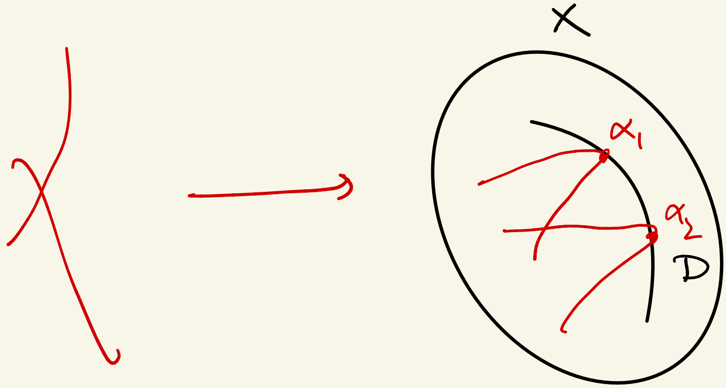
ETH Moduli Seminar

9/10/2020

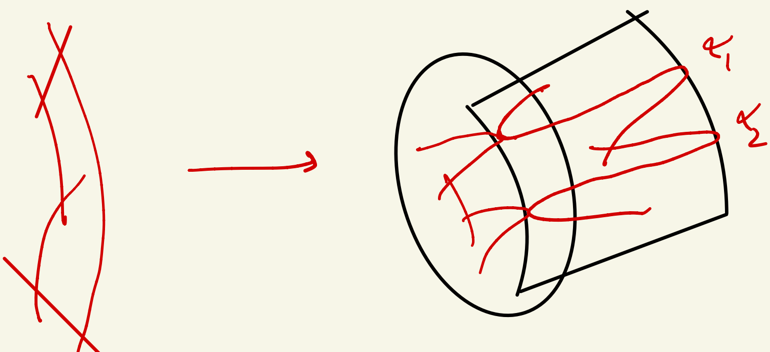
Log Stable Maps - An Overview

Motivating Problem: Compactify moduli space of relative stable maps



First solution (in alg. geometry) - Jun Li:

Relative stable maps to Expansions



Key condition: Predeformability

$$A[x,y]/xy=t \longleftarrow A[u,v,x_1,\dots,x_k]/uv=s$$

$$x^a \longleftarrow u$$

$$y^a \longleftarrow v$$

Predeformability in particular is a closed condition on space of maps. $\rightsquigarrow$ hard to understand obstruction theory.

By now, several other theories of rel. stable maps

Kim - log stable maps to expansions

AF - Orbifold stable maps to expansions

ACGS - log stable maps to X

(Ranganathan - Higher rank analogue of Kim)

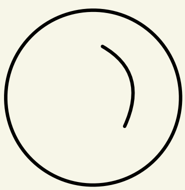
All variants give the same GW theory.

Goal for the talk: Explain how in fact all theories are log theories (Li, Kim, ACGS) and explain the relationship between the spaces.

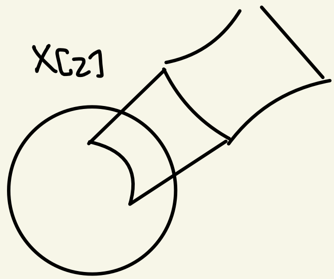
Conventions: Write Z/S for a family of targets. Those could be expansions, expanded degenerations, rubber, X itself, etc.

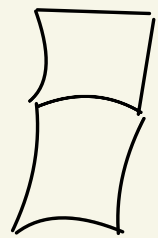
Statements about Z/S will apply to all cases. When necessary I'll specify and write X, $X[n]$, etc.

X



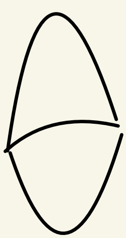
$X[2]$

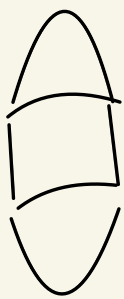




rubber

$X_1 \cup_D X_2$





$X_1 \cup_D X_2[1]$

Quick review of log geometry

Log scheme $(X, \mathcal{M}_X)$ $\quad \alpha : \mathcal{M}_X \longrightarrow \mathcal{O}_X$

$$\alpha^{-1}(\mathcal{O}_X^*) = \mathcal{O}_X^*$$

$\overline{\mathcal{M}}_X = \mathcal{M}_X / \mathcal{O}_X^*$ characteristic monoid.

Prototype of log scheme = spectrum of monoid algebra

$X = \operatorname{Spec} \mathbb{Z}[P]$

$\mathcal{M}_{\operatorname{Spec}\mathbb{Z}[P]} = \langle P, \mathbb{Z}[P]^* \rangle = $ Monomials $\subset \mathbb{Z}[P]$

We assume $P \subset P^{gp}$ + finitely gen.

("fine" monoid in log terminology)

Usually P is even <u>saturated</u> in P^{gp}

("f.s")

Then $\operatorname{Spec}\mathbb{Z}[P]$ is an affine toric variety

Meaning of word prototype:

All our log schemes are required to have <u>charts</u>.

