The Log Tautological Ring

joint w/ R. Pandharipande, J. Schmitt.

Motivation

$$\begin{array}{c} \overline{\mathcal{C}}_g \\ \big\downarrow \pi \\ \overline{\mathcal{M}}_g \end{array} \qquad \mathbb{E}_g = \pi_* \omega_\pi$$

rank g vector bundle

"Hodge bundle"

$$\lambda_g = c_g(\mathbb{E}_g)$$

Over the locus $\mathcal{M}_g^{ct}$ of curves of compact type, λ_g satisfies a remarkable formula.

$$\begin{tikzcd} \mathrm{Pic}^0(\mathcal{C}_g^{ct}) \arrow[r, "\tilde{t}"] \arrow[d] & \mathcal{X}_g \arrow[d] \\ \mathcal{M}_g^{ct} \arrow[u, bend left, "0"] \arrow[r, "t"'] & \mathcal{A}_g \end{tikzcd}$$

$\mathcal{X}_g \supset [Z_g]$ class of 0-section

Torelli map

$$\lambda_g = (-1)^g\, \tilde{t}^*[Z_g]$$

But

Theta divisor

$$Z_g = \Theta^g/g! \quad \text{(Denninger - Murre)}$$

Consequently,

Thm (Grushevsky - Zakharov)

$$\lambda_g = D^g/g! \quad \text{for some } D \in CH^1(\mathcal{M}_g^{ct})$$

Question: Does the formula extend to $\overline{\mathcal{M}}_g$?

For example, diagram does extend

$$\begin{tikzcd} \bar{\mathcal{X}}_g \times_{\bar{\mathcal{A}}_g} \bar{\mathcal{M}}_g \arrow[r] \arrow[d] & \bar{\mathcal{X}}_g \arrow[d] \\ \bar{\mathcal{M}}_g \arrow[r] & \bar{\mathcal{A}}_g \end{tikzcd}$$

But answer is no, in a strong sense.

<u>Thm</u>: For $g \geq 3$, $\lambda_g \notin \operatorname{div} CH(\bar{\mathcal{M}}_g)$, i.e

λ_g is not a product of divisors.

Argument is an explicit calculation using "admcycles" + a recursive argument.

(Delecroix-Schmitt-van Zelm, Pixton)

Intuition from point of view of logarithmic geometry is that such a formula should hold

$$\begin{array}{c} \mathrm{LogPic}^0(\overline{C}_g) \\ \big\downarrow \; \circlearrowleft 0 \\ \overline{M}_g \end{array}$$

proper, smooth, group

has Θ divisor

All ingredients necessary for DM argument seem to be there.

Except: Question does not make sense

$\mathrm{LogPic}(\overline{C}_g)$ is not algebraic, and has no Chow groups.

However, there is a (log) blowup

$$P \longrightarrow \mathrm{LogPic}(\overline{C}_g)$$

scheme

So the natural place to ask the question is in

$$\log CH^*(\overline{\mathcal{M}}_g) \text{ ``}:=\text{''} \varinjlim_{\tilde{\mathcal{M}}_g \to \overline{\mathcal{M}}_g} CH^*(\tilde{\mathcal{M}}_g)$$

iterated blowups along boundary strata

<u>Thm</u> $\lambda_g \in \operatorname{div} \log CH(\overline{\mathcal{M}}_g)$

Or, more concretely,

$\exists\ \tilde{\mathcal{M}}_g \longrightarrow \overline{\mathcal{M}}_g$ a (log) blowup s.t

$\lambda_g \in \operatorname{div} CH^*(\tilde{\mathcal{M}}_g)$.

