RESEARCH STATEMENT

SAM MOLCHO

My research interests lie broadly in algebraic geometry, though my expertise is more specifically in moduli theory. My work falls roughly under three circles of ideas, although the ideas often overlap significantly in my papers: moduli spaces of curves and stable maps ([AM14],[MR19],[MPS21],[MR21]), moduli spaces of line bundles ([MW22], [HMOP20], [MMUV22]), and resolution of singularities ([MT21], [Mol21], [GM15a], [GM15b]).

A recurring theme in my research is the study of moduli problems by degeneration methods, and especially by employing techniques from logarithmic geometry. A situation one encounters often in moduli theory is that one is interested in a certain moduli problem, parametrizing families of, say, smooth varieties with certain properties. As smooth varieties tend to degenerate into singular ones, the moduli problem is typically not compact. The philosophy of logarithmic geometry is that one must endow the objects in question with additional structure, called a logarithmic structure, and study the moduli problem of these logarithmic objects instead. Doing so correctly is often more difficult than it seems; once it is done, however, it gives access to a variety of additional tools with which one can study the smooth objects and the degenerate objects that appear in the limit uniformly. It gives access to categorical, deformation theoretic, and combinatorial methods: log schemes form a category, with notions of morphisms, fiber products, and so on; it makes sense to speak about smoothness in this setup, and deformation theory of logarithmic objects has almost identical formal behavior as ordinary deformation theory, controlled by the cohomology groups of the so called logarithmic cotangent complex; and every logarithmic scheme has a tropicalization – a certain piecewise linear object, consisting of a complex of cones, which is an object central to the field of tropical geometry – to aid in their study. The moduli spaces of logarithmic objects that arise tend to be automatically proper, and furthermore to have excellent formal properties: a functor of points, functorial desingularizations, or perfect obstruction theories when appropriate.

Below I present a sample of my results, which illustrate the scope of my research and my mathematical taste.

1. Jacobians of Nodal Curves and Double Ramification Cycles

While the Jacobian of a smooth family of curves $X \to S$ is an abelian scheme over S, the situation breaks down when $X \to S$ is nodal. There are two natural candidates with which one can replace the Jacobian – the space $\mathrm{Pic}^0(X/S)$ of line bundles of total degree 0, and the space $\mathrm{Pic}^{[0]}(X/S)$ of multidegree 0 line bundles, that is, degree 0 on every irreducible component of every fiber of $X \to S$. However, neither is proper: $\mathrm{Pic}^0(X/S)$ is both not universally closed and highly non-separated, while $\mathrm{Pic}^{[0]}(X/S)$ is semiabelian, i.e. an extension of an abelian scheme by a torus. Compactifications of $\mathrm{Pic}^{[0]}(X/S)$ have been the subject of a vast study ([OS79],[Sim94],[Cap94],[KP19],[Mel19],[Est01]). While the resulting spaces – the so called compactified Jacobians – are of course proper, they depend on auxiliary choices, and do not have a group structure.

A very interesting story runs in parallel on the moduli space of curves. Fix a vector of integers $A = (a_1, \cdots, a_n)$ with $\sum a_i = k(2g-2) := d$. The *double ramification cycle* $\mathsf{DR}^k_{g,A}$ is the virtual class of the locus of curves in $\overline{\mathcal{M}}_{g,n}$ which admit a k-differential with zeros and poles of orders dictated by A. More precisely, the $\mathsf{DR}^k_{g,A}$ is a codimension g algebraic cycle class in

$\mathsf{CH}^g(\overline{\mathcal{M}}_{g,n})$, supported on the locus of curves $(C, x_1, \cdots, x_n)$ for which $\omega_C^{\otimes k} \cong \mathcal{O}(\sum a_i x_i)$. While the afformentioned locus is unambiguous, properly defining the cycle class $\mathsf{DR}_{g,A}^k$, and moreso computing it in terms of more familiar classes (say, in the tautological ring of $\overline{\mathcal{M}}_{g,n}$) is subtle. Over the locus of smooth curves, the distinction is clear, and so is the connection with Jacobians. The difference between the locus and the class is the difference between the schematic intersection DRL of the section $\omega^{\otimes k}$ with the Abel-Jacobi section $\mathsf{aj}_A(C, x_1, \cdots, x_n) = \mathcal{O}(\sum a_i x_i)$, and the refined intersection $\mathsf{aj}_A^*([\omega^{\otimes k}])$, as in the diagram below:

$$\begin{tikzcd} \mathsf{DRL} \arrow[r] \arrow[d] & \mathcal{M}_{g,n} \arrow[d, "\omega^{\otimes k}"] \\ \mathcal{M}_{g,n} \arrow[r, "\mathsf{aj}_A"] & \mathrm{Pic}^d(\mathcal{C}_{g,n}/\mathcal{M}_{g,n}) \end{tikzcd}$$

