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Algebraic Certificates for Positivity and Kazhdan's Property (T)

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Thesis presented for the degree
Master of Science ETH in Mathematics

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August 2023

Department of Mathematics
ETH Zürich
Academic year 2022–2023

Abstract

Hilbert's 17th problem states that every polynomial with real coefficients that takes only positive values is in fact a sum of squares of rational functions. This thesis investigates analogues of this for noncommutative $*$-algebras following Schmüdgen, Cimprič, and Ozawa, culminating in a characterisation of positive elements in Archimedean $*$-algebras as those in the Archimedean closure of the cone of sums of hermitian squares. We improve this result following Netzer and Thom for algebras of virtually free groups to show that the only elements that are positive under finite dimensional representations are sums of hermitian squares. We then consider Kazhdan's property (T) and recover Ozawa's characterisation of discrete groups with property (T) as those for which there is some $\lambda > 0$ such that $\Delta^2 - \lambda\Delta$ is a sum of hermitian squares, where Δ denotes the group algebra Laplacian. We recount applications of this to verifying property (T) for some groups, and following Bader and Nowak, to higher dimensional cohomology vanishing.

Contents

Chapter 0

Introduction

Suppose we are given a positive real number, say 4. How can we convince ourselves algebraically that it is positive? That is, is there something inherent about 4 that tells us it *must* be positive?

With a bit of thought, an obvious answer pops out — $4 = 2^2$, and squares are clearly positive. So 4 must be positive. The same trick works for any real positive number r: $\sqrt{r}$ exists, and $\sqrt{r}^2 = r$, so r must be positive (foreshadowing a bit, by standard functional calculus – an 'analytic' process – the same trick works for any C^* algebra).

The same game fails when we move onto the rationals however — $\frac{1}{2}$ is clearly positive, but isn't the square of something rational. But with a bit of thought we can again convince ourselves of its positivity, by noting that $\frac{1}{2} = (\frac{1}{2})^2 + (\frac{1}{2})^2$, and a sum of two positive things must be positive.

So we've seen in the above examples that we can detect positivity (in some sense an analytic property) with an algebraic certificate — exhibiting our term as a sum of squares. We can ask this question in greater generality for $\mathbb{Q}$-algebras and rings: if something is positive in an analytic sense, can we algebraically prove that? As we examine in the first part, there are many instances with a positive solution, and many with a negative solution. In many cases the answer to the above is yes*. That is yes, up to an analytic error that we can't in general get rid of. We spend considerable effort in demonstrating instances when we don't need this analytic correction, namely for group algebras of free groups. We then spend even more effort to demonstrate cases where a similar process *might* be possible — when we have enough finite dimensional representations of our algebra.

In the second part, we are concerned primarily with Kazhdan's property (T), a powerful rigidity result, that turns out to be related to the positivity (in the analytic sense) of a certain term in the group algebra — and so our game kicks in again. The main breakthrough of Narutaka Ozawa in [Oza16] was to show that in this case, this analytic positivity can be verified algebraically, in fact over the rationals. As such proving property (T) can be made into a computational task, and this has allowed Mathematicians over the last five years to exhibit property (T) for a new class of groups (for which (T) was long conjectured), the automorphism groups of free groups with ≥ 4 generators.

In chapter 1 we survey classical Artin-Schreier theory, presenting the solution to Hilbert's 17th problem. That is, we will show that a polynomial takes non-negative values everywhere if and only if it can be written as a sum of squares of rational functions. We prove this by firstly considering the first order theory of the real numbers, and in fact observe that it is identical to the first order theory of a much larger class of fields. This model theoretical trickery can be recast as saying that we expect something like Hilbert's 17th to hold in any object which behaves like the reals, these are called *real closed fields*. This is an introductory chapter and serves as motivation to chapter 2.

In chapter 2 we ask the same question for an arbitrary algebra with involution, and show how we can nicely associate a C^*-algebra to it. As such we have access to a variety of tools coming from functional analysis, and we can show that for nice enough algebras (called *Archimedean*) we can characterise when an element is positive as sums of squares, up to a topological closure that in general is essential. That is, we prove

Theorem 2.7.3 ([Sch09], Proposition 15)**.** *Let $k \in \{\mathbb{R}, \mathbb{C}\}$ and $a \in k[\Gamma]$ be an element such that $\pi(a)$ is positive semidefinite for every unitary representation π of Γ. Then for all $\varepsilon > 0$, $a + \varepsilon 1$ is a sum of (hermitian) squares in the group algebra, where 1 denotes the group identity.*

In chapter 3 we start by surveying some cases where the topological closure (the $\varepsilon 1$ term for all $\varepsilon > 0$)

alluded to above indeed is essential, and see that it always is in high dimensions (≥ 3). However, for the group algebras of (virtually) free groups, we can get rid of this annoying relic — so far no other examples of this sort are known. In particular, we prove (following Tim Netzer and Andreas Thom)

Theorem 3.2.1. *Let Γ be a virtually free group, and let $a \in \mathbb{C}[\Gamma]$ be such that $\pi(a)$ is positive semidefinite for every* ***finite dimensional*** *unitary representation of Γ. Then a is a sum of (hermitian) squares in $\mathbb{C}[\Gamma]$.*

In chapter 4 we examine the structure of the proof of Theorem 3.2.1, and see that it follows from there being enough finite dimensional representations. We survey (a slightly weaker version of) this phenomenon, known as the RFD property, and explore its alternative characterisations and links to other important group properties, such as residual finiteness, and property (τ).

In chapter 5 we introduce property (T), and state many of its equivalent characterisations. We also explore finite dimensional representations of property (T) groups, and see that they are all isolated — in a sense opposite to the RFD property (although both can happen simultaneously). This chapter will be basic for those familiar with the theory of Kazhdan's Property (T).

In chapter 6 we exploit the fact that property (T) (for a finitely generated group Γ with finite symmetric generating set $S = S^{-1}$) is equivalent to there being some $\lambda > 0$ such that $\Delta^2 - \lambda\Delta \geq 0$ in $C^*(\Gamma)$, where $\Delta := |S| - \sum_{s \in S}$ denotes the group algebra Laplacian. Using the machinery developed in chapter 2, we show a famous recent characterisation of property (T) due to Narutaka Ozawa:

Theorem 6.1.1 (Ozawa's characterisation)**.** *Let Γ be a finitely generated group. Then Γ has property (T) if and only if there is some $\lambda > 0$ such that $\Delta^2 - \lambda\Delta$ is a sum of (hermitian) squares in $\mathbb{R}[\Gamma]$.*

We will then see how this can be rephrased as a statement about the first group cohomology vanishing, or equivalently it having a Hausdorff topology. This will allow us, following Uri Bader and Roman Sauer, to partially generalise this statement to higher dimensional cohomology vanishing.

In chapter 7 we survey the main applications of Theorem 6.1.1, including proving that the automorphism groups of free groups have Property (T). This involves a computer calculation which we make no effort to reproduce, but we explain how it is done and later verified.

In chapter 8 we survey some contemporary interest in cohomology vanishing, namely the idea of group stability. We prove, following De Chiffre-Glebsky-Lubotzky-Thom, that vanishing of 2-cohomology for all unitary representations (which may be detected using the Bader-Sauer machinery) implies that a group is *Frobenius stable*: namely that any function that is *almost* a homomorphism into some unitary group $U(n)$ is indeed *close* to some genuine homomorphism, this will all be made precise later. The term *Frobenius* is because we will be interested in the Frobenius norm on $U(n)$.

We are covering many different topics, and not all of them will be of interest to all readers. So the reader is encouraged to look at the leitfaden (Figure 1) to plot the desired path. Notice that not all of each chapter might be vital for other parts — so if one is only looking for some specific topics it is recommended to head to them directly, and then track back the prerequisites. An index is included for this purpose.

It was surprising to the author that many of these topics are as deeply connected as they are, and many of the same protagonists appear throughout. We therefore touch upon many interesting topics (Congruence Subgroup Problem, o-minimality, Connes Embedding Problem, ...) but don't go beyond general motivations and basic ideas to not distract from the main ideas in this thesis. However, it is hoped that our hints are enough to whet the reader's appetite, and as such we provide plenty of references from which to proceed.

Notation and prerequisites

We assume familiarity with functional analysis, in particular the theory of C^*-algebras and functional calculus (all of this can be found in [EW17]). We also assume some standard commutative algebra, but this isn't essential for most of the theory. We make heavy use of the theory of unitary representations — the appendix in [BdlHV08] more than suffices. We reference the facts we use as we go along, so the reader is encouraged to refer to this source only when needed. When saying *representation* we will always mean a unitary representation (or $*$-representation, this will be clear from context), we often still choose to stress this.

$\mathbb{N}$ will denote the natural numbers (without 0) and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. G will typically denote a topological group, and Γ a discrete one. We will use 1 to denote the group identity — unless the group is abelian, in which case we use additive notation and the identity will be denoted 0. Sometimes when there is a potential for confusion we will write 1_Γ to stress the ambient group, and in the case of matrix groups we use I_n to denote the identity matrix. For rings and fields, we will use $0, 1$ and additive/multiplicative notation as standard.

For a discrete group Γ, λ_Γ will denote the (left) regular representation, and 1_Γ the trivial representation. $\leq$ represents containment of representations, and $\prec$ weak containment. If Γ has generating set S, for any $g \in \Gamma$ and $n \in \mathbb{N}$, $B_\Gamma(g, n)$ is the (closed) n-ball in the word metric on S (which is implicit in the notation), centered at $g \in \Gamma$. For any set S, F_S indicates the free group on S; F_n is the standard free group on n generators.

$\mathcal{H}$ will typically denote a (real or complex) Hilbert space, and $\mathcal{L}(\mathcal{H})$ and $\mathbb{B}(\mathcal{H})$ will denote the linear operators and the bounded operators on $\mathcal{H}$ respectively. For a field k, $M_{m\times n}(k)$ will denote the algebra of $m\times n$ matrices with coefficients in k; $M_n(k)$ will be used to denote $M_{n\times n}(k)$.

Acknowledgements

First and foremost I would like to thank Marc Burger, who introduced me to the area and whose invaluable teaching, comments, and advice throughout have shaped this thesis immensely. I would also like to thank Uri Bader, Narutaka Ozawa, and Andreas Thom for their clear and helpful explanations to my questions, and for their engaging writings. Additionally, I thank my friends and family, and everyone doing anything vaguely related to dynamics or geometry at ETH or UZH for putting up with me over the last few months. Especially I wish to thank LB (BDT), ED, BH, RM, YM, DO, SO, ČS, PT, ATJ, and BU — you guys are the best!

Leitfaden

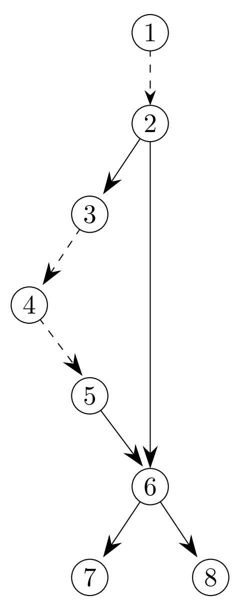


Figure 1: Leitfaden for this thesis. Dashed arrows represent ideas and motivations, whereas solid arrows represent necessary prerequisites

Part I

Algebraic Certificates for Positivity

Chapter 1

Hilbert's 17th Problem

We start with a survey of mostly classical theory — we define a natural notion positivity for fields (and polynomial rings over them), and then categorise it in terms of sums of squares.

We will assume familiarity with the language of model theory — see for example the appendix in [Pre] for a quick overview that is more than sufficient for our purposes.

1.1 Ordered Fields

We begin this survey by reviewing the algebraic background on ordered fields. All the results here are due to Artin and Schreirer, see chapter 1 in [BCR98] for a full introduction and historical account.

Taking inspiration from $\mathbb{R}$, there are some properties we want a reasonable order on a field to always satisfy.

Definition 1.1.1. An *ordered field* is field k equipped with a total order relation $\leq$ compatible with the algebraic structure of k — in particular

(a) If $x \leq y$ then $x + z \leq y + z$ for all z;

(b) If $0 \leq x$ and $0 \leq y$ then $0 \leq xy$.

Example 1.1.2. $\mathbb{R}$ is clearly an ordered field, and this ordering is unique. Similarly any subfield of $\mathbb{R}$, including $\mathbb{Q}$, and $\mathbb{R}_{\text{alg}}$ (the field of real algebraic numbers) have a unique ordering induced from the one on $\mathbb{R}$.

This might make you think that orderings are always unique if they exist — but this certainly isn't the case.

Proposition 1.1.3. *The set of orderings of* $\mathbb{R}(X)$ *can be naturally indexed by* $\mathbb{R} \cup \mathbb{R} \cup \{-\infty, +\infty\}$.

Proof. We begin by noting that there is exactly one ordering of $\mathbb{R}(X)$ such that X is positive, and smaller than any positive real number. The element

$$f(X) = a_n X^n + \cdots + a_m X^m \quad \text{with } m \leq n \text{ and } a_m \neq 0$$

we clearly need to set $f(X) > 0$ if and only if $a_m > 0$. Also, since squares have to be positive, we're forced to set $f(X)/g(X) > 0$ if and only if $f(X)g(X) > 0$. This indeed defines an ordering of $\mathbb{R}(X)$.

Now given any ordering of $\mathbb{R}(X)$, we get a *cut* (I_-, I_+) where $I_- = \{x \in \mathbb{R} \mid x < X\}$ and $I_+ = \{x \in \mathbb{R} \mid x > X\}$. The possible cuts are

(a) $(\varnothing, \mathbb{R})$;

(b) $\big((-\infty, a), [a, \infty)\big)$ for some $a \in \mathbb{R}$;

(c) $\big((-\infty, a], (a, \infty)\big)$ for some $a \in \mathbb{R}$;

(d) $(\mathbb{R}, \varnothing)$.

By setting $Y = -1/X$, $a - X$, $X - a$, and $1/X$ respectively, we obtain an ordering of $\mathbb{R}(Y)$ such that Y is positive and smaller than any positive real number — and by the above there is exactly one such ordering. Hence the orderings of $\mathbb{R}(X)$ are indexed by the possible cuts. □

In fact, the set of orderings of a field may have strictly larger cardinality than that of the field itself.

Proposition 1.1.4. *There is a continuum of distinct orderings of* $\mathbb{Q}(X)$.

Proof. Let us try to do the same thing we did with $\mathbb{R}(X)$, for any ordering of $\mathbb{Q}(X)$ we define the corresponding cuts $I_- = \{x \in \mathbb{Q} \mid x < X\}$ and $I_+ = \{x \in \mathbb{Q} \mid x > X\}$. All the possibilities (a) - (d) from the $\mathbb{R}(X)$ case can happen here too (for $a \in \mathbb{Q}$), and each gives a unique ordering — this gives countably many orderings.

But since $\mathbb{Q}$ isn't complete, there is also the possibility that the cuts will take the form $(-\infty, a)$ and (a, ∞) for some $a \in \mathbb{R}$. It is easy to see that in this case the number a must be *transcendental* over $\mathbb{Q}$, and in fact the order on $\mathbb{Q}(X)$ we have is the one obtained by the pull back of the ordering on $\mathbb{Q}(a)$ (as a subfield of $\mathbb{R}$) under the field isomorphism $\mathbb{Q}(X) \cong \mathbb{Q}(a)$. Since $X < q$ if and only if $a < q$, different transcendentals yield different orderings of $\mathbb{Q}(X)$, and there is a continuum of these. □

Remark 1.1.5. Note that the order on $\mathbb{R}(X)$ given by the cut $(-\infty, 0] \cup (0, \infty)$ is *non-Archimedean* — it contains infinitesimals, and for example $1/X$ is bigger than any real number.

Definition 1.1.6. An ordered field k is *Archimedean* if it doesn't contain infinitesimals. That is, for any $0 < x \in k$, there is some natural $n \in \mathbb{N}$ such that $1 < nx$.

Equivalently, this can be defined by the non-existence of infinite elements (if x is infinitesimal, then $1/x$ is larger than any natural number). In the next chapter we will define Archimedean algebras in general using the non-existence of infinite elements — but in that case the two possible definitions don't necessarily coincide.

Can we describe the subset of k that contains all the positive elements?

Definition 1.1.7. A *cone (of a field k)* is a subset $P \subset k$ such that

(a) $x, y \in P \Rightarrow x + y, xy \in P$;

(b) $x \in k \Rightarrow x^2 \in P$

It is called *proper* if $-1 \notin P$.

There are two natural cones to consider.

Definition 1.1.8. Let k be a field.

(a) The *cone of sums of squares* is

$$\Sigma^2 k := \{\sum_{i=1}^{n} x_i^2 \mid n \in \mathbb{N}, x_i \in k\}$$

It is clear from the definitions that $\Sigma^2 k$ is contained in any other positive cone.

(b) Suppose k is ordered by $\leq$. Then

$$k^+(\leq) := \{x \in k \mid x \geq 0\}$$

is called *the positive cone (associated to $\leq$).*

These two concepts are the motivating examples for the question we are asking in Part I. $k^+(\leq)$ is what we think of as an *analytically* defined positive cone, whereas $\Sigma^2 k$ is clearly *algebraic*.The interplay between analogues of $k+(\leq)$ and $\Sigma^2 k$, in the context of (non-commutative) algebras, is the focus of chapters 2 and 3.

Unsurprisingly it is easy to see that cones and orderings convey similar information.

Proposition 1.1.9. *Let $(k,\leq)$ be an ordered field, then $k^+(\leq)$ satisfies $k^+(\leq) \cup (-k^+(\leq)) = k$. Conversely, if P is any proper cone satisfying $P \cup -P = k$, then k is ordered by $x \leq y \Leftrightarrow y - x \in P$ (and in this case, $P = k^+(\leq)$).*

If $P \cup -P \subsetneq k$, then we have some choice for the remaining elements whether they are positive or negative.

Lemma 1.1.10. *Let P be a proper cone of k. Then*

(a) If $-a \notin P$, then $P[a] = \{x + ay \mid x, y \in P\}$ is a proper cone of k;

(b) There exists an ordering $\leq$ of k such that $P \subset k^+(\leq)$.

Proof. (a) $P[a]$ is clearly a cone, we just need to show that $-1 \notin P[a]$. Indeed, suppose that $-1 = x + ay$ with $x, y \in P$ — then either $y = 0$ and $-1 \in P$, or $y \neq 0$ and $-a = (1/y)^2 y(1 + x) \in P$, a contradiction.

(b) By Zorn's lemma there is a maximal proper cone Q containing P, and it suffices to show that $Q \cup -Q = k$. Let $a \notin Q$, then by (i) $Q[-a]$ is a proper cone and hence $Q[-a] = Q$ by maximality. So $-a \in Q$.

□

Putting this all together, we can characterise the relation between cones and orderings.

Theorem 1.1.11. *Let k be a field. The following are equivalent:*

(a) k can be ordered;

(b) The field k has a proper cone;

(c) $-1 \notin \Sigma^2 k$;

(d) For every $x_1, \ldots, x_n$ in k,

$$\sum_{i=1}^{n} x_i^2 = 0 \Rightarrow x_1 = \cdots = x_n = 0$$

*We call any field satisfying this last property **formally real**. .*

Proof. (a) ⇒ (b) ⇒ (c) ⇔ (d) are trivial. To show that (c) ⇒ (a) note that since $-1 \notin \Sigma^2 k$, $\Sigma^2 k$ is a proper cone, and so we conclude by using (b) of Lemma 1.1.10. □

We obtain from characterisation (d) that any formally real field must have characteristic zero (and in particular must contain $\mathbb{Q}$). Notice also that despite having characteristic zero, the fields of p-adic numbers can't be ordered. Indeed, $\mathbb{Q}_2$ contains a square root of -7, and for every prime $p \geq 3$ the field $\mathbb{Q}_p$ contains a square root of $1 - p$.

We are thus able to completely characterise a cone P in terms of the orderings it generates. Precisely,

Proposition 1.1.12. *Let $k \supset \mathbb{Q}$ be a field, and P a cone of k. Then*

$$P = \bigcap \{k^+(\leq) \mid \leq \text{ is an ordering with } P \subset k^+(\leq)\}$$

where the empty intersection is defined to be k.

Proof. The cone P is certainly contained in this intersection. If there is some $a \notin P$ then we firstly observe that P is proper. Indeed, consider the equation

$$a = \tfrac{1}{4}((1+a)^2 - (1-a)^2) \tag{1.1}$$

Hence if $-1 \in P$ then $a \in P$. Therefore by (a) in Lemma 1.1.10 $P[-a]$ is proper, and (b) gives us an ordering $\leq$ such that $P[-a]$ is positive, so a isn't in the intersection. □

Corollary 1.1.13 (Dimension zero Positivstellensatz)**.** *Let $k \supset \mathbb{Q}$ be a field. Then*

$$\Sigma^2 k = \bigcap \{k^+(\leq) \mid \leq \text{ is an ordering of } k\}$$

This is the type of result we want — it says that an element is positive in some analytic sense (that is, positive in *all*' the possible orderings) if and only if it is positive for a very obvious algebraic reason (it is a sum of squares).

Remark 1.1.14. In the above, we mean Krull dimension zero. We will see that in the commutative setting, positivstellensätze as above (where positivity has a purely algebraic characterisation) are a fundamentally low dimensional phenomenon. They can hold only up to dimension 2, as we will see in chapter 3.

1.2 Real Closed Fields

Note that $\mathbb{C}$ isn't formally real and so $\mathbb{R}$ has no proper formally real algebraic extensions, this is clearly an important property.

Definition 1.2.1. A field $\mathbb{K}$ is a *real closed field (RCF)* if it is formally real and has no proper formally real algebraic extensions.

We want to study the theory of real closed fields in $\mathcal{L}_r = \{+, \cdot, 0, 1\}$, the language of rings, but it is not a priori obvious that being an RCF is axiomatisable. Luckily, Artin-Schreier theory classifies RCFs as those that contain the expected square roots, and satisfy an intermediate value theorem for odd degree polynomials.

Theorem 1.2.2. *Let $\mathbb{K}$ be a formally real field. The following are equivalent:*

(a) $\mathbb{K}$ is an RCF;

(b) $\mathbb{K}[i]$ is algebraically closed (where $i = \sqrt{-1}$);

(c) For any $x \in \mathbb{K}$, either x or $-x$ is a square, and every polynomial of odd degree has a root.

For the proof of this see Theorem 1.2.2 in [BCR98].

Corollary 1.2.3. *The class of real closed fields is $\mathcal{L}_r$-axiomatizable.*

Proof. We axiomatise RCFs by including:

(i) The standard field axioms;

(ii) Being formally real, that is for every $n \in \mathbb{N}$ the axiom

$$\forall x_1 \cdots \forall x_n \quad (x_1^2 + \cdots + x_n^2 + 1 \neq 0)$$

(iii) The existence of square roots

$$\forall x \exists y \quad (y^2 = x \vee y^2 = -x)$$

(iv) The intermediate value theorem for odd polynomials, that is for each $n \in \mathbb{N}_0$ the axiom

$$\forall x_0 \cdots \forall x_{2n} \exists y \quad (y^{2n+1} + \sum_{i=0}^{2n} x_i y^i = 0)$$

□

Notice that $\Sigma^2\mathbb{K}$ is the only possible positive cone of an RCF $\mathbb{K}$, since for all $x \in \mathbb{K}$ either x or $-x$ is a square. In particular, for an RCF we can define an order predicate algebraically, by $x \leq y$ if and only if there exists some z such that $y - x = z^2$. So in fact we have the theory of real closed fields in the language of ordered rings, $\mathcal{L}_{\mathrm{or}}$.

Definition 1.2.4. Let **(RCF)** be the $\mathcal{L}_{\mathrm{or}}$-theory axiomatised by the axioms above and the axioms for ordered fields.

Since the ordering is in fact defined using an $\mathcal{L}_{\mathrm{r}}$-sentence, it shouldn't be too surprising that using the ordering doesn't change the definable sets. Indeed, suppose $X \subset \mathbb{K}^n$ is definable using an $\mathcal{L}_{\mathrm{or}}$-formula with terms $\{t_i\}$. Then replace all instances of $t_i < t_j$ by $\exists z \ (z \neq 0 \wedge t_i + z^2 = t_j)$ to obtain an $\mathcal{L}_{\mathrm{r}}$-sentence.

We hope that any ordered field will lie in some RCF, and for this we will need a Zorn argument. However in order to do this, we need to show how to transfer orderings to field extensions.

Lemma 1.2.5. *If $(k, \leq)$ is an ordered field and $0 < x \in k$, and x isn't a square in k, we can extend the ordering of k to $k(\sqrt{x})$.*

Proof. We simply define an order on $k(\sqrt{x})$ in the same way we get an order on $\mathbb{Q}(\sqrt{r})$ for any $r \in \mathbb{Q}$ that isn't a square. That is, we set $0 < a + b\sqrt{x}$ if and only if one of the following happens:

(a) $b = 0$, and $a > 0$;

(b) $b > 0$ and $(a > 0$ or $x > \frac{a^2}{b^2})$;

(c) $b < 0$ and $(a > 0$ and $x < \frac{a^2}{b^2})$.

□

Therefore by Zorn's lemma every ordered field has a maximal formally real algebraic extension, and indeed this is a real closed field — which we see by using (c) in Theorem 1.2.2.

Definition 1.2.6. For any ordered field, its *real closure* is a real closed algebraic extension of it.

Remark 1.2.7. The real closure may not be unique. Indeed let $k = \mathbb{Q}(X)$, and $K^{\pm} = k(\sqrt{\pm X})$. They can both be ordered by Lemma 1.2.5, so let $\mathbb{K}^{\pm}$ be the real closure of $K^{\pm}$. But there is no isomorphism between $\mathbb{K}^{+}$ and $\mathbb{K}^{-}$ because X is a square in $\mathbb{K}^{+}$ but not $\mathbb{K}^{-}$.

So we need to work a bit to show that any ordered field admits a real closure whose order extends the original one.

Theorem 1.2.8. *Let $(k, \leq)$ be an ordered field. Then there is a unique (up to isomorphism) real closure, whose canonical order extends that on k.*

Proof. By repeatedly applying Lemma 1.2.5 we get a field $(k', \leq)$ which is an algebraic extension of $(k, \leq)$, where every positive element of k has a square root. Now we use Zorn's lemma to find a maximal formally real algebraic extension $\mathbb{K}$ of k'. Since every positive element of k is a square in $\mathbb{K}$, the canonical order on $\mathbb{K}$ extends that on k, showing existence.

For the proof of uniqueness see for example Theorem 1.3.2 in [BCR98]. □

Remark 1.2.9. The original proof of the uniqueness of real algebraic closure due to Artin and Schreier uses Sturm's Theorem, which allows one to know how many distinct real roots a polynomial has in an interval. A different proof was found by Knebusch, which was related to the previous one by Becker and Spitzlay. This latter proof is found in [Pre84] for example.

RCFs therefore are somewhat well-behaved, so one might hope that they share similar properties between them.

Theorem 1.2.10. *The theory* (**RCF**) *admits quantifier elimination in* $\mathcal{L}_{\text{or}}$.

For the proof of this fact see the book of Marker ([Mar02]).

Corollary 1.2.11. (**RCF**) *is complete and model complete. In fact,* (**RCF**) *is the theory of* $(\mathbb{R}, +, \cdot, 0, 1, \leq)$.

Proof. Completeness is clear because $\mathbb{R}$ is a model of (**RCF**), and model completeness follows from quantifier elimination. Since every model of (**RCF**) has characteristic zero, the rational numbers are embedded in any such field, as is the real closure of the rationals (the algebraic numbers $\mathbb{R}_{\text{alg}}$). So for any $\mathbb{K} \models$ (**RCF**), we have $\mathbb{R}_{\text{alg}} \prec \mathbb{K}$ and hence $\mathbb{K} \equiv \mathbb{R}_{\text{alg}} \equiv \mathbb{R}$. □

Corollary 1.2.12 (Tarski transfer principle)**.** *Let* $\mathbb{K}$ *be an RCF and let* Φ *be a formula of* $\mathcal{L}_{\text{or}}$ *with parameters in* $\mathbb{K}$ *without a free variable. Suppose* $\mathbb{F}$ *is a real closed extension of* $\mathbb{K}$.

Then Φ *is true in* $\mathbb{F}$ *if and only if it is true in* $\mathbb{K}$.

As an example of the use of the transfer principle, we remark that many standard theorems in analysis hold over all real closed fields.

Theorem 1.2.13 (Rolle)**.** *Let* $\mathbb{K}$ *be a real closed field,* $f : \mathbb{K} \to \mathbb{K}$ *a definable function, and* $a < b$. *Suppose that* f *is continuous on* $[a, b]$, *differentiable on* (a, b) *and suppose that* $f(a) = f(b)$. *Then there exists* $c \in (a, b)$ *such that* $f'(c) = 0$.

Proof. If we can express the statement of Rolle's theorem as a first order sentence, then by Tarski transfer the fact that it holds over $\mathbb{R}$ immediately implies that it holds over $\mathbb{K}$. Notice that we haven't yet made clear what we mean by continuity and differentiability over real closed fields — these will be given by first order sentences.

(i) Since f is definable, there is a formal expression $\varphi(x, y, \underline{b})$ that defines "$f(x) = y$";

(ii) Since $|x| < \varepsilon$ if and only if $-\varepsilon < x < \varepsilon$, there is a first order formula $\psi(x, y, \varepsilon, \underline{c})$ that defines "$|f(x) - f(y)| < \varepsilon$";

(iii) The sentence "f is continuous at x" can be written as

$$C(f, x) = \forall \varepsilon \exists \delta \forall y \, \Big(\varepsilon > 0 \Rightarrow \big(\delta > 0 \wedge (|y - x| < \delta \Rightarrow |f(y) - f(x)| < \varepsilon)\big)\Big)$$

(iv) The sentence "f is continuous on $[a, b]$" can be written as

$$C(f, [a, b]) = \forall x \, \big(a \leq x \leq b \Rightarrow C(f, x)\big)$$

(v) The sentence "f is differentiable at x with derivative $f'(x)$" can be written as

$$D(f, x, f'(x)) = \forall \varepsilon \exists \delta \forall h \, \Big(\varepsilon > 0 \Rightarrow \big(\delta > 0 \wedge (0 < |h| < \delta \Rightarrow \big|\tfrac{f(x+h) - f(x)}{h} - f'(x)\big| < \varepsilon)\big)\Big)$$

(vi) The sentence "f is differentiable on (a, b)" can be written as

$$D(f, (a, b)) = \forall x \, \big(a < x < b \Rightarrow \exists f'(x) \; D(f, x, f'(x))\big)$$

Finally, Rolle's Theorem can be written as

$$\big(C(f,[a,b]) \wedge D(f,(a,b)) \wedge f(a) = f(b)\big) \Rightarrow \exists c\ \big((a < c < b) \wedge D(f,c,0)\big)$$

□

We've already observed that $\mathbb{R}$ and $\mathbb{R}_{\text{alg}}$ are examples of real closed fields. We collect some more examples to convince the reader that they can be quite exotic — the first order theory certainly doesn't capture everything about their behaviour.

The first example requires basic knowledge of ultrafilters and (model-theoretic) ultraproducts, see for example §2 in [Pre]. Ultrafilters will be needed again at the very end of this thesis, in chapter 8.

Example 1.2.14. Let $\omega \in \beta\mathbb{N}$ be a non-principal ultrafilter. Then the ultraproduct $\mathbb{R}^{\omega}$ is a field, often called the *hyperreal numbers*. It is a real closed field by applying Łos' Theorem — but note that again by applying Łos it has infinitesimals and in particular is non-Archimedean. We should also note that saying *the* hyperreal numbers is somewhat misleading — it has been shown that the construction being independent of the choice of ultrafilter is equivalent within ZFC to the continuum hypothesis.

Other examples include the *surreal numbers* and the *computable numbers* (we will see the latter again in section 4.6).

Definition 1.2.15. For $k \in \{\mathbb{R}, \mathbb{C}\}$, the *field of Puiseux series* is given by

$$k(X)^{\vee} := \{\sum_{i=m}^{\infty} a_i X^{i/q} \mid m \in \mathbb{Z}, q \in \mathbb{N}, a_i \in k\}$$

That is, they are Laurent series in fractional powers of X.

We can order $\mathbb{R}(X)^{\vee}$ so that $\sum_{i=m}^{\infty} a_i X^{i/q}$ with $a_m \neq 0$ is > 0 if and only if $a_m > 0$.

Since $\mathbb{R}(X)^{\vee}[i] = \mathbb{C}(X)^{\vee}$, and the latter is well known to be algebraically closed, we see that these form a real closed field (see page 11 in [BCR98]). These can readily be given a valuation and hence a natural metric, with respect to which they aren't complete — completeness is not a first order property. Their completion with respect to this metric, the *Levi-Civita field*, is also an RCF, see [BCS18] for an accessible introduction to many of these concepts.

1.3 Hilbert's 17th Problem

There is a natural analogue of Corollary 1.1.13 for polynomials. That is, if $f \in \mathbb{R}[X_1, \ldots, X_n]$ is positive everywhere ($f(a_1, \ldots, a_n) \geq 0$ for all $(a_1, \ldots, a_n) \in \mathbb{R}^n$) then must f be a sum of squares of polynomials?

This was asked by Minkowski in his thesis defence, and Hilbert (who sat in the audience) reportedly immediatley suspected the answer was no. Indeed, consider for example the polynomial

$$f(X,Y,Z) = Z^6 + X^4Y^2 + X^2Y^4 - 3X^2Y^2Z^2 \tag{1.2}$$

It is positive definite, since $X^2Y^2Z^2$ is the geometric mean of the other three terms — but it is not hard to check that it can't be written as a sum of squares of polynomials. Hilbert showed this phenomenon in greater generality — there exist homogeneous polynomials of degree $2m$ in n variables that aren't sums of squares if and only if $(n, 2m) \neq (3,4)$ and $n \geq 3$ or $2m \geq 4$. See chapter 6 in [BCR98] for more details, and for quantitative estimates for the number of squares needed. There are also non-homogeneous examples, for example the *Motzkin polynomial* $X^4Y^2 + X^2Y^4 - 3X^2Y^2 + 1$, see [Mot67] (Hilbert knew these existed in two variables, but this was the first explicit example).

But what if we ask the same question for rational functions? Now since we are working over a field, the Artin-Schreier machinery kicks in and both notions of positivity coincide — this was Hilbert's 17th question asked in his famous list of 23 open problems in the 1900 ICM. It was answered in the positive by Artin in 1927, but the proof we're presenting is due to Abraham Robinson. We also remark that Dubois in 1967 showed that the Theorem fails for general ordered fields.

Definition 1.3.1. Let $\mathbb{K}$ be a real closed field and $f \in \mathbb{K}(X_1, \ldots, X_n)$ a rational function. We say that f is *positive semidefinite* (denoted $f \geq 0$) if $f(a_1, \ldots, a_n) \geq 0$ for all $(a_1, \ldots, a_n) \in \mathbb{K}^n$.

Theorem 1.3.2 (Hilbert's 17th Problem)**.** *If f is a positive semidefinite rational function over a real closed field $\mathbb{K}$, then f is a sum of squares of rational functions.*

Proof. Suppose f isn't a sum of squares, so by the dimension 0 positivstellensatz (Corollary 1.1.13) there is an ordering on $\mathbb{K}(X_1, \ldots, X_n)$ such that $f(X_1, \ldots, X_n)$ is negative. Let $\mathbb{F}$ be the real closure of $\mathbb{K}(X_1, \ldots, X_n)$ extending this order. Then since $f(X_1, \ldots, X_n) < 0$ in $\mathbb{K}(X_1, \ldots, X_n) \subset \mathbb{F}$, we have that

$$\mathbb{F} \models \exists (a_1, \ldots, a_n) \left(f(a_1, \ldots, a_n) < 0 \right)$$

By the Tarski transfer principle therefore

$$\mathbb{K} \models \exists (a_1, \ldots, a_n) \left(f(a_1, \ldots, a_n) < 0 \right)$$

but this is a contradiction to f being positive semidefinite. □

This is the starting point to the rest of our investigations in the first part of this thesis. However, it would be a shame not to mention some of the major results in the same direction as Hilbert's 17th. There we deal with polynomials that are positive everywhere — but what if they are only positive on a suitable subset of $\mathbb{K}^n$?

Definition 1.3.3. Let $\mathbb{K}$ be a real closed field and $f_1, \ldots, f_r \in \mathbb{K}[X_1, \ldots, X_n]$ polynomials. A *basic closed semialgebraic set (in $\mathcal{R}^n$)* is a set of the form

$$\mathcal{W}(f_1, \ldots, f_r) := \{(a_1, \ldots, a_n) \in \mathbb{K}^n \mid f_1(a_1, \ldots, a_n) \geq 0, \ldots, f_r(a_1, \ldots, a_n) \geq 0\}$$

A *semialgebraic set (in $\mathcal{R}^n$)* is a **finite** Boolean combination of basic closed semialgebraic sets.

These are just the definable subsets of $\mathbb{K}^n$ in **(RCF)**, so we observe two facts:

(a) The semialgebraic subsets in $\mathcal{R}$ are finite unions of points and intervals. This, in modern terminology, means that **(RCF)** is an *o-minimal theory*, and these are an active area of research with ties to Diophantine Analysis and Hodge Theory. See [Dri98] as the standard reference for o-minimality, and [JW15] for some of the ties to Diophantine Analysis.

(b) A consequence of quantifier elimination is the *(Tarski-Seidenberg) Projection Theorem*: the image of a semialgebraic set under the projection $\mathcal{R}^{n+1} \to \mathcal{R}^n$ is again semialgebraic. This can also be proven without quantifier elimination, but it is certainly non-trivial.

What are the functions that are obviously going to be non-negative on $\mathcal{W}(f_1, \ldots, f_r)$?

Definition 1.3.4. Let $\mathcal{A}$ be a commutative ring, and $f_1, \ldots, f_r \in \mathcal{A}$. The *preordering* $\mathcal{P}(f_1, \ldots, f_r)$ is the smallest set closed under addition and multiplication containing $f_1, \ldots, f_r$ and all sums of squares.

We can write what this is explicitly:

$$\mathcal{P}(f_1, \ldots, f_r) = \left\{ \sum_{e \in \{0,1\}^r} \sigma_e \cdot f_1^{e_1} \cdots f_r^{e_r} \mid \sigma_e \text{ is a sum of squares in } \mathcal{A} \right\}$$

For $\mathcal{A} = \mathbb{K}[X_1, \ldots, X_r]$, $\mathcal{P}(f_1, \ldots, f_r)$ contains the functions that must be non-negative on $\mathcal{W}(f_1, \ldots, f_r)$. As with the case of Hilbert's 17th, up to denominators this is all of them.

Theorem 1.3.5 (Nichtnegativstellensatz)**.** *Let $\mathbb{K}$ be a real closed field, and $f, f_1, \dots, f_r \in \mathbb{K}[X_1, \dots, X_n]$. Then $f \geq 0$ on $\mathcal{W}(f_1, \dots, f_r)$ if and only if there are polynomials $p, q \in \mathcal{P}(f_1, \dots, f_r)$ where $p \neq 0$ and $e \in \mathbb{N}$ such that*

$$pf = f^{2e} + q$$

In particular, $\mathcal{W}(f_1, \dots, f_r) = \varnothing$ if and only if $-1 \in \mathcal{P}(f_1, \dots, f_r)$.

Compare this to Hilbert's Nullstellensatz, which provides an algebraic certificate for solvability of a polynomial equation over an algebraically closed field — the system $0 = f_1(X_1, \dots, X_n) = \cdots = f_1(X_1, \dots, X_r)$ has a solution if and only if $1 \notin \langle f_1, \dots, f_r \rangle$, the ideal generated by the f_i. The nichtnegativstellensatz tells us that there is a solution to a system of polynomial inequalities (over a real closed field) if and only if $-1 \notin \mathcal{P}(f_1, \dots, f_r)$.

Notice also that the case $r = 0$ is precisely Hilbert's 17th problem. The most general case is due to Krivine and Stengle (independently), which deals with general semialgebraic sets.

Sometimes we can even get rid of the denominator p. The first such result is due to Schmüdgen, and was the starting point for a wave of new developments.

Theorem 1.3.6 (Schmüdgen's Positivstellensatz, [Sch91])**.** *Let $f, f_1, \dots, f_r \in \mathbb{R}[X_1, \dots, X_n]$ be polynomials such that $\mathcal{W}(f_1, \dots, f_r) \subset \mathbb{R}^n$ is bounded. Then $f > 0$ on $\mathcal{W}(f_1, \dots, f_r)$ if and only if $f \in \mathcal{P}(f_1, \dots, f_r)$.*

Notice that this holds only for $\mathbb{R}$, not arbitrary RCFs, and that boundedness of $\mathcal{W}(f_1, \dots, f_r)$ and strict positivity of f are needed in general. Many other Positivstellensätze have been proven or disproven (Putinar, Vasilescu, ...), see [Mar08], [Net16], and the references therein. For example, one can also prove a Positivstellensatz for definable C^r-functions on o-minimal structures expanding some real closed field — see [AAB02].

Chapter 2

Non Commutative Real Algebraic Geometry

We have seen so far how positivity in some commutative object can sometimes be deduced from algebraic certificates — our next task is to generalise this to the noncommutative setting.

Firstly, our rings will come with an involution, and we will be dealing with *hermitian squares*. This is the natural thing to consider if we are interested in notions of positivity — for example for any $z \in \mathbb{C}$, z^2 isn't anything nice, but the hermitian square is $\overline{z}z = |z|^2 \geq 0$.

2.1 $\star$-algebras

In all that follows, $k \in \{\mathbb{R}, \mathbb{C}\}$. In the theory below we will mostly be following Cimprič ([Cim09]), Schmüdgen ([Sch09]), and Ozawa ([Oza12]). The $\mathbb{R}$ and $\mathbb{C}$ theories are similar in many ways and so we typically won't comment on this distinction, see for example section 7 in [Oza12].

Definition 2.1.1. An algebra $\mathcal{A}$ over k is a $*$-*algebra* if it is equipped with an involution $(\ )^* : \mathcal{A} \to \mathcal{A}$ satisfying

(a) $(xy)^* = y^* x^*$, and

(b) $(\lambda x + y)^* = \bar{\lambda} x^* + y^*$ for all $\lambda \in k$

(where $\bar{\lambda}$ denotes complex conjugation). If $\mathcal{A}$ is unital then we deduce also the requirement

(c) $1^* = 1$.

We will always denote the unit (if it exists) in an algebra by 1, and we reserve i for the imaginary unit.

As standard we consider the sets of *hermitian* and *unitary/orthogonal* elements:

$$\mathcal{A}^h := \{a \in \mathcal{A} \mid a^* = a\} \quad \& \quad U(\mathcal{A}) := \{a \in \mathcal{A} \mid a^* a = 1 = aa^*\}$$

Every $x \in \mathcal{A}$ decomposes uniquely as a sum of a hermitian element and a skew hermitian element, and $\mathcal{A}^h$ is itself an $\mathbb{R}$-vector space.

Definition 2.1.2. A subset $\mathcal{A}^+ \subset \mathcal{A}^h$ is a *($*$-)positive cone* (or *quadratic module*, or *m-admissable wedge* in [Sch90]) if it satisfies:

(a) $-1 \notin \mathcal{A}$;

(b) $\mathbb{R}_{\geq 0} \cdot 1 \subset \mathcal{A}^+$;

(c) $\lambda a + b \in \mathcal{A}^+$ for $a, b \in \mathcal{A}^+$ and $\lambda \in \mathbb{R}_{\geq 0}$;

(d) $x^* a x \in \mathcal{A}^+$ for every $a \in \mathcal{A}^+$ and $x \in \mathcal{A}$.

This should be reminiscent of the (proper) cones considered in section 1.1, with the appropriate noncommutative interpretation of 'squares' as 'hermitian squares' previously alluded to. When discussing a unital $*$-algebra $\mathcal{A}$, we assume it comes with some distinguished positive cone.

Example 2.1.3. $\mathbb{C}$ with conjugation is a $*$-algebra, and it is easy to show that the only possible $*$-positive cones are sets of the form $\{z \in \mathbb{C} \mid -\theta_1 \leq \arg z \leq \theta_2\}$ for $\theta_1, \theta_2 \in [0, \pi]$ (if both = 0 we get $\mathbb{R}_{\geq 0}$, if both = π we get $\mathbb{C}\}$, and $\mathbb{R}$.

Given a positive cone $\mathcal{A}^+$ we define as we did in chapter 1 a partial order on $\mathcal{A}^h$ by setting $a \preccurlyeq b$ if $b - a \in \mathcal{A}^+$, and say that b is *positive* if $b \in \mathcal{A}^+$.

With this structure $\mathcal{A}$ becomes an *ordered algebra.* If we forget the multiplicative structure (and requirement (d) in definition 2.1.2) $\mathcal{A}$ is simply an *ordered vector space.*

As in the case of positive cones in fields, there is a unique minimal cone.

