The Logarithmic Tautological Ring

The logarithmic tautological ring. (w/ Pandharipande-Schmitt)

Motivation $\left\{ \begin{array}{l} \text{Relative GW theory / DR cycles} \\ \\ \text{Degenerations of abelian varieties} \end{array} \right.$

$$\begin{array}{c} \mathcal{C}_g \\ \downarrow \pi \\ \mathcal{M}_g \end{array} \qquad \mathbb{E}_g = \pi_* \omega_\pi \quad \text{Hodge bundle}$$

$$\lambda_g := c_g(\mathbb{E}_g)$$

Formula (Hain, Grushevsky-Zakharov)

$$\begin{tikzcd} \mathrm{Pic}^0(\mathcal{C}_g) \arrow[r] \arrow[d] & \mathcal{U}_g \arrow[d] \\ \mathcal{M}_g \arrow[r, "t"'] \arrow[u, bend left, "AJ"] & \mathcal{A}_g \end{tikzcd}$$

$\mathcal{A}_g$ = moduli space of PPAVS

$$\lambda_g = (-1)^g AJ^*([0])$$

But $[0] \in CH^*(\mathcal{U}_g)$ satisfies

$$[0] = \frac{\Theta^g}{g!} \quad \text{(Denninger-Murre)}$$

So $\lambda_g = \dfrac{T^g}{g!}$ for some $T \in CH^1(\mathcal{M}_g)$

Can ask the same question over $\overline{\mathcal{M}}_g$

$$\begin{tikzcd}
\overline{\mathcal{U}}_g \times_{\overline{\mathcal{A}}_g} \overline{\mathcal{M}}_g \arrow[r] \arrow[d] & \overline{\mathcal{U}}_g \arrow[d] \\
\overline{\mathcal{M}}_g \arrow[r, "t"'] & \overline{\mathcal{A}}_g
\end{tikzcd}$$

any compactification for which the Torelli extends, e.g Alexeev/Olsson.

Does the formula for [0] section/λ_g extend?

Answer: No. In fact

$$\lambda_g \notin \operatorname{div} CH^*(\overline{\mathcal{M}}_g)$$

i.e λ_g is not a product of divisors for $g > 2$.

(Explicit calculation w/ admcycles + recursion)

Nevertheless

$$\begin{array}{ccl} & \mathcal{L}Pic^0(\bar{C}_g) & + \Theta \text{ divisor} \\ AJ & \big(\ \downarrow & \\ & \bar{M}_g & \end{array}$$

Still expect DM formula to hold

Difficulty: $\mathcal{L}Pic^0$ is not an algebraic stack.

So $CH(\mathcal{L}Pic^0)$ is meaningless.

However, $\mathcal{L}Pic^0$ has a (log) blowup which is algebraic.

So can consider question in

$$\log CH^*(\bar{M}_g) = \lim_{\substack{\tilde{M}_g \to \bar{M}_g \\ \text{blowups along} \\ \text{bdry strata}}} CH^*(\tilde{M}_g)$$

<u>Thm</u> (wP, S)

$$\lambda_g \in \operatorname{div} \log CH^*(\bar{M}_g)$$

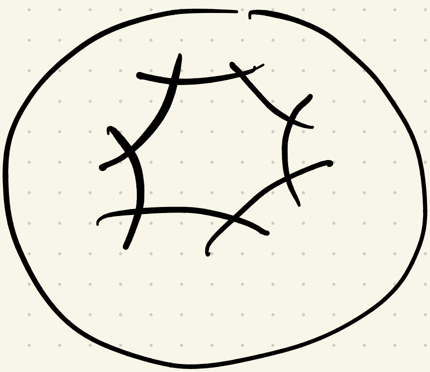
Log tautological ring

(X, D) a normal crossings pair
or toroidal embedding

(X, D) has a natural log structure

$$\mathcal{M}_X(U) = \{ f \in \mathcal{O}_X(U) \mid f \text{ a unit on } U \setminus D \}$$

Subtlety: Must consider on étale site

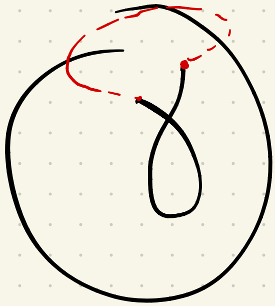


When irreducible components of D are smooth

"Toroidal embedding w/o self-intersection" (KKMSD)

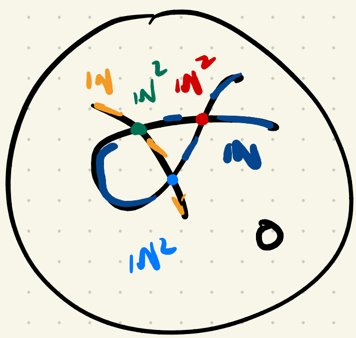
then $\mathcal{M}_X$ can be defined on Zariski site.

