Goal: Start with family $X \to B$ of complex projective varieties

Get

$$\begin{tikzcd}
X_3 \arrow[r] \arrow[d, "\text{s.s.}"'] & X_2 \arrow[r] \arrow[d, "\text{weakly s.s.}"'] & X_1 \arrow[r] \arrow[d, "\text{toroidal}"'] & X \arrow[d] \\
B_3 \arrow[r] & B_2 \arrow[r] & B_1 \arrow[r] & B
\end{tikzcd}$$

semistable

Horizontal maps are alterations or compositions of modifications + root stacks

After step 1, everything is combinatorics of cone complexes.

Dan will explain step 1.

Step 3 is the real purpose of the seminar.

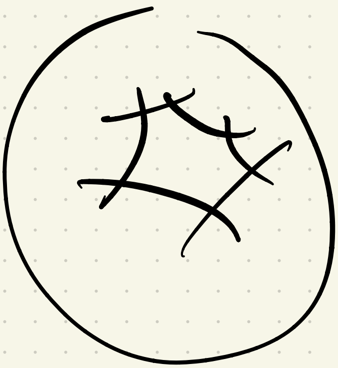
Today: Explain Step 2.

Recall: $(X,D)$ toroidal embedding

$\rightsquigarrow \Sigma(X,D)$ a cone complex

built from the local charts of $(X,D)$.

When $(X,D)$ has no self-intersection



For every stratum $S$,

w/ generic point $x_S$,

there exists local chart

$$\begin{array}{ccc} U_S & \longrightarrow & V(\sigma_{X,x_S}, N_{X,x_S}) \\ \downarrow & & \\ X & & \end{array}$$

If $x_S \rightsquigarrow x_T$ is a specialization, i.e

$x_T \in \overline{x_S}$, then

$$\sigma_{X,x_S} \subset \sigma_{X,x_T}$$

face.

$$\Sigma(X,D) = \varinjlim(\sigma_{X,x_S}, \sigma_{X,x_S} \cap N_{X,x_S})$$

Example: When $(X,D)$ = toric variety,

$\Sigma(X,D)$ = fan of $X$, without embedding into cocharacter lattice $N$.

$$\Sigma(X,D) = \Big( \underbrace{\Sigma_X = \{\sigma_{X,x_s}\}, N_X = \{N_{X,x_s}\}}_{} \Big)$$

collection of cones + integral structures

- Things like orbit-cone correspondence extend to strata-cone correspondence by construction.

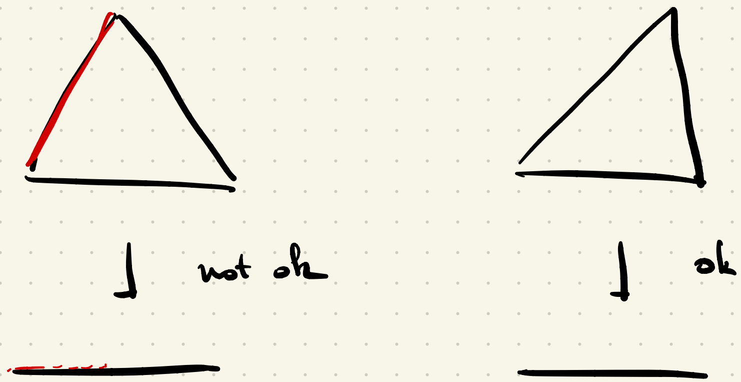
<u>Recall</u>. A map $(X,D) \xrightarrow{f} (Y,E)$ is combinatorially weakly s.s. if $\Sigma(X,D) \xrightarrow{\Sigma(f)} \Sigma(Y,E)$ takes