Example 2.1.4. For any unital $*$-algebra $\mathcal{A}$, the set of sums of hermitian squares

$$\Sigma^2 \mathcal{A} := \Big\{ \sum_{i=1}^{n} \xi_i^* \xi_i \mid n \in \mathbb{N}, \xi_i \in \mathcal{A} \Big\}$$

is a $*$-positive cone.

Čimpric considers $\mathbb{Q}$-algebras, and this suffices for many purposes; but we will later use a completion procedure (Corollary 2.5.1) that will introduce $\mathbb{R}$ anyway, so we don't lose much by considering only $\mathbb{R}$ (or $\mathbb{C}$) algebras (alternatively, given a $*$-algebra $\mathcal{A}$ over $\mathbb{Q}$ consider simply $\mathcal{A}_{\mathbb{R}} := \mathcal{A} \otimes_{\mathbb{Q}} \mathbb{R}$).

A priori one might be worried that this drastically changes the cone $\Sigma^2\mathcal{A}$, but notice that any positive rational p/q is a sum of squares, indeed

$$\frac{p}{q} = \sum_{i=1}^{pq} \frac{1}{q^2}$$

and so considering sums of squares over $\mathbb{Q}$ is not manifestly different than over $\mathbb{R}$.

Remark 2.1.5. In section 3.1 we will consider sums of hermitian squares in commutative algebras over $k = \mathbb{C}$. In this case, a sum of hermitian squares is in fact a sum of squares of hermitian elements is (that is, $\Sigma^2\mathcal{A} = \Sigma^2(\mathcal{A}^h)$), as was pointed out to the author by Andreas Thom.

More generally, let $\mathcal{A}$ be a $*$-algbera over $\mathbb{C}$, and $a \in \mathcal{A}$ a *normal* element (that is, $aa^* = a^*a$). Then

$$a^*a = \left(\frac{a+a^*}{2} + i\frac{a-a^*}{2i}\right)^*\left(\frac{a+a^*}{2} + i\frac{a-a^*}{2i}\right) = \left(\frac{a+a^*}{2}\right)^2 + \left(\frac{a-a^*}{2i}\right)^2$$

As such when we will discuss commutative $*$-algebras over $\mathbb{C}$, we will forget the existence of an involution.

Proposition 2.1.6. *Let $\mathcal{A}$ be a unital $*$-algebra. Then*

$$\mathcal{A}^h = \mathcal{A}^+ - \mathcal{A}^+$$

Proof. If $a \in \mathcal{A}^h$ then $a = \frac{1}{2}\big((1+a)^*(1+a) - (1+a^*a)\big)$. □

This is why we included $-1 \notin \mathcal{A}^+$ as part of the definition, since otherwise $\mathcal{A}^+ = \mathcal{A}^h$. From now on, any unital $*$-algebra will come equipped with a positive cone.

Example 2.1.7 (Key example). Let Γ be a discrete group. Then its *group algebra* $k[\Gamma]$ is the set of functions $f : \Gamma \to k$ with finite support. To define the product we linearly extend the equation $\delta_g * \delta_h := \delta_{gh}$ where δ_g denotes the Dirac delta at $g \in \Gamma$; this gives the *convolution product* which for $a, b \in k[\Gamma]$ and $g \in \Gamma$ is given by

$$a * b(g) := \sum_{h\in\Gamma} a(gh^{-1})b(h)$$

The involution is given by $a^*(g) := \overline{a(g^{-1})}$.

Under the identification $\Gamma \ni g \mapsto \delta_g$ we will typically identify $k[\Gamma]$ with the set of formal finite sums of elements of Γ with coefficients in k. We will always be interested in $k[\Gamma]$ equipped with the positive cone $\Sigma^2 k[\Gamma]$.

Example 2.1.8. The algebra $M_n(k)$, where for $A = (a_{i,j})_{i,j} \in M_n(k)$ the involution is $A^* = (\overline{a_{j,i}})_{i,j}$ is a $*$-algebra with positive cone $\Sigma^2 M_n(k)$. It is standard that the matrices $A \in \Sigma^2 M_n(k)$ are characterised by the fact that for any $x \in k^n$, $x^*Ax \geq 0$, where x^* denotes conjugate transpose. As such they are known as *positive semidefinite matrices.*

Definition 2.1.9 (Tensor product)**.** Let $(\mathcal{A}, \mathcal{A}^+)$, $(\mathcal{B}, \mathcal{B}^+)$ be two $*$-algebras. We can endow $\mathcal{A} \otimes_k \mathcal{B}$ with the involution $(x \otimes y)^* = (x^* \otimes y^*)$ and the $*$-positive cone $(\mathcal{A} \otimes \mathcal{B})^+$ generated by $\{a \otimes b \mid a \in \mathcal{A}^+, b \in \mathcal{B}^+\}$.

Example 2.1.10. If $\mathcal{B} = M_n(k)$ is a matrix algebra then the tensor product has a particularly nice characterisation; namely $\mathcal{A} \otimes M_n(k) =: M_n(\mathcal{A})$, the algebra of $n \times n$ matrices with coefficients in $\mathcal{A}$. The involution for $A = (a_{ij})_{ij} \in M_n(\mathcal{A})$ is then given by $A^* = (a^*_{ji})_{ij}$. We will consider matrix algebras with entries in $\mathcal{A}$ frequently in the sequel, and we refer to matrices in $(M_n(\mathcal{A}))^+$ as *positive semidefinite matrices* in analogy with the $M_n(k)$ case.

Example 2.1.11. If $\mathbb{R} \subset \mathbb{K}$ is an extension of real closed fields, then $\mathbb{C} \subset \mathbb{K}[i]$ is an extension of algebraically closed fields, and $\mathbb{K}[i]$ is a unital $*$-algebra over $\mathbb{C}$ with $\mathbb{K}[i]^h = \mathbb{K}$. Notice that $\Sigma^2\mathbb{K}[i] = \mathbb{K}_{\geq 0}$ and we find that $(\mathbb{K}[i] \otimes M_n(\mathbb{C}))^+$ is the set of positive semidefinite hermitian matrices over $\mathbb{K}[i]$.

By the Tarski transfer principle (Corollary 1.2.12), being positive semidefinite has the same characterisation over $\mathbb{K}[i]$ as it does over $\mathbb{C}$.

2.2 Positivity

Definition 2.2.1. Let $(\mathcal{A}, \mathcal{A}^+)$ and $(\mathcal{B}, \mathcal{B}^+)$ be ordered $*$-algebras. A linear map $\varphi : \mathcal{A} \to \mathcal{B}$ is $\mathcal{A}^+$*-positive* if it respects the involution and positivity. That is for all $a \in \mathcal{A}$, $\varphi(a^*) = \varphi(a)^*$, and $\varphi(\mathcal{A}^+) \subset \mathcal{B}^+$. It is called *faithful* if $\varphi(a^*a) = 0$ implies that $a = 0$.

If φ is positive, and in addition is an algebra homomorphism, we say it is a *homomorphism of ordered* $*$*-algebras.*

There is an important family of $*$-algebras to consider: the C^* algebras.

Definition 2.2.2. Let $\mathcal{A}$ be a C^*-algebra. Then an element $a \in \mathcal{A}$ is *positive*, denoted $a \geq 0$, if $a \in \mathcal{A}^h$ and $\mathrm{Sp}_{\mathcal{A}}(a) \subset [0, \infty)$, where $\mathrm{Sp}_{\mathcal{A}}(a)$ denotes the spectrum of a in $\mathcal{A}$.

The set of positive elements clearly forms a positive cone, which has an easy characterisation.

Proposition 2.2.3 (C^*-algebra positivstellensatz)**.** *Let $\mathcal{A}$ be a C^*-algebra (over k) and $a \in \mathcal{A}^h$. Then the following are equivalent:*

(a) a is positive;

(b) $a \in \Sigma^2\mathcal{A}$ (in fact, $a = b^*b$ for some $b \in \mathcal{A}$);

(c) For all $R \geq \|a\|$ we have $\|R - a\| \leq R$;

(d) For some $R \geq \|a\|$ we have $\|R - a\| \leq R$.

In addition, if $\mathcal{A}$ is unital then $a \geq 0$ is invertible if and only if there is some $\varepsilon > 0$ such that $a \geq \varepsilon \cdot 1$.

Proof. Consider the C^*-algebra generated by a; this is a commutative and so under the Gelfand representation is identified with the algebra $C_0(X)$ of functions vanishing at infinity on a locally compact Hausdorff space X. Then $a \in C_0(X)$ being positive just means that $a(x) \geq 0$ for all $x \in X$, and hence we get the equivalence of (a) and (b) by taking the square root function $\sqrt{a}$.

(a) $\Rightarrow$ (c) follows since for any $x \in X$ we have $0 \geq a(x) - R \geq -R$ and so $\|a(x) - R\| \leq R$. (c) $\Rightarrow$ (d) is immediate, and to show (d) $\Rightarrow$ (a) we must have that for any $x \in X$, $|a(x) - R| \leq \|a - R\| \leq R$ and hence $a(x) \geq 0$.

To prove the 'in addition' note that firstly if $\mathcal{A}$ is unital then the space X is compact. Hence if $a \geq \varepsilon \cdot 1$ then the inverse a^{-1} is a well-defined function everywhere. Conversely, if a is invertible and $a \geq 0$, let $\varepsilon := \inf_{x \in X}\{a(x)\}$. Positivity of a ensures that $\varepsilon \geq 0$, and invertibility (where the inverse is continuous) ensures that $\varepsilon > 0$, consequently $a - \varepsilon \cdot 1 \geq 0$. ☐

Recall that by the Gelfand-Naimark Theorem, every C^*-algebra $\mathcal{A}$ can in fact be realised as a C^*-subalgebra of the bounded linear operators on some Hilbert space $\mathcal{H}$ — that is we have a faithful $*$-homomorphism $\pi : \mathcal{A} \to \mathbb{B}(\mathcal{H})$. Under this identification, $a \in \mathcal{A}$ is positive if and only if for all $\xi \in \mathcal{H}$, $\langle \pi(a)\xi, \xi \rangle \geq 0$.

More generally we can define a positive cone in $\mathcal{L}(\mathcal{H})$, the $*$-algebra of linear operators on $\mathcal{H}$, to be

$$\mathcal{L}(\mathcal{H})^+ := \{T \in \mathcal{L}(\mathcal{H})^h \mid \langle T\xi, \xi \rangle \geq 0 \text{ for all } \xi \in \mathcal{H}\}$$

In the case that $\mathcal{H} = k^n$ is finite dimensional, this of course coincides with our previous definition of positive semidefinite matrices.

Definition 2.2.4. An $\mathcal{A}^+$*-positive* $*$*-representation* of $(\mathcal{A}, \mathcal{A}^+)$ is a homomorphism of ordered $*$-algebras $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H}_\pi)$ for some Hilbert space $\mathcal{H}_\pi$.

Two such representations π_1, π_2 are said to be *(unitarily) equivalent* if there is an isometric linear mapping $U : \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2}$ such that $\pi_2(a) = U\pi_1(a)U^{-1}$. A representation π is *irreducible* if $\mathcal{H}_\pi$ has no proper nontrivial closed $\mathcal{A}$-invariant subspaces.

Notice that any $*$-representation is automatically $\Sigma^2\mathcal{A}$-positive.

Definition 2.2.5. Let $\mathcal{R}$ be a family of (equivalence classes) of $*$-representations of $\mathcal{A}$. We say that $a \in \mathcal{A}^h$ is *positive semidefinite on* $\mathcal{R}$ if $\pi(a) \geq 0$ for all $\pi \in \mathcal{R}$. When the choice of $\mathcal{R}$ is obvious, we will denote this by $a \geq 0$.

If $\mathcal{R}$ is any family of $*$-representations of $\mathcal{A}$, then $\mathcal{A}(\mathcal{R})^+ := \{a \text{ is positive semidefinite on } \mathcal{R}\}$ is a $*$-positive cone (provided that $-1 \notin \mathcal{A}(\mathcal{R})^+$). We wish to understand how this might relate to an algebraically defined $*$-positive cone, such as $\Sigma^2\mathcal{A}$.

Example 2.2.6. For $\mathcal{A} = \mathbb{R}[X_1, \dots, X_d]$ with trivial involution consider the family $\mathcal{R} = \mathbb{R}^d$ given by evaluation. That is, $\mathcal{R} = \{\pi_t : t \in \mathbb{R}^d\}$ where $\pi_t(a) := a(t)$ for $a \in \mathcal{A}$. Hence being positive semidefinite in our new setting is simply a natural generalisation of the commutative case.

Let $(\mathcal{A}, \mathcal{A}^+)$ be a $*$-algebra. We define $\mathcal{R}(\mathcal{A}^+)$ to be *all* the $\mathcal{A}^+$-positive $*$-representations. In the case that $\mathcal{A}^+ = \Sigma^2\mathcal{A}$, we have that $\mathcal{R}(\Sigma^2\mathcal{A})$ is *all* the $*$-representations since they are automatically positive with respect to $\Sigma^2\mathcal{A}$.

Definition 2.2.7. In either case, $\mathcal{R}(\mathcal{A}^+)$ is called the *dual cone to* $\mathcal{A}^+$ and $\mathcal{A}(\mathcal{R}(\mathcal{A}^+))^+$ is the *double dual cone of* $\mathcal{A}^+$.

Clearly $\mathcal{A}^+ \subset \mathcal{A}(\mathcal{R}(\mathcal{A}^+))^+$; that is if $a \succcurlyeq 0$, then $a \geq 0$.

Remark 2.2.8. In general, the collection of all representations of an Archimedean algebra isn't a set, but we can make it one by restricting the cardinality of the target Hilbert space. For example if $\mathcal{A} = k[\Gamma]$ where Γ is a countable discrete group, we can ask for our target Hilbert spaces to all be separable. We won't worry about this in the sequel.

2.3 States and the Gelfand-Naimark-Segal Construction

Given a $*$-algebra $\mathcal{A}$ with positive cone $\mathcal{A}^+$, it makes sense to consider special types of linear functionals.

Definition 2.3.1. We say a map $\varphi : \mathcal{A} \to k$ is

(i) an *($\mathcal{A}^+$-positive) state* if it is a positive linear functional of norm 1;

(ii) a *pure state* if it is a state and if ψ is any state satisfying $\psi(a^*a) \leq \varphi(a^*a)$ for all $a \in \mathcal{A}$, then ψ is a multiple of φ;

(iii) a *trace* if it is a state and in addition for any $a, b \in \mathcal{A}$, $\varphi(ab) = \varphi(ba)$.

We will denote the set of states (endowed with the weak-$*$ topology) by $\mathcal{S}(\mathcal{A})$, and the set of pure states by $\mathcal{P}(\mathcal{A})$.

Example 2.3.2. Let $(\pi, \mathcal{H}_\pi)$ be an $\mathcal{A}^+$-positive $*$-representation of $\mathcal{A}$, and $\xi \in \mathcal{H}_\pi$ a unit vector. Then $\varphi(\cdot) = \langle \pi(\cdot)\xi, \xi\rangle$ is an $\mathcal{A}^+$-positive state of $\mathcal{A}$.

That is, states can come from $*$-representations. The famous Gelfand-Naimark-Segal construction shows that this is basically the only way that states can arise. In the context where $\mathcal{A} = k[\Gamma]$ is a group algebra, the state associated to a representation (at least when restricted to the group elements) is often referred to as the *matrix coefficient*.

Theorem 2.3.3 (GNS construction)**.** *Let φ be a state on a $*$-algebra $\mathcal{A}$. Then there is a $*$-representation π of $\mathcal{A}$ on some Hilbert space $\mathcal{H}_\pi$, and a cyclic unit vector $\xi \in \mathcal{H}_\pi$ such that $\varphi(\cdot) = \langle \pi(\cdot)\xi, \xi\rangle$.*

In our context, there is no continuity requirement on representations so this is easy. In the more general case of locally compact groups the GNS construction still works, but extra care is needed to verify continuity (in fact, a measurable version of GNS also exists). Similarly, we will assume that $\mathcal{A}$ is unital — the general case is slightly more involved.

Proof sketch when $\mathcal{A}$ is unital. Define the sesquilinear positive form

$$\langle \cdot, \cdot \rangle_\varphi : \mathcal{A} \times \mathcal{A} \to k : (a, b) \mapsto \varphi(b^* a)$$

Let $\mathcal{N} = \{a \in \mathcal{A} \mid \varphi(a^* a) = 0\}$, by the Cauchy-Schwarz inequality this is a vector subspace. Then let $\mathcal{H}$ be the completion of $\mathcal{A}$ with respect to the norm induced by $\langle \cdot, \cdot \rangle_\varphi$; indeed, $\langle a + \mathcal{N}, b + \mathcal{N} \rangle_\varphi := \varphi(b^* a)$ is well defined on the quotient space.

Left multiplication by an element of a on $\mathcal{A}$ descends to $\mathcal{A}/\mathcal{N}$ and extends to the completion, giving us the unitary representation $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$. Indeed if $b \in \mathcal{N}$ then

$$|\varphi((ab)^*(ab))|^2 = |\varphi(b^*(a^* ab))|^2 \leq \varphi(b^* b)\varphi((a^* ab)^* a^* ab) = 0$$

and so $ab \in \mathcal{N}$ too.

Finally, the vector $\xi = 1 + \mathcal{N}$ is clearly cyclic by the definition of $\mathcal{H}$, and $\varphi(a) = \langle a + \mathcal{N}, 1 + \mathcal{N} \rangle_\varphi = \langle \pi(a)\xi, \xi \rangle_\varphi$. □

Remark 2.3.4. We used the Cauchy-Schwarz inequality twice — once to show that $\mathcal{N}$ is a vector subspace, and another time to show that it is in fact a left ideal and the $\mathcal{A}$-action is defined on $\mathcal{A}/\mathcal{N}$. We will return to this point later in section 2.9.

We collect without proof some more facts, which can be found in any reference on C^*-algebras. In the group algebra case, chapter 1 of [EW23] provides detailed proofs in all generalities.

Theorem 2.3.5. *The following facts are true*

(a) $\mathcal{S}(\mathcal{A})$ is convex;

(b) $\mathcal{S}(\mathcal{A})$ is weak-$\star$ compact if and only if $\mathcal{A}$ is unital;

(c) The extremal points of $\mathcal{S}(\mathcal{A})$ are precisely $\mathcal{P}(\mathcal{A})$;

(d) A state is pure if and only if the corresponding GNS representation is irreducible.

In particular the Krein-Milman Theorem and Choquet's Theorem help us approximate any state by pure states, and hence any representation by irreducible ones. The statement of (d) and the definition we gave for pure states should be reminiscent of Schur's lemma, we won't discuss this further.

Notice also that by the GNS construction, we can view any family of representations $\mathcal{R}$ as a family of states on $\mathcal{A}$. Thus following on from remark 2.2.8, to get around issued of cardinality we can just consider the set of GNS representations. This is a set, since in particular $\mathcal{S}(\mathcal{A})$ is a subset of the dual space of $\mathcal{A}$.

2.4 Bounded Elements and Archimedean Algebras

When doing Functional Analysis we are interested in *bounded* operators on a Hilbert space $\mathcal{H}$ say (because these are precisely the continuous ones). Say $T : \mathcal{H} \to \mathcal{H}$ is a bounded operator, then in particular its spectrum is bounded and for some $R \geq 0$ the operator $R^2 \cdot 1 - T^* T$ is positive. This motivates the following definition, for a general unital $\star$-algebra $\mathcal{A}$ with positive cone $\mathcal{A}^+$.

Definition 2.4.1. The set of *bounded elements* is

$$\mathcal{A}^b := \{a \in \mathcal{A} \mid \exists R > 0 \text{ such that } a^* a \preccurlyeq R1\}$$

The set of *infinitesimal elements* is

$$\mathcal{A}^0 := \{a \in \mathcal{A} \mid a^* a \preccurlyeq \varepsilon 1 \text{ for all } \varepsilon > 0\}$$

Definition 2.4.2. An *Archimedean algebra* (*semi-pre-C^*-algebra* in [Oza12]) is a unital $\star$-algebra $\mathcal{A}$ that comes equipped with a $\star$-positive cone $\mathcal{A}^+$ satisfying the *Combes axiom* (also called the *Archimedean property*) that $\mathcal{A}^b = \mathcal{A}$.

As with the spectrum of elements in the C^*-algebra $\mathbb{B}(\mathcal{H})$, it is worth trying to quantify this.

Definition 2.4.3. Let $\mathcal{A}$ be a $*$-algebra over k. Then for any $a \in \mathcal{A}$ let

$$||a|| := \inf\{R \in \mathbb{R}_{\geq 0} \mid a^* a \preccurlyeq R^2 \cdot 1\}$$

where we define the infimum over the empty set to be ∞. So $\mathcal{A}$ is Archimedean if and only if $||\cdot||$ is real valued.

For hermitian elements, there is an easy way to check if they are bounded.

Lemma 2.4.4 ([Cim09], Lemma 3.1). *Let $\mathcal{A}$ be a $*$-algebra over k with positive cone $\mathcal{A}^+$. For every $b \in \mathcal{A}^h$ and every $R \in \mathbb{R}_{\geq 0}$, we have*

$$R^2 \cdot 1 - b^2 \in \mathcal{A}^+ \Leftrightarrow R \cdot 1 \pm b \in \mathcal{A}^+$$

Proof. If $R \cdot 1 \pm b \in \mathcal{A}^+$ then we calculate

$$R^2 \cdot 1 - b^2 = \frac{1}{2R}\big((R\cdot 1 - b)^*(R\cdot 1 + b)(R\cdot 1 - b) + (R\cdot 1 + b)^*(R\cdot 1 - b)(R\cdot 1 + b)\big) \in \mathcal{A}_+$$

where we use that b is Hermitian. Conversely, if $R^2 \cdot 1 - b^2 \in \mathcal{A}_+$ then

$$R \cdot 1 \pm b = \frac{1}{2R}\big((R\cdot 1 \pm b)^2 + (R^2 \cdot 1 - b^2)\big) \in \mathcal{A}^+$$

□

Corollary 2.4.5. *A $*$-algebra $\mathcal{A}$ is Archimedean if and only if for any $b \in \mathcal{A}^h$, there is some $R > 0$ such that $b + R \cdot 1 \in \mathcal{A}^+$.*

The function $||\cdot||$ is remarkably well-behaved.

Theorem 2.4.6 ([Cim09], Theorem 3.2). *Let $\mathcal{A}$ be a $*$-algebra over k. Then for every $a, b \in \mathcal{A}$ and $\lambda \in k$ we have*

(a) $||\lambda a|| = |\lambda| \cdot ||a||$;

(b) $||a|| = ||a^*||$;

(c) $||ab|| \leq ||a|| \cdot ||b||$;

(d) $||a + b|| \leq ||a|| + ||b||$;

(e) $||a^* a|| = ||a||^2$;

(f) $||a||^2 \leq ||a^* a + b^* b||$.

Proof. (a) is trivial, as are (b),(c), and (d) if either $||a|| = \infty$ or $||b|| = \infty$. So assume this isn't the case, and pick any $R, S \in \mathbb{R}_{\geq 0}$ such that $||a|| < R$, $||b|| < S$.

(b) We just need to prove that $||a^*|| \leq ||a||$, so it suffices to show $||a^*|| \leq R$. We calculate

$$\Big(\frac{R^2}{2}\Big)^2 - \Big(\frac{R^2}{2} - aa^*\Big)^2 = a(R^2 - a^* a)a^* \in \mathcal{A}^+$$

we therefore see that $\frac{R^2}{2} \pm (\frac{R^2}{2} - aa^*) \in \mathcal{A}^+$ by Lemma 2.4.4, and hence $||a^*|| \leq R$.

(c) We need to show that $||ab|| \leq RS$ and this follows from the computation

$$(RS)^2 - (ab)^*(ab) = S(R^2 - b^* b)S + b^*(S^2 - a^* a)b \in \mathcal{A}^+$$

(d) By (b) and (c) we have that $\|a^*b\| = \|b^*a\| < RS$ and so

$$(2RS)^2 - (a^*b + b^*a)^2 = 2(R^2S^2 - a^*bb^*a) + 2(R^2S^2 - b^*aa^*b) + (a^*b - b^*a)^*(a^*b - b^*a) \in \mathcal{A}^+$$

By Lemma 2.4.4 we get that $2RS \pm (a^*b + b^*a) \in \mathcal{A}^+$ and so

$$(R+S)^2 - (a \pm b)^*(a \pm b) = R^2 - a^*a + S^2 - b^*b + 2RS \pm (a^*b + b^*a) \in \mathcal{A}^+$$

and so $\|a \pm b\| \leq R + S$.

(e) Follows immediately from Lemma 2.4.4. Indeed, we have that $R \pm a^*a \in \mathcal{A}^+$ if and only if (since a^*a is self-adjoint) $R^2 - (a^*a)(a^*a) \in \mathcal{A}^+$, and so the result follows.

(f) If $\|a^*a + b^*b\| < R$ then by Lemma 2.4.4 again we have that $R - (a^*a + b^*b) \in \mathcal{A}^+$. Since $b^*b \in \mathcal{A}^+$ it follows that $R - a^*a \succcurlyeq 0$ and so $\|a\| \leq \sqrt{R}$.

□

Remark 2.4.7. Condition (f) in this Theorem is important in the theory of real C^*-algebras. Indeed if $\mathcal{A}$ is a real C^*-norm that satisfies this property, then this norm extends to a C^*-norm on the complexification, see Theorem 1 in [Pal70]; but we won't comment on this further.

Corollary 2.4.8. *$\mathcal{A}^b$ is a subalgebra of $\mathcal{A}$, and $\mathcal{A}^0$ is a two sided $*$-ideal in $\mathcal{A}^b$.*

Definition 2.4.9. Let $\mathcal{A}$ be a (unital) $*$-algebra. We say that a subset $S \subset \mathcal{A}$ *generates $\mathcal{A}$ as a (unital) $*$-algebra* if the smallest (unital, with the same unit) $*$-subalgebra of $\mathcal{A}$ containing S is $\mathcal{A}$ itself. Alternatively, every element in $\mathcal{A}$ can be written as a (non-commutative) polynomial in $S \cup S^* \cup \{1\}$ with coefficients in k.

So we see that if $\mathcal{A}$ is generated as a unital $*$-algebra by some set S, then $S \subset \mathcal{A}^b$ implies that $\mathcal{A} = \mathcal{A}^b$.

Definition 2.4.10. We say that $a \in \mathcal{A}$ is a *partial isometry* if $aa^*a = a$.

Partial isometries are bounded, indeed

$$a^*a \preccurlyeq a^*a + (1 - a^*a)^*(1 - a^*a) = a^*a + 1 - 2a^*a + a^*(aa^*a) = 1 \tag{2.1}$$

Corollary 2.4.11. *If $\mathcal{A}$ is generated as a unital $*$-algebra by partial isometries and $-1 \notin \mathcal{A}^+$, then it is Archimedean.*

Example 2.4.12. $M_n(k)$ is generated by δ_{ij}, and these are partial isometries — hence $M_n(k)$ is Archimedean.

Example 2.4.13. Let Γ be a discrete group, and consider its group algebra $k[\Gamma]$. It is generated by unitary elements (the $g \in \Gamma$) and so is Archimedean. The infinitesimal ideal of $k[\Gamma]$ is $\{0\}$.

In both of these examples in fact the generation happens as a vector space, but this needn't hold in general. For example $M_n(k)$ is generated as unital $*$-algebra by the δ_{ij} with $i \geq j$.

It is worthwhile to prove that $k[\Gamma]$ is Archimedean more directly, since it allows us to bound the norm of any element.

Lemma 2.4.14 ([Cim09], Example 3)**.** *$k[\Gamma]$ is an Archimedean algebra. In particular, for any $a \in k[\Gamma]$, $a^*a \preccurlyeq \|a\|_1^2 \cdot 1$ (where $\|a\|_1 := \sum_g |a_g|$ is the ℓ^1-norm).*

Proof. Choose any total order on the support of a (or just on all of Γ). We now calculate

$$\begin{aligned}
\|a\|_1^2 \cdot 1 - a^*a &= \sum_{g \neq h} |a_g| \cdot |a_h| \cdot 1 - (a_g g)^*(a_h h) \\
&= \sum_{g<h} \left(2|a_g a_h| \cdot 1 - \overline{a_g} a_h g^{-1}h - a_g \overline{a_h} h^{-1} g\right) \\
&= \sum_{g<h} |a_g a_h| \left(1 - \frac{\overline{a_g} a_h}{|a_g a_h|} g^{-1}h\right)^* \left(1 - \frac{\overline{a_g} a_h}{|a_g a_h|} g^{-1}h\right) \in \Sigma^2 k[\Gamma]
\end{aligned} \tag{$*$}$$

□

Remark 2.4.15. At first sight this seems like a lucky guess, but it follows by examining the proof of Lemma 2.4.4. Indeed, by Corollary 2.4.5 it suffices to prove that for a hermitian $b = \sum_g b_g g \in k[\Gamma]^h$, $b + \|b\|_1 \cdot 1 \in \Sigma^2 k[\Gamma]$ where $\|b\|_1$ is the ℓ^1 norm of $(b_g)_g$. Since b is hermitian in fact $b = \sum_g \frac{1}{2}(b_g g + \overline{b_g} g^*)$. Let $z = b_g/|b_g|$, then we calculate

$$b + \|b\|_1 \cdot 1 = b + \sum_g |b_g| \cdot 1 = \sum_g \frac{|b_g|}{2}(2 + zg + (zg)^*) = \sum_g \frac{|b_g|}{2}((1+zg)^*(1+zg)) \in \Sigma^2 k[\Gamma]$$

With $b = -a^* a$ for any $a \in \mathcal{A}$, this is exactly the computation ($*$).

We will shortly see (Proposition 2.5.14) that the tensor product of two Archimedean algebras is Archimedean in full generality — but again using the technique of Corollary 2.4.5 we are able to conclude this explicitly in a special case (in fact, the only one that will concern us anyway).

Proposition 2.4.16. *If $\mathcal{A}$ is Archimedean and is generated (as a vector space) by partial isometries, then $M_n(\mathcal{A})$ is Archimedean for all $n \in \mathbb{N}$. In particular $M_n(k[\Gamma])$ is Archimedean.*

Proof. We condition on n, and for $n = 1$ as above we have that for $b \in \mathcal{A}^h$, $b + \|b\|_1 \cdot 1 \in \mathcal{A}^+$ (here $\|b\|_1$ denotes the ℓ^1-norm with respect to the basis of partial isometries), where we use the fact that for a partial isometry x, $x^* x \preccurlyeq 1$ (see 2.1). For $n = 2$, it clearly suffices to consider matrices of the form $A = \left(\begin{smallmatrix} 0 & a \\ a^* & 0 \end{smallmatrix}\right) \in M_2(\mathcal{A})^h$ where $a = \sum a_k x_k$ with the x_k partial isometries.

$$\begin{aligned} A + \|a\|_1 I_2 &= \sum_k |a_k| \left(I_2 + \left(\begin{smallmatrix} 0 & y_k \\ y_k^* & 0 \end{smallmatrix}\right)\right) \qquad \left(y_k = \tfrac{a_k}{|a_k|} x_k\right) \\ &\succcurlyeq \sum_i \frac{|a_k|}{2} \left(I_2 + \left(\begin{smallmatrix} 0 & y_k \\ y_k^* & 0 \end{smallmatrix}\right)\right)^* \left(I_2 + \left(\begin{smallmatrix} 0 & y_k \\ y_k^* & 0 \end{smallmatrix}\right)\right) \\ &\in \Sigma^2 M_2(\mathcal{A}) \end{aligned}$$

For $n \geq 3$ we have a matrix such that $A^* = A$, and so we can decompose $A = (a_{i,j}) = \sum_{i,j} A_{i,j}$ where $A_{i,j} = \frac{1}{2}(a_{i,j}\delta_{i,j} + a_{j,i}\delta_{j,i})$. For $i \neq j$, let $I_{i,j} := \delta_{i,i} + \delta_{j,j}$, then we see from the $n = 2$ case that

$$A_{i,j} + \|A_{i,j}\|_1 I_{i,j} \in \Sigma^2 M_n(\mathcal{A})$$

If $i = j$ then $A_{i,i} + \|A_{i,i}\|_1 \delta_{i,i} \in \Sigma^2 M_n(\mathcal{A})$, and thus we have

$$\begin{aligned} A + \|A\|_1 I_n &= \sum_{i,j} (A_{i,j} + \|A_{i,j}\|_1 I_n) \\ &\succcurlyeq \sum_{i,j} (A_{i,j} + \|A_{i,j}\|_1 I_{i,j}) \in \Sigma^2 M_n(\mathcal{A}) \end{aligned}$$

In fact, we have shown that $A + \|A\|_1 I_n \in (M_n(\mathcal{A}))^+$ for any $A \in M_n(\mathcal{A})^h$, and so by Corollary 2.4.5 $M_n(\mathcal{A})$ is Archimedean. ☐

2.5 The Universal C^*-algebra

We see from Theorem 2.4.6 that $\|\cdot\|$ behaves like a C^* norm and therefore immediately obtain the following corollary.

Corollary 2.5.1. *$\|\cdot\|$ induces a norm on $\mathcal{A}^b/\mathcal{A}^0$ and the completion of $\mathcal{A}^b/\mathcal{A}^0$ with respect to this norm is a (real or complex) C^*-algebra.*

Definition 2.5.2. The *universal C^*-algebra of $\mathcal{A}$*, denoted $C^*_{\mathrm{u}}(\mathcal{A})$, is this completion of $\mathcal{A}^b/\mathcal{A}^0$ with respect to $\|\cdot\|$.

Calling this C^*-algebra *universal* is no accident. We firstly need to find the correct generalisation of states for more general target $*$-algebras (rather than just k).

Definition 2.5.3. Let $\mathcal{A}$ and $\mathcal{B}$ be two $*$-algebras, and let $\varphi : \mathcal{A} \to \mathcal{B}$ be a linear map. We say it is *unital completely positive (u.c.p.)* if it is unital and for all $n \in \mathbb{N}$ the matrix map

$$(\varphi \otimes I_n) : \mathcal{A} \otimes M_n(k) \to \mathcal{B} \otimes M_n(k)$$

is positive (without the unital assumption it is just called *completely positive*).

We have already seen maps of this sort.

Proposition 2.5.4. *Let $\mathcal{A}$ be a $*$-algebra with a positive cone $\mathcal{A}^+$. Then*

(a) If $(\mathcal{B}, \mathcal{B}^+)$ is another $*$-algebra and $\varphi : \mathcal{A} \to \mathcal{B}$ is a homomorphism of ordered $*$-algebras, then φ is completley positive. If both are unital and φ preserves the unit, it is u.c.p.

(b) Any state $\varphi : \mathcal{A} \to k$ is completely positive (and u.c.p. when $\mathcal{A}$ is unital).

Proof. To show (a) since $\varphi \otimes I_n$ is homomorphism, it suffices (by the definition of the tensor product positive cone) to check that it preserves positivity for the generators of the positive cone $(\mathcal{A} \otimes M_n(k))^+$, that is we need to show that for $a \in \mathcal{A}^+$ and $M \in M_n(k)^+$, $\varphi(a \otimes M)$ is positive. But this is just $\varphi(a) \otimes M$ which is clearly positive.

Suppose now for (b) that φ is a state, and that $\mathcal{A}$ is unital (otherwise consider the unitisation). Let $A = (a_{ij})_{ij} \in M_n(\mathcal{A})^+$ is positive semidefinite, then for any vector $x = (x_i)_i \in k^n$ we calculate

$$x^* \big((\varphi \otimes I_n)(A) \big) x = \sum_{i,j} \varphi(a_{ij}) \overline{x_i} x_j = \sum_{i,j} \varphi\big((x_j \cdot 1)^* a_{ij} (x_i \cdot 1) \big) = \varphi\big((x_1, \ldots, x_n)^* A (x_1, \ldots, x_n) \big) \geq 0$$

where we used that φ is a state, and so $(\varphi \otimes I_n)(A)$ is positive semidefinite. ☐

We now have (following Proposition 2.7 in [ANT19]):

Theorem 2.5.5 (universality)**.** *Let $\mathcal{A}$ be Archimedean, and $\imath : \mathcal{A} \to \mathcal{A}/\mathcal{A}^0 \hookrightarrow C^*_{\mathrm{u}}(\mathcal{A})$ the canonical map. Let $\mathcal{B}$ be a C^*-algebra. Then*

(a) $\imath(\mathcal{A})$ is dense in $C^*_{\mathrm{u}}(\mathcal{A})$;

(b) $\imath$ respects the (semi-)norm;

(c) $\imath$ is a positive homomorphism of Archimedean algebras;

(d) There is a one-to-one correspondence between u.c.p. morphisms $\varphi : \mathcal{A} \to \mathcal{B}$ and u.c.p. morphisms $\overline{\varphi} : C^*_{\mathrm{u}}(\mathcal{A}) \to \mathcal{B}$, where $\overline{\varphi} \circ \imath = \varphi$. This correspondence maps homomorphisms to homomorphisms.

Proof. (a) and (b) follow from the construction. For (c), $\imath$ is clearly a homomorphism — so we just need to show it respects positivity. Suppose for some $a \in \mathcal{A}^h$ we have $\|a\| < R$, that is $R^2 \cdot 1 - a^* a \in \mathcal{A}^+$, and so by Lemma 2.4.4 that means that $R \cdot 1 - a \in \mathcal{A}^+$. In particular, $2R \cdot 1 - a \in \mathcal{A}^+$ and

$$R^2 \cdot 1 - (R \cdot 1 - a)^* (R \cdot 1 - a) = \tfrac{1}{2R} \big((2R \cdot 1 - a)^* a (2R \cdot 1 - a) + a^* (2R \cdot 1 - a) a \big) \in \mathcal{A}^+$$

and so whenever $\|a\| < R$, we have that $\|R - a\| \leq R$. In particular, the same holds in $C^*_{\mathrm{u}}(\mathcal{A})$ and this implies positivity by Proposition 2.2.3.

For (d) note that $\imath$ is u.c.p., and that any u.c.p. map on $C^*_{\mathrm{u}}(\mathcal{A})$ is automatically continuous. Let now $\varphi : \mathcal{A} \to \mathcal{B}$ be a u.c.p. map, and suppose that $\|a\| = 0$ — that is $\varepsilon \cdot 1 - a^* a \in \mathcal{A}^+$ for all $\varepsilon > 0$. So

$$\begin{pmatrix} 1 & a \\ a^* & \varepsilon \end{pmatrix} = \begin{pmatrix} 1 \\ a^* \end{pmatrix} \begin{pmatrix} 1 & a \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} (\varepsilon - a^* a) \begin{pmatrix} 0 & 1 \end{pmatrix} \in (\mathcal{A} \otimes M_2)^+$$

and so since φ is u.c.p. and the cone $\Sigma^2\mathcal{B} \subset \mathcal{B}^h$ is closed (in the topology τ_{st} as we shall see in section 2.6) we see that

$$\begin{pmatrix} 1 & \varphi(a) \\ \varphi(a)^* & 0 \end{pmatrix} \geq 0$$

In particular, the determinant of this matrix must be ≥ 0, but this is $-\varphi(a)^*\varphi(a)$ so we must have that $\varphi(a) = 0$, and φ extends to a map on $C^*_{\text{u}}(\mathcal{A})$.

Since $(\imath \otimes \text{Id})(\mathcal{A} \otimes M_n)^+$ is dense in the positive elements of $C^*_{\text{u}}(\mathcal{A}) \otimes M_n$, we must have that $\overline{\varphi}$ is u.c.p. too. The fact that this process carries homomorphisms to homomorphisms is easy. ☐

Remark 2.5.6. Alternatively, we could have defined the C^*-semi-norm

$$\|a\|' := \sup\{\|\pi(a)\|_{\text{op}} \mid \pi \text{ is an } \mathcal{A}^+\text{-positive } *\text{-representation of } \mathcal{A}\}$$

If $\mathcal{A}$ is Archimedean then this supremum is finite, because if $a^*a \preccurlyeq R^2 \cdot 1$ then for any representation π we have

$$(\|\pi(a)\|')^2 = \|\pi(a^*a)\|' \leq R^2$$

We will see in Corollary 2.7.2 these norms coincide for Archimedean algebras, and so we will drop the $'$ and use $\|\cdot\|$ for either in this case.

Hence for an Archimedean $*$-algebra $\mathcal{A}$ all $*$-representations are bounded. For example

Corollary 2.5.7. *Let $\pi : k[\Gamma] \to \mathcal{L}(\mathcal{H}_\pi)$ be a $*$-representation, then in fact the image lies in $\mathbb{B}(\mathcal{H}_\pi)$.*

Let Γ be a discrete, countable group. The norm $\|\cdot\|'$ on $k[\Gamma]$ typically has a different name.

Definition 2.5.8. The *universal norm* of $a \in k[\Gamma]$ is defined to be

$$\|a\|_{\text{u}} := \sup\{\|\pi(a)\|_{\text{op}} \mid \pi \text{ is a unitary representation of } \Gamma\}$$

The *universal group C^*-algebra* is the completion of $k[\Gamma]$ with respect to $\|\cdot\|_{\text{u}}$, denoted $C^*(\Gamma)$.

Alternatively we could define the *universal representation* to be the sum over all the cyclic representations of Γ (which must therefore be on separable Hilbert spaces), and take the norm with respect to that.

Example 2.5.9. By universality it is clear that $C^*_{\text{u}}(k[\Gamma]) = C^*(\Gamma)$.

The following two examples are examples 2 and 3 in [Oza12].

Example 2.5.10. Consider the $*$-algebra $\mathcal{A} = k[X_1, \ldots, X_d]$ of polynomials where the X_i are hermitian ($X_i^* = X_i$). This is Archimedean when equipped with the $*$-positive cone

$$\mathcal{A}^+ = *\text{-positive cone generated by } \{1 - X_i^2 \mid i = 1, \ldots, d\}$$

We have that $C^*_{\text{u}}(\mathcal{A}) = C([-1,1]^d)$, the algebra of continuous functions on $[-1,1]^d$, with X_i identified with the i^{th} coordinate projection.

Example 2.5.11. Consider the $*$-algebra $\mathcal{A} = k\langle X_1, \ldots, X_d\rangle$ of polynomials in d non-commuting variables $X_1, \ldots, X_d$ with $X_i^* = X_i$. This is Archimedean when equipped with the $*$-positive cone

$$\mathcal{A}^+ = *\text{-positive cone generated by } \{1 - X_i^2 \mid i = 1, \ldots, d\}$$

We have that $C^*_{\text{u}}(\mathcal{A}) = C([-1,1]) *^{\text{sup}} \cdots *^{\text{sup}} C([-1,1])$, the *unital full free product* of d copies of $C([-1,1])$.

Definition 2.5.12. Let $\mathcal{A}$ and $\mathcal{B}$ be two unital C^* algebras. Their *unital full free product* $\mathcal{A} *^{\text{sup}} \mathcal{B}$ is the completion of $\mathcal{A} * \mathcal{B}$ (the coproduct in the category of $*$-algebras) with respect to the norm

$$\|c\| := \sup\{\|\pi(c)\| \mid \pi \text{ is a } *\text{-representation of } \mathcal{A} * \mathcal{B}\}$$

Note that $*$-representations of $\mathcal{A} * \mathcal{B}$ are in one-to-one correspondence with pairs of $*$-representations of $\mathcal{A}$ and $\mathcal{B}$.

Note also that any Archimedean algebra has a state, given by pulling back any state on $C^*_u(\mathcal{A})$. Clearly, if $\mathcal{A}$ has a state then it must be true that $-1 \notin \mathcal{A}^+$. Hence we have the following observation:

Corollary 2.5.13. *If a $*$-algebra is generated by partial isometries and has a state, then it is Archimedean.*

This allows us to fulfill a promise and show that tensor products of Archimedean algebras are again Archimedean.

Proposition 2.5.14. *Let $\mathcal{A}$, $\mathcal{B}$ be Archimedean algebras, then so is $\mathcal{A} \otimes \mathcal{B}$.*

Proof. Let φ, ψ be states on $\mathcal{A}, \mathcal{B}$ correspondingly, and so $\varphi \cdot \psi$ is a state on $\mathcal{A} \otimes \mathcal{B}$ and thus $-1 \otimes 1 \notin (\mathcal{A} \otimes \mathcal{B})^+$. Now note that $\mathcal{A} \otimes \mathbb{Q} = \mathcal{A}^b \otimes \mathbb{Q} \leq (\mathcal{A} \otimes \mathcal{B})^b$ and similarly $\mathbb{Q} \otimes \mathcal{B} \leq (\mathcal{A} \otimes \mathcal{B})^b$. Hence by (c) in Theorem 2.4.6 (the submultiplicativity of the norm) we have that $\mathcal{A} \otimes \mathcal{B} \leq (\mathcal{A} \otimes \mathcal{B})^b$, whence the result. ☐

Similarly one can show that $\mathcal{A} \oplus \mathcal{B}$ is Archimedean, see Lemma 7 in [BN20].