<u>Def</u> : A chart for a log scheme X is a map $X \xrightarrow{f} \operatorname{Spec}\mathbb{Z}[P]$ such that $f^*\mathcal{M}_{\operatorname{Spec}\mathbb{Z}[P]} \simeq \mathcal{M}_X$

(In particular $\overline{\mathcal{M}}_X = P/_{P^*}$)

Log Smoothness

Smooth scheme $X \simeq \mathbb{A}^n$ étale locally

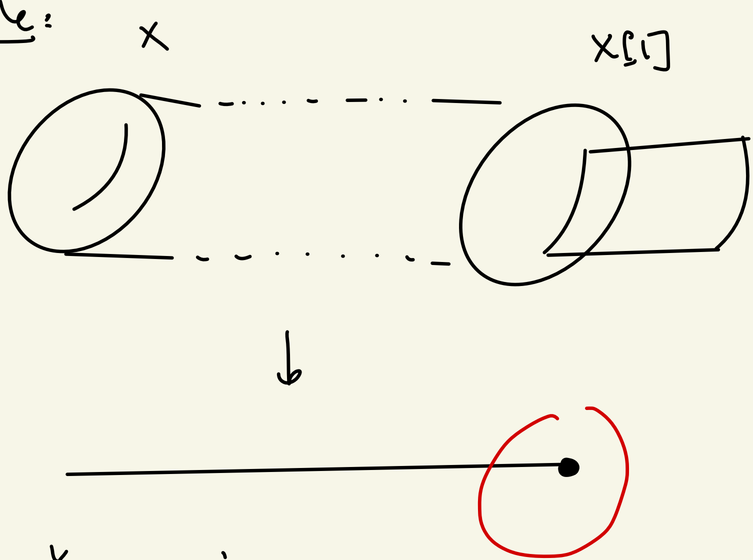
Smooth map $X \to Y \simeq Y \times \mathbb{A}^n \to Y$

Log smooth scheme $X \xrightarrow[\text{étale}]{} \operatorname{Spec}\mathbb{Z}[P]$ so this is a chart

<u>Def</u> $X \to Y$ is log smooth if

$$\begin{tikzcd} X \arrow[r, "\text{sm/étale}"'] \arrow[rr, bend left, "\text{chart}"] & Y \times_{\operatorname{Spec}\mathbb{Z}[Q]} \operatorname{Spec}\mathbb{Z}[P] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec}\mathbb{Z}[P] \arrow[d] \\ & Y \arrow[r, "\text{chart}"'] & \operatorname{Spec}\mathbb{Z}[Q] \end{tikzcd}$$

Example:



$\mathbb{A}^2 \times \mathbb{A}^k \longrightarrow \mathbb{A}^1$

$(x, y, z_i) \longrightarrow xy$

$M_{\mathbb{A}^1} = k^* \cdot t^n \simeq k^* \oplus \mathbb{N}$

$$Y = (\operatorname{Spec} k \,,\; k^* \oplus \mathbb{N})$$

$$\begin{array}{cc} & t \\ \downarrow & \downarrow \\ k & 0 \end{array}$$

$$\begin{tikzcd} X[1] \arrow[r, "sm"] & Y \times_{\mathbb{A}^1} \mathbb{A}^2 \arrow[r] \arrow[d] & \mathbb{A}^2 \arrow[d] \\ & Y \arrow[r] & \mathbb{A}^1 \end{tikzcd}$$

i.e: $\exists$ log str. on $X[1]$, and on $\operatorname{Spec} k$, such that

$X[1] \longrightarrow \operatorname{Spec} k$

is log smooth

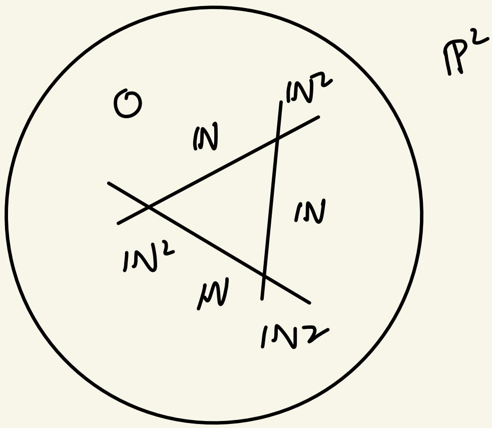
Tropicalization

(For now we restrict to f.s. log schemes)

X is stratified by $\overline{M}_X$.

$$\text{Strata} = \{x \in X \mid \overline{M}_{X,x} \text{ is constant}\}$$

Ex:



For $x \in X$,

$$\sigma_{X,x} := \mathrm{Hom}(\overline{M}_{X,x}, \mathbb{R}_{\geq 0})$$

$$N_{X,x} := \mathrm{Hom}(\overline{M}_{X,x}^{gp}, \mathbb{Z})$$

For generization $x \rightsquigarrow y$, $\overline{M}_{X,y} = \overline{M}_{X,x}/\mathcal{F}$.

So $\sigma_{X,y} \hookrightarrow \sigma_{X,x}$ is inclusion of <u>face</u>.

Def: $\Sigma(X) = \varinjlim (\sigma_{X,x}, N_{X,x})$

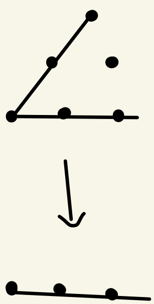
Gluing along generizations

$\Sigma(X)$ is a (generalized) cone complex with integral structure

Example: If $X = V(\Delta, N)$ is a toric variety, defined by fan Δ in lattice N,

$\Sigma(X) = (\Delta, N)$

Example: $X[1] \longrightarrow (\mathrm{Spec}\, k, \mathbb{N} \oplus k^*) = Y$



$$\begin{array}{c} \Sigma(X[1]) \\ \downarrow \\ \Sigma(Y) \end{array}$$

In general, $X \longrightarrow Y$ induces a piecewise linear map $\Sigma(X) \longrightarrow \Sigma(Y)$

Essentially, $\Sigma(X)$ organizes the combinatorial structure of X in a covariant manner