(i) Cones onto Cones

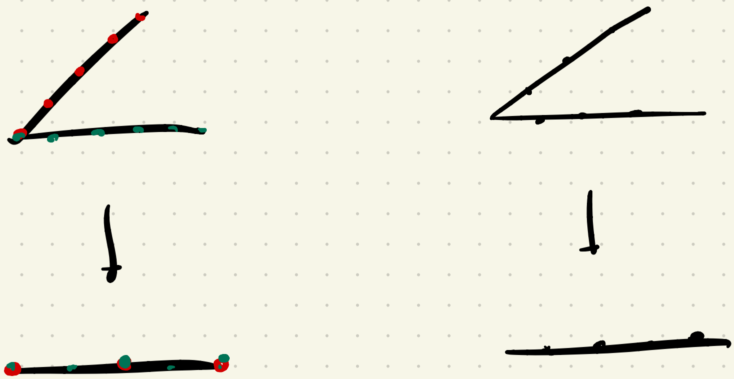
(ii) When $\Sigma(f)(\sigma) = \tau$, $\Sigma(f)(\sigma \cap N_\sigma) = \tau \cap N_\tau$.

("Lattices onto lattices")

(i)




(ii)



Consequences:

- When $X \to Y$ is dominant, weakly s.s. $\iff X \to Y$ is flat w/ reduced fibers.

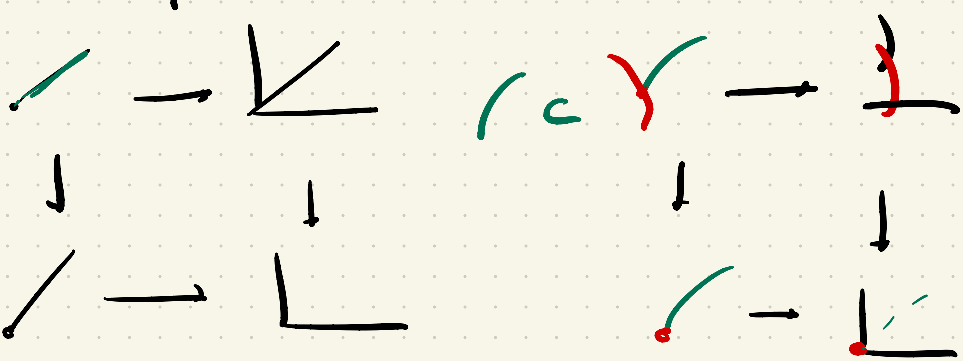
- For any diagram

$$\begin{tikzcd} & (X, D) \arrow[d, "f"] \\ (Z, F) \arrow[r] & (Y, E) \end{tikzcd}$$

$f$ weakly s.s. $\Rightarrow X \times_Y Z$ is toroidal

In general, this fails badly

Example diagram of toric varieties.



In general, normalization of closure of main component is toroidal. For weakly s.s. maps the two notions agree.

The cone complex of this main component is the fiber product of cone complexes:

$$C(X \times_Y^{f.s.} Z) = C(X) \times_{C(Y)} C(Z)$$

So when $(X, D) \to (Y, E)$ is weakly s.s., there is no ambiguity (so e.g. important in moduli)

(3) weakly s.s. maps are stable under base change.

Goal: Given $(X,D) \to (Y,E)$, turn

$\Sigma(X,D) \to \Sigma(Y,E)$ weakly s.s. in

"less intrusive way possible".

<u>Operations</u>. Fix $(X,D)$.

- Subdivision $\tilde{\Sigma} \longrightarrow \Sigma(X,D) = \{\Sigma_X, N_X\}$

is a compatible choice of subdivisions

$$(\tilde{\Sigma}_{x_s}, N_{X,x_s}) \longrightarrow (\sigma_{X,x_s}, N_{X,x_s})$$

of each cone $\sigma_{X,x_s} \in \Sigma_X$.

$$\begin{tikzcd} \tilde{\mathcal{U}}_s \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V(\tilde{\Sigma}_{x_s}, N_{X,x_s}) \arrow[d] \\ \mathcal{U}_s \arrow[r] \arrow[d] & V(\sigma_{X,x_s}, N_{X,x_s}) \\ X & \end{tikzcd}$$

