# TAUTOLOGICAL PROJECTION FOR CYCLES ON THE MODULI SPACE OF ABELIAN VARIETIES 

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#### Abstract

We define a tautological projection operator for algebraic cycle classes on the moduli space of principally polarized abelian varieties $\mathcal{A}_{g}$ : every cycle class decomposes canonically as a sum of a tautological and a non-tautological part. The main new result required for the definition of the projection operator is the vanishing of the top Chern class of the Hodge bundle over the boundary $\overline{\mathcal{A}}_{g} \backslash \mathcal{A}_{g}$ of any toroidal compactification $\overline{\mathcal{A}}_{g}$ of the moduli space $\mathcal{A}_{g}$. We prove the vanishing by a careful study of residues in the boundary geometry.

The existence of the projection operator raises many natural questions about cycles on $\mathcal{A}_{g}$. We calculate the projections of all product cycles $\mathcal{A}_{g_{1}} \times \ldots \times \mathcal{A}_{g_{\ell}}$ in terms of Schur determinants, discuss Faber's earlier calculations related to the Torelli locus, and state several open questions.


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## 1. Introduction

1.1. Tautological rings of $\mathcal{A}_{g}$ and $\overline{\mathcal{A}}_{g}$. Let $\mathcal{A}_{g} \subset \overline{\mathcal{A}}_{g}$ be a toroidal compactification of the moduli space of principally polarized abelian varieties. The space $\mathcal{A}_{g}$ is a nonsingular Deligne-Mumford stack of dimension $\binom{g+1}{2}$, and the compactification $\overline{\mathcal{A}}_{g}$ is a reduced and irreducible (but possibly singular) proper Deligne-Mumford stack, see [FC]. The Hodge bundle

$$
\mathbb{E} \rightarrow \mathcal{A}_{g}
$$

is defined as the pullback to $\mathcal{A}_{g}$ via the zero section $s$ of the relative cotangent bundle of the universal family $\mathcal{X}_{g}$ of abelian varieties,

$$
p: \mathcal{X}_{g} \rightarrow \mathcal{A}_{g}, \quad s: \mathcal{A}_{g} \rightarrow \mathcal{X}_{g}, \quad \mathbb{E} \cong s^{*} \Omega_{p}
$$

There is a canonical extension of the Hodge bundle over $\overline{\mathcal{A}}_{g}$ by FC, Theorems V.2.3, VI.1.1, VI.4.2],

$$
\mathbb{E} \rightarrow \overline{\mathcal{A}}_{g}
$$

By vdG EV, the Chern classes $\lambda_{i}$ of the Hodge bundle satisfy Mumford's relation ${ }^{11}$

$$
\begin{equation*}
\left(1+\lambda_{1}+\lambda_{2}+\ldots+\lambda_{g}\right)\left(1-\lambda_{1}+\lambda_{2}-\ldots+(-1)^{g} \lambda_{g}\right)=1 \in \mathrm{CH}^{\mathrm{op}}\left(\overline{\mathcal{A}}_{g}\right), \tag{1}
\end{equation*}
$$

for all $g \geq 1$. Van der Geer vdG defined the tautological rings

$$
\mathrm{R}^{*}\left(\mathcal{A}_{g}\right) \subset \mathrm{CH}^{*}\left(\mathcal{A}_{g}\right), \quad \mathrm{R}^{*}\left(\overline{\mathcal{A}}_{g}\right) \subset \mathrm{CH}^{\mathrm{op}}\left(\overline{\mathcal{A}}_{g}\right)
$$

to be the $\mathbb{Q}$-subalgebras generated by the $\lambda$-classes. Both tautological rings are calculated by a fundamental result of vdG .

Theorem 1 (van der Geer). The following properties hold:
(i) The kernel of the quotient

$$
\mathbb{Q}\left[\lambda_{1}, \lambda_{2}, \lambda_{3}, \ldots, \lambda_{g}\right] \rightarrow \mathrm{R}^{*}\left(\overline{\mathcal{A}}_{g}\right) \rightarrow 0
$$

is generated as an ideal by Mumford's relation (1).
(ii) $\mathrm{R}^{*}\left(\overline{\mathcal{A}}_{g}\right)$ is a Gorenstein local ring with socle in codimension $\binom{g+1}{2}$,

$$
\mathrm{R}^{\left(\frac{g+1}{2}\right)}\left(\overline{\mathcal{A}}_{g}\right) \cong \mathbb{Q} .
$$

The class $\lambda_{1} \lambda_{2} \lambda_{3} \cdots \lambda_{g}$ is a generator of the socle.
(iii) $\mathrm{R}^{*}\left(\mathcal{A}_{g}\right) \cong \mathrm{R}^{*}\left(\overline{\mathcal{A}_{g}}\right) /\left(\lambda_{g}\right)$ is a Gorenstein local ring with socle in codimension $\binom{g}{2}$,

$$
\mathrm{R}^{\left(\frac{g}{2}\right)}\left(\mathcal{A}_{g}\right) \cong \mathbb{Q}
$$

The class $\lambda_{1} \lambda_{2} \lambda_{3} \cdots \lambda_{g-1}$ is a generator of the socle.
1.2. Tautological projection for $\overline{\mathcal{A}}_{g}$. The idea of tautological projection on $\overline{\mathcal{A}}_{g}$ (Definition 2 below) appears in work of Faber [Fa and of Grushevsky and Hulek GH. Since $\overline{\mathcal{A}}_{g}$ is proper of dimension $\binom{g+1}{2}$, we obtain an evaluation

$$
\left.\epsilon^{\mathrm{cpt}}: \mathrm{R}^{(g+1} 2\right)\left(\overline{\mathcal{A}}_{g}\right) \rightarrow \mathbb{Q}, \quad \alpha \mapsto \int_{\overline{\mathcal{A}}_{g}} \alpha,
$$

and a pairing between classes on $\overline{\mathcal{A}}_{g}$,

$$
\begin{equation*}
\langle,\rangle^{\mathrm{cpt}}: \mathrm{CH}^{k}\left(\overline{\mathcal{A}}_{g}\right) \times \mathrm{R}^{(g+1)-k}\left(\overline{\mathcal{A}}_{g}\right) \rightarrow \mathbb{Q}, \quad\langle\gamma, \delta\rangle^{\mathrm{cpt}}=\int_{\overline{\mathcal{A}}_{g}} \gamma \cdot \delta . \tag{2}
\end{equation*}
$$

Here, cpt stands for compact.
By Theorem 1 (ii), the socle of $\mathrm{R}^{*}\left(\overline{\mathcal{A}}_{g}\right)$ is spanned by the class $\lambda_{1} \lambda_{2} \lambda_{3} \cdots \lambda_{g}$. Equivalently,

$$
\begin{equation*}
\gamma_{g}=\epsilon^{\mathrm{cpt}}\left(\lambda_{1} \lambda_{2} \lambda_{3} \cdots \lambda_{g}\right) \neq 0 . \tag{3}
\end{equation*}
$$

The exact evaluation ${ }^{2}$,

$$
\begin{equation*}
\gamma_{g}=\prod_{i=1}^{g} \frac{\left|B_{2 i}\right|}{4 i} \tag{4}
\end{equation*}
$$

[^0]is computed in vdG, page 9]. By the Gorenstein property of $\mathrm{R}^{*}\left(\overline{\mathcal{A}}_{g}\right)$, the pairing of tautological classes
$$
\left.\mathrm{R}^{k}\left(\overline{\mathcal{A}}_{g}\right) \times \mathrm{R}^{(g+1}{ }_{2}^{g+1}\right)-k\left(\overline{\mathcal{A}}_{g}\right) \rightarrow \mathrm{R}^{(g+1}{ }_{2}^{g+1}\left(\overline{\mathcal{A}}_{g}\right) \cong \mathbb{Q}
$$
is non-degenerate (where the last isomorphism is via $\epsilon^{\mathrm{cpt}}$ ).
Definition 2. Let $\gamma \in \mathrm{CH}^{*}\left(\overline{\mathcal{A}}_{g}\right)$. The tautological projection $\operatorname{taut}^{\mathrm{cpt}}(\gamma) \in \mathrm{R}^{*}\left(\overline{\mathcal{A}}_{g}\right)$ is the uniqu$\}^{3}$ tautological class which satisfies
$$
\left\langle\operatorname{taut}^{\mathrm{cpt}}(\gamma), \delta\right\rangle^{\mathrm{cpt}}=\langle\gamma, \delta\rangle^{\mathrm{cpt}}
$$
for all classes $\delta \in \mathrm{R}^{*}\left(\overline{\mathcal{A}}_{g}\right)$.

- If $\gamma \in \mathbf{R}^{*}\left(\overline{\mathcal{A}}_{g}\right)$, then $\gamma=\operatorname{taut}^{\mathrm{cpt}}(\gamma)$, so we have a $\mathbb{Q}$-linear projection operator:

$$
\operatorname{taut}^{\mathrm{cpt}}: \mathrm{CH}^{*}\left(\overline{\mathcal{A}}_{g}\right) \rightarrow \mathrm{R}^{*}\left(\overline{\mathcal{A}}_{g}\right), \quad \text { taut }^{\mathrm{cpt}} \circ \text { taut }^{\mathrm{cpt}}=\text { taut }^{\mathrm{cpt}} .
$$

- From the point of view of $\mathcal{A}_{g}$, a difficulty with the theory on $\overline{\mathcal{A}}_{g}$ is the dependence upon compactification. Given a subvariety

$$
V \subset \mathcal{A}_{g},
$$

we can define a projection

$$
\begin{equation*}
\operatorname{taut}^{\mathrm{cpt}}([\bar{V}]) \in \mathrm{CH}^{*}\left(\overline{\mathcal{A}}_{g}\right) \tag{5}
\end{equation*}
$$

with respect to the Zariski closure $V \subset \bar{V}$ in a toroidal compactification $\mathcal{A}_{g} \subset \overline{\mathcal{A}}_{g}$, but the projection (5) will depend upon the choice of $\overline{\mathcal{A}}_{g}$. In order to study cycles on the moduli of abelian varieties, we would like to construct a canonical projection operator depending just upon $\mathcal{A}_{g}$.
1.3. Top Chern class of the Hodge bundle. To define a tautological projection operator on the interior $\mathcal{A}_{g}$, we will define a pairing similar to (2). The theory depends upon a new vanishing result for the top Chern class of Hodge bundle on $\overline{\mathcal{A}}_{g}$.

We recall the $\lambda_{g}$-pairing on tautological classes ${ }^{4} \mathrm{~B}$ n the moduli space of curves of compact type $\mathcal{M}_{g}^{\text {ct }} \subset \overline{\mathcal{M}}_{g}$. On the tautological ring of $\mathcal{M}_{g}^{\text {ct }}$, the $\lambda_{g}$-pairing is given by

$$
\mathrm{R}^{k}\left(\mathcal{M}_{g}^{\mathrm{ct}}\right) \times \mathrm{R}^{2 g-3-k}\left(\mathcal{M}_{g}^{\mathrm{ct}}\right) \rightarrow \mathrm{R}^{2 g-3}\left(\mathcal{M}_{g}^{\mathrm{ct}}\right) \cong \mathbb{Q}, \quad(\alpha, \beta) \mapsto \int_{\overline{\mathcal{M}}_{g}} \bar{\alpha} \cdot \bar{\beta} \cdot \lambda_{g},
$$

where $\bar{\alpha}$ and $\bar{\beta}$ are arbitrary lifts of $\alpha$ and $\beta$ to $\overline{\mathcal{M}}_{g}$. The $\lambda_{g}$-pairing is well-defined, independent of the lifts, because $\lambda_{g} \in \mathrm{R}^{g}\left(\overline{\mathcal{M}}_{g}\right)$ restricts trivially to the boundary

$$
\left.\lambda_{g}\right|_{\overline{\mathcal{M}}_{g} \backslash \mathcal{M}_{g}^{\mathrm{ct}}}=0,
$$

see [FP2. A parallel boundary vanishing for $\lambda_{g} \in \mathrm{R}^{*}\left(\overline{\mathcal{A}}_{g}\right)$ is our first result.

[^1]Theorem 3. The restriction of $\lambda_{g}$ to $\overline{\mathcal{A}}_{g} \backslash \mathcal{A}_{g}$ vanishes for every toroidal compactification $\overline{\mathcal{A}}_{g}$.
In characteristic $p$, the Theorem 3 can be proven ${ }^{5}$ by considering the $p$-rank zero locus in $\overline{\mathcal{A}}_{g}$. The $p$-rank zero locus avoids the boundary and has class in $\mathrm{R}^{*}\left(\overline{\mathcal{A}}_{g}\right)$ equal to a multiple of $\lambda_{g}$, vdG, Theorem 2.4]. In all characteristics, the vanishing of $\lambda_{g}$ over the boundary of the partial compactification $\mathcal{A}_{g}^{\text {part }}$ of torus rank 1 degenerations follows from the discussion of vdG, page 6]. The statement for the entire boundary is new.

Our proof of Theorem 3 is obtained as a consequence of the following statements about semistable degenerations of abelian varieties:
(i) The sheaf of relative log differentials has a trivial rank 1 quotient on the singularities of the fibers of the universal family. The trivial quotient statement is true for any semistable family, independently of abelian structure (also applying, for example, to families of curves).
(ii) For abelian schemes, the sheaf of relative $\log$ differentials is isomorphic to the pullback of the Hodge bundle FC].
The full proof is presented in Section 2 after a review of $\log$ structures, semistable degenerations, and residues.