As there is no good candidate for a compactification of $\mathrm{Pic}^d(\mathcal{C}_{g,n}/\mathcal{M}_{g,n})$ – the compactified Jacobians may sound like a natural choice, but there is no compactified Jacobian to which the Abel-Jacobi section extends –, extending the class away from the locus of compact type curves is subtle. The question of how to do so goes back to Eliashberg, and by now has been studied by a great number of people – see for instance [GV05], [KZ03], [BCG+18], [FP18], [Hol21], [MW20] for just a sample. At the end, answers have been proposed which lie in the intersection of Gromov-Witten theory, complex geometry, Abel-Jacobi theory and logarithmic geometry. In a sense, the definitive approach to the question was provided in the papers [JPPZ17], [BHP+21], which together culminate with an explicit formula for $\mathsf{DR}_{g,A}^k$ in the tautological ring of $\overline{\mathcal{M}}_{g,n}$, called Pixton's formula. However, even these approaches reach a stumbling block: for instance, if we were to study the analogous problem for two vectors $A = (a_1, \cdots, a_n), B = (b_1, \cdots, b_n)$, one can define the analogous higher ramification cycle "$\mathsf{DDR}_{g,A,B}^k$" supported on the locus $(C, x_1, \cdots, x_n)$ where $\mathcal{O}(\sum a_i x_i) \cong \omega_C^{\otimes k} \cong \mathcal{O}(\sum b_i x_i)$, but the methods of [JPPZ17] do not generalize to provide a formula. In particular, the naive expectation $\mathsf{DDR}_{g,A,B}^k = \mathsf{DR}_{g,A}^k \mathsf{DR}_{g,B}^k$, while valid in the interior of $\overline{\mathcal{M}}_{g,n}$, fails over the boundary. Yet the classes $\mathsf{DDR}_{g,A,B}$ (and their even-higher dimensional analogues) play a central role in the Gromov-Witten theory of higher dimensional varieties, and so are of significant interest.

1.1. **The logarithmic Jacobian.** While no compactification of $\mathrm{Pic}^{[0]}(X/S)$ can retain both smoothness and group structure, such a best possible compactification is possible in the logarithmic setting. Following ideas of Illusie-Kato, and Kajiwara-Kato-Nakayama, Jonathan Wise and I construct the logarithmic Picard stack $\mathbf{LogPic}(X/S)$ of a family $X \to S$ of nodal (or more generally, log) curves and study its structure and main properties [MW22]. The stack $\mathbf{LogPic}(X/S)$ is in fact a gerbe, and a rigidified version, the logarithmic Picard group $\mathrm{LogPic}(X/S)$ is also available. The logarithmic Picard stack/group have several remarkable properties, which can be summarized as follows:

Theorem 1.1.1. [MW22, Theorems A,B,D,E] *The stack $\boldsymbol{LogPic}(X/S)$ (resp. sheaf $\mathrm{LogPic}(X/S)$) is a logarithmically smooth, proper commutative group object. It is a stack (resp. sheaf) in the log étale topology, and it has a log étale cover by an algebraic stack with a log structure (resp. log scheme). It coincides with the classical Picard stack $\boldsymbol{Pic}$ (resp. Picard scheme Pic) on the smooth locus of $X \to S$. The sheaf $\mathrm{LogPic}(X/S)$ is a logarithmic abelian variety in the sense of* [KKN08].

In particular, given a family of nodal curves $X \to S$ which is smooth over some open subscheme $U \subset S$, $\mathbf{LogPic}(X/S)$ provides a log smooth, proper, group compactification of $\mathbf{LogPic}(X_U/U) = \mathrm{Pic}(X_U/U)$. For instance, applying the construction to the universal curve

$\overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ over the moduli space of curves produces a compatification of the universal Jacobian $\mathbf{Pic}(\mathcal{C}_{g,n}/\overline{M}_{g,n})$.

The biggest difficulty in the theory is that **LogPic** is not representable by an algebraic stack with a logarithmic structure. To explain what this entails is technical, but, informally, it means that in order to define the T points $\mathbf{LogPic}(X/S)(T)$, it is necessary to specify a log structure on T. This may give the impression that the logarithmic Picard group is complicated, but in fact it is possible to get a good grasp on it simply on account of its simple functor of points, and through its tropicalization $\mathbf{TroPic}(X/S)$. This tropicalization is the solution of an analogous moduli problem on the tropicalization $\mathfrak{X}$ of $X \to S$ – a tropical curve –, and is a fully combinatorial object that can be understood explicitly: it is a constructible sheaf of abelian groups varying over strata in an appropriate stratification of S. We relate the combinatorial object TroPic to $\mathbf{LogPic}(X/S)$ by observing that $\mathbf{LogPic}(X/S)$ contains the multidegree 0 part $\mathrm{Pic}^{[0]}(X)$ of the Jacobian of X as a subgroup, and that we have:

Theorem 1.1.2. [MW22, Theorem C.] *There is an exact sequence of group stacks*

$$0 \longrightarrow \boldsymbol{Pic}^{[0]}(X) \longrightarrow \boldsymbol{LogPic}(X/S) \longrightarrow \boldsymbol{TroPic}(\mathfrak{X}) \longrightarrow 0$$

and an analogous sequence of sheaves for $\mathrm{LogPic}(X/S)$.