Compatibility: The $\tilde{\mathcal{U}}_s$ descend to modification (proper, birational)

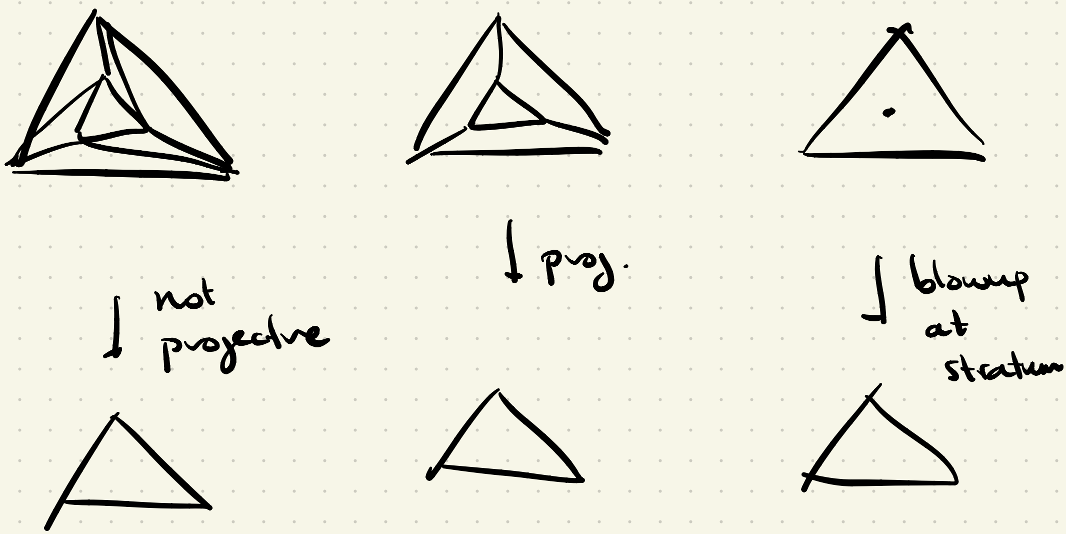
$(\tilde{X},\tilde{D}) \longrightarrow (X,D)$ with $\Sigma(\tilde{X},\tilde{D}) = \tilde{\Sigma}$.

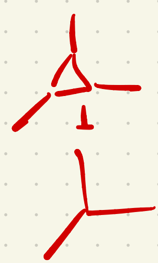
Geometric meaning

proper, birational.

- A blowup if $\tilde{\Sigma}$ is domain of linearity of some PL function.

- Star subdivision at a cone = blowup of stratum corresponding to cone.



• Root construction.

$$\tilde{\Sigma} = (\Sigma_X, \mathcal{L}_X, N_X)$$

with $\{\sigma_{X,x_S}\}$ $\{\mathcal{L}_{X,x_S}\}$ $\{N_{X,x_S}\}$

$\{\sigma_S\}$ $\{\mathcal{L}_S\}$ $\{N_S\}$

with $\mathcal{L}_S \subset N_S$ a finite index inclusion s.t

(*) if $\sigma_S \subset \sigma_T$,

$$\mathcal{L}_S = \mathcal{L}_T \cap N_S.$$

Given $(\sigma, \mathcal{L}, N)$,

$$T_{\mathcal{L}} \longrightarrow T_N \quad \text{finite map of tori}$$

$k_{\mathcal{L}N}$ = Kernel of this map.

$$V(\sigma, \mathcal{L}, N) := \left[ V(\sigma, \mathcal{L}) / K_{\mathcal{L}/N} \right]$$

This is a toric stack w/ coarse moduli space

$V(\sigma, N)$.