The vanishing of Theorem 3 of the top Chern class of the Hodge bundle for the moduli of abelian varieties implies the parallel vanishing for the moduli of curves,

$$
\left.\lambda_{g}\right|_{\overline{\mathcal{A}}_{g} \backslash \mathcal{A}_{g}}=\left.0 \quad \Longrightarrow \quad \lambda_{g}\right|_{\overline{\mathcal{M}}_{g} \backslash \mathcal{M}_{g}^{\text {ct }}}=0,
$$

via the Torelli map from $\overline{\mathcal{M}}_{g}$ to a suitabl $]^{6}$ toroidal compactification $\overline{\mathcal{A}}_{g}$. We can hope for an even deeper connection: a lifting of Pixton's formula HMPPS JPPZ for $\lambda_{g}$ on $\overline{\mathcal{M}}_{g}$ to $\overline{\mathcal{A}}_{g}$. The natural context for such a lifting should be the logarithmic Chow ring of the moduli space of abelian varieties. A discussion of these ideas is presented in Section 2.6.
1.4. Tautological projection for $\mathcal{A}_{g}$. Given $\alpha \in \mathrm{CH}^{*}\left(\mathcal{A}_{g}\right)$, we define an evaluation,

$$
\epsilon^{\mathrm{ab}}: \mathrm{CH}^{\binom{g}{2}}\left(\mathcal{A}_{g}\right) \rightarrow \mathbb{Q}, \quad \alpha \mapsto \int_{\overline{\mathcal{A}}_{g}} \bar{\alpha} \cdot \lambda_{g}=\operatorname{deg}\left(\lambda_{g} \cap \bar{\alpha}\right),
$$

where $\bar{\alpha}$ is a lift to the toroidal compactification $\overline{\mathcal{A}}_{g}$. The answer is well-defined (independent of lift) by the vanishing of Theorem 3. We also have an induced pairing between classes on $\mathcal{A}_{g}$,

$$
\begin{equation*}
\langle,\rangle: \mathrm{CH}^{k}\left(\mathcal{A}_{g}\right) \times \mathrm{R}^{\left(\frac{g}{2}\right)-k}\left(\mathcal{A}_{g}\right) \rightarrow \mathbb{Q}, \quad\langle\gamma, \delta\rangle=\int_{\overline{\mathcal{A}}_{g}} \bar{\gamma} \cdot \bar{\delta} \cdot \lambda_{g}=\operatorname{deg}\left(\bar{\delta} \cdot \lambda_{g} \cap \bar{\gamma}\right), \tag{6}
\end{equation*}
$$

which we call the $\lambda_{g}$-pairing for the moduli of abelian varieties. Here, $\bar{\delta}$ is a lift of

$$
\delta \in \mathrm{R}^{*}\left(\mathcal{A}_{g}\right)=\mathrm{R}^{*}\left(\overline{\mathcal{A}}_{g}\right) /\left\langle\lambda_{g}\right\rangle
$$

to a tautological class on $\overline{\mathcal{A}}_{g}$, while $\bar{\gamma}$ is an arbitrary lift of $\gamma$. The lift of $\delta$ is well-defined up to the class $\lambda_{g}$, while the lift of $\gamma$ is well-defined up to cycles supported on the boundary. The vanishing of $\lambda_{g}^{2}$ on $\overline{\mathcal{A}}_{g}$ and Theorem 3 ensure that the $\lambda_{g}$-pairing is well-defined.

[^2]By Theorem 1 1 (iii), the socle of $\mathrm{R}^{*}\left(\mathcal{A}_{g}\right)$ is spanned by the class $\lambda_{1} \lambda_{2} \lambda_{3} \cdots \lambda_{g-1}$. The Gorenstein property of $\mathrm{R}^{*}\left(\mathcal{A}_{g}\right)$ together with the non-vanishing (3) implies that the restriction of the $\lambda_{g}$-pairing to tautological classes

$$
\mathrm{R}^{k}\left(\mathcal{A}_{g}\right) \times \mathrm{R}^{\left(\frac{g}{2}\right)-k}\left(\mathcal{A}_{g}\right) \rightarrow \mathrm{R}^{\left(\frac{g}{2}\right)}\left(\mathcal{A}_{g}\right) \cong \mathbb{Q}
$$

is non-degenerate (where the last isomorphism is via $\epsilon^{\mathrm{ab}}$ ).
Definition 4. Let $\gamma \in \mathrm{CH}^{*}\left(\mathcal{A}_{g}\right)$. The tautological projection $\operatorname{taut}(\gamma) \in \mathrm{R}^{*}\left(\mathcal{A}_{g}\right)$ is the unique ${ }^{7}$ tautological class which satisfies

$$
\langle\operatorname{taut}(\gamma), \delta\rangle=\langle\gamma, \delta\rangle
$$

for all classes $\delta \in \mathrm{R}^{*}\left(\mathcal{A}_{g}\right)$.

- If $\gamma \in \mathrm{R}^{*}\left(\mathcal{A}_{g}\right)$, then $\gamma=\operatorname{taut}(\gamma)$, so we have a $\mathbb{Q}$-linear projection operator:

$$
\text { taut : } \mathrm{CH}^{*}\left(\mathcal{A}_{g}\right) \rightarrow \mathrm{R}^{*}\left(\mathcal{A}_{g}\right), \quad \text { taut } \circ \text { taut }=\text { taut } .
$$

- For $\gamma \in \mathrm{CH}^{*}\left(\mathcal{A}_{g}\right)$, tautological projection provides a canonical decomposition

$$
\gamma=\operatorname{taut}(\gamma)+(\gamma-\operatorname{taut}(\gamma))
$$

into purely tautological and purely non-tautological parts.

- Tautological projection commutes with restriction: for every toroidal compactification $\mathcal{A}_{g} \subset \overline{\mathcal{A}}_{g}$ and every class $\gamma \in \mathrm{CH}^{*}\left(\overline{\mathcal{A}}_{g}\right)$,

$$
\left.\operatorname{taut}^{\mathrm{cpt}}(\gamma)\right|_{\mathcal{A}_{g}}=\operatorname{taut}\left(\left.\gamma\right|_{\mathcal{A}_{g}}\right) .
$$

To prove the restriction property, consider classes

$$
\gamma \in \mathrm{CH}^{*}\left(\overline{\mathcal{A}}_{g}\right) \text { and } \delta \in \mathrm{R}^{*}\left(\mathcal{A}_{g}\right) .
$$

Equations (2) and (6) imply the compatibility between pairings

$$
\begin{equation*}
\left\langle\left.\gamma\right|_{\mathcal{A}_{g}}, \delta\right\rangle=\left\langle\gamma, \bar{\delta} \lambda_{g}\right\rangle^{\mathrm{cpt}} \tag{7}
\end{equation*}
$$

where $\bar{\delta}$ is any lift of $\delta$ to the compactification $\overline{\mathcal{A}}_{g}$. Then,

$$
\begin{aligned}
\left\langle\operatorname{taut}^{\mathrm{cpt}}(\gamma), \bar{\delta} \lambda_{g}\right\rangle^{\mathrm{cpt}}=\left\langle\gamma, \bar{\delta} \lambda_{g}\right\rangle^{\mathrm{cpt}} & \Longrightarrow \quad\left\langle\left.\operatorname{taut}^{\mathrm{cpt}}(\gamma)\right|_{\mathcal{A}_{g}}, \delta\right\rangle=\left\langle\left.\gamma\right|_{\mathcal{A}_{g}}, \delta\right\rangle \\
& \left.\Longrightarrow \quad \operatorname{taut}^{\mathrm{cpt}}(\gamma)\right|_{\mathcal{A}_{g}}=\operatorname{taut}\left(\left.\gamma\right|_{\mathcal{A}_{g}}\right) .
\end{aligned}
$$

Here, we have used Definition 2, equation (7), and Definition 4 (and the argument is not possible without Theorem (3).

[^3]1.5. Tautological projection of product classes. As an application of the theory, we consider the tautological projections of product loci. For $g=g_{1}+g_{2}$ with $g_{i} \geq 1$, the product map
$$
\mathcal{A}_{g_{1}} \times \mathcal{A}_{g_{2}} \rightarrow \mathcal{A}_{g}
$$
defines a class $\left[\mathcal{A}_{g_{1}} \times \mathcal{A}_{g_{2}}\right] \in \mathrm{CH}^{*}\left(\mathcal{A}_{g}\right)$. More generally, for every partition $g=\sum_{i=1}^{\ell} g_{i}$ in positive parts, we have a product map and an associated class:
\[

$$
\begin{equation*}
\prod_{1=1}^{\ell} \mathcal{A}_{g_{i}} \rightarrow \mathcal{A}_{g}, \quad\left[\prod_{1=1}^{\ell} \mathcal{A}_{g_{i}}\right] \in \mathrm{CH}^{*}\left(\mathcal{A}_{g}\right) \tag{8}
\end{equation*}
$$

\]

Whether these product maps and classes (8) naturally extend to a compactification $\overline{\mathcal{A}}_{g}$ depends upon the choice of toroidal compactification. Toroidal compactifications of $\mathcal{A}_{g}$ correspond to choices of admissible fans $\Sigma_{g}$ in $\operatorname{Sym}_{\mathrm{rc}}^{2}\left(\mathbb{R}^{g}\right)$, the rational closure of the positive-definite symmetric forms on $\mathbb{R}^{g}$. Following GHT], a collection of such fans $\left\{\Sigma_{g}\right\}_{g \in \mathbb{N}}$ is additive if the sum $\sigma_{1} \oplus \sigma_{2}$ of any two cones $\sigma_{1} \in \Sigma_{g_{1}}$ and $\sigma_{2} \in \Sigma_{g_{2}}$ is a cone in $\Sigma_{g_{1}+g_{2}}$. Let $\overline{\mathcal{A}}_{g}^{\Sigma_{g}}$ be a toroidal compactification corresponding to an additive collection of fans $\left\{\Sigma_{g}\right\}$. The perfect cone compactification satisfies these properties, see $\overline{\mathrm{SB}}$. In the additive case, the product maps extend,

$$
\prod_{i=1}^{\ell} \overline{\mathcal{A}}_{g_{i}}^{\Sigma_{g_{i}}} \rightarrow \overline{\mathcal{A}}_{g}^{\Sigma_{g}}
$$

and we can therefore define cycles

$$
\left[\prod_{i=1}^{\ell} \overline{\mathcal{A}}_{g_{i}}^{\Sigma_{g_{i}}}\right] \in \mathrm{CH}^{*}\left(\overline{\mathcal{A}}_{g}^{\Sigma_{g}}\right)
$$

While the definition of tautological projection is independent of toroidal compactification, natural compactifications can be used for the calculation. We prove a closed formula for the tautological projection of the product cycles. The result extends calculations of GH for $g \leq 5$.

Theorem 5. For $g_{1}+\ldots+g_{\ell}=g$, the tautological projection of the product locus $\overline{\mathcal{A}}_{g_{1}}^{\Sigma_{g_{1}}} \times \cdots \times \overline{\mathcal{A}}_{g_{\ell}}^{\Sigma_{\ell}}$ in $\overline{\mathcal{A}}_{g}^{\Sigma_{g}}$ is given by a $g \times g$ determinant,

$$
\operatorname{taut}^{\mathrm{cpt}}\left(\left[\overline{\mathcal{A}}_{g_{1}}^{\Sigma_{g_{1}}} \times \cdots \times \overline{\mathcal{A}}_{g_{\ell}}^{\Sigma_{g_{\ell}}}\right]\right)=\frac{\gamma_{g_{1}} \cdots \gamma_{g_{\ell}}}{\gamma_{g}}\left|\begin{array}{cccc}
\lambda_{\alpha_{1}} & \lambda_{\alpha_{1}+1} & \ldots & \lambda_{\alpha_{1}+g-1} \\
\lambda_{\alpha_{2}-1} & \lambda_{\alpha_{2}} & \ldots & \lambda_{\alpha_{2}+g-2} \\
\ldots & \ldots & \ldots & \ldots \\
\lambda_{\alpha_{g}-g+1} & \lambda_{\alpha_{g}-g+2} & \ldots & \lambda_{\alpha_{g}}
\end{array}\right|
$$

for the vector

$$
\alpha=(\underbrace{g-g_{1}, \ldots, g-g_{1}}_{g_{1}}, \underbrace{g-g_{1}-g_{2}, \ldots, g-g_{1}-g_{2}}_{g_{2}}, \ldots, \underbrace{g-g_{1}-\ldots-g_{\ell}, \ldots, g-g_{1}-\ldots-g_{\ell}}_{g_{\ell}}) .
$$

We set $\lambda_{k}=0$ for $k<0$ or $k>g$ and $\lambda_{0}=1$.
In the above determinant, $\alpha_{i}$ denotes the $i^{\text {th }}$ component of the vector $\alpha$. The last $g_{\ell}$ entries of $\alpha$ are 0 , and contribute rows with 1 's on the main diagonal and 0 's below the main diagonal. These last entries do not change the determinant, but are included for a more symmetric formulation. The constants $\gamma_{g}$ are defined in (4).

The proof of Theorem 5 in Section 3 relies on the connections between the tautological ring of $\overline{\mathcal{A}}_{g}^{\Sigma_{g}}$ and the Chow ring of the Lagrangian Grassmannian $\mathrm{LG}_{g}$ of $\mathbb{C}^{2 g}$ as explained in vdG2. The argument combines properties of tautological projection, the Hirzebruch-Mumford proportionality principle, and the geometry of $\mathrm{LG}_{g}$.

Using the restriction property of tautological projection and the relations in Theorem 1 (iii), we prove the following result in Section 3.4.