1.2. **Compactified Jacobians.** Due to its lack of representability, the logarithmic Picard stack/group may be thought of as an organizing formalism instead of a genuine space as a first approximation – just as a usual algebraic stack can be thought of as an organizing formalism instead of a space. Regardless, we have found the formalism useful rather than vacuous. It is a general principle of logarithmic geometry that given a log scheme X, with tropicalization Σ_X, which is a cone complex, *subdivisions* Σ'_X of the cone complex Σ_X correspond to a very specific subset of blowups $X' \to X$ called *logarithmic blowups*. The significance of this observation in this context is that using the exact sequence of 1.1.2 we can classify toroidal compactifications of $\mathrm{Pic}^{[0]}(X)$ by subdivisions of $\mathrm{TroPic}(X/S)$ – put otherwise, they are all log blowups of $\mathrm{LogPic}(X/S)$. In [MMUV22, MMU⁺on] we carry out this analysis more carefully. We show that all the usual spaces that arise in the study of compactified Jacobians – the so called universal and generalized Jacobians, the Caporaso, Esteves and Melo compactified Jacobians, and the compactified Jacobians of Kass-Pagani all have tropicalizations, which we compute, and are determined relative to $\mathbf{LogPic}(X/S)$ by these tropicalizations. For example, for any choice of an appropriate "stability condition" ϕ (which we do not define here), Kass and Pagani construct a compactified Jacobian $\mathrm{Pic}(\phi)$ over $\overline{\mathcal{M}}_{g,n}$. We show

Theorem 1.2.1. [MMUV22, MMU⁺on] *The compactified Jacobian* $\mathrm{Pic}(\phi)$ *has a tropicalization* $\mathrm{Pic}^{\mathrm{trop}}(\phi)$*, which comes with a map* $\pi : \mathrm{Pic}^{\mathrm{trop}}(\phi) \to \mathrm{TroPic}(\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n})$*. We have*

$$\mathrm{Pic}(\phi) = \mathrm{Pic}^{\mathrm{trop}}(\phi) \times_{\mathrm{TroPic}(\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n})} \mathrm{LogPic}(\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n})$$

The map π is a subdivision of cone complexes. Analogous statements hold for the universal and generalized Jacobian, other compactified Jacobians, and the analogous versions for the corresponding (unrigidified) stacks.

Thus, the theory of compactified Jacobians on a family of nodal curve $X \to S$ can be seen as specifying subdivisions of $\mathrm{TroPic}(X/S)$. This point of view allows us to get new constructions of compactified Jacobians; for instance, the constructions of Caporaso, or of Kass-Pagani, depend on the choice of a stability condition but yield a proper space only when the stability condition turns out to have no strictly semistable objects. In the ongoing work [MMU⁺on], we show

Theorem 1.2.2. *For any stability condition ϕ, which may contain strictly semistable objects, there is a way to associate a subdivision $\pi : \widetilde{\mathrm{Pic}}^{\mathrm{trop}}(\phi) \to \mathrm{TroPic}(\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n})$. The fiber product*

$$\widetilde{\mathrm{Pic}}(\phi) = \widetilde{\mathrm{Pic}}^{\mathrm{trop}}(\phi) \times_{\mathrm{TroPic}(\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n})} \mathrm{LogPic}(\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n})$$

provides a new compactification of the universal Jacobian, which coincides with the usual compactification $\mathrm{Pic}(\phi)$ *when ϕ does not contain strictly semistable objects.*

1.3. **Double Ramification Cycle.** The discussion on the log Jacobian immediately provides a suggested route out of the difficulties the double ramification cycle runs over the boundary of $\overline{\mathcal{M}}_{g,n}$: One should instead look at the diagram

$$\begin{tikzcd} \mathsf{DRL} \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{g,n} \arrow[d, "\omega^{\otimes k}"] \\ \overline{\mathcal{M}}_{g,n} \arrow[r, "\mathsf{aj}_A"] & \mathrm{LogPic}^d(\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n}) \end{tikzcd}$$

and define $\mathsf{DR}^k_{g,A}$ as $\mathsf{aj}_A^*[\omega^{\otimes k}]$, for $[\omega^{\otimes k}]$ the class of the section determined by $\omega^{\otimes k}$ in $\mathsf{CH}^g(\mathrm{LogPic}(\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n})$. The problem is that, as LogPic is not algebraic, this suggestion is nonsense! There is simply no way to make sense of $\mathsf{CH}^g(\mathrm{LogPic})$ directly. There is however, an alternative cohomology theory one may use. Given a log smooth logarithmic scheme X, the log Chow ring of X is defined as the colimit

$$\mathsf{logCH}(X) := \varinjlim_{X' \to X} \mathsf{CH}^*(X')$$

where $X' \to X$ ranges through all *log* blowups of X with X' smooth. This is a (non-finitely generated) ring that contains $\mathsf{CH}(X)$. Moreover, as $\mathrm{LogPic}(\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n})$, while non-algebraic, has log blowups which are algebraic (the compactified Jacobians), it does make sense to ask about the class $[\omega^{\otimes k}] \in \mathsf{logCH}(\mathrm{LogPic}(\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n}))$.