The cones of $\Sigma(X)$ capture the local charts

$$(\sigma_{X,x}, N_{X,x}) \rightsquigarrow \mathrm{Hom}(\sigma_{X,x} \cap N_{X,x}, \mathbb{N}) = P$$

and $\exists$ chart

$\mathcal{U} \longrightarrow \mathrm{Spec}\,\mathbb{Z}[P]$

around x

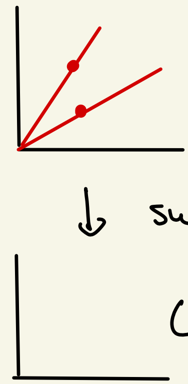
$$\left(\underset{\substack{\shortparallel \\ \sigma_{X,x}}}{\mathrm{Hom}(P, \mathbb{R}_{\geq 0})}, \underset{\substack{\shortparallel \\ N_{X,x}}}{\mathrm{Hom}(P^{gp}, \mathbb{Z})}\right) \leftarrow\!\!\!\sim\!\!\sim \text{Chart } \mathcal{U} \longrightarrow \mathrm{Spec}\,\mathbb{Z}[P]$$

What else is $\Sigma(X)$ good for?

(A) Defines two key classes of maps

(i) Locally

$$\begin{tikzcd} \mathcal{U}(\Delta, N_{X,x}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V(\Delta, N_{X,x}) \arrow[d] \\ \mathcal{U} \arrow[r] & \operatorname{Spec} \mathbb{Z}[P] \end{tikzcd} \sim\!\sim$$



$(\Delta, N_{X,x})$

$\downarrow$ subdivision

$(\sigma_{X,x}, N_{X,x})$

Subdividing $\Sigma(X)$ (i.e. every cone compatibly) globalizes this

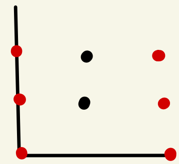
$\Delta \longrightarrow \Sigma(X)$ determines $X(\Delta) \longrightarrow X$

<u>Def</u>: A map $X(\Delta) \longrightarrow X$ of this form is called a <u>subdivision</u> or <u>log modification</u> (or log blowup if one is lazy)

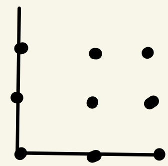
If X is log smooth, subdivisions are birational

(ii) locally

$$
\begin{tikzcd}
\mathcal{U}(N') \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & {[\mathrm{Spec}\,\mathbb{Z}[P']/G]} \arrow[d] \\
\mathcal{U} \arrow[r] & \mathrm{Spec}\,\mathbb{Z}[P]
\end{tikzcd}
$$

$\rightsquigarrow$  $(\sigma_{X,x}, N')$

$\downarrow$ $\qquad\qquad$ $\downarrow$

$\rightsquigarrow$  $(\sigma_{X,x}, N_{X,x})$

$$P = \{ u \in \overbrace{\mathrm{Hom}(N_{X,x}, \mathbb{Z})}^{P^{gp}} : u \geq 0 \text{ on } \sigma_{X,x} \}$$

$$\cap$$

$$P' = \{ u \in \mathrm{Hom}(N', \mathbb{Z}) : u \geq 0 \text{ on } \sigma_{X,x} \}$$

$\mathrm{Spec}\,\mathbb{Z}[P'^{gp}] \longrightarrow \mathrm{Spec}\,\mathbb{Z}[P^{gp}]$ is map of tori with kernel a fin. group G.

Refining the integral structure on $\Sigma(X)$ gives to give global $X' \longrightarrow X$

Such maps are called Kummer

These two types of maps are remarkably log étale. In fact:

Thm (Nakayama): The class of log étale maps is generated by classically étale maps, subdivisions, and Kummer maps.

So everything that log geometry adds to étale maps can be seen in $\Sigma(X)$.

Also remarkably, log modifications/Kummer maps are <u>monomorphisms</u> in log category.

$$\mathrm{Hom}_{LSch}(X, \mathrm{Spec}\,\mathbb{Z}[P]) = \mathrm{Hom}_{Mon}(P, M_X(X))$$

$$= \mathrm{Hom}_{Mon}(M_X(X)^\vee, \sigma_{X,x} \cap N_{X,x})$$

$$\cup$$

and

$$\mathrm{Hom}_{LSch}(X, V(\Delta)) = \{ \phi : M_X(X)^\vee \longrightarrow \sigma_{X,x} \cap N_{X,x}$$

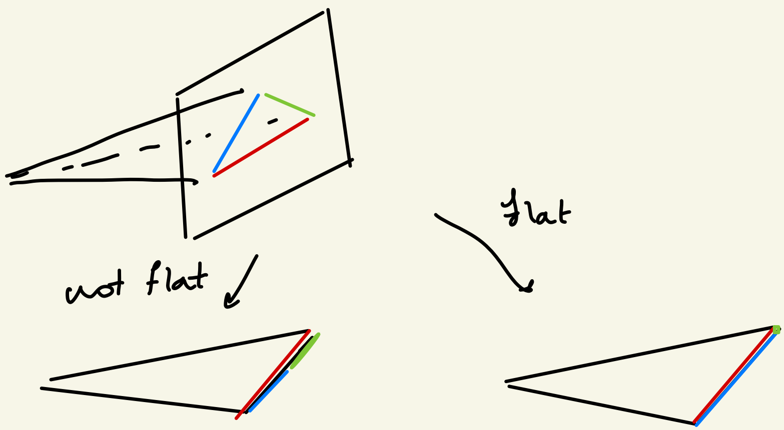
$$\text{which factor throug } \Delta \}$$