Example 2.5.15. Two important examples of the tensor product are

(a) For two groups Γ, Δ under the identification $k[\Gamma \times \Delta] \cong k[\Gamma] \otimes_k k[\Delta]$ we get that $\Sigma^2 k[\Gamma \times \Delta] = (k[\Gamma] \otimes_k k[\Delta])^+$;

(b) For $M_{n \times n}(k)$ with positive cone $\Sigma^2 M_{n \times n}(k)$, then $\mathcal{A} \otimes M_n(k) \cong M_n(\mathcal{A})$ with involution $(x_{ij})^* = (x^*_{ji})$ and the $*$-positive cone

$$(\mathcal{A} \otimes M_n(k))^+ = \left\{ \sum_{k=1}^{r} ((a^k_i)^* q_k a^k_j)_{i,j} \mid a^k_i \in \mathcal{A}, q_k \in \mathcal{A}^+ \right\}$$

This is Archimedean since both $\mathcal{A}$ and $M_n(k)$ are.

We end this section by giving some examples of how standard results from the theory of C^*-algebras carry over to (complex) Archimedean algebras. We won't require these results in the sequel, so we content ourselves with the statements and the remark that they follow more or less immediately from the versions for C^*-algebras. See [ANT19] for the details.

The first result gives a justification for why u.c.p. maps, rather than just positive maps, are the correct generalisation of positive functionals.

Corollary 2.5.16 (Stinespring's Dilation Theorem)**.** *Let $\mathcal{A}$ be Archimedean and let*

$$\rho : \mathcal{A} \to \mathbb{B}(\mathcal{H}_\rho)$$

be a u.c.p. morphism. Then there is a representation $\pi : \mathcal{A} \to \mathbb{B}(\mathcal{H}_\pi)$ and an isometric embedding $U : \mathcal{H}_\rho \hookrightarrow \mathcal{H}_\pi$ such that $\rho(a) = U^* \circ \pi(a) \circ U$.

This is a generalisation of the GNS construction — when the dimension of $\mathcal{H}_\rho$ is 1, we recover the former.

Similarly, we have

Corollary 2.5.17 (Arveson's Extension Theorem)**.** *Let $\mathcal{A}$ be Archimedean and $\mathcal{V} \subset \mathcal{A}$ a unital $*$-subspace. Then any u.c.p. map $\rho : \mathcal{V} \to \mathbb{B}(\mathcal{H})$ extends to a u.c.p. map $\rho : \mathcal{A} \to \mathbb{B}(\mathcal{H})$.*

Proposition 2.5.18 (Choi's Theorem)**.** *Let $\mathcal{A}$ be Archimedean and $\mathcal{V} \subset \mathcal{A}$ a unital $*$-subspace. Let $\mathcal{H}$ be a Hilbert space of dimension $n < \infty$ and $\rho : \mathcal{V} \to \mathbb{B}(\mathcal{H})$ unital and $*$-linear. Then the following are equivalent:*

(a) ρ is u.c.p.

(b) $\rho \otimes I_n$ maps $(\mathcal{A} \otimes M_n(\mathbb{C}))^+|_{\mathcal{V}}$ to positive operators;

(c) The functional $\mathcal{V} \otimes M_n(\mathbb{C}) \to \mathbb{C} : a \otimes A \mapsto \mathrm{Tr}(\rho(v)A)$ is nonnegative on $(\mathcal{A} \otimes M_n(\mathbb{C}))^+|_{\mathcal{V}}$.

2.6 The Finest Locally Convex Topology

There is an intrinsic way to define a topology for our algebras. In what follows, V will be an $\mathbb{R}$ vector space; for us this will always be $\mathcal{A}^h$.

Definition 2.6.1. The *finest locally convex topology* on V is the topology τ_{st} generated by the family of all seminorms.

Note that all linear functionals are automatically continuous, and every linear subspace is closed, see chapters II and III of [Bar02] for these facts and more. It is a standard fact that any two Hausdorff linear topologies on finite dimensional $\mathbb{R}$-vector spaces coincide; so on $\mathbb{R}^n$ we have that τ_{st} is just the standard Euclidean topology.

Luckily there is also a nice algebraic characterisation of τ_{st} (hence it is sometimes referred to as the *algebraic topology*).

Let $C \subset V$ be a convex set. We define its *algebraic interior* as

$$\text{int}(C) := \{c \in C \mid \forall v \in V, \exists t \in (0,1] \text{ with } tv + (1-t)c \in C\}$$

It is easy to see that for a convex set C, $\text{int}(\text{int}(C)) = \text{int}(C)$, and a basis for τ_{st} is given by convex sets C such that $\text{int}(C) = C$.

Suppose our vector space comes with a distinguished cone $C \subset V$ (of course, in our cases this will be $\mathcal{A}^+ \subset \mathcal{A}^h$), we similarly define a linear partial order $\preccurlyeq$ on V by setting $u \preccurlyeq v$ if $v - u \in C$.

An element $e \in C$ is called an *order unit for C in V* if for every $x \in V$ there exists some $R > 0$ such that $x + Re \in C$ (equivalently $e \in \text{int}(C)$). Notice also that $\text{int}(C) = \text{int}(\overline{C})$ where the closure is taken with respect to τ_{st}.

Definition 2.6.2. We say a cone C in a real vector space V is *algebraically solid* if $\text{int}(C) \neq \varnothing$. We say it is *Archimedean closed* if $\overline{C} = C$.

Example 2.6.3. Let $\mathcal{A}$ be a C^*-algebra. Then $\Sigma^2\mathcal{A} \subset \mathcal{A}^h$ is Archimedean closed, as follows from Proposition 2.2.3.

In the case of unital $*$-algebras, $\mathcal{A}$ being Archimedean is just the assertion that 1 is an order unit for $\mathcal{A}^+$ in $\mathcal{A}^h$, this is what Corollary 2.4.5 says.

We make note of an important separation theorem which follows by an easy application of the Hahn-Banach Theorem.

Lemma 2.6.4. *Let $C \subset V$ be a convex subset with an interior point $e \in \text{int}(C)$, and $x \notin C$. Then there is a non-zero linear functional φ on V such that*

$$\varphi(x) \leq \inf_{c \in C} \varphi(c)$$

In particular, $\varphi(x) < \varphi(c)$ for any algebraic interior point $c \in C$.

Proof. We suppose that $C = \text{int}(C) \neq \varnothing$ is open; the general result will follow by a standard limiting procedure. Consider the open set $e - C$, this is non-empty, convex, and contains 0 in its interior. Hence its associated *Minkowski functional* (or *gauge*)

$$\mu = \mu_{(e-C)} : V \to \mathbb{R}_{\geq 0} : v \mapsto \inf\{R \in \mathbb{R}_{\geq 0} \mid v \in R(e - C)\}$$

is subadditive and positive homogeneous — and so is suitable to use in the Hahn-Banach Theorem.

Consider the subspace $H = \mathbb{R} \cdot (e - x)$ and the linear functional $\varphi : H \to \mathbb{R} : \lambda(e - x) \mapsto \lambda$. Since $x \notin C$, $e - x \notin e - C$, and hence for $\lambda > 0$

$$\varphi(\lambda(e - x)) = \lambda \leq \lambda\mu(e - x) = \mu(\lambda(e - x))$$

and so $\varphi \leq \mu|_H$ (this is clearly true for $\lambda \leq 0$). So by the Hahn-Banach Theorem, φ extends to a functional on V, which we also denote by φ, such that $\varphi \leq \mu$ everywhere.

For any $c \in C$ we have that $\varphi(e-c) \leq \mu(e-c) < 1 = \varphi(e-x)$, and so $\varphi(c) > \varphi(x)$ as required. □

We can also separate convex sets; we technically don't need this, but include it for completeness. This typically goes by the name *Hahn-Banach separation Theorem* or the *Hyperplane separation lemma*.

Lemma 2.6.5. *Let $C, K \subset V$ be two convex subsets of V. Suppose that $C \cap K = \varnothing$ and that $\mathrm{int}(C) \neq \varnothing$. Then there is a non-zero linear functional φ on V such that*

$$\sup_{k \in K} \varphi(k) \leq \inf_{c \in C} \varphi(c)$$

Proof. Consider the set $Z = C - K$, this is convex and by disjointness of C and K we have that $0 \notin Z$. Since Z has an interior point Z and 0 satisfy the conditions of Lemma 2.6.4 and so we have a non-zero linear functional $\varphi : V \to \mathbb{R}$ such that $0 = \varphi(0) \leq \varphi(z)$ for all $z \in Z$. Since any $z \in Z$ can be written as $z = c - k$ for $c \in C$ and $k \in K$ we see that

$$0 \leq \varphi(z) = \varphi(c) - \varphi(k)$$

and the result follows. □

Applying both of the above in the special case when C is a cone we obtain

Corollary 2.6.6 (Eidelheit-Kakutani)**.** *Let V be an $\mathbb{R}$-vector space, C an algebraically solid cone and $x \in V \backslash C$. Then there is a non-zero linear functional $\varphi : V \to \mathbb{R}$ such that*

$$\varphi(x) \leq \inf_{c \in C} \varphi(c)$$

In particular, $\varphi(x) < \varphi(c)$ for any algebraic interior point $c \in C$.

Similarly if $K \subset V$ is a convex set such that $K \cap C = \varnothing$ then there is a non-zero linear functional φ on V such that

$$\sup_{k \in K} \varphi(k) \leq \inf_{c \in C} \varphi(c)$$

It is also possible to prove the Eidelheit-Kakutani Theorem by considering the *order interval*

$$[-e, e] = \{v \in V \mid -e \preccurlyeq v \preccurlyeq e\} = (C - e) \cap (e - C)$$

as illustrated in Figure 2, instead of the set $e - C$. In this case since $[-e, e]$ is balanced, the associated Minkowski functional is a genuine seminorm.

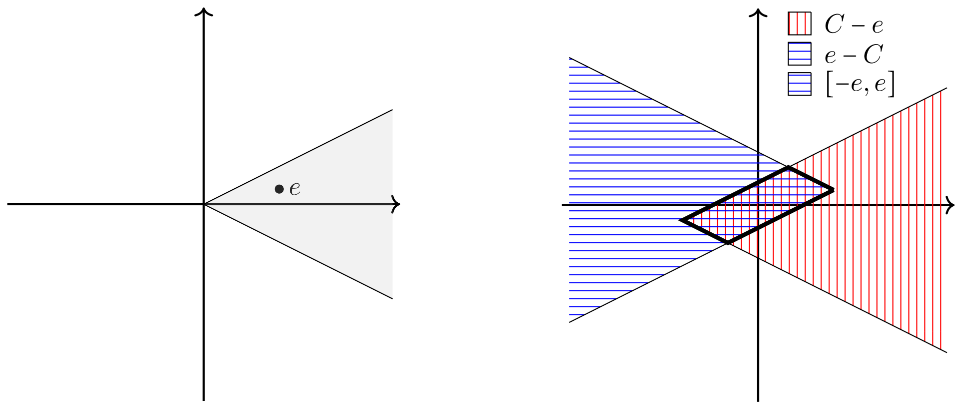
2.7 The Abstract Positivstellensatz

We will now restrict to the $k = \mathbb{C}$ case for simplicity, but similar statements hold over $\mathbb{R}$ (and $\mathbb{Q}$) too.

Lemma 2.7.1 ([Sch09], Lemma 13)**.** *Let $\mathcal{A}^+$ be a $*$-positive cone such that $\mathcal{A}$ is Archimedean (in particular, $1 \in \mathrm{int}(\mathcal{A}^+)$ in the finest locally convex topology of $\mathcal{A}^h$). Suppose that C is a convex subset of $\mathcal{A}^h$ such that $\mathrm{int}(\mathcal{A}^+) \cap C = \varnothing$. Then there exists a state φ of $\mathcal{A}$ such that $\varphi(b) \leq 0$ for all $b \in C$, and the associated GNS $*$-representation is $\mathcal{A}^+$-positive.*

Proof. By the Eidelheit-Kautani separation Theorem (Corollary 2.6.6) there exists a nonzero $\mathbb{R}$-linear functional φ on $\mathcal{A}^h$ such that $\sup\{\varphi(c) \mid c \in C\} \leq \inf\{\varphi(a) \mid a \in \mathcal{A}^+\}$. Since $\mathcal{A}^+$ is a $*$-positive cone, we must have $\varphi(a) \geq 0$ for all $a \in \mathcal{A}^+$, and $\varphi(c) \leq 0$ for $c \in C$.

Since 1 is an internal point of $\mathcal{A}^+$ and $\varphi \neq 0$ we scale so that $\varphi(1) = 1$. For any $a \in \mathcal{A}$ we can write $a = \frac{a+a^*}{2} + i\frac{a-a^*}{2i} \in \mathcal{A}^h + i\mathcal{A}^h$, and hence we can extend φ $\mathbb{C}$-linearly to all of $\mathcal{A}$ — we also denote this by φ.



(a) The order unit $e \in \operatorname{int}(C)$

(b) The order interval $[-e, e]$

Figure 2: An illustration of the order interval $[-e, e]$ for an algebraically solid cone C in $\mathbb{R}^2$

Since $\Sigma^2\mathcal{A} \subset \mathcal{A}^+$, φ is a state of the $*$-algebra $\mathcal{A}$, with corresponding GNS representation π and unit vector ξ. For $a \in \mathcal{A}^+$ and any $b \in \mathcal{A}$ we have $b^*ab \in \mathcal{A}^+$ and hence $\varphi(b^*ab) \geq 0$. So

$$0 \leq \varphi(b^*ab) = \langle \pi(b^*ab)\xi, \xi\rangle = \langle \pi(a)\pi(b)\xi, \pi(b)\xi\rangle$$

Since ξ is cyclic and the above holds for any $b \in \mathcal{A}^+$, we have that $\pi(a) \geq 0$ and so π is positive on $\mathcal{A}^+$. □

Corollary 2.7.2. *The two norms $\|\cdot\|$ and $\|\cdot\|'$ defined in remark 2.5.6 coincide for an Archimedean $*$-algebra $\mathcal{A}$.*

Proof. Recall that for $a \in \mathcal{A}$ we defined

$$\|a\| := \inf\{R \in \mathbb{R}_{\geq 0} \mid a^*a \preccurlyeq R^2 \cdot 1\}$$
$$\|a\|' := \sup\{\|\pi(a)\|_{\text{op}} \mid \pi \text{ is an } \mathcal{A}^+\text{-positive } *\text{-representation of } \mathcal{A}\}$$

If $a^*a \preccurlyeq R^2 \cdot 1$ then $\|\pi(a)\|_{\text{op}}^2 = \|\pi(a^*a)\|_{\text{op}} \leq R^2$ whence $\|\cdot\|' \leq \|\cdot\|$. Suppose now that there is some $S \in \mathbb{R}$ such that $\|a\|' < S < \|a\|$. In particular $S^2 \cdot 1 - a^*a \notin \operatorname{int}(\mathcal{A}^+)$ and so by Lemma 2.7.1 there is some $\mathcal{A}^+$-positive $*$-representation π with $\pi(S^2 \cdot 1 - a^*a) \leq 0$. In particular, $\|\pi(a)\|_{\text{op}}^2 = \|\pi(a^*a)\|_{\text{op}} \geq S^2$, a contradiction, and so $\|\cdot\| = \|\cdot\|'$. □

Using this separation theorem, we are able to fully characterise positive semidefinite elements in Archimedean algebras. They are *almost* just the elements in $\mathcal{A}^+$.

Proposition 2.7.3 (Abstract Positivstellensatz, [Sch09] Proposition 15)**.** *Let $\mathcal{A}$ be Archimedean. For any element $a \in \mathcal{A}^h$, the following are equivalent:*

(a) $a + \varepsilon \cdot 1 \in \mathcal{A}^+$ for all $\varepsilon > 0$;

(b) $\pi(a) \geq 0$ for each $\mathcal{A}^+$-positive $*$-representation of $\mathcal{A}$ (that is, $a \in \mathcal{A}(\mathcal{R}(\mathcal{A}^+))^+$);

(c) $\varphi(a) \geq 0$ for each $\mathcal{A}^+$-positive state φ of $\mathcal{A}$.

Proof. (a) ⇒ (b) ⇒ (c) are clear. To prove (c) ⇒ (a), we assume that $a+\varepsilon\cdot 1 \notin \mathcal{A}^+$ for some $\varepsilon > 0$. Consider the convex set $C = \{a+\varepsilon\cdot 1\}$, and apply Lemma 2.7.1 — so we obtain an $\mathcal{A}^+$-positive state φ with $\varphi(a+\varepsilon\cdot 1) \leq 0$. But then $\varphi(a) < 0$, contradicting (c). □

Remark 2.7.4. The statement '$a + \varepsilon \cdot 1 \in \mathcal{A}^+$ for all $\varepsilon > 0$' just says that $a \in \overline{\mathcal{A}^+}$, the closure in $\mathcal{A}^h$ with respect to τ_{st}. It is therefore interesting to see when this closure condition can be dispensed of. Furthermore, we only really need the existence of *some* interior point for this description, it needn't necessarily be 1.

Corollary 2.7.5. *Let $\mathcal{A}$ be Archimedean with respect to $\Sigma^2\mathcal{A}$. Then for any element $a \in \mathcal{A}^h$, the following are equivalent*

(a) $a + \varepsilon \cdot 1 \in \Sigma^2\mathcal{A}$ *for all* $\varepsilon > 0$*;*

(b) $\pi(a) \geq 0$ *for each* $*$*-representation of* $\mathcal{A}$*.*

Proof. Indeed, any $*$-representation is automatically positive on $\Sigma^2\mathcal{A}$. □

Example 2.7.6. For a C^*-algebra $\mathcal{A}$, $\pi(a) \geq 0$ for all $*$-representations of $\mathcal{A}$ if and only if $a \geq 0$ in $\mathcal{A}$. In particular, the positive cone $\Sigma^2\mathcal{A}$ is closed with respect to τ_{st} in $\mathcal{A}^h$.

Similarly, there is a criterion for strict positivity.

Proposition 2.7.7 (Strict abstract Positivstellensatz, [Sch09] Proposition 16)**.** *Let $\mathcal{A}$ be Archimedean. For any element $a \in \mathcal{A}^h$, the following are equivalent:*

(a) There exists $\varepsilon > 0$ *such that* $a - \varepsilon \cdot 1 \in \mathcal{A}^+$*;*

(b) For each $\mathcal{A}^+$*-positive* $*$*-representation of* $\mathcal{A}$*, there exists a number* $\delta_\pi > 0$ *such that* $\pi(a - \delta_\pi \cdot 1) \geq 0$*;*

(c) For each $\mathcal{A}^+$*-positive state* φ *of* $\mathcal{A}$ *there exists a number* $\delta_\varphi > 0$ *such that* $\varphi(a - \delta_\varphi \cdot 1) \geq 0$*.*

Proof. (a) ⇒ (b) ⇒ (c) are clear. To prove (c) ⇒ (a), we assume that (a) doesn't hold. Consider the positive cone $\widetilde{\mathcal{A}^+} := \mathbb{R}_{>0} \cdot 1 + \mathcal{A}^+$, by the assumption, $a \notin \operatorname{int}(\widetilde{\mathcal{A}^+})$ and so using Lemma 2.7.1 we find an $\widetilde{\mathcal{A}^+}$-positive state φ with $\varphi(a) \leq 0$. The result now follows since $\varphi(a - \delta \cdot 1) < 0$ for all $\delta > 0$. □

Using easy spectral theory we reformulate this result in terms of the map $\imath : \mathcal{A} \to C^*_{\mathrm{u}}(\mathcal{A})$.

Corollary 2.7.8 ([BN20], Proposition 10)**.** *Let $\mathcal{A}$ be Archimedean and $a \in \mathcal{A}^+$. Then $\imath(a)$ is invertible in $C^*_{\mathrm{u}}(\mathcal{A})$ if and only if there is some $\varepsilon \in \mathbb{R}_{>0}$ such that $a \succcurlyeq \varepsilon \cdot 1$.*

Proof. Note that $a \succcurlyeq \varepsilon \cdot 1$ if and only if $\imath(a) \succcurlyeq \varepsilon \cdot 1$, and $\imath(a) \geq 0$ as a C^*-algebra element. The result then follows from Proposition 2.2.3. □

We can rephrase this by using the universality of $C^*_{\mathrm{u}}(\mathcal{A})$.

Corollary 2.7.9 ([BN20], Corollary 11)**.** *Let $\mathcal{A}$ be Archimedean and $a \in \mathcal{A}^+$. Then $\pi(a)$ is invertible for every $\mathcal{A}^+$-positive $*$-representation $\pi : \mathcal{A} \to \mathbb{B}(\mathcal{H}_\pi)$ if and only if there is some $\varepsilon > 0$ such that $a \succcurlyeq \varepsilon \cdot 1$.*

Proof. By Corollary 2.7.8 we want to show that $\pi(a)$ is positive for every $\mathcal{A}^+$-positive $*$-representation if and only if $\imath(a)$ is invertible in $C^*_{\mathrm{u}}(\mathcal{A})$; by universality we consider $*$-representations of $C^*_{\mathrm{u}}(\mathcal{A})$. If $\imath(a)$ is invertible then clearly $\pi(a)$ is invertible for all $*$-representations. Conversely by the Gelfand-Naimark Theorem we have a faithful representation $\pi : C^*_{\mathrm{u}}(\mathcal{A}) \to \mathbb{B}(\mathcal{H}_\pi)$, by assumption $\pi(a)$ is invertible in $\mathbb{B}(\mathcal{H}_\pi)$. But since $\pi(C^*_{\mathrm{u}}(\mathcal{A}))$ is a C^*-subalgebra of $\mathbb{B}(\mathcal{H}_\pi)$ and contains the unit, $\pi(a)$ is already invertible in $\pi(C^*_{\mathrm{u}}(\mathcal{A}))$, and hence in $C^*_{\mathrm{u}}(\mathcal{A})$. □

So far we've talked about positivity of $\pi(a)$ with respect to *all* the vectors $\xi \in \mathcal{H}_\pi$ — but what if we only want positivity at one vector? We think of this as asking that a matrix have *at least one* positive eigenvalue, rather than all eigenvalues being nonnegative. We can also characterise this thanks to to Jaka Cimprič ([Cim09]), but this won't be used in the sequel.

Proposition 2.7.10. *For $a \in \mathcal{A}^h$ the following are equivalent:*

(a) There exist nonzero elements $x_1, \ldots, x_r$ *of* $\mathcal{A}$ *such that* $\sum_i x_i^* a x_i \in 1 + \mathcal{A}^+$*;*

(b) For any $\mathcal{A}^+$*-positive* $*$*-representation, there exists a vector* η *such that* $\langle \pi(a)\eta, \eta\rangle > 0$*.*

Proof. to show that (a) ⇒ (b), suppose that $\sum_i x_i^* a x_i = 1 + b$ for some $b \in \mathcal{A}^+$ and let π be an $\mathcal{A}^+$-positive representation and $0 \neq \xi \in \mathcal{H}_\pi$ be any vector. Then

$$\begin{aligned}\sum_i \langle \pi(a)\pi(x_i)\xi, \pi(x_i)\xi\rangle &= \sum_i \langle \pi(x_i^* a x_i)\xi, \xi\rangle \\ &= \langle \pi(1+b)\xi, \xi\rangle \\ &\geq \langle \pi(1)\xi, \xi\rangle = \|\xi\|^2 > 0\end{aligned}$$

and hence at least one summand $\langle \pi(a)\pi(x_i)\xi, \pi(x_i)\xi\rangle > 0$. Conversely, suppose that (a) doesn't hold. We set

$$C = \{\sum_{i=1}^n x_i^* a x_i \mid n \in \mathbb{N}, x_i \in \mathcal{A}\}$$

and let $\tilde{\mathcal{A}}^+ = 1 + \mathcal{A}^+$. By assumption $C \cap \tilde{\mathcal{A}}^+ = \emptyset$, and so there is a state φ of $\mathcal{A}$ such that the corresponding GNS representation π (with cyclic vector ξ) is $\tilde{\mathcal{A}}^+$-positive and $\varphi(C) \leq 0$, that is $\varphi(x^* a x) = \langle \pi(a)\pi(x)\xi, \pi(x)\xi\rangle \leq 0$ for all $x \in \mathcal{A}$. Since ξ is cyclic for π, (b) isn't satisfied. □

Remark 2.7.11. It was observed by Schmüdgen that these methods can give a much more elementary proof of his strict positivstellensatz for bounded real semialgebraic sets (Theorem 1.3.6), see section 5.3 in [Sch09].

Example 2.7.12. Consider again example 2.5.11 with $k = \mathbb{R}$. That is, $\mathcal{A} = \mathbb{R}\langle X_1, \ldots, X_n\rangle$ is the $*$-algebra of polynomials in d non-commuting hermitian (so $X_i^* = X_i$) variables, and this is Archimedean when equipped with the $*$-positive cone

$$\mathcal{A}^+ = *\text{-positive cone generated by } \{1 - X_i^2 \mid i = 1, \ldots, d\}$$

A positivstellensatz due to Helton and McCullough ([HM04]) states that $f(A_1, \ldots, A_d) \geq 0$ for all $A_1, \ldots, A_d \in M_n(\mathbb{R})^h$ of operator norm at most 1 if and only if $f + \varepsilon \cdot 1 \in \mathcal{A}^+$ for all $\varepsilon > 0$.

Suppose instead that f is *trace positive*, that is we only ask that $\mathrm{Tr}(f(A_1, \ldots, A_d)) \geq 0$ for all $A_1, \ldots, A_d$ as above. Clearly any positive polynomial in the Helton-McCullough sense is trace positive, but we can also add finite sums of commutators $[A, B] := AB - BA$. We can now ask if a positivstellensatz still holds — that is, if f is trace positive must we have for all $\varepsilon > 0$ that $f + \varepsilon \cdot 1$ differs from an element in $\mathcal{A}^+$ by a sum of commutators? As was proven by Klep and Schweighofer ([KS08], see also the erratum by Burgdorf, Dykema, and the two aforementioned authors), this turns out to be equivalent to the Connes Embedding Problem. We discuss this briefly in section 4.6.

2.8 Matrix Algebras

We have already discussed matrix algebras a bit, and noted that if $\mathcal{A}$ is Archimedean then so is $M_n(\mathcal{A})$ for all n. However it will be useful for us to reformulate a positivstellensatz for this specific case. In order to do this, we firstly need to understand how representations of $\mathcal{A}$ relate to those of $M_n(\mathcal{A})$.

Given any representation $\pi : \mathcal{A} \to \mathbb{B}(\mathcal{H}_\pi)$ we get the associated representation $M_n(\pi) := \pi \otimes I_n : M_n(\mathcal{A}) \to \mathbb{B}(\mathcal{H}_\pi^n)$. But in fact, any representation of $M_n(\mathcal{A})$ arises this way. Indeed, suppose we have a representation

$$\rho : M_n(\mathcal{A}) \to \mathbb{B}(\mathcal{H}_\rho)$$

Let $\mathcal{H}_i = \delta_{ii}\mathcal{H}_\rho$, and note that since $M_n(k)$ and $\mathcal{A}$ commute we have that each $\mathcal{H}_i$ is $\mathcal{A}$-invariant and hence gives us a $*$-representation of $\mathcal{A}$. By using the permutation matrices, we in fact see that all the $\mathcal{H}_i$ are isomorphic representations, call them $\mathcal{H}_\pi$. Since

$$\sum_i \delta_{ii} = I_n \quad \text{and} \quad \delta_{ii}\delta_{jj} = \begin{cases} \delta_{ii} & \text{if } i = j \\ 0 & \text{otherwise}\end{cases}$$

we have that $\mathcal{H}_\rho \cong \mathcal{H}_\pi^n$, and this is an isomorphism as both an $\mathcal{A}$ and an $M_n(k)$ representation — and hence as an $M_n(\mathcal{A})$ representation.

We can thus rephrase Corollary 2.7.9 in this specific case.

Corollary 2.8.1 ([BN20], Corollary 15). *Let $\mathcal{A}$ be an Archimedean algebra, and $A \in (M_n(\mathcal{A}))^+$ a positive element. Then $M_n(\pi)(A)$ is invertible for every $\mathcal{A}^+$-positive $*$-representation $\pi : \mathcal{A} \to \mathbb{B}(\mathcal{H}_\pi^n)$ if and only if there is some $\varepsilon > 0$ such that $A \succcurlyeq \varepsilon I_n$.*

We will use this later with $\mathcal{A} = k[\Gamma]$ and positive cone $\Sigma^2 k[\Gamma]$, and so all of these $*$-representations are simply the unitary representations of the group. Something unique about matrix algebras is that we can (sometimes) multiply matrices of different sizes — and a priori this might ruin our notion of being a sum of hermitian squares. This turns out not to be the case, as noted in Lemma 14 of [BN20]. Indeed, suppose we have a matrix $A \in M_{m\times n}(\mathcal{A})$, and let e_i^m denote the standard i'th basis vector in $M_{m\times 1}(k)$. Observe that $(e_1^n)^* e_1^n = 1 \in k$ and $\sum_{i=1}^m e_i^m (e_i^m)^* = I_m \in M_m(k)$. So we calculate that

$$A^*A = A^*\Big(\sum_{i=1}^m e_i^m (e_i^m)^*\Big)A = \sum_{i=1}^m A^* e_i^m (e_i^m)^* A = \sum_{i=1}^m A^* e_i^m (e_1^n)^* e_1^n (e_i^m)^* A = \sum_{i=1}^m \xi_i^* \xi_i \in \Sigma^2 M_n(\mathcal{A}) \tag{2.2}$$

where $\xi_i = e_1^n (e_i^m)^* A \in M_n(\mathcal{A})$.

2.9 A Real Closed Positivstellensatz

In fact, we can completely detect positivity as a sum of squares if we allow for functionals to real closed extensions of $\mathbb{R}$, and the corresponding *generalised* ($*$-)representations — this was done in [NT13] and extended in [ANT19]. By a generalised representation we simply mean a $*$-representation of $\mathcal{A}$ where the target Hilbert space is over some RCF extension $\mathbb{K}$ of $\mathbb{R}$ (and there is no completeness assumption). This is natural since being self-adjoint/unitary/positive semidefinite all have the same characterisation over any RCF $\mathbb{K}$ as they do over $\mathbb{R}$.

Notice that the abstract positivstellensatz follows by using Eidelheit-Kakutani separation, but here the topological closure is vital. Luckily, Netzer and Thom show a stronger separation Theorem with virtually no assumptions — the trade-off is that we need to allow functionals to RCF extensions of $\mathbb{R}$.

Theorem 2.9.1 ([NT13], Theorem 2.1). *Let V be an $\mathbb{R}$-vector space, $C \subset V$ a convex cone, and $x \notin C$. Then there exists an RCF extension $\mathbb{R} \subset \mathbb{K}$ and an $\mathbb{R}$-linear functional $\varphi : V \to \mathbb{K}$ such that*

$$\varphi(x) < 0 \quad \& \quad \varphi(y) \geq 0 \text{ for all } y \in C$$

If $y \in C \backslash (C \cap -C)$, we can even ensure that $\varphi(y) > 0$. Also, $\mathbb{K}$ depends only on V, not on x or C.

Let us explain the idea behind the finite dimensional case of the above. We can find (by Eidelheit-Kakutani separation for example, Corollary 2.6.6) a functional φ that is nonnegative on C, and negative on x. We only don't get complete separation when this functional might be zero, so we consider the (codimension 1) subspace H where this happens, and find a functional ψ that does the separation on $H \cap C$. Now for any RCF extension of $\mathbb{R}$ we have infinitesimals, say ε — so $\varphi + \varepsilon\psi$ separates completely outside of $\ker\varphi \cap \ker\psi$. That is, we are using infinitesimals in the RCF extension of $\mathbb{R}$ to 'zoom into' the zero sets where we don't get perfect separation, without ruining the separation at the previous step. The infinite dimensional case follows by taking a suitable ultrafilter on the set of finite-dimensional subspaces of V and then using Łos' Theorem.

This is good, but we don't want functionals to separate our points — we want to upgrade these to representations. Suppose $\mathcal{A}$ is a $*$-algebra over $\mathbb{C}$, and we have an $\mathbb{R}$-linear functional $\varphi : \mathcal{A}^h \to \mathbb{K}$. We can extend it uniquely to a $\mathbb{C}$-linear functional $\varphi : \mathcal{A} \to \mathbb{F} := \mathbb{K}[i]$ fulfilling $\varphi(a^*) = \overline{\varphi(a)}$. In order to obtain a representation from this we want to follows the GNS construction, so we define $\langle a, b\rangle_\varphi := \varphi(b^* a)$, and we need to consider the quotient of $\mathcal{A}$ by the set $\mathcal{N} := \{a \in \mathcal{A} \mid \langle a, a\rangle_\varphi = 0\}$.

Unfortunately, in general this needn't be a subspace or a left ideal — in the real case we used Cauchy-Schwarz (see remark 2.3.4), but this doesn't held for functionals to general RCFs. The following example is from [NT13].

Example 2.9.2. Consider the algebra $\mathbb{C}[X]$ where X is hermitian, so $\Sigma^2\mathbb{C}[X]$ is the cone of nonnegative real polynomials. Let $\mathbb{K}$ be an RCF extension of $\mathbb{R}$, and pick an infinitesimal $\varepsilon \in \mathbb{K}$. Consider the functional $\varphi : \mathbb{R}[t] \to \mathbb{K} : f \mapsto f(0) + \varepsilon f''(0)$. φ is positive, but for $a = 1 + X^2$ and $b = 1$ we see that

$$|\varphi(a^*b)|^2 = 1 + 4\varepsilon + 4\varepsilon^2 > 1 + 4\varepsilon = \varphi(a^*a)\varphi(b^*b)$$

We see moreover that φ is not completely positive. Indeed, with $A = \left(\begin{smallmatrix} 1 & a \\ 0 & 0 \end{smallmatrix}\right)$ we have

$$(\varphi \otimes I_2)(A^*A) = \begin{pmatrix} 1 & \varphi(a) \\ \varphi(a^*) & \varphi(a^*a) \end{pmatrix}$$

This has negative determinant (in $\mathbb{K}$), and so in particular is not positive semidefinite.

In fact, this is the only thing that can go wrong.

Proposition 2.9.3 ([NT13], Corollary 3.7)**.** *If $\varphi : \mathcal{A} \to \mathbb{F}$ be completely positive (in fact, $\varphi \otimes I_2$ being positive suffices) then it fulfills the Cauchy-Schwarz inequality*

$$|\varphi(a^*b)|^2 \leq \varphi(a^*a)\varphi(b^*b)$$

Proof. We have that $\varphi \otimes I_2$ is positive, and so in particular the determinant of $(\varphi \otimes I_2)(A^*A)$ is positive for all matrices $A \in \mathcal{A} \otimes M_2(\mathbb{C})$. Hence we calculate

$$0 \leq \det\left((\varphi \otimes I_2)\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\right) = \det\begin{pmatrix} \varphi(a^*a) & \varphi(a^*b) \\ \varphi(b^*a) & \varphi(b^*b) \end{pmatrix} = \varphi(a^*a)\varphi(b^*b) - |\varphi(a^*b)|^2$$

as required. □

We need to verify that $||a||_\varphi := \varphi(a^*a)^{1/2}$ indeed defines a semi-norm. It is easy to see that for $\lambda \in \mathbb{C}$, $||\lambda a||_\varphi = |\lambda| \cdot ||a||_\varphi$, and the triangle inequality follows as for the standard inner product on $\mathbb{C}^n$ (see Corollary 3.9 in [NT13]).

So now, the natural question is whether or not we can separate using a completely positive functional — if we can, then by the GNS construction we get separation by a representation. Clearly if we have that $\sum_{i=1}^n a_i^* a_i = 0$ for some non-zero a_i we can't hope for this. This turns out to be the only obstruction.

Definition 2.9.4. Let $\mathcal{A}$ be a $*$-algebra over $\mathbb{C}$. We say that $\mathcal{A}$ is *formally real* if $\forall a_1, \ldots, a_n \in \mathcal{A}$

$$\sum_{i=1}^n a_i^* a_i = 0 \quad \Rightarrow \quad a_1 = \cdots = a_n = 0$$

Theorem 2.9.5 ([NT13], Theorem 3.11)**.** *Let $\mathcal{A}$ be a formally real $*$-algebra (over $\mathbb{C}$), and $b \in \mathcal{A}^h \backslash \Sigma^2\mathcal{A}$. Then there is an RCF extension $\mathbb{R} \subset \mathbb{K}$ and a completely positive $\mathbb{C}$-linear functional $\varphi : A \to \mathbb{F} = \mathbb{K}[i]$ with $\varphi(a^*) = \overline{\varphi(a)}$ such that*

$$\varphi(b) < 0 \quad \textit{and} \quad \varphi(a^*a) > 0 \quad \textit{for } a \in \mathcal{A}\backslash\{0\}$$

Proof. For any finite dimensional subspace $H \leq \mathcal{A}$, let $\Sigma^2 H = \{\sum_i a_i^* a_i \mid a_i \in H\}$ be the set of sums of hermitian squares of elements from H. This is a closed convex cone in a finite dimensional subspace of $\mathcal{A}^h$ (see for example Lemma 2.7 in [PS01]).

Also, we have that $\Sigma^2 H \cap -\Sigma^2 H = \{0\}$ since $\mathcal{A}$ is formally real. So for each such H we have an $\mathbb{R}$-linear functional $\varphi_H : \mathcal{A}^h \to \mathbb{R}$ with $\varphi_H(b) < 0$ and $\varphi_H(a^*a) > 0$ for all $a \in H\backslash\{0\}$.

Let $\mathcal{V}$ be the set of finite dimensional subspaces of $\mathcal{A}$, and equip it with an ultrafilter ω containing all the sets $\{H \in \mathcal{V} \mid c \in H\}$ for $c \in \mathcal{A}$. Then define

$$\varphi : \mathcal{A}^h \to \mathbb{R}^\omega : a \mapsto (\varphi_H(a))_{H \in \mathcal{V}}$$

This does the separation as desired.

We just need to show that the $\mathbb{C}$-linear extension of φ to $\mathcal{A}$, which will also be denoted φ, is completely positive. That is, we need to show that for all $(a_i) \in \mathbb{C}^n$ the matrix $(\varphi(a_i^* a_j))_{i,j} \in M_n(\mathbb{R}^\omega[i])$ is positive semidefinite, and by Łos' Theorem we can do this component-wise. But it is clear that the $\mathbb{C}$-linear extension of φ_H to $\mathcal{A}$ maps a matrix of the form $(a_i^* a_j)_{i,j} \in M_n(\mathcal{A})$ (where all $a_i \in H$) to a positive semidefinite matrix, so the result follows. □

So suppose we have a self-adjoint completely positive map $\varphi : \mathcal{A} \to \mathbb{F}$. As in the GNS construction we want to use it to define a seminorm on $\mathcal{A} \otimes_{\mathbb{C}} \mathbb{F}$ — but then we need the $\mathbb{F}$-linear extension of φ to be positive still. Luckily this holds, see Lemma 3.6 in [NT13].

So by doing the GNS construction (we don't need to take the completion of $(\mathcal{A} \otimes_{\mathbb{C}} \mathbb{F})/\mathcal{N}$ since we don't assume our target Hilbert space is complete), we have the characterisations (for a formally real Archimedean $*$-algebra $\mathcal{A}$ over $\mathbb{C}$) and $a \in \mathcal{A}^h$:

$$\begin{aligned} a \in \overline{\Sigma^2 \mathcal{A}} \quad &\Leftrightarrow \quad a \text{ is positive semidefinite under every } *\text{-representation} \\ a \in \Sigma^2 \mathcal{A} \quad &\Leftrightarrow \quad a \text{ is positive semidefinite under every \textit{generalised} } *\text{-representation} \end{aligned} \tag{2.3}$$

We will refer to this as the *real closed positivstellensatz* or the *Netzer-Thom positivstellensatz*.

It is important to stress again that complete positivity of φ is what allows us to ensure that $\mathcal{N}$ is indeed a subspace of $\mathcal{A} \otimes_{\mathbb{C}} \mathbb{F}$, and that the Hilbert space we get from the GNS construction is a quotient of $\mathcal{A} \otimes_{\mathbb{C}} \mathbb{F}$.

Chapter 3

Archimedean Closures

We saw in the previous chapter that if $\mathcal{A}^+$ is Archimedean closed, then we can completely algebraically certify if an element is positive semidefinite with respect to some family of representations. We already saw that this is the case in 'dimension 0', that is, for an ordered field (Corollary 1.1.13).

We might hope that it holds in greater generality. Suppose $\mathcal{A}$ is a regular local ring, then in dimension 1 $\mathcal{A}$ is a discrete valuation ring and here the Theorem holds true — in fact it suffices for $\mathcal{A}$ to be an arbitrary valuation ring. This was probably first proved by Kneser and Colliot-Thélène, see page 250 in [CLRR80] for Kneser's proof. In dimension 2 both positive and negative results exist following mostly Scheiderer ([Sch00], [Sch03], [Sch06]). Of particular interest to us is [Sch06], which shows that the result holds for $\mathbb{C}[\mathbb{Z}^2]$.

However, for dimensions 3 and above this no longer holds — as we shall now investigate. In particular, $\Sigma^2\mathbb{C}[\mathbb{Z}^3]$ isn't Archimedean closed and so we can't dispense of the analytic aspect.

On the other end of the spectrum, we can show that for (virtually) free groups, the cone of hermitian sums of squares is closed, and so positivity may be detected purely algebraically. We will see that this is strongly linked to a the property of being 'residually finite dimensional', which means that groups have enough finite dimensional representations — see the next chapter for the precise definitions. This property is still an active area of research, with links to the Connes Embedding Problem, we comment on this later.

We firstly note that being Archimedean closed passes to subgroups, as was essentially observed by Narutaka Ozawa in [Oza22].

Proposition 3.0.1. *Suppose we have a subgroup $\Gamma \leq \Delta$ and $\Sigma^2 k[\Delta]$ is Archimedean closed. Then so is $\Sigma^2 k[\Gamma]$.*

Proof. Suppose that $b \in \overline{\Sigma^2 k[\Gamma]}$, so $b \geq 0$ in $C^*(\Gamma)$. Since $C^*(\Gamma) \leq C^*(\Delta)$ (we can induce any representation of Γ to one of Δ) we must have that $b \geq 0$ in $C^*(\Delta)$, and so by the Positivstellensatz (Corollary 2.7.5) $b \in \overline{\Sigma^2 k[\Delta]}$.

By our assumption $\Sigma^2 k[\Delta]$ is Archimedean closed and so b is a sum of squares — say $b = \sum_\ell \xi_\ell^* \xi_\ell$.

Choose a left transversal T for Γ in Δ, so any $\xi_\ell \in k[\Delta]$ can be written as $\xi_\ell = \sum_i a_{g_i}^{(\ell)} x_i^{(\ell)} h_i^{(\ell)}$, where $x_i^{(\ell)} \in T$, $h_i^{(\ell)} \in \Gamma$ and $g_i^{(\ell)} = x_i^{(\ell)} h_i^{(\ell)}$. Now compute

$$\begin{aligned}\sum_\ell \xi_\ell^* \xi_\ell &= \sum_\ell \sum_{i,j} \overline{a_{g_j}^{(\ell)}} a_{g_i}^{(\ell)} (h_j^{(\ell)} x_j^{(\ell)})^* x_i^{(\ell)} h_i^{(\ell)} \\ &= \sum_\ell \Big(\sum_i a_{g_i}^{(\ell)} h_i^{(\ell)} \Big)^* \Big(\sum_i a_{g_i}^{(\ell)} h_i^{(\ell)} \Big) + \sum_\ell \sum_{i \neq j} (\text{——})\end{aligned}$$

Since $\Gamma x_i x_j^{-1} \Gamma = \varnothing$ if $x_i \neq x_j$, the second term in the above vanishes (since $b \in k[\Gamma]$), and we have a sums of squares decomposition for $b \in \Sigma^2 k[\Gamma]$. □

Remark 3.0.2. The proof above is slightly misleading, since we seem to be assuming that the $x_i^{(\ell)}$ must be distinct — this of course need not be the case, but it is too notationally cumbersome to spell this out.

Remark 3.0.3. Trivially if Γ is finite then $\Sigma^2 k[\Gamma]$ is Archimedean closed. But we note that this proposition doesn't give us any new examples of groups for which the cone of sums of hermitian squares is Archimedean closed — indeed, subgroups of finite/virtually free groups are finite/virtually free.