Theorem 6. For $g_{1}+\ldots+g_{\ell}=g$, the tautological projection of the product locus $\mathcal{A}_{g_{1}} \times \cdots \times \mathcal{A}_{g_{\ell}}$ in $\mathcal{A}_{g}$ is given by a $(g-\ell) \times(g-\ell)$ determinant,

$$
\operatorname{taut}\left(\left[\mathcal{A}_{g_{1}} \times \cdots \times \mathcal{A}_{g_{\ell}}\right]\right)=\frac{\gamma_{g_{1}} \cdots \gamma_{g_{\ell}}}{\gamma_{g}} \cdot \lambda_{g-1} \cdots \lambda_{g-\ell+1} \cdot\left|\begin{array}{cccc}
\lambda_{\beta_{1}} & \lambda_{\beta_{1}+1} & \ldots & \lambda_{\beta_{1}+g^{*}-1} \\
\lambda_{\beta_{2}-1} & \lambda_{\beta_{2}} & \ldots & \lambda_{\beta_{2}+g^{*}-2} \\
\ldots & \ldots & \ldots & \ldots \\
\lambda_{\beta_{g^{*}-g^{*}+1}} & \lambda_{\beta_{g^{*}}-g^{*}+2} & \ldots & \lambda_{\beta_{g^{*}}}
\end{array}\right|,
$$

for the vector

$$
\beta=(\underbrace{g^{*}-g_{1}^{*}, \ldots, g^{*}-g_{1}^{*}}_{g_{1}^{*}}, \underbrace{g^{*}-g_{1}^{*}-g_{2}^{*}, \ldots, g^{*}-g_{1}^{*}-g_{2}^{*}}_{g_{2}^{*}}, \ldots, \underbrace{g^{*}-g_{1}^{*}-\ldots-g_{\ell}^{*}}_{g_{\ell}^{*}}),
$$

where $g^{*}=g-\ell$ and $g_{i}^{*}=g_{i}-1$.
The tautological projections of the product loci in $\mathcal{A}_{g}$ from Theorem 6 in the simplest cases are:

$$
\begin{gather*}
\operatorname{taut}\left(\left[\mathcal{A}_{1} \times \mathcal{A}_{g-1}\right]\right)=\frac{g}{6\left|B_{2 g}\right|} \lambda_{g-1},  \tag{9}\\
\operatorname{tant}\left(\left[\mathcal{A}_{2} \times \mathcal{A}_{g-2}\right]\right)=\frac{1}{360} \cdot \frac{g(g-1)}{\left|B_{2 g}\right|\left|B_{2 g-2}\right|} \cdot \lambda_{g-1} \lambda_{g-3},  \tag{10}\\
\operatorname{taut}\left(\left[\mathcal{A}_{3} \times \mathcal{A}_{g-3}\right]\right)=\frac{1}{45360} \cdot \frac{g(g-1)(g-2)}{\left|B_{2 g}\right|\left|B_{2 g-2}\right|\left|B_{2 g-4}\right|} \cdot \lambda_{g-1}\left(\lambda_{g-4}^{2}-\lambda_{g-3} \lambda_{g-5}\right),  \tag{11}\\
\operatorname{taut}\left(\left[\mathcal{A}_{1} \times \mathcal{A}_{2} \times \mathcal{A}_{g-3}\right]\right)=\frac{1}{90} \cdot \frac{g(g-1)(g-2)}{\left|B_{2 g}\right|\left|B_{2 g-2}\right|\left|B_{2 g-4}\right|} \cdot \lambda_{g-1} \lambda_{g-2} \lambda_{g-4},  \tag{12}\\
\operatorname{taut}([\underbrace{\mathcal{A}_{1} \times \ldots \times \mathcal{A}_{1}}_{k} \times \mathcal{A}_{g-k}])=\left(\prod_{i=g-k+1}^{g} \frac{i}{6\left|B_{2 i}\right|}\right) \lambda_{g-1} \cdots \lambda_{g-k} \tag{13}
\end{gather*}
$$

In genus $g=4$, formula 13) yields

$$
\operatorname{taut}\left(\left[\mathcal{A}_{1} \times \mathcal{A}_{1} \times \mathcal{A}_{2}\right]\right)=420 \lambda_{3} \lambda_{2}, \quad \operatorname{taut}\left(\left[\mathcal{A}_{1} \times \mathcal{A}_{1} \times \mathcal{A}_{1} \times \mathcal{A}_{1}\right]\right)=4200 \lambda_{3} \lambda_{2} \lambda_{1}
$$

In fact, $\left[\mathcal{A}_{1} \times \mathcal{A}_{1} \times \mathcal{A}_{2}\right]$ and $\left[\mathcal{A}_{1} \times \mathcal{A}_{1} \times \mathcal{A}_{1} \times \mathcal{A}_{1}\right]$ are tautological [COP].
An interesting case of (13) occurs when $k=g-1$ since the class $\lambda_{g-1} \cdots \lambda_{1}$ generates the socle of the tautological ring $\mathrm{R}^{\left(\frac{g}{2}\right)}\left(\mathcal{A}_{g}\right)$. A speculation of COP is that the $g$-fold product

$$
\left[\mathcal{A}_{1} \times \cdots \times \mathcal{A}_{1}\right] \in \mathrm{CH}^{\left(\frac{g}{2}\right)}\left(\mathcal{A}_{g}\right)
$$

also lies in the socle of the tautological ring. If the speculation is correct, then we obtain an exact evaluation

$$
\begin{equation*}
\left[\mathcal{A}_{1} \times \cdots \times \mathcal{A}_{1}\right]=\left(\prod_{i=1}^{g} \frac{i}{6\left|B_{2 i}\right|}\right) \lambda_{g-1} \cdots \lambda_{1} . \tag{14}
\end{equation*}
$$

Question A. When is the non-tautological part of the product locus nonzero:

$$
\left[\prod_{1=1}^{\ell} \mathcal{A}_{g_{i}}\right]-\operatorname{taut}\left(\left[\prod_{1=1}^{\ell} \mathcal{A}_{g_{i}}\right]\right) \neq 0 ?
$$

The cycles $\left[\mathcal{A}_{1} \times \mathcal{A}_{g-1}\right] \in \mathrm{CH}^{*}\left(\mathcal{A}_{g}\right)$ are studied in COP via the Torelli map

$$
\text { Tor : } \mathcal{M}_{g}^{\mathrm{ct}} \rightarrow \mathcal{A}_{g}
$$

A central result of $[\mathrm{COP}]$ is that $\left[\mathcal{A}_{1} \times \mathcal{A}_{5}\right]$ is not tautological on $\mathcal{A}_{6}$, so the purely non-tautological part is nonzero,

$$
\left[\mathcal{A}_{1} \times \mathcal{A}_{5}\right]-\operatorname{taut}\left(\left[\mathcal{A}_{1} \times \mathcal{A}_{5}\right]\right) \neq 0
$$

Detection of the non-vanishing of the non-tautological part is subtle since the class

$$
\Delta_{g}=\operatorname{Tor}^{*}\left(\left[\mathcal{A}_{1} \times \mathcal{A}_{g-1}\right]-\frac{g}{6\left|B_{2 g}\right|} \lambda_{g-1}\right)
$$

lies in the kernel of the $\lambda_{g}$-pairing

$$
\mathrm{R}^{g-1}\left(\mathcal{M}_{g}^{\mathrm{ct}}\right) \times \mathrm{R}^{g-2}\left(\mathcal{M}_{g}^{\mathrm{ct}}\right) \rightarrow \mathrm{R}^{2 g-3}\left(\mathcal{M}_{g}^{\mathrm{ct}}\right) \cong \mathbb{Q}
$$

for all genera $g$ by an argument of Pixton, see COP.
The Noether-Lefschetz loci in $\mathcal{A}_{g}$ parametrize abelian varieties whose Picard rank jumps. The rank 2 Noether-Lefschetz loci have been classified in DL (the products $\mathcal{A}_{g_{1}} \times \mathcal{A}_{g_{2}}$ for $g_{1}+g_{2}=g$ arise in the classification, but there are other loci as well). After Theorem 6, we can hope for a more general result.

Question B. Calculate the tautological projections of all Noether-Lefschetz loci in $\mathcal{A}_{g}$.
Beyond product and Noether-Lefschetz cycles, we can consider the tautological projection of the locus of Jacobians of genus $g$ curves of compact type,

$$
\left[\mathcal{J}_{g}\right] \in \mathrm{CH}^{*}\left(\mathcal{A}_{g}\right)
$$

Faber Fa determined these explicitly for $g \leq 7$. For all genera, in the basis of monic square-free monomials in the $\lambda^{\prime} s$, the leading term is given by

$$
\operatorname{taut}\left(\left[\mathcal{J}_{g}\right]\right)=\left(\frac{1}{g-1} \prod_{i=1}^{g-2} \frac{2}{(2 i+1)\left|B_{2 i}\right|}\right) \lambda_{1} \cdots \lambda_{g-3}+\ldots
$$

as proposed in Fa, Conjecture 1] and confirmed via FP1, Theorem 4]. A more complicated formula for the coefficient of the term $\lambda_{2} \ldots \lambda_{g-4} \lambda_{g-2}$ was predicted by Fa, Conjecture 2] and subsequently proven in [FP2, Section 5.2].

For each genus $g$, the class taut $\left(\left[\mathcal{J}_{g}\right]\right) \in \mathrm{R}^{*}\left(\mathcal{A}_{g}\right)$ can be computed algorithmically by a finite number of Hodge integral evaluations Fa. Finding expressions for the coefficients of the remaining terms of $\operatorname{taut}\left(\left[\mathcal{J}_{g}\right]\right)$ is an open question, but we could hope for more structure.

Question C. Is there a simpler way to understand the tautological projection

$$
\operatorname{taut}\left(\left[\mathcal{J}_{g}\right]\right) \in \mathrm{R}^{*}\left(\mathcal{A}_{g}\right) ?
$$

When is the non-tautological part nonzero?
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## 2. The top Chern class of the Hodge bundle

2.1. Overview. Logarithmic geometry provides a convenient tool for considering all toroidal compactifications of $\mathcal{A}_{g}$ simultaneously and plays an important role in the proof of Theorem 3 A quick review of the basic language of log geometry is given in Section 2.2. The proof of Theorem 3 relies on the residue map constructed in Section 2.3. After a discussion of the Hodge bundle on toroidal compactifications of $\mathcal{A}_{g}$ in Section 2.4 , the proof of Theorem 3 is presented in Section 2.5 Conjectures and future directions related to tautological classes on toroidal compactifications of $\mathcal{A}_{g}$ are discussed in Section 2.6.
2.2. The logarithmic Chow ring for toroidal embeddings. We will use the language of $\log$ geometry and assume some rudimentary familiarity with the central definitions as given in $[\mathrm{K}$. A summary of the relevant background information can be found in MPS.

For a log scheme $\left(S, M_{S}\right)$, we write

$$
\epsilon: \underset{9}{M_{S}} \rightarrow \mathcal{O}_{S}
$$

for the structure morphism from the monoid $M_{S}$. Let $M_{S}^{\mathrm{gp}}$ be the associated group, and let $\bar{M}_{S}$ be the characteristic monoid

$$
\bar{M}_{S}=M_{S} / \mathcal{O}_{S}^{*} .
$$

The sheaf $\bar{M}_{S}$ is constructible, and thus stratifies $S$.
For a toroidal embedding $(S, D)$, the log structure is given by the étale sheaf of monoids

$$
M_{S}=\left\{f \in \mathcal{O}_{S}: f \text { is a unit on } S \backslash D\right\} .
$$

For toroidal embeddings, we will denote the $\log$ structure by either $(S, D)$ or ( $S, M_{S}$ ) depending upon context.

An important special case is that of a normal crossings pair ( $S, D$ ): a smooth Deligne-Mumford stack $S$ with a normal crossings divisor $D$. These are precisely the log smooth log Deligne-Mumford stacks with smooth underlying stack. The normal crossings condition is equivalent to

$$
\bar{M}_{S, s}=\mathbb{N}^{k}
$$

for each $s \in S$ and for some $k$ depending on $s$. The integer $k$ is the number of local branches of $D$ passing though $s$.

For a morphism of $\log$ structures $f: X \rightarrow S$, let

$$
\bar{M}_{X / S}=\bar{M}_{X} / \bar{M}_{S}
$$

be the relative characteristic monoid. The morphism $f$ is strict if $\bar{M}_{X / S}=0$. For a log scheme ( $X, M_{X}$ ) defined over a field $K$, a (global) chart for the $\log$ structure of $X$ is a finitely generated, saturated monoid $P$ and a strict map

$$
X \rightarrow \operatorname{Spec}(K[P]),
$$

where the target carries the canonical log structure (coming from the torus invariant divisor). We require all $\log$ schemes to admit charts étale locally.

A morphism $f: X \rightarrow S$ is $\log$ smooth if, étale locally on $X$, we can find a map of monoids $Q \rightarrow P$ such that $\operatorname{Ker}\left(Q^{\mathrm{gp}} \rightarrow P^{\mathrm{gp}}\right)$ and the torsion part of $\operatorname{Coker}\left(Q^{\mathrm{gp}} \rightarrow P^{\mathrm{gp}}\right)$ are finite groups of order invertible in $K$, and a diagram

with $\alpha_{X}, \alpha_{S}$ strict and $\alpha_{X}$ smooth. If we can find such a diagram with $Q \rightarrow P$ of finite index and $\alpha_{X}$ étale, the morphism is log étale. In particular, toroidal embeddings ( $S, D$ ) are exactly the log smooth log schemes over (Spec $K, K^{*}$ ).

To a toroidal embedding $(S, D)$, we can associate a cone complex $\Sigma_{(S, D)}$. We refer the reader to MPS, Section 4.3] for an outline of the construction and further references. Each cone has an integral structure, and the cone complex is built by gluing the cones together with their integral
structure. A strata blowup is a blowup of $(S, D)$ along a smooth closed stratum. The result is a new toroidal embedding $\left(S^{\prime}, D^{\prime}\right)$ with $D^{\prime}$ the total transform of $D$, so the procedure can be iterated indefinitely. A $\log$ modification of $(S, D)$ is a proper birational map $S^{\prime} \rightarrow S$ that can be dominated by an iterated strata blowup. More intrinsically, the log modifications of $S$ are precisely the proper, representable, log étale monomorphisms $S^{\prime} \rightarrow S$. Combinatorially, log modifications of $S$ correspond exactly to subdivisions of the cone complex $\Sigma_{(S, D)}$.