In general, can have self-intersection + monodromy



and $\mathcal{M}_X$ is an étale sheaf

$\overline{\mathcal{M}}_X := \mathcal{M}_X / \mathcal{O}_X^*$ is constructible and stratifies X



Construction:

Let S be a stratum

$\tilde{S}$ normalization of $\bar{S}$

$$\tilde{S} \xrightarrow{\varepsilon} X$$

Normal bundle N_ε étale locally splits

If $S = D_1 \cap \dots \cap D_k$ étale locally,

$$N_\varepsilon = \varepsilon^* \mathcal{O}(D_1) \oplus \dots \oplus \mathcal{O}(D_k)$$

But globally, monodromy obstructs this.

Let G = monodromy group acting on the D_i.

$$N_\varepsilon = \bigoplus_{\gamma = G\text{-orbits}} N_\gamma \longleftarrow \text{rk} = \text{size of orbit.}$$

There is a G-torsor

$$\begin{tikzcd} T \arrow[r, "p"] \arrow[dr, "i"'] & \tilde{S} \arrow[d, "\varepsilon"] \\ & X \end{tikzcd}$$

and $p^* N_\varepsilon = N_1 \oplus \dots \oplus N_k$

Let P be a G-invariant polynomial.

<u>Def</u>: $\iota_* P(c_1(N_1), \dots c_1(N_k)) \in CH^*(X)$

is a <u>normally</u> <u>decorated</u> <u>strata class</u>

<u>Def</u> The logarithmic tautological ring is

$$R^*_{\log}(X, D) := \langle \text{normally decorated strata classes} \rangle$$

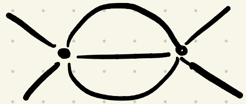
<u>Caution</u>

$$R^*_{\log}(\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n}) \subsetneq R^*(\overline{\mathcal{M}}_{g,n})$$

Connection with DR:

<u>Pixton's formula</u>

$\Gamma = (V(\Gamma), H(\Gamma))$, $A = (a_1, \dots, a_n)$, $\sum a_i = 0$



Weighting mod r:

$$w : H(\Gamma) \longrightarrow \mathbb{Z}/r\mathbb{Z}$$

- $w(h) = -w(h')$ if $\{h, h'\}$ is an edge
- $w(l_i) = a_i$ for i-th leg
- $\sum_{h \ni v} w(h) = 0 \quad \forall v \in V(\Gamma)$

To any Γ, assign

$$\mathcal{M}_\Gamma = \prod \mathcal{M}_{v, n(v)} \xrightarrow{\xi_\Gamma} \overline{\mathcal{M}}_{g,n}$$

monodromy torsor from before

Pixton's formula

$$P^r_{g,A} = \sum_{\Gamma} \sum_{\substack{\text{weightings } w \\ \text{of } \Gamma}} \frac{1}{\mathrm{Aut}(\Gamma)} \frac{1}{r^{h_1(\Gamma)}}$$

$$(\xi_\Gamma)_* \left\{ \prod_{h \text{ legs}} \exp(a_i^2 \psi_{l_i}) \cdot \prod_{\substack{e=(h,h') \\ \text{edges}}} \frac{1 - \exp(-w(h)w(h'))(\psi_h + \psi_{h'})}{\psi_h + \psi_{h'}} \right\}$$

Degree d piece

$P^{r,d}_{g,A}$ = polynomial in r for $r \gg 0$

Constant term of $P^{r,g}_{g,A} = DR_{g,A}$

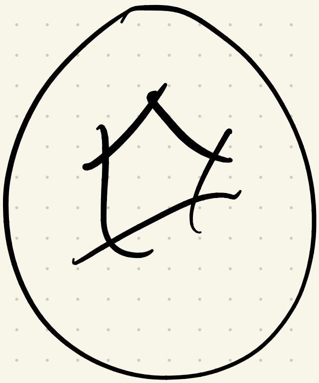
For $A = \emptyset$, no leg contributions

So $\lambda_g \in R^*_{\log}(\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n})$

(in fact $R^*_{\log}(\overline{\mathcal{M}}_{g,n}, \Delta_0)$)