We will need to approach λ_g from different direction $\rightsquigarrow$ GW-theory

$A = (a_1, \dots, a_n) \in \mathbb{Z}^n$ s.t $\sum a_i = 0$

$$\overline{\mathcal{M}}_{g,A}(\mathbb{P}^1)^{\sim} = \left\{ C \longrightarrow \mathbb{P}^1 \text{ w/ ramification profile } A \text{ over } 0 \text{ and } \infty \right\} \big/ \mathbb{C}^* \text{ action}$$

$\overline{\mathcal{M}}_{g,A}(\mathbb{P}^1)^{\sim}$ has a virtual fundamental class

$$\overline{\mathcal{M}}_{g,A}(\mathbb{P}^1)^{\sim} \xrightarrow{p} \overline{\mathcal{M}}_{g,n}$$

$$\boxed{\begin{aligned} DR_{g,A} &:= p_*\left[\mathcal{M}_{g,A}(\mathbb{P}^1)^{\sim}\right]^{vir} \\ \lambda_g &= (-1)^g \, DR_{g,\emptyset} \end{aligned}}$$

(X, D) ~ smooth variety w/ normal crossings divisor

(more generally: toroidal embedding)

(X, D) has a "logarithmic structure".

System of local charts

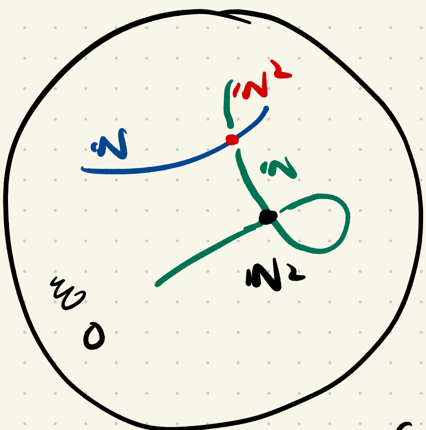
$\forall x \in X$

$$\begin{array}{ccc} \mathcal{U}_x & \xrightarrow{f = \text{smooth}} & \mathbb{A}^n \\ i \downarrow \text{étale} & & \\ X & & \end{array}$$

(or $V(\sigma_x)$ arbitrary affine toric variety)

s.t $f^{-1}(\mathbb{A}^n \setminus \mathbb{G}_m^n)$

$= \mathcal{U}_x \setminus i^{-1}(D)$



Stratifies X

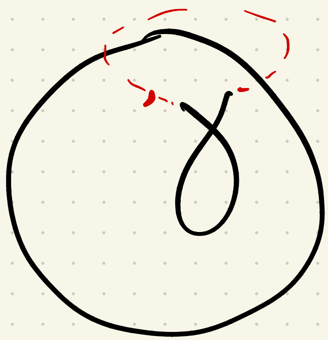
$\mathcal{M}_X(\mathcal{U}) = \{ f \in \mathcal{O}_X(\mathcal{U}) \mid f \text{ a unit on } \mathcal{U} \setminus D \}$

$\overline{\mathcal{M}}_X := \mathcal{M}_X / \mathcal{O}_X^*$

constructible

Strata where $\overline{\mathcal{M}}_X$ is locally constant

Subtlety: Strata can self-intersect and have monodromy



For experts: $\mathcal{M}_X$ is a sheaf on étale site of X, and can't be pulled back from Zariski site.

When (X,D) has no self intersection ("toroidal embedding w/o self-intersection") (X,D) were extensively studied in "KKMSD".

But for $\overline{\mathcal{M}}_{g,n}$ dealing w/ monodromy is essential.

Construction 1.

Assume (X, D) has no self-intersection

$$\begin{array}{ccc} \mathcal{U}_x & \longrightarrow & V(\sigma_x) \\ \downarrow & & \text{"}\overline{M}_{X,x}^{\vee}\text{"} \\ X & & \end{array}$$

For the generic point x_S of each stratum,

have a cone σ_x

When $x_S \overset{\text{specializes}}{\rightsquigarrow} x_T$,

$\sigma_{x_S} \subset \sigma_{x_T}$ is a face.

$$C(X, D) = \varinjlim_{S \text{ strata}} \sigma_{x_S} \quad \text{a cone complex}$$

<u>Ex</u>: $(X, D) =$ (toric variety, complement of torus)

$C(X, D) =$ fan of toric variety (w/o embedding in lattice)