Log modifications form a filtered system. Indeed, two log modifications

$$
S^{\prime} \rightarrow S \quad \text { and } \quad S^{\prime \prime} \rightarrow S
$$

can always be dominated by a third: the log modification corresponding to the common refinement of the subdivisions corresponding to $S^{\prime}$ and $S^{\prime \prime}$ together with the intersection of the integral structures. To a toroidal embedding $(S, D)$, we can thus associate refined operational Chow groups

$$
\log \mathrm{CH}^{*}(S, D)=\underset{\longrightarrow}{\lim } \mathrm{CH}^{\mathrm{op}}\left(S^{\prime}\right)
$$

where $S^{\prime}$ ranges over log modifications of $S$.
Toroidal compactifications of $\mathcal{A}_{g}$ correspond to admissible decompositions of the rational closure $\operatorname{Sym}_{\mathrm{rc}}^{2}\left(\mathbb{R}^{g}\right)$ of the cone of positive-definite symmetric quadratic forms on $\mathbb{R}^{g}$. Any two admissible decompositions can be refined by a third. Hence, $\log \mathrm{CH}^{*}\left(\overline{\mathcal{A}}_{g}, \partial \overline{\mathcal{A}}_{g}\right)$ is independent of the choice of compactification. We define

$$
\log \mathrm{CH}^{*}\left(\mathcal{A}_{g}\right)=\log \mathrm{CH}^{*}\left(\overline{\mathcal{A}}_{g}, \partial \overline{\mathcal{A}}_{g}\right)
$$

for any toroidal compactification $\overline{\mathcal{A}}_{g}$.
2.3. Semistable families and residues. For suitable families of $\log$ schemes, we prove the existence of a residue map in Theorem 16 below. The residue map will be applied in Section 2.4 to the universal family over the moduli space of abelian varieties in order to prove Theorem 3 ,

The sheaf of relative logarithmic differentials $\Omega_{X / S}^{\log }$ is defined as the quotient of

$$
\Omega_{X / S} \oplus\left(\mathcal{O}_{X} \otimes_{\mathbb{Z}} M_{X}^{\mathrm{gp}}\right)
$$

by the subsheaf locally generated by sections of the form

$$
(d \epsilon(m), 0)-(0, \epsilon(m) \otimes m) \quad \text { and } \quad(0,1 \otimes n)
$$

where $m \in M_{X}$ and $n \in \operatorname{Im}\left(M_{S}^{\mathrm{gp}}\right) \subset M_{X}^{\mathrm{gp}}$, see K . As usual, we write

$$
d \log m=(0,1 \otimes m)
$$

which we view as $d \epsilon(m) / \epsilon(m)$.
For a strict map $f: X \rightarrow S$, we have $\Omega_{X / S}^{\log }=\Omega_{X / S}$, and for a log étale map $f: X \rightarrow S$, we have $\Omega_{X / S}^{\log }=0$.

Definition 7. The sheaf of residues is defined to be the quotient

$$
\mathcal{R}=\Omega_{X / S}^{\log } / \Omega_{X / S}
$$

Definition 8. (M, Definition 2.1.2]) A logarithmic family $X \rightarrow S$ is a $\log$ smooth, surjective, integral and saturated map of log schemes.

Families of stable curves and families of toroidal compactifications of semi-abelian schemes are all examples of $\log$ families. The condition that $f$ is integral and saturated - called weak semistability in $M$ - is a technical condition that, for $\log$ smooth $f$, implies that $f$ is flat with reduced fibers $M$, Lemma 3.1.2], [Ts, Theorem II.4.2] $]^{8}$ Being integral and saturated is local on $X$ and can be understood in terms of the cone complexes $\Sigma_{X}$ and $\Sigma_{S}$. Integrality combined with saturatedness say, locally on $X$, that the associated map $\Sigma_{X} \rightarrow \Sigma_{S}$ maps cones of $\Sigma_{X}$ surjectively onto cones of $\Sigma_{S}$ and that the integral structure of a cone $\sigma$ surjects onto the integral structure of its image cone.

Given a $\log$ scheme $\left(S, M_{S}\right)$ and a finite index extension of sheaves $M_{S} \rightarrow M_{S}^{\prime}$, there is a universal log DM stack $\left(S^{\prime}, M_{S}^{\prime}\right)$ with a log map to $\left(S, M_{S}\right)$ whose map on log structures is given by the extension $M_{S} \rightarrow M_{S}^{\prime}$. The stack $S^{\prime}$ is called the root of $S$ along $M_{S} \rightarrow M_{S}^{\prime}$. The simplest instance of this operation is taking a root along a boundary stratum of a normal crossings pair $(S, D)$. We call a composition of logarithmic modifications and roots a logarithmic alteration. Log alterations of toroidal embeddings are isomorphisms on $S \backslash D$, but are not necessarily representable. Logarithmic alterations are furthermore log étale. See MW for a lengthier discussion.

Remark 9. Because we work with $\mathbb{Q}$-coefficients, pullbacks via root maps induce isomorphisms on Chow groups. Therefore, for a toroidal embedding $(S, D)$, the logarthmic Chow groups can be equivalently defined as

$$
\log \mathrm{CH}^{*}(S, D)=\underset{\longrightarrow}{\lim } \mathrm{CH}^{\mathrm{op}}\left(S^{\prime}\right)
$$

where $S^{\prime}$ ranges over logarithmic alterations of $S$.

Definition 10. Let $f: X \rightarrow S$ be a log map. A logarithmic alteration of $f$ is a $\log$ map $f^{\prime}: X^{\prime} \rightarrow S^{\prime}$ and a commutative diagram

such that $S^{\prime} \rightarrow S$ and $X^{\prime} \rightarrow X$ are logarithmic alterations.

Theorem 11 (Semistable Reduction Theorem, ALT, AK, M). Let $f: X \rightarrow S$ be a dominant log smooth morphism of logarithmic schemes. Then there is a log alteration $f^{\prime}: X^{\prime} \rightarrow S^{\prime}$ of $f$ which is a log family. Furthermore, if $S$ is $\log$ smooth, then one can take $X^{\prime}$ and $S^{\prime}$ smooth.

Definition 12. A pair $(X, D)$ is called simplest normal crossings if $D \subset X$ is normal crossings in the Zariski topology and each intersection of components of $D$ is connected.

[^4]Remark 13. In more geometric terms, a toroidal embedding $(X, D)$ has simplest normal crossings if the following conditions are all satisfied:
(i) $(X, D)$ is a normal crossings pair (with $X$ smooth),
(ii) the components of $D$ have no self-intersection,
(iii) intersections of components of $D$ are connected.

We note that properties (ii) and (iii) are stable under logarithmic alterations, but (i) is not.
Corollary 14. Let $f:\left(X, D_{X}\right) \rightarrow\left(S, D_{S}\right)$ be a log family with $\left(S, D_{S}\right)$ toroidal. Then there is a log alteration $f^{\prime}:\left(X^{\prime}, D_{X^{\prime}}\right) \rightarrow\left(S^{\prime}, D_{S^{\prime}}\right)$ which is a log family where both $S^{\prime}$ and $X^{\prime}$ have simplest normal crossings.

Proof. We can make $S$ and $X$ simplest normal crossings by suitable log modifications

$$
S_{1} \rightarrow S, \quad X_{1} \rightarrow X \times_{S} S_{1}
$$

For example, we can take the log modifications corresponding to double barycentric subdivisions (see the discussion in MPS, Section 5.6]). Since logarithmic alterations are log étale and surjective, the morphism $X_{1} \rightarrow S_{1}$ remains $\log$ smooth and surjective. Moreover, as noted in Remark 13, any further $\log$ alteration of $X_{1}$ or $S_{1}$ which is smooth will have simplest normal crossings. Therefore, we may apply Theorem 11 to $X_{1} \rightarrow S_{1}$ to get the desired $X^{\prime} \rightarrow S^{\prime}$.

Let $\left(X, M_{X}\right)$ be a log scheme. An orientation on $M_{X}$ is an ordering of the sections of $\bar{M}_{X}(U)$ for all $U \subset X$ compatible with the restriction maps. We say $M_{X}$ is orientable if it admits an orientation.

Lemma 15. Suppose $(X, D)$ is a pair with simplest normal crossings. Then the log structure of $(X, D)$ is orientable.

Proof. We choose an ordering of the components $D_{i}$ of the divisor $D$. Every stratum is the intersection

$$
D_{i_{1}} \cap \cdots \cap D_{i_{k}}
$$

where the components appear in the ordering we have chosen. For each $x$ in the stratum, we have a canonical isomorphism

$$
\bar{M}_{X, x}=\bigoplus \mathbb{N} m_{i_{k}}
$$

Here, $m_{i_{k}}$ is the image in $\bar{M}_{X, x}$ of any element $\widetilde{m}_{i_{k}} \in M_{X, x}$ that maps to a local defining equation for $D_{i_{k}}$. We order the sections as $m_{i_{1}}<m_{i_{2}} \cdots<m_{i_{k}}$.

Theorem 16. Let $f: X \rightarrow S$ be a log family with $X$ and $S$ simplest normal crossings pairs. Every choice of orientation on $\bar{M}_{X}$ yields a map

$$
\begin{equation*}
\mathcal{R} \rightarrow \underset{13}{\oplus_{H}} \mathcal{O}_{H} \tag{15}
\end{equation*}
$$

where $H$ ranges over the irreducible components of the stratification determined by $\bar{M}_{X / S}$ on the locus where $\operatorname{rank} \bar{M}_{X / S} \geq 1$. Furthermore, the projection

$$
\mathcal{R} \rightarrow \mathcal{O}_{H}
$$

of (15) to each summand $\mathcal{O}_{H}$ is surjective.
Proof. Choose an orientation of $\bar{M}_{X}$ as in Lemma 15. We will construct the map 15) locally on $X$, and then we will prove the gluing compatibility required for the global definition.

Let $x$ be a point of $X$. Choose an ordered basis

$$
m_{1}, \ldots, m_{n} \in \bar{M}_{X, x}
$$

of $\bar{M}_{X}$ at $x$. Let $\widetilde{m}_{i}$ be arbitrary lifts in $M_{X}$, and write $x_{i}=\epsilon\left(\widetilde{m}_{i}\right)$ for their images in $\mathcal{O}_{X}$. In other words, $x_{i}$ are local defining equations for the divisor $D_{i}$ of $X$ at $x$. Similarly, write $t_{i}$ for the corresponding images in $S$ near $f(x)$. Then, without loss of generality, we may assume that the map of characteristic monoids has the form

$$
\mathbb{N}^{k}=\bar{M}_{S, f(x)} \rightarrow \bar{M}_{X, x}=\mathbb{N}^{n_{1}} \oplus \mathbb{N}^{n_{2}} \oplus \cdots \oplus \mathbb{N}^{n_{k}} \oplus \mathbb{N}^{\ell}
$$

with the $j^{\text {th }}$ basis element of $\mathbb{N}^{k}$ mapping to the vector $(\underbrace{1, \ldots, 1}_{n_{j}})$ of the summand $\mathbb{N}^{n_{j}}$ on the right. We have equations

$$
t_{1}=u_{1} \prod_{\alpha \in A_{1}} x_{\alpha}, \ldots, t_{k}=u_{k} \prod_{\alpha \in A_{k}} x_{\alpha}
$$

for disjoint sets $A_{1}, \ldots, A_{k} \subset\{1,2, \ldots, n\}$ with $n_{1}, \ldots, n_{k}$ elements respectively and units $u_{i} \in$ $\mathcal{O}_{X, x}$. By the orientation assumption, the sets $A_{1}, \ldots, A_{k}$ are ordered. For convenience, we write

$$
A=A_{1} \cup \ldots \cup A_{k} .
$$

The additional $\ell$ parameters $y_{1}, \cdots, y_{\ell}$ have vanishing loci $V\left(y_{i}\right)$ representing horizontal divisor $[9$ over $S$.

The logarithmic differentials $\Omega_{X, x}^{\log }$ are generated by $\Omega_{X, x}$ and $\frac{d x_{\alpha}}{x_{\alpha}}, \alpha \in A$, and $\frac{d y_{1}}{y_{1}}, \ldots, \frac{d y \ell}{y_{\ell}}$. We have the relations

$$
\frac{d u_{i}}{u_{i}}+\sum_{\alpha \in A_{i}} \frac{d x_{\alpha}}{x_{\alpha}}=\frac{d t_{i}}{t_{i}}, \quad 1 \leq i \leq k
$$

The quotient $\mathcal{R}=\Omega_{X / S}^{\log } / \Omega_{X / S}$ has a presentation as an $\mathcal{O}_{X}$-module with generators

$$
\begin{equation*}
\frac{d x_{\alpha}}{x_{\alpha}}, \frac{d y_{1}}{y_{1}}, \ldots, \frac{d y_{\ell}}{y_{\ell}}, \tag{16}
\end{equation*}
$$

where $\alpha \in A$. The relations are

$$
\sum_{\alpha \in A_{i}} \frac{d x_{\alpha}}{x_{\alpha}}=0, \quad 1 \leq i \leq k
$$

[^5](since we are working with relative differentials and the $d u_{i} / u_{i}$ are in $\Omega_{X / S}$ ) and additionally
$$
x_{\alpha} \frac{d x_{\alpha}}{x_{\alpha}}=0, y_{1} \frac{d y_{1}}{y_{1}}=0, \ldots, y_{\ell} \frac{d y_{\ell}}{y_{\ell}}=0
$$
where $\alpha \in A$. The irreducible components of the stratification of $M_{X / S}$ at $x$ (with rk $\geq 1$ ) are given either by

- $t_{i}=0, x_{\beta}=0, x_{\gamma}=0$ for triples $(i, \beta, \gamma)$ with $\beta<\gamma$ elements in $A_{i}$, or by
- $y_{j}=0$ for some $1 \leq j \leq \ell$.