Log Chow

Instead of $\overline{\mathcal{M}}_X$, build combinatorial space that recovers $\overline{\mathcal{M}}_X$



(X, D) toroidal w/o self-intersection

For generic point x_S of stratum S

$$\sigma_{X,x_S} = \overline{\mathcal{M}}^{\vee}_{X,x_S}$$

If $x_T \rightsquigarrow x_S$ (i.e $x_S \in \overline{\{x_T\}}$)

$\sigma_{X,x_T} \subset \sigma_{X,x_S}$ is a face

KKMSD $\quad C(X,D) = \varinjlim_{\text{strata } S} \sigma_{X,x_S}$

rational polyhedral cone complex

Ex: When X = toric variety,

$C(X, X \setminus T)$ = fan of X, without embedding into cocharacter lattice

In the presence of monodromy presentation

$$\mathcal{U} \rightrightarrows \mathcal{V} \to (X, D)$$

as small as possible

$$C(X,D) = [\, C(\mathcal{U}) \rightrightarrows C(\mathcal{V}) \,]$$

Stack over category of rat. polyhedral cones

Think about it as colimit of cones $\sigma_{x,\bar{x}} = \overline{M}^{\vee}_{x,\bar{x}}$

- maps: isomorphisms onto a face
- potentially more than one map between 2 objects.

Ex above:

Def: A subdivision $C' \to C(X,D)$ is a subdivision of each cone compatible w/ all maps in the colimit.

Construction: For $C' \to C(X,D)$, get

étale locally

$$\begin{tikzcd} U' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V(C'|_{\sigma_{x,x}}) \arrow[d] \\ U \arrow[r] \arrow[d] & \operatorname{Spec} k[\bar{M}_{x,\bar{x}}] = V(\sigma_{x,x}) \\ X & \end{tikzcd}$$

Compatibility means the local U' glue to

$$(X', D') \longrightarrow (X, D)$$

with $C' = C(X', D')$

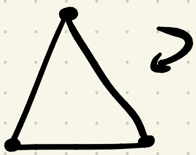
Def Such a map is called a log modification.

- For a stratum S w/ $\bar{S}$ normal,

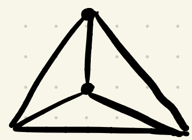
  star subdivision of core of S =

  blowup of X at $\bar{S}$. ← "Simple blowup"

<u>Example</u>

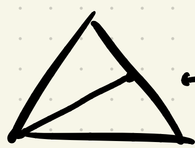


$\mathbb{Z}/3\mathbb{Z}$



blowup at point

triple point



doesn't work

because not equivariant

<u>Def</u>

$$\log CH^*(X,D) = \varinjlim_{\substack{(X',D') \to (X,D) \\ \text{log} \\ \text{modification} \\ X' \text{ smooth}}} CH^*(X')$$

- Maps in colimit are indexed by Gysin pullback
- Simple blowups are cofinal among log modifications: For any

$$(X',D') \longrightarrow (X,D) \text{ log mod., there}$$

exists

$$\begin{tikzcd} (X'',D'') \arrow[rr, "f''"] \arrow[dr] & & (X,D) \\ & (X',D') \arrow[ur] & \end{tikzcd}$$

s.t f'' is a composition of simple blowups.

Alternate point of view

Artin fan (Abramovich-Wise)

For monoid P,

$$\mathcal{A}_P := \left[\operatorname{Spec} k[P] / \operatorname{Spec} k[P^{gp}]\right]$$

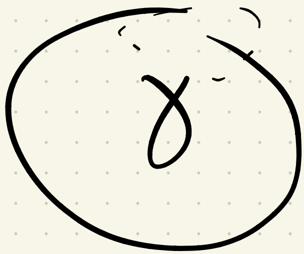
alg. stack w/ log structure

$$\operatorname{Hom}_{LogSt}(\mathcal{A}_P, \mathcal{A}_Q) = \operatorname{Hom}_{Mon}(Q, P) = \operatorname{Hom}_{Cone}(P^\vee, Q^\vee)$$

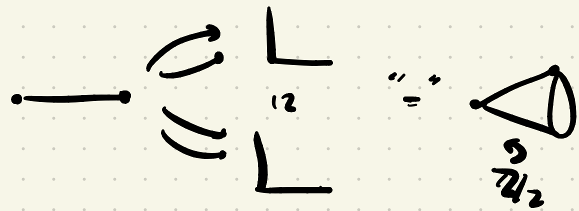
i.e. $P^\vee \longrightarrow \mathcal{A}_P$ embeds cat. of cones into cat. of log stacks.

