The strict transform in log geometry

based on ongoing work w/

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+ Pandharipande-Pixton-Schmitt,

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Interested in nice compactifications of a space

torus $T \rightsquigarrow$ toric variety X

smooth $U \rightsquigarrow$ smooth X w/ $X \setminus U = D$ normal crossings

From moduli:

$\mathcal{M}^{rel}(X,D) = \{ C \longrightarrow X$ w/ prescribed tangency behavior along $D \}$

D smooth:
Relative stable maps
(Li-Ruan, J. Li)

Log stable maps $K_\Gamma(X,D)$
ACGS

Ranganathan

- Toroidal compactifications of Jac on nodal curves (Oda-Seshadri)
- $A_g \subset \overline{A}_g$ toroidal compactifications

These compactifications are not unique

So how do we pick one?

- Maybe there is a minimal one

(If so, usually a great advantage)

E.g. $\overline{\mathcal{M}}_{g,n}$, $K_r(X,D)$

But often, no minimal compactification exists

e.g. no minimal compactification of torus.

It's better to try and work with all compactifications at once.

This has advantages even when a minimal compactification exists (but can be much harder)

Advantages: • For any given geometric problem, it lets us pick a compactification best adapted to the problem.

• Uncovers extra structure.

Sample problem

$$A = (a_1, \dots, a_n) \qquad \sum a_i = 0$$

$$B = (b_1, \dots, b_m) \qquad \sum b_i = 0$$

$$DR_{g,A} \text{ ``=''} \left\{ C \in \overline{\mathcal{M}}_{g,n} \mid \exists\, C \to \mathbb{P}^1 \text{ w/ ramification profile } A \right\}$$

$$= p_* \left[\mathcal{M}_{g,A}^{rel} (\mathbb{P}^1)_{\sim} \right]^{vir}$$

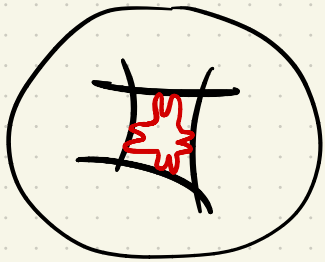
rubber

$$\begin{array}{c} \mathcal{M}_{g,A}^{rel}(\mathbb{P}^1) \\ \downarrow \\ \overline{\mathcal{M}}_{g,n} \end{array}$$

$\leadsto$ parametrizes maps to $\mathbb{P}^1$ w/ ramification profile A up to $\mathbb{C}^*$-action

Similarily

$$DDR_{g,A,B} \text{ "="} \left\{ C \to \mathbb{P}^1 \times \mathbb{P}^1 \text{ w/ ram. profile } A, B \right\}$$



$$= p_*\left[\mathcal{M}_{g,A,B}(\mathbb{P}^1 \times \mathbb{P}^1)^{\sim}_{rel} \right]^{vir}$$

Question: Is $DDR_{g,A,B} \in R^*(\overline{\mathcal{M}}_{g,n+m})$?

Proof: $DDR_{g,A,B} = DR_{g,A} \cdot DR_{g,B}$

$\in R^*(\overline{\mathcal{M}}_{g,n+m})$ by Faber-Pand. Pixton's formula

So yes.

But the proof is wrong, because

$$DDR_{g,A,B} \neq DR_{g,A} \cdot DR_{g,B}$$

But there are related classes

$$DR_{g,A}^{\log} \in CH(\tilde{\mathcal{M}}_{g,n}),\ DR_{g,B}^{\log},\ DDR_{g,A,B}^{\log}$$

$\tilde{\mathcal{M}}_{g,n} \xrightarrow{\pi} \overline{\mathcal{M}}_{g,n}$ some blowup along strata

s.t. $\bullet$ $\pi_* DR_{g,A}^{\log} = DR_{g,A}$

$\bullet$ $DDR_{g,A,B}^{\log} = DR_{g,A}^{\log} \cdot DR_{g,B}^{\log}$

We have

$$DR_{g,A}^{\log} = \pi^* DR_{g,A} + \text{correction}$$

So if we can understand the correction term, the wrong proof could succeed.