We won't comment much on this, but there is a rich theory of whether or not $\mathcal{A}^+$ is closed also when $\mathcal{A}$ isn't an Archimedean algebra, and the Positivstellensätze that follow. In that case, there is more of an art in picking the correct set of representations $\mathcal{R}(\mathcal{A}^+)$ — for Archimedean algebras we simply picked all the ones that are $\mathcal{A}^+$-positive.

In fact, just one representation can suffice.

Example 3.0.4 (Weyl algebras). Let $d \in \mathbb{N}$. The Weyl algebra $\mathcal{A} = \mathcal{W}(d)$ is the unital $*$-algebra with generators $a_1, \ldots, a_d, a_{-1}, \ldots, a_{-d}$ and defining relations

$$a_k a_{-k} - a_{-k} a_k = 1 \quad \& \quad a_k a_l = a_l a_k \text{ if } k \neq l$$

and with involution $(a_k)^* = a_{-k}$.

The 'correct' set $\mathcal{R}(\mathcal{A})$ consists of a single representation π_0, the *Bargmann-Fock representation.*

There doesn't seem to be a method in general to pick the correct set of representations. The reader is encouraged to head to section 3.4 in [Sch09], and the references therein.

There is also an interesting theory already in the case of commutative $*$-algebras, see [CMN11].

3.1 Failure in Dimension ≥ 3

We are really only interested in the $\mathcal{A} = \mathbb{C}[\mathbb{Z}^3]$ case — but there is no difficulty gained in doing the general case. We will see that the failure comes precisely from the existence of homogeneous polynomials in degree ≥ 3 that are positive semidefinite but not sums of squares — with the correct setup, we just push this down to our ring $\mathcal{A}$. The fact that the ring is formally real is integral, as it was in the dimension 0 case we investigated in chapter 1. Also, as in remark 2.1, since we will only be considering commutative things, we ignore the existence of an involution, and everything will be genuine squares.

Let first $\mathcal{A}$ be a regular local ring — say its maximum ideal is $\mathfrak{m}$, its field of fractions is K, and its residue field is $k = A/\mathfrak{m}$, with $\operatorname{char}(k) \neq 2$.

We will always assume that K is formally real, otherwise every element is a sum of squares (see Lemma 0.1 in [Sch00]).

Recall that we have the standard graded ring associated to $\mathcal{A}$,

$$\operatorname{Gr}(\mathcal{A}) := \bigoplus_{n \geq 0} \operatorname{Gr}_n(\mathcal{A})$$

where $\operatorname{Gr}_n(\mathcal{A}) := \mathfrak{m}^n / \mathfrak{m}^{n+1}$. We define the *valuation* $\mu : \mathcal{A} \to \mathbb{N}_0$ and *leading term* $\ell : \mathcal{A} \to \operatorname{Gr}(\mathcal{A})$ by (for $a \in \mathcal{A}$)

$$\mu(a) := \sup\{n \geq 0 \mid a \in \mathfrak{m}^n\} \quad \text{and} \quad \ell(a) := a \pmod{\mathfrak{m}^{\mu(a)+1}}$$

Remark 3.1.1. Fix a regular system of parameters $x_1, \ldots, x_d$ for $\mathcal{A}$ (so that $\mathfrak{m}$ is the minimal prime ideal containing $\langle x_1, \ldots, x_d \rangle$), and write $\xi_i = \ell(x_i)$. Then $\operatorname{Gr}(\mathcal{A})$ is the polynomial ring $k[\xi_1, \ldots, \xi_d]$ with the standard grading, and $\ell(a)$ is a non-zero form (homogeneous polynomial) of degree $\mu(a)$ in $k[\xi_1, \ldots, \xi_d]$.

Firstly, we need a suitable way to interpret positivity in these rings.

Definition 3.1.2. The *real spectrum* of a ring $\mathcal{A}$ is the set of all orderings on the ring, topologised as a subset of the prime spectrum. It will be denoted $\operatorname{Sper}(\mathcal{A})$.

Alternatively, it is the set of $(\mathfrak{p}, \leq)$ where $\mathfrak{p}$ is a prime ideal and $\leq$ is an ordering on $\operatorname{Frac}(\mathcal{A}/\mathfrak{p})$, with a basis for the topology given by sets of the form

$$\mathcal{U}(a) := \{(\mathfrak{p}, \leq) \in \operatorname{Sper}(\mathcal{A}) \mid a + \mathfrak{p} \geq 0\}$$

where $a \in \mathcal{A}$. For $\alpha = (\mathfrak{p}, \leq) \in \operatorname{Sper}(\mathcal{A})$ we write $\operatorname{supp}(\alpha) := \mathfrak{p}$.

We say that a is *positive semidefinite on* $\operatorname{Sper}(\mathcal{A})$ if $a + \mathfrak{p} \geq 0$ for all $(\mathfrak{p}, \leq) \in \operatorname{Sper}(\mathcal{A})$. We denote the set of positive semidefinite elements by $\mathcal{A}(\operatorname{Sper}(\mathcal{A}))^+$.

Remark 3.1.3. If $\operatorname{Sper}(\mathcal{A}) = \varnothing$ then $-1 \in \Sigma^2\mathcal{A}$ and $\Sigma^2\mathcal{A} = \mathcal{A} = \mathcal{A}^+$, see [KS89] III section 2.

Lemma 3.1.4 ([Sch00], Lemma 1.1)**.** *Being a sum of squares passes nicely to the leading term. That is, let $\mathcal{A}$ be a regular local ring for which $k = A/\mathfrak{m}$ is formally real, and suppose that $0 \neq a \in \Sigma^2\mathcal{A}$ is a sum of r squares (say $a = \sum_i a_i^2$). Then the valuation $\mu(a) = 2s$ is even and $\ell(a) \in \operatorname{Gr}_{2s}(\mathcal{A})$ is a sum of r squares of elements in $\operatorname{Gr}_s(\mathcal{A})$.*

Proof. Note firstly that the residue field of the valuation μ is a purely transcendental extension of k of dimension $\dim(\mathcal{A}) - 1$, and in particular it must be formally real. It is then standard that $\mu(a) = \mu(\sum_i a_i^2) = 2\min_i(\mu(a_i)) =: 2s$. It also follows that $\ell(a)$ is the sum of the $\ell(a_j)^2$, for indices j such that $\mu(f_j) = s$. □

Proposition 3.1.5 ([Sch00], Proposition 1.2)**.** *Let $\mathcal{A}$ be a regular local ring with $\dim \mathcal{A} \geq 3$ for which $k = A/\mathfrak{m}$ is formally real. Then there is an element $a \in \mathcal{A}(\operatorname{Sper}(\mathcal{A}))^+ \backslash \Sigma^2\mathcal{A}$.*

Proof. Choose a form $f \in \mathbb{Z}[X_1, \ldots, X_d]$ which is positive semidefinite but isn't a sum of squares in $\mathbb{R}[X_1, \ldots, X_d]$ (see example 1.2). Let $a = f(x_1, \ldots, x_d)$, then $\ell(a) = h(\xi_1, \ldots, \xi_d)$ isn't a sum of squares in $\operatorname{Gr}(\mathcal{A})$ and so $a \notin \Sigma^2\mathcal{A}$ by the previous lemma.

However, a is the image of a positive semidefinite element f under the ring homomorphism

$$\mathbb{Z}[X_1, \ldots, X_d] \to \mathcal{A} : X_i \mapsto x_i$$

We claim that any ring homomorphism maps positive semidefinite elements to positive semidefinite ones, whence $a \in \mathcal{A}(\operatorname{Sper}(\mathcal{A}))^+$.

Suppose we have a ring homomorphism $\varphi : \mathcal{B} \to \mathcal{B}'$, then we get the pullback $\varphi^* : \mathrm{Sper}(\mathcal{B}') \to \mathrm{Sper}(\mathcal{B})$, this map is continuous. This map is characterised by the fact that for $b \in \mathcal{B}$ and $\beta \in \mathrm{Sper}(\mathcal{B}')$ the sign of b at $\varphi^*(\beta)$ is the sign of $\varphi(b)$ at β, so clearly $\varphi(\mathcal{B}^+) \subset (\mathcal{B}')^+$ as required. □

Corollary 3.1.6 ([Sch00], Corollary 1.3)**.** *Let $\mathcal{A}$ be a noetherian ring, and suppose that there is a real prime ideal $\mathfrak{p}$ such that $\mathcal{A}_\mathfrak{p}$ is regular of dimension ≥ 3. Then $\mathcal{A}(\mathrm{Sper}(\mathcal{A}))^+ \neq \Sigma^2\mathcal{A}$.*

Proof. By the previous proposition, we can find some $a \in \mathcal{A}$ which is positive semidefinite in $\mathcal{A}_\mathfrak{p}$ but isn't in $\Sigma^2\mathcal{A}_\mathfrak{p}$. Consider

$$I = \bigcap_\alpha \{\mathrm{supp}(\alpha) \mid \alpha \in \mathrm{Sper}(\mathcal{A}) \text{ and } a(\alpha) < 0\}$$

Clearly $I \notin \mathfrak{p}$, so pick any $s \in I \backslash \mathfrak{p}$, then $s^2 f$ is positive semidefinite in $\mathcal{A}$ but isn't in $\Sigma^2\mathcal{A}$, because it isn't in $\Sigma^2\mathcal{A}_\mathfrak{p}$. □

See remark 1.4 in [Sch00] to see how this result relates to Hilbert's 17th Problem.

Theorem 3.1.7 ([Sch00], Theorem 6.2)**.** *Let k be a field, and $\mathcal{A}$ a formally real connected k-algebra of finite type. Suppose that $\dim \mathcal{A} \geq 0$, and $\mathrm{Sper}(\mathcal{A}) \neq \varnothing$. Then $\mathcal{A}(\mathrm{Sper}(\mathcal{A}))^+ \neq \Sigma^2\mathcal{A}$.*

Note that if $\mathcal{A}$ isn't formally real, then we can find an a which is identically 0 on $\mathrm{Sper}(\mathcal{A})$ but isn't in $\Sigma^2\mathcal{A}$, see Lemma 6.3 in [Sch00].

Proof. We want to show that under the assumptions of the Theorem we can use Corollary 3.1.6. Indeed, there is some $s \in \mathcal{A}$ such that $\mathcal{A}_s$ is a regular domain with formally real quotient field and $\dim(\mathcal{A}_s) \geq 3$. By the Artin-Lang Theorem (see for example chapter 4 of [BCR98]) $\mathcal{A}_s$ has (plenty of) maximal ideals with formally real residue field. □

We finally remark that $\mathbb{C}[\mathbb{Z}^3]$ satisfies the requirements of the Theorem. For any (not necessarily commutative) group that contains $\mathbb{Z}^3$, the result also holds since being Archimedean closed passes down to subgroups (Lemma 3.0.1).

3.2 (Virtually) Free Groups

At the opposite extreme, we can prove that the cone $\Sigma^2\mathbb{C}[\Gamma]$ is closed in $\mathbb{C}[\Gamma]^h$ with the topology τ_{st} if Γ is free (or virtually free). We do this by using the real closed positivstellensatz (2.3) and approximating generalised $*$-representations by finite dimensional ones — where we can use the Tarski transfer principle (Corollary 1.2.12).

Theorem 3.2.1. *Let $\Gamma = F_n$ be a free group on n generators, and let $\mathcal{R}$ denote its family of finite dimensional $*$-representations (over $\mathbb{C}$).*

*(a) $\mathbb{C}[\Gamma](\mathcal{R})^+ = \Sigma^2\mathbb{C}[\Gamma]$. That is, if $a \in \mathbb{C}[\Gamma]^h$ is mapped to a positive semidefinite matrix under each **finite dimensional** $*$-representation of $\mathbb{C}[\Gamma]$, then $a \in \Sigma^2\mathbb{C}[\Gamma]$;*

(b) The same conclusion holds for a virtually free group $\tilde{\Gamma}$.

Remark 3.2.2. Both (a) and (b) in the above Theorem were known to Schmüdgen, in a private communication to Netzer and Thom. (a) is Theorem 6.1 in [NT13], to our knowledge this is the first place where (b) appears in full.

Proof. Consider an element $a \in \mathbb{C}[\Gamma]^h \backslash \Sigma^2\mathbb{C}[\Gamma]$. By the real closed positivstellensatz (2.3) we have some RCF extension $\mathbb{K}$ of $\mathbb{R}$, and a ($\mathbb{C}$-linear) generalised $*$-representation

$$\pi : \mathbb{C}[\Gamma] \to \mathcal{L}(\mathcal{H})$$

where $\langle \pi(a)\xi, \xi\rangle < 0$ for some unit vector ξ and $\mathcal{H}$ is a $\mathbb{F} = \mathbb{K}[i]$ vector space with a $\mathbb{F}$-valued inner product. Recall that by construction $\mathcal{H}$ is a quotient of $\mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{F}$.

Let the generators of Γ be $s_1, \dots, s_n$, and let $\mathcal{H}_r$ be a finite dimensional $*$-subspace of $\mathcal{H}$, containing the residue classes of all of $B_\Gamma(1, r)$ (in particular, by the GNS construction it also contains ξ which is the residue class of 1_Γ). We can find a basis of $\mathcal{H}_r$ using the standard Gram-Schmidt procedure over $\mathbb{F}$, and use it to define a projection map $P_r : \mathcal{H} \to \mathcal{H}_r$.

Define $X_i^{(r)} := P_r \circ \pi(s_i) \in \mathcal{L}(\mathcal{H}_r)$, these are contractions and so the linear operators $\sqrt{I_r - (X_i^{(r)})^* X_i^{(r)}}$ and $\sqrt{I_r - X_i^{(r)}(X_i^{(r)})^*}$ exist on $\mathcal{H}_r$. We use Choi's unitary trick ([Cho80], Theorem 7. See also [BO08], Theorem 7.4.1) — namely we consider the operators

$$U_i^{(r)} := \begin{pmatrix} X_i^{(r)} & \sqrt{I_r - X_i^{(r)}(X_i^{(r)})^*} \\ \sqrt{I_r - (X_i^{(r)})^* X_i^{(r)}} & -(X_i^{(r)})^* \end{pmatrix} \in \mathcal{L}(\mathcal{H}_r \oplus \mathcal{H}_r)$$

These are unitary operators, and thus (since Γ is free) we get a $\mathbb{C}$-linear $*$-representation π_r of $\mathbb{C}[\Gamma]$ on the space $\mathcal{H}_r \oplus \mathcal{H}_r$. Taking r large enough so that all the (residue classes of) words occurring in b lie in $\mathcal{H}_r$, and setting $\xi' = (\xi, 0) \in \mathcal{H}_r \oplus \mathcal{H}_r$, we have that

$$\langle \pi_r(a)\xi', \xi'\rangle_{\mathcal{H}_r \oplus \mathcal{H}_r} = \langle \pi(a)\xi, \xi\rangle < 0$$

Since $\mathcal{H}_r \oplus \mathcal{H}_r$ is finite dimensional the existence of such a representation over $\mathbb{F}$ implies the existence over $\mathbb{C}$ by Tarski's transfer principle (Corollary 1.2.12), and we are done.

Consider now a virtually free group $\tilde{\Gamma} \geq \Gamma = F_n$, and choose a left transversal $T = \{x_1, \dots, x_s\}$ for $\tilde{\Gamma}/\Gamma$. As before, we have a $\mathbb{C}$-linear generalised $*$-representation $\tilde{\pi} : \mathbb{C}[\tilde{\Gamma}] \to \mathcal{L}(\mathcal{H})$, with $\langle \tilde{\pi}(a)\xi, \xi\rangle < 0$. Consider $\pi = \tilde{\pi}|_\Gamma$, and consider the representations π_r of Γ on $\mathcal{L}(\mathcal{H}_r \oplus \mathcal{H}_r)$, where this time $\mathcal{H}_r$ is generated by the residue classes of $B_\Gamma(1, r)$ and all of the x_i.

The idea is to use the fact that $\mathrm{Ind}_\Gamma^{\tilde{\Gamma}}(\tilde{\pi}|_\Gamma) \cong \tilde{\pi} \otimes \lambda_{\tilde{\Gamma}/\Gamma}$, where $\lambda_{\tilde{\Gamma}/\Gamma}$ denotes the left quasiregular representation of $\tilde{\Gamma}$ on $\ell^2(\tilde{\Gamma}/\Gamma)$, and the fact that the $\mathrm{Ind}_\Gamma^{\tilde{\Gamma}}(\pi_r)$ are all finite dimensional. Since we have continuity of induction (with respect to the Fell topology) these finite dimensional representations converge to our original one, and in particular one of them detects the negativity of a. For the reader's convenience we reproduce the necessary arguments in full below.

Recall that the underlying space of $\mathrm{Ind}_\Gamma^{\tilde{\Gamma}}(\pi)$ is $\mathcal{K} = \{f : \tilde{\Gamma} \to \mathcal{H} \mid f(gh) = \pi(h)^* f(g) \ \forall h \in \Gamma, g \in \tilde{\Gamma}\}$, and $\mathrm{Ind}_\Gamma^{\tilde{\Gamma}}(\pi)(g)f(h) := f(g^{-1}h)$. Consider the function $f_\xi : g \mapsto \pi(g)^*\xi$, and note that its matrix coefficient is given by

$$\begin{aligned} \langle \mathrm{Ind}_\Gamma^{\tilde{\Gamma}}(\pi)(g) f_\xi, f_\xi\rangle &= \bigoplus_{i=1}^{s} \langle \mathrm{Ind}_\Gamma^{\tilde{\Gamma}}(\pi)(g) f_\xi(x_i), f_\xi(x_i)\rangle \\ &= \bigoplus_{i=1}^{s} \langle f_\xi(g^{-1}x_i), f_\xi(x_i)\rangle \\ &= \bigoplus_{i=1}^{s} \langle \pi(x_i)^* \pi(g)\xi, \pi(x_i)^*\xi\rangle \\ &= s\langle \pi(g)\xi, \xi\rangle < 0 \qquad (*) \end{aligned}$$

We now need to show that this matrix coefficient is approximated by the finite dimensional $\mathrm{Ind}_\Gamma^{\tilde{\Gamma}}(\pi_r)$, and so in particular one of them is negative. This is a specialised form of the continuity of induction argument, see for example Theorem F.3.5 in [BdlHV08]. Consider the function $f_i : \Gamma \to \mathcal{H}$ where

$$f_i(g) := \sum_{h \in H} \delta_{x_i}(gh)\pi(h)\pi(x_i)^*\xi = \begin{cases} \pi(g)^*\xi & g \in x_i H \\ 0 & \text{otherwise} \end{cases}$$

and the functions $f_i^{(r)} : \Gamma \to \mathcal{H}_r \oplus \mathcal{H}_r$ given by

$$f_i^{(r)}(g) \coloneqq \Big(\sum_{h \in H} \delta_{x_i}(gh)\pi_r(h)P_r(\pi(x_i)^*\xi), 0 \Big)$$

(recall that $\pi(x_i)^*\xi \in \mathcal{H}_r$ by definition and so $P_r(\pi(x_i)^*\xi) = \pi(x_i)^*\xi$). We calculate

$$\begin{aligned}
\langle \mathrm{Ind}_\Gamma^{\tilde{\Gamma}}(\pi)(g)f_i, f_i \rangle &= \sum_{j=1}^{s} \langle f_i(g^{-1}x_j), f_i(x_j) \rangle \\
&= \sum_{j=1}^{s} \sum_{h,k \in H} \langle \delta_{x_i}(g^{-1}x_j h)\pi(h)\pi(x_i)^*\xi, \delta_{x_i}(x_j k)\pi(k)\pi(x_i)^*\xi \rangle \\
&= \sum_{j=1}^{s} \sum_{h,k \in H} \delta_{x_i}(g^{-1}x_j h)\delta_{x_i}(x_j k)\langle \pi(h)\pi(x_i)^*\xi, \pi(k)\pi(x_i)^*\xi \rangle \\
&= \sum_{j=1}^{s} \sum_{h \in H} \delta_{x_i}(g^{-1}x_j h)\langle \pi(h)\pi(x_i)^*\xi, \pi(x_i)^*\xi \rangle
\end{aligned}$$

Similarly we compute that

$$\langle \mathrm{Ind}_\Gamma^{\tilde{\Gamma}}(\pi)(g)f_i, f_i \rangle - \langle \mathrm{Ind}_\Gamma^{\tilde{\Gamma}}(\pi_r)(g)f_i^{(r)}, f_i^{(r)} \rangle = \sum_{j=1}^{s} \sum_{h \in H} \delta_{x_i}(g^{-1}x_j h)\Big\langle \big(\pi(h) - \pi_r(h)\big)\pi(x_i)^*\xi, \pi(x_i)^*\xi \Big\rangle$$

and this is zero when $h \in B_\Gamma(1, r)$. Now observe that $\sum_i f_i = f_\xi$, and let $f_\xi^{(r)} = \sum_i f_i^{(r)}$.
So we see that for big enough r,

$$\langle \mathrm{Ind}_\Gamma^{\tilde{\Gamma}}(\pi_r)(g)f_\xi^{(r)}, f_\xi^{(r)} \rangle = \langle \mathrm{Ind}_\Gamma^{\tilde{\Gamma}}(\pi)(g)f_\xi, f_\xi \rangle < 0$$

and so negativity of a is detected in a finite dimensional generalised representation — and so by Tarski's transfer principle again, in a finite dimensional $\mathbb{C}$-representation of $\mathbb{C}[\tilde{\Gamma}]$. □

Remark 3.2.3. Some words are required on the use of the transfer principle here — the first order statement involves the generators of Γ and the transversal $\{x_i\}$, and reads 'when this finite dimensional representation of Γ is induced, this particular matrix coefficient is negative'. This avoids the problem of having to worry if $\tilde{\Gamma}$ is finitely presented or not.

Notice that being Archimedean closed also holds for any countably generated free group, by considering the finitely generated free group that our element b is contained in, or by recalling that this property passes to subgroups (Lemma 3.0.1).

Example 3.2.4. $\mathrm{SL}(2, \mathbb{Z})$ is virtually free, indeed we have the subgroup

$$\Lambda = \langle \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right) \rangle \cong F_2$$

and $[\mathrm{SL}(2, \mathbb{Z}) : \Lambda] = 12$.

Example 3.2.5. Free products of finite groups are virtually free, indeed, suppose $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ are finite groups, and consider the canonical epimorphism $\Gamma_1 * \Gamma_2 * \cdots * \Gamma_n \to \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n$. By a Theorem of Nielsen (see for example Theorem 2 in [Lyn73]) the kernel of this is free.

There is a similar positivstellensatz for any *operator valued* free group algebra due to McCullough ([McC01]) and Bakonyi-Timotin ([BT07]). See section 14 in [Oza12] for a simpler proof.

Chapter 4

Residual Finite Dimensionality

We saw in the previous chapter that the strong positivstellensatz came about from approximating generalised representations by finite dimensional ones — so it is natural to consider when we can do this, at least over $\mathbb{R}$ (so just normal unitary representations). This is an active area of research, and so we present some of the theory below.

4.1 Definitions and Equivalent Characterisations

Definition 4.1.1. A group Γ is *Residually Finite Dimensional (RFD)* if its finite dimensional representations are dense in the set of representations of Γ with respect to the Fell topology. In this case we also say that $C^*(\Gamma)$ is RFD.

Proposition 4.1.2. *The following are equivalent:*

(a) $C^*(\Gamma)$ *is RFD;*

(b) Finite dimensional representations separate the points of $C^*(\Gamma)$*;*

(c) There exists a faithful representation

$$C^*(\Gamma) \hookrightarrow \bigoplus_n M_n(\mathbb{C})$$

Proof. The equivalence of (b) and (c) is trivial, so assume (a) and consider any $a \in C^*(\Gamma)$ and a representation π such that $\pi(a) \neq 0$. In particular, there is some (unit) vector $\xi \in \mathcal{H}_\pi$ such that $\pi(a)\xi \neq 0$, so consider the matrix coefficient $\langle \pi(a)\xi, \xi\rangle$. By assumption we can approximate this uniformly on $\{a^*a\}$ by functions of positive type associated to a finite dimensional representation of Γ — that is, for any $\varepsilon > 0$ there exists a finite dimensional $\rho : C^*(\Gamma) \to \mathbb{B}(\mathcal{H}_\rho)$ and (unit) vectors $\xi_1, \ldots, \xi_n \in \mathcal{H}_\rho$ such that

$$\left|\langle \pi(a^*a)\xi, \xi\rangle - \sum_{i=1}^{n} \langle \rho(a^*a)\xi_i, \xi_i\rangle\right| < \varepsilon$$

Take $0 < \varepsilon < \|\pi(a)\xi\|^2 = \langle \pi(a^*a)\xi, \xi\rangle$, and so for some ξ_i we have $\|\rho(a)\xi_i\|^2 = \langle \rho(a^*a)\xi_i, \xi_i\rangle \neq 0$ — in particular $\rho(a) \neq 0$.

Conversely recall that we have the state space $\mathcal{S}(C^*(\Gamma))$, and denote by $\mathcal{S}_{\text{fd}}(C^*(\Gamma))$ the set of finite dimensional states. Notice that $\mathcal{S}_{\text{fd}}(C^*(\Gamma))$ is convex — indeed if $\varphi_1, \varphi_2 \in \mathcal{S}_{\text{fd}}(C^*(\Gamma))$ then the GNS representation of $t\varphi_1 + (1-t)\varphi_2$ is equivalent to a subrepresentation of the sum of the GNS representations of φ_1 and φ_2.

Assume for contradiction that there is some $\varphi \in \mathcal{S}(C^*(\Gamma)) \backslash \overline{\mathcal{S}_{\text{fd}}(C^*(\Gamma))}$ (where we've taken the weak-$*$ closure). We can identify $C^*(\Gamma)^h$ with $(C^*(\Gamma)^*)^h$, where the latter is the space of self-adjoint continuous linear functionals endowed with the weak-$*$ topology. So by the Hahn-Banach Theorem we have an element $a \in C^*(\Gamma)^h$ and some $R \in \mathbb{R}$ such that $\varphi(a) > R$, but $\psi(a) \leq R$ for all $\psi \in \mathcal{S}_{\text{fd}}(C^*(\Gamma))$. That is, for any finite dimensional representations π of $C^*(\Gamma)$ and any unit vector $\xi \in \mathcal{H}_\pi$, $\langle \pi(a)\xi, \xi\rangle \leq R$, and in particular $\pi(a) \leq R$. Since by assumption the direct sum of all finite-dimensional representations of $C^*(\Gamma)$ is faithful we must have that $a \leq R$, but this contradicts $\varphi(a) > R$. $\square$

The latter is more frequently given as the definition of RFD algebras, see for example chapter 7 in [BO08]. As such, we say that a general C^* algebra $\mathcal{A}$ is RFD if its finite dimensional representations separate points. There is again an equivalent characterisation in terms of density in the spectrum, where the topology is the pullback of the hull-kernel topology on the primitive spectrum $\mathrm{Prim}(\mathcal{A})$ under the surjection $\pi \mapsto \ker \pi$. We don't describe this (since the topology coincides with the Fell topology for C^* algebras of locally compact groups anyway) but we remark that the proof is not much more involved (also in the case that $\mathcal{A}$ isn't unital). For all of this, see the book of Dixmier [Dix69].

Example 4.1.3. Commutative and finite dimensional C^* algebras are clearly RFD. Theorem 3.2.1 in particular shows that $C^*(F_n)$ is RFD for any $n \in \mathbb{N}$, this was originally shown by Choi ([Cho80]). The proof can also be modified (and simplified) to show that if $\Gamma \leq \tilde{\Gamma}$ is of finite index and Γ is RFD, then so is $\tilde{\Gamma}$.

Example 4.1.4. We've already remarked that free products of finite groups are virtually free, and in particular their C^* algebras are also RFD (clearly finite groups are RFD). More generally, Exel and Loring showed that the free product of RFD algebras is again RFD, see [EL92]. Many amalgamated free products of RFD algebras are again RFD, see [Shu22].

Remark 4.1.5. Satisfying a strong positivstellensatz is (much) stronger than being RFD — for example $\mathbb{Z}^3$ is commutative and hence RFD, but we saw that it doesn't satisfy anything like Theorem 3.2.1.

Remark 4.1.6. Is $F_n \times F_m$ RFD for any n, m? Even for $n = m = 2$ this question is very open. In fact, the following are equivalent:

(a) There is a unique C^* norm on $C^*[F_n] \otimes C^*[F_n]$;

(b) $C^*[F_n \times F_n]$ is RFD.

and both are equivalent to the Connes Embedding Problem (CEP), as was shown by Kirchberg in his seminal paper [Kir93], see for example Proposition 7.4.4 in [BO08] for the above equivalence. Notice however that it is easy to see that $C^*[F_n \times F_n]$ is *quasidiagonal*, which can be thought of as 'approximately RFD', we won't comment on this further.

Proposition 4.1.7 ([Cho80], Corollary 9)**.** *Let $\mathcal{A}$ be an RFD C^*=algebra. Then it has a finite faithful trace.*

Proof. Since $\mathcal{A}$ is RFD there is faithful representation $\pi : \mathcal{A} \hookrightarrow \bigoplus_{n\geq 1} M_n(\mathbb{C})$. Let Tr_n be the finite faithful normalised trace on $M_n(\mathbb{C})$ and define

$$\mathrm{Tr} : \bigoplus_{n\geq 1} M_n(\mathbb{C}) \to \mathbb{C} : (A_n)_n \mapsto \sum_{n\geq 1} 2^{-n}\, \mathrm{Tr}_n(A_n)$$

Then $\mathrm{Tr} \circ \pi$ is a finite faithful trace on $\mathcal{A}$. □

So following on from remark 4.1.6 we naturally ask

Question ([Kir93])**.** *Does there exist a faithful trace on $C^*(F_2 \times F_2)$?*

Remark 4.1.8. $F_2 \times F_2$ is a subgroup of $\mathrm{SL}(4, \mathbb{Z})$, and so $C^*(F_2 \times F_2) \subset C^*(\mathrm{SL}(4, \mathbb{Z}))$. So if there is a faithful trace on the latter we would get one on $C^*(F_2 \times F_2)$; however, Bekka proved (unpublished) that $C^*(\mathrm{SL}(n, \mathbb{Z}))$ doesn't have a faithful trace for $n \geq 3$.

We will discuss CEP a bit more in section 4.6.

4.2 A Lifting Characterisation

Suppose that our C^* algebra $\mathcal{A}$ is separable. We might expect that in this case it might be easier to lift any faithful representation of $\mathcal{A}$ (which exists by the Gelfand-Naimark Theorem) to a direct sum of finite dimensional ones, by using separability of the target Hilbert space. We now investigate this.

Consider a separable Hilbert space $\mathcal{H}$ with orthonormal basis $\{e_1, e_2, \ldots\}$, let P_n be the projection onto $\mathrm{Span}\{e_1, \ldots, e_n\}$, and denote by ℓ^2 the standard ℓ^2 space on the basis. Consider the space $M_n = P_n \mathbb{B}(\ell^2) P_n$ (this is just the natural inclusion of $M_n(\mathbb{C})$ into ℓ^2), and consider the spaces

$$\mathcal{B} = \left\{ (T_n)_n \in \prod_{n=1}^{\infty} M_n \mid \exists T \in \mathbb{B}(\ell^2) \text{ with } T_n \to T \ (*\text{-SOT}) \right\}$$
$$\mathcal{I} = \{ (T_n)_n \in \mathcal{B} \mid T_n \to 0 \ (*\text{-SOT}) \}$$

We then have the obvious projection map

$$P : \mathcal{B} \to \mathbb{B}(\mathcal{H}) : (T_n) \mapsto (*\text{-SOT})\text{-}\lim_n T_n$$

Lemma 4.2.1 ([Had14], Lemma 1)**.** *We have that*

(a) $\mathcal{B}$ is a unital C^*-algebra;

(b) $\mathcal{I}$ is a closed two sided ideal in $\mathcal{B}$;

(c) If $T \in \mathbb{B}(\mathcal{H})$, then $(P_n T P_n)_n \in \mathcal{B}$ and $P((P_n T P_n)_n) = T$;

(d) P is a unital surjective $*$-homomorphism;

(e) If $U \in \mathbb{B}(\mathcal{H})$ is unitary, then there exists a unitary $(U_n)_n \in \mathcal{B}$ such that $P((U_n)_n) = U$

Proof. (a) - (d) are easy to show. To prove (e), note that by standard functional calculus there is a self-adjoint element $A = A^* \in \mathbb{B}(\mathcal{H})$ such that $U = \exp(iA)$. By (c) we can find an element $(A_n)_n = (A_n)_n^* \in \mathcal{B}$ with $P((A_n)_n) = A$. If we let $U_n = \exp(iA_n) \in M_n$, we get the element we want. □

Suppose we have a separable C^*-algebra such that some faithful representation $\mathcal{A} \to \mathbb{B}(\mathcal{H})$ lifts to a representation from $\mathcal{A} \to \mathcal{B}$, then clearly $\mathcal{A}$ must be RFD.

Remark 4.2.2. Notice that this is in some ways a unified technique for doing what we did when proving that free groups are RFD. In particular, if we allow for any net of projections P_α with $(*\text{-SOT}) \lim P_\alpha = \mathrm{Id}$ we can modify this to show that *any* free group is RFD.

This technique for proving that an algebra is RFD was used by Goodearl and Menal in [GM90], see also [Lor97]. It was conjectured by Loring that every separable RFD C^*-algebra has this lifting property; this was shown in the positive by Hadwin, see [Had14], as we now sketch.

Definition 4.2.3. For a (unital or nonunital) C^* algebra $\mathcal{A}$, the *unitization* of $\mathcal{A}$ is defined to be

$$\mathcal{A}^e := \mathcal{A} \oplus \mathbb{C}$$

equipped with the obvious addition, and multiplication $(x, z) \cdot (y, w) := (xy + zy + wx, zw)$.

Recall that for nonunital commutative C^*-algebras, this corresponds to taking a one-point compactification by the Gelfand transform. The reason we need it here, even when $\mathcal{A}$ is already unital, is that it guarantees the existence of a unital one-dimensional representation.

Theorem 4.2.4 ([Had14], Theorem 11)**.** *Suppose $\mathcal{A}$ is a separable C^*-algebra. Then $\mathcal{A}$ is RFD if and only if for every unital $*$-homomorphism $\pi : \mathcal{A}^e \to \mathbb{B}(\ell^2)$ there is a unital $*$-homomorphism $\rho : \mathcal{A}^e \to \mathcal{B}$ such that $P \circ \rho = \pi$.*

Proof sketch. We remarked the 'if' direction above. Suppose that $\mathcal{A} = C^*(\{a_1, a_2, \ldots\})$ and $\pi : \mathcal{A}^e \to \mathbb{B}(\ell^2)$ is a unital $*$-homomorphism.

By Theorem 6 in [Had14] there is a strictly increasing sequence $(n_k)_k$ and unital $*$-homomorphisms π_{n_k} : $\mathcal{A} \to M_{n_k}$, such that

$$\|(\pi_{n_k}(a_j) - \pi(a_j))e_i\| < \tfrac{1}{k}$$

for $1 \le i, j, \le k$. In particular, $\pi_{n_k}(a) \to \pi(a)$ in $*$-SOT for every $a \in \mathcal{A}^e$. Let $\varphi : \mathcal{A}^e \to \mathbb{C}$ be the unique $*$-homomorphism with $\ker\varphi = \mathcal{A}$ — we use it to define π_n for all n. Indeed, set $\pi_n : \mathcal{A}^e \to M_n$ for $n_k < n < n_{k+1}$ to be

$$\pi_n(a) = \begin{pmatrix} \pi_{n_k}(a) & & & \\ & \varphi(a) & & \\ & & \ddots & \\ & & & \varphi(a) \end{pmatrix}$$

relative to the decomposition

$$P_n(\ell^2) = P_{n_k}(\ell^2) \oplus \mathbb{C}e_{n_k+1} \oplus \cdots \oplus \mathbb{C}e_{n_{k+1}-1}$$

Clearly $\pi_n(a) \to \pi(a)$ in $*$-SOT for every $a \in \mathcal{A}^e$, and so simply define $\rho(a) = (\pi_n(a))_n$. □

4.3 Residual Finiteness and Amenability

The RFD property for groups says that we have separation of $C^*(\Gamma)$ by finite dimensional representations; but we can just ask for separation of Γ itself. In all that follows Γ will be a countable discrete group, although many of the definitions and results can be stated in greater generality.

Definition 4.3.1. A group Γ is *maximally almost periodic (MAP)* if its finite dimensional representations separate points of Γ.

Definition 4.3.2. A group Γ is *residually finite (RF)* if its finite quotients separate points of Γ.

Clearly RFD groups are MAP. Clearly also RF groups are MAP, we remark that the converse holds in the finitely generated case.

Proposition 4.3.3. *A finitely generated MAP group is RF.*

Proof. Recall that *Mal'cev's Theorem* tells us that finitely generated linear groups are RF — the result clearly follows. □

Before we continue studying when $C^*(\Gamma)$ might be RFD, we note that it isn't the only C^* algebra that can naturally be associated to a group.

Definition 4.3.4. The *reduced group C^*-algebra* $C^*_\lambda(\Gamma)$ is the norm closure of the linear span of $\{\lambda_\Gamma(g) \mid g \in \Gamma\} \subset \mathbb{B}(\ell^2(\Gamma))$.

Equivalently it is the C^*-completion of $\mathbb{C}[\Gamma]$ with respect to the norm $\|f\|_\lambda := \sup\{\|f * g\|_2 \mid \|g\|_2 = 1\}$ (where f and g are finitely supported functions on Γ). To see this note that $\lambda_\Gamma : \mathbb{C}[\Gamma] \to \mathbb{B}(\ell^2(\Gamma))$ is an isometry into a complete metric space, and hence extends to an isometry of the completion.

We therefore also see that $C^*_\lambda(\Gamma)$ is a quotient of the full group C^* algebra, again by extending the map λ_Γ to a surjective isomorphism $\lambda_\Gamma : C^*(\Gamma) \to C^*_\lambda(\Gamma)$.

Remark 4.3.5. It turns out to be an interesting question whether or not $C^*_\lambda(\Gamma)$ has a *unique* trace, compare with Corollary 4.1.7. See [BKKO17] for some recent progress on this, and the notion of C^* *simplicity* (which asks if $C^*_\lambda(\Gamma)$ is simple).

Definition 4.3.6. A group Γ is *amenable* if $1_\Gamma \prec \lambda_\Gamma$.

It is not too hard to see that amenability is equivalent to the fact that $C^*(\Gamma) \cong C^*_\lambda(\Gamma)$, using the standard fact that for two representations π and ρ of $C^*(\Gamma)$,

$$\pi \prec \rho \qquad \Leftrightarrow \qquad \ker\rho \subset \ker\pi \tag{4.1}$$

(Theorem F.4.4 in [BdlHV08]). See section G in [BdlHV08] for all of this.

We can also ask when $C^*_\lambda(\Gamma)$ is RFD — but this turns out to be relatively easier.

Theorem 4.3.7 ([BL00], Theorem 4.3)**.** *Let Γ be a countable discrete group. Then $C^*_\lambda(\Gamma)$ is RFD if and only if Γ is an amenable MAP group.*

We need an easy lemma, that will be useful for us a few times.

Lemma 4.3.8. *Let π be a finite dimensional representation of Γ, and let $\overline{\pi}$ denote its contragredient representation. Then $1_\Gamma \leq \pi \otimes \overline{\pi}$.*

Proof. We can realise $\pi \otimes \overline{\pi}$ on $\mathrm{End}(M_n(\mathbb{C}))$ by

$$(\pi \otimes \overline{\pi})(g)A = \pi(g)A\pi(g)^{-1}$$

for $A \in \mathrm{End}(M_n(\mathbb{C}))$ and $g \in \Gamma$, then clearly the identity is a fixed point. □

Proof of Theorem 4.3.7. If $C^*_\lambda(\Gamma)$ is RFD then clearly Γ is MAP since Γ injects in $C^*_\lambda(\Gamma)$, we now show it is amenable. In fact more generally, we claim that if $C^*_\lambda(\Gamma)$ has *any* finite dimensional representation, then Γ is amenable.

Indeed, let π be a finite dimensional representation of $C^*_\lambda(\Gamma)$, then by Lemma 4.3.8 we have that $1_\Gamma \leq \pi \otimes \overline{\pi}$. So $1_\Gamma \leq \pi \otimes \overline{\pi} \prec \lambda_\Gamma \otimes \overline{\lambda_\Gamma} \cong \infty\lambda_\Gamma$, and in particular $1_\Gamma \prec \lambda_\Gamma$.

Conversely, since λ_Γ is a cyclic representation, it suffices to show that the matrix coefficient $\langle \lambda_\Gamma(\cdot)\delta_e, \delta_e \rangle$ can be uniformly approximated on finite sets by matrix coefficients associated to finite dimensional representations of Γ.

Let $g_1, \ldots, g_n \in \Gamma$, $g_i \neq 1$, so since Γ is MAP there exists a finite dimensional representation π of Γ with $\pi(g_i) \neq 1$ for all $i = 1, \ldots, n$.

Choose unit vectors $\xi_i \in \mathcal{H}_\pi$ with $\pi(g_i)\xi_i \neq \xi_i$, and let

$$\varphi(g) = \frac{1}{n+1}\Big(\sum_{i=1}^{n} \langle \pi(g)\xi_i, \xi_i \rangle + 1\Big)$$

This is a positive definite function on Γ associated to a finite dimensional representation, and $|\varphi(g_i)| < 1$ for all $i = 1, \ldots, n$.

Hence $\varphi^k(g_i) \to 0$ as $k \to \infty$ (whereas $\varphi^k(1) = 1$), and as φ^k is a matrix coefficient associated to a finite dimensional representation, the claim follows. □

Corollary 4.3.9. *An amenable group is RFD if and only if it is MAP. A finitely generated amenable group is RFD if and only if it is RF.*

Proof. As noted before, $C^*_\lambda(\Gamma) \cong C^*(\Gamma)$ if and only if Γ is amenable. □

The finitely generated result above can also be deduced from the work of T. Shulman [Shu22], see Corollary 6.10.

4.4 High Rank Phenomena

So far we haven't seen any examples of non RFD groups, we turn to this now. In this section we blackbox a lot of results but provide references.

Theorem 4.4.1 ([Bek99], main Theorem)**.** *Let k be a number field, $\mathcal{O}_k$ the corresponding ring of integers, S a set of places including all the Archimedean ones, and $\mathcal{O}_S = \{x \in k \mid \nu(x) \geq 0$ for all $v \notin S\}$ the S-arithmetic integers. Let $\mathbb{G}$ be a simple, simply connected k-algebraic group, and let $\Gamma = \mathbb{G}(\mathcal{O}_S)$. Assume that*

(a) k-$\mathrm{rank}(\mathbb{G}) \geq 1$;

(b) $\sum_{\nu \in S} k_v\text{-rank}(\mathbb{G}) \geq 2$*;*

Then Γ *isn't RFD.*

Remark 4.4.2. The expression in (b) is called the S-rank *of* $\mathbb{G}$.

Examples covered by this Theorem are $\mathrm{SL}(n, \mathbb{Z})$ for $n \geq 3$ and $\mathrm{Sp}(n, \mathbb{Z})$ for $n \geq 2$ — both of these examples have Property (T). However, the Theorem also covers groups such as $\mathrm{SL}(2, \mathbb{Z}[\sqrt{2}])$ which doesn't have Property (T).

Notice also that the rank condition is necessary — we saw already that $\mathrm{SL}(2, \mathbb{Z})$ is RFD (in fact it satisfies a strong Positivstellensatz).

Before we sketch how to prove this Theorem, let us see a general example that illustrates the argument.

Definition 4.4.3. A group Γ has *property* (τ) if the trivial representation is isolated (with respect to the Fell topology) in the set of finite dimensional representations.

This was defined by Lubotzky-Zimmer in [LZ89] to generalise property (T), which in turn is equivalent (as we shall see) to the trivial representation being isolated in the set of all irreducible representations. Probably all the examples covered by Theorem 4.4.1 have property (τ), but this isn't known.

Lemma 4.4.4 ([Bek99], Lemma 2)**.** *Let* Γ *be a discrete group. Assume that there is a subgroup* $\Lambda \leq \Gamma$ *such that*

(a) Λ *doesn't have property (T);*

(b) 1_Λ *is isolated in* $\{\pi|_\Lambda : \pi$ *is a finite dimensional unitary representation of* $\Gamma\}$.

Then $C^*(\Gamma)$ *is not RFD.*

Corollary 4.4.5. *If* Γ *has property* (τ) *but not property (T) then it is not RFD.*

Proof. This is just $\Lambda = \Gamma$ in the Lemma above. □

For example, this holds for $\mathrm{SL}(2, \mathbb{Z}[1/p])$ for any prime p.