$\Sigma(X, D)$ = a colimit of cones $\sigma_{x,\bar{x}} = \overline{M}_{X,\bar{x}}^\vee$

$\mathcal{A}_X = \mathcal{A}_{(X,D)}$ = <u>same</u> colimit of $\mathcal{A}_{\overline{M}_{X,\bar{x}}}$



$\rightsquigarrow$ $\mathcal{C}(X,D) =$



$$\mathcal{A}_X = \quad [\mathbb{A}^1/\mathbb{G}_m] \rightrightarrows [\mathbb{A}^2/\mathbb{G}_m^2]_{(x,y)} \cong [\mathbb{A}^2/\mathbb{G}_m^2]^{(y,x)}, \qquad t \mapsto (t,1),\ (1,t)$$

" = "

$B(\mathbb{G}_m^2 \rtimes \mathbb{Z}/2)$ $\quad B\mathbb{G}_m \quad$ pt

- $\mathcal{A}_X$ is universal log space w/ combinatorics governed by $\mathcal{C}(X,D)$.
- There is a <u>smooth</u> map $X \to \mathcal{A}_X$
- Subdivisions $\mathcal{C}' \to \mathcal{C}(X,D)$ also give proper birational map $\mathcal{A}_X' \to \mathcal{A}_X$

and

$$\begin{tikzcd}
X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{A}_{X'} \arrow[d] \\
X \arrow[r, "\alpha"'] & \mathcal{A}_X
\end{tikzcd}$$

Thm $R^*_{\log}(X,D) = \operatorname{Im}(CH^*_{op}(\mathcal{A}_X) \to CH^*_{op}(X))$

Proof: Every step in the construction of a normally decorated strata class is pulled back from $\mathcal{A}_X$

$$\begin{tikzcd}
& T \arrow[rr, "b"] \arrow[dl, "p"'] \arrow[dd, "i"] & & \mathcal{T} \arrow[dl, "q"'] \arrow[dd, "j"] \\
\tilde{S} \arrow[rr, dashed, "\gamma"] \arrow[dr, "\varepsilon"'] & & \tilde{\mathcal{S}} \arrow[dr, "\delta"'] & \\
& X \arrow[rr, "\alpha"'] & & \mathcal{A}_X
\end{tikzcd}$$

$p^* N_\varepsilon = b^* q^* N_\delta$

and so n.d.s.c $\iota_* x$ is of form

$\iota_* b^* y = \alpha^* j_* y$

Consequences:

- For a log modification $(\tilde{X}, \tilde{D}) \xrightarrow{f} (X, D)$,

$$f_* R^*_{\log}(\tilde{X}, \tilde{D}) = R^*_{\log}(X, D)$$

$$f^* R^*_{\log}(X, D) \subset R^*_{\log}(\tilde{X}, \tilde{D})$$

Connection w/ combinatorics

<u>Thm</u>: $CH^*_{op}(A_X) = PP^*(C(X))$

Ingredients in the proof

- For a cone $\sigma = \mathbb{N}^k$,

$$CH^*_{op}(A_{\mathbb{N}^k}) = PP^*(\mathbb{N}^k) = \mathbb{Q}[x_1, \dots, x_k]$$

easy

- When X is smooth,

$$A_X = \varinjlim A_{N^k}$$

and one can show

$$CH^*_{op}(\varinjlim A_{N^k}) = \varinjlim CH^*_{op}(A_{N^k})$$

- For general X, need descent for CH^*_{op} for blowups

⟶ For schemes : Kimura

stacks : Bae - Park

- From the descent we can reduce to case of smooth X,

or even the theorems of Brion/Payne

which prove the analogous statement for toric varieties.

In any case, $CH^*(\ \ ,1)$ play a key role.

Some consequences

- The dichotomy of $\mathcal{M}_X$

Zariski	Étale
No self-intersection	Self-intersection + monodromy

can be seen on $C(X,D)$

No self-intersection $\iff$

$C(X,D)$ can be PL embedded in a vector space.

When

$C(X,D)$ is <u>also</u> simplicial

(i.e each cone in $C(X,D)$ is simplicial)

$$PP^*(C(X,D)) = \text{Stanley-Reisner ring of } C(X,D)$$

$$= k[\text{rays}] \Big/ \text{ideal of non-faces}$$

$$= k[x_\rho] \Big/ \langle x_{i_1} \cdots x_{i_k} : (i_1, \dots, i_k) \text{ do not span a face} \rangle$$

In particular,

$R^*_{\log}(X,D)$ is then generated by divisors.