Before taking the suggestion of working with a cohomology theory of the logarithmic Jacobian seriously, we can explore some of its consequences. First of all, it suggests that the double ramification cycle should be more naturally defined as a class in $\mathsf{logCH}(\overline{\mathcal{M}}_{g,n})$. What this means, in practice, is that it should be studied as a collection of classes $(\mathsf{DR}^{k,\alpha}_{g,A})_{\alpha \in \mathcal{C}} \in \mathsf{CH}(\overline{\mathcal{M}}^{\alpha}_{g,n})$, where $f_\alpha : \overline{\mathcal{M}}^{\alpha}_{g,n} \to \overline{\mathcal{M}}_{g,n}$, $\alpha \in \mathcal{C}$ range through a class of (log) blowups of $\overline{\mathcal{M}}_{g,n}$, so that the classes $\mathsf{DR}^{k,\alpha}_{g,A}$ are compatible: for any log blowup $\overline{\mathcal{M}}'_{g,n}$ of $\overline{\mathcal{M}}_{g,n}$ dominating two of the given blowups $\overline{\mathcal{M}}^{\alpha}_{g,n}, \overline{\mathcal{M}}^{\beta}_{g,n}$, the pullbacks of $\mathsf{DR}^{k,\alpha}_{g,A}$ and $\mathsf{DR}^{k,\beta}_{g,A}$ must coincide. In fact, such a class was already constructed by Holmes in [Hol21]. The classes $(\mathsf{DR}^{k,\alpha}_{g,A})_{\alpha \in \mathcal{C}}$ are somewhat remarkable. First, they form a refinement of $\mathsf{DR}^k_{g,A}$, i.e. they contain strictly more information: we have

$$(f_\alpha)_* \mathsf{DR}^{k,\alpha}_{g,A} = \mathsf{DR}^k_{g,A}$$

but

$$f_\alpha^* \mathsf{DR}^k_{g,A} \neq \mathsf{DR}^{k,\alpha}_{g,A}$$

But, more importantly, the classes $\mathsf{DR}^{k,\alpha}_{g,A}$ controll the problematic higher ramification cycles $\mathsf{DDR}_{g,A,B}$ discussed above:

$$\mathsf{DDR}_{g,A,B} = (f_\alpha)_*(\mathsf{DR}^{k,\alpha}_{g,A} \mathsf{DR}^{k,\beta}_{g,B})$$

for appropriate α.

Thus, the study of the refinement $(\mathsf{DR}^{\alpha,k}_{g,A})_{\alpha \in \mathcal{C}}$ – elliptically refered to as $\mathsf{logDR}^k_{g,A} \in \mathsf{logCH}(\overline{\mathcal{M}}_{g,n})$ – becomes the key to the study of the higher ramification problems.

A special case deserves mentioning. When $A = (0, \cdots, 0)$ and $k = 0$, the class $\mathsf{DR}^k_{g,A}$ is nothing but the class $(-1)^g\lambda_g$: the top Chern class of the Hodge bundle $\mathbb{E}_g$. In this special case, we also happen to have that $\mathsf{DR}^{k,\alpha}_{g,A} = f^*_\alpha(-1)^g\lambda_g$, i.e. we are in one of the rare cases where the class $\mathsf{logDR}^k_{g,A}$ is determined by $\mathsf{DR}^k_{g,A}$. Nevertheless, studying $\lambda_g \in \mathsf{logCH}(\overline{\mathcal{M}}_{g,n})$ rather than $\mathsf{CH}(\overline{\mathcal{M}}_{g,n})$ still reveals additional structure. In [MPS21], together with Pandharipande and Schmitt, we study whether – as happens over the locus of compact type curves, where λ_g is the pullback $\Theta^g/g!$ of the universal Θ-divisor on the moduli space of Abelian varieties under the Torelli map – λ_g is a product of divisors. We show

Theorem 1.3.1. [MPS21] *The class λ_g cannot be written as a product of divisor classes in $\mathsf{CH}(\overline{M}_g)$ for $g \geq 2$. However, it is a product of divisor classes in $\mathsf{logCH}(\overline{M}_g)$.*

Another special case is studied in joint work [MR21] with Dhruv Ranganathan. There we still study the case when $k = 0$, but $A = (a_1, \cdots, a_n)$ an arbitrary vector of integers summing to 0. The resulting cycle $\mathsf{DR}^k_{g,A}$, elliptically denoted as simply $\mathsf{DR}_{g,A}$, is the case of primary interest from the perspective of Gromov-Witten theory, as it can be interpreted as the virtual fundamental class of the moduli space of rubber relative stable maps. The work [MR21] devotes a significant amount of effort into developing foundations for logarithmic intersection theory. In our setup, we recognize the refined classes $\mathsf{DR}^\alpha_{g,A}$ as arising from a general construction in logarithmic intersection theory, which essentially expresses the difference of $f^*_\alpha\mathsf{DR}_{g,A}$ and $\mathsf{DR}^\alpha_{g,A}$ as the difference between the (virtual) total and strict transforms of a subscheme. We show that, working on a log scheme X, such differences are in general controlled by a specific subalgebra of the Chow ring of X, the algebra $\mathrm{PP}(\Sigma_X)$ of *piecewise polynomial* functions on the tropicalization of X, for which we develop the foundations. This algebra is very simple – essentially a combinatorial object – but was already a crucial ingredient in the proof of 1.3.1 in [MPS21]. It is further crucially used here to show that

Theorem 1.3.2. [MR21] *The classes $\mathsf{DR}^\alpha_{g,A}$ are in the subring of $\mathsf{CH}(\overline{\mathcal{M}}^\alpha_{g,n})$ generated by the pullbacks of tautological classes on $\overline{\mathcal{M}}_{g,n}$ and piecewise polynomial functions on the tropicalization of $\overline{\mathcal{M}}^\alpha_{g,n}$. In particular, for vectors A,B, the higher ramification cycles*

$$\mathsf{DDR}_{g,A,B}$$

are in the tautological ring of $\overline{\mathcal{M}}_{g,n}$.