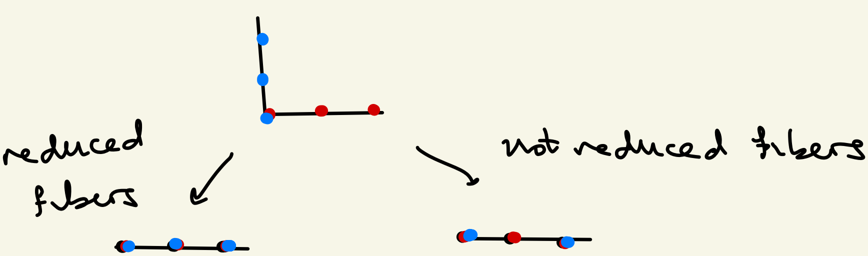
(B) Can read flatness/multiplicities of fibers from $\Sigma(X)$

<u>Thm</u>: A log smooth map $X \to Y$ is ~~flat~~ integral with reduced fibers iff the map $\Sigma(X) \to \Sigma(Y)$

(i) Sends cones onto cones

(ii) Sends lattices onto lattices





Aside - Artin Fans

The most elegant way to translate combinatorics is via the "Artin fan" $\mathcal{A}_X$ of X.

$$\Sigma(X) = \varinjlim_{P \text{ loc. charts}} \left(\mathrm{Hom}(P, \mathbb{R}_{\geq 0}), \mathrm{Hom}(P^{gp}, \mathbb{Z}) \right)$$

$$\mathcal{A}_X := \varinjlim \left[\operatorname{Spec}\mathbb{Z}[P] / \operatorname{Spec}\mathbb{Z}[P^{gp}] \right]$$

Advantage: There is an honest map $X \to \mathcal{A}_X$

log modification / Kummer map corresponds to

$$\begin{tikzcd} X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{A}_X' \arrow[d] \\ X \arrow[r] & \mathcal{A}_X \end{tikzcd}$$

log étale map

Ex: $X = V(\Delta, N)$

$$\mathcal{A}_X = [V(\Delta, N)/\text{Torus}]$$

Ex: For $X[1] \longrightarrow (Y = \operatorname{Spec} k, \mathbb{N} \oplus k^*)$

$$\mathcal{A}_{X[1]} = [\mathbb{A}^2/\mathbb{G}_m^2] \longrightarrow [\mathbb{A}^1/\mathbb{G}_m] = \mathcal{A}_Y$$

$\boxed{\text{Log Curves}}$

<u>Def</u> A log curve is a proper, ~~flat~~ integral, log smooth map $C \to S$ with reduced fibers which are reduced and purely 1-dimensional.

Structure.

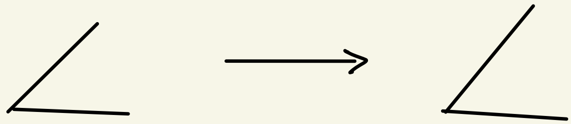
<u>1</u>. Log smooth means

$$\begin{tikzcd}
C \arrow[r, "\text{sm.}"'] \arrow[dr, bend right, "\text{rel dim } 1"'] & \operatorname{Spec}\mathbb{Z}[P] \times_{\operatorname{Spec}\mathbb{Z}[Q]} S \arrow[r] \arrow[d] & \operatorname{Spec}\mathbb{Z}[P] \arrow[d, "\text{rel dim } \leq 1"] \\
& S \arrow[r] & \operatorname{Spec}\mathbb{Z}[Q]
\end{tikzcd}$$

2. $\Sigma(C) \to \Sigma(S)$ must satisfy conditions of theorem, with rel. dim ≤ 1.

Rel dim 0

(i)



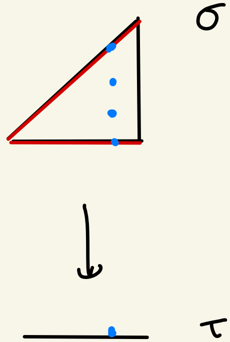
$\Sigma(C) \ni \sigma \longrightarrow \tau \in \Sigma(S)$

necessarily an iso.

Rel dim 1

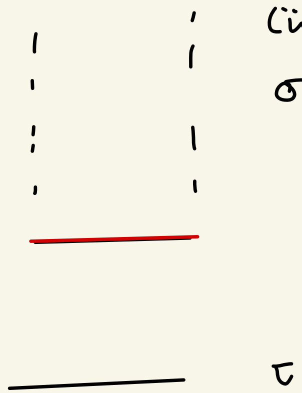
Either

(ii)



or

(iii)



2 faces mapping isomorphically

1 face mapping isomorphically

$\ell(v)$ = "distance" between 2 preimages

$\sigma \simeq \tau \times \mathbb{N}$

$$\sigma = \mathbb{N}^2 \times_{\mathbb{N}} \tau = \{ (a, b, v) : a + b = \ell(v) \}$$

It will be convenient to reorganize a bit.

Given $C \to S$, fix a fiber $C_s \to s$;

Set $\overline{M} = \overline{M}_{S,s}$

$\tau = \mathrm{Hom}(\overline{M}, \mathbb{R}_{\geq 0})$ the corresponding cone in $\Sigma(S)$.

Type (i) points correspond to generic points v of irred components.

