} \pi \text{ log curve} \\ f \text{ log map w/ data } \Gamma \end{array} \right\}$$

Thm $K_\Gamma(V)$ is representable by a proper algebraic stack with a log structure

- $K_\Gamma(V)$ is singular, but has a virtual fundamental class.

In GW theory of a smooth target W

$$\begin{tikzcd} \text{universal family} \arrow[dr] & & \\ & \mathcal{U}_\Gamma(W) \arrow[r, "f"] \arrow[d, "\pi"'] & W \\ & \mathcal{M}_\Gamma(W) & \end{tikzcd}$$

$\mathcal{E}^\bullet = (R\pi_* f^* \underset{\nearrow\ \text{v. bundle}}{T_W})^\vee$ is a perfect obstruction theory relative to $\underset{\nwarrow\ \text{smooth}}{\overline{\mathcal{M}}_{g,n}}$

i.e, once one fixes C, deformations and obstructions to deformations of maps $C \to W$ are governed by T_W

In the log setting,

- want to deform log map
- The log structure can also deform in families.

Analogous statement

$$\begin{tikzcd}
\mathcal{U}_\Gamma(V) \arrow[r, "f"] \arrow[d, "\pi"'] & V \\
K_\Gamma(V)
\end{tikzcd}$$

$\mathcal{E}^\bullet = \left(R\pi_* f^* T_V^{\log}\right)^\vee$ is p.o.t relative to $\mathrm{Log}_{\overline{\mathcal{M}}_{g,n}}$ ← prestable

v. bundle because V log smooth

log smooth

i.e.

fixing source + log structure,

deformations/obstructions of log maps are governed by $T_V^{\log}$

remember

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & \mathrm{Log} \\ & \mathcal{A}_X \arrow[u] \end{tikzcd}$$

$$= \lim_{\to} \left[\mathrm{Spec}\, k[M_{X,x}] \big/ \mathrm{Spec}\, k[M_{X,x}^{gp}] \right]$$

$\mathrm{Log}_{\overline{\mathcal{M}}_{g,n}}$ is a huge stack

$K_\Gamma(V) \longrightarrow \mathrm{Log}_{\overline{\mathcal{M}}_{g,n}}$ factors through

$$\begin{tikzcd} K_\Gamma(V) \arrow[r] \arrow[dr, "p"'] & \mathrm{Log}_{\overline{\mathcal{M}}_{g,n}} \\ & K_\Gamma(\mathcal{A}_V) \arrow[u, dashed] \end{tikzcd}$$