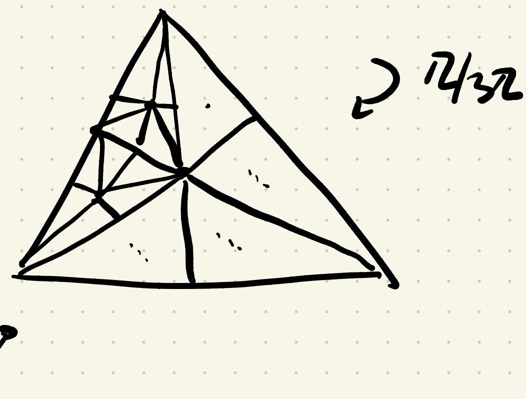
Thm λ_g is in $\operatorname{dir}\mathcal{L}og\,CH(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g)$

Proof: $\lambda_g \in R^*_{\log}(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g)$

But for every (X, D), $\exists$ log modification s.t $(\tilde{X}, \tilde{D})$ is Zariski + simplicial

E.g. Canonical choice

second barycentric subdivision



Every cone is generated by a unique collection of rays, so there is embedding in $\mathbb{R}^{|\text{rays}|}$

Therefore,

$$\lambda_g \in R^*_{\log}(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g)$$

$$\subset R_{\log}(\widetilde{\mathcal{M}}_g, \partial\widetilde{\mathcal{M}}_g) =$$

$$\mathrm{div}\, R_{\log}(\widetilde{\mathcal{M}}_g, \partial\widetilde{\mathcal{M}}_g)$$

and so is in div Log CH.

Same argument: $DR_{g,A}$ is in

div Log CH

<u>Proof</u>: Pixton's formula puts $DR_{g,A}$ in ring generated by

$$\langle R^*_{\log}(\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n}),\ \psi \text{ classes} \rangle$$

Open Questions:

We have done no calculations!

Compute:

- $R^*_{\log}(\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n})$

at least for small g, n

- $R^*(\overline{\mathcal{M}}_{g,n}) / \langle R^*_{\log}(\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n}) \rangle$

I expect some interesting pattern

For a log modification $\widetilde{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$

define

$$R^*(\widetilde{\mathcal{M}}_{g,n}) := \langle R^*(\overline{\mathcal{M}}_{g,n}), R^*_{\log}(\widetilde{\mathcal{M}}_{g,n}, \partial\widetilde{\mathcal{M}}_{g,n}) \rangle$$

Compute:

- $R^*_{\log}(B^2\overline{\mathcal{M}}_{g,n}, \partial B^2\overline{\mathcal{M}}_{g,n})$

double barycentric subdivision

<u>and</u> $R^*(B^2\overline{\mathcal{M}}_{g,n}) \big/ R^*_{\log}(B^2\overline{\mathcal{M}}_{g,n}, \partial B^2\overline{\mathcal{M}}_{g,n})$

conj: they are =

<u>Dream</u>: What is

$$\lim_{\substack{\longrightarrow \\ \widetilde{\mathcal{M}}_{g,n} \to \mathcal{M}_{g,n} \\ \text{log mod}}} R^*(\widetilde{\mathcal{M}}_{g,n}) \big/ R^*_{\log}(\widetilde{\mathcal{M}}_{g,n}, \partial\widetilde{\mathcal{M}}_{g,n}) \ ?$$

Finally: Run the same program

for $\bar{A}_g$.

For example:

Is there a subdivision $\tilde{A}_g \to \bar{A}_g$

s.t the 0-section of

$\tilde{\mathcal{U}}_g = \bar{\mathcal{U}}_g \times_{\bar{A}_g} \tilde{A}_g$ is a product of divisors?

• The combinatorics of $C(\bar{A}_g, \partial\bar{A}_g)$ and

$$C(\bar{\mathcal{M}}_{g,n}, \partial\bar{\mathcal{M}}_{g,n}) \xrightarrow{\text{torelli}} C(\bar{A}_g, \partial\bar{A}_g)$$

are understood (tropical abelian varieties, tropical torelli)

First Guess: Find any subdivision

$\tilde{A}_g \to A_g$ which is smooth

and for which the composition

$$C(B^2\bar{\mathcal{M}}_g, \partial B^2\bar{\mathcal{M}}_g) \longrightarrow C(\bar{A}_g, \partial\bar{A}_g)$$

lifts to $C(\tilde{A}_g, \partial\bar{A}_g)$.