In recent work with Holmes, Pandharipande, Pixton and Schmitt, we go further and obtain explicit formulas for $\mathsf{DR}^{k,\alpha}_{g,A}$. This way, we can also get formulas for the $\mathsf{DDR}^k_{g,A,B}$, and so on. To do so, we take the suggestion that $\mathsf{DR}^k_{g,A}$ should be considered as the pullback of $[\omega^{\otimes k}] \in \mathsf{logCH}(\mathrm{LogPic}(\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n})$ seriously. This suggestion entails that, for *any* compactified Jacobian – say, for concreteness, any of the Jacobians $\mathrm{Pic}(\phi)$ defined by Kass and Pagani for a stability condition ϕ –, there must be a log blowup $\overline{\mathcal{M}}_{g,n}(\phi)$ of $\overline{\mathcal{M}}_{g,n}$, and a resolution of the Abel-Jacobi section

$$\overline{\mathcal{M}}_{g,n}(\phi) \xrightarrow{aj_\phi} \mathrm{Pic}(\phi)$$

and the representative $\mathsf{DR}^k_{g,A}(\phi)$ of the class $\mathsf{logDR}^k_{g,A}$ on $\overline{\mathcal{M}}_{g,n}(\phi)$ must be $aj^*_\phi[\omega^{\otimes k}]$. We show that this is indeed the case, and show that $\overline{\mathcal{M}}_{g,n}(\phi)$ is determined by an explicit subdivision of the tropicalization of $\overline{\mathcal{M}}_{g,n}$. Using this, we show

Theorem 1.3.3. [HMP+22] *There is an explicit class*

$$P^k_{g,A}(\phi) \in \mathsf{CH}_*(\overline{\mathcal{M}}_{g,n}(\phi))$$

written in terms of tautological classes on $\overline{\mathcal{M}}_{g,n}$ and piecewise polynomial classes on the tropicalization of $\overline{\mathcal{M}}_{g,n}(\phi)$. Its codimension g part computes $\mathsf{DR}^k_{g,A}(\phi)$.

The formula is in fact amenable to implementation to a computer, and allows us to compute several examples of higher ramification cycles which were previously out of reach.

1.4. **Néron Models.** The results above illustrate that the (non-algebraic) space $\mathrm{LogPic}(X/S)$ controls the theory of compactified Jacobians, via (algebraic) subdivisions of its tropicalization. As mentioned above, the source of non-algebraicity of $\mathrm{LogPic}(X/S)$ is that, in order to define its T points, it is necessary to specify a log structure on T. A more technical way to say this is that, $\mathrm{LogPic}(X/S)$ is a sheaf on the category **LogSch/S** of log schemes over S, and that there is no way to descend $\mathrm{LogPic}(X/S)$ to an algebraic space on **Sch/S**, under the natural forgetful morphism **LogSch/S** $\to$ **Sch/S**. On the other hand, the category **Sch/S** also embeds into **LogSch/S**, by giving a scheme $T \to S$ its log structure induced from S (called the "strict" log structure). Thus, instead of trying to resolve non-algebraicity by blowing up, it is possible to address it by restricting $\mathrm{LogPic}(X/S)$ to a sheaf $\mathrm{sLogPic}(X/S)$ (with s standing for "strict") on **Sch/S**. The sheaf $\mathrm{sLogPic}(X/S)$ is now representable by an algebraic space, and turns out to be intimately connected with the Néron model of the Jacobian of $X \to S$. Assume that $X \to S$ is a family of nodal curves over a toroidal base S, smooth over a dense open set U. Recall that for an abelian scheme $A_U \to U$, the Néron model of $A_U \to U$ is defined to be a smooth, separated algebraic space $A \to S$ restricting to $A_U \to U$ over U, with the *Néron mapping property*: Given a smooth map $f : T \to S$ with $f^{-1}(U) = V$, and a map $m : V \to A_U$, there is a unique extension of m to a map $T \to A$. Néron's theorem asserts that a Néron model exists when S is a Dedekind scheme. In [HMOP20], we show

Theorem 1.4.1. [HMOP20] *Let $X \to S$ be a family of nodal curves over a toroidal base S of arbitrary dimension, smooth over a dense open $U \subset S$. Then $\mathrm{sLogPic}(X/S)$ is a smooth algebraic space which satisfies the Néron mapping property.*

In other words, the algebraic space $\mathrm{sLogPic}(X/S)$ is in a sense the Néron model of $\mathrm{Jac}(X_U/U)$ – it is the unique object that can possibly be the Néron model, and the only property in the definition that potentially fails is separatedness. However, non-separatedness of $\mathrm{sLogPic}(X/S)$ can be again detected tropically. More precisely, we show that the tropicalization $\mathrm{sTroPic}(X/S)$ of $\mathrm{sLogPic}(X/S)$ controls the smooth separated group models of $\mathrm{Jac}(X_U/U)$):

Theorem 1.4.2. [HMOP20] *There is an equivalence of categories between smooth separated group models of $\mathrm{Jac}(X_U/U)$ and quasi-finite open subgroups of $\mathrm{sTroPic}(X/S)$, which, to such a subgroup $G \subset \mathrm{Jac}(X_U/U)$ assigns the group model*

$$\mathcal{G} = G \times_{\mathrm{sTroPic(X/S)}} \mathrm{sLogPic(X/S)}$$

In particular, there is always a maximal smooth separated group model of $\mathrm{Jac}(X_U/U)$, corresponding to the torsion subgroup $\mathrm{sTroPic}(X/S)$.

