# Introduction to Non-Linear Analysis - Seminar 

Fabian Schulte and Floris Koster

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#### Abstract

In this short summary we want to provide a brief overview over the proof of the Strichartz estimates. The Strichartz estimates are averaged temporal bounds for the solution of the Schrödinger equation. For more details we refer to Chapter 5.2 Strichartz space-time bounds Rap.


## 1 Definitions

We start by defining some kind of "mixed norm" in space and time using $L^{p}$-norms.
Definition 1.1 (mixed $L^{q}$ - $L^{r}$-norm ). Let $(q, r) \in[1, \infty]^{2}$. For every $u \in L^{q}\left(\mathbb{R} ; L^{r}\left(\mathbb{R}^{d}\right)\right)$ define

$$
\|u\|_{L_{t}^{q} L_{x}^{r}}=\left\{\begin{array}{l}
\left(\int_{\mathbb{R}}\|u(t, \cdot)\|_{L_{x}^{r}}^{q} d t\right)^{\frac{1}{q}} \text { for } 1 \leq q<+\infty \\
\sup _{t \in \mathbb{R}}\|u(t, \cdot)\|_{L_{x}^{r}} \text { for } q=\infty
\end{array}\right.
$$

One can show that the following restriction on pairs $(q, r)$ is necessary for the Strichartz estimates.

Definition 1.2 (Admissible pair). We say $(q, r) \in[2, \infty]^{2}$ is admissible if

$$
\frac{2}{q}+\frac{d}{r}=\frac{d}{2} \text { and }(q, r, d) \neq(2, \infty, 2)
$$

where $d$ denotes the dimension of our space $\mathbb{R}^{d}$. We say that $(q, r)$ is strictly admissible if in addition $(q, r) \neq\left(2, \frac{2 d}{d-2}\right)$.

## 2 Theorem

We now state the main theorem of Chapter 5.2. in Rap.
Theorem 2.1 (Strichartz estimates). Let $d \geq 1$.
i) Homogenous case: For all admissible pair $(q, r)$ there exists $C \geq 0$ such that for all $u_{0} \in L^{2}\left(\mathbb{R}^{d}\right)$, the solution $S(t) u_{0} \in C\left(\mathbb{R} ; L^{2}\left(\mathbb{R}^{d}\right)\right)$ to the homogenous Schrödinger equation

$$
\left\{\begin{array}{l}
i \partial_{t} u+\Delta u=0 \\
u_{\mid t=0}=u_{0}
\end{array}\right.
$$

satisfies

$$
\begin{equation*}
\left\|S(t) u_{0}\right\|_{L_{t}^{q} L_{x}^{r}} \leq C\left\|u_{0}\right\|_{L^{2}} \tag{1}
\end{equation*}
$$

ii) Inomogenous case: For all admissible pairs $\left(q_{1}, r_{1}\right),\left(q_{2}, r_{2}\right)$ there exists $C \geq 0$ such that for all $f \in L^{q_{2}^{\prime}}\left(\mathbb{R} ; L^{r_{2}^{\prime}}\left(\mathbb{R}^{d}\right)\right)$, the unique solution $v \in C\left(\mathbb{R} ; L^{2}\left(\mathbb{R}^{d}\right)\right)$ given by the Duhamel formula (see Rap, p. 73]) to

$$
\left\{\begin{array}{l}
i \partial_{t} u+\Delta u=f \\
u_{\mid t=0}=0
\end{array}\right.
$$

satisfies

$$
\begin{equation*}
\|v\|_{L_{t}^{q_{1}} L_{x}^{r_{1}}} \leq C\|f\|_{L_{t}^{q_{2}^{\prime}} L_{x}^{r_{2}^{\prime}}} . \tag{2}
\end{equation*}
$$

Remark 2.1. Note that (1) shows that $u(t)=S(t) u_{0} \in L^{r}\left(\mathbb{R}^{d}\right)$ for a.e. $t$ even if $u_{0} \in L^{2}\left(\mathbb{R}^{d}\right)$.

## 3 Proof Idea

The proof of (1), (2) consists of the following four steps.

1. First, one proves an elementary lemma called $T T^{*} L e m m a$. The lemma allows to use the following equivalence

$$
T \in \mathcal{L}(\mathcal{H} ; \mathcal{B}) \Longleftrightarrow T^{*} \in \mathcal{L}\left(\mathcal{B}^{\prime}, \mathcal{H}\right) \Longleftrightarrow T T^{*} \in \mathcal{L}\left(\mathcal{B}^{\prime} ; \mathcal{B}\right)
$$

where $\left(\mathcal{H},(\cdot, \cdot)_{\mathcal{H}}\right)$ is a Hilbert space, $\mathcal{B}$ denotes a Banach space with its corresponding dual $\mathcal{B}^{\prime}$ and $T^{*}: \mathcal{B}^{\prime} \rightarrow \mathcal{B}$ denotes the adjoint of $T$.
2. Define the following linear operator

$$
T: \mathcal{H} \rightarrow \mathcal{B}, u_{0} \mapsto\left[t \mapsto S(t) u_{0}\right]
$$

and then compute $T^{*}, T T^{*}$.
3. Show that

$$
\left\|T T^{*}\right\|_{\mathcal{L}\left(\mathcal{B}^{\prime} ; \mathcal{B}\right)}<\infty
$$

and therefore $\|T\|_{\mathcal{L}(\mathcal{H} ; \mathcal{B})}<\infty$ and $\left\|T^{*}\right\|_{\mathcal{L}\left(\mathcal{B}^{\prime} ; \mathcal{H}\right)}<\infty$ by the $T T^{*}$ Lemma. Applied to the spaces

$$
\mathcal{H}=L^{2}\left(\mathbb{R}^{d}\right) \quad \mathcal{B}=L^{q}\left(\mathbb{R} ; L^{r}\left(\mathbb{R}^{d}\right)\right) \quad \mathcal{B}^{\prime}=L^{q^{\prime}}\left(\mathbb{R} ; L^{r^{\prime}}\left(\mathbb{R}^{d}\right)\right)
$$

one obtains the desired estimate (11), using the pointwise in time estimate and the Hardy-Littlewood-Sobolev Theorem (see Rap, p. 74 and p. 9]), in the case $q<\infty$. The case $q=\infty$ follows from $L^{2}$ conservation of mass of $S(t)$ which was also proved in Rap, Ch. 5.1].
4. Lastly on shows that the linear map $U: f \rightarrow v$, which is given by the Duhamel integral, is a bounded map from

$$
\begin{array}{rll}
L_{t}^{q^{\prime}} L_{x}^{r^{\prime}} & \text { into } & L_{t}^{q} L_{x}^{r}, \\
L_{t}^{q^{\prime}} L_{x}^{r^{\prime}} & \text { into } L_{t}^{\infty} L_{x}^{2}, \\
L_{t}^{1} L_{x}^{2} & \text { into } L_{t}^{q} L_{x}^{r} .
\end{array}
$$

Applying the Riesz-Thorin interpolation theorem for space-time Lebesgue spaces (see Rap p. 7]) to the target space and to the domain space separately yields the estimate (22).

## References

[Rap] Pierre Raphaël Raphaël Danchin. An introduction to the study of non linear waves. URL: https://www.dpmms.cam.ac.uk/study/III/Introductiontononlinearanalysis/ 2022-2023/cours-camb.pdf.