← étale, strict

$\underbrace{K_\Gamma(\mathcal{A}_V)}$

log maps from prestable curves to Artin fan $\mathcal{A}_V$

This stack is quasicompact

Conclusion: $\mathcal{E}^\bullet$ is p.o.t for $K_\Gamma(V)$ relative

to $K_\Gamma(\mathcal{A}_V)$

$\nearrow$ log smooth, so equidimensional

$$[K_\Gamma(V)]^{vir} := p_E^! [K_\Gamma(\mathcal{A}_V)]$$

Remark: When $V = (V, D)$ $\leftarrow$ smooth

there is familiar compactification

due to Jun Li

$\mathcal{M}_\Gamma^{rel}(V)$ = maps to expansions

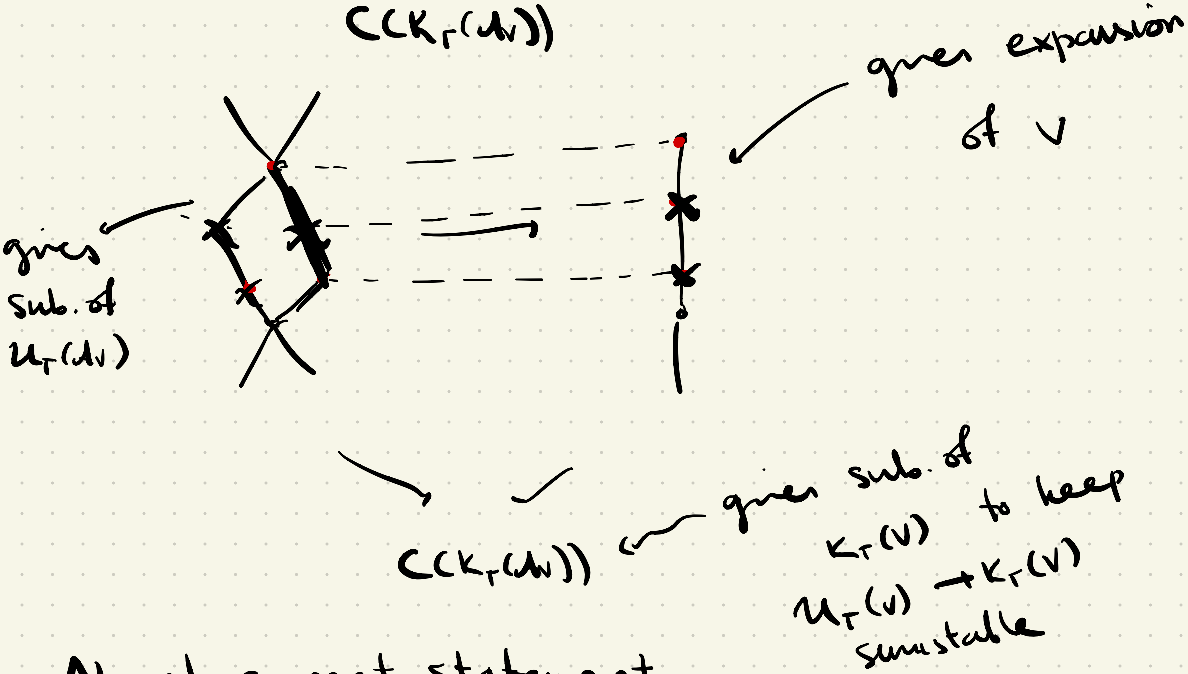
If C tends into D, expand V and remember

contact order there

In fact, this is essentially the same solution.

Idea: look at map of cone complexes

$$\begin{tikzcd}
C(\mathcal{U}_\Gamma(\mathcal{A}_V)) \arrow[r, "f"] \arrow[d, "\pi"] & C(V) \times C(K_\Gamma(\mathcal{A}_V)) \arrow[dl] \\
C(K_\Gamma(\mathcal{A}_V)) &
\end{tikzcd}$$



Almost correct statement

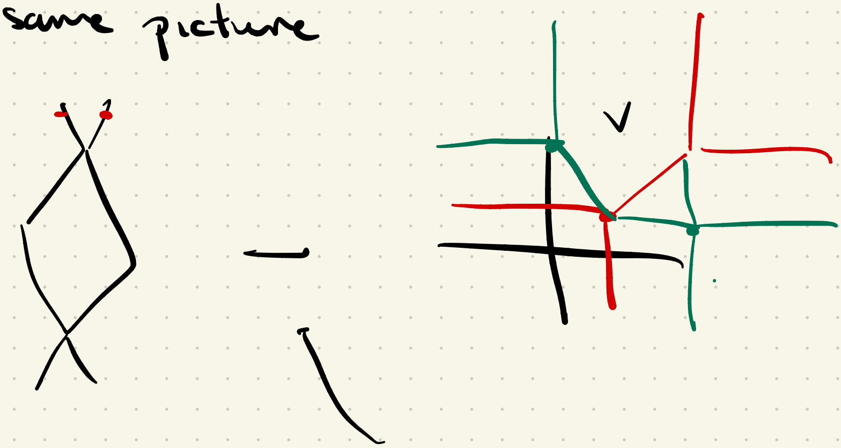
$\mathcal{M}_\Gamma^{rel}(V)$ is obtained from $K_\Gamma(V)$ by performing weak semistable reduction to the universal map.

So $\mathcal{M}_\Gamma^{\mathrm{rel}}(V) \xrightarrow{p} K_\Gamma(V)$ is a subdivision

(in fact map only exists after saturation/ normalization)

So its log étale.