To summarize, if we relax the hypothesis that the Néron model is separated, then the Jacobian of a family of nodal curves X over a toroidal base S of arbitrary dimension always has a Néron model – $\mathrm{sLogPic}(X/S)$ – and if we insist on separatedness, a Néron model exists if and only if sTroPic(X/S) is torsion. Either way, this analysis provides the Néron model with a modular interpretation, which as far as we know is novel even for one dimensional bases.

2. Resolution of Singularities

2.1. **Weak Semistable Reduction.** One of the main problems is moduli theory is the following: given a family of varieties $X \to S$, can one modify X and S to obtain a family $X' \to S'$ that is "as nice as possible"? A classical result [KKMSD73] shows that when the base S is one-dimensional, one can take a finite cover $S' \to S$ such that for a blowup $Y \to X \times_S S'$ the map is $Y \to S'$ is flat, has reduced fibers, and Y is smooth. When S is higher dimensional, Abramovich and Karu proved that by taking a blowup of a finite cover of S – an alteration –, one can at least always ensure a blowup Y of $X \times_S S'$ will be flat with reduced fibers over S'. The proof in either [AK00] or [KKMSD73] is done by reduction to the case of toroidal embeddings, by appeal to Hironaka's desingularization, and then studying the combinatorics of their associated cone complexes – the tropicalization in the language above. In [Mol21], I study the toroidal part of the problem from the viewpoint that its functorial aspects are important. I define a weakly semistable morphism of toroidal embeddings to be a log smooth morphism which is flat with reduced fibers, and I show that this condition is essentially combinatorial in nature. Furthermore, it is precisely the condition necessary to ensure that the pullback of toroidal embeddings remains toroidal. The main result of [Mol21] shows that the toroidal part of [AK00] can be done canonically if one works with Deligne-Mumford stacks instead of schemes. Specifically, I show:

Theorem 2.1.1. [Mol21, Theorem 1.0.1] *Let $X \to S$ be a proper, surjective, log smooth morphism of toroidal embeddings. There is a commutative diagram*

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d] & X \arrow[d] \\ \mathcal{S} \arrow[r] & S \end{tikzcd}$$

where $\mathcal{X} \to \mathcal{S}$ is a weakly semistable morphism of toroidal Deligne-Mumford stacks, and any weakly semistable $Y \to T$ mapping to $X \to S$ factors uniquely through $\mathcal{X} \to \mathcal{S}$.

The theorem is essentially an analogue of the Raynaud-Gruson theorem in the toroidal category. While the shift in perspective is innocuous, the observation that semistablization should be thought of as an operaration seems to appear in various guises in logarithmic geometry – it appears for instance in the work of Ranganathan [Ran20] on expansions of higher dimensional targets, in the comparisons between strict and total transforms in [MR21], or in [AM14].

2.2. **Logarithmic Differential Geometry.** The project [GM15a] (my first one) has a slightly different flavor than the works discussed above. It can be seen as a response to the papers [KM11] and [Joy12] on manifolds with corners. Manifolds with corners do not form a good category. Firstly, even though a definition of manifolds with corners is more or less accepted, there is no consensus on what a morphism between manifolds with corners should be. Secondly, the category of manifolds with corners is not closed under inverse limits: for example, fiber products of transverse maps of manifolds with corners do not exist as manifolds with corners. In [GM15a], we propose that manifolds with corners should be studied in the enlarged category of differentiable spaces – a category which is roughly speaking built from the category of manifolds in the same way as the category of schemes is built out of the category of smooth varieties. In fact, we enlarge the category even more by allowing differentiable spaces to carry log structures. This allows us to give a natural definition of a morphism of manifolds with corners – they are precisely the log morphisms. We characterize manifolds with corners in terms of their log structure:

Theorem 2.2.1. [GM15a, Theorem 6.7.8] *Manifolds with corners are positive log differentiable spaces which are log smooth with a free log structure.*

The payoff of working with differentiable spaces, which form a nicer category, is that many objects obtain singularities – for instance, the motivating example of fiber products of manifolds with corners. We thus discuss resolution of singularities in a rather general context. We start from any category of "log spaces", which can be for example the category of log schemes, log analytic spaces or log differentiable spaces. We discuss a notion of logarithmic smoothness for any log space and we describe a procedure to assign combinatorial data to a log space, which we call a monoidal space. This is analogous to the data of the fan of a toric variety. We discuss how operations on this data, analogous to subdivisions of fans, have a geometric realization that produces a new log space with a log smooth map to the original one. Following techniques from the algebraic theory of resolution of singularities, we show how to resolve any monoidal space, so that the geometric realization of this resolution gives a space with a free log structure. In the example of positive differentiable spaces, it yields

Theorem 2.2.2. [GM15a, Theorem 10.4.1] *Any (fs) positive log differentiable space X admits a log smooth, surjective, locally projective map $X' \to X$ from a positive log differentiable space with a free log structure, which is an isomorphism over the locus where the log structure of X is free.*

In other words, X' is a resolution of singularities of X. If we begin with a log smooth space, the resulting resolved space will be log smooth and free - a manifold with corners. As a corollary, we obtain a different proof of the main result of [KM11]: the fiber product of transversal maps of manifolds with corners can be resolved to a manifold with corners.