When (X,D) does have self-intersection/

monodromy

presentation — as nice as possible

$$(V,D_V) \rightrightarrows (U,D_U) \to (X,D)$$

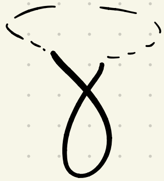
and $C(X,D) = [\,C(V,D_V) \rightrightarrows C(U,D_U)\,]$

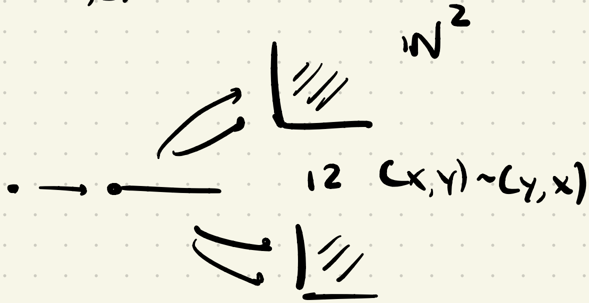
This is now a stack over category of cone complexes (CCUW)

In practical terms, think of $C(X,D)$ as a colimit

- One cone σ_{x_S} for each stratum
- For strata S,T, isomorphisms $\sigma_{x_S} \longrightarrow \sigma_{x_T}$ onto a face

Ex. $\mathcal{C}(X,D)$





For $\overline{\mathcal{M}}_{g,n}$, $\mathcal{C}(\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n})$

= moduli space of tropical curves.

Strata $\longleftrightarrow$ stable graphs Γ

Cone corresponding to Γ = $\mathbb{N}^{E(\Gamma)}$

Specializations $\longleftrightarrow$ Edge contractions

Monodromy $\longleftrightarrow$ Automorphisms of Γ

Construction 2.

A subdivision $C(\widetilde{X,D})$ of $C(X,D)$ is a compatible subdivision of each σ_x.

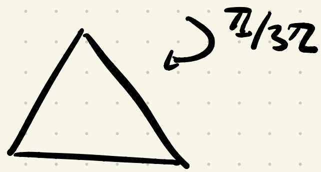
$$\begin{tikzcd} \tilde{\mathcal{U}}_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V(\tilde{\sigma}_x) \arrow[d] \\ \mathcal{U}_x \arrow[r] \arrow[d] & V(\sigma_x) \\ X & \end{tikzcd}$$

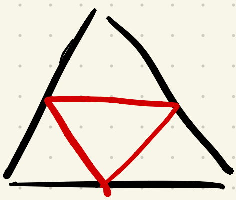
proper, birational, equivariant map

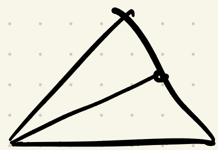
Compatibility $\Rightarrow$ $\tilde{\mathcal{U}}_x$ glue to proper birational $\tilde{X} \longrightarrow X$

(a "log modification")

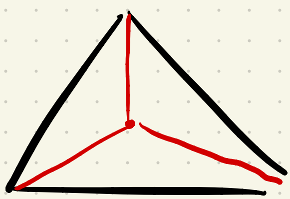
Example (triple pt. w/ $\mathbb{Z}/3$ monodromy)







Important case: Star Subdivision



- Can only do if $\bar{S}$ is normal. Then, corresponds to Blowup of X at $\bar{S}$ (when X smooth)

Def Given (X,D),

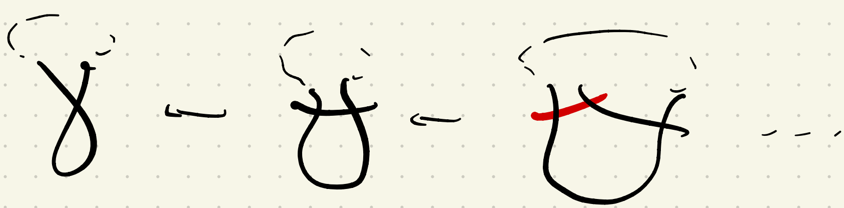
$$\log CH^*(X,D) = \varinjlim_{\substack{\tilde{X} \to X \\ \text{log modification}}} CH^*(\tilde{X})$$

Star subdivisions are cofinal

$$\begin{tikzcd} C(X,D)_1 \arrow[rr] & & C(X,D) \\ & C(X,D)_2 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

composition of star subdivisions.