On $X$:

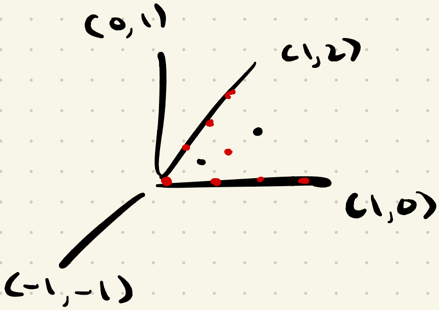
$$\begin{tikzcd}
\tilde{\mathcal{U}}_s \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V(\sigma_{X,x_s}, \ell_{X,x_s}, N_{X,x_s}) \arrow[d] \\
\mathcal{U}_s \arrow[r] \arrow[d] & V(\sigma_{X,x_s}, N_{X,x_s}) \\
X &
\end{tikzcd}$$

and compatibility means $\tilde{\mathcal{U}}_s$ glue to

$$\tilde{X} \longrightarrow X$$

DM stack over $X$ w/ coarse space $X$.

<u>Example</u> $(X, D)$ = simplicial toric variety




red lattice misses $(1,1)$.

For each $\sigma_{x,x_s}$, $\mathcal{L}_{x,x_s}$ = lattice generated by extreme rays of $\sigma_{x,x_s}$

$\mathcal{V}(\sigma_{x,x_s}, \mathcal{L}_{x,x_s}, N_{x,x_s})$ smooth DM stack w/ coarse space $(X, D)$

"canonical smoothing"

This works for arbitrary $(X, D)$ with $\Sigma(X, D)$ simplicial.

<u>Thm</u>: Let $(X,D) \to (Y,E)$ be a proper, dominant, toroidal morphism. Then $\exists$ a root + modification

$$
\begin{tikzcd}
 & (X,D) \arrow[d] \\
(Z,F) = V(\tilde{\Sigma}_Y, \tilde{\mathcal{L}}_Y, \tilde{N}_Y) \arrow[r] & (Y,E)
\end{tikzcd}
$$

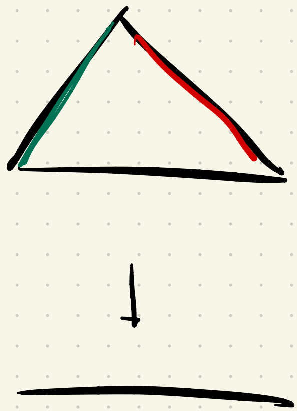
s.t
$$
\begin{tikzcd}
X \times_Y^{fs} Z \arrow[d] \\
(Z,F)
\end{tikzcd}
$$
is weakly s.s.

Furthermore, $(Z,F)$ is universal: for any other $(W,G)$ w/ this property, there exists a unique map

$$(W,G) \longrightarrow (Z,F).$$

**Proof**: We are given $\Sigma(X,D) \longrightarrow \Sigma(Y,E)$ and we must make it combinatorally weakly s.s.

**Idea**



**Algorithm**

(i) Local on $Y$, so can assume

$\Sigma(Y,E) = (\tau, Q)$ is a single cone, 1 lattice.

For $v \in \tau$, let

$$N_0(v) = \left\{ \sigma \in \Sigma_X \;\middle|\; \begin{array}{l} \sigma \text{ maps to } \tau \text{ w/ rel. dim } 0 \\ \exists \text{ lift } v_\sigma \text{ of } v \text{ on } \sigma \end{array} \right\}$$