3. Moduli of Maps

3.1. **Broken Toric Varieties.** In [AM14], Kenneth Ascher and I study the moduli space of broken toric varieties in a toric variety V. Specifically, for a fixed toric variety V and a fixed subtorus H of its torus, we study the stack $\mathcal{AB}$, introduced in the work [AB06] of Alexeev and Brion, which parametrizes families of equivariant maps $f : X \to V$ from a broken toric variety with torus H to V. The stack $\mathcal{AB}$ is shown to be a proper Deligne-Mumford stack in [AB06], but it is not irreducible. In the logarithmic setting, we study instead the logarithmic version $\mathcal{K}$ of $\mathcal{AB}$; we show that the stack $\mathcal{K}$ is a proper, log smooth Deligne-Mumford stack, and we obtain:

Theorem 3.1.1. [AM14, Theorem 5.12] *The forgetful morphism $\mathcal{K} \to \mathcal{AB}$ is the normalization of the closure of the main component of $\mathcal{AB}$.*

Furthermore, we give an explicit description of $\mathcal{K}$ as a toric stack, by describing its stacky fan – we do this by observing that the Chow quotient $V \mathbin{/\!\!/} H$ of Kapranov-Sturmfels-Zelevinsky [KSZ91] has a natural structure structure of a "KM fan" [1] which carries a universal family, and we show:

Theorem 3.1.2. [AM14, Proposition 5.11] *The stack $\mathcal{K}$ is isomorphic to the Chow quotient stack $[V \mathbin{/\!\!/}_C H]$.*

3.2. **Gromov-Witten Theory.** The space of relative stable maps $\mathcal{M}^{\mathrm{Li}}(X)$, constructed by Jun Li [Li01], parametrizes maps from nodal curves to a singular target $X = Y_1 \cup_D Y_2$ that have prescribed tangency conditions along D. The invariants one extracts from $\mathcal{M}^{\mathrm{Li}}(X)$ are called relative Gromov-Witten invariants, and are important in Gromov-Witten theory due to their deformation invariance: the Gromov-Witten invariants of a family X_t of smooth varieties degenerating to X coincide with the relative Gromov-Witten invariants of X. The paper [Li01] is groundbreaking, but introduces one technical difficulty: the space of relative stable maps to X is a locally closed subset of the space of all maps to X, and hence the ordinary deformation/obstruction theory of maps cannot be used; this makes its virtual fundamental class hard to understand, and thus the invariants hard to compute. One of the main computational tools is the localization formula of Graber-Vakil, available when there is an action of $\mathbf{C}^*$ on X. In [Kim10], the author introduced

[1]The theory of KM fans is developed in joint work with W.Gillam in [GM15b], with main motivation to understand this construction, and its precursor in [GM], but also certain mysterious differential geometric constructions that appeared in the work [KM11] of Kottke and Melrose, to which the notion owes the name.

the space $\mathcal{M}^{\mathrm{Kim}}(X)$ of logarithmic stable maps to X. This space produces the same invariants as $\mathcal{M}^{\mathrm{Li}}(X)$, but has several technical advantages, and in particular a virtual class governed by standard log deformation theory. In [MR19], we study $\mathcal{M}^{\mathrm{Kim}}(X)$ when X carries a $\mathbf{C}^*$-action, and show:

Theorem 3.2.1. [MR19, Theorem 5.1] *There is a localization formula for $\mathcal{M}^{\mathrm{Kim}}(X)$. This formula recovers the formula of Graber-Vakil by pushing forward via the forgetful map $\pi : \mathcal{M}^{\mathrm{Kim}}(X) \to \mathcal{M}^{\mathrm{Li}}(X)$.*

4. Future and Ongoing Work

The results above suggest several new routes of investigation. I outline here a sample of some that I am pursuing currently or plan to pursue in the future.

4.1. **Double Ramification Cycles.** The geometry of the compactified Jacobians is very rich. The compactified Jacobians carry in particular a collection of algebraic cycles, called the Brill-Noether loci. The arguments of [HMP+22] show that the logarithmic double ramification cycle is the pullback of a special Brill-Noether locus (the 0-th degree 0 Brill-Noether locus). The Brill-Noether loci can be realized as degeneracy loci. Consequently, the logarithmic double ramification cycle can be realized as a degeneracy locus as well. This provides a different way of calculating it, via more traditional algebro-geometric techniques, such as the Grothendieck-Riemann-Roch theorem. In ongoing work with Abreu and Pagani, we give formulas for the double ramification cycle that come from its interpretation as a degeneracy locus. The calculation turns out to be very interesting: it requires developing a way to calculate cycles on $\overline{\mathcal{M}}_{g,n}$ which are a mix of tautological and *tropical* classes – classes which come from the combinatorics of stable graphs. The methods should be applicable to many other problems, and, remarkably, the formulas that arise are totally different from Pixton's formula. In particular, as they give two formulas for the same class, this process produces relations in the Chow ring of $\overline{\mathcal{M}}_{g,n}$.