Proof of Lemma 4.4.4. Suppose on the contrary that $C^*(\Gamma)$ is RFD, so in particular the finite dimensional representations of $C^*(\Gamma)$ separate the elements of $C^*(\Lambda)$. So any unitary representation of Λ is weakly contained in the set

$$\{\pi|_\Lambda \mid \pi \text{ is a finite dimensional unitary representation of } \Gamma\}$$

Since Λ doesn't have property (T) we see that in particular 1_Λ isn't isolated in this set — but this contradicts assumption (b). □

Proof sketch of Theorem 4.4.1. We make note of the following facts about Γ:

(i) Since $k\text{-rank}(\mathbb{G}) \geq 1$, Γ has a non-trivial unipotent element u such that the length of u^n with respect to a fixed finite set of generators of Γ is $\mathcal{O}(\log n)$ (see [LMR00]);

(ii) By the Margulis Normal Subgroup Theorem, every normal subgroup of Γ is either finite or is of finite index (see [Mar91] and [Zim84]);

(iii) Any representation π of Γ with a kernel of finite index factors through a finite quotient (this is an easy consequence of the fact that Γ satisfies the congruence subgroup property, we discuss this briefly in section 8.3.4).

Suppose we have a finite dimensional representation π of Γ. Then there is some k such that $\pi(u^k)$ is a unipotent element ([LMR00]). But the unitary group $U_n(\mathbb{C})$ has no nontrivial unipotents, so we must have that $\pi(u^k) = I_n$.

Hence $\ker(\pi)$ is infinite, and so by (ii) it is of finite index, and by (iii) factors through a finite quotient. So we've shown that any finite dimensional representation of Γ factors through a finite quotient.

On the other hand, since k-rank$(\mathbb{G}) \geq 1$ there is an algebraic homomorphism with finite kernel from SL_2 into $\mathbb{G}$, so up to commensurability either $\Lambda = \mathrm{SL}(2,\mathbb{Z})$ (or $\Lambda = \mathrm{PSL}(2,\mathbb{Z})$) is embedded in Γ in a way that the congruence topology on Γ induces the congruence topology on $\mathrm{SL}(2,\mathbb{Z})$ (this is the topology induced by $\mathrm{SL}(2,\mathbb{A})$, where $\mathbb{A}$ denotes the adeles).

Suppose now for contradiction that the finite dimensional representations of Γ separate the points of $C^*(\Gamma)$ — in particular, their restrictions to Λ separate the points of $C^*(\Lambda)$. But in our case, the finite dimensional representations of Γ restricted to Λ only give the congruence quotient representations of $\mathrm{SL}(2,\mathbb{Z})$. By Selberg's Theorem ([LZ03], §4.1 and [Sel65]) the latter are bounded away from the trivial representation. Λ doesn't have (T), so the finite congruence representations cannot be dense in the unitary dual of Λ. □

We remark that Theorem 4.4.1 also holds in the $\mathrm{char}(k) = p > 0$ case, with $\mathrm{SL}(2,\mathbb{Z})$ being replaced by $\mathrm{SL}(2,\mathbb{F}_p[t])$. The analogous Selberg property is replaced by a result of Drinfel'd, see [Dri88]. The proof also works in many cases where the k-rank = 0, when the congruence subgroup property holds. See the remarks in section 9.1 of [LZ03] for more details.

4.5 Property FD

We continue this discussion with some more examples in which RFD is known. In fact, following the work of Lubotzky-Shalom ([LS04]), a stronger property is known.

Definition 4.5.1. Let Γ be a discrete group. We say a representation Γ is *finite* if it factors through a finite quotient. We say Γ has the *finitary density property (FD)* if its finite representations are dense in the set of all unitary representations.

Clearly, FD implies RFD, and FD passes to subgroups. In fact, some of the examples of RFD groups we've seen satisfy this stronger property.

Proposition 4.5.2. *Let Γ be an amenable, residually finite group. Then Γ has property FD.*

Proof. It is a theorem of Shalom that for a residually finite group Γ, $\ell^2(\Gamma)$ is in the closure of the finite representations — see Theorem 4.13 in [LZ03]. If Γ is amenable, then the closure of $\ell^2(\Gamma)$ contains all the unitary representations. □

Similarly, the free groups in fact have property FD.

Theorem 4.5.3 ([LS04], Theorem 2.2)**.** *Let Γ be a countably generated free group. Then it has property FD.*

The nice thing about property FD is that it often passes from a subgroup to the main group (in RFD groups this is not necessarily the case, but as we've noted this happens if the subgroup is of finite index).

Since finite representations are intimately related to their corresponding finite quotients, it makes sense that we will need to consider the set of all of these at once. There is a neat algebraic way to do this.

Definition 4.5.4. Let Γ be a discrete group. Its *profinite completion* is

$$\hat{\Gamma} \coloneqq \varprojlim \Gamma/N$$

where we've taken the inverse limit over the directed system of the normal subgroups of Γ, ordered by reverse inclusion (in the general topological group case, we take the *open* normal subgroups).

In this section only, $\hat{\Gamma}$ will denote the profinite completion and not the unitary dual.

Remark 4.5.5. There is a natural map $\Gamma \to \hat{\Gamma}$, which is injective if and only if Γ is residually finite.

Definition 4.5.6. The *profinite topology* on Γ is the topology which has as a basis of open sets the cosets of finite index normal subgroups of Γ.

Remark 4.5.7. For any inclusion $\imath : \Lambda \hookrightarrow \Gamma$, we get a corresponding extension $\hat{\imath} : \hat{\Lambda} \to \hat{\Gamma}$. This is injective if and only if the profinite topology of Γ induces the profinite topology on Λ. Unpacking the definitions this just says that for any finite index subgroup $N \triangleleft \Lambda$, there is some finite index subgroup $M \triangleleft \Gamma$ such that $M \cap \Lambda \leq N$.

Proposition 4.5.8 ([LZ03], Proposition 9.11)**.** *Let Γ be a countable discrete group and $\Lambda \triangleleft \Gamma$ a normal subgroup such that*

(a) Λ is finitely generated;

(b) $Z(\hat{\Lambda}) = 1$;

(c) Γ/Λ is amenable and residually finite.

Then if Λ is FD, so is Γ.

Proof sketch. Conditions (a) and (b) guarantee that Γ induces the profinite topology on Λ — this uses the fact that $\mathrm{Aut}(\hat{\Lambda})$ is profinite (since Λ is finitely generated, see [DDSMS99]).

Consider now any representation π of Γ — by assumption, $\pi|_\Lambda$ is a limit of finite representations $\{\pi_i\}$ of Λ. Since Γ induces the profinite topology on Λ, for each i there is some normal subgroup $M_i \triangleleft \Gamma$ of finite index such that $N_i = M_i \cap \Lambda \leq \ker \pi_i$.

Therefore by recalling (4.1) we see that $\pi_i \prec \ell^2(\Lambda/\ker \pi_i) \prec \ell^2(\Lambda/N_i)$. By continuity of induction (see Theorem F.3.5 in [BdlHV08]) we therefore have that $\mathrm{Ind}_\Lambda^\Gamma(\pi_i|_\Lambda)$ is a limit of $\mathrm{Ind}_\Lambda^\Gamma(\ell^2(\Lambda/N_i)) = \ell^2(\Gamma/N_i)$.

By amenability of Γ/Λ, we have that $1_{\Gamma/\Lambda} \prec \ell^2(\Gamma/\Lambda)$ and hence

$$\pi \prec \pi \otimes \ell^2(\Gamma/\Lambda) = \mathrm{Ind}_\Lambda^\Gamma(\pi|_\Lambda)$$

So π is a limit of the $\ell^2(\Gamma/N_i)$'s, but each of the Γ/N_i is residually finite (since Γ/Λ is, and by our choice of N_i) and hence each $\ell^2(\Gamma/N_i)$ is a limit of finite representations, and so is π. ☐

This is what was used by Lubotzky and Shalom in [LS04] to show that some groups that aren't covered by Theorem 4.4.1 are FD, and hence RFD.

It is well known that the *Picard group* $\mathrm{SL}(2, \mathbb{Z}[i])$ is commensurable to a group Γ, which has a subgroup Λ such that

(a) Λ is finitely generated and free;

(b) In particular, $Z(\hat{\Lambda}) = 1$ (see [LvdD81]);

(c) Γ/Λ is abelian, and hence amenable and residually finite.

Since Λ is free, it is FD by Theorem 4.5.3, and so the fact that Γ (and hence $\mathrm{SL}(2, \mathbb{Z}[i])$) are FD follows from the above proposition.

A similar argument works also for $\mathrm{SL}(2, \mathbb{Z}[\sqrt{-3}])$, see Theorem 2.8 in [LS04] and the preceding comments for the details of both.

Definition 4.5.9. The *surface groups* are the fundamental groups of the connected closed orientable surfaces (of genus $g \geq 1$). They are given by the presentation $T_g = \langle a_1, \ldots, a_g, b_1, \ldots, b_g \mid \prod [a_i, b_i] = 1 \rangle$.

Following [Mac86] and [CLR97] it is possible to embed these into the groups $\mathrm{SL}(2, \mathbb{Z}[i])$ and $\mathrm{SL}(2, \mathbb{Z}[\sqrt{-3}])$ and hence they are also FD, see Theorem 9.12 in [LZ03] or [LS04] for the details.

4.6 Computability of Norms

It would be a shame not to expand slightly our conversation about the Connes Embedding Problem, given that we have already come across it twice: the group algebraic version of it is equivalent to $F_2 \times F_2$ being RFD (see remark 4.1.6), and as in example 2.7.12 it is also equivalent to a tracial positivestellensatz for the algebra of polynomials in noncommuting hermitian variables. In fact we will see the CEP again in chapter 8.

We consciously haven't stated Connes' original formulation of the Problem, and we even more consciously ignore the fact that in full generality, it has (probably) been proven false in 2021 ([JNV+21]), by showing the equivalence of two classes of algorithms in quantum complexity theory (MIP* = RE). It is however still open in the case of group algebras. We refer the reader to the excellent recent survey article [Gol21], where the many different formulations of CEP are explored to really whet the reader's appetite. We also recommend to [Oza04], [BO08], and [Oza12] for detailed expositions that touch upon the different aspects (all written by Ozawa). The latter is of particular interest for us, being a key milestone in the development of non commutative Real Algebraic Geometry.

In this section, we further show the deep links between RFD groups and the CEP, following Fritz, Netzer, and Thom ([FNT14]) — all that follows is taken from there. As noted above, CEP has links to various computational questions, so we investigate some of these.

Definition 4.6.1. A real number $\alpha \in \mathbb{R}$ is *computable* if it can be approximated to any precision with rational numbers by a Turing machine.

This is equivalent to asking for two sequences of rational numbers: (p_n) which is monotone increasing, and (q_n) which is monotone decreasing, such that $\sup_n p_n = \alpha = \inf_n q_n$. Since there are countably many algorithms, there are only countably many computable numbers, and they form a field (in fact, a real closed field!). Quantifier elimination in **(RCF)** says that all numbers defined in the first order language of **(RCF)** are computable. By definition, most numbers we typically think of are computable.

Example 4.6.2 (Non computable number). Consider a list (indexed by the naturals) of all Turing machines, and set $\varepsilon_n \in \{0,1\}$ depending on whether or not the n^{th} machine halts. Then *Chaitin's constant* $\sum_{n \in \mathbb{N}} \varepsilon_n 2^{-n}$ is not computable.

Definition 4.6.3. Let $\Gamma = \langle S \mid R \rangle$ be a (not-necessarily finite) presentation for a group, and denote by $P : \mathbb{Z}[F_S] \to \mathbb{Z}[\Gamma]$ the canonical projection. A function $f : \mathbb{Z}[\Gamma] \to \mathbb{R}$ is *computable* if there is an algorithm that takes as input any $a \in \mathbb{Z}[F_S]$ and outputs two sequences of rational numbers: (p_n) which is monotone increasing, and (q_n) which is monotone decreasing, such that $\sup_n p_n = f(P(a)) = \inf_n q_n$.

We remark that the values of a computable function are clearly computable, but the converse need not hold.

Example 4.6.4. The *word problem* for a finitely presented group is to find for any $g \in F_S$, whether or not $P(g) = 1_\Gamma$. The major difficulty often lies in getting a certificate that $P(g) \neq 1_\Gamma$.

Computability of the norm functions $a \mapsto \|a\|_u$ (definition 2.5.8) or $a \mapsto \|a\|_\lambda$ (definition 4.3.4) would both imply decidability of the word problem — indeed, either $\|P(g) - 1_\Gamma\| = 0$ if $P(g) = 1_\Gamma$, or $\|P(g) - 1_\Gamma\| \geq 1$. Since the word problem isn't decidable in general, we can't hope that the norm functions are computable in general.

We remark that for amenable groups this is an if and only if, indeed

Theorem 4.6.5 ([FNT14], Theorem 1.3)**.** *Let $\Gamma = \langle S \mid R \rangle$ be a finitely presented amenable group. Then the word problem is decidable if and only if $a \mapsto \|a\|_\lambda = \|a\|_u$ is computable.*

In general amenable groups need not have decidable word problem, as shown by Kharlampovich ([Kha82]).

Theorem 4.6.6. *Every finitely presented residually finite group has decidable word problem.*

Proof. We describe an algorithm that decides the word problem for a word ω in the generators (we also write $\omega \in \Gamma$). We do two parallel searches:

(a) We enumerate the elements of $\langle\langle R\rangle\rangle$ and compare with ω — this will terminate if $\omega \in \langle\langle R\rangle\rangle$, that is $\omega = 1_\Gamma$ in Γ;

(b) We enumerate all S-tuples of permutations, check if they satisfy the relations of R, and if they do compute ω on the S tuple. In this way we enumerate the image of ω under all finite quotients of Γ, and this will terminate if $\omega \neq 1_\Gamma$.

□

We've studied a strengthening of residual finiteness — namely RFD, so we might hope that a stronger result holds for this class of groups. Indeed

Theorem 4.6.7. *Let $\Gamma = \langle S \mid R\rangle$ be a finitely presented RFD group. Then Γ has a computable norm function $\|\cdot\|_u : \mathbb{Z}[\Gamma] \to \mathbb{R} : a \mapsto \|a\|_u$.*

Remark 4.6.8. In particular, if the norm function is not computable then Γ can't be RFD — this gives a potential route to disprove Kirchberg's conjecture (and hence CEP). This might be a reasonable approach since there are many computational problems about $F_2 \times F_2$ which are known to be unsolvable.

To prove Theorem 4.6.7 we need to show how to compute upper bounds for the norm, this can be done for any group. The lower bounds come from the RFD property — we can approximate the norm from below by finite dimensional things. We start with the problem of computing an upper bound.

Consider the $\star$-positive cone in $\mathbb{C}[F_S]$ generated by $\{1 - r \mid r \in R\}$, that is

$$Q(S,R) := \left\{ \sum_{r\in R\cup\{0\}} \sum_{k=1}^{n_r} b_{r,k}^*(1-r)b_{r,k} \;\middle|\; n_r \in \mathbb{N}, b_{r,k} \in \mathbb{C}[F_S] \text{ for all } r \in R\cup\{0\} \right\}$$

Functionals which are positive on $Q(S,R)$ are in bijection with positive functionals on $\mathbb{C}[\Gamma]$, and hence by Corollary 2.7.2 we see that

$$\|P(a)\|_u = \inf\{\lambda \in \mathbb{R}_{\geq 0} \mid \lambda^2 - a^*a \in Q(S,R)\}$$

Let $Q_n(S,R)$ be the set of elements in $Q(S,R)$ that can be represented such that all $b_{i,k} \in \mathbb{C}[B_{F_S}(1,n)]$. Then each $Q_n(S,R)$ lies in a finite dimensional subspace, and their union is $Q(S,R)$. We can write

$$Q_n(S,R) = \left\{ \sum_{r\in R\cup\{0\}} \sum_{g,h\in B_{F_S}(1,n)} C_{r,g,h}\, g^{-1}(1-r)h \;\middle|\; (C_{r,g,h})_{g,h\in B_\Gamma(1,n)} \in \bigoplus_{r\in R\cup\{0\}} M_{B_{F_S}(1,n)}(\mathbb{C})^+ \right\}$$

Consider any $a \in \mathbb{Z}[F_S]$, and consider all the $n \geq n_0$, where $a \in \mathbb{C}[B_{F_S}(e,n_0)]$ and $\Lambda - a^*a \in Q_{n_0}(S,R)$ for some $\Lambda \in \mathbb{R}$. Then

$$\|a\|_u^2 \leq \min\{\Lambda \in \mathbb{R} \mid \Lambda - a^*a \in Q_n(S,R)\}$$

This is now a semidefinite programming problem in the finite dimensional space $\bigoplus_{r\in R\cup\{0\}} M_{B_{F_S}(1,n)}(\mathbb{C})$, and this bound becomes tight as $n \to \infty$ — so if we say compute the value of each semidefinite program by bounding it from above by an accuracy of $\frac{1}{n}$ say, we can obtain a convergent sequence of upper bounds on $\|a\|_u$. What we've shown is

Corollary 4.6.9. *For any finitely presented group Γ there is an algorithm that computes a convergent sequence of upper bounds $\|\cdot\|_u$ on $\mathbb{Z}[\Gamma]$.*

Remark 4.6.10. We see that semidefinite programming fits nicely into the framework of group algebras — this will be vital in the next part.

For more remarks, including an alternative description of the above in terms of a *dual SDP problem*, see the discussion after Corollary 2.2 in [FNT14] and the references therein.

Let $\|a\|_{fd} := \sup\{\|\pi(a)\| \mid \pi$ is a finite dimensional representation of $\Gamma\}$, and now let $\|\cdot\| = \|\cdot\|_u$. We've seen that Γ is RFD if and only if $\|a\|_{fd} = \|a\|$ for all $a \in \mathbb{C}[\Gamma]$ — but in fact it suffices to consider $a \in \mathbb{Z}[\Gamma]$:

Lemma 4.6.11. *Γ is RFD if and only if $\|\cdot\|_{\mathrm{fd}} = \|\cdot\|$ on $\mathbb{Z}[\Gamma]$.*

Proof. The only if direction is clear, so suppose that $\|\cdot\|_{\mathrm{fd}} = \|\cdot\|$ on $\mathbb{Z}[\Gamma]$. By homogeneity of norms, and the density of $\mathbb{Q} \subset \mathbb{R}$, we can conclude that the same holds on $\mathbb{R}[\Gamma]$. If we decompose $\mathbb{C}[\Gamma] = \mathbb{R}[\Gamma] + i\mathbb{R}[\Gamma]$ then by assumption the two norms coincide on each of the two summands. Furthermore, for $a, b \in \mathbb{R}[\Gamma]$

$$\|a + ib\| = \|a - ib\| \qquad \& \qquad \|a + ib\|_{\mathrm{fd}} = \|a - ib\|_{\mathrm{fd}}$$

since every representation of π has its complex conjugate representation, and taking complex conjugates preserves finite-dimensionality. Now note that

$$\begin{aligned} \|a + ib\| &\le \|a\|_{\mathrm{fd}} + \|b\|_{\mathrm{fd}} \\ &= \|\tfrac{1}{2}(a + ib) + \tfrac{1}{2}(a - ib)\|_{\mathrm{fd}} + \|\tfrac{1}{2}(a + ib) + \tfrac{1}{2}(-a + ib)\|_{\mathrm{fd}} \\ &\le \|a + ib\|_{\mathrm{fd}} + \|a - ib\|_{\mathrm{fd}} = 2\|a + ib\|_{\mathrm{fd}} \end{aligned}$$

and so

$$\|a + ib\|_{\mathrm{fd}} \le \|a + ib\| \le 2\|a + ib\|_{\mathrm{fd}}$$

hence the corresponding C^*-completions $C^*(\Gamma)$ and $C^*_{\mathrm{fd}}(\Gamma)$ are canonically isomorphic, and the canonical surjection $C^*(\Gamma) \to C^*_{\mathrm{fd}}(\Gamma)$ is an isomorphism. By the uniqueness of the norm on a C^*-algebra, we are done. ☐

Proof of Theorem 4.6.7. We just need to provide a convergent series of lower bounds on the operator norm in the universal representation, and we do this in a similar way to the solution of the word problem in RF groups. Enumerate $S = \{s_1, \dots, s_d\}$ and pick any $a \in \mathbb{Z}[F_S]$. For any $n \in \mathbb{N}$ consider the set

$$X(n) = \{(u_1, \dots, u_d) \in U(n)^d \mid r(u_1, \dots, u_d) = 1 \ \forall r \in R\} \subset U(n)^d$$

This is the space of all the n-dimensional unitary representations of Γ, and since it is bounded and closed (it is defined by polynomial equations in noncommuting variables) it is in fact a compact real algebraic subset of $\mathbb{R}^{d \cdot 2n^2}$. Denote by $D(n)$ the closed $2n$-dimensional ball in $\mathbb{C}^n$, that is $D(n) := \{\xi \in \mathbb{C}^n \mid \|\xi\| \le 1\}$, and consider the norm evaluation function

$$\text{norm-ev}_n : X(n) \times D(n) \to \mathbb{R} : ((u_s)_{s \in S}, \xi) \mapsto \|a((u_s)_{s \in S})\xi\|^2$$

Let the maximum of norm-ev$_n$ on $X(n) \times D(n)$ (a compact set) be α_n, and so (since Γ is RFD) $\|\pi(a)\| = \sup\{\alpha_n^{1/2} \mid n \in \mathbb{N}\}$. By quantifier elimination in **(RCF)** each α_n is computable, and hence we have obtained the required sequence of lower bounds. ☐

In fact, it can be shown that for any number α defined in **(RCF)**, it is decidable whether or not for any $a \in \mathbb{Z}[F_S]$ we have $\|a\| = \alpha$. In particular, it is decidable whether or not an element $a \in \mathbb{Z}[F_S]$ is invertible in $C^*(F_S)$ (see Theorem 1.6 and Corollary 2.8 in [FNT14]).

Part II

Kazhdan's Property (T)

Chapter 5

Introduction to Property (T)

Kazhdan's property (T) is in essence a rigidity result, and the fact that any infinite groups satisfy it is somewhat counterintuitive. It it of no real surprise therefore that actually exhibiting it for a group is often difficult.

We make no attempt here to list examples of property (T) groups, or to give descriptions of many of the ways that property (T) can be proved; let alone any historical account of the material. We also don't describe the many applications of property (T), for all of this we refer to the wide ranging monograph [BdlHV08].

We also refer to the appendix of [BdlHV08] for the standard material on the theory of unitary representations, we will state some results without proof.

5.1 Definitions and Equivalent Characterisations

We will use G to denote a general topological group (sometimes not even locally compact), and Γ to denote a countable discrete group.

Definition 5.1.1. Let $(\pi, \mathcal{H}_\pi)$ be a unitary representation of a topological group G.

(a) For $Q \subset G$ and $\varepsilon > 0$, we say that $\xi \in \mathcal{H}_\pi$ is (Q, ε)-*invariant* if

$$\sup_{g \in Q} \|\pi(g)\xi - \xi\| < \varepsilon \|\xi\|$$

(b) π has *almost invariant vectors* if it has (Q, ε)-invariant vectors for every *compact* subset Q of G and every $\varepsilon > 0$;

(c) π has *(non-zero) invariant vectors* if there is some $0 \neq \xi \in \mathcal{H}_\pi$ such that $\pi(g)\xi = \xi$ for all $g \in G$.

Remark 5.1.2. Condition (b) in the above definition is simply the statement that $1_G \prec \pi$ (where $\prec$ denotes weak containment of representations). (c) is the statement that $1_G \leq \pi$.

Definition 5.1.3. A group G has *(Kazhdan's) property (T)* if for any unitary representation π

$$1_G \prec \pi \quad \Leftrightarrow \quad 1_G \leq \pi$$

It might seem a pain to check the requirements for *every* compact subset $Q \subset G$ (even in the discrete case). Luckily checking one set suffices.

Definition 5.1.4. A subset $Q \subset \Gamma$ is a *Kazhdan set* if there is some $\varepsilon > 0$ such that every unitary representation that has a (Q,ε)-invariant vector also has a (non-zero) invariant vector.

Proposition 5.1.5. *A topological group G has property (T) if and only if it has a compact Kazhdan set.*

This was the definition of Property (T) given by Kazhdan in [Kaz67].

Proof. The 'if' direction is trivial. To show the converse, suppose that G doesn't have a compact Kazhdan set. Let

$$I = \{(Q,\varepsilon) \mid Q \subset G \text{ is compact}, \varepsilon > 0\}$$

Then by assumption, for every $\alpha = (Q,\varepsilon) \in I$ there is some representation π_α on $\mathcal{H}_{\pi_\alpha}$ without non-zero invariant vectors, but with a (Q,ε)-invariant vector ξ_α. Consider now the representation

$$\pi = \bigoplus_\alpha \pi_\alpha$$

Clearly $1_G \prec \pi$, but 1_G isn't contained in π. Indeed, suppose we have a vector $\eta = \bigoplus_\alpha \eta_\alpha$ which is invariant for π, then $\eta_\alpha = 0$ for all α and hence $\eta = 0$. □

Remark 5.1.6. If G is σ-compact it suffices in the above proof to consider the set $\{(Q_n, \frac{1}{n})\}$ where (Q_n) is an increasing union of compact sets that cover G.

Definition 5.1.7. Let G be a topological group. For a compact subset $Q \subset G$ and a unitary representation π of G, define the *Kazhdan constant associated to Q and π* to be

$$\kappa(G,Q,\pi) \coloneqq \inf\{\max_{g\in Q} \|\pi(g)\xi - \xi\| \mid \|\xi\| = 1\}$$

In particular, $1_G \prec \pi$ if and only if $\kappa(G,Q,\pi) = 0$ for all compact subsets $Q \subset \Gamma$.

Definition 5.1.8. We define the *Kazhdan constant associated to Q* to be

$$\kappa(G,Q) \coloneqq \inf\{\kappa(G,Q,\pi) \mid \pi \text{ has no nonzero invariant vectors}\}$$

Thus by the above lemma, G has (T) if and only if $\kappa(G,Q) > 0$ for some compact $Q \subset G$.

Remark 5.1.9. For any property (T) group with a compact generating set Q, this set itself will be a Kazhdan set. Indeed, suppose not, then for every $\varepsilon > 0$ there is some unitary representation π_ε of G without non-zero invariant vectors, and with a (Q,ε)-invariant vector. Set $\pi = \bigoplus_\varepsilon \pi_\varepsilon$, then π has a (Q,ε)-vector for all $\varepsilon > 0$.

Since Q generates G we see (for example by Proposition F.1.7 in [BdlHV08]) that $1_G \prec \pi$, and hence since G has (T) by assumption we have that $1_G \leq \pi$ — but this is a contradiction.

In fact, in finding a Kazhdan set for a property (T) group, the generating set suffices.

Theorem 5.1.10. *Let G be a locally compact group with property (T). Then G is compactly generated. Furthermore*

(a) If Q is a generating set for G, then it is a Kazhdan set;

(b) If Q is a Kazhdan set with a non-empty interior, then Q is a generating set.

The first statement is Theorem 1.3.1 in [BdlHV08], the 'furthermore' is Proposition 1.3.2 in the same reference.

Remark 5.1.11. In particular, discrete property (T) groups are finitely generated, and subsets are Kazhdan sets if and only if they are generating sets.

We now fix a finitely generated group Γ with finite symmetric generating set $S = S^{-1}$, and collect a few equivalent characterisations of property (T) for it.

Firstly, let $\Delta = |S| - \sum_{s\in S} s = \frac{1}{2}\sum_{s\in S}(1-s)^*(1-s) \in \mathbb{R}[\Gamma]$ be the (unnormalised) group algebra Laplacian; we will later see why this is a sensible definition (for a few different reasons).

Theorem 5.1.12 (Characterisations of property (T))**.** *Let Γ be a finitely generated group, with finite symmetric generating set $S = S^{-1}$. The following are equivalent:*

(a) *Γ has Kazhdan's property (T);*

(b) *$\overline{H^1}(\Gamma,\pi) = H^1(\Gamma,\pi)$ for all unitary representations π;*

(c) *$\overline{H^1}(\Gamma,\pi) = 0$ for all unitary representations π;*

(d) *$\overline{H^1}(\Gamma,\pi) = 0$ for all* ***irreducible*** *unitary representations π;*

(e) *$H^1(\Gamma,\pi) = 0$ for all unitary representations π;*

(f) *$H^1(\Gamma,\pi) = 0$ for all* ***irreducible*** *unitary representations π;*

(g) *The Laplacian (as a self-adjoint element of $C^*(\Gamma)$) has spectral gap (that is, there is some $\lambda > 0$ such that $\mathrm{Sp}_{C^*(\Gamma)}(\Delta) \subset \{0\} \cup [\lambda,\infty)$, where Sp denotes the spectrum);*

(h) *There is some $\lambda > 0$ such that $\Delta^2 - \lambda\Delta \geq 0$ in $C^*(\Gamma)$.*

We haven't defined either the group cohomology $H^1(\Gamma,\pi)$, or the *reduced* group cohomology $\overline{H^1(\Gamma,\pi)}$ — we will see these in section 6.4, and the reader is encouraged to come back to this Theorem then.

Remark 5.1.13. (Remarks on Theorem 5.1.12)

(i) Characterisation (h) is a question about positivity — so it shouldn't be too surprising given the previous part that this is the property we will investigate further.

(ii) The equivalence of (g) and (h) is standard.

(iii) Characterisation (b) is nothing more than the definition we gave for property (T) in a different language.

(iv) The fact that (d) implies (c) follows from the nice integrability properties of reduced cohomology, see for example Lemma 3.2.4 in [BdlHV08]. The same isn't true for standard group cohomology.

(v) We will motivate why (a) and (g) are equivalent when discussing group cohomology.

(vi) Characterisation (e) clearly implies (c) and (b), since $\overline{H^1(\Gamma,\pi)}$ is a quotient of $H^1(\Gamma,\pi)$.

(vii) Characterisation (e) is equivalent to Serre's Property (FH) (see section 6.4), and the fact that this is equivalent to Property (T)[1] is known as the *Delorme-Guichardet Theorem*. See chapter 2 in [BdlHV08] for more details, and the original references.

(viii) The fact that characterisations (c) (and (d)) are equivalent to (T) was observed by Yehuda Shalom in [Sha00], in this paper he also notes the fact that (f) suffices[2]. We will see why (c) and (h) are equivalent in section 6.4, and we will see a similar result in higher dimensions in section 6.5.

[1] In fact for any σ-compact locally compact group, see Theorem 2.12.4 in [BdlHV08].

[2] All of these hold for locally compact group which is second countable and compactly generated, see Theorem 3.2.1 in [BdlHV08].

To summarise what we will prove in this paper we have the following schematic:

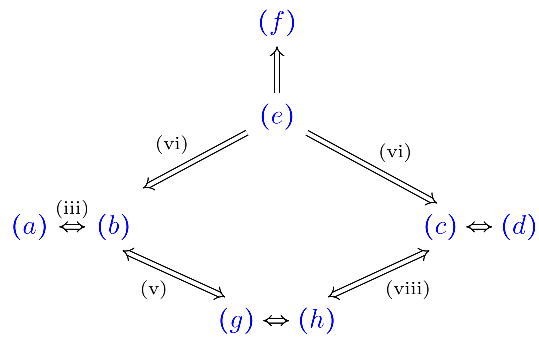


Figure 3: A schematic of the different implications in Theorem 5.1.12 that we will prove in this thesis. Short arrows indicate that these results are somewhat trivial. Note that the obvious implication (f) ⇒ (d) is missing for clarity.

So we see that we will only miss things implying (e). For these, see Theorem 2.12.4 in [BdlHV08].

Remark 5.1.14. We can relate the spectral gap λ for the Laplacian (associated to some finite symmetric generating set S) to the Kazhdan constant. In fact,

$$\sqrt{\frac{2\lambda}{|S|}} \leq \kappa(\Gamma, S) \tag{5.1}$$

See remark 5.4.7 in [BdlHV08] for this (notice that they use the normalised Laplacian, hence we have included the $|S|$ factor).

Remark 5.1.15. In many ways, the most natural definition for the Laplacian Δ arises from graph theory (and this ties in to the group algebra Laplacian when considering the Cayley Graph). We will see this definition (for a specific graph) in section 7.2, but notice that it is somewhat cohomological in nature, so we won't expand on it too much — instead referring to chapter 5 of [BdlHV08].

Remark 5.1.16. Property (τ) can similarly be characterised in terms of the first cohomology — it is equivalent to

$$\overline{H^1}(\Gamma, \bigoplus \ell^2(\Gamma/N_i)) = H^1(\Gamma, \bigoplus \ell^2(\Gamma/N_i))$$

for the family $\{N_i\}$ of finite index subgroups of Γ, see Theorem 2.16 in [LZ03].

The machinery we will develop later works in this case too, and recovers a well-known classification of property (τ) in terms of the Laplacian of the graph of G/N_i (which is equivalent to these graphs forming a family of expanders), see section 2.1 in the same book.

5.2 Finite Dimensional Representations of Property (T) Groups

In the previous part we investigated RFD groups, and in this part we are interested in property (T). It is nontrivial (and perhaps surprising) that these are opposite behaviours in some sense.

$\hat{\Gamma}$ will denote the set of (equivalence classes of) irreducible unitary representations of Γ endowed with the Fell topology, and we denote by $\hat{\Gamma}_{\text{fd}}$ the set of finite dimensional unitary representations.

The results in this section will be true for all locally compact groups, in fact a lot will be true for general topological groups — as before we will denote these general cases with G instead of Γ. In the remainder of this work we are only concerned with finitely generated discrete groups — but there is no real difficulty in proving results in their full generality. We start with another reformulation of property (T) in terms of the Fell topology. The standard reference for all of this is section 1.2 in [BdlHV08].

Proposition 5.2.1. *Let G be a topological group. The following are equivalent:*

(a) G has property (T);

(b) For any set $\mathcal{R}$ of equivalence classes of unitary representations of G without nonzero invariant vectors, 1_G is isolated in $1_G \cup \mathcal{R}$.

Proof. To show that (a) implies (b), suppose that G has property (T), and there exists a set $\mathcal{R}$ of equivalence classes of unitary representations without nonzero invariant vectors such that 1_G isn't isolated in $\mathcal{R} \cup \{1_G\}$. So there is a net $(\pi_i)_{i\in I}$ in $\mathcal{R}$ with $\pi_i \neq 1_G$ for all $i \in I$ converging to 1_G. That is, $1_G \prec \bigoplus_{i\in I} \pi_i$, and so by property (T) $1_G \leq \bigoplus_{i\in I}$ and we have that $1_G \leq \pi_i$ for some i, a contradiction.

Conversely if G doesn't have property (T) then there is some representation π such that $1_G \prec \pi$ but $1_G \nleq \pi$. Then 1_G isn't isolated in $\{1_G\} \cup \{\pi\}$. □

Corollary 5.2.2. *If G is locally compact, then G has property (T) if and only if 1_G is isolated in $\hat{G}$.*

In fact, in more generality we can characterise open subsets of the (unitary) dual (often called the spectrum) of a C^*-algebra (in particular, for the unitary dual of a group C^*-algebra). The following is due to Wang, see [Wan75].

Definition 5.2.3. Let $\mathcal{A}$ be a C^*-algebra and π a representation of it. The *support of* π is

$$\operatorname{supp}(\pi) := \{\rho \in \hat{\mathcal{A}} \mid \rho \text{ is weakly contained in } \pi\}$$

Theorem 5.2.4. *Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{R} \subset \hat{\mathcal{A}}$ a countable subset in its unitary dual such that $\mathcal{H}_\pi$ is separable for every $\pi \in \mathcal{R}$. Then the following are equivalent:*

(a) $\mathcal{R} \subset \hat{\mathcal{A}}$ is open;

(b) For any representation ρ of $\mathcal{A}$, if $\mathcal{R} \cap \operatorname{supp}(\rho) \neq \varnothing$ then in fact some $\pi \in \mathcal{R}$ is contained in ρ.

Proof. Suppose therefore that $\mathcal{R}$ is open, and let I be the closed two sided ideal of $\mathcal{A}$ such that $\hat{I} = \mathcal{R}$.

Since $\mathcal{R} \cap \operatorname{supp}(\rho) \neq \varnothing$, $\rho|_I \neq 0$. So $\rho|_I$ is a direct sum of irreducible representations (see for example Lemma 1.4 in [Wan75] — this is where the separability assumption is needed), and in particular some $\pi \in \mathcal{R}$ is contained in ρ.

For the converse suppose that $\mathcal{R}$ isn't open in $\hat{\mathcal{A}}$, and consider the representation $\rho = \bigoplus_{\pi\in\hat{\mathcal{A}}\setminus\mathcal{R}} \pi$. Since $\mathcal{R}$ isn't open, $\hat{\mathcal{A}}\setminus\mathcal{R}$ isn't closed — and so $\mathcal{R} \cap \operatorname{supp}(\rho) \neq \varnothing$. Hence by our assumption some $\pi \in \mathcal{R}$ is contained in ρ, but this is clearly a contradiction to the definition of ρ. □

In the case of group C^* algebras, we can read off some nice corollaries. Let G be a locally compact group, and $\mathcal{R}$ a countable subset of $\hat{G}$ such that $\mathcal{H}_\pi$ is separable for all $\pi \in \mathcal{R}$. Then the previous Theorem just tells us that $\mathcal{R}$ is open if and only if for any continuous unitary representation ρ of G, if $\mathcal{R} \cap \operatorname{supp}(\rho) \neq \varnothing$ then some $\pi \in \mathcal{R}$ is contained in ρ.

In particular the corollary we need is

Corollary 5.2.5. *Let G be a locally compact group and $\pi \in \hat{G}$ such that $\mathcal{H}_\pi$ is separable. Then π is an isolated point if and only if for any continuous representation ρ of Γ, $\rho \prec \pi$ if and only if $\rho \leq \pi$.*

For a different proof of this fact (under the assumption that π is finite dimensional), see Lemma 1.2.4 in [BdlHV08]. The below Theorem appears as Theorem 2.1 in [Wan75], see also Theorem 1.2.5 in [BdlHV08].

Theorem 5.2.6. *Let G be a locally compact group. Then the following are equivalent:*

(a) G has property (T);

(b) 1_G *is isolated in* $\hat{G}$*;*

(c) Every finite dimensional $\pi \in \hat{G}$ *is isolated in* $\hat{G}$*;*

(d) Some finite dimensional $\pi \in \hat{G}$ *is isolated in* $\hat{G}$*;*

If G *is separable then these are all equivalent to*

(e) $\hat{G}_{\mathrm{fd}}$ *is countable and open in* $\hat{G}$*.*

Proof. The equivalence of (a) and (b) has already been established. To show that (b) implies (c), consider a finite dimensional irreducible representation π of G, and another representation ρ such that $\pi \prec \rho$. Then it is standard (see for example F.3.2 in [BdlHV08]) that

$$\pi \otimes \overline{\pi} \prec \rho \otimes \overline{\pi}$$

Since π is finite dimensional, $1_G \leq \pi \otimes \overline{\pi}$ (Lemma 4.3.8), and so in particular $1_G \prec \rho \otimes \overline{\pi}$, and by property (T) $1_G \leq \rho \otimes \overline{\pi}$. Since π is irreducible, we must have that $\pi \leq \rho$ (by Corollary A.1.13 in [BdlHV08]) and so π is isolated in $\hat{\Gamma}$ by Corolalry 5.2.5.

Clearly (c) implies (d), so assume (d) — we want to show (b). Let π be a finite dimensional representation of G that is isolated in $\hat{G}$, and let $(\rho_i)_{i \in I}$ be any net in $\hat{G}$ such that $\lim_i \rho_i = 1_G$. Equivalently, for any subnet $(\sigma_j)_{j \in J}$ we have $1_G \prec \bigoplus_{j \in J} \rho_j$, and therefore

$$\pi \prec \bigoplus_{j \in J} \rho_j \otimes \pi$$

and so $\pi \leq \bigoplus_{j \in J} \rho_j \otimes \pi$.

By irreducibility of π, we therefore have some j such that $\pi \leq \rho_j \otimes \pi$, and $\pi \otimes \overline{\pi} \leq \rho_j \otimes \pi \otimes \overline{\pi}$. By using Lemma 4.3.8 twice we see that $1_\Gamma \leq \rho_j \otimes \pi \otimes \overline{\pi}$, and $\rho_j \leq \pi \otimes \overline{\pi}$. Since $\pi \otimes \overline{\pi}$ is finite dimensional, we can find some irreducible finite dimensional representations $\pi_1, \dots, \pi_n$ such that

$$\pi \otimes \overline{\pi} = \pi_1 \oplus \cdots \oplus \pi_n$$

and so (after relabeling) ρ_j is unitarily equivalent to π_1.

So we've seen that there is a subnet $(\rho_k)_{k \in K}$ of $(\rho_i)_{i \in I}$ such that ρ_k is unitarily equivalent to π_1 for all $k \in K$. Since $1_G \prec \bigoplus_{k \in K} \rho_k$, it follows that $1_G \prec \pi_1$.

For finite dimensional irreducible representations, weak containment and containment coincide (see Corollary F.2.9 in [BdlHV08]), and hence in fact we have that $1_G = \pi_1$. So ρ_k is unitarily equivalent to 1_G for all $k \in K$. We've shown that for any net with $\lim_i \rho_i = 1_G$, we have a subnet $(\rho_k)_{k \in K}$ such that ρ_k is unitarily equivalent to 1_G. This clearly proves (b).

Now, suppose that G is separable, and $\hat{G}$ is separable. So (e) clearly follows from (c), and conversely (e) implies (d) by Theorem 1.6 in [Wan75]. ☐

Remark 5.2.7. Condition (e) of Theorem 5.2.6 warrants some more words. A completely different proof of this fact (for Γ a countable, discrete group — we might as well assume finitely generated) is given in section 17.2 of [BO08] — although some knowledge of central covers of representations is needed. The proof is roughly as follows:

To each (nondegenerate) representation $\pi : C^*(\Gamma) \to \mathbb{B}(\mathcal{H}_\pi)$ there is a way (using a universal property of $C^*(\Gamma)^{**}$, which is isometrically isomorphic to the eneveloping von Neumann algebra of $C^*(\Gamma)$) to define a (unique) normal extension $\tilde{\pi} : C^*(\Gamma)^{**} \to \mathbb{B}(\mathcal{H}_\pi)$. The kernel of this extension admits a unit e_π which is a central projection in $C^*(\Gamma)^{**}$ — and we define the *central cover of* π to be

$$cc(\pi) := e_\pi^\perp = 1_{C^*(\Gamma)^{**}} - e_\pi$$

Then one notes that two representations π_1, π_2 are unitarily equivalent if and only if $cc(\pi_1) = cc(\pi_2)$. So far everything holds for any group — in fact for any C^*-algebra. But for property (T) groups we can extract (nonzero) invariant vectors from almost-invariant ones — which via Schur's lemma allows one to show that in fact for finite dimensional representations π, $cc(\pi) \in C^*(\Gamma)$. Now we simply observe that under the assumptions $C^*(\Gamma)$ must be separable, and so can have at most countably many orthogonal projections.

In this context, the $cc(\pi)$ are called *Kazhdan projections*.

We also note that a different method to prove (e) was developed by Wassermann in [Was91], that was further extended by de la Harpe, Robertson, and Valette in [dlHRV93]. They are able to asymptotically bound the number of finite dimensional representations up to a certain dimension.

Theorem 5.2.8 ([dlHRV93], Proposition IV)**.** *Let Γ be a discrete property (T) group with finite symmetric generating set $S = S^{-1}$. Then*

$$|\{\textit{irreducible representations of } \Gamma \textit{ of dimension } \leq m\}| \leq \mathcal{O}(e^{Cm^2})$$

for some constant $C = C(\Gamma, S)$.

We remark in passing that in this paper they use the *Markov operator* $M = \frac{1}{|S|}\sum_{s\in S} s \in \mathbb{C}[\Gamma]$ — this is just $1 - \frac{1}{|S|}\Delta$, and they show that spectral gap of this operator (at 1, rather than at 0 for Δ) is equivalent to (T) (Proposition III part (2)) — that is, they prove the arrow labelled (v) in figure 3. We should note that they build on the seminal papers of Kesten ([Kes59b], [Kes59a]) who considers (amongst other things) the operator $\lambda_\Gamma(M)$ and relates it to amenability, following the work of Følner ([Føl55]). The interplay between property (T) and amenability isn't too interesting however:

Theorem 5.2.9. *A locally compact group G is compact if and only if it is amenable and has property (T). In particular, a discrete amenable group with property (T) is finite.*

Proof sketch. Compactness implies both amenability and property (T) by suitable averaging arguments. Conversely, amenability implies that $1_G \prec \lambda_G$, and property (T) implies that $1_G \leq \lambda_G$, and so G must be compact. For more details see Theorem 1.1.6 in [BdlHV08]. □

One final result of interest is, which we state without proof, says that an RFD group with countable unitary dual must in fact be compact.