Thus, we find ${ }^{10}$

$$
\oplus_{H} \mathcal{O}_{H}=\bigoplus_{(i, \beta, \gamma)} \mathcal{O}_{X} /\left(t_{i}, x_{\beta}, x_{\gamma}\right) \oplus \bigoplus_{j} \mathcal{O}_{X} /\left(y_{j}\right)
$$

We define a map

$$
\mathcal{R} \rightarrow \mathcal{O}_{X} /\left(t_{i}, x_{\beta}, x_{\gamma}\right)
$$

by sending all the generators in (16) to 0 , with the exception of

$$
\frac{d x_{\beta}}{x_{\beta}} \mapsto 1, \quad \frac{d x_{\gamma}}{x_{\gamma}} \mapsto-1 .
$$

Similarly, we define a map

$$
\mathcal{R} \rightarrow \mathcal{O}_{X} /\left(y_{j}\right)
$$

by sending all generators in (16) to 0 , with the exception of

$$
\frac{d y_{j}}{y_{j}} \mapsto 1 .
$$

We must verify that the map is well-defined. First, for each $(i, \beta, \gamma)$, we see that

$$
\sum_{\alpha \in A_{i}} \frac{d x_{\alpha}}{x_{\alpha}} \mapsto 0
$$

since $\frac{d x_{\beta}}{x_{\beta}}$ and $\frac{d x_{\gamma}}{x_{\gamma}}$ map to opposite elements in $\mathcal{O}_{X} /\left(t_{i}, x_{\beta}, x_{\gamma}\right)$, and the other terms map to 0 . The fact that

$$
x_{\alpha} \frac{d x_{\alpha}}{x_{\alpha}}, y_{1} \frac{d y_{1}}{y_{1}}, \ldots, y_{\ell} \frac{d y_{\ell}}{y_{\ell}}, \alpha \in A
$$

map to 0 in $\mathcal{O}_{X} /\left(t_{i}, x_{\beta}, x_{\gamma}\right)$ and $\mathcal{O}_{X} /\left(y_{j}\right)$ is immediate from the definitions. Surjectivity of the map to any summand $\mathcal{O}_{H}$ is also clear, as the generator 1 is in the image.

We now inspect how our map depended on choices; the only choices involved were the lifts $\widetilde{m}_{i}$ of $m_{i}$, and the choice of ordering of the $m_{i}$. A different choice of $\widetilde{m}_{i}^{\prime}$ differs from the original one by a unit, and we have

$$
\frac{d(u x)}{u x}=u^{-1} d u+\frac{d x}{x} .
$$

The term $u^{-1} d u$ is an ordinary differential, and thus the residue of the logarithmic form is independent of lift. On the other hand, the map does depend on the ordering of the coordinates. Since we assume that the ordering is global, however, the local maps patch uniquely to all of $X$.

[^6]Remark 17. In case $S$ is a point, $(X, D)$ is a simplest normal crossings pair, and only the horizontal divisors $H=\mathcal{O}_{X} /\left(y_{i}\right)$ are present in our analysis. These are precisely the components $D_{i}$ of the divisor $D$. Our residue map then reduces to the classical residue homomorphism

$$
0 \longrightarrow \Omega_{X} \longrightarrow \Omega_{X}^{\log } \longrightarrow \oplus_{i} \mathcal{O}_{D_{i}} \longrightarrow 0,
$$

see F2, Chapter 4, Proposition 1].
2.4. The Hodge bundle. Let $\overline{\mathcal{A}}_{g}$ be a toroidal compactification of the moduli space of principally polarized abelian varieties. The compactification $\overline{\mathcal{A}}_{g}$ carries a universal family of semi-abelian schemes

$$
q: \mathcal{U}_{g} \rightarrow \overline{\mathcal{A}}_{g}
$$

together with a zero section $s: \overline{\mathcal{A}}_{g} \rightarrow \mathcal{U}_{g}$. The Hodge bundle is the rank $g$ vector bundle on $\overline{\mathcal{A}}_{g}$ defined by

$$
\mathbb{E}=s^{*} \Omega_{q},
$$

with Chern classes $\lambda_{i}=c_{i}(\mathbb{E})$.
Definition 18. A compactification of $q: \mathcal{U}_{g} \rightarrow \overline{\mathcal{A}}_{g}$ is a diagram

where $p$ is a proper $\log$ smooth morphism, $\mathcal{U}_{g}$ is open and dense in $\mathcal{X}_{g}$, and $\mathcal{U}_{g}$ acts on $\mathcal{X}_{g}$ extending the natural action of $\mathcal{U}_{g}$ on itself (and commuting with $p$ ). A compactification $p$ is a compactified universal family if in addition $p$ is a $\log$ family.

The fiber $\left(\mathcal{X}_{g}\right)_{t}$ of a compactified universal family $p$ over a point $t \in \overline{\mathcal{A}}_{g}$ contains the semiabelian scheme $\left(\mathcal{U}_{g}\right)_{t}$ as an open subscheme. More precisely, write

$$
\begin{equation*}
0 \longrightarrow T_{t} \longrightarrow\left(\mathcal{U}_{g}\right)_{t} \longrightarrow A_{t} \longrightarrow 0 \tag{17}
\end{equation*}
$$

with $T_{t}$ a torus and $A_{t}$ an abelian scheme. Then $\left(\mathcal{X}_{g}\right)_{t}$ admits a fibration

$$
\begin{equation*}
X\left(T_{t}\right) \longrightarrow\left(\mathcal{X}_{g}\right)_{t} \longrightarrow A_{t} \tag{18}
\end{equation*}
$$

where $X\left(T_{t}\right)$ is a union of complete toric varieties with torus $T_{t}$.
An arbitrary toroidal compactification $\overline{\mathcal{A}}_{g}$ may not carry a compactified universal family. However, toroidal compactifications $\overline{\mathcal{A}}_{g}$, with compactified universal families

$$
p: \mathcal{X}_{g} \rightarrow \overline{\mathcal{A}}_{g}
$$

can be constructed, see FC, Chapter VI, Section 1]. Compactifications of $q$ correspond to $\mathrm{GL}_{g} \ltimes N$ admissible decompositions $\widetilde{\Sigma}_{g}$ of a certain subcone of $\operatorname{Sym}_{r c}^{2}\left(\mathbb{R}^{g}\right) \times \operatorname{Hom}(N, \mathbb{R})$ for a rank $g$ lattice $N$. The decomposition is required to have the property that every cone in $\widetilde{\Sigma}_{g}$ maps into a cone of
the admissible decomposition $\Sigma_{g}$ of $\operatorname{Sym}_{r c}^{2}\left(\mathbb{R}^{g}\right)$ defining $\overline{\mathcal{A}}_{g}$. A compactification $p$ is a compactified universal family if the map

$$
\widetilde{\Sigma}_{g} \rightarrow \Sigma_{g}
$$

satisfies the additional hypotheses of Definition 8 (the cones of $\widetilde{\Sigma}_{g}$ map onto cones of $\Sigma_{g}$, and surjectivity also holds for their integral structure).

Both notions of compactification are stable under arbitrary base change $\overline{\mathcal{A}}_{g}^{\prime} \rightarrow \overline{\mathcal{A}}_{g}$. For a compactification $p: \mathcal{X}_{g} \rightarrow \overline{\mathcal{A}}_{g}$, an arbitrary log alteration $\mathcal{X}_{g}^{\prime} \rightarrow \mathcal{X}_{g}$ of the domain of $p$ remains a compactification. On the other hand, log alterations of $\mathcal{X}_{g}$ are not compactified families, even if the original $p$ is a family, as the composed map $\mathcal{X}_{g}^{\prime} \rightarrow \overline{\mathcal{A}}_{g}$ is rarely a $\log$ family (flatness and reducedness of fibers is typically destroyed). Nevertheless, semistable reduction by Theorem 11 ensures that there is a $\log$ alteration of the $\operatorname{map} \mathcal{X}_{g}^{\prime} \rightarrow \overline{\mathcal{A}}_{g}$ which is a compactified family.

The sheaf of relative logarithmic differentials of $q$ and $p$ are fiberwise trivial of rank $g$ (FC, Chapter VI, Theorem 1.1]. Triviality follows from the fibration descriptions (17) and 18) since the sheaf of differentials on a semi-abelian variety and the sheaf of logarithmic differentials on a toric variety are both trivial.

When $\overline{\mathcal{A}}_{g}$ has a compactified family $p$, we can use $\Omega_{p}^{\log }$ to define the Hodge bundle, as

$$
\mathbb{E}=s^{*} \Omega_{q}=s^{*} \Omega_{q}^{\log }=\left.s^{*} \Omega_{p}^{\log }\right|_{\mathcal{U}_{g}}=s^{*} \Omega_{p}^{\log }
$$

since the section $s$ factors through $\mathcal{U}_{g}$ and the map $q$ is strict.
A second approach to the Hodge bundle is available. The following result can be found in FC, Chapter VI, Theorem 1.1].

Lemma 19. Suppose $\overline{\mathcal{A}}_{g}$ carries a compactified universal family $p: \mathcal{X}_{g} \rightarrow \overline{\mathcal{A}}_{g}$. Then,

$$
\Omega_{p}^{\log }=p^{*} \mathbb{E} \quad \text { and } \quad \mathbb{E}=p_{*} \Omega_{p}^{\log }
$$

Proof. Since $\Omega_{p}^{\log }$ is fiberwise trivial, cohomology and base change implies that

$$
\Omega_{p}^{\log }=p^{*} p_{*} \Omega_{p}^{\log }
$$

Since $s^{*} p^{*}=\mathrm{id}$, we have

$$
\mathbb{E}=s^{*} \Omega_{p}^{\log }=s^{*} p^{*} p_{*} \Omega_{p}^{\log }=p_{*} \Omega_{p}^{\log },
$$

and the result follows.

Lemma 20. The classes $\lambda_{i}$ extend to $\log \mathrm{CH}^{*}\left(\mathcal{A}_{g}\right)$.
Proof. In light of Remark 6, we check compatibility of Hodge classes under logarithmic alterations. Suppose $\tau: \overline{\mathcal{A}}_{g}^{\prime} \rightarrow \overline{\mathcal{A}}_{g}$ is a logarithmic alteration. Then we have a Cartesian diagram


Since $s \circ \tau=\rho \circ s^{\prime}$ for the respective zero sections, we have

$$
\tau^{*} \mathbb{E}=\left(s^{\prime}\right)^{*} \rho^{*} \Omega_{q}=\left(s^{\prime}\right)^{*} \Omega_{q^{\prime}}=\mathbb{E}^{\prime}
$$

which implies the required compatibility for $\lambda_{i}$.
Remark 21. Toroidal compactifications of $\mathcal{A}_{g}$ with a compactified universal family

$$
p: \mathcal{X}_{g} \rightarrow \overline{\mathcal{A}}_{g}
$$

form a cofinal system among all toroidal compactifications: given an arbitrary toroidal compactification $\overline{\mathcal{A}}_{g}^{\prime}$, we can choose a toroidal compactification $\overline{\mathcal{A}}_{g}$ with a compactified universal family, and then any common refinement $\overline{\mathcal{A}}_{g}^{\prime \prime}$ of both compactifications carries a compactified universal family.
Lemma 22. The collection of simplest normal crossings compactifications $\mathcal{A}_{g} \subset \overline{\mathcal{A}}_{g}$ that carry a compactified universal family with simplest normal crossings is cofinal among the toroidal compactifications $\overline{\mathcal{A}}_{g}$.

Proof. Starting with an arbitrary compactified universal family

$$
p: \mathcal{X}_{g} \rightarrow \overline{\mathcal{A}}_{g}
$$

we may apply Corollary 14 to $p$, to obtain the desired family.
2.5. Proof of Theorem 3. Let $\overline{\mathcal{A}}_{g}$ be a toroidal compactification of $\mathcal{A}_{g}$. By Lemma 22, there exists a toroidal compactification $\overline{\mathcal{A}}_{g}^{\prime}$ satisfying the following conditions:
(i) there is a proper surjection $\overline{\mathcal{A}}_{g}^{\prime} \rightarrow \overline{\mathcal{A}}_{g}$ that is an isomorphism on the interior,
(ii) $\overline{\mathcal{A}}_{g}^{\prime}$ admits a universal family $\mathcal{X}_{g}^{\prime} \rightarrow \overline{\mathcal{A}}_{g}^{\prime}$ of toroidal compactifications of semi-abelian schemes,
(iii) both $\overline{\mathcal{A}}_{g}^{\prime}$ and $\mathcal{X}_{g}^{\prime}$ have simplest normal crossings.

Because proper surjections are Chow envelopes, it suffices to show that

$$
\left.\lambda_{g}\right|_{\partial \overline{\mathcal{A}}_{g}^{\prime}}=0 .
$$

Hence, after replacing $\overline{\mathcal{A}}_{g}$ by $\overline{\mathcal{A}}_{g}^{\prime}$, we may assume that $\overline{\mathcal{A}}_{g}$ has properties (ii) and (iii) above.
Let $T$ be a component of the boundary divisor of $\overline{\mathcal{A}}_{g}$ and denote by $p_{T}: \mathcal{X}_{T} \rightarrow T$ the base change to $T$ of the universal family

$$
p: \mathcal{X}_{g} \rightarrow \overline{\mathcal{A}}_{g} .
$$

Over $T$, either $\mathcal{X}_{T} \rightarrow T$ is smooth, or we can find a nonempty component $H$ of the singular locus $\mathcal{X}_{T}^{\text {sing }}$ of $p_{T}$. In the first case, $\left.\lambda_{g}\right|_{T}=0$ because $\left.\lambda_{g}\right|_{\mathcal{A}_{g}}=0$.