(ii) marked points l

(iii) Nodes e (in particular, a log curve is always nodal)

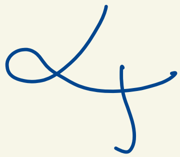
We can recover $\Sigma(C)$ over τ from fiber over a point, and the homomorphisms

$$l(e) : \tau \cap N \longrightarrow \mathbb{N}$$

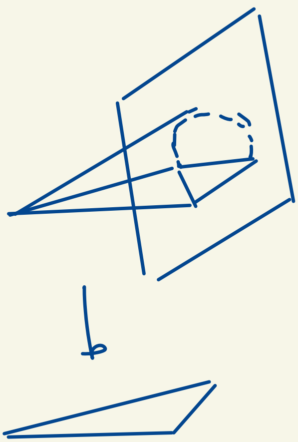
$$\rightsquigarrow l(e)^\vee : \mathbb{N} \to \overline{M}$$

$$\rightsquigarrow \delta(e) \in \overline{M}$$

So, given



Instead of drawing $\Sigma(C)$ over τ:



(this picture is wrong anyway)

We draw a fiber, marked by $\delta(e) \in \overline{M}$

This is what we will refer to as the tropicalization.

It's a system of dual graphs over the points $s \in S$, marked by $\delta(e) \in \overline{M}_{S,s}$ for e the edges in each fiber.

(When $C \longrightarrow S$ is a one parameter degeneration one gets a usual "tropical curve")

This is the combinatorial data for $C \to S$, i.e. what's captured by $\overline{M}_C, \overline{M}_S$.

Full log structures contain a bit more.

Again, work locally around $s \in S$, so write $A = \mathcal{O}_{S,\bar{s}}$, $M = M_{S,\bar{s}} \xrightarrow{\alpha} A$

$$\overline{M} = \overline{M}_{S,\bar{s}}$$

Dualizing (i), (ii), (iii), we find

$$\overline{\mathcal M}_{C,v} = \overline{\mathcal M} \qquad \mathcal M_{C,v} \simeq \pi^*\mathcal M$$

$$\overline{\mathcal M}_{C,\ell} = \overline{\mathcal M} \oplus \mathbb N \qquad \mathcal M_{C,\ell} \simeq \langle \pi^*\mathcal M, \mathcal I(\ell) \rangle$$

(ideal sheaf of marked point)

$$\overline{\mathcal M}_{C,e} = \overline{\mathcal M} \oplus_{\mathbb N} \mathbb N^2 \qquad \delta(e) \mapsfrom 1$$

$$\mathcal O_{C,e} \simeq A[x(e), y(e)] / x(e)y(e) = t(e) \in m_A$$

$\mathcal M$ contains element $\log t(e)$ such that

$$\alpha(\log t(e)) = t(e)$$

Image $\overline{\log t(e)}$ in $\overline{\mathcal M}$ equal to $\delta(e)$

and $\mathcal M_{C,e} = \langle \mathcal M, x(e), y(e) \rangle / x(e)y(e) = \log t(e)$

What about moduli?

- From the beginning, run into difficulty

$$\mathcal{M}_{g,n}^{\log} \longrightarrow \mathcal{L}og\,Sch$$

$$\{\text{Log curves } C \to S\} \leftarrow\!\!\rightsquigarrow S$$

But we want to descend to schemes

$$\mathcal{M}_{g,n}^{\log} \longrightarrow Sch$$

$$\{\text{Log curves } (\underline{C}, ?) \to (\underline{S}, ?)\ \leftarrow\!\!\rightsquigarrow \underline{S}$$

If we use every log structure, get something enormous.

Observe: If $(\underline{C}, M_C) \longrightarrow (\underline{S}, M_S)$ is a log curve,

for any $M_S \to M_S'$ get

$$\begin{array}{ccc} (\underline{C}, M_C') & \longrightarrow & (\underline{C}, M_C) \\ \downarrow & & \downarrow \\ (\underline{S}, M_S') & \longrightarrow & (\underline{S}, M_S) \end{array}$$

Idea: Use only minimal families, those from which every other is pulled back.

I.e. For each $\underline{C} \longrightarrow \underline{S}$, want collection

$$\{ (\underline{C}, \mathcal{M}_{C,i}) \longrightarrow (\underline{S}, \mathcal{M}_i) \} \quad \text{s.t}$$

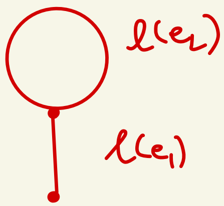
(*) $\forall$ log curves $(\underline{C}, \mathcal{M}_C) \to (\underline{S}, \mathcal{M}_S)$, $\exists\, i$ and map $\mathcal{M}_i \xrightarrow{\phi} \mathcal{M}_S$ s.t

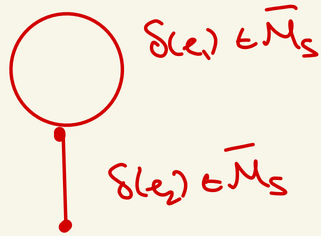
$$\begin{tikzcd}
(\underline{C}, \mathcal{M}_C) \arrow[r] \arrow[d] & (\underline{C}, \mathcal{M}_{C,i}) \arrow[d] \\
(\underline{S}, \mathcal{M}_S) \arrow[r, "(id, \phi)"'] & (\underline{S}, \mathcal{M}_i)
\end{tikzcd}$$

Tropically, this is very easy

Given dual graph, find monoid $\overline{M}$ and length $l(e) \in \overline{M}$ such that $\forall\ \overline{M}_S$, $\delta(e) \in \overline{M}_S$, $\exists$ map $\overline{M} \longrightarrow \overline{M}_S$

$l(e) \longmapsto \delta(e)$





$\overline{\mathcal{M}} = \mathbb{N}^{|E(G)|}$

$\ell(e)$ = e-th generator

<u>Thm</u> (F. Kato) Every family of nodal curves $\underline{C} \longrightarrow \underline{S}$ has a unique structure of a minimal family.