Question 4.1.1. *What do the relations that arise from comparing Pixton's formula with the Grothendieck-Riemann-Roch formula look like? How do they fit with the Faber-Zagier and Pixton relations?*

On the other hand, from the point of view of the compactified Jacobians, the double ramification cycle is a special case of the classes of all Brill-Noether loci. One can pull back these loci via the Abel-Jacobi section and calculate their classes on $\overline{\mathcal{M}}_{g,n}$ via Grothendieck-Riemann-Roch.

Question 4.1.2. *Is there an intepretation of the pulled back Brill-Noether loci in terms of Gromov-Witten theory? Are they related to the other codimension pieces in Pixton's formulas?*

One can in fact go further. Exploiting the vanishing of Brill-Noether loci in appropriate codimensions for negative degrees gives yet another large set of relations. We suspect the study of these relations will be very fruitful, and plan to carry out this study in the future.

4.2. **Abelian Varieties.** The moduli space of principally polarized Abelian varieties of dimenson g is not compact. Isolated toroidal compactifications exist in the literature [Ale02],[Ols08], such as the compactification $\mathcal{A}_g^{\mathrm{vor}}$ corresponding to the Voronoi decomposition. While a minimal toroidal compactification cannot exist, the existence of a minimal, non-representable compactification is established in recent work of Kajiwara, Kato and Nakayama, and indicates that the toroidal compactifications should be studied all together. This study should resolve the indeterminacy of certain maps – for instance, the Prym-Torelli map – and suggest that the truly interesting invariant should be the logarithmic Chow group of any (hence every) toroidal compactification.

Question 4.2.1. *What can we say about $\mathsf{logCH}(\mathcal{A}_g^{\mathrm{vor}})$? Does it have any interesting cycles?*

4.2.2. *Higher dimensional Picard Functors.* In [MW22], we studied the logarithmic Picard group of a family of curves. We expect that the analogous theory for appropriate families is within reach, and we plan to undertake the study in the future.

Definition 4.2.3. A logarithmic family is a proper, log smooth, flat map $X \to S$ with reduced fibers. A log blowup of a logarithmic family is a logarithmic family $Y \to T$ such that $T \to S$ and $Y \to X \times_S T$ are log blowups.

In particular, starting with a logarithmic family $X \to S$, we expect that there is a stack $\mathbf{LogPic}(X/S)$, parametrizing a certain collection of M_X^{gp} torsors on X, satisfying the following properties:

- $\mathbf{LogPic}(X/S)$ is a stack in the étale topology; it is a stack in the log étale topology when S is log flat.
- $\mathbf{LogPic}(X/S)$ has a cover by the ordinary $\mathbf{Pic}(Y/T)$ where $Y \to T$ ranges through all log blowups of T, and Y through all log blowups of $X \times_S T$.
- The stack $\mathbf{LogPic}(X/S)$ has a tropicalization, which is (after rigidification) a family of tropical abelian varieties.

In particular, the sheaf of isomorphism classes of $\mathbf{LogPic}(X/S)$ will be a sheaf $\mathrm{LogPic}(X/S)$. Assuming the construction goes through as expected, one can aim to study duality for logarithmic abelian varieties. We expect that given a log abelian variety A/S, the dual logarithmic abelian variety $A^\vee/S$ will be identified with $\mathrm{LogPic}(X/S)$. Evidence in this direction is provided in ongoing work [MUWon] with Ulirsch and Wise, in which we prove that the logarithmic Jacobian satisfies duality: Following ideas introduced by Deligne in his study of duality for $\mathbf{Pic}$, we construct a "Deligne" pairing $\mathbf{LogPic}(X/S) \times \mathbf{LogPic}(X/S) \to \mathbf{LogPic}(S)$ for any family of nodal curves, showing that $\mathbf{LogPic}(X/S)$ is its own dual log abelian variety. The perspective is of independent interest, as it allows to approach theta divisors on compactified Jacobians in a systematic way.

4.2.4. *Néron Models.* While this discussion is admittedly technical, it should have immediate applications. For example, if one were successful in proving the above results, the techniques of [HMOP20] would go through to provide necessary and sufficient conditions for finding Néron models of abelian schemes A_U/U over arbitrary dimensional bases U.

4.2.5. *Moduli of Abelian Varieties.*

4.3. **Compactifications of Moduli Spaces.** Framing the problem of finding toroidal compactifications of $\mathsf{Jac}(C/S)$ as that of finding blowups of $\mathsf{LogJac}(C/S)$ turns the problem of finding such compactifications into a combinatorial question: finding compatible subdivisions of a system of tropical abelian varieties. Using that, in [MMU$^+$on], we construct novel compactifications of $\mathsf{Jac}(C/S)$, which apply to a wider class of families of curves C/S – for instance, families which do not require a marked point on C.

Furthermore, the idea of using the log Jacobian to transport stability conditions on Jacobians to other problems should have several applications. Already, in ongoing work with di Lorenzo, Gross, Horn, and Ulirsch, we use the idea to produce modular compactifications of the Prym variety of a family of étale covers $C \to B$ over a base S. On the other hand, at the moment we do not know:

Question 4.3.1. *Do the compactified Pryms carry any interesting cycles?*

I suspect that the answer is positive – the compactified Pryms should carry natural Brill-Noether loci, and, moreso, the methods developed with Abreu-Pagani should be applicable to allow us to calculate them.

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