Theorem 5.2.10 ([Wan75], Theorem 6.3)**.** *Let G be a separable locally compact group. Then G is compact if and only if $\hat{G}$ is countable and $\hat{G}_{\mathrm{fd}}$ is dense in $\hat{G}$.*

Chapter 6

The Laplacian and Sums of Squares

6.1 Setup

Using characterisation (h) in Theorem 5.1.12, we recall that a (finitely generated, with symmetric finite generating set $S = S^{-1}$) group Γ has property (T) if and only if for some $\lambda > 0$, $\Delta^2 - \lambda\Delta \geq 0$ in $C^*(\Gamma)$. Since this is an element of $\mathbb{R}[\Gamma]$, in our notation this is just the statement that $\Delta^2 - \lambda\Delta \succcurlyeq 0$ in $\mathbb{R}[\Gamma]$, which by the abstract positivstellensatz (Proposition 2.7.3, see also (2.3)) is equivalent to $\Delta^2 - \lambda\Delta \in \overline{\Sigma^2\mathbb{R}[\Gamma]}$ (the closure is taken with respect to the finest locally convex topology τ_{st} on $\mathbb{R}[\Gamma]^h$).

This just means that Γ has property (T) if and only if for any $\varepsilon > 0$,

$$\Delta^2 - \lambda\Delta + \varepsilon 1 \in \Sigma^2\mathbb{R}[\Gamma] \tag{6.1}$$

However this means we need to solve an infinite number of equations to verify property (T) — and so isn't too useful.

In a seminal work, building on earlier observations by Netzer and Thom ([NT13]), Narutaka Ozawa was able to improve this to a single explicit sums of square decomposition.

Theorem 6.1.1 ([Oza16], Main Theorem)**.** *A finitely generated group Γ (with finite symmetric generating set S) has property (T) if and only if there is a some $\lambda > 0$ such that $\Delta^2 - \lambda\Delta \in \Sigma^2\mathbb{R}[\Gamma]$. That is, there is a finite sequence $\xi_1, \ldots, \xi_n \in \mathbb{R}[\Gamma]$ such that*

$$\Delta^2 - \lambda\Delta = \sum_i \xi_i^* \xi_i \tag{6.2}$$

Corollary 6.1.2. *If Γ has property (T) and $\lambda > 0$ is a rational number such that the spectrum of Δ in $C^*(\Gamma)$ is contained in $\{0\} \cup (\lambda, +\infty)$ then in fact $\Delta^2 - \lambda\Delta \in \Sigma^2\mathbb{Q}[\Gamma]$. That is, there is a finite sequence $\xi_1, \ldots, \xi_n \in \mathbb{Q}[\Gamma]$ such that*

$$\Delta^2 - \lambda\Delta = \sum_i \xi_i^* \xi_i$$

Proof of Corollary 6.1.2 from Theorem 6.1.1. If λ is rational then for a given finite sequence $\xi_1, \ldots, \xi_n \in \mathbb{Q}[\Gamma]$, then the linear equation $\Delta^2 - \lambda\Delta = \sum_{i=1}^n r_i \xi_i^* \xi_i$ in $(r_i)_{i=1}^n$ has a positive real solution if and only if it has a positive rational one. □

Remark 6.1.3. Ozawa in fact shows the same result for

$$\Delta_\mu := \frac{1}{2} \sum_{x \in \Gamma} \mu(x)(1-x)^*(1-x) = 1 - \sum \mu(x)x$$

for μ, any symmetric finitely supported measure such that $\langle \operatorname{supp} \mu \rangle = \Gamma$, our case above just corresponds to μ being the counting measure on $S = S^{-1}$.

We remark that such a decomposition was already known in a few cases — see [Zuk03], [BŚ97], and the references therein. Also, this implies that property (T) is semidecidable as was already observed by Silberman in [Sil11].

This characterisation is especially useful, since it means we can feasibly use computers to verify property (T) for certain groups — we investigate this in the next chapter.

6.2 The Augmentation Ideal

Equation (6.1) followed since 1 is an order unit for $\Sigma^2\mathbb{R}[\Gamma]$ in $\mathbb{R}[\Gamma]^h$, in order to improve it we would like to be able to say something similar about the Laplacian — but this isn't necessarily the case.

Example 6.2.1. Consider $\Gamma = \mathbb{Z} = \langle t \rangle$ with the symmetric generating set $S = \{t, t^{-1}\}$. The Laplacian is then $\Delta = 2 - t - t^{-1}$, but $R\Delta \not\geq 1$ for all $R > 0$. Indeed consider the trivial representation 1_Γ:

$$1_\Gamma(R\Delta - 1) = 1_\Gamma(2R - 1) - R1_\Gamma(t) - R1_\Gamma(t^{-1}) = -1 < 0$$

and in particular $R\Delta - 1 \notin \Sigma^2\mathbb{R}[\Gamma]$.

But recall that for a discrete group Γ with symmetric generating set $S = S^{-1}$, the Laplacian is defined to be $\Delta = \frac{1}{2}\sum_s (1-s)(1-s)^*$. So it makes sense to consider (for $k \in \{\mathbb{R}, \mathbb{C}\}$) the smallest subalgebra of $k[\Gamma]$ containing all the $(1-s)_{s\in S}$.

Definition 6.2.2. The *augmentation homomorphism* is the map that sends $\varepsilon : k[\Gamma] \to k : \sum a_g g \to \sum a_g$. Its kernel, the *augmentation ideal*, will be denoted by $I[\Gamma]$.

Note that the augmentation ideal is spanned (as a vector space) by $\{g - 1 \mid g \neq 1\}$, and we have the formula

$$(g-1)(h-1) = (gh-1) - (g-1) - (h-1) \tag{6.3}$$

Using this equation we see that $I[\Gamma]$ is generated as a k-algebra by $(1-s)_{s\in S}$. This also allows us to obtain information on the structure of the grading $I[\Gamma] \supset I^2[\Gamma] \supset \cdots$, where $I^n[\Gamma] := \text{span}_k\{a_1 \cdots a_n \mid a_i \in I[\Gamma]\}$. Our positive cone $\Sigma^2 I[\Gamma]$ is clearly a subset of $I^2[\Gamma]$.

Proposition 6.2.3. *We have that $I[\Gamma]/I^2[\Gamma] \cong \Gamma' \otimes_{\mathbb{Z}} k$, where $\Gamma' := \Gamma/[\Gamma,\Gamma]$ is the abelianisation of Γ.*

Proof. Define a $\mathbb{Z}$-bilinear map

$$\Phi : \Gamma' \times k \to I[\Gamma]/I^2[\Gamma] : (g, \lambda) \mapsto \lambda(g-1) + I^2[\Gamma]$$

To show that that Φ is well defined, consider any $g, h \in \Gamma$ — we need to show that $\Phi(gh, 1) = \Phi(hg, 1)$. But this follows immediately from (6.3):

$$\Phi(gh, 1) = (gh-1) + I^2[\Gamma] = (g-1) + (h-1) + I^2[\Gamma] = \Phi(hg, 1)$$

Hence by the universal property of tensor products we get a linear map $\phi : \Gamma' \otimes_{\mathbb{Z}} k \to I[\Gamma]/I^2[\Gamma]$, which is an isomorphism since it admits an inverse given by linearly extending $(g-1) + I^2[\Gamma] \mapsto g \otimes 1$. ☐

Corollary 6.2.4. $I[\Gamma]/I^2[\Gamma] \cong H_1(\Gamma, k)$.

Proof. This is immediate since $\Gamma' \otimes_{\mathbb{Z}} k$ is canonically identified with $H_1(\Gamma, k)$, see for example [Löh19, Section 1.4].

Another way to see this is to observe that the homology of Γ can be computed using a suitable classifying space X with $\pi_1(X) = \Gamma$, and then using the Hurewicz Theorem (and the Universal Coefficients Theorem). See [Bro82] for the construction of suitable spaces X. ☐

This observation is important — the augmentation ideal contains homological data about the group, and it is precisely this that ties into property (T) as we mentioned in Theorem 5.1.12.

We also remark that being a sum of squares is the same in the augmentation ideal as it is in the group algebra — indeed

Lemma 6.2.5. *For any group Γ, $\Sigma^2 I[\Gamma] = \Sigma^2 k[\Gamma] \cap I[\Gamma]$.*

Proof. $\Sigma^2 I[\Gamma] \subset \Sigma^2 k[\Gamma] \cap I[\Gamma]$ is obvious. For the reverse inclusion note that if $\sum a_i^* a_i \in I[\Gamma]$ where $a_i \in k[\Gamma]$, then $0 = \varepsilon(\sum_i a_i^* a_i) = \sum_i |\varepsilon(a_i)|^2$ and hence $\varepsilon(a_i) = 0$ for all i. □

In particular, our notion of $\preccurlyeq$ coincides for $k[\Gamma]$ and $I[\Gamma]$.

It is natural to ask now whether $I[\Gamma]$ is also Archimedean. The answer is no — at least not in the way that we've considered it. This shouldn't be too surprising, since it isn't unital. However, recall that being Archimedean is just the statement that 1 is an order unit (an algebraic interior point) for $\mathcal{A}^+$ in $\mathcal{A}^h$. So we can ask if there a natural order unit for $\Sigma^2 I[\Gamma]$ in $I[\Gamma]^h$?

This is where the Laplacian comes in.

6.3 The Laplacian as an Order Unit

Proposition 6.3.1 ([Oza16], Proposition 4)**.** *For every $a \in I[\Gamma]^h$ the following are equivalent.*

- *(a) $a \geq 0$ in $C^*[\Gamma]$;*
- *(b) $a + \epsilon\Delta \in \Sigma^2 I[\Gamma]$ for every $\epsilon > 0$.*

Proof of Theorem 6.1.1. As noted in (6.1), Γ has (T) if and only if there is some $\lambda > 0$ such that $\Delta^2 - \lambda\Delta \in \overline{\Sigma^2\mathbb{R}[\Gamma]}$. In particular, if $\Delta^2 - \lambda\Delta \in \Sigma^2\mathbb{R}[\Gamma]$ for some λ, then Γ has (T).

Conversely if Γ has (T) then there is some $\lambda > 0$ such that $\Delta^2 - \lambda\Delta \geq 0$ in $C^*[\Gamma]$. Take any $\epsilon < \lambda$, and then by Proposition 6.3.1 $\Delta^2 - (\lambda - \epsilon)\Delta \in \Sigma^2 I[\Gamma] \subset \Sigma^2\mathbb{R}[\Gamma]$. □

We now prove the Proposition. We've already noted that for $a \in \mathbb{R}[\Gamma]^h$, $a \geq 0$ in $C^*(\Gamma)$ is equivalent to the condition that $a \in \overline{\Sigma^2\mathbb{R}[\Gamma]}$. If $a \in I[\Gamma]^h$ in fact, we would like for this to be equivalent to $a \in \overline{\Sigma^2 I[\Gamma]}$.

Lemma 6.3.2 ([Oza16], Lemma 3)**.** *We have that $\overline{\Sigma^2 I[\Gamma]} = I[\Gamma] \cap \overline{\Sigma^2\mathbb{R}[\Gamma]}$ in $\mathbb{R}[\Gamma]$.*

Proof. It suffices to prove that for every positive linear functional $\phi : I[\Gamma] \to \mathbb{R}$, there is a sequence of positive linear functionals $\phi_n : \mathbb{R}[\Gamma] \to \mathbb{R}$ such that $\phi_n \to \phi$ pointwise on $I[\Gamma]$.

We claim that $\varphi(g) \coloneqq \phi(1 - g)$ defines a function conditionally of negative type on Γ (see Appendix C in [BdlHV08] or Appendix D in [BO08]), indeed for every $a \in I[\Gamma]$ one has

$$\sum_{g,h\in\Gamma} \varphi(h^{-1}g)a(h)a(g) = -\phi(a^*a) \leq 0$$

By Schoenberg's theorem the functions $\phi_t(g) \coloneqq \exp(-t\psi(g))$ are of positive type for all $t \in \mathbb{R}_{\geq 0}$ and extend to positive linear functionals on $\mathbb{R}[\Gamma]$. Since

$$\lim_{t\to 0} t^{-1}\phi_t(1-g) = \lim_{t\to 0} \frac{1 - \exp(-t\phi(1-g))}{t} = \phi(1-g)$$

for every $g \in \Gamma$, we see that $t^{-1}\phi_t \to \phi$ on $I[\Gamma]$, as desired. □

So now we just need to show that the Laplacian is such that

$$\overline{\Sigma^2 I[\Gamma]} = \{\xi \in I[\Gamma]^h \mid \xi + \varepsilon\Delta \in \Sigma^2 I[\Gamma] \text{ for every } \varepsilon > 0\}$$

In other words, it is an *order unit* for $\Sigma^2 I[\Gamma]$ on $I[\Gamma]^h$.

Lemma 6.3.3. *The Laplacian is an order unit for $\Sigma^2 I[\Gamma]$ in $I[\Gamma]^h$*

Proof. Note that $I[\Gamma]^h$ is spanned by $(1-g)^*(1-g) = 2 - g - g^{-1}$ so it suffices to show that

$$\{g \in \Gamma \mid \exists R > 0 \text{ such that } (1-g)^*(1-g) \preccurlyeq R\Delta\} = \Gamma$$

The left hand side clearly contains S, so we need to show it is closed under multiplication. For this we calculate

$$\begin{aligned}(1-gh)^*(1-gh) &= (1-g+g(1-h))^*(1-g+g(1-h)) \\ &= 2(1-g)^*(1-g) + 2(1-h)^*(1-h) - (1-g-g(1-h))^*(1-g-g(1-h)) \\ &\preccurlyeq 2(1-g)^*(1-g) + 2(1-h)^*(1-h)\end{aligned}$$

as required. ☐

We are thus done with the proof of Proposition 6.3.1 and Theorem 6.1.1.

There is another method of proving that the Laplacian is an order unit for $\Sigma^2 I[\Gamma]$ in $I[\Gamma]^h$ in many cases, due to Mizerka and Nowak in the preprint [MN23]. We sketch it out below, as it provides extra motivation for why the Laplacian is special. It also motivates the study of similar objects, such as Δ^2, and $\square$ (the latter plays a key role in [Oza22]).

Consider the dual of the real group algebra, $\mathbb{R}[\Gamma]^*$. This can be identified with the space of all functions $f : \Gamma \to \mathbb{R}$, and Γ has both a left and right action on $\mathbb{R}[\Gamma]^*$ — where for $g, h \in \Gamma$, $f \in \mathbb{R}[\Gamma]^*$, and $\eta \in \Gamma$ we denote by $(g \cdot f \cdot h)(\eta) := f(g^{-1}\eta h^{-1})$. We extend this action linearly to give $\mathbb{R}[\Gamma]^*$ the structure of an $\mathbb{R}[\Gamma]$-bimodule.

Recall that the augmentation ideal $I[\Gamma]$ is the kernel of the augmentation map ε, whose dual $\varepsilon^* : \mathbb{R} \to \mathbb{R}[\Gamma]^*$ is given by inclusion of constants. Hence the dual of the augmentation ideal is $I[\Gamma]^* = \mathbb{R}[\Gamma]^*/\text{const}$.

Consider the inclusion $\imath : \mathbb{R} \to \mathbb{R}[\Gamma]^*$ of the trivial Γ-module $\mathbb{R}$ into $\mathbb{R}[\Gamma]^*$, it induces a map $\imath_* : H^1(\Gamma, \mathbb{R}) \to H^1(\Gamma, \mathbb{R}[\Gamma]^*)$. We haven't defined group cohomology yet but in this case these objects admit an easy description, for example $H^1(\Gamma, \mathbb{R})$ is just the set of functions $c : \Gamma \to \mathbb{R}$ such that $c(gh) = c(g) + c(h)$ (called *additive characters*).

Let $S = \{s_1, \dots, s_n\}$ now be a finite generating set of Γ and consider the 0-codifferential $d = d^0 : \mathbb{R}[\Gamma] \to \mathbb{R}[\Gamma]^n$ given by the matrix $(1-s_1, \dots, 1-s_n)^T \in M_{n\times 1}(\mathbb{R}[\Gamma])$. Define the map

$$D : M_n(\mathbb{R}[\Gamma]) \to I[\Gamma] : A \mapsto d^* A d$$

Indeed, the image of D is in $I[\Gamma]$, since ε is a homomorphism. D is a positive map between these two $*$-algebras (equipped with the cones of sums of hermitian squares). Indeed, if $A = \sum_i \xi_i^* \xi_i$ is a sum of squares then $D(A) = \sum_i d^* \xi_i^* \xi_i d$ is too.

Theorem 6.3.4 ([MN23], Theorem 1)**.** *Let Γ be generated by a finite symmetric generating set $S = S^{-1}$. Then the following are equivalent:*

- *(a) The augmentation ideal $I[\Gamma]$ is idempotent, namely $I^2[\Gamma] = I[\Gamma]$ (that is, $H_1(\Gamma, \mathbb{R}) = 0$);*
- *(b) The map $\imath_* : H^1(\Gamma, \mathbb{R}) \to H^1(\Gamma, \mathbb{R}[\Gamma]^*)$ is injective;*
- *(c) The map $D : M_n(\mathbb{R}[\Gamma]) \to I[\Gamma]$ is surjective.*

Corollary 6.3.5. *Let Γ be a finitely generated group with $H^1(\Gamma, \mathbb{R}) = 0$. Then D is surjective.*

This holds if Γ has property (T) — or more generally if Γ has finite abelianisation. Indeed, there are no finite additive subgroups of $\mathbb{R}$, so any additive character from a group with finite abelianisation must be trivial.

Lemma 6.3.6. *Let Γ be a group as above such that D is surjective. Then $\Delta = D(I_n)$ is an order unit for $I[\Gamma]$.*

Proof. Let $a \in I[\Gamma]^h$, then since $a = d^* A d$ for some $A \in M_n(\mathbb{R}[\Gamma])$ we have that $d^*(A - A^*)d = 0$ and hence

$$a = d^*\Big(\frac{A + A^*}{2}\Big)d + d^*\Big(\frac{A - A^*}{2}\Big)d = d^*\Big(\frac{A + A^*}{2}\Big)d$$

Since I_n is always an order unit for $\Sigma^2 M_n(\mathbb{R}[\Gamma])$ in $M_n(\mathbb{R}[\Gamma])^h$ there exists an $R > 0$ such that

$$\frac{A + A^*}{2} + RI_n \in \Sigma^2 M_n(\mathbb{R}[\Gamma])$$

As we noted, applying D preserves the property of being a sum of squares, and so

$$a + R\Delta = D\Big(\frac{A + A^*}{2} + RI_n\Big) \in \Sigma^2 I[\Gamma]$$

□

Remark 6.3.7. The observation that Δ is an order unit for $\Sigma^2 I[\Gamma]$ in $I[\Gamma]^h$ (when $k = \mathbb{C}$) whenever the first homology $H_1(\Gamma, \mathbb{C})$ vanishes was already made by Netzer and Thom — see [NT13], Theorem 4.11.

In fact, they show that Δ is always an interior point of the cone $\Sigma^2 I[\Gamma]$ in $I^2[\Gamma]^h$, and note that $I[\Gamma] = I^2[\Gamma]$ precisely when $H_1(\Gamma, \mathbb{C}) = 0$.

6.4 Group Cohomology

We briefly introduce the theory of group cohomology with unitary coefficients for a discrete countable group Γ. There are several equivalent ways to define this — we refer to the book of Brown [Bro82] as the standard reference. We will define group cohomology in terms of a specific chain complex, called the *inhomogeneous chain complex* (and the corresponding *bar resolution*). The differential at first sight appears surprising, but it is the natural thing to consider once we inhomogenise 'homogeneous cochains' — see the book of Frigerio ([Fri17]) and the notes of Löh ([Löh19]) for good references for this.

So suppose we have a we have a unitary (or orthogonal) representation $(\pi, \mathcal{H}_\pi)$ of Γ. Since Γ is countable we might as well take $\mathcal{H}_\pi$ to be separable.

Definition 6.4.1. The *inhomogeneous chain complex* $C^\bullet(\Gamma, \pi)$ is defined by taking the groups $C^n(\Gamma, \pi) = 0$ for $n < 0$, $C^n(\Gamma, \pi) = \{f : \Gamma^n \to \mathcal{H}_\pi\}$ for $n \geq 1$ and $C^0(\Gamma, \pi) = \mathcal{H}_\pi$ (viewed as the constant maps) with the coboundary map $d^n : C^n(\Gamma, \pi) \to C^{n+1}(\Gamma, \pi)$ given by the linear extension of

$$d^n f(g_0, \dots, g_n) \coloneqq \pi(g_0) f(g_1, \dots, g_n) + \sum_{i=0}^{n-1} (-1)^{i+1} f(g_0, \dots, g_i g_{i+1}, \dots, g_n) + (-1)^{n+1} f(g_0, \dots, g_{n-1}) \quad (6.4)$$

It is easy to check that $d^{n+1} \circ d^n = 0$ and so we have a cochain complex and we we can define cohomology.

Definition 6.4.2. We set $Z^n(\Gamma, \pi) \coloneqq \ker d^n$ to be the *space of n-cocycles* and $B^n(\Gamma, \pi) = \operatorname{Im} d^{n-1}$ the *space of n-coboundaries* as standard, and define the n^{th} *cohomology group (with coefficients in π)* to be

$$H^n(\Gamma, \pi) \coloneqq Z^n(\Gamma, \pi) / B^n(\Gamma, \pi)$$

We can define the topology of uniform convergence on compact sets on $C^n(\Gamma, \pi)$, this has the structure of a Fréchet space. Since $Z^n(\Gamma, \pi)$ is closed in $C^n(\Gamma, \pi)$ it is also a Fréchet space. We have the quotient topology on $H^n(\Gamma, \pi)$, however generally $B^n(\Gamma, \pi)$ isn't closed in $Z^n(\Gamma, \pi)$ and the topology on $H^n(\Gamma, \pi)$ is non-Hausdorff. To remedy this, we can do the obvious thing.

Definition 6.4.3. The n^{th} *reduced cohomology (with coefficients in π)* is

$$\overline{H^n}(\Gamma, \pi) \coloneqq Z^n(\Gamma, \pi) / \overline{B^n(\Gamma, \pi)}$$

Since this is clearly a quotient of $H^n(\Gamma,\pi)$, the arrows labelled (vi) in Figure 3 follow.

We now give useful characterisations of the cohomology in degrees 1 and 2. Degree 1 cohomology is intricately related to property (T), and is the only thing we will need for the remainder of this chapter and the next. Understanding dimension 2 is useful in chapter 8, where we see that vanishing of this cohomology tells us something about asymptotic homomorphisms.

Remark 6.4.4. Clearly the cohomology is reduced (in all degrees) whenever π is finite dimensional. In fact, this happens whenever we have *any* linear Γ-action (where Γ is countable) on a finite dimensional vector space over any topological field, see [Aus18].

6.4.1 Dimension 1

Let $c \in C^1(\Gamma,\pi)$, then $d^1c(g,h) = \pi(g)c(h) - c(gh) + c(h)$. So any function $c : \Gamma \to \mathcal{H}_\pi$ satisfying $c(gh) = \pi(g)c(h) + c(g)$ (called the *cocycle equation*) is a 1-cocycle. Similarly the 1-couboundaries are given by functions of the form $d^0\xi(g) = \pi(g)\xi - \xi$ for some $\xi \in \mathcal{H}_\pi$.

What to these represent? Suppose we have an *isometric* action of our group Γ on some Hilbert space $\mathcal{H}_\pi$. This is a homomorphism

$$\alpha : \Gamma \to \mathrm{Iso}(\mathcal{H}_\pi) = U(\mathcal{H}_\pi) \ltimes \mathcal{H}_\pi$$

So we can write, for any $\xi \in \mathcal{H}_\pi$, the action of Γ as some rotation and some translation — that is

$$\alpha(g)(\xi) = \pi(g)\xi + c(g)$$

for some function $c : \Gamma \to \mathcal{H}_\pi$, and some unitary representation π called the *linear part of* α.

This is an action so we need $\alpha(gh) = \alpha(g)\alpha(h)$. For any $\xi \in \mathcal{H}_\pi$ we compute

$$\begin{aligned}\pi(gh)\xi + c(gh) &= \alpha(gh)\xi \\ &= \alpha(g)\alpha(h)\xi \\ &= \alpha(g)(\pi(h)\xi + c(h)) \\ &= \pi(g)\pi(h)\xi + (\pi(g)c(h) + c(g))\end{aligned}$$

That is, α defines an action if and only if c is a cocycle.

Suppose now that α has a non-zero fixed point, say ξ. So for all $g \in \Gamma$, we have

$$\xi = \alpha(g)\xi = \pi(g)\xi + c(g)$$

and so $c(g) = \pi(g)(-\xi) - (-\xi)$. That is, α has a fixed point if and only if c is a coboundary. Putting this all together, we deduce that

Theorem 6.4.5. *Let π be a unitary representation of Γ. Then $H^1(\Gamma,\pi) = 0$ if and only if every isometric action of Γ on $\mathcal{H}_\pi$ with linear part π has a fixed point.*

Definition 6.4.6. We say that Γ has *(Serre's) property (FH)* if the conditions in the Theorem are satisfied for all unitary representations π of Γ.

So the Delorme-Guichardet Theorem just says that properties (T) and (FH) are equivalent for all locally compact, σ-compact topological groups.

But in fact as we've alluded to, reduced cohomology is in many ways more natural to study property (T). Firstly, we notice that we can detect exactly when $B^1(\Gamma,\pi)$ is closed.

Proposition 6.4.7. *Let Γ be a countable discrete group and π a representation without non-zero invariant vectors. Then $B^1(\Gamma,\pi)$ is closed in $Z^1(\Gamma,\pi)$ if and only if π does not almost have invariant vectors.*

Corollary 6.4.8. *Γ has property (T) if and only if $H^1(\Gamma,\pi) = \overline{H^1}(\Gamma,\pi)$ for every unitary representation π.*

This is arrow (iii) in Figure 3.

Proof of proposition 6.4.7. Since π has no non-zero invariant vectors, we observe that d^0 is injective. If π does not almost have invariant vectors, then there exists a finite subset $Q \subset \Gamma$ and $\varepsilon > 0$ such that for all $\xi \in \mathcal{H}_\pi$

$$\max_{g \in Q} \|\pi(g)\xi - \xi\| \geq \varepsilon \|\xi\|$$

Let $(\xi_i)_i$ be a net of vectors in $\mathcal{H}_\pi$ such that $(d^0\xi_i)_i$ converges to some $b \in Z^1(\Gamma, \pi)$. In particular there is some subsequence $(\xi_n)_n$ such that

$$\lim_n \max_{g \in Q} \|d^0\xi_n - b\| = 0$$

hence we see that $(\xi_n)_n$ is a Cauchy sequence, converging to some $\xi \in \mathcal{H}_\pi$. Therefore $d^0\xi = b$ and $B^1(\Gamma, \pi)$ is closed.

Conversely if Γ is countable then $Z^1(\Gamma, \pi)$ is a Fréchet space, and so is $B^1(\Gamma, \pi)$ since it is closed. So by the open mapping theorem d^0 is bicontinuous, and there is some finite subset $Q \subset \Gamma$ and $\varepsilon > 0$ with

$$\max_{g \in Q} \|\pi(g)\xi - \xi\| \geq \varepsilon \|\xi\|$$

for all $\xi \in \mathcal{H}_\pi$ — so π does not almost have invariant vectors. □

This leads to two natural generalisations of property (T) in higher dimensions.

Definition 6.4.9. We say that Γ is *n-Kazkhdan* if $H^n(\Gamma, \pi) = 0$ for every unitary representation π, and that it is *strongly n-Kazhdan* if it is k-Kazhdan for all $k \leq n$ (this is called *property* $[T_n]$ in [BLSW23]).

Definition 6.4.10. We say that Γ has property (T_n) if for every unitary representation π of Γ, $H^{n+1}(\Gamma, \pi) = \overline{H^{n+1}}(\Gamma, \pi)$.

So a group has property (T) if and only if it is (strongly) 1-Kazhdan, or has property (T_0) (by the Delorme-Guichardet Theorem). However in higher dimensions these need not be the same, see for example Proposition 19 in [BN20] which shows an example due to Dymara-Januszkiewicz.

Remark 6.4.11. Our definition of property (T_n) is taken from the work of Bader and Nowak [BN14]. Note that it is different to the definition of property (T_n) given in [BLSW23].

6.4.2 The Cohomological Laplacian

Where does the Laplacian tie into this picture? We start with the 1-dimensional case first.

Let Γ be a group with a finite symmetric generating set S. A cocycle $c \in Z^1(\Gamma, \pi)$ is uniquely determined by its values on $Z^1(\Gamma, \pi)$, so the latter can naturally be viewed as a closed subspace of $\mathcal{H}_\pi^{|S|}$. Under this identification, the degree 0 codifferential is given by $d^0 = \bigoplus_{s \in S}(\pi(s) - \mathrm{Id})$. It is easy to verify that its adjoint $\partial_1 = (d^0)^*$ is given by $\sum_{s \in S} \pi(s)^* - \mathrm{Id}$.

We define the *degree 0 cohomological Laplacian* to be $\Delta_\pi^0 := (d^0)^* d^0 = \partial_1 \circ d^0$, and note this is a function on $\mathcal{H}_\pi$. In fact we can calculate

$$\Delta_\pi^0 = \sum_{s \in S} (\mathrm{Id} - \pi(s)^*)(\mathrm{Id} - \pi(s)) = \pi(\sum_{s \in S} (1 - s)^*(1 - s)) = 2\pi(\Delta)$$

where Δ is the group algebra Laplacian we've introduced earlier, and we've extended the Γ-action linearly to make $\mathcal{H}_\pi$ an $\mathbb{R}[\Gamma]$-module.

Now we can readily see why spectral gap for Δ has anything to do with property (T) — for any representation π and any $\xi \in \mathcal{H}_\pi$ we have that

$$\|d^0\xi\|^2 = \langle d^0\xi, d^0\xi\rangle = 2\langle \pi(\Delta)\xi, \xi\rangle$$

and so $\pi(\Delta)$ having spectral gap is just the statement that d^0 either doesn't move things at all, or moves them enough (that is, there are no almost invariant vectors). Since this holds for all π, this is just the property that Δ has spectral gap in $C^*(\Gamma)$, and so we've recovered arrow (v) in Figure 3.

Now we ask: why does spectral gap of Δ have anything to do with reduced cohomology?

Remark 6.4.12. We will later see that positivity of (the higher dimensional analogue of) $\Delta^2 - \lambda\Delta$ detects property (T_n) for all n.

Identifying again $Z^1(\Gamma, \pi)$ with a subspace of $\mathcal{H}_\pi^{|S|}$, we note that

$$\overline{H^1}(\Gamma, \pi) \cong Z^1(\Gamma, \pi) \cap (\operatorname{Im} d^0)^\perp = Z^1(\Gamma, \pi) \cap \ker(d^0)^*$$

Suppose that we have a 1-cocycle c, it gives rise to an associated function $\varphi' : \Gamma \to \mathbb{R} : g \mapsto -\frac{1}{2}\|c(g)\|^2$, and we get an associated functional $\varphi : I[\Gamma] \to \mathbb{R} : g - 1 \mapsto \varphi'(g)$. Note that φ' is a function *conditionally of positive type* — that is, the function

$$\langle \cdot, \cdot \rangle_\varphi : I[\Gamma] \otimes I[\Gamma] \to \mathbb{R} : a \otimes b \mapsto \varphi(b^* a)$$

is positive semidefinite. $\langle \cdot, \cdot \rangle_\varphi$ is automatically invariant under left multiplication of Γ on $I[\Gamma]$. similarly, any Γ-invariant positive semidefinite bilinear form $\langle \cdot, \cdot \rangle_\varphi$ on $I[\Gamma]$ gives rise to a cocycle — what we've done is nothing more than the GNS construction.

Suppose that our cocycle $c \in \overline{H^1}(\Gamma, \pi)$. In particular,

$$0 = (d^0)^* c = \sum_{s \in S} (\pi(s)^* c(s) - c(s)) = -\sum_{s \in S} (c(s^{-1}) + c(s)) = -2 \sum_{s \in S} c(s)$$

where we've used that c is a cocycle, $c(1) = 0$, and S is symmetric. We call a cocycle that satisfies $\sum c(s) = 0$ *harmonic* — so $\overline{H^1}(\Gamma, \pi)$ classifies (nontrivial) harmonic cocycles.

Since c comes from some function conditionally of positive type φ, harmonicity can be written as

$$0 = \|\sum_{s \in S} c(s)\|^2 = \|\sum_{s \in S} c(1) - c(s)\|^2 = \|\sum_{s \in S} 1 - s\|_\varphi^2 = \|\Delta\|_\varphi^2 = \varphi(\Delta^2)$$

Definition 6.4.13. We won't need this, but $\varphi(\Delta^2)$ is called the *curvature of* c.

How can we enforce that c is nontrivial? We can declare that $\|c\|^2_{\mathcal{H}^{|S|}} = 2$, that is

$$2 = \|c\|^2_{\mathcal{H}^{|S|}} = \sum_{s \in S} \|c(s)\|^2 = \sum_{s \in S} \|c(1) - c(s)\|^2 = \sum_{s \in S} \|1 - s\|_\varphi^2 = 2\varphi(\Delta)$$

So $\overline{H^1}(\Gamma, \pi)$ vanishing is equivalent to the non-existence of a positive semidefinite functional $\varphi : I[\Gamma] \to \mathbb{R}$ with

$$\varphi(\Delta^2) = 0 \quad \text{and} \quad \varphi(\Delta) = 1 \tag{6.5}$$

This is clearly equivalent to $\Delta^2 - \lambda\Delta \geq 0$ for some $\lambda > 0$, so we've recovered arrow (viii) in figure 3. This discussion can also be found in [Nit22] and [Oza22].

6.4.3 Dimension 2

Now we similarly characterise 2-cohomology, as this will be useful in chapter 8.3. However, this is certainly not needed for Kazhdan's property (T), and thus the reader only interested in that may readily skip this bit.

We saw that degree 1 cohomology captures affine actions with linear part π. We will now see that the second cohomology group characterises something else, namely *extensions* of Γ. Everything we say below works for a general Γ-module, but we will again only consider $\mathcal{H}_\pi$.

Definition 6.4.14. An *extension* of Γ by $\mathcal{H}_\pi$ is a short exact sequence

$$1 \longrightarrow \mathcal{H}_\pi \xrightarrow{\imath} \tilde{\Gamma} \xrightarrow{q} \Gamma \longrightarrow 1$$

By surjectivity we have a section $\sigma : \Gamma \to \tilde{\Gamma}$ (so $q \circ \sigma = \mathrm{Id}_\Gamma$), this generally won't be a homomorphism — but when such a section exists, we say the sequence *splits*.

We have a natural Γ-action on $\mathcal{H}_\pi$, given by conjugation of $\tilde{\Gamma}$ on $\mathcal{H}_\pi$, and we assume it coincides with π — that is, $\imath(\pi(g)\xi) = \sigma(g)\imath(\xi)\sigma(g)^{-1}$. We will identify $\mathcal{H}_\pi$ with $\imath(\mathcal{H}_\pi)$ as standard.

Consider the function

$$c_\sigma(g,h) = c(g,h) := \sigma(g)\sigma(h)\sigma(gh)^{-1}$$

Notice that $c(g,h) \in \ker(q) = \mathcal{H}_\pi$, so c is an element of $C^2(\Gamma,\pi)$. In fact, we claim this is in $Z^2(\Gamma,\pi)$, namely

$$0 = (d^2c)(g_0,g_1,g_2) = \pi(g_0)c(g_1,g_2) - c(g_0g_1,g_2) + c(g_0,g_1g_2) - c(g_0,g_1)$$

We show this by proving that

$$-\pi(g_0)c(g_1,g_2) + c(g_0,g_1) = c(g_0,g_1g_2) - c(g_0g_1,g_2)$$

These are elements of $\mathcal{H}_\pi$ (which is abelian) so we will use additive notation — but once we want to compare elements of $\tilde{\Gamma}$ we will use multiplicative notation, a potential source of confusion in the below computations. The left hand side is

$$\begin{aligned}
-\pi(g_0)c(g_1,g_2) + c(g_0,g_1) &= -\big(\sigma(g_0)\sigma(g_1)\sigma(g_2)\sigma(g_1g_2)^{-1}\sigma(g_0)^{-1}\big) + \sigma(g_0)\sigma(g_1)\sigma(g_0g_1)^{-1} \\
&= \sigma(g_0)\sigma(g_1g_2)\sigma(g_2)^{-1}\sigma(g_1)^{-1}\sigma(g_0)^{-1}\sigma(g_0)\sigma(g_1)\sigma(g_0g_1)^{-1} \\
&= \sigma(g_0)\sigma(g_1g_2)\sigma(g_2)^{-1}\sigma(g_0g_1)^{-1}
\end{aligned}$$

and the right hand side is

$$\begin{aligned}
c(g_0,g_1g_2) - c(g_0g_1,g_2) &= \sigma(g_0)\sigma(g_1g_2)\sigma(g_0g_1g_2)^{-1} - \sigma(g_0g_1)\sigma(g_2)\sigma(g_0g_1g_2)^{-1} \\
&= \sigma(g_0)\sigma(g_1g_2)\sigma(g_0g_1g_2)^{-1}\sigma(g_0g_1g_2)\sigma(g_2)^{-1}\sigma(g_0g_1)^{-1} \\
&= \sigma(g_0)\sigma(g_1g_2)\sigma(g_2)^{-1}\sigma(g_0g_1)^{-1}
\end{aligned}$$

These agree, and so $c \in Z^2(\Gamma,\pi)$ as required.

Example 6.4.15. If π is the trivial action, then we have that $\mathcal{H}_\pi \le Z(\tilde{\Gamma})$ and this is known as a *central extension.*

Lemma 6.4.16. *The class of c in $H^2(\Gamma,\pi)$ doesn't depend on the section. Furthermore, if $[c] = [c']$ are two representatives for the same cohomology class, they come from equivalent extensions.*

So we have a well-defined map from extensions as above to $H^2(\Gamma,\pi)$.

Proof. Suppose $\tau : \Gamma \to \tilde{\Gamma}$ is another section — so in particular, $\tau(g) = \alpha(g)\sigma(g)$ for some $\alpha : \Gamma \to \mathcal{H}_\pi$. We now compute

$$\begin{aligned}
c_\tau(g,h) &= \alpha(g)\sigma(g)\alpha(h)\sigma(h)\sigma(gh)^{-1}\alpha(gh)^{-1} \\
&= \alpha(g)\big(\sigma(g)\alpha(h)\sigma(g)^{-1}\big)\big(\sigma(g)\sigma(h)\sigma(gh)^{-1}\big)\alpha(gh)^{-1} \\
&= \alpha(g)\big(\pi(g)\cdot\alpha(h)\big)c_\sigma(g,h)\alpha(gh)^{-1} \\
&= c_\sigma(g,h) + \big(\pi(g)\cdot\alpha(h) - \alpha(gh) + \alpha(g)\big) \\
&= c_\sigma(g,h) + (d^1\pi\alpha)(g,h)
\end{aligned}$$

as required (where again we use additive notation and commutativity when the terms lie in $\mathcal{H}_\pi$). The 'furthermore' part comes from essentially the same computation in reverse. ☐

Corollary 6.4.17. *If $\tilde{\Gamma} \cong \mathcal{H}_\pi \rtimes_\pi \Gamma$ then the corresponding class c is trivial.*

Proof. It is standard that $\tilde{\Gamma} \cong \mathcal{H}_\pi \rtimes_\pi \Gamma$ if and only if there exists a section $\sigma : \Gamma \to \tilde{\Gamma}$ that is a homomorphism. In this case, c_σ is clearly trivial. □

Suppose conversely we are given a class $[c] \in H^2(\Gamma, \pi)$, we show how to obtain a suitable extension from it.

Lemma 6.4.18. *$[c]$ admits a representative c_0 with $c_0(g,1) = c_0(1,g) = 0$.*

Proof. Since c is a cocycle we have the identity

$$\pi(g_0)c(g_1,g_2) - c(g_0g_1,g_2) + c(g_0,g_1g_2) - c(g_0,g_1) = 0$$

Setting $g_1 = 1$ we have that

$$\pi(g_0)c(1,g_2) = c(g_0,1) \quad \text{and so} \quad c(1,g_2) = \pi(g_0)^* c(g_0,1)$$

changing either g_0 or g_2 we see that $c(1,g) = \pi(g)^* c(g,1) = \xi$ is constant. Consider the map

$$b : \Gamma \to \mathcal{H}_\pi : g \mapsto -\xi$$

so $d^1 b(g,h) = -\pi(g)\xi$. Then set $c_0 = c + d^1 b$ and note that

$$c_0(g,1) = c(g,1) - \pi(g)\xi = 0 \quad \text{and} \quad c_0(1,g) = c(1,g) - \xi = 0$$

So c_0 is the representative we desired. □

Now define the group $\tilde{\Gamma}_{[c]}$ corresponding to a cohomology class $[c]$ as follows. As a set, it is just $\mathcal{H}_\pi \times \Gamma$, and define multiplication by

$$(\xi, g) \cdot (\eta, h) := (\xi + \pi(g)\eta + c_0(g,h), gh)$$

where c_0 is the representative for $[c]$ found in the above Lemma.

It is instructive to see why this indeed defines a group — the facts that c_0 is a cocycle and that $c_0(g,1) = c_0(1,g) = 0$ are very important. It is easy to check that $(0,1)$ is the identity. Inverses are given by $(\xi,g)^{-1} = (-\pi(g)^*\xi - c_0(g^{-1},g), g^{-1})$. To see this note that since c_0 is a cocycle we have that

$$0 = \pi(g)c_0(g^{-1},g) - c_0(gg^{-1},g) + c_0(g,g^{-1}g) - c_0(g,g^{-1}) = \pi(g)c_0(g^{-1},g) - c_0(g,g^{-1})$$

and similarly $\pi(g^{-1})c(g,g^{-1}) = c_0(g^{-1},g)$. So we can compute

$$(\xi,g)(-\pi(g)^*\xi - c_0(g^{-1},g), g^{-1}) = (-\pi(g)c_0(g^{-1},g) + c_0(g,g^{-1}), 1) = (0,1)$$
$$(-\pi(g)^*\xi - c_0(g^{-1},g), g^{-1})(\xi,g) = (-\pi(g)^*\xi + \pi(g^{-1})\xi, 1) = (0,1)$$

Finally, associativity just amounts to the cocycle equation for c_0.

Define the maps $\imath : \mathcal{H}_\pi \to \tilde{\Gamma}_{[c]} : \xi \mapsto (\xi,1)$ and $q : \tilde{\Gamma}_{[c]} \to \Gamma : (\xi,g) \mapsto g$, these clearly give us the short exact sequence. Consider the section $\sigma : g \to (0,g)$, and calculate

$$\begin{aligned} c_\sigma(g,h) &= (0,g)(0,h)(0,gh)^{-1} \\ &= (c_0(g,h), gh)(-c_0(h^{-1}g^{-1}, gh), h^{-1}g^{-1}) \\ &= (c_0(g,h),1) = \imath(c_0)(g,h) \end{aligned}$$

and so this is the 'correct' section. Notice also that

$$\sigma(g)\imath(\xi)\sigma(g^{-1}) = (0,g)(\xi,1)(0,g^{-1}) = (0,g)(\xi,g^{-1}) = (\pi(g)\xi + c_0(g,g^{-1}), 1)$$

Remark 6.4.19. σ is a homomorphism precisely when $\sigma(g)\sigma(h) = (0,g)(0,h) = (c_0(g,h), gh) = (0,gh) = \sigma(gh)$ for all g,h — that is, when c_0 is trivial.

Remark 6.4.20. It is possible to define $\Gamma_{[c]}$ using any other cocycle representative, however using c_0 gives the simplest computations. Explicitly, suppose we defined $\Gamma_{[c]}$ using two different cocycle representatives c and c', where $c' = c + d^1 b$ as before. Then we have an isomorphism $\Gamma_{[c]} \cong \Gamma_{[c']}$ via $(\xi,g) \mapsto (\xi + b(g), g)$.

Putting this all together, we've shown

Theorem 6.4.21. *$H^2(\Gamma,\pi)$ is in bijection with the set of extensions of Γ (up to equivalence), such that the natural action of Γ on $\mathcal{H}_\pi$ given by the extension coincides with the π-action.*

6.4.4 Norm Bounds

We will need the following results later.