<u>Idea</u>: Work locally on $\underline{C}, \underline{S}$

Around node e, write

$$\mathcal{O}_{\underline{C},e} = A[x(e), y(e)] \Big/ _{x(e)y(e) = t(e)}$$

Define $\mathcal{M}_S = \langle t(e) \rangle$

$$\mathcal{M}_{C,e} = \langle x(e), y(e), t(e) \rangle \Big/ _{x(e)y(e) = t(e)}$$

This is unique up to unique iso, so globalizes.

Kato's theorem can be rephrased as

$$\mathcal{M}_{g,n}^{\log} \simeq \{\text{Minimal log curves}\}$$

$$\overset{12}{}$$

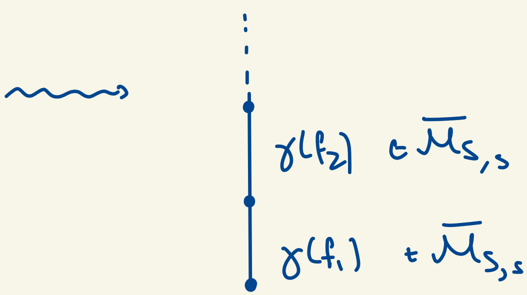
$$\overline{\mathcal{M}}_{g,n}$$

This gives $\overline{\mathcal{M}}_{g,n}$ a log structure, which is the one coming from the boundary.

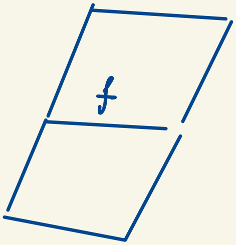
Exactly the same works for expansions.

Proper, log smooth, flat, reduced fiber family $Z \longrightarrow S$ with fibers expansions (or variants)

- Tropicalization: Around $s \in S$, w/ fiber $Z_s = X[n]$



- Log structure



Around f, Z looks like

$$\mathcal{O}_{S,\bar{s}}[u(f), v(f), z_i(f)] / u(f)v(f) = s(f)$$

$$\in \mathcal{M}_{S,\bar{s}}$$

$\mathcal{M}_{S,\bar{s}}$ contains lift $\log s(f)$ of $s(f)$,

$$\begin{array}{ccc} & \mathcal{M}_{S,\bar{s}} \ni \log s(f) & \\ \swarrow & & \searrow \\ \mathcal{O}_{S,\bar{s}} \ni s(f) & & \overline{\mathcal{M}}_{S,\bar{s}} \ni \gamma(f) \end{array}$$

$$\mathcal{M}_{Z,f} = \langle \mathcal{M}_S, u(f), v(f) \rangle / u(f)v(f) = \log s(f)$$

- Every family $\underline{Z} \longrightarrow \underline{S}$ of expansions has canonical minimal log structure, corresponding to divisorial log structure on stack of expansions T.

Minimal log structure looks like

$$\overline{\mathcal{M}}_S = \mathbb{N}^n$$

$$\mathcal{M}_S = \langle s(f) \rangle$$

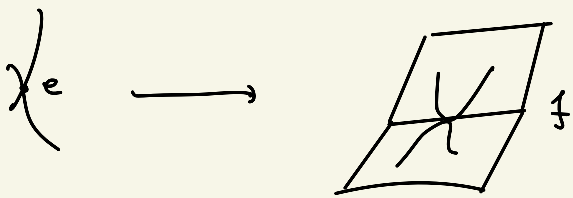
$$\mathcal{M}_{Z,f} = \langle u(f), v(f), s(f) \rangle / u(f)v(f) = s(f)$$

Log Maps

Structure of non-degenerate log map

$$\begin{tikzcd} C \arrow[r] \arrow[dr] & Z \arrow[d] \\ & S \end{tikzcd}$$

- On scheme level, locally looks like



$$A[x(e), y(e)]/x(e)y(e) = t(e) \longleftarrow A[u(f), v(f), z_i]/uv = s(f)$$

$$x^a \longleftarrow u$$

$$y^b \longleftarrow v$$

But $\mathcal{M}_{Z,f} \longrightarrow \mathcal{M}_{C,e}$

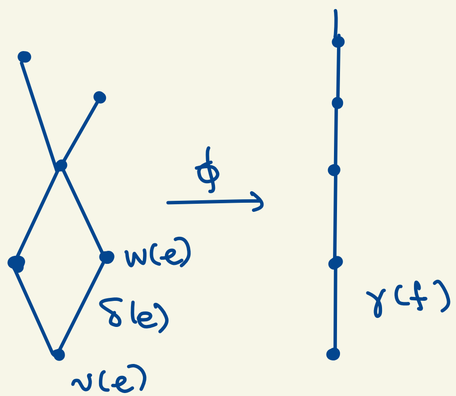
$$u(f)v(f) \longrightarrow x(e)^a \, y(e)^b$$

$$= \log s(f) = \log t(e)^a \cdot y(e)^{b-a}$$

- So $a = b$

$\rightsquigarrow$ map is automatically predeformable.

- Tropically, get a PL function (valued in $\overline{M}_S^{gp}$):



$$\gamma(f) = \phi(w(e)) - \phi(v(e)) = \overset{\mathbb{Z}}{a(e)}\,\delta(e)$$

slope = contact order

So have relations: if e, e' map to f,

corresponding to nodes w/ contact

order $a(e), a(e)'$

need $\delta(e)\,a(e) = \delta(e')\,a(e')$

Or even

$$\log t(e)^{a(e)} = \log t(e')^{a(e')}$$

For moduli, need to identify minimal log structures for map.