$$\begin{tikzcd}
 & \widetilde{\mathcal{M}}_{g,n+m} \arrow[dl, "\nu"'] \arrow[d, "\sigma"] \arrow[r, "\mu"] & \widetilde{\mathcal{M}}_{g,n} \arrow[d, "\pi"] \\
\widetilde{\mathcal{M}}_{g,m} \arrow[d, "\rho"'] & \overline{\mathcal{M}}_{g,n+m} \arrow[dl, "\nu"] \arrow[r, "\mu"'] & \overline{\mathcal{M}}_{g,n} \\
\overline{\mathcal{M}}_{g,m} & &
\end{tikzcd}$$

$$DDR_{g,A,B} = \sigma_*\left(DDR^{\log}_{g,A,B}\right)$$

$$= \sigma_*\left(\mu^* DR^{\log}_{g,A} \cdot \nu^* DR^{\log}_{g,B}\right)$$

$$= \sigma_*\left(\left(\mu^*\pi^* DR_{g,A} + \text{corr.}\right)\left(\nu^*\rho^* DR_{g,B} + \text{corr}\right)\right)$$

$$= \sigma_*\left(\left(\sigma^* \mu^* DR_{g,A} + \text{corr}\right)\left(\sigma^*\nu^* DR_{g,B} + \text{corr}\right)\right)$$

$$= \mu^* DR_{g,A} \cdot \nu^* DR_{g,B} + \mu^* DR_{g,A} \cdot \cancel{\sigma_*(\text{corr})}^{\,0}$$

$$+ \nu^* DR_{g,B} \cdot \cancel{\sigma_*(\text{corr})}^{\,0}$$

$$+ \sigma_*(\text{corr.} \cdot \text{corr.})$$

Context : Logarithmic geometry

Extra data on scheme X

Log Scheme = $(X, \mathcal{M}_X)$ sheaf of monoids

$$\varepsilon : \mathcal{M}_X \longrightarrow \mathcal{O}_X$$

$$\varepsilon^{-1}(\mathcal{O}_X^*) = \mathcal{O}_X^*$$

Map $(X, \mathcal{M}_X) \longrightarrow (Y, \mathcal{M}_Y)$

$$f : X \longrightarrow Y$$

$+$

$$\begin{tikzcd} f^{-1}\mathcal{M}_Y \arrow[r] \arrow[d] & \mathcal{M}_X \arrow[d] \\ f^{-1}\mathcal{O}_Y \arrow[r] & \mathcal{O}_X \end{tikzcd}$$

$\overline{\mathcal{M}}_X = \mathcal{M}_X / \mathcal{O}_X^*$ "characteristic monoid"

For geometry : extra assumptions

- $\overline{\mathcal{M}}_X$ finitely generated, integral, saturated (f.s.)
- "Coherent" (has atlas of local charts)

Consequences, all I'll use:

- Log scheme comes w/ stratification $\{\overline{M}_{X,x} = \text{constant}\}$

- $\forall x \in X$,

$$\begin{tikzcd} U_x \arrow[r, "f_x"] \arrow[d, "\text{ét}"'] & V(\sigma_x) \\ X & \end{tikzcd}$$

affine toric variety of cone $\sigma_x \simeq \overline{M}_{X,x}^\vee$

stratified map.

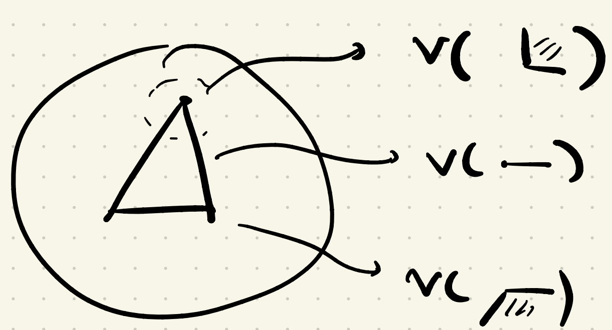
The choice of σ_x is locally constant on strata.