Proposition 6.4.22 ([CGLT20], Proposition 4.2)**.** *Let Γ be a countable group and π a unitary representation on $\mathcal{H}_\pi$ such that $H^n(\Gamma,\pi) = \{0\}$. Then for each finite set $Q \subset \Gamma^{n-1}$ there exist a finite set $F = F(\pi, Q) \subset \Gamma^n$ and a constant $C_{\pi,Q} \geq 0$, such that for every cocycle $c \in Z^n(\Gamma,\pi)$ there is an element $b \in C^{n-1}(\Gamma,\pi)$ with $c = d^{n-1}b$ and $||b||_Q < C_{\pi,Q}||z||_F$.*

Proof. Recall that the basic open sets in $C^n(\Gamma,\pi)$ are given by

$$U_{\delta,A} = \{f \in C^n(\Gamma,\mathcal{H}_\pi) \mid ||f||_A < \delta\}$$

for a finite subset $A \subset \Gamma^n$ and $\delta > 0$. Since $d^{n-1} : C^{n-1}(\Gamma,\pi) \to Z^n(\Gamma,\pi)$ is linear, bounded, and surjective, we may apply the open mapping theorem — so we can find $C_{\pi,Q} > 0$ and F $= F(\pi,Q) \subset \Gamma^n$ such that

$$U_{C^{-1}_{\pi,Q},F} \cap Z^n(\Gamma,\pi) \subset d^{n-1}(U_{1,Q})$$

That is, for any $c \in Z^n(\Gamma,\pi)$ with $||c||_F = 1$ we have that $C^{-1}_{\pi,Q}c \in U_{C^{-1}_{\pi,Q},F}$ so there is some $b \in C^{n-1}(\Gamma,\pi)$ with $d^{n-1}b = c$, and $||b||_Q < C_{\pi,Q} = C_{\pi,Q}||c||_F$ as required. ☐

By an easy diagonalisation argument, in the case that $H^n(\Gamma,\pi)$ vanishes for all unitary representations we can choose $F(\pi,Q)$ and $C_{\pi,Q}$ uniformly over π.

Corollary 6.4.23. *Let Γ be a countable n-Kazhdan group. Then for each finite set $Q \subset \Gamma^{n-1}$ there exist a finite set $F = F(Q) \subset \Gamma^{n-1}$ and a constant $C_Q \geq 0$, such that for any unitary representation π and every cocycle $c \in Z^n(\Gamma,\pi)$ there is an element $b \in C^{n-1}(\Gamma,\pi)$ with $c = d^{n-1}b$ and $||b||_Q < C_Q||z||_F$.*

We also note that cohomology vanishing can easily be carried to extensions of groups.

Proposition 6.4.24. *Suppose we have a short exact sequence of groups*

$$1 \longrightarrow N \longrightarrow \Gamma \longrightarrow Q \longrightarrow 1$$

If N is strongly n-Kazhdan and Q is n-Kazhdan, then Γ is n-Kazhdan.

This follows from the Hochschild-Serre spectral sequence ([HS53]), see Proposition 4.4 in [CGLT20].

6.5 Hilbert Complexes

We've seen that in degree 0, the cohomological Laplacian tells us something about $H^1(\Gamma,\pi)$ being reduced. In fact, we can do this in greater generality, which will allow us to prove higher dimensional variants of Theorem 6.1.1.

We survey the mostly classical theory of Laplace operators on Hilbert spaces, as in section 3 of [BN20]. See also [Arn18] and [BL91]. We will do this for a general cochain complex of Hilbert spaces — but recall we are only interested in the inhomogeneous complex with coefficients in some representation π for a countable discrete group Γ. So suppose we have the complex

$$\cdots \longrightarrow C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \longrightarrow \cdots$$

where $d^\bullet$ comprises of bounded maps in every degree, and let $\partial_n = (d^{n-1})^*$ be the Hermitian dual.

Recall the standard fact that for a linear map $T : U \to V$ between Hilbert spaces we have $\ker T = (\operatorname{Im} T^*)^\perp$ and so $(\ker T)^\perp = \left((\operatorname{Im} T^*)^\perp\right)^\perp = \overline{\operatorname{Im} T^*}$. Thus $(\ker \partial_n)^\perp = \overline{\operatorname{Im} d^{n-1}} \leq \ker d^n$ and in particular we see that $\ker d^n + \ker \partial_n = C^n$.

We define the spaces

$$\begin{aligned} C_0^n &:= \ker d^n \cap \ker \partial_n \\ C_-^n &:= \ker d^n \cap (\ker \partial_n)^\perp \\ C_+^n &:= (\ker d^n)^\perp \cap \ker \partial_n \end{aligned}$$

so that we have the orthogonal decomposition

$$C^n = C_-^n \oplus C_0^n \oplus C_+^n$$

By definition it is clear that

$$\ker d^n = C_-^n \oplus C_0^n \quad \text{and} \quad \ker \partial_n = C_0^n \oplus C_+^n \tag{6.6}$$

By taking the adjoint of these (and shifting indices) we also see that

$$(\ker \partial_{n+1})^\perp = \overline{\operatorname{Im} d^n} = C_-^{n+1} \quad \text{and} \quad (\ker d^{n-1})^\perp = \overline{\operatorname{Im} \partial_n} = C_+^{n-1} \tag{6.7}$$

As such, $d^n : C^n \to C^{n+1}$ decomposes as

$$d^n : C^n \longrightarrow\!\!\!\!\rightarrow C_+^n \xrightarrow{\bar{d}^n} C_-^{n+1} \hookrightarrow C^{n+1}$$

where $C^n \twoheadrightarrow C_+^n$ is the orthogonal projection, $C_-^{n+1} \hookrightarrow C^{n+1}$ is the inclusion and $\bar{d}^n$ is injective with dense image.

Similarly we have the decomposition

$$\partial_n : C^n \longrightarrow\!\!\!\!\rightarrow C_-^n \xrightarrow{\bar{\partial}_n} C_+^{n-1} \hookrightarrow C^{n-1}$$

where $\bar{\partial}_n = (\bar{d}^{n-1})^*$ is injective with dense image.

Now define

$$\begin{aligned} \Delta_+^n &= \partial_{n+1} d^n \\ \Delta_-^n &= d^{n-1} \partial_n \\ \Delta^n &= \Delta_-^n + \Delta_+^n \end{aligned}$$

We see that clearly these are positive self adjoint operators, and $\Delta_+^n \Delta_-^n = 0$ and $\Delta_-^n \Delta_+^n = (\Delta_+^n \Delta_-^n)^* = 0$.

We will need some technical lemmas.

Lemma 6.5.1. *Let $T : U \to V$ be a linear map between Hilbert spaces. Then $\ker T = \ker T^*T$ and $\overline{\operatorname{Im} T} = \overline{\operatorname{Im} T^*T}$.*

Proof. Clearly $\ker T \leq \ker T^*T$, and we get equality by noting that for for $v \in \ker T^*T$, $||Tv||^2 = \langle T^*Tv, v\rangle = 0$. The second statement follows by taking the orthogonal complement of the first. □

We immediately apply this to (6.6) and (6.7) to see that

$$\ker \Delta_+^n = C_-^n \oplus C_0^n \quad \text{and} \quad \ker \Delta_-^n = C_0^n \oplus C_+^n \tag{6.8}$$

$$\overline{\operatorname{Im} \Delta_+^n} = C_+^n \quad \text{and} \quad \overline{\operatorname{Im} \Delta_-^n} = C_-^n \tag{6.9}$$

hence if we define

$$\bar{\Delta}_+^n = \bar{\partial}_{n+1}\bar{d}^n : C_+^n \to C_+^n \quad \text{and} \quad \bar{\Delta}_-^n = \bar{d}^{n-1}\bar{\partial}_n : C_-^n \to C_-^n$$

these are both injective, positive, self-adjoint operators with dense images. We have the orthogonal decomposition

$$\Delta^n = \bar{\Delta}_-^n \oplus 0 \oplus \bar{\Delta}_+^n : C_-^n \oplus C_0^n \oplus C_+^n \to C_-^n \oplus C_0^n \oplus C_+^n \tag{6.10}$$

In particular, $\ker \Delta^n = C_0^n$.

As always define the n^{th} *cohomology group* of $C^\bullet$ to be $H^n = \ker d^n / \operatorname{Im} d^{n-1}$. As noted in the group cohomology case, in general $\operatorname{Im} d^{n-1}$ isn't closed, and so we will also consider the n^{th} *reduced cohomology group* of $C^\bullet$, defined to be $\overline{H^n} = \ker d^n / \overline{\operatorname{Im} d^{n-1}}$.

Definition 6.5.2. The n^{th} cohomology is said to be *reduced* if $H^n = \overline{H^n}$.

Equivalently, H^n is Hausdorff in the quotient topology.

Proposition 6.5.3 ([BN20], Proposition 16)**.** *Cohomology vanishing, or being reduced, can be detected using the cohomological Laplacian. In particular*

(a) The reduced n^{th} cohomology $\overline{H^n}$ is isomorphic to $\ker \Delta^n$ (in particular $\overline{H^n} = 0$ if and only if Δ^n is injective);

(b) H^n is reduced if and only if $\bar{\Delta}_-^n$ is invertible if and only if there exists $\lambda > 0$ such that $\Delta_-^n(\Delta_-^n - \lambda) \geq 0$;

(c) H^{n+1} is reduced if and only if $\bar{\Delta}_+^n$ is invertible, if and only if there exists some $\lambda > 0$ such that $\Delta_+^n(\Delta_+^n - \lambda) \geq 0$;

(d) Both H^n and H^{n+1} are reduced if and only if both $\bar{\Delta}_-^n$ and $\bar{\Delta}_+^n$ are invertible, if and only if there exists $\lambda > 0$ such that $\Delta^n(\Delta^n - \lambda) \geq 0$;

(e) $H^n = 0$ and H^{n+1} is reduced if and only if Δ^n is invertible, if and only if there exists $\lambda > 0$ such that $\Delta^n - \lambda 1 \geq 0$.

Remark 6.5.4. Notice that (b) in 6.5.3 doesn't give a checkable condition for property (T_{n-1}) in the same sense as Ozawa's characterisation for property (T_0); there the extra step is that the Laplacian is an order unit. It would be interesting to see if there are analogues in higher dimensions.

Before we can prove the proposition, we need two technical lemmas about operators on Hilbert spaces.

Lemma 6.5.5. *Let $T : U \to V$ be a map between Hilbert spaces, and $S : V \to V$ a positive self-adjoint map. Then*

(i) S is invertible if and only if there exists some $\lambda > 0$ such that $S - \lambda 1$ is positive;

(ii) If in addition S is injective then (i) happens if and only if $S(S - \lambda)$ is positive;

(iii) T is an isomorphism if and only if T^* is an isomorphism if and only if T^*T is invertible.

Proof. (i) and (ii) follow easily from standard spectral theory. For (iii), note that clearly T is an isomorphism if and only if T^* is, and both imply that T^*T is invertible. To see the converse, note that T^* is surjective, and since $\overline{\operatorname{Im} T} = \overline{\operatorname{Im} T^*T}$ we see that T has dense image and thus T^* is injective. So T^* is an isomorphism by the open mapping theorem. □

Proof of Proposition 6.5.3. (a) Since we have $\ker d^n = C_0^n \oplus C_-^n$ and $\overline{\operatorname{Im} d^{n-1}} = C_-^n$, we have that $\overline{H^n} \cong C_0^n = \ker \Delta^n$;

(b) H^n is reduced if and only if d^{n-1} is onto $\overline{\operatorname{Im} d^{n-1}} = C^n_-$, if and only if $\bar{d}^{n-1}$ is surjective, if and only if it is an isomorphism (by the Open Mapping Theorem). This happens if and only if $\bar{\Delta}^n_-$ is invertible (by (iii) in Lemma 6.5.5), of and only if there is some $\lambda > 0$ such that $\bar{\Delta}^n_-(\bar{\Delta}^n_- - \lambda)$ is positive (by (ii) in Lemma 6.5.5) if and only if there is some $\lambda > 0$ such that $\Delta^n_-(\Delta^n_- - \lambda)$ is positive (using the decomposition $C^n_- \oplus C^n_0 \oplus C^n_+$);

(c) This is similar to (b). H^{n+1} is reduced if and only if d^n is onto $\overline{\operatorname{Im} d^n} = C^{n+1}_-$, if and only if $\bar{d}^n$ is surjective, if and only if it is an isomorphism (by the open mapping Theorem). This in turn happens if and only if $\bar{\Delta}^n_+$ is invertible (by (iii) in Lemma 6.5.5), if and only if there is some $\lambda > 0$ such that $\bar{\Delta}^n_+(\bar{\Delta}^n_+ - \lambda)$ is positive (by (ii) in Lemma 6.5.5), if and only if there exists some $\lambda > 0$ such that $\Delta^n_+(\Delta^n_+ - \lambda)$ is positive (using the decomposition $C^n_- \oplus C^n_0 \oplus C^n_+$);

(d) This follows from (b) and (c), since $\Delta^n_+\Delta^n_- = 0 = \Delta^n_-\Delta^n_+$;

(e) By (i) in 6.5.5, Δ^n is invertible if and only if there is some $\lambda > 0$ such that $\Delta - \lambda 1$ is positive. Assuming that Δ^n is invertible, we have that $C^n_0 = \ker \Delta^n = 0$, and so Δ^n decomposes as $\bar{\Delta}^n_- \oplus \bar{\Delta}^n_+$, and both are invertible as Δ^n is. By (d), both H^n and H^{n+1} are reduced, and by (a) $H^n = 0$.

Conversely assume that $H^n = 0$, and H^{n+1} is reduced. Then by (a) Δ^n decomposes as $\bar{\Delta}^n_- \oplus \bar{\Delta}^n_+$. Since both H^n and H^{n+1} are reduced we use (d) to deduce that both Δ^n_- and Δ^n_+ are invertible — hence so is Δ^n.

□

Using Proposition 6.5.3 (c) we once again recover arrow (viii) in Figure 3. In this case the 0 degree Laplacian lives in the group algebra $\mathbb{Q}[\Gamma]$, and so this allows us to say a lot — but annoyingly we can't immediately say something similar here for the higher degree Laplacians. However, for groups satisfying nice enough finiteness property, we can always compute their cohomology with unitary coefficients by using a nicer complex, where Δ lies in some matrix algebra.

We can do this uniformly over all $\mathbb{Q}[\Gamma]$-modules V — and this is vital to detect vanishing of these cohomologies using just one computation.

Proposition 6.5.6 ([BN20], Proposition 1)**.** *Let Γ be a group which acts with finite stabilizers by automorphisms on a contractible simplicial complex X, and suppose $X^{(n)}$ has finitely many orbits for $n \le N+1$. Let $k_n := |X^{(n)}/G|$ be this number of orbits.*

Then for every $n \le N$ there exist matrices $D^n \in M_{k_n \times k_{n+1}}(\mathbb{Q}[\Gamma])$ such that $D^n D^{n-1} = 0$, and for every $\mathbb{Q}[\Gamma]$-module V, the cohomology groups $H^n(\Gamma, V)$ are isomorphic to the cohomology of the complex

$$\cdots \longrightarrow V^{k_{n-1}} \xrightarrow{D^{n-1}} V^{k_n} \xrightarrow{D^n} V^{k_{n+1}} \longrightarrow \cdots$$

Let $C^n = C^n(V)$ be the vector space of alternating Γ-equivariant maps from $X^{(n)}$ to V, and for $t = 0, \dots, n+1$ let $f^n_t : C^n \to C^{n+1}$ be the t'th face map, that is

$$f^n_t(\phi)(\sigma_0, \dots, \sigma_{n+1}) := \phi(\sigma_0, \dots, \hat{\sigma}_t, \dots, \sigma_{n+1})$$

Notice then that $d^n = \sum_{t=0}^n (-1)^t f^n_t : C^n \to C^{n+1}$ is the standard boundary map in cohomology. We also have the following well known fact.

Theorem 6.5.7. *If X is contractible and the Γ stabilizers are finite, then the chain complex $(C^\bullet, d^\bullet)$ computes the cohomology of Γ with coefficients in V.*

The chain complex $(C^\bullet, d^\bullet)$ is called the *equivaraiant chain complex* associated with the Γ action on X. If we choose a fundamental domain $Y^{(n)} \subset X^{(n)}$, we introduce an isomorphic chain complex, called the *nonequivariant chain complex.*

Let $\overline{C}^n$ be the vector space of maps from $Y^{(n)}$ to V; the restriction map $C^n \to \overline{C}^n$ is clearly a linear isomorphism, and by conjugating the boundary maps we get a new, isomorphic chain complex $(\overline{C}^\bullet, \overline{d}^\bullet)$.

Remark 6.5.8. This is basically the same process that is done in obtaining the inhomogeneous complex to compute group cohomology, as hinted to at the start of section 6.4.

For an explicit description of the boundary maps see the proof of Proposition 1 in [BN20]. We are now ready to use this construction to detect cohomology vanishing.

Theorem 6.5.9 ([BN20], Main Theorem). *Let Γ be a group which acts with finite stabilizers by automorphisms on a contractible simplicial complex X, and suppose $X^{(n)}$ has finitely many orbits for $n \leq N+1$. Let $k_n := |X^{(n)}/G|$ be this number of orbits.*

For $n \leq N$ let D^n be the matrix in $M_{k_n \times k_{n+1}}(\mathbb{Q}[\Gamma])$ given in Proposition 6.5.6, and let

$$\Delta^n = (D^n)^* D^n + D^{n-1}(D^{n-1})^* \in M_{k_n}(\mathbb{Q}[\Gamma])$$

Then for a fixed $1 \leq n \leq N$, the following are equivalent:

(a) Γ is n-Kazhdan and has property (T_n);

(b) There exists some rational $\lambda > 0$ such that $\Delta^n - \lambda I_{k_n} \in \Sigma^2 M_{k_n}(\mathbb{Q}[\Gamma])$. That is, there are elements $\xi_1, \dots, \xi_m \in M_{k_n}(\mathbb{Q}[\Gamma])$ such that

$$\Delta^n - \lambda I_{k_n} = \sum_{i=1}^{m} \xi_i^* \xi_i$$

Proof. In fact we show that both are equivalent to

(c) For every unitary representation π of Γ, the image of Δ^n under $M_{k_n}(\pi)$ is invertible.

By Proposition 6.5.6, For every unitary representation π of Γ on a Hilbert space $\mathcal{H}_\pi$, the cohomology groups $H^n(\Gamma, \pi)$ are isomorphic to the cohomology groups of the complex

$$\cdots \longrightarrow \mathcal{H}_\pi^{k_{n-1}} \xrightarrow{D^{n-1}} \mathcal{H}_\pi^{k_n} \xrightarrow{D^n} \mathcal{H}_\pi^{k_{n+1}} \longrightarrow \cdots$$

and hence, (a) and (c) are equivalent by (e) in Proposition 6.5.3.

The equivalence of (b) and (c) follows from Corollary 2.8.1 and equation (2.2). □

Chapter 7

Implementations of Ozawa's Criterion

Now that we've built up the machinery, we will survey the successful implementations of it that have been found so far.

7.1 Finitely Presented Coverings

We start with an almost immediate consequence of Ozawa's characterisation, as shown in [Oza16]. This recovers a result of Shalom.

Theorem 7.1.1 ([Sha00], Theorem 6.7)**.** *Every discrete property (T) group is a quotient of a finitely presented property (T) group.*

Proof. Let Γ be a property (T) group with finite symmetric generating set $S = S^{-1}$. Then, by Theorem 6.1.1 we have $\lambda > 0$ and elements $\xi_1, \dots, \xi_n \in \mathbb{R}[\Gamma]$ such that

$$\Delta^2 - \lambda\Delta = \sum_i \xi_i^* \xi_i$$

We have the canonical quotient map $q : F_S \twoheadrightarrow \Gamma$, and we denote by Δ the group algebra Laplacian in F_S too. Let $\tilde{\xi_i} \in \mathbb{R}[F_S]$ be lifts of ξ_i through q, so $q(\Delta^2 - \lambda\Delta - \sum_i \tilde{\xi}_i^* \tilde{\xi}_i) = 0$.

Consider the finite subset

$$R = \{x^{-1}y \in F_S \mid x, y \in \operatorname{supp}(\Delta^2 - \lambda\Delta - \sum_i \tilde{\xi}_i^* \tilde{\xi}_i) \text{ such that } q(x) = q(y)\}$$

and the corresponding finitely presented group $\Lambda = \langle S \mid R \rangle$. By definition q factors through Λ and so Γ is a quotient of Λ. Let $q' : F_S \twoheadrightarrow \Lambda$ be the canonical quotient map for Λ and again denote by Δ the group algebra Laplacian in Λ, so we see that

$$\Delta^2 - \lambda\Delta = \sum_i q'(\tilde{\xi}_i)^* q'(\tilde{\xi}_i)$$

and Λ also has property (T). □

Remark 7.1.2. Similarly we also recover the standard fact that quotients of (discrete) property (T) groups also have property (T), and that the spectral gap of the Laplacian doesn't decrease.

We define the *Kazhdan radius* (as in [KMN22]) to be the smallest r such that there is a $\lambda > 0$ and a sum of squares decomposition $\Delta^2 - \lambda\Delta = \sum \xi_i^* \xi_i$ where all the $\xi_i \in \mathbb{R}[B_\Gamma(1, r)]$.

This notion will be useful in this chapter but we won't comment on it too much. It is easy to see that the Kazhdan radius of $\mathrm{SL}(2, \mathbb{F}_p)$ increases as $p \to \infty$, otherwise one could obtain a sum of squares decomposition of $\Delta^2 - \lambda\Delta$ for $\mathrm{SL}(2, \mathbb{Z})$ — but this group doesn't have property (T). See section 2 in [KMN22] for the details.

7.2 Zuk's Criterion

As another illustration we recover Zuk's criterion for a group to have property (T), see Theorem 1 in [Zuk03] and section 5.6 in [BdlHV08].

Suppose Γ is generated by a finite symmetric set $S = S^{-1}$ and $1 \notin S$. Its *link* is the (directed) graph with vertex set S and edge set $E = \{(s, t) \mid s^{-1}t \in S\}$.

Put on E the uniform probability measure, and on S the probability measure μ where

$$\mu(s) = \frac{|\{t \in S \mid (s, t) \in E\}|}{|E|}$$

Which is just the (normalised) degree of the vertex s.

Define the map $d : L^2(S, \mu) \to L^2(E)$ by $(d\xi)(s, t) = \xi(t) - \xi(s)$ and consider the operator $\Lambda = d^* d/2$ (this is the *graph theoretic Laplacian* which can be defined for any directed graph, potentially with loops).

Theorem 7.2.1 (Zuk's criterion)**.** *If the link graph is connected and the spectrum of* Λ *is contained in* $\{0\} \cup [\lambda, +\infty)$ *then there are* $\xi_i \in \mathbb{R}[\Gamma]$*,* $i \in S$*, such that*

$$\Delta_\mu^2 - (2 - \lambda^{-1})\Delta_\mu = \sum_i \xi_i^* \xi_i$$

where $\Delta_\mu = \frac{1}{2}\sum_{s \in S} \mu(s)(1-s)^*(1-s)$ *is as in remark 6.1.3.*

In particular, Γ *has property (T) if* $\lambda < 1/2$*, with Kazhdan constant* $\sqrt{2(2-\lambda)^{-1}}$*.*

Proof. Let P be the orthogonal projection of $L^2(S,\mu)$ onto the one dimensional subspace of constant functions. By assumption $\lambda^{-1}\Lambda + P - I \geq 0$ and so we can find an operator T on $L^2(S,\mu)$ such that $\lambda^{-1}\Lambda + P - I = T^*T$.

For any operator A on $L^2(S,\mu)$ consider its entries $A_{s,t} (= \langle A\delta_t, \delta_s\rangle)$ and calculte that

$$\sum_{s,t \in S^2} (\lambda^{-1}\Lambda_{s,t} + P_{s,t} - I_{s,t}) s^{-1} t = \sum_{s,t \in S^2} \sum_{i \in S} \mu(i)^{-1} T_{s,i}^* T_{i,t} s^{-1} t = \sum_{i \in S} \xi_i^* \xi_i$$

where $\xi_i = \mu(i)^{-1/2} \sum_s T_{i,s} s \in \mathbb{R}[\Gamma]$. Now observe that $\Lambda_{s,s} = \mu(s)$, $\Lambda_{s,t} = |E|^{-1}$ if $(s,t) \in E$ and $\Lambda_{s,t} = 0$ otherwise, $P_{s,t} = \mu(s)\mu(t)$ and $I_{s,t} = \delta_{s,t}\mu(s)$. So the left hand side is

$$\begin{aligned}
&\lambda^{-1}\Big(1 - \frac{1}{|E|} \sum_{(s,t) \in E} s^{-1} t\Big) + \Big(\sum_{s,t \in S} \mu(s)\mu(t) s^{-1} t\Big) - 1 \\
&= \lambda^{-1}\Big(1 - \sum_s \mu(s) s\Big) + \Big(\sum_s \mu(s)\Big)^2 - 1 \\
&= \Delta_\mu^2 - (2 - \lambda^{-1})\Delta_\mu
\end{aligned}$$

□

7.3 SDP and Certification

Now we want to use Ozawa's characterisation for explicit examples of groups. Recall, we want to show that for a particular element $b \in \mathbb{R}[\Gamma]$, we can find $\xi_1, \ldots, \xi_n$ such that

$$b = \sum_{i=1}^{n} \xi_i^* \xi_i \tag{7.1}$$

For property (T) we only care about $b = \Delta^2 - \lambda\Delta$ for some $\lambda > 0$.

Let $E \subset \Gamma$ be a finite subset (in practice we are only interested in $E = B_\Gamma(1,r)$, and typically only $r = 2$ and potentially $r = 3$ are computationally feasible).

Let $\mathbf{x}$ be a marked basis of $\mathbb{R}[E] \subset \mathbb{R}[\Gamma]$. Then (7.1) has a solution in $\mathbb{R}[E]$ if and only if there exists a positive definite matrix P such that

$$b = \mathbf{x}^* P \mathbf{x}^T \tag{7.2}$$

Indeed, then $P = QQ^T$ by positive semidefiniteness, and so if $Q = \big(q_1|\cdots|q_k\big)$ where $k = |E|$ we see that

$$\mathbf{x}^* P \mathbf{x}^T = (\mathbf{x}Q)^*(\mathbf{x}Q)^T = \sum_{i=1}^{k} (\mathbf{x}q_i)^*(\mathbf{x}q_i)$$

Let $M_E(\mathbb{R})$ be the set of matrices with rows and columns indexed by E, and for $g \in \Gamma$ define the matrix $\delta_g \in M_E(\mathbb{R})$ via

$$(\delta_g)_{x,y} = \begin{cases} 1 & \text{if } x^{-1}y = g \\ 0 & \text{otherwise} \end{cases}$$

Equivalently, this is $g \in \Gamma$ viewed as an endomorphism of $\mathbb{R}[E]$ given by the left regular representation of Γ on $\mathbb{R}[\Gamma]$ (the so-called *Toeplitz operators*, see for example section 14 in [Oza12]).

If we consider the standard inner product on $M_E(\mathbb{R})$, where for $A, B \in M_E(\mathbb{R})$

$$\langle A, B \rangle := \mathrm{Tr}(A^T B)$$

We see that $\langle \delta_g, P \rangle = \sum_{x^{-1}y=g} P_{x,y}$ and so for any $g \in E^{-1}E$, $b_g = \langle \delta_g, P \rangle$.

So we've reduced the existence of a solution to 7.1 to the following semidefinite optimization problem:

$$\begin{cases} \text{minimize} & -\lambda \\ \text{subject to} & P \geq 0, P \in M_E(\mathbb{R}) \\ & \langle \delta_g, P \rangle = (\Delta^2 - \lambda\Delta)_g \text{ for all } g \in E^{-1}E \end{cases} \tag{SDP}$$

There are solvers specialised in numerical solutions of such problems.

Suppose now that we carry out (SDP) numerically, and obtain an approximate solution (P, λ_0) — that is, we have that

$$\Delta^2 - \lambda_0 \Delta \approx \mathbf{x} P \mathbf{x}^T$$

There are some potential issues

- The matrix P may not be positive semidefinite — it might have (small, up to the requested precision) negative eigenvalues;
- The linear constraints defined by $\Delta^2 - \lambda_0\Delta$ may be slightly violated;
- Some solvers claim to certify the solution but this is done in floating point arithmetic — which provides no mathematical certainty (see [Neu06]).

Luckily, we can turn an approximate solution into an exact one — at the cost of decreasing λ_0. Recall that to prove property (T), if we don't care about the Kazhdan constant, we only need some $\lambda > 0$.

Firstly, we find the real part of the square root of P, say Q — so that $QQ^T \approx P$. From each element we subtract the mean value of the elements in that column to obtain a matrix $\overline{Q}$ — this is equivalent to projecting onto the augmentation ideal $I[\Gamma]$. So now we have $\mathbf{x}\overline{Q}(\mathbf{x}\overline{Q})^T = \sum_i \xi_i^* \xi_i$, with $\overline{Q}$ a matrix whose columns correspond to elements in the augmentation ideal.

Recall that the Laplacian is an order unit in the augmentation ideal (Lemma 6.3.3), so for any $r \in I[\Gamma]^h$ there is some some $R_0 > 0$ such that $r + R\Delta \in \Sigma^2 I[\Gamma]$ for all $R \geq R_0$. So if we let $r = \Delta^2 - \lambda_0\Delta - \sum \xi_i^* \xi_i$, we can dominate our error by some R and turn our approximate solution into a mathematically rigorous proof of (T). Luckily, following Netzer and Thom ([NT15]), we can bound how big this R_0 has to be.

Lemma 7.3.1. *Let $r \in I[\Gamma]^h$ be supported on $B_\Gamma(1, 2^m)$. Then $R_0 \leq 2^{2m-1}\|r\|_1$.*

In fact if S has no self-inverse elements (in particular Γ has no 2-torsion) then $R_0 \leq 2^{2m-2}$.

Proof. Note firstly that for $s \in S$, we clearly have that $(1-s)^*(1-s) \preccurlyeq 2\Delta$. As in the proof of Lemma 6.3.3 we have the equation

$$(1-gh)^*(1-gh) \preccurlyeq 2(1-g)^*(1-g) + 2(1-h)^*(1-h) \tag{$*$}$$

So suppose that $g = s_1 \cdots s_{2^m} \in B_\Gamma(1, 2^m)$ where $s_i \in S \cup \{1\}$. Iterating ($*$) we immediatley see that

$$(1-g)^*(1-g) = (1 - s_1 \cdots s_{2^m})^*(1 - s_1 \cdots s_{2^m}) \preccurlyeq 2^m \sum_{i=1}^{2^m} (1-s_i)^*(1-s_i) \preccurlyeq 2^{2m+1}\Delta$$

Write $r = \sum_g r_g g$, and we have $\sum_g r_g = 0$ and $r_g = r_{g^{-1}}$ for all g. So

$$-r = \sum_g r_g - \sum_g \frac{r_g}{2}(g + g^{-1}) = \sum_{g \neq e} \frac{r_g}{2}(2 - g - g^{-1}) = \sum_{g \neq e} \frac{r_g}{2}(1-g)^*(1-g)$$

Since by assumption every $g \in B_{2^m}(e,S)$, we calculate

$$-r \preccurlyeq \sum_{g\neq e, r_g>0} \frac{r_g}{2}(1-g)^*(1-g) \preccurlyeq \Big(\sum_{g\neq e, r_g>0} r_g\Big) 2^{2m}\Delta$$

Since $\sum r_g = 0$ we see that $\sum_{g\neq e, r_g>0} r_g \leq \frac{1}{2}\|r\|_1$ which gives the required result.

If S has no 2-torsion then we have for $s \in S$ that $(1-s)^*(1-s) \preccurlyeq \Delta$, which yields the improved statement. □

Corollary 7.3.2. *If following the computation for Q as above we have that*

$$\|r\|_1 < 2^{-(2m-1)}\lambda_0$$

Then the group Γ has property (T), with spectral gap $\lambda \geq \lambda_0 - 2^{2m-1}\|r\|_1$.

We have the corresponding improvement by a factor of 2 if S has no 2-torsion.

See also section 4.2 in [KKN21] for an in depth analogue of this step for the augmentation ideal.

7.4 SL$(n,\mathbb{Z})$ and Aut(F_n)

Here we survey two concrete examples in which Ozawa's framework has been put to use: property (T) for the groups SL$(n,\mathbb{Z})$, $n \geq 3$ (a classical result already due to Kazhdan), and property (T) for Aut(F_n), $n \geq 4$.

This process began with Netzer and Thom ([NT15]) for SL$(3,\mathbb{Z})$, where they showed a spectral gap of $\frac{1}{6}$ for the Laplacian given by the standard generating set of elementary matrices and their inverses (this was later improved by Fujiwara-Kabaya to 0.2155 in [FK19], where they study lattices in $\tilde{A}_2$-buildings in general). See also [KN18] for more on the certification process we sketched above.

The next main step was proving property (T) for groups for which it wasn't previously known — in [KNO19] it is shown that Aut(F_5) has (T), and this method is expanded in [KKN21] to cover Aut(F_n) for $n \geq 6$ and SL$(n,\mathbb{Z})$ for $n \geq 3$. In particular also the lower bounds known on Kazhdan constants for SL$(n,\mathbb{Z})$ are improved when $n \geq 7$. We remark that these results (for Aut(F_n)) explain the fast convergence of the Product Replacement Algorithm, see [LP01].

We give a brief sketch of the approach of Kaluba-Kielak-Nowak, which makes heavy use of the high symmetry present in the groups SL$(n,\mathbb{Z})$ and Aut(F_n). But first, let us recall why Aut(F_2) and Aut(F_3) don't have (T).

Let Γ be a discrete group.

Definition 7.4.1. The *automorphism group* Aut(Γ) is the group of automorphisms of Γ under composition. The *inner automorphism group* Inn(Γ) is the group of automorphisms of the form $x \mapsto gxg^{-1}$ for some $g \in \Gamma$, that is automorphisms that are conjugations. The *outer automorphism group* Out(Γ) is the quotient Aut$(\Gamma)/$Inn(Γ).

It is clear that Aut(F_n) surjects onto Out(F_n), and this latter surjects onto GL$(n,\mathbb{Z})$ by abelianisation. For $n = 2$ in fact we have that Out$(F_2) \cong$ GL$(2,\mathbb{Z})$, and since this is virtually free it doesn't have property (T) and hence neither does Aut(F_2) (we've already noted that property (T) is inherited by quotients — or see Theorem 1.3.4 in [BdlHV08]).

There are two ways to see that Aut(F_3) doesn't have (T). Firstly, McCool ([McC89]) showed that Out(F_3) is virtually residually torsion-free nilpotent, and hence Out(F_3), and Aut(F_3), can't have property (T). Alternatively, Grunewald and Lubotzky ([GL09]) showed that Aut(F_3) virtually maps onto F_2, hence again can't have property (T).

Returning to Aut(F_n) with $n \geq 6$, we will work with SAut(F_n), the (index 2) kernel of the determinant map

$$\text{Aut}(F_n) \to \text{Out}(F_n) \to \text{GL}(n,\mathbb{Z}) \to \{\pm 1\}$$

For $\mathrm{SL}(n,\mathbb{Z})$ we take as generators the set of elementary matrices $E_{ij}^{\pm 1}$ $(i \neq j)$, where E_{ij} differs from the identity only with a 1 in the (i,j) spot. $\mathrm{SL}(n,\mathbb{Z})$ acts naturally on $\mathbb{Z}^n$, where E_{ij} sends the i'th basis vector e_i to $e_i + e_j$, and fixes every other basis vector. Since $\mathrm{SAut}(F_n)$ acts naturally on $F_n = \langle a_1, \dots, a_n \rangle$, by analogy a natural choice of generating set is given by the *Nielsen transvections* and their inverses. Namely let

$$\rho_{ij}(a_k) = \begin{cases} a_k & k \neq i \\ a_i a_j & k = i \end{cases} \quad \text{and} \quad \lambda_{ij}(a_k) = \begin{cases} a_k & k \neq i \\ a_j a_i & k = i \end{cases}$$

and consider the finite symmetric generating set $S = \{\rho_{ij}^{\pm 1}, \lambda_{i,j}^{\pm 1} \mid i \neq j\}$ for $\mathrm{SAut}(F_n)$.

Both $\mathrm{SAut}(F_n)$ and $\mathrm{SL}(n,\mathbb{Z})$ act on a set of n-basis vectors/generators, and have generators corresponding (not 1-1) to all pairs i, j. So in a sense we are considering complete graphs on n vertices. We will denote the basis vectors/generating set by $\{a_1, \dots, a_n\}$, the groups by Γ_n, and the generating sets of $\mathrm{SL}(n,\mathbb{Z})/\mathrm{Aut}(F_n)$ (elementary matrices/Nielsen transvections) by S_n — when confusion might occur, we will specify which one we mean. We will let Δ_n denote the group algebra Laplacian of Γ_n.

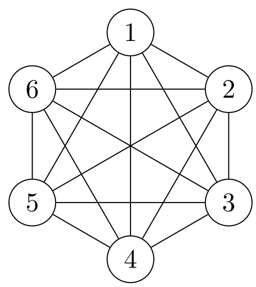
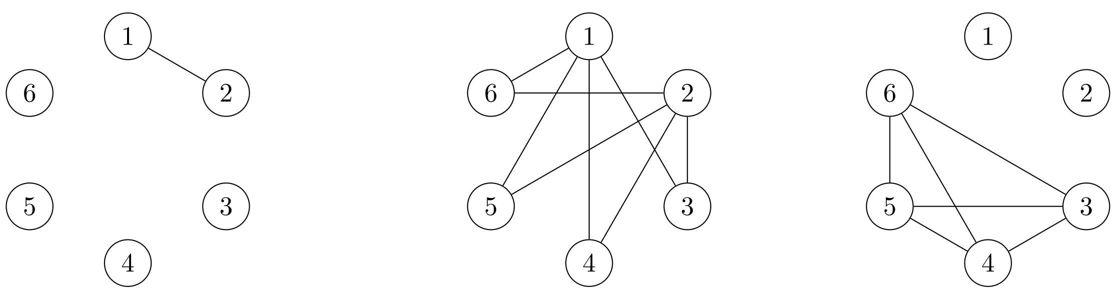


Figure 4: The complete graph on 6 vertices

When considering the Laplacian squared, we need to pair up generators. So suppose we fix an edge — there are three different types of edges related to it. Firstly we have the edge itself, we have edges adjacent to it, and edges opposite to it — as shown in Figure 5.



(a) The edge $\{1,2\}$ (b) The adjacent edges (c) The opposite edges

Figure 5: The edge $\{1,2\}$ and the edges adjacent and opposite to it in the complete graph on 6 vertices.

So in some sense it makes sense to consider the same edge, the adjacent, and the opposite edges separately, as they all have different behaviour. This was the idea in [KNO19].

In [KKN21] they use the fact that inclusion of the basis $\{a_1, \dots, a_n\} \hookrightarrow \{a_1, \dots, a_n\} \cup \{a_{n+1}\}$ coincides with the inclusion $\Gamma_n \hookrightarrow \Gamma_{n+1}$, and sends $S_n \hookrightarrow S_{n+1}$. We also observe that the *behaviour* of adjacent and opposite edges doesn't change — only the number of them. Thus they are able to simultaneously verify property (T) for all groups with large enough n. We now explain these two ideas in more detail.

Note that A_n, the alternating group of n letters, acts on N_n and hence induces an action of A_n on Γ_n (and $\mathbb{R}[\Gamma_n]$), and this action preserves the generating set S_n. We will denote this action by $\sigma(\cdot)$ for $\sigma \in A_n$.

This action respects the inclusion — the A_{n+1}-action on Γ_{n+1} restricts to the A_n-action on Γ_n (that is, the inclusion $\Gamma_n \hookrightarrow \Gamma_{n+1}$ is A_n-equivariant). We also get an A_n action on the set of (unoriented) edges of the complete graph $E_n = \{\{i,j\} \mid 1 \le i,j \le n, i \neq j\}$, and this coincides with the A_n-action on S_n after we remember only the labelling. That is, define the obvious function

$$\ell_n : S_n \to E_n : \begin{cases} E_{ij}^{\pm 1} \mapsto \{i,j\} \\ \rho_{ij}^{\pm 1}, \lambda_{ij}^{\pm 1} \mapsto \{i,j\} \end{cases}$$

and then for $\sigma \in A_n$ and $s \in S_n$ we have that $\ell_n(\sigma(s)) = \sigma(\ell_n(s))$. Now comes the key observation — if $\ell_n(s)$ and $\ell_n(t)$ are opposite edges, then s and t commute in Γ_n.

For any edges $e = \{i,j\} \in E_n$, let $S_e := \{s \in S_n \mid \ell_n(s) = e\}$, and define the Laplacian corresponding to e to be

$$\Delta_e := |S_e| - \sum_{s \in S_e} s$$

Our observations before tell us that $\sigma(\Delta_e) = \Delta_{\sigma(e)}$, and that each Δ_e is just the Laplacian of Γ_2 (up to relabeling). We identify $\Delta_{\{1,2\}}$ with Δ_2, the Laplacian of Γ_2. By a simple count, we see that

Lemma 7.4.2. *For any $n \ge 3$ we have that*

$$\Delta_n = \sum_{e \in E_n} \frac{1}{(n-2)!} \sum_{\sigma \in A_n} \sigma(\Delta_2)$$

Corollary 7.4.3. *For $m \ge n \ge 3$ we have that*

$$\sum_{\sigma \in A_m} \sigma(\Delta_n) = \binom{n}{2}(m-2)!\Delta_m \tag{7.3}$$

For an edge $e \in E_n$, let $\mathrm{Adj}(e)$ be the set of edges adjacent (but not equal) to e, and $\mathrm{Op}(e)$ the set of edges opposite e. Then we see that

$$\begin{aligned} \Delta_n^2 &= \sum_{e \in E_n} \sum_{f \in E_n} \Delta_e \Delta_f \\ &= \sum_{e \in E_n} \Delta_e^2 + \sum_{e \in E_n} \Delta_e \Big(\sum_{f \in \mathrm{Adj}(e)} \Delta_f \Big) + \sum_{e \in E_n} \Delta_e \Big(\sum_{f \in \mathrm{Op}(e)} \Delta_f \Big) \\ &=: \mathrm{Sq}_n + \mathrm{Adj}_n + \mathrm{Op}_n \end{aligned}$$

(by convention empty sums are 0). It is also possible to express each of these in terms of Δ_e for any $e \in E_n$ and the A_n action — indeed.

$$\begin{aligned} \mathrm{Sq}_n &= \frac{1}{(n-2)!} \sum_{\sigma \in A_n} \sigma(\Delta_e^2) \\ \mathrm{Adj}_n &= \frac{1}{(n-2)!^2} \sum_{\sigma \in A_n} \sigma(\Delta_e) \Big(\sum_{\tau(e) \in \mathrm{Adj}(\sigma(e))} \tau(\Delta_e) \Big) \\ \mathrm{Op}_n &= \frac{1}{(n-2)!^2} \sum_{\sigma \in A_n} \sigma(\Delta_e) \Big(\sum_{\tau(e) \in \mathrm{Op}(\sigma(e))} \tau(\Delta_e) \Big) \end{aligned}$$

Notice also that $\mathrm{Sq}_n, \mathrm{Op}_n \in \Sigma^2 \mathbb{R}[\Gamma_n]$. The first is obvious, and the latter comes from recalling that if $\ell_n(s)$ and $\ell_n(t)$ are opposite, then s and t commute. Hence the part that causes problems when trying to show that for some $\lambda > 0$, $\Delta_n^2 - \lambda \Delta_n$ is a sum of squares will be the Adj_n part. Firstly, let us collect without proof (see Lemmas 3.7, 3.8 in [KKN21]) the two following identities:

$$\sum_{\sigma \in A_m} \sigma(\mathrm{Adj}_n) = n(n-1)(n-2)\frac{(m-3)!}{2} \mathrm{Adj}_m \qquad \text{for } m \ge n \ge 3 \tag{7.4}$$

$$\sum_{\sigma \in A_m} \sigma(\mathrm{op}_n) = 2\binom{n}{2}\binom{n-2}{2}(m-4)!\,\mathrm{Op}_m \qquad \text{for } m \ge n \ge 4 \tag{7.5}$$

We are now ready to show how Kaluba, Kielak, and Nowak use Property (T) for some Γ_n to bootstrap property (T) for Γ_m with m sufficiently large.

Proposition 7.4.4 ([KKN21], Proposition 4.1)**.** *Let $n \geq 3$ and suppose that*

$$\mathrm{Adj}_n + k\,\mathrm{Op}_n - \lambda\Delta_n \in \Sigma^2\mathbb{R}[\Gamma_n]$$

for some $k \geq 0$ and $\lambda > 0$. Then for every $m \geq n$ with $k(n-3) \leq (m-3)$, the group Γ_m has property (T).