So you can think iterated blowups along strata



Construction 3.

Assume X smooth here.

- S a stratum.
- $$\begin{tikzcd} \tilde{S} \arrow[rr, "\varepsilon"] \arrow[dr] & & X \\ & \bar{S} \arrow[ur] & \end{tikzcd}$$ normalization of closure is an immersion

So has normal bundle N_ε.

Etale locally, $\bar{S} = D_1 \cap \dots \cap D_k$

and $N_\varepsilon = \varepsilon^*(\mathcal{O}(D_1) \oplus \dots \oplus \mathcal{O}(D_k))$

But globally, monodromy obstructs this

Let G = monodromy group acting on $D_1, \dots D_k$.

$\exists$ G-torsor

$$\begin{tikzcd} T \arrow[rr, "p"] \arrow[dr, "i"'] & & \tilde{S} \arrow[dl] \\ & X & \end{tikzcd}$$

$$N_\Sigma = \bigoplus_\gamma N_\gamma \ , \ \gamma \text{ monodromy orbit}$$

$$\mathrm{rk}\, N_\gamma = |\gamma|.$$

$$p^* N_\Sigma = N_1 \oplus \cdots \oplus N_k$$

Let P be a G-invariant polynomial

Def: A normally decorated strata class is a class of form

$$\iota_* P(c_i(N_1), \ldots, c_i(N_k))$$

Def. The log tautological ring is $\langle$normally decorated strata classes$\rangle$ $\subset CH^*(X)$

Caution

$$R^*_{\log}(\overline{M}_{g,n}, \partial \overline{M}_{g,n}) \subsetneq R^*(\overline{M}_{g,n})$$

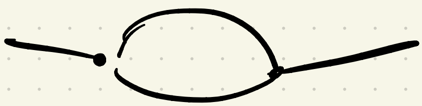
Example and connection w/ λ_g.

Fix $A = (a_1, \dots, a_n)$, $\sum a_i = 0$

$\Gamma = \left(V(\Gamma),\ \mathcal{H}(\Gamma) = \{\text{legs}, \text{edges}\}\right)$

a stable graph

$e = \{h, h'\}$



Each $\Gamma \leftrightarrow$ stratum on $\overline{\mathcal{M}}_{g,n}$

$$\mathcal{M}_\Gamma := \prod_{v \in V(\Gamma)} \mathcal{M}_{g(v),n(v)} \xrightarrow{\xi_\Gamma} \overline{\mathcal{M}}_{g,n}$$

valence

$\mathcal{M}_\Gamma$ is the monodromy torsor

associated to Γ

Pixton's formula

A weighting mod r is $w: H(\Gamma) \to \mathbb{Z}/r\mathbb{Z}$

- $w(l_i) = a_i \quad \forall l \in L(\Gamma)$
- $w(h) = -w(h') \quad$ for $\{h, h'\} \in E(\Gamma)$
- $\sum_{v \in h} w(h) = 0 \quad \forall v \in V(\Gamma)$.

$$P_{g,A}^{r} := \sum_{\Gamma} \sum_{w} \frac{1}{\mathrm{Aut}(\Gamma)} \frac{1}{r^{h_1(\Gamma)}}$$

$$(\xi_\Gamma)_* \left\{ \prod_{l_i \in L(\Gamma)} \exp(a_i^2 \psi_{l_i}) \prod_{e = \{h, h'\}} \frac{1 - \exp(-w(h)w(h')(\psi_h + \psi_{h'}))}{(\psi_h + \psi_{h'})} \right\}$$

Degree d piece $P_{g,A}^{d,r}$ is a polynomial in r for $r \gg 0$

$DR_{g,A}$ = constant term of $P_{g,A}^{r,g}$

$A = \emptyset$ (no ψ_{li})

$\rightsquigarrow \lambda_g \in R^*_{\log}(\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n})$

In general

$$DR_{g,A} \in \langle R^*_{\log}(\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n}), \psi\text{-classes} \rangle$$

<u>Thm</u>: $\forall\ (X, D)$,

$$R^*_{\log}(X, D) \subset \operatorname{div} \operatorname{Log} CH^*(X, D)$$

"Universal" geometry

To each (X,D), we assigned cone complex $C(X,D)$.