Even for arbitrary V, you can look at the same picture



perform s.s. reduction

Gives (non-canonically) moduli space to expansions of V

This is the approach of Ranganathan

We still have a log étale map

$$\mathcal{M}_\Gamma^{rel}(V) \xrightarrow{\phi} K_\Gamma(V)$$

subdivision root stack

These maps respect virtual classes:

$$\phi_*[\mathcal{M}_\Gamma^{rel}(V)]^{vir} = [K_\Gamma(V)]^{vir}$$

In fact, the same is true for any

subdivision $W \longrightarrow V$

$$K_\Gamma(W) \xrightarrow{\phi} K_\Gamma(V)$$

$$\phi_*[K_\Gamma(W)]^{vir} = [K_\Gamma(V)]^{vir}$$

The reason is that all these maps are log étale, so they do not change

$$\mathcal{E}^\bullet = \underline{\underline{(R\pi_* f^* T_V^{\log})^\vee}}$$

More technically: For any subdivision

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & K_\Gamma(V) \arrow[d] \\ \mathfrak{X} \arrow[r] & K_\Gamma(\mathcal{A}_V) \end{tikzcd}$$

can give $X \to \mathfrak{X}$ the pullback P.O.T,

and $\phi_*[X]^{vir} = [K_\Gamma(V)]^{vir}$

When $\mathfrak{X} \to K_\Gamma(V)$ has a modular interpretation as in the previous cases, the pullback obstruction theory coincides with the modular one.

Most compact way to phrase this:

$$[K_\Gamma(V)]^{vir} \in A_*^{log}(K_\Gamma(V)) := \varprojlim_{\substack{X \to K_\Gamma(V) \\ \text{log blowups}}} A_*(X)$$

Double Ramification cycle.

$V = (\mathbb{P}^1, 0, \infty)$

$\Gamma = (g, n, b, \vec{\alpha})$ n determined by g,

$\alpha = (a_1, \dots, a_n)$, $\sum a_i = 0$.

Consider $\mathcal{M}_\Gamma^{rub}(V) = \left\{ f \in \mathcal{M}_\Gamma^{rel}(V) \text{ up to } \mathbb{G}_m\text{-action on } \mathbb{P}^1 \right\}$

Same as before, $\mathcal{M}_\Gamma^{rub}$ has a virtual class

$$\underset{\sim}{DR} = \underset{\sim}{DR}_{g,\alpha} := [\mathcal{M}_\Gamma^{rub}]^{vir} \in A_*(\mathcal{M}_\Gamma^{rub})$$

and $DR := \varepsilon_* \underset{\sim}{DR} \in A_*(\overline{\mathcal{M}}_{g,n})$,

where $\varepsilon : \mathcal{M}_\Gamma^{rub} \longrightarrow \overline{\mathcal{M}}_{g,n}$ the obvious map.

DR is a cycle of codimension g,

supported on locus

$$\left\{ (C, x_1, \dots, x_n) \mid \mathcal{O}(\sum a_i x_i) = \mathcal{O} \text{ in } \mathcal{L}og\mathcal{P}ic(\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n}) \right\}$$

By previous remark, for every log blowup

$\pi : \widetilde{\mathcal{M}}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}$, can get class

$$\begin{tikzcd}
\widetilde{M}^{rub}(v) \arrow[r] \arrow[d, "\delta"'] \arrow[dr, phantom, "f.s."] & M^{rub}(v) \arrow[d, "\varepsilon"] \\
\widetilde{\mathcal{M}}_{g,n} \arrow[r, "\pi"'] & \overline{\mathcal{M}}_{g,n}
\end{tikzcd}$$

$\boldsymbol{\mathcal{M}}$ = st

M = prest.

$\widetilde{DR} = \delta_* [\widetilde{M}_{rub}(v)]^{vir}$; so there is

finer class

$DR \in A_*^{log}(\overline{\mathcal{M}}_{g,n})$.

Reasonable question: Is $\pi^* DR = \widetilde{DR}$?

Answer is no.

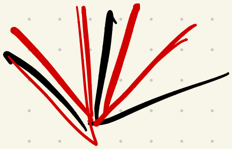
Reason is that the diagram is not

Cartesian; Cartesian only in f.s.