In the second case, $H$ is an irreducible component of the locus where rank $\bar{M}_{\mathcal{X}_{g} / \overline{\mathcal{A}}_{g}} \geq 1$. Let

$$
i: H \rightarrow \mathcal{X}_{T}, \quad p_{T} \circ i: H \rightarrow T
$$

be the inclusion and the projection. The map $p_{T} \circ i$ is proper and surjective because $p$ is a log family, so it suffices to show that $i^{*} p_{T}^{*}\left(\left.\lambda_{g}\right|_{T}\right)=0$. Using Lemma 19, we have $\Omega_{p}^{\log }=p^{*} \mathbb{E}$. After base change and pullback by $i$, we find

$$
\left.i^{*} p_{T}^{*} \mathbb{E}\right|_{T}=i^{*} \Omega_{p_{T}}^{\log } .
$$

It remains to check that $c_{g}\left(i^{*} \Omega_{p_{T}}^{\log }\right)=0$. By Definition 7 and Theorem 16 , we have surjections

$$
i^{*} \Omega_{p_{T}}^{\log } \rightarrow i^{*} \mathcal{R} \rightarrow 0, \quad i^{*} \mathcal{R} \rightarrow \mathcal{O}_{H} \rightarrow 0
$$

We therefore have an exact sequence of vector bundles on $H$,

$$
0 \longrightarrow K \longrightarrow i^{*} \Omega_{p_{T}}^{\log } \longrightarrow \mathcal{O}_{H} \longrightarrow 0
$$

We conclude $c_{g}\left(i^{*} \Omega_{p_{T}}^{\log }\right)=c_{g-1}(K) c_{1}\left(\mathcal{O}_{H}\right)=0$.
2.6. Log geometry and $\lambda_{g}$. There is a distinguished subalgebra of classes coming from the boundary in the logarithmic Chow theory defined by the image of the algebra PP of $\mathrm{GL}_{g}$-invariant piecewise polynomials ${ }^{11}$ on $\operatorname{Sym}_{\mathrm{rc}}^{2}\left(\mathbb{R}^{g}\right)$,

$$
\begin{equation*}
\mathrm{PP} \rightarrow \log \mathrm{CH}^{*}\left(\mathcal{A}_{g}\right) \tag{19}
\end{equation*}
$$

We refer the reader to MPS, MR for further details regarding the construction of the map 19). Our main conjecture concerning $\lambda_{g}$ in the logarithmic theory is the following claim.

Conjecture 23. The class $\lambda_{g} \in \log \mathrm{CH}^{*}\left(\mathcal{A}_{g}\right)$ lies in the image 19 of the algebra of piecewise polynomials.

Our motivation for Conjecture 23 comes from a parallel study of $\lambda_{g}$ in the logarithmic Chow theory of the moduli space of curves $\mathcal{M}_{g}$. Using Pixton's formula HMPPS,JPPZ, the class $\lambda_{g}$ is proven in MPS to lie in the image of the algebra of piecewise polynomials in $\log \mathrm{CH}^{*}\left(\mathcal{M}_{g}\right)$.

Question D. Find a lifting to $\log \mathrm{CH}^{*}\left(\mathcal{A}_{g}\right)$ of Pixton's formula for $\lambda_{g} \in \log \mathrm{CH}^{*}\left(\mathcal{M}_{g}\right)$ which is compatible with the Torelli map.

The definition by van der Geer of the tautological ring is best suited for studying classes on the moduli space of abelian varieties $\mathcal{A}_{g}$. There is a larger tautological ring which takes the boundaries of the various compactifications into account,

$$
\log \mathrm{R}^{*}\left(\mathcal{A}_{g}\right) \subset \log \mathrm{CH}^{*}\left(\mathcal{A}_{g}\right)
$$

defined to be generated by all possible piecewise polynomial and Hodge classes on all boundary strata of all toroidal compactifications $\mathcal{A}_{g} \subset \overline{\mathcal{A}}_{g}$.

The investigation of the structure of $\log \mathrm{R}^{*}\left(\mathcal{A}_{g}\right)$ is an interesting future direction. For example, pushing forward powers of the polarization, we can define $\kappa$ classes over $\overline{\mathcal{A}}_{g}$, see MOP and A2 for similar constructions in the context of the moduli of K3 surfaces and over KSBA moduli respectively. In the case of abelian varieties, we expect ${ }^{12}$ that the $\kappa$ classes lie in $\log \mathrm{R}^{*}\left(\mathcal{A}_{g}\right)$. Can a precise formula be found?

[^7]
## 3. Tautological projection of product classes

3.1. Product cycles. We compute here the tautological projections of all product cycles

$$
\mathcal{A}_{g_{1}} \times \ldots \times \mathcal{A}_{g_{\ell}} \rightarrow \mathcal{A}_{g}
$$

for all $g$. Calculations for product cycles for genus $g \leq 5$ can be found in GH.
Fix toroidal compactifications $\overline{\mathcal{A}}_{g}$ corresponding to an additive collection of fans. The product maps

$$
\prod_{g_{1}+\cdots+g_{\ell}=g} \mathcal{A}_{g_{1}} \times \cdots \times \mathcal{A}_{g_{\ell}} \rightarrow \mathcal{A}_{g}
$$

then extend to maps

$$
\begin{equation*}
\prod_{g_{1}+\cdots+g_{\ell}=g} \overline{\mathcal{A}}_{g_{1}} \times \cdots \times \overline{\mathcal{A}}_{g_{\ell}} \rightarrow \overline{\mathcal{A}}_{g} . \tag{20}
\end{equation*}
$$

For example, we could take the perfect cone compactification for every $g$ by [SB, Lemma 2.8].
The Hodge bundle splits canonically over the product (20). Indeed, the universal semiabelian variety restricts in the natural fashion over the product, and the splitting of the Hodge bundle then follows by restricting the relative cotangent bundle to the zero section.
3.2. Lagrangian Grassmannian. As remarked in vdG, a consequence of Theorem 1 is the $\mathbb{Q}$ algebra isomorphism

$$
\mathrm{R}^{*}\left(\overline{\mathcal{A}}_{g}\right) \simeq \mathrm{CH}^{*}\left(\mathrm{LG}_{g}\right)
$$

where $\mathrm{LG}_{g}$ denotes the Lagrangian Grassmannian of $\mathbb{C}^{2 g}$ with respect to a symplectic form $\omega$. An overview of the cohomology of the Lagrangian Grassmannian from the point of Schubert calculus can be found, for instance, in $\mathrm{FPr}, \mathrm{KT}, \mathrm{PR}$.

The spaces $\overline{\mathcal{A}}_{g}$ and $\mathrm{LG}_{g}$ are further connected by the Hirzebruch-Mumford proportionality principle. Let $\mathrm{S} \rightarrow \mathrm{LG}_{g}$ be the universal rank $g$ subbundle, and let $x_{i}=c_{i}\left(\mathrm{~S}^{*}\right)$. Then,

$$
\begin{equation*}
\int_{\overline{\mathcal{A}}_{g}} \lambda_{I}=\gamma_{g} \int_{\mathrm{LG}_{g}} x_{I} \tag{21}
\end{equation*}
$$

for every $I \subset\{1,2, \ldots, g\}$. Here, we use the multindex notation

$$
\lambda_{I}=\prod_{i \in I} \lambda_{i}, \quad x_{I}=\prod_{i \in I} x_{i} .
$$

The proportionality constant $\gamma_{g}$ was computed in vdG, page 9]:

$$
\gamma_{g}=\int_{\overline{\mathcal{A}}_{g}} \lambda_{1} \ldots \lambda_{g}=\prod_{i=1}^{g} \frac{\left|B_{2 i}\right|}{4 i} .
$$

For any partition

$$
g_{1}+\ldots+g_{\ell}=g
$$

we can consider the product

$$
\begin{equation*}
\mathrm{LG}_{g_{1}} \times \ldots \times \mathrm{LG}_{g_{\ell}} \rightarrow \mathrm{LG}_{g} \tag{22}
\end{equation*}
$$

Finding the class of

$$
\overline{\mathcal{A}}_{g_{1}} \times \ldots \times \overline{\mathcal{A}}_{g_{\ell}} \rightarrow \overline{\mathcal{A}}_{g}
$$

is equivalent to finding the class of the product cycle (22) in $\mathrm{CH}^{*}\left(\mathrm{LG}_{g}\right)$ in terms of the Chern classes $x_{i}=c_{i}\left(\mathrm{~S}^{*}\right)$ of the dual subbundle. More precisely, if

$$
\begin{equation*}
\left[\mathrm{LG}_{g_{1}} \times \ldots \times \mathrm{LG}_{g \ell}\right]=\mathrm{P}\left(x_{1}, \ldots, x_{g}\right) \in \mathrm{CH}^{*}\left(\mathrm{LG}_{g}\right) \tag{23}
\end{equation*}
$$

then we have

$$
\begin{equation*}
\operatorname{taut}^{\mathrm{cpt}}\left(\left[\overline{\mathcal{A}}_{g_{1}}^{\Sigma_{g_{1}}} \times \ldots \times \overline{\mathcal{A}}_{g_{\ell}}^{\Sigma_{g_{\ell}}}\right]\right)=\frac{\gamma_{g_{1}} \cdots \gamma_{g_{\ell}}}{\gamma_{g}} \cdot \mathrm{P}\left(\lambda_{1}, \ldots, \lambda_{g}\right) \in \mathrm{R}^{*}\left(\overline{\mathcal{A}}_{g}^{\Sigma_{g}}\right) . \tag{24}
\end{equation*}
$$

To derive (24) from (23), we use the Gorenstein property of $\mathrm{R}^{*}\left(\overline{\mathcal{A}}_{g}^{\Sigma_{g}}\right)$. We need only check that polynomials of complementary degrees in the $\lambda$ classes pair equally with both sides of 24 :
(i) When restricted to the product loci in $\overline{\mathcal{A}}_{g}^{\Sigma_{g}}$ and $\mathrm{LG}_{g}$, both the Hodge bundle $\mathbb{E}$ and the dual subbundle S* split as direct sums.
(ii) By the Hirzebruch-Mumford proportionality principle, integrals in the $\lambda$ 's over $\overline{\mathcal{A}}_{g}^{\Sigma_{g}}$ can be evaluated in terms of integrals in $x$ 's over $\mathrm{LG}_{g}$. The answers are always proportional (21), with proportionality constant $\gamma_{g}$.
Combining (i) and (ii), we see that the constant $\gamma_{g_{1}} \cdots \gamma_{g_{\ell}}$ arises for all factors on the left hand side, while the constant $\gamma_{g}$ arises for all terms on the right hand side, showing that (23) implies (24). The purely non-tautological part of the cycle in (24) plays no role in the argument.
3.3. Proof of Theorem 5. For $g_{1}+\ldots+g_{\ell}=g$, we must show that the tautological projection of the product locus $\overline{\mathcal{A}}_{g_{1}} \times \ldots \times \overline{\mathcal{A}}_{g_{\ell}}$ in $\overline{\mathcal{A}}_{g}$ is given by the $g \times g$ determinant

$$
\operatorname{taut}^{\mathrm{cpt}}\left(\left[\overline{\mathcal{A}}_{g_{1}} \times \ldots \times \overline{\mathcal{A}}_{g_{\ell}}\right]\right)=\frac{\gamma_{g_{1}} \ldots \gamma_{g_{\ell}}}{\gamma_{g}}\left|\begin{array}{cccc}
\lambda_{\alpha_{1}} & \lambda_{\alpha_{1}+1} & \ldots & \lambda_{\alpha_{1}+g-1} \\
\lambda_{\alpha_{2}-1} & \lambda_{\alpha_{2}} & \ldots & \lambda_{\alpha_{2}+g-2} \\
\ldots & \ldots & \ldots & \ldots \\
\lambda_{\alpha_{g}-g+1} & \lambda_{\alpha_{g}-g+2} & \ldots & \lambda_{\alpha_{g}}
\end{array}\right|
$$

for the vector

$$
\alpha=(\underbrace{g-g_{1}, \ldots, g-g_{1}}_{g_{1}}, \underbrace{g-g_{1}-g_{2}, \ldots, g-g_{1}-g_{2}}_{g_{2}}, \ldots, \underbrace{g-g_{1}-\ldots-g_{\ell}, \ldots, g-g_{1}-\ldots-g_{\ell}}_{g_{\ell}}) .
$$

By the connection between product cycles on $\overline{\mathcal{A}}_{g}$ and $\mathrm{LG}_{g}$ proven in Section 3.2, it suffices to show that the class of the product $\mathrm{LG}_{g_{1}} \times \ldots \times \mathrm{LG}_{g_{\ell}}$ in $\mathrm{LG}_{g}$ is given by the determinant

$$
\left|\begin{array}{cccc}
x_{\alpha_{1}} & x_{\alpha_{1}+1} & \ldots & x_{\alpha_{1}+g-1} \\
x_{\alpha_{2}-1} & x_{\alpha_{2}} & \ldots & x_{\alpha_{2}+g-2} \\
\ldots & \ldots & \ldots & \ldots \\
x_{\alpha_{g}-g+1} & x_{\alpha_{g}-g+2} & \ldots & x_{\alpha_{g}}
\end{array}\right|
$$

We will prove this determinantal formula using the geometry of $\mathrm{LG}_{g}$.
Let $V=\mathbb{C}^{2 g}$ with symplectic form $\omega$. We consider a splitting

$$
\begin{equation*}
(V, \omega) \simeq\left(V_{1}, \omega_{1}\right) \oplus \ldots \oplus\left(V_{\ell}, \omega_{\ell}\right) \tag{25}
\end{equation*}
$$

where $V_{1}, \ldots, V_{\ell}$ are symplectic subspaces of $V$ with $\operatorname{dim} V_{i}=2 g_{i}$. The splitting (25) defines an embedding

$$
j: \mathrm{LG}_{g_{1}} \times \ldots \times \mathrm{LG}_{g_{\ell}} \rightarrow \mathrm{LG}_{g}, \quad\left(P_{1}, \ldots, P_{\ell}\right) \mapsto P=P_{1} \oplus \ldots \oplus P_{\ell}
$$

Consider the embedding of $\mathbf{L G}_{g}$ into the usual Grassmannian $\mathbf{G}=\mathbf{G}(g, 2 g)$ :

$$
\iota: \mathrm{LG}_{g} \rightarrow \mathrm{G} .
$$

Let $S \rightarrow G$ be the universal subbundle (which restricts to the universal subbundle $S \rightarrow L_{g}$ via the embedding $\iota$ ). Similarly, let $x_{i}$ be the Chern classes of $\mathrm{S}^{*}$ on G (which restrict to the classes $x_{i}$ on $\mathrm{LG}_{g}$ ). Let $\Sigma$ be the Schubert cycle in the Grassmannian $G$ associated to the partition $\alpha$ with respect to any complete flag satisfying the property

$$
F_{2\left(g_{1}+\ldots+g_{i}\right)}=V_{1} \oplus \ldots \oplus V_{i}, \quad 1 \leq i \leq \ell
$$

By definition, $P \in \Sigma$ provided

$$
\operatorname{dim}\left(P \cap F_{g+j-\alpha_{j}}\right) \geq j, \quad 1 \leq j \leq g .
$$

For $1 \leq i \leq \ell$, let $j=g_{1}+\ldots+g_{i}$, so that $\alpha_{j}=g-\left(g_{1}+\ldots+g_{i}\right)$. We see that for $P \in \Sigma$ we have

$$
\begin{equation*}
\operatorname{dim}\left(P \cap F_{2\left(g_{1}+\ldots+g_{i}\right)}\right) \geq g_{1}+\ldots+g_{i}, \quad 1 \leq i \leq \ell \tag{26}
\end{equation*}
$$

The converse is also true. While there are additional requirements about dimensions of intersections with other members of the flag, these are automatically fulfilled by elementary linear algebra considerations.