Instead, can assign a stack.

$$\sigma_x = \bar{M}^{\vee}_{x,x} \rightsquigarrow \left[V(\sigma_x) / T_{\sigma_x} \right] =: \mathcal{A}_{\sigma_x}$$

toroidal stack

$$\mathcal{H}om_{\text{Cones}}(\sigma_x, \sigma_y) = \mathrm{Hom}_{\text{Tor. stacks}}(\mathcal{A}_{\sigma_x}, \mathcal{A}_{\sigma_y})$$

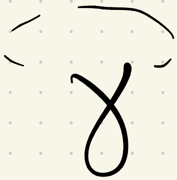
So cone complexes $\subset$ Alg. Stacks.

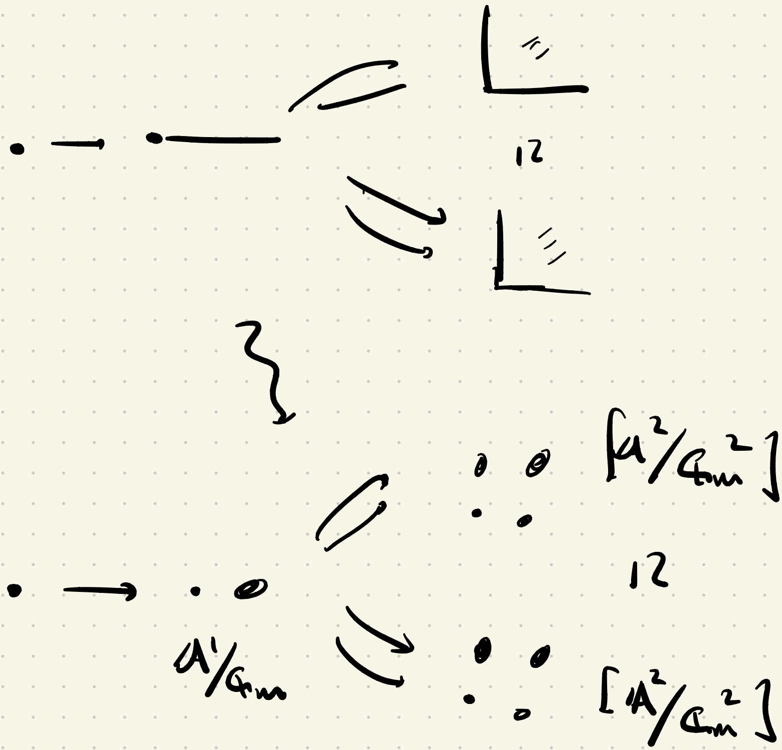
<u>Def</u> (Abramovich–Wise, Olsson)

$$X \rightsquigarrow C(X,D) = \{ \varinjlim \sigma_{x_s} \}$$

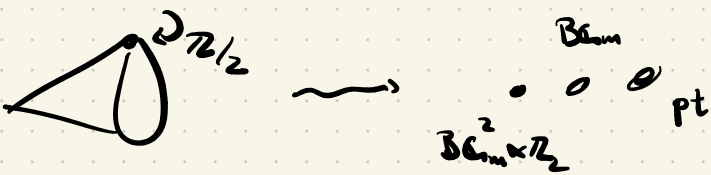
$$\rightsquigarrow \mathcal{A}_{C(X,D)} = \{ \varinjlim \mathcal{A}_{\sigma_{x_s}} \}$$

$\mathcal{A}_{C(X,D)}$ is the universal space w/ combinatorics given by $C(X,D)$.