Given
$$\begin{tikzcd} \underline{C} \arrow[r] \arrow[dr] & \underline{Z} \arrow[d] \\ & \underline{S} \end{tikzcd}$$

must identify $\mathcal{M}_S, \mathcal{M}_C, \mathcal{M}_Z$ from which

all others are pulled back.

But we already have

$$\mathcal{M}_S = \langle \log t(e) \rangle \big/ \log t(e)^{a(e)} = \log t(e')^{a(e')}$$

if e, e' map to same f

$$\left(\overline{\mathcal{M}}_S = \mathbb{N}^{|E(G)|} \Big/ a(e)\cdot e\text{-th gen} = a(e')\cdot e'\text{-th gen.} \right)$$

Caveat: This is not saturated (but its monoid algebra is l.c.i.)

At any rate, every predeformable map to an expansion can be promoted uniquely to a (fine) minimal log map.

Let $\mathcal{Z} \longrightarrow \mathcal{T}$ universal expansion.

$$\mathcal{L}og\mathcal{M}ap \longrightarrow Sch$$

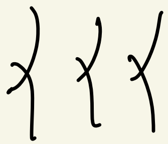
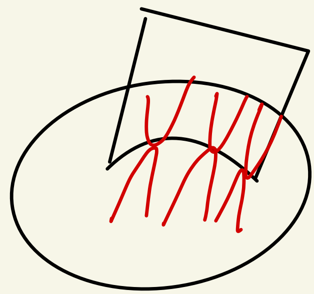
$$\left\{ \begin{array}{ccc} C & \to & \mathcal{Z} \\ \downarrow & & \downarrow \\ S & \to & \mathcal{T} \end{array} \right. \quad \longleftarrow\!\!\sim\!\!\sim \underline{S}$$

log map w/ minimal log structure + stability $\Big\}$

Thm $\mathcal{L}og\mathcal{M}ap \simeq \mathcal{R}_i$

If I insist on working with f.s. log structures minimal log structure on family no longer unique

Ex:

e mapping to f

w/ contact order $a(e)$

Set $d = \operatorname{lcm} a(e)$

$$\overline{M}_S = \left(\mathbb{N}^{|E(G)|} \big/ a(e) b_e = a(e') b_{e'} \right)^{sat}$$ still unique

$$\simeq \mathbb{N}$$

$$\begin{tikzcd}
\mathbb{N}^{|E(G)|} \arrow[r] & \overline{M}_S & \mathbb{N}^{|E(\Sigma(Z))|} = \mathbb{N} \arrow[l] \\
b_e \arrow[dr] & d & b_f \arrow[l] \\
& \frac{d}{a(e)} &
\end{tikzcd}$$

To lift to log structure need to choose roots for $\log s(f)$, $\log(t(e))$

If we define

$$\mathrm{LogMaps} \longrightarrow \mathrm{Sch}$$

$$\left\{ \begin{array}{ccc} C & \to & \mathcal{Z} \\ \downarrow & & \downarrow \\ S & \to & \mathcal{T} \end{array} \right\} \leftsquigarrow S$$

non-degenerate log maps
w/ f.s. minimal
log str.
+ stability

Then: (a) $\mathrm{LogMaps} = \mathrm{Kim}$

(b) $\mathrm{Kim} \longrightarrow \mathrm{Li}$ is finite

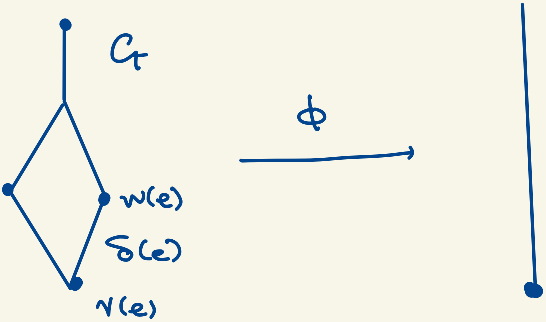
In fact, for a fine log scheme/stack X, $\exists$ universal $X^{sat} \to X$ which is f.s.

It is in particular log étale. Geometrically, it's a partial normalization.

(c) $\mathrm{Kim} \to \mathrm{Li}$ is the saturation.

What about degenerate maps to X?

Tropically



The PL function still records contact orders

$a(e)$, as the slopes $\phi(w(e)) - \phi(v(e)) = a(e)\delta(e)$

- However, the $\delta(e)$ need to satisfy much fewer relations in $\overline{M}_S$.

- Interesting relations come from $H_1(G)$

- Minimal log structure

$$\mathbb{N}^{V(G)} \oplus \mathbb{N}^{E(G)} \Big/ \phi(w(e)) - \phi(v(e)) = a(e)\delta(e)$$

Can be pretty singular (e.g. not lci)

Saturating P gives the "basic" log structures of ACGS.

$$\mathrm{LogMap} \rightsquigarrow \mathrm{Sch}$$

$$\left\{ \begin{array}{ccc} C & \longrightarrow & X \\ \downarrow & & \\ S & & \end{array} \right\} \quad \longleftarrow\!\!\!\rightsquigarrow \quad \underline{S}$$

log maps w/ sat. minimal log str + stability

Thm $\mathrm{LogMap} \simeq \mathrm{ACGS}$

If one does not saturate, get a new space N, with finite log étale map

$$\mathrm{ACGS} \longrightarrow N$$

Relationship between expanded and non-expanded maps.