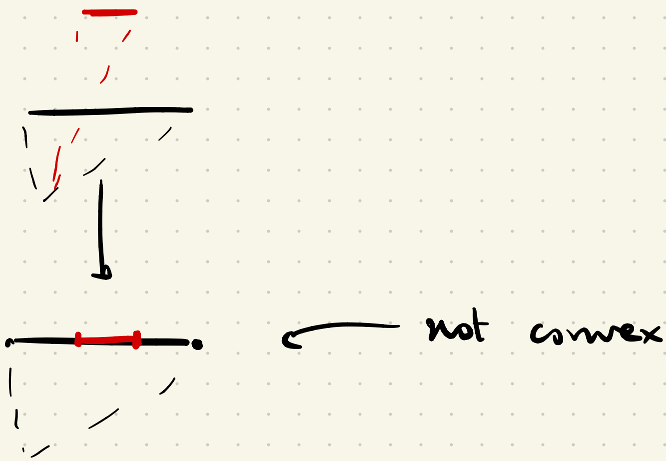
## Claim

$\{v \in \tau \mid Nb(v) = \text{fixed}\}$ form a subdivision of $\tau$

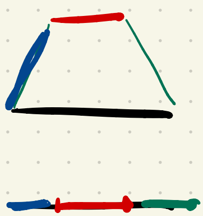
It is clear that the sets partition $\tau$, and are conical.

What's not clear is that they are convex.

E.g.

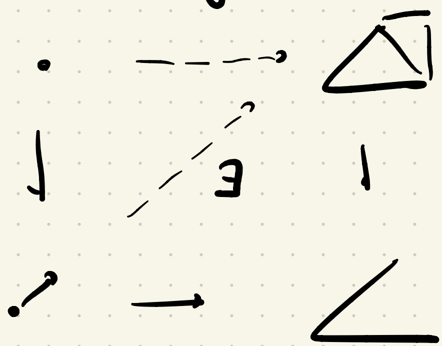



Properness saves this



In fact, properness is not strictly necessary.

Need something like "tropical smoothness"

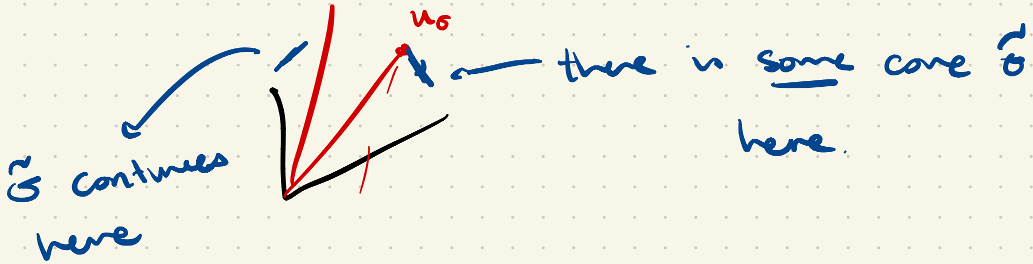


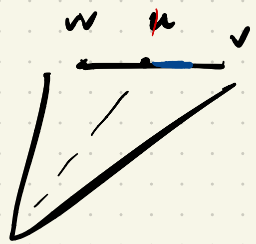
Properness or this condition give convexity

Argument: Suppose $v, w$ have $N_0(v) = N_0(w)$


Let $u$ first point in $\overline{vw}$ w/ $N_0(u) \neq N_0(v)$.

This means $\exists$ cone $\sigma \in \Sigma(X, D)$ w/ lift of $u$, but s.t. $v, w$ do not lift to $\sigma$.




path from $u$ to $v$ lifts to path $u_\sigma \tilde v$ for some $\tilde v$ in a core $\tilde\sigma$ in $N_\sigma(v)$.

By def., $\exists$ lift $\tilde w$ on $\tilde\sigma$ as well.

But then $\sigma$ and $\tilde\sigma$ do not meet along mutual faces, which is impossible.

So only possibility is $v=u$ or $u=w$.

with this subdivision, call it $\tilde{\Sigma}_Y$,

$$\begin{tikzcd} \{\tilde{\Sigma}_X = \Sigma_X \times_{\Sigma_Y} \Sigma_{\tilde{Y}}, \tilde{N}_X\} \arrow[r] \arrow[d] & \{\Sigma_X, N_X\} \arrow[d] \\ \{\tilde{\Sigma}_Y, \tilde{N}_Y\} \arrow[r] & \{\Sigma_Y, N_Y\} \end{tikzcd}$$

Cones map onto cones.