<u>Examples</u>

- X toric variety

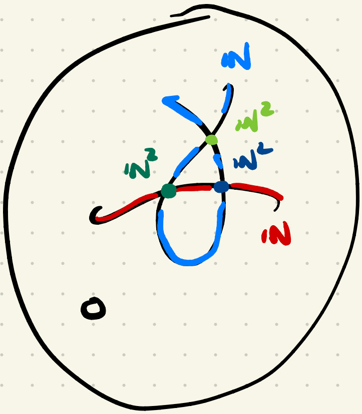
  M_X generated by units/monomials.

$\mathbb{P}^2$



Normal crossings pair (X,D)

$\mathcal{M}_X(U) = \{ f \in \mathcal{O}_X(U) \mid f$ is a unit on $U \setminus D \}$

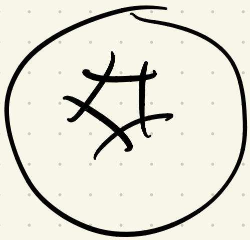


Choice of log structure

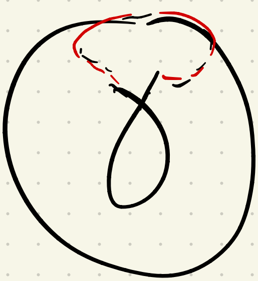
$\Longleftrightarrow$

Choice of D.

Simpler (no self-intersection)

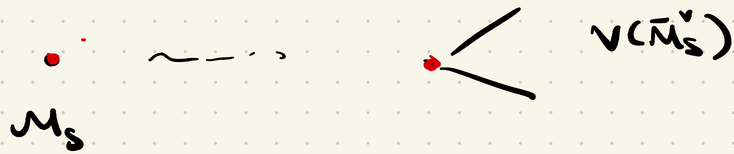


Maximally complicated (monodromy)

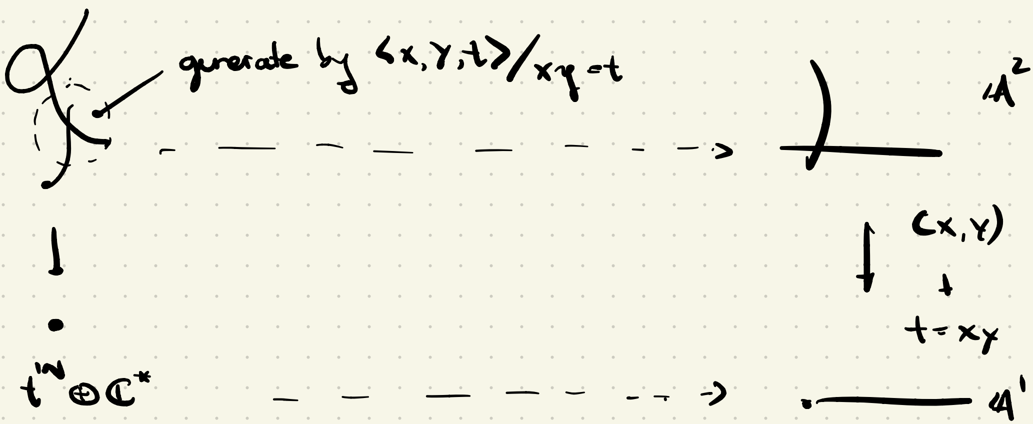


Also less geometric examples

- Log point : $S = \operatorname{Spec} \mathbb{C}$, interesting log structure.



+ family examples



<u>Def</u> : X is log smooth/toroidal if

charts f_x are smooth

I'll stick with that for the rest of the talk.