Nevertheless, we can take a sufficiently fine log blowup $\tilde{M}_\Gamma \to \bar{M}_\Gamma$ s.t

$$\begin{tikzcd}
\tilde{\tilde{M}}_\Gamma^{rub} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] \arrow[dr, phantom, "f.s = sch."] & \tilde{M}_\Gamma^{rub} \arrow[r] \arrow[d] & \boxed{M_\Gamma^{rub}} \arrow[d] \\
\tilde{\tilde{M}}_\Gamma \arrow[r] & \tilde{M}_\Gamma \arrow[r] & \bar{M}_\Gamma
\end{tikzcd}$$

weakly s.s.



Then everything further along stabilizes, and we get a class $DR^{log} \in A^*_{log}(\bar{M}_{g,n})$
$= \varinjlim A^*(\tilde{M}_{g,n})$

which is not pulled back from $A^*(\bar{M}_{g,n})$.

Language for the situation

$f : X \to Y$ map of log smooth and smooth log schemes (more generally: log lci map)

Define new operation $f^*_{\log}$ by

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "p"'] & Y' \arrow[d] \arrow[r, phantom, "f.s."] & V' \arrow[l] \arrow[d] \\ X \arrow[r, "f"] & Y & V \arrow[l, "\phi"] \end{tikzcd}$$

Disclaimer

$f^*, f^!$

with g weakly semistable; g becomes lci

$$f^*_{\log} V = p_* g^! V' \in A^*(X)$$

Case of interest: $X \xrightarrow{f} Y$ log blowup. In this case

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, equal] & X \arrow[d] \arrow[r, phantom, "f.s."] & V' \arrow[l] \arrow[d] \\ X \arrow[r, "f"] & Y & V \arrow[l] \end{tikzcd}$$

$V \leftarrow$ generically 0 log structure

$f^!_{\log} V$ = strict transform.

<u>Caution</u> (i) $f^!_{\log}$ depends on choice of $\phi : V \to Y$

and <u>does not respect</u> rational equivalence

(ii) Nevertheless, $f^!_{\log}$ respects rational equivalence as long as the combinatorial type of $C(V) \longrightarrow C(Y)$ does not change.

(iii) We have

$$\pi^!_{\log} DR = \widetilde{DR}$$ in previous notation.

(iv) Once f is semistable with respect to ϕ, $f^!_{\log}$ "stabilises" and $= f^!$ giving a class in $A^*_{\log}(X)$.

"Double" double ramification locus.

Now fix 2 vectors of contact orders

$$\vec{a} = (a_1, \dots, a_n), \quad \vec{b} = (b_1, \dots, b_m)$$

$$\sum a_i = 0 \qquad \sum b_i = 0$$

Γ all discrete data.

Have

$$\mathcal{M}_\Gamma^{rub}(\mathbb{P}^1 \times \mathbb{P}^1) = \left\{ \text{rel. maps } C \to \mathbb{P}^1 \times \mathbb{P}^1 \text{ up to } \mathbb{G}_m^2 \text{ action} \right\}$$

$$\downarrow \varepsilon$$

$$\overline{\mathcal{M}}_\Gamma$$

$$\mathrm{DDR} := \mathrm{DDR}_\Gamma = [\mathcal{M}_\Gamma^{rub}(\mathbb{P}^1 \times \mathbb{P}^1)^{vir}]$$

$$\mathrm{DDR} = \varepsilon_* \mathrm{DDR} \in A^{2g}(\overline{\mathcal{M}}_\Gamma).$$

Supported on locus

$$\left\{ (C, x_1, \dots x_n, y_1, \dots y_m) \mid \mathcal{O}(\Sigma a_i x_i) = \mathcal{O}(\Sigma b_i y_i) = 0 \text{ in } \mathcal{L}og\mathrm{Pic}(\overline{C}_\Gamma / \overline{\mathcal{M}}_\Gamma) \right\}$$

As before:

- $\mathrm{DDR} \in A^{\log}_*(\overline{\mathcal{M}}_\Gamma)$
- For a log blowup $\pi : \widetilde{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_\Gamma$,

$$\begin{tikzcd}
\mathcal{M}_\Gamma^{\mathrm{rub}}(\mathbb{P}^1\times\mathbb{P}^1) \arrow[r] \arrow[d, "\delta"'] & \mathcal{M}_\Gamma^{\mathrm{rub}}(\mathbb{P}^1\times\mathbb{P}^1) \arrow[d] \\
\widetilde{\mathcal{M}}_\Gamma \arrow[r, "\pi"] & \overline{\mathcal{M}}_\Gamma
\end{tikzcd}$$

$$\pi^*\mathrm{DDR} \neq \widetilde{\mathrm{DDR}} := \delta_*[\mathcal{M}_\Gamma^{\mathrm{rub}}(\mathbb{P}^1\times\mathbb{P}^1)]^{\mathrm{vir}}$$

$$\pi^!_{\log}\mathrm{DDR} = \widetilde{\mathrm{DDR}}$$

- $\pi^!_{\log}$ eventually stabilises to π^*,

giving class $\qquad \varinjlim A^*(\widetilde{\mathcal{M}}_\Gamma)$

$\qquad\qquad\qquad\qquad \shortparallel$

$\mathrm{DDR}^{\log} \in A^*_{\log}(\overline{\mathcal{M}}_\Gamma)$, not

pulled back by $\underline{A^*(\overline{\mathcal{M}}_\Gamma)}$.

Reasonable question:

$$\begin{tikzcd}
\bar{\mathcal{M}}_\Gamma \arrow[r, "a"] \arrow[dr, bend right, "b"'] & \vec{\mathcal{M}}_{g,n} \\
& \vec{\mathcal{M}}_{g,m}
\end{tikzcd}$$

Is $a^* DR_{\vec{\alpha}} \cdot b^* DR_{\vec{\beta}} = DDR_\Gamma$?

For example, over $\mathcal{M}_\Gamma$,

$$\{(C, x_1, \dots, x_n, y_1, \dots y_m) \mid \mathcal{O}(\Sigma a_i x_i) \simeq \mathcal{O}(\Sigma b_i y_i) \simeq \mathcal{O}\}$$

$$\simeq \{(C, x_1, \dots x_n, y_1, \dots y_m) \mid \mathcal{O}(\Sigma a_i x_i) \simeq \mathcal{O} \text{ and } \mathcal{O}(\Sigma b_i y_i) \simeq \mathcal{O}\}$$

is almost tautology.

Answer: $\underline{\text{No}}$

Naive product formula fails in log GW theory.

$$\begin{tikzcd}
K_\Gamma(V\times W) \arrow[r, "h"] \arrow[d, "f.s"] & Q \arrow[r] \arrow[d] \arrow[dr, phantom, "f.s"] & K_\Gamma(V)\times K_\Gamma(W) \arrow[d] \\
\mathcal{D} \arrow[r] & Q' \arrow[r] \arrow[d] \arrow[dr, phantom, "f.s"] & \overline{\mathcal{M}}_\Gamma \times \overline{\mathcal{M}}_\Gamma \arrow[d] \\
 & \overline{\mathcal{M}}_\Gamma \arrow[r, "\Delta"] & \overline{\mathcal{M}}_\Gamma \times \overline{\mathcal{M}}_\Gamma
\end{tikzcd}$$

Then $\Delta^! [K_\Gamma(V)^{vir} \times K_\Gamma(W)^{vir}] \neq h_* [K_\Gamma(V\times W)]^{vir}$

Need

$$\underline{\Delta^!_{log}}$$

Put otherwise, what is true is that $\exists$ some $\pi : \widetilde{\mathcal{M}}_\Gamma \longrightarrow \overline{\mathcal{M}}_\Gamma$ s.t

$$\pi^!_{log}\, DDR = \pi^!_{log}\, DR \;\; \pi^!_{log}\, DR$$

i.e.

$$DDR^{log} = DR^{log} . DR^{log}$$

if we are interested in difference

$$DDR - DR \cdot DR \quad \text{in} \quad A^*(\overline{\mathcal{M}}_{g,n})$$

need some calculus for $f^!_{\log}$ and $A^*_{\log}$.

Fulton tells us how to do it.