In $\mathrm{CH}^{*}(\mathrm{G})$, we have the standard expression [F1, Chapter 14]:

$$
[\Sigma]=\left|\begin{array}{cccc}
x_{\alpha_{1}} & x_{\alpha_{1}+1} & \ldots & x_{\alpha_{1}+g-1} \\
x_{\alpha_{2}-1} & x_{\alpha_{2}} & \ldots & x_{\alpha_{2}+g-2} \\
\ldots & \ldots & \ldots & \ldots \\
x_{\alpha_{g}-g+1} & x_{\alpha_{g}-g+2} & \ldots & x_{\alpha_{g}}
\end{array}\right|
$$

Moreover, we have

$$
\operatorname{codim}(\Sigma, \mathrm{G})=|\alpha|=\sum_{i=1}^{\ell}\left(g-g_{1}-\ldots-g_{i}\right) g_{i}=\sum_{i>j} g_{i} g_{j}
$$

which agrees with

$$
\operatorname{codim}\left(\mathrm{LG}_{1} \times \ldots \times \mathrm{LG}_{g_{\ell}}, \mathrm{LG}_{g}\right)=\binom{g+1}{2}-\sum_{i=1}^{\ell}\binom{g_{i}+1}{2}=\sum_{i>j} g_{i} g_{j} .
$$

The scheme-theoretic claim

$$
\begin{equation*}
\mathrm{LG}_{g_{1}} \times \ldots \times \mathrm{LG}_{g_{\ell}}=i^{-1} \Sigma=\Sigma \cap \mathrm{LG}_{g} \tag{27}
\end{equation*}
$$

then implies

$$
\left[\mathrm{LG}_{g_{1}} \times \ldots \times \mathrm{LG}_{g_{\ell}}\right]=\iota^{*}[\Sigma]=\left|\begin{array}{cccc}
x_{\alpha_{1}} & x_{\alpha_{1}+1} & \ldots & x_{\alpha_{1}+g-1}  \tag{28}\\
x_{\alpha_{2}-1} & x_{\alpha_{2}} & \ldots & x_{\alpha_{2}+g-2} \\
\ldots & \ldots & \ldots & \ldots \\
x_{\alpha_{g}-g+1} & x_{\alpha_{g}-g+2} & \ldots & x_{\alpha_{g}}
\end{array}\right|
$$

as required.
We first establish (27) set-theoretically. The left to right containment is clear for split subspaces $P=P_{1} \oplus \ldots \oplus P_{\ell}$, so we show the converse. Let $P \in \Sigma \cap \mathrm{LG}_{g}$. For convenience, write

$$
h_{i}=g_{1}+\ldots+g_{i} .
$$

We set $P_{i}=P \cap V_{i}$. Note that $P \cap F_{2 h_{i}}$ is isotropic in $F_{2 h_{i}}$, hence $\operatorname{dim}\left(P \cap F_{2 h_{i}}\right) \leq h_{i}$. By the Schubert condition (26), we must have

$$
\begin{equation*}
\operatorname{dim}\left(P \cap F_{2 h_{i}}\right)=h_{i} . \tag{29}
\end{equation*}
$$

We will prove that $\operatorname{dim} P_{i}=g_{i}$ for all $1 \leq i \leq \ell$.
The case $i=1$ is clear by (29) since $V_{1}=F_{2 h_{1}}$. For the general case, we induct on $i$. We assume that

$$
\operatorname{dim}\left(P \cap V_{1}\right)=g_{1}, \ldots, \operatorname{dim}\left(P \cap V_{i}\right)=g_{i},
$$

and show that

$$
\operatorname{dim}\left(P \cap V_{i+1}\right)=g_{i+1}
$$

To this end, let $Q=P \cap F_{2 h_{i+1}}$, so that

$$
\operatorname{dim} Q=h_{i+1}, \quad \operatorname{dim}\left(Q \cap F_{2 h_{i}}\right)=h_{i}
$$

by 29). Furthermore, $Q$ is isotropic hence Lagrangian in $\left(F_{2 h_{i+1}}, \eta\right)$ where $\eta$ is the restriction of the symplectic form $\omega$. To show

$$
\operatorname{dim}\left(P \cap V_{i+1}\right)=\operatorname{dim}\left(Q \cap V_{i+1}\right)=g_{i+1},
$$

we compute

$$
\operatorname{dim}\left(Q \cap V_{i+1}\right)=\operatorname{dim} Q+\operatorname{dim} V_{i+1}-\operatorname{dim}\left(Q+V_{i+1}\right)=h_{i+1}+2 g_{i+1}-\operatorname{dim}\left(Q+V_{i+1}\right) .
$$

It suffices then to show that $\operatorname{dim}\left(Q+V_{i+1}\right)=h_{i+1}+g_{i+1}$, or equivalently,

$$
\begin{equation*}
\operatorname{dim}\left(Q+V_{i+1}\right)^{\eta}=2 h_{i+1}-\left(h_{i+1}+g_{i+1}\right)=h_{i} . \tag{30}
\end{equation*}
$$

Here, the complement is taken in $F_{2 h_{i+1}}$. Since $Q$ is Lagrangian, $Q^{\eta}=Q$. By construction, $V_{i+1}^{\eta}=F_{2 h_{i}}$. We can therefore rewrite (30) as

$$
\operatorname{dim}\left(Q \cap F_{2 h_{i}}\right)=h_{i},
$$

which is correct by the Schubert condition (29). The inductive step is proven.
Since $P_{1} \oplus \ldots \oplus P_{\ell} \subset P$, equality must hold for dimension reasons. Therefore, $P \in \mathrm{LG}_{g_{1}} \times \ldots \times \mathrm{LG}_{g_{\ell}}$, and the proof of the set-theoretic equality 27 is complete.

To show (27) holds scheme-theoretically, it suffices to prove that the scheme-theoretic intersection $\Sigma \cap \mathrm{LG}_{g}$ is nonsingular at all points $P \in \Sigma \cap \mathrm{LG}_{g}$. Equivalently, we will show

$$
\begin{equation*}
\operatorname{dim} T_{P}\left(\Sigma \cap \mathbf{L G}_{g}\right)=\operatorname{dim}\left(T_{P} \Sigma \cap T_{P} \mathbf{L G}_{g}\right) \leq \operatorname{dim} \mathrm{LG}_{g_{1}} \times \ldots \mathrm{LG}_{g_{\ell}}=\sum_{i=1}^{\ell}\binom{g_{i}+1}{2} \tag{31}
\end{equation*}
$$

We claim first that all $P \in \Sigma \cap \mathrm{LG}_{g}$ are nonsingular points of the Schubert variety $\Sigma \subset \mathrm{G}$. We use here a result due to LSS, [C, Corollary 2.5]: singular points of $\Sigma$ must lie in Schubert varieties for singular partitions associated to $\alpha$, see [C, Definition 2.1] for the terminology. In our case, nonsingularity at $P \in \Sigma \cap \mathrm{LG}_{g}$ is due to the fact that equality holds in (29). Equality (29) prevents $P$ from satisfying the Schubert conditions for any of the singular partitions associated to $\alpha$.

The tangent space of $\Sigma$ at nonsingular points is computed in EH, Theorem 4.1]: $T_{P} \Sigma$ is identified with a subspace of the space of linear maps

$$
\Phi: P \rightarrow \mathbb{C}^{2 g} / P
$$

satisfying the property

$$
\Phi:\left(P \cap F_{2 h_{i}}\right) \rightarrow\left(P+F_{2 h_{i}}\right) / P .
$$

For tangent space $T_{P} \mathrm{LG} g$, we require

$$
\Phi: P \rightarrow P^{*}
$$

to be symmetric, where we identify $\mathbb{C}^{2 g} / P \simeq P^{*}$ using the symplectic form.
Assume $\Phi \in T_{P} \Sigma \cap T_{P} \mathrm{LG}$. A straighforward check shows that

$$
\left(P+F_{2 h_{i}}\right) / P \simeq F_{2 h_{i}} /\left(P \cap F_{2 h_{i}}\right)
$$

gets identified with $\left(P \cap F_{2 h_{i}}\right)^{*}$, so that

$$
\Phi:\left(P \cap F_{2 h_{i}}\right) \rightarrow\left(P \cap F_{2 h_{i}}\right)^{*} .
$$

We have shown above that

$$
\left(P \cap F_{2 h_{i}}\right)=\left(P \cap V_{1}\right) \oplus \ldots \oplus\left(P \cap V_{i}\right) .
$$

Therefore, $\Phi$ must be symmetric block diagonal with blocks of size $g_{1}, \ldots, g_{\ell}$. Equation (31) then follows.

Example 24. For $g_{1}+g_{2}=g$, the tautological projection of the product locus $\overline{\mathcal{A}}_{g_{1}} \times \overline{\mathcal{A}}_{g_{2}}$ is given by the $g_{1} \times g_{1}$ determinant

$$
\operatorname{taut}^{\mathrm{cpt}}\left(\left[\overline{\mathcal{A}}_{g_{1}} \times \overline{\mathcal{A}}_{g_{2}}\right]\right)=\frac{\gamma_{g_{1}} \gamma_{g_{2}}}{\gamma_{g}}\left|\begin{array}{cccc}
\lambda_{g_{2}} & \lambda_{g_{2}+1} & \ldots & \lambda_{g-1}  \tag{32}\\
\lambda_{g_{2}-1} & \lambda_{g_{2}} & \ldots & \lambda_{g-2} \\
\ldots & \ldots & \ldots & \ldots \\
\lambda_{g_{2}-g_{1}+1} & \lambda_{g_{2}-g_{1}+2} & \ldots & \lambda_{g_{2}}
\end{array}\right| .
$$

The right hand side is the Schur determinant associated to the partition $(\underbrace{g_{2}, \ldots, g_{2}}_{g_{1}})$. The Schur determinant is in general not preserved by exchanging $g_{1}$ and $g_{2}$ (which amounts to transposing the partition), but it is so in the presence of Mumford's relation by precisely [F1, Lemma A.9.2].

- In case $g_{1}=1$, we obtain

$$
\operatorname{taut}^{\mathrm{cpt}}\left(\left[\overline{\mathcal{A}}_{1} \times \overline{\mathcal{A}}_{g-1}\right]\right)=\frac{g}{6\left|B_{2 g}\right|} \lambda_{g-1}
$$

- In case $g=2$, we obtain

$$
\operatorname{taut}^{\mathrm{cpt}}\left(\left[\overline{\mathcal{A}}_{2} \times \overline{\mathcal{A}}_{g-2}\right]\right)=\frac{1}{360} \cdot \frac{g(g-1)}{\left|B_{2 g}\right|\left|B_{2 g-2}\right|} \cdot\left(\lambda_{g-2}^{2}-\lambda_{g-1} \lambda_{g-3}\right) .
$$

Example 25. For $g_{1}=\ldots=g_{k}=1, g_{k+1}=g-k$, Theorem 5 yields

$$
\operatorname{taut}^{\mathrm{cpt}}([\underbrace{\overline{\mathcal{A}}_{1} \times \ldots \times \overline{\mathcal{A}}_{1}}_{k} \times \overline{\mathcal{A}}_{g-k}])=\frac{\gamma_{1}^{k} \gamma_{g-k}}{\gamma_{g}}\left|\begin{array}{ccccc}
\lambda_{g-1} & \lambda_{g} & 0 & \ldots & 0  \tag{33}\\
\lambda_{g-3} & \lambda_{g-2} & \lambda_{g-1} & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\lambda_{g_{2}-2 k+1} & \lambda_{g-2 k+2} & \lambda_{g-2 k+3} & \ldots & \lambda_{g-k}
\end{array}\right| .
$$

For example, we have

$$
\operatorname{taut}^{\mathrm{cpt}}\left(\left[\overline{\mathcal{A}}_{1} \times \overline{\mathcal{A}}_{1} \times \overline{\mathcal{A}}_{g-2}\right]\right)=\frac{1}{36} \cdot \frac{g(g-1)}{\left|B_{2 g}\right|\left|B_{2 g-2}\right|} \cdot\left(\lambda_{g-1} \lambda_{g-2}-\lambda_{g} \lambda_{g-3}\right)
$$

3.4. Proof of Theorem 6. Our goal is to prove that after restriction to $\mathcal{A}_{g}$, the tautological projections of the product cycles admit further factorization:

$$
\operatorname{taut}\left(\left[\mathcal{A}_{g_{1}} \times \cdots \times \mathcal{A}_{g_{\ell}}\right]\right)=\frac{\gamma_{g_{1}} \ldots \gamma_{g_{\ell}}}{\gamma_{g}} \cdot \lambda_{g-1} \cdots \lambda_{g-\ell+1} \cdot\left|\begin{array}{cccc}
\lambda_{\beta_{1}} & \lambda_{\beta_{1}+1} & \ldots & \lambda_{\beta_{1}+g^{*}-1} \\
\lambda_{\beta_{2}-1} & \lambda_{\beta_{2}} & \ldots & \lambda_{\beta_{2}+g^{*}-2} \\
\ldots & \ldots & \ldots & \ldots \\
\lambda_{\beta_{g^{*}}-g^{*}+1} & \lambda_{\beta_{g^{*}}-g^{*}+2} & \ldots & \lambda_{\beta_{g^{*}}}
\end{array}\right|,
$$

for the vector

$$
\beta=(\underbrace{g^{*}-g_{1}^{*}, \ldots, g^{*}-g_{1}^{*}}_{g_{1}^{*}}, \underbrace{g^{*}-g_{1}^{*}-g_{2}^{*}, \ldots, g^{*}-g_{1}^{*}-g_{2}^{*}}_{g_{2}^{*}}, \ldots, \underbrace{g^{*}-g_{1}^{*}-\ldots-g_{\ell}^{*}}_{g_{\ell}^{*}})
$$

where $g^{*}=g-\ell$ and $g_{i}^{*}=g_{i}-1$.
The term $\lambda_{g-1} \cdots \lambda_{g-\ell+1}$ is expected to appear in the formula of Theorem 6 by the following reasoning. First,

$$
\lambda_{g-m} \cdot\left[\mathcal{A}_{g_{1}} \times \ldots \times \mathcal{A}_{g_{\ell}}\right]=0, \quad 1 \leq m \leq \ell-1
$$