Our Tools:

(1) To each log scheme, we assign two combinatorial objects (tropicalization)

$\swarrow$ $\searrow$

$C(X)$ = cone complex of X (KKMSD)

$\mathcal{A}_X$ = "Artin fan" (Abramovich-Wise)

For X toroidal, no self-intersection:

- For each stratum S, w/ generic point x_S, look at cone σ_{x_S}.
- When S specializes to T ($x_T \in \overline{x_S}$) $\sigma_{x_S} \subset \sigma_{x_T}$ is a face

- $C(X) = \varinjlim_{\text{strata}} \sigma_{X_S}$

e.g. when X is a toric variety, $C(X)$ = fan of X

- $\mathcal{A}_X$ is built the same way

Instead of σ_X, look at $\bar{\mathcal{M}}_{X,x} = \sigma_X^\vee$

$$\mathcal{A}_{\bar{\mathcal{M}}_{X,x}} := \left[V(\sigma_X) / T_{V(\sigma_X^\vee)} \right]$$

Stack w/ log structure

$$\mathcal{H}om_{\mathrm{LogSt}}(\mathcal{A}_P, \mathcal{A}_Q) = \mathcal{H}om(Q, P) = \mathcal{H}om(P^\vee, Q^\vee)$$

So $\mathrm{Hom}(\sigma_{X_S}, \sigma_{X_T}) = \mathcal{H}om(\mathcal{A}_{\bar{\mathcal{M}}_{X,x_S}}, \mathcal{A}_{\bar{\mathcal{M}}_{X,x_T}})$

$$C(X) = \underset{\text{strata}}{\underline{\lim}}\ \sigma_{X_s}$$

$$\mathcal{A}_X = \varinjlim_{\text{strata}} \mathcal{A}_{\overline{N}_{X, X_s}}$$

$C(X)$ more geometric

$\mathcal{A}_X$ easier to relate to X

because it comes with a map

$$X \xrightarrow{\alpha_X} \mathcal{A}_X$$

<u>Example</u>: X is toroidal $\iff$ α_X is smooth

<u>Example</u> $(\underset{\text{smooth}}{X}, D)$

$$C(X) = \bullet\!\!-\!\!-\!\!-\!\!-\!\!-$$

$\mathcal{A}_X = [\mathbb{A}^1/\mathbb{G}_m]$ = moduli space of line bundles w/ section

$(X, D) \longrightarrow \mathcal{A}^1$ tautological map given by D

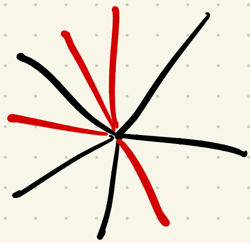
The constructions unlock some basic geometry.

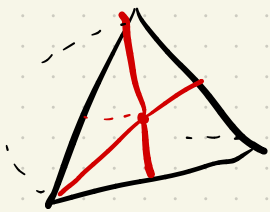
(2)

Log Modifications

- Subdivide

  $C(X)$





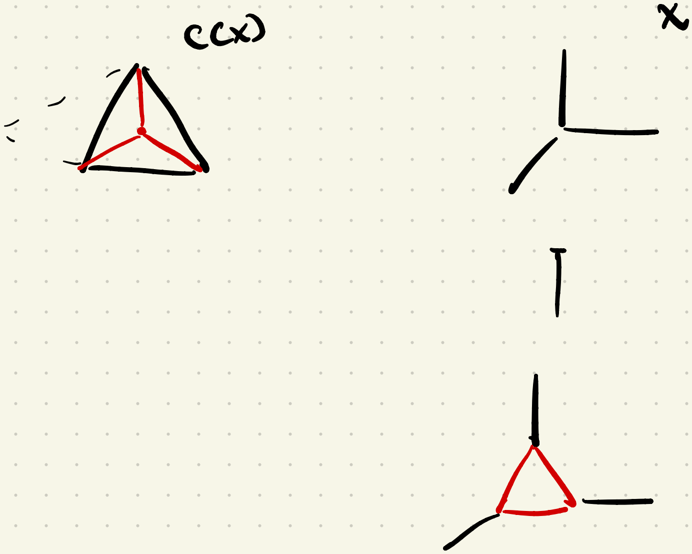
$$\begin{tikzcd} \tilde{\mathcal{U}}_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V(\tilde{\sigma}_x) \arrow[d] \\ \mathcal{U}_x \arrow[r] \arrow[d] & V(\sigma_x) \\ X & \end{tikzcd}$$