From here, we fix the lattices: let $\tau \in \tilde{\Sigma}_Y$,

$\sigma_1, \dots, \sigma_k$ the cones which map

onto $\tau$.

We take $\tilde{\mathcal{L}}_\tau = \bigcap \Sigma(f)(N_{\sigma_i})$, and $\tilde{\mathcal{L}}_{\sigma_i} = \Sigma(f)^{-1}(\mathcal{L}_\tau)$

$$\begin{tikzcd} \{\tilde{\Sigma}_X, \tilde{\mathcal{L}}_X, \tilde{N}_X\} \arrow[r] \arrow[d] & \{\Sigma_X, N_X\} \arrow[d] \\ \{\tilde{\Sigma}_Y, \tilde{\mathcal{L}}_Y, \tilde{N}_Y\} \arrow[r] & \{\Sigma_Y, N_Y\} \end{tikzcd}$$

weakly s.s.

not representable

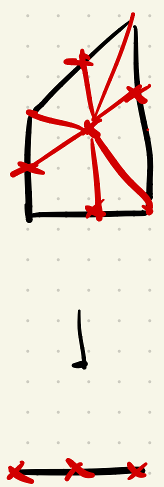
Comment 1: weak semistability is stable under base change, so from here we can modify the base at will. For example, we could take a resolution of $(Z, F)$.

Comment 2: For $\Sigma(X, D) \longrightarrow \Sigma(Y, E)$ weakly s.s., the total barycentric subdivision

$B\Sigma(X, D) \longrightarrow \Sigma(Y, E)$ factors through

$B\Sigma(Y, E)$ and $B\Sigma(X, D) \longrightarrow B\Sigma(Y, E)$

remains weakly s.s.



Thus, from $\Sigma(X,D) \to \Sigma(Y,E)$

we can construct

$$\begin{tikzcd} \Sigma(W,G) \arrow[r] \arrow[d] \arrow[d, bend right, "\text{weakly s.s.}"'] & \Sigma(X,D) \arrow[d] \\ \Sigma(Z,F) \arrow[r] & \Sigma(Y,E) \end{tikzcd}$$

simplicial

smooth

- If we take the canonical smoothing of $\Sigma(W,G)$, we get non-representable s.s. reduction.

$$\begin{tikzcd} \tilde{\Sigma}(W,G) \arrow[d] \arrow[dd, bend right, "\text{s.s.}"'] \\ \Sigma(W,G) \arrow[d] \\ \Sigma(Z,F) \end{tikzcd}$$

Honest s.s. reduction:

Find a further lattice altered subdivision

$$\begin{tikzcd} (\tilde{\Sigma}_W, \mathcal{L}_W \times_{\mathcal{L}_Z} \tilde{\mathcal{L}}_Z, N_W \times_{N_Z} \tilde{N}_Z) \arrow[r] \arrow[d] & (\Sigma_W, \mathcal{L}_W, N_W) \arrow[d] \\ (\tilde{\Sigma}_Z, \tilde{\mathcal{L}}_Z, \tilde{N}_Z) \arrow[r] & (\Sigma_Z, \mathcal{L}_Z, N_Z) \end{tikzcd}$$

with $\tilde{\Sigma}_W$ unimodular.

Again, this is local. So can assume $\Sigma_Z$

= cone over polytope $Q$.

Then fiber of $\Sigma_W$ over $Q$ = polytopal

complex $P \to Q$.

A root $\tilde{\mathcal{L}}_Z \to \mathcal{L}_Z$ corresponds to dilating

$Q$; pulling back, to dilating $P$.

So, problem becomes:

Given map $P \longrightarrow Q$ of polytopal complexes,

s.t faces of $P$ map onto faces,

find

- Dilation of $P$ and $Q$
- unimodular triangulation of dilation of $Q$

s.t pullback to $P$ has unimodular triangulation.

Classical KKMSD result: $Q = pt$.

So simplifies to find dilation of $P$ w/ unimodular triangulation.