Indeed, the splitting of the Hodge bundle distributes a top Hodge class on at least one of the $\ell$ factors $\mathcal{A}_{g_{i}}$, yielding the vanishing by Theorem 1 (iii). Second, we compute the annihilator ideal

$$
\operatorname{Ann}\left\langle\lambda_{g-1}, \ldots, \lambda_{g-\ell+1}\right\rangle=\left\langle\lambda_{g-1} \ldots \lambda_{g-\ell+1}\right\rangle
$$

The right to left containment follows from the relations

$$
\begin{equation*}
\lambda_{j}^{2} \lambda_{j+1} \ldots \lambda_{g-1}=0, \quad 1 \leq j \leq g-1 \tag{34}
\end{equation*}
$$

on $\mathcal{A}_{g}$ noted in vdG, page 4]. The left to right inclusion can be justified by expressing an arbitrary element $z$ of the annihilator in terms of the square-free monomial basis in the $\lambda$ 's. Using (34), in particular $\lambda_{g-1}^{2}=0$, it follows that all monomials that appear in $z$ must contain $\lambda_{g-1}$. If not, $z \cdot \lambda_{g-1}$ would contain nonzero terms in the square-free monomial basis, corresponding to the monomials of $z$ not containing $\lambda_{g-1}$. This contradicts that $z$ is in the annihilator ideal. Successively, we see that $\lambda_{g-2}, \ldots, \lambda_{g-\ell+1}$ must also appear in each of the monomials of $z$, proving the claim.

Proof. We only indicate the proof of Theorem 6 when $\ell=2$. The general case is an $\ell$-fold iteration of the same argument. To start, we restrict to $\mathcal{A}_{g}$ the expression provided by Theorem 5 , see (32). Then, we must prove

$$
\left|\begin{array}{cccc}
\lambda_{g_{2}} & \lambda_{g_{2}+1} & \ldots & \lambda_{g-1} \\
\lambda_{g_{2}-1} & \lambda_{g_{2}} & \ldots & \lambda_{g-2} \\
\ldots & \ldots & \ldots & \ldots \\
\lambda_{g_{2}-g_{1}+1} & \lambda_{g_{2}-g_{1}+2} & \ldots & \lambda_{g_{2}}
\end{array}\right|=\lambda_{g-1} \cdot\left|\begin{array}{cccc}
\lambda_{g_{2}-1} & \lambda_{g_{2}} & \ldots & \lambda_{g-3} \\
\lambda_{g_{2}-2} & \lambda_{g_{2}-1} & \ldots & \lambda_{g-4} \\
\ldots & \ldots & \ldots & \ldots \\
\lambda_{g_{2}-g_{1}+1} & \lambda_{g_{2}-g_{1}+2} & \ldots & \lambda_{g_{2}-1}
\end{array}\right|
$$

after setting $\lambda_{g}=0$. The parallel identity for the Lagrangian Grassmannian is equivalent:

$$
\left|\begin{array}{cccc}
x_{g_{2}} & x_{g_{2}+1} & \ldots & x_{g-1}  \tag{35}\\
x_{g_{2}-1} & x_{g_{2}} & \ldots & x_{g-2} \\
\ldots & \ldots & \ldots & \ldots \\
x_{g_{2}-g_{1}+1} & x_{g_{2}-g_{1}+2} & \ldots & x_{g_{2}}
\end{array}\right|=x_{g-1} \cdot\left|\begin{array}{cccc}
x_{g_{2}-1} & x_{g_{2}} & \ldots & x_{g-3} \\
x_{g_{2}-2} & x_{g_{2}-1} & \ldots & x_{g-4} \\
\ldots & \ldots & \ldots & \ldots \\
x_{g_{2}-g_{1}+1} & x_{g_{2}-g_{1}+2} & \ldots & x_{g_{2}-1}
\end{array}\right| \quad \bmod x_{g} .
$$

The identity does not hold in the absence of the Mumford relations.
We will derive identity (35) geometrically via an excess intersection calculation on $\mathrm{LG}_{g}$. Fix a symplectic splitting

$$
V=W_{1} \oplus L_{1} \oplus W_{2} \oplus L_{2}, \quad \operatorname{dim} W_{i}=2\left(g_{i}-1\right), \quad \operatorname{dim} L_{i}=2 .
$$

In addition, fix Lagrangian subspaces $P_{1} \subset L_{1}$ and $P_{2} \subset L_{2}$. Let

$$
\iota: \mathrm{LG}_{g-1} \rightarrow \mathrm{LG}_{g}, \quad P \rightarrow P \oplus P_{2}
$$

Here $\mathbf{L G}_{g-1}$ is the Lagrangian Grassmannian of $W_{1} \oplus L_{1} \oplus W_{2}$. For this embedding, we have

$$
\begin{equation*}
\iota_{*}\left[\mathrm{LG}_{g-1}\right]=x_{g} \cap\left[\mathrm{LG}_{g}\right], \tag{36}
\end{equation*}
$$

as can be seen by a normal bundle calculation. The reader can verify that

$$
\iota^{-1}\left(\mathrm{LG}_{g_{1}} \times \mathrm{LG}_{g_{2}}\right)=\mathrm{LG}_{g_{1}} \times \mathrm{LG}_{g_{2}-1}
$$

The left hand side has codimension $g_{1} g_{2}$ in $\mathbf{L G} g$, while the right hand side has codimension $g_{1}\left(g_{2}-1\right)$ in $\mathrm{LG}_{g-1}$. Write

$$
j: \mathrm{LG}_{g_{1}} \times \mathrm{LG}_{g_{2}-1} \rightarrow \mathrm{LG}_{g-1}
$$

for the natural map determined by the pair $\left(W_{1} \oplus L_{1}, W_{2}\right)$. The class $\iota^{*}\left(\mathrm{LG}_{g_{1}} \times \mathrm{LG}_{g_{2}}\right)$ can be computed via excess intersection. The excess bundle is the dual tautological subbundle $\mathrm{S}_{g_{1}}^{*}$. Therefore,

$$
\begin{equation*}
\iota^{*}\left(\left[\mathrm{LG}_{g_{1}} \times \mathrm{LG}_{g_{2}}\right]\right)=j_{*}\left(\left(x_{g_{1}} \times 1\right) \cap\left[\mathbf{L G}_{g_{1}} \times \mathrm{LG}_{g_{2}-1}\right]\right)=j_{*} k_{*}\left(\left[\mathrm{LG}_{g_{1}-1} \times \mathrm{LG}_{g_{2}-1}\right]\right), \tag{37}
\end{equation*}
$$

after using (36) again. The embedding

$$
k: \mathrm{LG}_{g_{1}-1} \times \mathrm{LG}_{g_{2}-1} \rightarrow \mathrm{LG}_{g_{1}} \times \mathrm{LG}_{g_{2}-1}
$$

is defined by taking sum with $P_{1}$ on the first factor. Consider

$$
u: \mathrm{LG}_{g_{1}-1} \times \mathrm{LG}_{g_{2}-1} \rightarrow \underset{26}{\mathrm{LG}_{g-2}}, \quad v: \mathrm{LG}_{g-2} \rightarrow \mathrm{LG}_{g-1},
$$

where the first map is determined by the pair $\left(W_{1}, W_{2}\right)$ and the second map is determined by taking sum with $P_{1}$. The equality $j \circ k=v \circ u$ follows from the definitions. By $(28)$ in the proof of Theorem 5, we find

$$
u_{*}\left(\left[\mathrm{LG}_{g_{1}-1} \times \mathrm{LG}_{g_{2}-1}\right]\right)=v^{*}\left|\begin{array}{cccc}
x_{g_{2}-1} & x_{g_{2}} & \ldots & x_{g-3}  \tag{38}\\
x_{g_{2}-2} & x_{g_{2}-1} & \ldots & x_{g-4} \\
\ldots & \ldots & \ldots & \ldots \\
x_{g_{2}-g_{1}+1} & x_{g_{2}-g_{1}+2} & \ldots & x_{g_{2}-1}
\end{array}\right|
$$

Then, using (37) and (38), we have

$$
\begin{aligned}
\iota^{*}\left(\left[\mathrm{LG}_{g_{1}} \times \mathrm{LG}_{g_{2}}\right]\right) & =j_{*} k_{*}\left(\left[\mathrm{LG}_{g_{1}-1} \times \mathrm{LG}_{g_{2}-1}\right]\right)=v_{*} u_{*}\left(\left[\mathrm{LG}_{g_{1}-1} \times \mathrm{LG}_{g_{2}-1}\right]\right) \\
& =v_{*} v^{*}\left|\begin{array}{cccc}
x_{g_{2}-1} & x_{g_{2}} & \ldots & x_{g-3} \\
x_{g_{2}-2} & x_{g_{2}-1} & \ldots & x_{g-4} \\
\ldots & \ldots & \ldots & \ldots \\
x_{g_{2}-g_{1}+1} & x_{g_{2}-g_{1}+2} & \ldots & x_{g_{2}-1}
\end{array}\right|=x_{g-1}\left|\begin{array}{cccc}
x_{g_{2}-1} & x_{g_{2}} & \ldots & x_{g-3} \\
x_{g_{2}-2} & x_{g_{2}-1} & \ldots & x_{g-4} \\
\ldots & \ldots & \ldots & \ldots \\
x_{g_{2}-g_{1}+1} & x_{g_{2}-g_{1}+2} & \ldots & x_{g_{2}-1}
\end{array}\right|,
\end{aligned}
$$

which recovers the right hand side of (35). On the other hand, by (28), the class on the left hand side equals

$$
\iota^{*}\left|\begin{array}{cccc}
x_{g_{2}} & x_{g_{2}+1} & \ldots & x_{g-1} \\
x_{g_{2}-1} & x_{g_{2}} & \ldots & x_{g-2} \\
\ldots & \ldots & \ldots & \ldots \\
x_{g_{2}-g_{1}+1} & x_{g_{2}-g_{1}+2} & \ldots & x_{g_{2}}
\end{array}\right|
$$

while the pullback $\iota^{*}: \mathrm{CH}^{*}\left(\mathrm{LG}_{g}\right) \rightarrow \mathrm{CH}^{*}\left(\mathrm{LG}_{g-1}\right)$ has the effect of setting $x_{g}=0$.

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[^0]:    ${ }^{1}$ Since $\overline{\mathcal{A}}_{g}$ is possibly singular, even as a stack, some care must be taken with the Chow theories. Here, $\mathrm{CH}^{\text {op }}$ is the $\mathbb{Q}$-algebra of operational Chow classes. Usual Chow cycle theory, indexed by codimension, is denoted by $\mathrm{CH}^{*}$.
    ${ }^{2}$ Here, $B_{2 i}$ is the Bernoulli number.

[^1]:    ${ }^{3}$ The existence and uniqueness of taut ${ }^{\mathrm{cpt}}(\gamma)$ follows from the Gorenstein property of $\mathrm{R}^{*}\left(\overline{\mathcal{A}}_{g}\right)$ applied to the functional $\delta \mapsto\langle\gamma, \delta\rangle^{\text {cpt }}$ on $\mathrm{R}^{*}\left(\overline{\mathcal{A}}_{g}\right)$.
    ${ }^{4}$ We refer the reader to FP3 P for a review of the theory of tautological classes on the moduli spaces of curves. Unlike the case of $\mathcal{A}_{g}$, the tautological ring $\mathrm{R}^{*}\left(\mathcal{M}_{g}^{\mathrm{ct}}\right)$ is not a Gorenstein local ring, see CLS Pix and Pet for the pointed case.

[^2]:    ${ }^{5}$ We thank van der Geer for the characteristic $p$ argument.
    ${ }^{6}$ The second Voronoi compactification can be taken here A1, N.

[^3]:    ${ }^{7}$ The existence and uniqueness of $\operatorname{taut}(\gamma)$ follows from the Gorenstein property of $\mathbf{R}^{*}\left(\mathcal{A}_{g}\right)$.

[^4]:    ${ }^{8}$ When $S$ and is also log smooth, the integrality condition is equivalent to $f$ being flat, and the conditions that $f$ is integral and saturated together are equivalent to $f$ being flat with reduced fibers.

[^5]:    ${ }^{9}$ For example, when $X \rightarrow S$ is a family of curves, the $V\left(y_{i}\right)$ correspond to markings. In our study of the moduli of abelian varieties, $\ell=0$.

[^6]:    ${ }^{10}$ Of course, $t_{i} \in\left(x_{\beta}, x_{\gamma}\right)$, but we have chosen to keep $t_{i}$ in the notation to emphasize that $\beta, \gamma$ belong to the same part $A_{i}$.

[^7]:    ${ }^{11}$ By definition, a piecewise polynomial on $\operatorname{Sym}_{\mathrm{rc}}^{2}\left(\mathbb{R}^{g}\right)$ is an admissible decomposition together with a continuous $\mathrm{GL}_{g}$-invariant function on the decomposition that is polynomial on each cone.
    ${ }^{12}$ We thank V. Alexeev for a discussion about $\kappa$ classes at the conference Higher Dimensional Algebraic Geometry in La Jolla in January 2024.