Set up.

blowup along X

exceptional

$$\begin{tikzcd} \tilde{X} \arrow[r, "j"] \arrow[d, "g"'] & \tilde{Y} \arrow[d, "f"] \\ X \arrow[r, "i"] & Y \end{tikzcd}$$

variety

center of blowup

$X \subset Y$ is regularly embedded

For $V \subset Y$,

class supported on $X \cap V$

$$f^* V = \text{strict transform} + j_*\left(c(E) \cdot g^* \boxed{s(X \cap V, V)}\right)$$

piece of correct dim.

$E = g^* N_{X/Y} / N_{\tilde{X}/\tilde{Y}}$ = excess bundle

$s(X \cap V, V)$ = Segre class

In our language

$$\begin{tikzcd}
\tilde{X} \arrow[r] \arrow[d] & \tilde{Y} \arrow[d] & \\
X \arrow[r] & Y & V \arrow[l, "\phi"']
\end{tikzcd}$$

smooth

log blowup

(DR)

generic log str. 0.

$(\bar{M}_Y)$

smooth

$X \subset Y$ corresponds to <u>monomial</u> ideal

$$I \subset \bar{M}_X$$

Form

$$\begin{tikzcd}
V \times_Y \tilde{X} \arrow[r, "j"] \arrow[d, "g"'] & V \times_Y \tilde{Y} \arrow[d, "f"] \\
V \times_Y X \arrow[r, "i"] & V
\end{tikzcd}$$

$$f^*[V] = f^!_{\log} V + j_*\left(c(E) \cdot g^* \underbrace{s(X \times_Y V, V)}\right)$$

in principle Segre class is hard to compute.

But $X \times_Y V$ is <u>monomial</u> in V

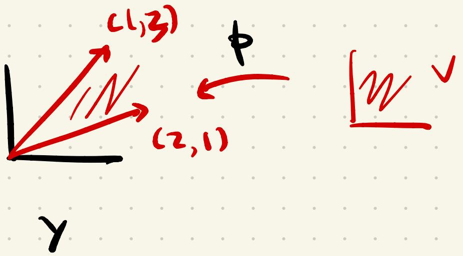
Aluffi: Amazing formula.

For a monomial subscheme S w.r.t normal crossings divisors $D_1, \dots, D_n$ in smooth V, form Newton region for S

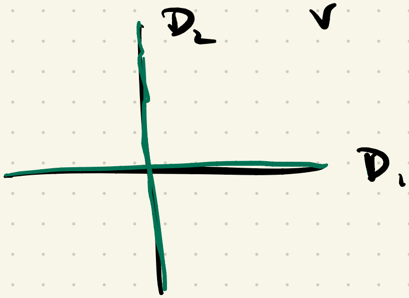
$X \subset$

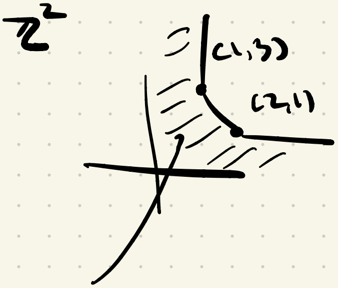
$\|$

(x,y)



$$\phi^*(x,y) = (D_1^2 D_2, D_1 D_2^3)$$





N = Newton region

$$s(S,V) = \int_N \frac{n!\, D_1 \cdots D_n}{(1 + x_1 D_1 + \cdots + x_n D_n)^{n+1}} dx_1 \cdots dx_n$$

$$= \sum (\text{coefficient}) \cdot \text{Monomial in } D_i$$

$$D_1^{\alpha_1} \cdots D_k^{\alpha_k}$$

How to interpret formula: each monomial picks out a stratum of V supported on S, and an excess class supported on stratum.

For $s(X \times_Y V, V)$, strata correspond to components of $\overline{C(X)} \times_{\overline{C(Y)}} \overline{C(V)}$, and each stratum is a projective bundle over a stratum of V. Each term in $s(X \times_Y V, V)$ gives an excess class on such a stratum.