Proof. Suppose firstly that $n \geq 4$, then by equations (7.3), (7.4), and (7.5) we have that

$$\begin{aligned}\sum_{\sigma\in A_m}\sigma(\mathrm{Adj}_n + k\,\mathrm{Op}_n - \lambda\Delta_n) &= n(n-1)(n-2)\frac{(m-3)!}{2}\mathrm{Adj}_m \\ &\quad + 2k\binom{n}{2}\binom{n-2}{2}(m-4)!\,\mathrm{Op}_m - \lambda\binom{n}{2}(m-2)!\Delta_m \\ &= \frac{n(n-1)(n-2)}{2}(m-3)!\Big(\mathrm{Adj}_m + \frac{k(n-3)}{m-3}\mathrm{Op}_m - \frac{\lambda(m-2)}{n-2}\Delta_m\Big) \in \Sigma^2\mathbb{R}[\Gamma_m]\end{aligned}$$

Hence we see that

$$\begin{aligned}\Delta_m^2 - \frac{\lambda(m-2)}{n-2}\Delta_m &= \mathrm{Sq}_m + \mathrm{Adj}_m + \mathrm{Op}_m - \frac{\lambda(m-2)}{n-2}\Delta_m \\ &= \mathrm{Sq}_m + \Big(1 - \frac{k(n-3)}{m-3}\Big)\mathrm{Op}_m + \frac{2}{n(n-1)(n-2)(m-3)!}\sum_{\sigma\in A_m}\sigma(\mathrm{Adj}_n + k\,\mathrm{Op}_n - \lambda\Delta_n)\end{aligned}$$

Since Sq_m and Op_m are sums of squares, we see that if $1 - \frac{k(n-3)}{m-3} \geq 0$, we have that

$$\Delta_m^2 - \frac{\lambda(m-2)}{n-2}\Delta_m \in \Sigma^2\mathbb{R}[\Gamma_m]$$

and hence Γ_m has (T) by Theorem 6.1.1.

If $n = 3$ then $\mathrm{Op}_3 = 0$ and the relevant calculation is

$$\Delta_m^2 - (m-2)\lambda\Delta = \mathrm{Sq}_m + \mathrm{Op}_m + \frac{1}{3(m-3)!}\sum_{\sigma\in A_m}\sigma(\mathrm{Adj}_3 - \lambda\Delta_3) \in \Sigma^2\mathbb{R}[\Gamma_m]$$

□

In particular, we see that the spectral gap for Δ_m is bounded below by $\frac{\lambda(m-2)}{n-2}$, and so by (5.1) we get the lower bound for the Kazhdan constant

$$\sqrt{\frac{2\lambda(m-2)}{(n-2)|S_m|}} \leq \kappa(\Gamma_m, S_m)$$

Remark 7.4.5. In fact, the above proof also shows that the elements needed to exhibit the sum of squares decomposition of $\Delta_m^2 - \frac{\lambda(m-2)}{n-2}\Delta_m$ have the same maximal length as those needed for $\Delta_n^2 - \lambda\Delta_n$. That is, the Kazhdan radius of m is no greater than that of n.

Using a computer calculation and the certification argument detailed previously, in [KKN21] it is shown that for $\mathrm{SL}(3,\mathbb{Z})$

$$\mathrm{Adj}_3 - 0.157999\Delta_3 \in \Sigma^2 I[\mathrm{SL}(3,\mathbb{Z})]$$

Using Proposition 7.4.4, property (T) for $\mathrm{SL}(m,\mathbb{Z})$, $m \geq 3$ immediately follows, as do their estimates for Kazhdan constants (actually, better estimates are obtained when considering $\mathrm{SL}(5,\mathbb{Z})$).

Similarly, they are able to show that for $\mathrm{SAut}(F_5)$ we have

$$\mathrm{Adj}_5 + 2\,\mathrm{Op}_5 - 0.278\Delta_5 \in \Sigma^2 I[\mathrm{SAut}(F_5)]$$

For $n \geq 7$, Proposition 7.4.4 then tells us that $\mathrm{SAut}(F_n)$ has property (T).

Remark 7.4.6. The proof for $\mathrm{SAut}(F_6)$, while similar, turns out to be more involved. See method II in [KKN21].

We note that in all of these, the Kazhdan radius was found to be at most 2. Notice also that the method above doesn't prove property (T) for $\mathrm{Aut}(F_5)$, this was done in [KNO19]. Both the methods there and in [KN18] failed when applied to $\mathrm{SAut}(F_4)$, leading to some believing that $\mathrm{Aut}(F_4)$ didn't have property (T), as somewhat anticipated by [BV10]. The issue is that it seems that a Kazhdan radius of 2 isn't sufficient for $\mathrm{SAut}(F_4)$ (or at least the spectral gap this gave wasn't large enough for the certification step), and Kazhdan radius 3 was already far too much for any feasible computation.

However, Martin Nitsche ([Nit22]) was recently able to certify that $\mathrm{Aut}(F_4)$ also has (T) (and also to simplify the SDP proofs for $\mathrm{SL}(n,\mathbb{Z})$). He does this by using the dual SDP problem, namely finding functionals as in (6.5), and a similar symmetrisation procedure to the above. By exploiting the geometry of the group it is possible to massively reduce the necessary computation, whence the result.

7.5 Higher Dimensions

We now present the first use (to the authors' knowledge) of the machinery of Bader-Nowak ([BN20]) to investigate higher dimensional cohomology. Consider the group $\mathrm{SL}(3,\mathbb{Z})$, and recall that $H^1(\mathrm{SL}(3,\mathbb{Z}),\pi) = 0$ for all unitary representations π as $\mathrm{SL}(3,\mathbb{Z})$ has property (T). Firstly, we use the *Fox calculus* to simplify the SDP we need to solve.

Let $\Gamma = \langle S \mid R\rangle$ be a finitely presented group, we consider a family of derivations into the group algebra, as described by Fox ([Fox53, Fox54]). For $s,t \in S$ and any $g,h \in \Gamma$ set

$$\frac{\partial 1}{\partial s} = 0\ , \quad \frac{\partial s}{\partial t} = \delta_{st} \quad \text{and} \quad \frac{\partial (gh)}{\partial s} = \frac{\partial g}{\partial s} + g\frac{\partial h}{\partial s}$$

For any left Γ-module V consider the cochain complex

$$0 \longrightarrow V \xrightarrow{d^0} V^{|S|} \xrightarrow{d^1} V^{|R|} \xrightarrow{d^2} \cdots$$

where the differentials are given by

$$d^0 = (1-s)_{s\in S} \qquad \text{and} \qquad d^1 = \left(\frac{\partial r}{\partial s}\right)_{r\in R, s\in S}$$

This complex computes the cohomology of Γ with coefficients in V, as noted in [Lyn50]. The maps d^0 and d^1 are the cochain maps for the cohomology of the presentation complex (e.g. the 2-skeleton of $K(G,1)$). The cohomological Laplacian Δ^1 is then

$$\Delta^1 = d^0(d^0)^* + (d^1)^* d^1$$

We compute that $(d^1)^* d^1$ has the form

$$\left(\sum_{r\in R}\left(\frac{\partial r}{\partial s}\right)^*\frac{\partial r}{\partial t}\right)_{s,t} = \sum_{r\in R}\left(\left(\frac{\partial r}{\partial s}\right)^*\frac{\partial r}{\partial t}\right)_{s,t} \tag{7.6}$$

Consider the element

$$J(r) = \begin{pmatrix} \frac{\partial r}{\partial s_1} & \cdots & \frac{\partial r}{\partial s_n} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

and observe that (7.6) is of the form $\sum_{r\in R} J(r)^* J(r)$ and so is a sum of squares. So we obtain:

Lemma 7.5.1 ([KMN22], Lemma 2.1)**.** *If we have that* $d^0(d^0)^* + \sum_{r\in R'} J(r)^* J(r) - \lambda I \in \Sigma^2 M_n(\mathbb{R}\Gamma)$ *for some subset of relations* $R' \subset R$*, then* $\Delta^1 - \lambda I \in \Sigma^2 M_n(\mathbb{R}\Gamma)$*.*

Let us now restrict to the case of $\Gamma = SL(3,\mathbb{Z})$, generated by the six elementary matrices $E_{i,j} = I + \delta_{i,j}$ for $i \neq j$. The relations are given by (for i, j, k distinct)

$$\begin{cases} r_{i,j,k} = [E_{i,j}, E_{i,k}] \\ r'_{i,j,k} = [E_{i,j}, E_{j,k}]E_{i,k}^{-1} \\ r = (E_{1,2}E_{2,1}^{-1}E_{1,2})^4 \end{cases}$$

see for example [Mil71], Corollary 10.3. The Fox derivatives can then be computed explicitly, see section 2.1 in [KMN22] for these.

Then following a computation, and a certification argument as discussed earlier, Kaluba-Mizerka-Nowak obtain:

Theorem 7.5.2 ([KMN22], Theorem 1.1)**.** *Let* Δ^1 *be the cohomological Laplacian for* $\mathrm{SL}(3,\mathbb{Z})$*. Then for some* $\lambda > 0$*,* $\Delta^1 - \lambda I_n \in \Sigma^2 M_n(\mathbb{R}\,\mathrm{SL}(3,\mathbb{Z}))$*. In particular by Theorem 6.5.9 the second cohomology* $H^2(\mathrm{SL}(3,\mathbb{Z}), \pi)$ *is reduced for every unitary representation* π *of* $\mathrm{SL}(3,\mathbb{Z})$*,*

For Δ^1 of the presentation complex the explicit estimate they obtain is $\lambda \geq 0.32$.

Remark 7.5.3. Uri Bader and Roman Sauer shortly after the announcement of [KMN22] announced a much broader result with a similar conclusion.

Chapter 8

Group (in)stability

A topic that has gained significant interest in recent years is the notion of *group stability*, we will explore its relation to various cohomology vanishing (and non-vanishing) results. The basic question is: given a function between groups that is *almost* homomorphism, is it *close* to a true homomorphism?

Many different cases of this question have been studied, with varying definitions of '*almost*' and '*close*'. There are also links to the theory of *language testability*, a problem of considerable importance in computer science. See for example [Arz14], [BLM22], [BL20], [GR09], and [Tho18] and the references within for different aspects of the theory.

8.1 Definitions and Examples

When trying to investigate the stability of a group Γ, we need to pick three things:

(a) The (family of) group(s) $\mathcal{C}$ into which the almost homomorphisms go;

(b) What it means to be an *almost* homomorphism — this involves picking a metric on the G_n, but also whether we want some uniform bound on our group, or bound just on the relations;

(c) What it means to be *close* to a homomorphism, which as before depends on the metrics d_n, but also on whether we measure this uniformly over Γ, or just on the generators.

In some cases, for example for $U(n) \subset M_n(\mathbb{C})$, it also makes sense to ask what happens if our group is mapped to things that are *almost* unitary, but we won't consider this.

We will now fix the notion of stability that will be relevant for us. Let $\Gamma = \langle S \mid R \rangle$ be a finitely presented group, and let $\mathcal{C}$ be a class of groups with bi-invariant metrics. Typically $\mathcal{C}$ will be a sequence of groups of a given type, see the examples below. We also fix a non-principal ultrafilter $\omega \in \beta\mathbb{N}$.

Note that any map $\varphi : S \to G \in \mathcal{C}$ determines a homomorphism $F_S \to G$, we will again denote this by φ.

Definition 8.1.1. Let $\varphi, \psi : S \to G \in \mathcal{C}$ be maps.

(a) The *defect of* φ is

$$\operatorname{def}(\varphi) := \max_{r \in R} d_G(\varphi(r), 1_G)$$

(b) The *distance between* φ *and* ψ is defined by

$$\|\varphi - \psi\|_S := \max_{s \in S} d_G(\varphi(s), \psi(s))$$

(c) The *homomorphism distance of* ψ is defined by

$$\operatorname{HomDist}(\varphi) := \inf_{\pi \in \hom(\Gamma, G)} \|\varphi - \pi|_S\|_S$$

(d) For $\varepsilon > 0$ we say that φ is an ε-*almost homomorphism* if $\operatorname{def}(\varphi) < \varepsilon$.

Suppose we now have sequences of maps $\varphi = (\varphi_n : S \to G_n)_n$ and $\psi = (\psi_n : S \to G_n)_n$ with $G_n \in \mathcal{C}$.

(e) φ is an *asymptotic homomorphism* if $\omega\text{-}\lim_n \operatorname{def}(\varphi_n) = 0$;

(f) φ and ψ are *(asymptotically) equivalent* if $\omega\text{-}\lim_n \|\varphi_n - \psi_n\|_S = 0$;

(g) φ is *trivial (or liftable)* if it is equivalent to a sequence of homomorphisms (that is, $\omega\text{-}\lim_n \operatorname{HomDist}(\varphi) = 0$).

Definition 8.1.2. The group Γ is called $\mathcal{C}$-*stable* if every asymptotic homomorphism $\varphi = (\varphi_n)_n$ to $\mathcal{C}$ is trivial.

Definition 8.1.3. The group Γ is called $\mathcal{C}$-*approximated* if there is an asymptotic homomorphism $(\varphi_n)_n$ to $\mathcal{C}$ that is *separating* — that is, for ever $x \in F_S \backslash \langle\!\langle R \rangle\!\rangle$

$$\omega\text{-}\lim_n d_n(\varphi_n(x), 1_{G_n}) > 0$$

We remark the following useful fact, which says that for each $r \in \langle\!\langle R \rangle\!\rangle$ we can uniformly bound how far it can be from the identity for any map $\varphi : S \to G$. That is

Lemma 8.1.4 ([CGLT20], Lemma 1.13)**.** *For any $r \in R$ there is a constant $K = K(r)$ such that for any group G and any map $\varphi : S \to G$, we have that*

$$d_G(\varphi(r), 1_G) \le K \operatorname{def}(\varphi)$$

Proof. Write $r = x_1 r_1 x_1^{-1} \cdots x_k r_k x_k^{-1}$ where $r_1, \ldots, r_k \in R \cup R^{-1}$ and $x_1, \ldots, x_k \in F_S$. By bi-invariance we clearly have that $d_G(\varphi(r_j), 1_G) = d_G(\varphi(r_j^{-1}), 1_G) \le \operatorname{def}(\varphi)$, and by using bi-invariance again and the triangle inequality we are done by setting $K(r) = k$. ☐

Example 8.1.5. The following examples are taken from [CGLT20], see the references therein for more details.

(a) When $G_n = \mathrm{Sym}(n)$ and d_n is the normalised Hamming distance, $\mathcal{C}$-approximated groups are called *sofic*. It is an important open problem due to Gromov whether all groups are in fact sofic;

(b) When $G_n = U(n)$ (the n-dimensional unitary groups) with d_n induced by the *normalised Hilbert-Schmidt norm* ($\|T\|_{\mathrm{HS}} := \sqrt{n^{-1}\sum_{i,j=1}^n |T_{ij}|^2}$), $\mathcal{C}$-approximated groups are called *hyperlinear*. The existence of a non-hyperlinear group would answer the Connes Embedding Problem (for group algebras) in the negative;

(c) When $G_n = U(n)$ with d_n induced by the unnormalised Hilbert-Schmidt norm (also called the *Frobenius norm*) $\mathcal{C}$-stable groups are called *Frobenius stable*. We will study these in more detail shortly;

(d) When $G_n = U(n)$ with d_n induced by the operator norm, $\mathcal{C}$-approximated groups are called *Operator (or matricially) stable*.

Remark 8.1.6. It was shown by Elek and Szabó that sofic groups are in fact hyperlinear, see [ES05]. They were also the first to characterise sofic groups as those that embed in the metric ultraproduct of the $\mathrm{Sym}(n)$.

We will use ultraproduct Banach spaces and metric groups in the sequel, which we define below.

Let $(V_n)_{n\in\mathbb{N}}$ be a sequence of Banach spaces, and consider the space $\mathcal{V}$ of bounded sequences in $\prod_n V_n$ and

$$\mathcal{I} = \{(v_n)_n \in \mathcal{V} \mid \omega\text{-}\lim_n \|v_n\|_{V_n} = 0\}$$

The *ultraproduct Banach space* is

$$\textstyle\prod^{\omega}(V_n, \|\cdot\|_{V_n}) := \mathcal{V}/\mathcal{I}$$

This is a Banach space (in fact a Banach algebra/C^*-algebra/Hilbert space if all the V_n are).

Similarly if (G_n) is a sequence of groups equipped with bi-invariant metrics d_n, then let

$$\mathcal{N} = \{(g_n) \in \prod_n G_n \mid \omega\text{-}\lim_n d_n(g_n, 1) = 0\}$$

This is a normal subgroup of $\prod_n G_n$, and so define the *metric ultraproduct group* to be

$$\textstyle\prod^{\omega}(G_n, d_n) := \displaystyle\prod_n G_n/\mathcal{N}$$

Notice that in this case we don't require sequences to be bounded.

Remark 8.1.7. These definitions differ from the definition of the model theoretic ultraproduct — but they are the correct definitions in the field of *continuous logic*, which handles metric structures. Standard facts like Łos' Theorem can be suitably interpreted in this setting too, see [YBHU08] for the details.

We also use the Landau notation: for two sequences $(x_n)_n$ and $(y_n)_n$ of nonnegative real numbers we write

- $x_n = \mathcal{O}_\omega(y_n)$ if there is some $C > 0$ such that $x_n < Cy_n$ for ω-a.e. $n \in \mathbb{N}$;
- $x_n = o_\omega(y_n)$ if there is a sequence $(\varepsilon_n)_n$ of non-negative real numbers such that $\omega\text{-}\lim_n \varepsilon_n = 0$ and $x_n = \varepsilon_n y_n$.

8.2 Operator Stability

We state without proofs some recently proved facts about groups which are stable with respect to the operator norm, as alluded to above, see [ESS20] for a systematic study of these groups. This has links to cohomology vanishing due to Marius Dadarlat:

Theorem 8.2.1 ([Dad21], Corollary 1.3)**.** *Suppose* Γ *is a countable linear group, and for* ***some*** *even* n *the cohomology group with trivial coefficients* $H^n(\Gamma, \mathbb{R})$ *is nonzero. Then* Γ *is not operator stable.*

This was exploited by Bader-Lubotzky-Sauer-Weinberger to show that many cocompact lattices in higher rank semisimple groups aren't operator stable, indeed

Theorem 8.2.2 ([BLSW23], Theorem 1.5)**.** *Let* G *be a semisimple real Lie group not locally isomorphic to* $\mathrm{SL}(3, \mathbb{R})$ *or* $\mathrm{SO}(n, 1)$ *for odd* n*, and* $\Gamma \leq G$ *a* ***cocompact*** *lattice. Then* Γ *is not operator stable.*

This Theorem follows from the classification of Riemannian Symmetric Spaces which are odd dimensional rational homology spheres, and *Matsushima's formula* which allows us to inject the continuous cohomology of G into the cohomology of Γ.

We compare this sense of stability which is a local property with a *uniform* version (often called *Ulam stability*). The definitions are similar, but the distances depend on the whole group. In this case, there is for example a positive result obtained by Burger-Ozawa-Thom in [BOT13], namely

Theorem 8.2.3 ([BOT13], Theorem 1.3)**.** *Let* $\mathcal{O}$ *be the ring of integers of a number field,* $S \subset \mathcal{O}$ *a multiplicatively closed subset and* $\mathcal{O}_S$ *the corresponding localization. Then, for every* $n \geq 3$*, the group* $\mathrm{SL}(n, \mathcal{O}_S)$ *is uniformly operator stable.*

We won't discuss this further, instead diverting the interested reader to the cited paper, in which the result follows by a nice application of the fact that these group are *boundedly generated*.

8.3 Frobenius Stability

In this section, we sketch the construction of some non-Frobenius approximable groups, following the work of De Chiffre-Glebsky-Lubotzky-Thom, [CGLT20]. We do this by using the following key observation, which states that stability and approximability somewhat counter each other, as was already noted by Arzhantseva and by Glebsky-Rivera.

Definition 8.3.1. A group Γ is *residually* $\mathcal{C}$ if for all $x \in \Gamma \backslash \{1\}$ there is some $G \in \mathcal{C}$ and a homomorphism $\pi : \Gamma \to G$ such that $\pi(x) \neq 1$.

Proposition 8.3.2. *Let* Γ *be finitely presented, and* $\mathcal{C}$ *a class of groups (all equipped with bi-invariant metrics). If* Γ *is* $\mathcal{C}$*-stable and* $\mathcal{C}$*-approximable, then it must be residually* $\mathcal{C}$*. In particular, if* $\mathcal{C}$ *consists of finite-dimensional unitary groups, then* Γ *is residually finite.*

Proof. The first result is clear from the definitions, and the second follows as standard by Mal'cev's Theorem. □

Hence if we find a Frobenius stable but non-residually finite group, it can't be Frobenius approximable. So we wish to find a criterion for Frobenius stability, and we will see that being 2-Kazhdan (that is, $H^2(\Gamma, \pi) = 0$ for all unitary representations π) suffices.

From now, $\mathcal{C} = \{(U(n), \|\cdot\|_{\mathrm{Frob}})\}$, we will typically denote the norm by $\|\cdot\|$.

The main idea in [CGLT20] is that to an asymptotic representation $(\varphi_n : \Gamma \to U(k_n))_n$, we can associate an element $\alpha \in H^2(\Gamma, \prod^{\omega}(M_{k_n}(\mathbb{C}), \|\cdot\|))$. Then we show that if α vanishes, then in fact we can find an asymptotic homomorphism with effectively better defect. Thus for 2-Kazhdan groups we can always do this, and we use this to obtain stability.

It might be a priori surprising that this stability question has anything to do with cohomology. However as we saw in section 6.4.3 triviality of 2-cohomology is equivalent to the splitting of some short exact sequence. We will use the section we get to construct a new asymptotic homomorphism. First we need two results about unitarily-invariant norms.

Proposition 8.3.3. *Let $\|\cdot\|$ be any unitarily invariant norm (such as the Frobenius norm), and $A, B, C \in M_n(\mathbb{C})$ for some n. Then*

(a) $\|ABC\| \leq \|A\|_{\text{op}} \cdot \|B\| \cdot \|C\|_{\text{op}}$*;*

(b) $\|A\| = \|A^*\| = \||A|\|$ *where $|A|$ denotes the unique self-adjoint matrix such that $|A|^2 = A^*A$;*

(c) If in addition A and B are positive semidefinite matrices with $A \leq B$ (that is, $B - A$ is positive semidefinite), then $\|A\| \leq \|B\|$

Proposition 8.3.4. *Let $A \in U(n)$. Then there is some unitary matrix $B \in U(n)$ such that $B^2 = I_n$ and $\|B - A\| \leq \|I_n - A^2\|$.*

Proof. By unitary invariance one can assume that $A = \text{diag}(a_1, \ldots, a_n)$ is diagonal, so set $b_j = (-1)^{\text{sign}(\text{Re}\, a_j)}$ so that $|b_j - a_j| \leq |1 - a_j| \cdot |-1 - a_j| = |1 - a_j^2|$. Then $B = \text{diag}(b_1, \ldots, b_n)$ is self-adjoint and unitary, and by (b) and (c) in the above proposition we see that $\|B - A\| = \||B - A|\| \leq \||I_n - A^2|\| = \|I_n - A^2\|$. □

Remark 8.3.5. Note also that the Frobenius norm is *submultiplicative*, whereas the normalised Hilbert-Schmidt norm isn't; this is the key difference that means we can't apply these techniques to study hyperlinear groups (and hence attack the CEP directly).

Suppose now that we are given an asymptotic homomorphism $\varphi = (\varphi_n : S \to U(k_n))_n$.

Notice that $U_\omega := \prod^\omega (U(k_n), d_{\|\cdot\|})$ acts unitarily on $M_\omega = \prod^\omega (M_{k_n}(\mathbb{C}), \|\cdot\|)$, and since φ defines a homomorphism $\varphi_\omega : \Gamma \to U_\omega$ we also have that Γ acts unitarily on M_ω.

Fix now a section $\sigma : \Gamma \to F_S$ of the natural surjection $F_S \to \Gamma$, such that $\sigma(1_\Gamma) = 1_{F_S}$ and $\sigma(g^{-1}) = \sigma(g)^{-1}$ whenever $g^2 \neq 1$. Note that $\sigma(g)\sigma(h)\sigma(gh)^{-1} \in \langle\langle R \rangle\rangle$ for all g, h.

Set $\tilde{\varphi}_n := \varphi_n \circ \sigma : \Gamma \to U(k_n)$ when $g^2 \neq e$, and note that

(a) $\tilde{\varphi}_n(1_\Gamma) = I_{k_n}$;

(b) $\tilde{\varphi}_n(g^{-1}) = \tilde{\varphi}_n(g)^*$ whenever $g^2 \neq 1$.

If $g^2 = 1_\Gamma$, then by Lemma 8.1.4 we see that

$$\|\varphi_n(\sigma(g))^2 - I_{k_n}\| = \mathcal{O}_\omega(\text{def}(\varphi_n))$$

and so by Proposition 8.3.4 there are self-adjoint unitaries $B_n \in U(k_n)$ such that

$$\|B_n - \varphi_n(\sigma(g))\| = \mathcal{O}_\omega(\text{def}(\varphi_n))$$

If we set $\tilde{\varphi}_n(g) := B_n$ in this case, we have a sequence $\tilde{\varphi} = (\tilde{\varphi}_n : \Gamma \to U(k_n))_n$ such that for all $g \in \Gamma$

$$\|\tilde{\varphi}_n(g) - \varphi_n(\sigma(g))\| = \mathcal{O}_\omega(\text{def}(\varphi_n))$$

8.3.1 Associating a Cocycle

Taking inspiration from how we associated a 2-cocycle to a section in section 6.4.3, we define the map $a_n := a_n(\varphi_n) : \Gamma \times \Gamma \to M_{k_n}(\mathbb{C})$ via

$$a_n(g, h) = \frac{\tilde{\varphi}_n(g)\tilde{\varphi}_n(h) - \tilde{\varphi}_n(gh)}{\text{def}(\varphi_n)} \tag{8.1}$$

whenever $\text{def}(\varphi_n) > 0$, and 0 otherwise.

Proposition 8.3.6 ([CGLT20], Proposition 3.1)**.** *For all $g,h,k \in \Gamma$, we have that*

$$\begin{cases} \tilde{\varphi}_n(g)a_n(h,k) - a_n(gh,k) + a_n(g,hk) - a_n(g,h)\tilde{\varphi}_n(k) = 0 \\ a_n(g,g^{-1}) = a_n(1_\Gamma,g) = a_n(g,1_\Gamma) = 0 \\ a_n(g,h)^* = a_n(h^{-1},g^{-1}) \end{cases} \tag{8.2}$$

Furthermore for every $g,h \in \Gamma$, we have that $\|a_n(g,h)\| = \mathcal{O}_\omega(1)$. In particular, $(a_n(g,h))_n$ is a bounded sequence, and so it defines a map $a = (a_n) : \Gamma \to M_\omega$.

The first line in (8.2) should be familiar — it is reminiscent of the cocycle equation (and indeed, it shows that a represents a cocycle in *Hochschild cohomology*). To correct this to a normal cocycle, by looking at (8.1) we see that we should define

$$c(g,h) = a(g,h)\varphi_\omega(gh)^*$$

Corollary 8.3.7. *The map $c \in C^2(\Gamma, M_\omega)$ is a 2-cocycle with respect to the isometric action $\pi(g)T = \varphi_\omega(g)T\varphi_\omega(g)^*$ for $g \in \Gamma$, $T \in M_\omega$.*

Proof. We need to check the cocycle equation for $g_0, g_1, g_2 \in \Gamma$. Indeed, using (8.2) we calculate

$$\begin{aligned} d^2c(g_0,g_1,g_2) =& \varphi_\omega(g_0)c(g_1,g_2)\varphi(g_0)^* - c(g_0g_1,g_2) + c(g_0,g_1g_2) - c(g_0,g_1) \\ =& \varphi_\omega(g_0)a(g_1,g_2)\varphi_\omega(g_1g_2)^*\varphi(g_0)^* - a(g_0g_1,g_2)\varphi_\omega(g_0g_1g_2)^* \\ &+ a(g_0,g_1g_2)\varphi_\omega(g_0g_1g_2)^* - a(g_0,g_1)\varphi_\omega(g_0g_1)^* \\ =& (\varphi_\omega(g_0)\alpha(g_1,g_2) - \alpha(g_0g_1,g_2) + \alpha(g_0,g_1g_2) - \alpha(g_0,g_1)\varphi_\omega(g_2))\varphi_\omega(g_0g_1g_2)^* \\ =& 0 \end{aligned}$$

as required. ☐

So to a given asymptotic homomorphism we can associate a cocycle. Now we want to explore what happens if this cocycle is trivial.

Proposition 8.3.8 ([CGLT20], Proposition 3.3)**.** *Suppose that $[c] = 0$ in cohomology, that is $c = d^1b$ for some $b : \Gamma \to M_\omega$. Then we have*

$$b(1_\Gamma) = 0 \tag{8.3}$$

$$b(g) = -\varphi_\omega(g)b(g^{-1})\varphi_\omega(g)^* \tag{8.4}$$

$$a(g,h) = \varphi_\omega(g)b(h)\varphi_\omega(h) - b(gh)\varphi_\omega(gh) + b(g)\varphi_\omega(gh) \tag{8.5}$$

Furthermore, we can choose b to be skew-symmetric for all $g \in \Gamma$.

Proof. These all follow from the previous calculations. Indeed, $c = d^1b$ just tells us that

$$c(g,h) = \varphi_\omega(g)b(h)\varphi_\omega(g)^* - b(gh) + b(g)$$

Using the definition of c we immediately get (8.5), and then (8.3) follows using Proposition 8.3.6 with $g = h = 1_\Gamma$. Similarly, (8.4) follows from (8.3), (8.5), and Proposition 8.3.6 and $h = g^{-1}$.

To get the last claim, notice that we can replace b with $\frac{1}{2}(b(g) - b(g)^*)$ and verifying that it also satisfies (8.5), see [CGLT20] for the full calculation. ☐

8.3.2 Correcting the Asymptotic Representation

Let b be as in Proposition 8.3.8, and $(b_n)_n : \Gamma \to M_{k_n}(\mathbb{C})$ a skew-symmetric lift of b.

Notice that $\exp(-\operatorname{def}(\varphi_n)b_n(g))$ is unitary, and so we can define a sequence of maps $\psi = (\psi_n : \Gamma \to U(k_n))_n$ by

$$\psi_n(g) = \exp(-\operatorname{def}(\varphi_n)b_n(g))\tilde{\varphi}_n(g)$$

Proposition 8.3.9 ([CGLT20], Proposition 3.4 and Lemma 3.5)**.** *ψ satisfies for every $g, h \in \Gamma$*

(a) $\|\tilde{\varphi}_n(g) - \psi_n(g)\| = \mathcal{O}_\omega(\operatorname{def}(\varphi_n))$;

(b) $\|\psi_n(gh) - \psi_n(g)\psi_n(h)\| = o_\omega(\operatorname{def}(\varphi_n))$.

So now we just let $\varphi'_n = \psi_n|_S$, and we get the following result:

Theorem 8.3.10 ([CGLT20], Theorem 3.6)**.** *Let $\Gamma = \langle S \mid R \rangle$ be a finitely presented group, and $\varphi = (\varphi_n : S \to U(k_n))_n$ an asymptotic representation with respect to a family of submultiplicative, unitarily invariant norms. If the associated 2-cycle is trivial in $H^2(\Gamma, M_\omega)$, then there is an asymptotic representation $\varphi' = (\varphi'_n)_n : S \to U(k_n)$ such that*

(a) $\|\varphi_n - \varphi'_n\|_S = \mathcal{O}_\omega(\operatorname{def}(\varphi_n))$, *and*

(b) $\operatorname{def}(\varphi'_n) = o_\omega(\operatorname{def}(\varphi_n))$

Remark 8.3.11. The connection between improving the defect in φ_n and finding a splitting for the short exact sequence can be made explicit, yielding a neater (bur non-constructive) proof of Theorem 8.3.10. For this, see section 3.4 in [CGLT20].

8.3.3 2-Kazhdan Groups

We are now ready to prove the aforementioned criterion for Frobenius stability.

Theorem 8.3.12 ([CGLT20], Theorem 5.1)**.** *Let $\Gamma = \langle S \mid R \rangle$ be a finitely presented group. If Γ is 2-Kazhdan, then it is Frobenius stable.*

Proof. Since in particular $H^2(\Gamma, M_\omega)$ vanishes, by Corollary 6.4.23 and the boundedness result in Proposition 8.3.6 there is some constant C such that for any asymptotic representation $\varphi = (\varphi_n : \Gamma \to U(k_n))_n$ with respect to $\|\cdot\| = \|\cdot\|_{\mathrm{Frob}}$, we can choose the associated 1-cocycle b so that

$$2 \max_{s \in S} \|b(s)\| \leq C$$

For any map $\varphi_n : S \to U(k_n)$ we let $\theta(\varphi_n) \coloneqq \operatorname{HomDist}(\varphi_n) - 2C \operatorname{def}(\varphi_n)$ — observe that $\omega\text{-}\lim_n \theta(\varphi_n) \geq 0$ if $\varphi = (\varphi_n)_n$ is an asymptotic representation, with equality if and only if φ is trivial.

Let now $(\varepsilon_n)_n$ be a sequence of strictly positive real numbers with $\omega\text{-}\lim_n \varepsilon_n = 0$, and (k_n) a sequence of natural numbers. We need therefore to show that for any sequence of ε_n-almost representations $(\psi_n : S \to U(k_n))_n$, $\theta(\psi_n)$ tends to 0.

Since for each $n \in \mathbb{N}$ the space of ε_n-almost representations on $U(k_n)$ is compact, by continuity of θ there is some sequence $\varphi = (\varphi_n : S \to U(k_n))_n$ with $\operatorname{def}(\varphi_n) \leq \varepsilon_n$, such that φ_n maximises θ for every n. By Theorem 8.3.10 there is an asymptotic representation $\varphi'_n : S \to U(k_n)$ such that $\|\varphi_n - \varphi'_n\|_S \leq C \operatorname{def}(\varphi_n)$, and

$$\operatorname{def}(\varphi'_n) \leq \tfrac{1}{4} \operatorname{def}(\varphi_n)$$

In particular φ'_n is also an ε_n-almost representation, and for ω-a.e. $n \in \mathbb{N}$ we have

$$\operatorname{HomDist}(\varphi_n) \leq \operatorname{HomDist}(\varphi'_n) + C \operatorname{def}(\varphi_n)$$

By our choice of φ_n we have

$$\operatorname{HomDist}(\varphi'_n) - 2C \operatorname{def}(\varphi'_n) = \theta(\varphi'_n) \leq \theta(\varphi_n) = \operatorname{HomDist}(\varphi_n) - 2C \operatorname{def}(\varphi_n)$$

and so putting these two equations together we see that

$$\operatorname{HomDist}(\varphi'_n) - 2C \operatorname{def}(\varphi'_n) \leq \operatorname{HomDist}(\varphi'_n) - C \operatorname{def}(\varphi_n)$$

and so

$$\mathrm{def}(\varphi_n) \leq 2\,\mathrm{def}(\varphi'_n) \leq \tfrac{1}{2}\,\mathrm{def}(\varphi_n)$$

so for ω-a.e. $n \in \mathbb{N}$, $\mathrm{def}(\varphi_n) = 0$ — so in fact φ_n is a representation for most n, and so $\lim_\omega \theta(\varphi_n) = 0$. By the maximality of our choice of φ_n, in fact $\lim_\omega(\psi_n) = 0$ for all ε_n-almost representations ψ_n. □

For completeness, we list some examples of groups to which the above Theorem applies.

- Let k be a non-Archimedean local field of residue class q, and $\mathbb{G}$ a simple k-algebraic group of k-rank $r \geq 1$. Then for $r \geq 3$ and sufficiently large q, any uniform lattice $\Gamma \leq \mathbb{G}(k)$ is 2-Kazhdan. This is due to Świątkowski ([BŚ97]), building on previous work of Garland ([Gar73]) — see section 4.1 in [CGLT20] for a further discussion. Note that these groups have property (T), and so their first cohomology vanishes also.
- Extensions of the examples above by suitable groups. A non-residually finite (and hence as noted previously, non Frobenius-approximable) example is constructed in section 5.2 of [CGLT20].
- Upcoming work of Bader and Sauer ([BS23]) shows that an irreducible lattice Γ in a higher rank semisimple Lie Group has $H^i(\Gamma, \pi) = 0$ for $1 \leq i \leq n$ whenever π has no non-zero Γ-invariant vectors, where n is the minimal rank of each non-compact factor of G. This strongly suggests that such a Γ is Frobenius stable if the rank of each non-compact factor is at least 3, however there might be nonzero second cohomology with trivial coefficients, $H^2(\Gamma, \mathbb{R})$.

  To get around this, Bader-Lubotzky-Sauer-Weinberger ([BLSW23]) show that $\tilde{\Gamma}$ is strongly 2-Kazhdan, where $\tilde{\Gamma} \leq \tilde{G}$ is the universal central extension. They then deduce Frobenius stability for Γ from that of $\tilde{\Gamma}$ (apart from the case that one of the non-compact factors is of Hermitian type, and Γ doesn't have the congruence subgroup property) by building on results of Deligne ([Del78]). We include this final step for general interest below, but redirect the interested reader to [BLSW23] for the rest.

8.3.4 The Congruence Subgroup Property

A full and careful introduction of arithmetic lattices and the congruence subgroup property (CSP) is too far outside the scope of this paper for us to do it proper justice. The reader is encouraged to head to [Zim84] or [Bor19] for the former, and [Rag76], [Rag86] for the latter. Instead, we try to motivate the definitions of both, and justify why CSP has anything to do with Frobenius stability.

Let us consider a semisimple Lie group $G \leq \mathrm{GL}(n, \mathbb{R})$ defined over $\mathbb{Q}$, for example $\mathrm{SL}(n, \mathbb{R})$. How can we find a lattice in it?

A good attempt would be to consider the $\mathbb{Z}$-points, that is $G(\mathbb{Z}) := G \cap \mathrm{GL}(n, \mathbb{Z})$. It is a classical Theorem of Borel and Harish-Chandra that this indeed gives a lattice — for example, $\mathrm{SL}(n, \mathbb{Z}) \leq \mathrm{SL}(n, \mathbb{R})$ is a lattice. How else can we find a lattice? clearly any finite index subgroup of $G(\mathbb{Z})$ or any group in which $G(\mathbb{Z})$ is of finite index must be a lattice.

Definition 8.3.13. Two subgroups $\Gamma, \Gamma' \leq G$ are *commensurable* if $\Gamma \cap \Gamma'$ is of finite index in both Γ and Γ'.

Hence we readily see that if Γ is a lattice and Γ' is commensurable to it, then Γ' must also be a lattice.

Finally, if we have a lattice $\Gamma \leq G$ and an epimorphism $\varphi : G \to H$ for some other algebraic group H, and the kernel is compact, the image $\varphi(\Gamma)$ must also be a lattice. Iterating these constructions (for general number fields), we obtain the definition of arithmetic lattices — that is, lattices which we can always find and 'easily' construct in semisimple Lie groups. It is a deep Theorem of Margulis (that follows from *Margulis superrigidity*) that all lattices in higher rank (≥ 2) semisimple groups are arithmetic.

Suppose we are now given an arithmetic group, say $\mathrm{SL}(n, \mathbb{Z})$. Can we describe all of its finite index subgroups easily?

There are obvious ways to construct finite index subgroups: for every $k \in \mathbb{Z}$ we have a ring homomorphism $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/k\mathbb{Z}$, which induces a homomorphism $\mathrm{SL}(n, \mathbb{Z}) \to \mathrm{SL}(n, \mathbb{Z}/k\mathbb{Z})$. The kernel of this is clearly of finite index, called the *principal congruence subgroup* Γ_k (those familiar with the theory of modular forms should recognise these in the case $n = 2$).

Definition 8.3.14. An arithmetic lattice satisfies the *congruence subgroup property (CSP)* if all of its finite index subgroups contain some principal congruence subgroup.

It is classical due to Bass-Lazard-Serre (1964) and Mennicke (1965) that for $k \geq 3$, $\mathrm{SL}(k,\mathbb{Z})$ satisfies CSP. In general the question of whether a lattice satisfies CSP is still an active field of interest.

Example 8.3.15. $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ doesn't satisfy CSP. Indeed, suppose it does, and let H be a finite quotient of it. In particular H is a quotient of Γ/Γ_k for some k. Notice that since $\mathbb{Z}/k\mathbb{Z} \cong \bigoplus \mathbb{Z}/p_i^{k_i}\mathbb{Z}$ for some primes p_i and $k_i \geq 0$, we have that

$$\Gamma/\Gamma_k \cong \bigoplus \Gamma/\Gamma_{p_i^{k_i}}$$

and hence the only non-abelian factors in a composition series for H are of the form $\mathrm{SL}(2,\mathbb{Z}/p\mathbb{Z})$. However as we saw (in example 3.2.4) $\mathrm{SL}(2,\mathbb{Z})$ is virtually free, so its finite quotients include *every* finite simple group in their composition series — this is clearly a contradiction.

So CSP tells us something about the finite index subgroups of Γ, and allows us to pass Frobenius stability to some quotient groups.

Definition 8.3.16. The *profinite radical of* Γ, denoted $\mathrm{PR}(\Gamma)$, is the intersection of all finite index subgroups of Γ.

The following Theorem is due to Deligne.

Theorem 8.3.17 (Deligne)**.** *Let $\Gamma \leq G$ be a lattice satisfying CSP, and assume that $\pi_1(G)$ is infinite. Let $\tilde{G} \to G$ be the universal cover, which is the universal central extension of G. Then the preimage $\tilde{\Gamma}$ is not residually finite, and furthermore $\mathrm{PR}(\Gamma) \subset \ker(\tilde{\Gamma} \to \Gamma)$ is a subgroup of finite index.*

Lemma 8.3.18 ([BLSW23], Corollary 2.14)**.** *Let Γ be a finitely presented group and let $N \triangleleft \Gamma$. Assume that $N_0 := N \cap \mathrm{PR}(\Gamma)$ is of finite index in N. If Γ is Frobenius stable, then so is Γ/N.*

Proof. Note that Γ/N_0 is the extension of Γ/N by the finite group N/N_0 and hence is finitely presented. We now proceed in two steps:

Step 1. Γ/N_0 is Frobenius stable: Indeed, we will show that for any normal subgroup $K \triangleleft \Gamma$ contained in $\mathrm{PR}(\Gamma)$, Frobenius stability descends from Γ to Γ/K. Let $\varphi = (\varphi_n)_n$ be an asymptotic homomorphism of Γ/K, and let $p : \Gamma \to \Gamma/K$ be the projection. So $\varphi \circ p$ is an asymptotic homomorphism of Γ, by assumption it is approximated by a sequence of true homomorphisms $\psi = (\psi_n)$.

For every $n \in \mathbb{N}$, $\psi_n(\Gamma)$ is a finitely generated linear group, hence by Mal'cev's theorem is residually finite. Thus each ψ_n factors through $\Gamma/\mathrm{PR}(\Gamma)$ and in particular through Γ/K, and these induced homomorphisms approximate φ.

Step 2. Frobenius stability descends from Γ/N_0 to Γ/N: Indeed, we will show more generally that if Γ is finitely presented and $K \triangleleft \Gamma$ is finite, then Frobenius stability descends from Γ to Γ/K.

Let $F_S \to \Gamma = \langle S \mid R \rangle$ be the canonical surjection, without loss of generality we can pick a subset $\tilde{K} \subset S$ with image K. Let $T = R \cup \tilde{K} \subset F_S$, this is a finite set of relations for Γ/K.

For every $\varepsilon > 0$, there is some $\delta > 0$ such that every Frobenius δ-almost representation of Γ is Frobenius ε-close to a representation.

Let $\alpha := \max_{k \in K} \|\pi(k) - 1\|$ ranging over all the non-trivial irreducible representations π of K. Fix now some $\varepsilon > 0$ (with $\varepsilon < \alpha/2$), then pick a corresponding δ (again with $\delta < \alpha/2$). Let $\varphi_n : S \to U(n)$ be a map such that for all $r \in R$, $\|\varphi_n(r) - 1\| < \delta$, in particular we get a map $\psi_n : S \to U(n)$ such that $\|\varphi_n - \psi_n\|_S < \varepsilon$, and ψ_n extends to a homomorphism from Γ. Restricting to K, we get that for every $k \in K$

$$\|\psi_n(k) - 1\| \leq \|\varphi_n(k) - \psi_n(k)\| + \|\varphi_n(k) - 1\| < \varepsilon + \delta \leq \alpha$$

Since the Frobenius norm of a representation majorizes the Frobenius norm of any subrepresentation, we get that $\psi_n|_K$ contains no non-trivial irreducible subrepresentation. So for every $t \in T$, $\psi_n(t) = 1$ and ψ_n descends to a representation of Γ/K which is ε-close to φ_n.

□

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