Local subdivisions glue to $\tilde{X} \longrightarrow X$

a proper birational map

"a log modification"

Ex: X is smooth, toroidal



star subdivision = blowup at stratum

- Star subdivisions are cofinal

Any $\tilde{X} \to X$ is dominated by

$$\begin{tikzcd} \tilde{\tilde{X}} \arrow[rr] \arrow[dr] & & \tilde{X} \arrow[dl, "\checkmark"] \\ & X & \end{tikzcd}$$

sequence of star subdivisions = sequence of blowups along non-singular strata

(3) Also get a distinguished subring

$$\alpha^* CH^*_{op}(\mathcal{A}_X) \subset CH^*_{op}(X)$$

"the logarithmic tautological ring".

<u>Thm</u>: $\alpha^* CH^*_{op}(\mathcal{A}_X)$ generated by Chern roots of normal bundles of strata of X

<u>Example</u>

- If X is a smooth toric variety,

$$\alpha^* CH^*_{op}(\mathcal{A}_X) = CH^*_{op}(X) = CH_*(X)$$

- If $X = (\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n})$

$$\lambda_g \in \alpha^* CH^*_{op}(\mathcal{A}_X)$$

$$\psi_i \notin \alpha^* CH^*_{op}(\mathcal{A}_X)$$

Theorem

$$CH^*_{op}(\mathcal{A}_X) = PP(C(X))$$

is a combinatorial object.

Proof requires some understanding

of $CH_*(-,1)$

some stack like $B\mathbb{G}_m^k$.

(A consequence is that $CH^*_{op}(\mathcal{A}_X)$

satisfies étale descent)

In summary, two tools to analyze X

- Its combinatorial shadow $C(X)$
- Simple piece $\alpha^* CH^*_{op}(X)$ of its cohomology