Cartoon version



- Every operation on $\mathcal{C}(X,D)$ translates to an operation on $\mathcal{A}_{(X,D)}$.

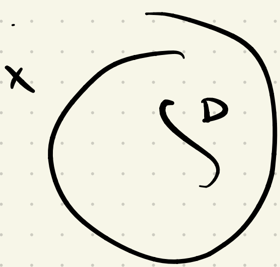
e.g. subdivision $\mathcal{C}(\widetilde{X,D}) \longrightarrow \mathcal{C}(X,D)$

gives $\widetilde{\mathcal{A}}_{(X,D)} \longrightarrow \mathcal{A}_{(X,D)}$

- There is a <u>smooth</u> map

$$\alpha : (X,D) \longrightarrow \mathcal{A}_{(X,D)}$$

Example:



$$\longrightarrow \mathcal{A}_{(X,D)} = [\mathbb{A}^1/\mathbb{G}_m]$$

= moduli space of divisors

<u>Thm</u> | $R^*_{\log}(X,D) = \alpha^* CH_*(\mathcal{A}_X)$

Point is that everything in the construction of a ndsc is pulled back from $\mathcal{A}_{(X,D)}$

$$\begin{tikzcd}
T \arrow[rr, "p"] \arrow[dr, "i"'] \arrow[rrrr, dashed, bend right=15, "\gamma"] & & \tilde{S} \arrow[rrr, bend left, "f"] \arrow[dl, "\varepsilon"] & & & \\
& & & & \mathcal{X} \arrow[r, "q"] \arrow[d, "j"] & \tilde{S} \arrow[dl, "\delta"] \\
& X \arrow[rrr, "\alpha"'] & & & \mathcal{A}_{(X,D)} &
\end{tikzcd}$$

$$p^* N_\varepsilon = p^* f^* N_\delta = \gamma^* q^* N_\delta$$

So ncds =

$$i_* \gamma^* x = \alpha^* j_* x$$

Thm 2

$$CH^*_{op}(\mathcal{A}_{(X,D)}) = PP^*(C(X))$$

algebra of piecewise polynomials

Idea · For $[\mathbb{A}^n/\mathbb{G}_m^n]$, easy calculation.

- For more general toric variety, result of Payne

- Show that $CH^*_{op}(\mathcal{A}_{(X,D)})$ satisfies

a sheaf-type property for

topology generated by stars of strata.

Amounts to controlling

$$CH_*(B\mathbb{G}_m^n \times G, 1)$$

But from here the result is easy:

if $C(X,D)$ is a simplicial complex,

$PP(C(X,D))$ is generated by divisors

(Stanley - Reisner ring:

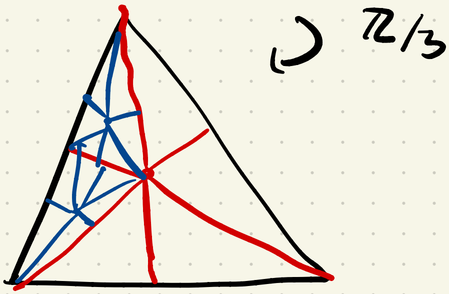
$$k[\text{rays}] / \text{Ideal of non-faces}$$

$$= k[x_\rho] / \langle x_{i_1}, \dots x_{i_k} \rangle \text{ if } \langle i_1, \dots, i_k \rangle \text{ are not a cone in } C(X,D) \text{)}$$

<u>Every</u> cone complex has a subdivision

that's a simplicial complex

Double barycentric subdivision will do.



So $R^*_{\log}(X,D) \subset \operatorname{div} CH^*(B^2(X,D))$

$\Rightarrow \lambda_g \in \operatorname{div} \operatorname{Log} CH^*(\overline{M}_g, \partial\overline{M}_g)$

and

$DR_{g,A} \in \operatorname{div} \operatorname{Log} CH^*(\overline{M}_{g,n}, \partial\overline{M}_{g,n})$