Question

$$\begin{tikzcd}
\tilde{M}_\Gamma \arrow[rrr] \arrow[d] & & & M_\Gamma \arrow[d] \\
\tilde{M}_{\Gamma_1} \arrow[r] & \tilde{M}_{\Gamma_2} \arrow[r, dashed] & \cdots \arrow[r] & \tilde{M}_{\Gamma_N}
\end{tikzcd}$$

a blowup along smooth center

Question to the audience

$$\begin{tikzcd}
\tilde{X} \arrow[r] \arrow[d] & \tilde{Y} \arrow[d] & \text{sm.} \arrow[l] \\
X \arrow[r] & Y & \text{sm.} \arrow[l] \\
V(J) \arrow[u] & &
\end{tikzcd}$$

Does $X \to Y$ have to be lci?

More careful analysis (still not optimal), using the map $C(V) \longrightarrow C(Y)$ gives that the image of

$f^* v - f^*_{\log} v$ in $A^*(\tilde{Y})$ has form

$v \in A^q(Y)$

$$\boxed{PP(\tilde{Y}) \cdot (g^* PP(Y) \cdot [v])}$$

Corollary: DDR is in tautological ring.

Proof Let $\tilde{M}_T \longrightarrow \boxed{\bar{M}_T}$ be a log blowup where $\widetilde{DDR} = \widetilde{DR}\,\widetilde{DR}$

$\widetilde{DR} = \pi^!_{\log} DR$

Then $\underline{\widetilde{DDR}} = (\boxed{\pi^* DR - \alpha})(\boxed{\pi^* DR - b})$

with α, b of form $PP(\tilde{M}_T)(g^* PP(\bar{M}_T) \cdot DR)$

Then $DDR = \underline{DR^2} - \underbrace{\pi_*(\alpha \cdot b)}_{\in R(\bar{M}_{g,n})}$

$\pi_*(\pi^* DR \times b)$

$= 0$

Open Questions.

(1) Streamline the analysis of computing

$$f^! - f^!_{\log}$$

In principle, every piece is explicitly computable, but too difficult to do by hand.

(2) Write a program to compute.

Need to combine combinatorics of fans, admcycles, Aluffi's formula

(3) Write Aluffi's formula in a form that's in line with decomposition of $X \times_Y Z$ in terms of $\overline{C(X)} \times_{\overline{C(Y)}} \overline{C(Z)}$ instead of Newton region.

$$A^*_{\log} = \log Ch / PP$$

$$//$$

From log point of view,

$$\mathrm{Log}\,A^*(X) = \varinjlim_{\tilde{X} \to X} A^*(\tilde{X}) / PP(\tilde{X})$$

is more natural — e.g., for log curve X/S

$$\mathrm{Log}\,A^1(X) \simeq \mathrm{Log}\,\mathrm{Pic}(X)$$

and

$A^*_{\log}(X)$ corresponds to "cover"

$$\varinjlim_{Y \to X} \mathrm{Pic}(Y) \Big/ \underline{PL(Y)} \quad \text{“=”} \quad \mathrm{Log}\,\mathrm{Pic}$$

(4) Can define

$$\mathrm{Log}\,R(\overline{\mathcal{M}}_{g,n}) = \varinjlim_{\tilde{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}} \frac{\langle R(\tilde{\mathcal{M}}_{g,n}), PP(\tilde{\mathcal{M}}_{g,n}) \rangle}{PP(\tilde{\mathcal{M}}_{g,n})}$$

$$\uparrow$$

What is this? ←

(no idea)

- $f^!_{\log} = f^!$ in $\log A^*(X)$

so $DDR = DR^2$ is easy in $\log A^*(X)$

(also related statements such that

$PR_{\vec a} DR_{\vec b} = DR_{\vec a + \vec b} DR_{\vec b}$ for HPS)

- Expect $\dfrac{\Theta^g}{g!} = DR$ to hold in $\log A^*$ (g-wise)

(5)
But if you are interested in $A^*(\bar{M}_{g,n})$, not clear how to lift equalities there - e.g, the result $\lambda_g \in \mathrm{div}\, \mathrm{Log}\, \mathrm{CH}$ is more precise than corresponding statement $\lambda_g = 0$ in $\log A^*$.

(6) What is the structure of $\log A^*(X)$? Is it finite dimensional? Does it satisfy Poincaré duality for X log smooth?

(7) What is the corresponding homology theory $\log A_*(X)$ of

log cycles / log rational equivalence?

$\uparrow$ Hard

THANKS