The trouble

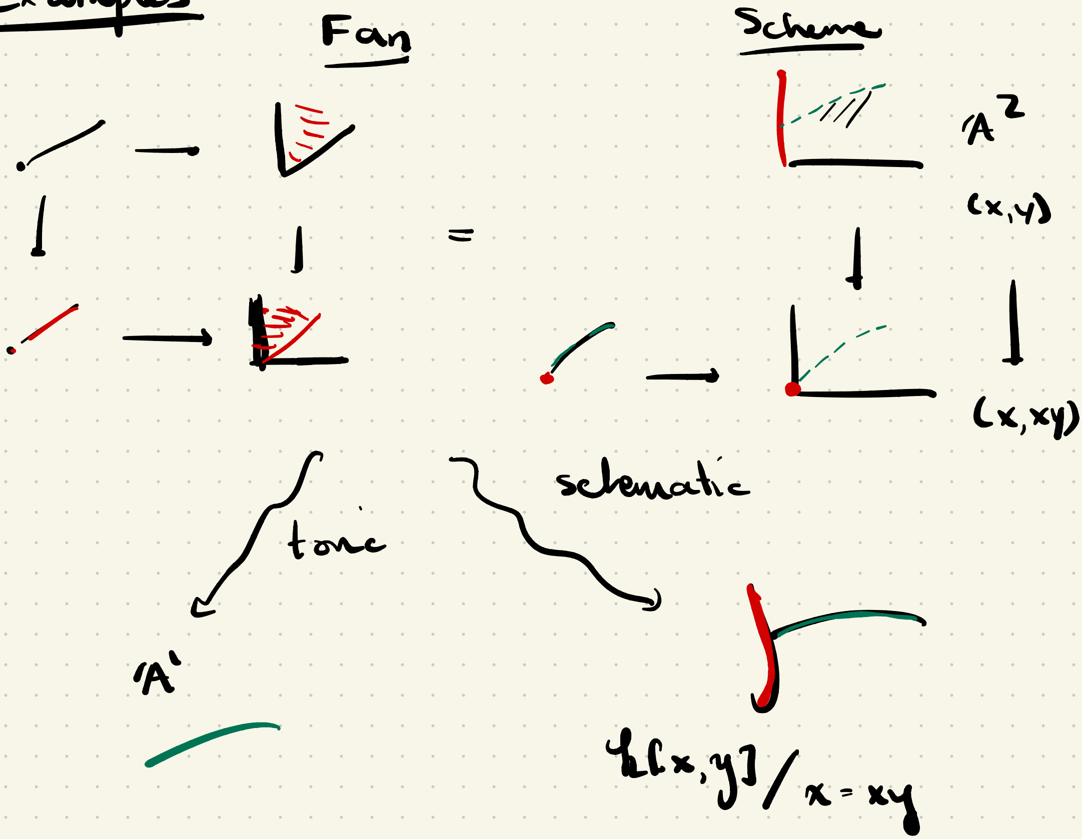
Elementary observation

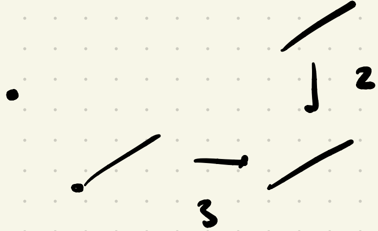
Fiber product in category of toric varieties

$\neq$

Fiber product in category of schemes

Examples





$$\begin{array}{ccc} & \mathbb{A}^1 & t \\ & \downarrow & \downarrow \\ \mathbb{A}^1 & \longrightarrow \ \mathbb{A}^1 & \\ t & \longrightarrow \ t^2 & t^3 \end{array}$$

$\mathbb{A}^1$

$k[x,y]/y^2 = x^3$ {

Theorem

Let $\begin{tikzcd} & Z \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$ be a toric diagram.

(i) Then the toric fiber product $X \times_Y^{fs} Z$ ← notation

is the normalization of the closure

of $T_X \times_{T_Y} T_Z$

(ii) In fact, $X \times_Y Z$ has an inductive description.

Irreducible components have following form:

Look at strata $O(\sigma) \times_{O(\kappa)} O(\tau)$

Normalization of closure $= \overline{O(\sigma)} \times^{fs}_{\overline{O(\kappa)}} \overline{O(\tau)}$

Irreducible components are maximal)

(iii) When $X \to Y$ is a log modification, no normalization is needed.

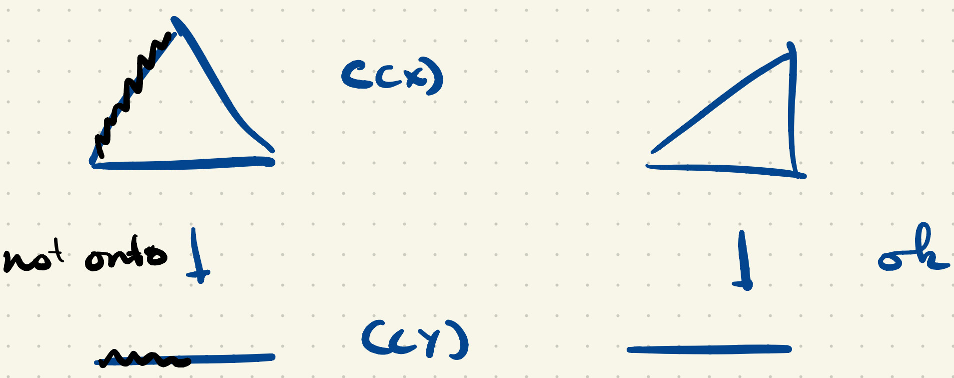
There is one case where everything works

$$\begin{tikzcd} & X \arrow[d] \\ Z \arrow[r] & Y \end{tikzcd}$$

If in $C(Z) \longrightarrow C(Y)$

"weakly semistable" $\begin{cases} \bullet \text{ cones map } \underline{\text{onto}} \text{ cones} \\ \bullet \text{ integral structure maps onto integral structure} \end{cases}$

Then $X \times_Y^{fs} Z = X \times_Y Z$



The geometric meaning is

- flat with reduced fibers when $X \to Y$ is dominant

- Intersecting strata transversely for $X \to Y$ closed embedding

The discussion generalizes verbatim to toroidal schemes.