Let
$$\begin{tikzcd} C \arrow[r] \arrow[d] & Z \arrow[dl] \\ S & \end{tikzcd}$$
be a map to an expansion

Compose

$$\begin{tikzcd} C \arrow[r] \arrow[d] & Z \arrow[r] \arrow[dl] & X \\ S & & \end{tikzcd}$$

Stabilize

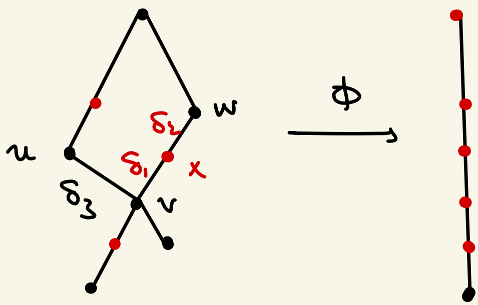
$$\begin{tikzcd} \overline{C} \arrow[r] \arrow[d] & X \\ S & \end{tikzcd}$$

Gives map Kim $\longrightarrow$ ACGS

(Li $\longrightarrow$ N)

What does the map look like?

Tropically, a map $C \to X$ in ACGS



$$\phi(w) - \phi(v) = m(\delta_1 + \delta_2)$$
$$\phi(x) - \phi(v) = m\delta_1 = m'\delta_3$$

Must adjoin $\dfrac{\phi(u) - \phi(v)}{m}$

- Values $\{\phi(v)\}$ are elements of $\overline{M}_S$
- For an expanded map, values are always ordered.
- Every ordering determines an expansion, but to have morphism need to blow up $C' \to C$.
- Compatibility with slopes requires adjoining roots.

Process identifies Kim as log modification/root stack over ACGS. In particular, log étale.

So have

$$(*) \qquad \begin{tikzcd} \mathrm{Kim} \arrow[r] \arrow[d] & \mathrm{ACGS} \arrow[d] \\ \mathrm{Li} \arrow[r] & \mathrm{N} \end{tikzcd}$$

everything is log étale.

Obstruction Theory

All variants have a P.O.T restricted

from the space of all maps

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[d, "\pi"'] & \mathcal{Z} \arrow[d] \\ \mathrm{LogMap} \arrow[r] & \mathcal{T} \end{tikzcd}$$

$$\left(R\pi_* f^* T^{\log}_{\mathcal{Z}/\mathcal{T}}\right)^\vee = E^\bullet$$

This is perfect relative to $\underbrace{\mathcal{L}og(\mathcal{M}_{g,n})}$

huge stack parametrizing a log curve on S w/ arbitrary log structure

Equality of GW invariants of all spaces is a consequence of $c_*[\mathcal{L}_1]^{vir} = [\mathcal{L}_2]^{vir}$ where $c : \mathcal{L}_1 \to \mathcal{L}_2$ is any of the comparison maps.

- In turn, this is a consequence of the fact that $c^* E_2^\bullet = E_1^\bullet$ for any of the obstruction theories, because log étale maps induce isomorphisms on log tangent bundles.

- If you are interested in localization arguments, you need global obstruction theory.

This works fine for Kim/Li, since they live over smooth/lci locus of $\mathcal{L}og(M_{g,n})$

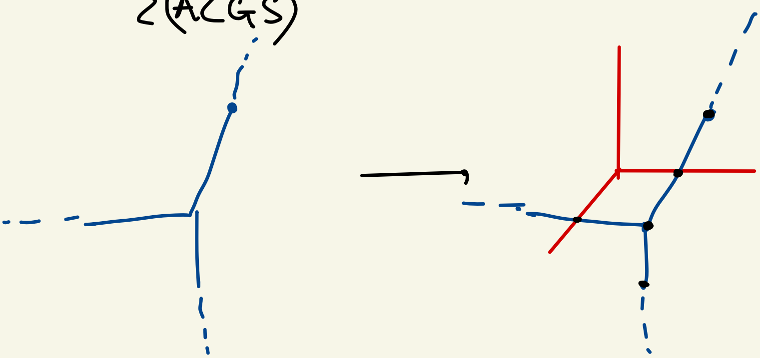
This is not the case for ACGS/N

A few words about higher rank.

When X is a general log smooth scheme, the ACGS space works, essentially with no changes.

The analogue of Kim was defined by Ranganathan. The idea is the same as the one we used to exhibit $\mathrm{Kim} \to \mathrm{ACGS}$ as a log modification.

$$\begin{tikzcd} \Sigma(\mathcal{C}) \arrow[r] \arrow[d] & \Sigma(X \times \mathrm{ACGS}) \arrow[dl] \\ \Sigma(\mathrm{ACGS}) & \end{tikzcd}$$



non-canonical!

Choose a decomposition of

$\Sigma(X \times ACGS)$ so that $\Sigma(\mathcal{C})$ is

a subcomplex.

$$
\begin{tikzcd}
\Sigma(D) \arrow[r] \arrow[dd] & C' \arrow[r] \arrow[d] \arrow[dd, bend right] & \Sigma' \arrow[d] \\
 & \mathcal{C} \arrow[r] \arrow[d] & \Sigma(ACGS \times X) \\
\Sigma(R) \arrow[r] & \Sigma(ACGS) &
\end{tikzcd}
$$

make this a log curve (i.e. flat + reduced fiber)

R is the log étale map determined by

$\Sigma(R) \longrightarrow \Sigma(ACGS)$

R depends on choice, but $[R]^{vir}$ does not, by same argument.