The upshot of this discussion is

bunch of components, indexed by strata of $V, Y, \tilde{Y}$

$$\begin{tikzcd}
V \times^{fs}_Y \tilde{Y} \subset V \times_Y \tilde{Y} \arrow[r] \arrow[d] & \tilde{Y} \arrow[d, "\text{log modification}"] \\
V \arrow[r, "\text{log map}"'] & Y
\end{tikzcd}$$

$\boxed{\text{the strict transform of } V}$

$\boxed{\text{for any diagram of log maps between toroidal varieties}}$

The operation

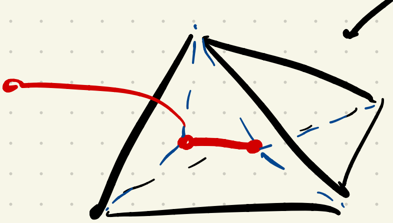
$$V \longrightarrow V \times_Y^{fs} \tilde{Y} \text{ eventually stabilizes}$$

Look at

$$C(V) \longrightarrow C(Y)$$

slice of 4d core

slice of 2d core $C(V)$



Any subdivision $C(\tilde{Y})$ that contains $C(V)$

For any $\tilde{\tilde{Y}} \longrightarrow \tilde{Y}$, $\quad V \times_Y^{fs} \tilde{\tilde{Y}} = (V \times_Y^{fs} \tilde{Y}) \times_{\tilde{Y}} \tilde{\tilde{Y}}$

Consequence

Given $V \to Y$, the class

$[V \times^{fs}_Y \tilde{Y}]$ stabilizes in

system of blowups $\tilde{Y} \to Y$

Meaning

$$\begin{tikzcd} & & & V \arrow[dl] \\ \tilde{\tilde{Y}} \arrow[r, "f"] & \tilde{Y} \arrow[r] & Y & \end{tikzcd}$$

$C(\tilde{Y})$ contains $C(V)$ as subcomplex

$$f^*[V \times^{fs}_Y \tilde{Y}] = [V \times^{fs}_Y \tilde{\tilde{Y}}]$$

Def: $[V]^{\log} = [V \times^{fs}_Y \tilde{Y}] \in CH_*(\tilde{Y})$.

(Fancy statement: $[V]^{\log} \in \mathrm{LogCH}^*(Y)$

$$:= \varinjlim_{\substack{\tilde{Y} \to Y \\ \text{log modif.}}} CH^*(\tilde{Y})$$)

Think

$$\begin{tikzcd} DR^{\log} = V^{\log} \arrow[r] \arrow[d] & V = DR \arrow[d] \\ \widetilde{\mathcal{M}}_{g,n} = \tilde{Y} \arrow[r] & Y = \overline{\mathcal{M}}_{g,n} \end{tikzcd}$$

some blowup determined by combinatorics

of $C(V) \longrightarrow C(Y)$

"tropical stable maps"

"tropical curves"

In reality, picture is a bit more complicated because of the virtual structure

not log smooth

carries p.o.t E

p.o.t pulls back

log smooth!

a log mod.

$$
\begin{tikzcd}
\mathcal{M}^{\log} \arrow[r] \arrow[d, "f"] \arrow[dd, bend right, "p"'] \arrow[dr, phantom, "\text{f.s. } \square"] & \mathcal{M}^{rel}_{g,A}(\mathbb{P}^1)^{\sim} \arrow[d] \\
V^{\log} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{f.s. } \square"] & \mathcal{M}^{rel}_{g,A}(\mathcal{A}_{\mathbb{P}^1}) = V \arrow[d] \\
\widetilde{\mathcal{M}}_{g,n} \arrow[r] & \overline{\mathcal{M}}_{g,n} = Y
\end{tikzcd}
$$

$$DR^{\log}_{g,A} := p_* [\mathcal{M}^{\log}]^{vir}$$

$$= p_* f^!_E [V^{\log}]$$

So how do I compare $V^{\log}$ with V?

$$\tilde{Y} \xrightarrow{f} Y$$

$f_{\log}^{*} V = V^{\log}$ = strict transform

$f^{*} V$ = total transform

Fulton: Suppose $\tilde{Y} \to Y$ blowup along smooth center X

$$\begin{tikzcd} \text{exceptional} \arrow[r] & \tilde{X} \arrow[r, "j"] \arrow[d, "g"'] & \tilde{Y} \arrow[d, "f"] \\ & X \arrow[r, "i"'] & Y \end{tikzcd}$$

$\exists$ <u>excess bundle</u>

$$E = g^{*} N_{X/Y} / N_{\tilde{X}/\tilde{Y}}$$

Pull back to V:

$$\begin{tikzcd} V \times_Y \tilde{X} \arrow[r, "j"] \arrow[d, "g"'] & V \times_Y \tilde{Y} \arrow[d, "f"] \\ V \times_Y X \arrow[r, "i"'] & V \end{tikzcd} , E$$

Then $f^* V - V^{\log}$

$$= j_*\left(c(E) \cdot g^* s(X \times_Y V, V)\right)$$

Segre Class

generally hard to compute

A class in

$V \times_Y \tilde{Y} = \bigcup$ (Components indexed by strata of V that map into X)

But because $V \longrightarrow Y$ is a log map

$V \times_Y X \subset V$ is a monomial subscheme.

$$
\begin{tikzcd}
V \times_Y X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{A}_V \times_{\mathcal{A}_Y} \mathcal{A}_X \arrow[d] \\
V \arrow[r, "\alpha"'] & \mathcal{A}_V
\end{tikzcd}
$$

$$s(V \times_Y X, V) = \alpha^* s(\mathcal{A}_V \times_{\mathcal{A}_Y} \mathcal{A}_X, \mathcal{A}_V)$$

$$= \text{a piecewise polynomial class.}$$

In fact, there is a remarkable formula

<u>Thm</u> (Aluffi) Assume (V, D) normal crossings pair

$$s(X \times_Y V, V) = \sum_{\substack{\text{strata of} \\ V}} (\text{coefficient}) \cdot D_1^{\alpha_1} \cdots D_n^{\alpha_n}$$

Consequence

$\mathcal{V}^{\log}$ is the subring of $CH^*(\tilde{Y})$ generated by $f^*\mathcal{V}$ and image of $PP(V)$

(For DR, because of virtual structure, get ring generated by f^*DR, $PP(\tilde{Y})$, lower dimensional DRs)

<u>Proof</u>: After replacing $\tilde{Y}$ with finer one, can factor

$\tilde{Y} \to Y$ as

$$\tilde{Y} = Y_n \to Y_{n-1} \to Y_{n-2} \to \cdots \to Y_1 \to Y_0 = Y$$

blowups along smooth centers.

Then apply Fulton + Aluffi.

Corollary:

$DDR_{g,A,B}$ is tautological

$$\sigma : \tilde{\mathcal{M}}_{g,n+m} \longrightarrow \overline{\mathcal{M}}_{g,n+m}$$

$$DDR_{g,A,B} = DR_{g,A} \cdot DR_{g,B}$$
$$+ \sigma_*(PP \cdot \pi(\text{smaller } DRs))$$
$$\in R^*(\overline{\mathcal{M}}_{g,n+m})$$

Questions

More or less everything is open.

- Effectwize algorithm.

e.g $DR_{g,A}$ has a formula

Can we write formula for $DR_{g,A}^{\log}$?

We are trying with PPS, but using other methods; algorithm too complicated.

- Can we instead relax assumptions in Fulton's formula? Relying on lci center seems artificial

- Is $DR_{g,A}^{\log}$ an honest strict transform using the right model?

- Write down structure of $DR^{\log s}$ in $\log CH(\overline{M}_{g,n})$

- Develop some calculus for $\log CH^*$