TROPICAL COORDINATES ON THE SPACE OF PERSISTENCE BARCODES

A DISSERTATION
SUBMITTED TO THE DEPARTMENT OF MATHEMATICS
AND THE COMMITTEE ON GRADUATE STUDIES
OF STANFORD UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

Sara Kališnik Verovšek

March 2016

I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

(Gunnar Carlsson) Principal Adviser

I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

(Gregory Brumfiel)

I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

(Steven Kerchoff)

Approved for the Stanford University Committee on Graduate Studies

Abstract

In the last two decades applied topologists have developed numerous methods for 'measuring' and building combinatorial representations of the shape of the data. The most famous example of the former is *persistent homology*. This adaptation of classical homology assigns a barcode, i.e. a collection of intervals with endpoints on the real line, to a finite metric space.

Unfortunately, barcodes are not well-adapted for use by practitioners in machine learning tasks. In this dissertation, I identify classes of max-plus polynomials and tropical rational functions that can be used as coordinates on the space of barcodes. All of these are stable with respect to standard distance functions (bottleneck distance, Wasserstein distances) used on the barcode space. I demonstrate how these coordinates can be used by combining persistent homology with SVM to classify numbers from the MNIST dataset.

In order to identify functions on the barcode space, I find generators for the semirings of tropical polynomials, max-plus polynomials and tropical rational functions invariant under the action of the symmetric group. The fundamental theorem of ordinary symmetric polynomials has an equivalent in the tropical and max-plus semirings. There are interesting differences if we consider the tropical polynomial semiring with nr variables that come in n blocks of r variables each and are permuted by the symmetric group S_n. As opposed to the ordinary polynomial case, the semiring of r-symmetric tropical polynomials is not finitely generated, but the semiring of r-symmetric tropical rational functions is.

Acknowledgments

I would like to thank my Ph.D. advisor, Gunnar Carlsson, for his mentorship and for introducing me to the world of applied topology. Every time I got stuck while working on a problem and was on the verge of giving up on it, he encouraged me to keep going and proposed ways of improving my results. Whenever I was anxious, he was quick to respond and offer comfort. But most of all, I am thankful to him for responding to an email I sent him six years ago and giving me a chance to work with him for three months before even applying to Stanford. If it was not for him, I would have been deprived of one of the most invaluable experiences in my life.

Gregory Brumfiel spent numerous hours learning new topics and discussing math and life with me. Many graduate students work primarily with their mentor, but I was lucky enough to be able to learn from two brilliant mathematicians. I would like to thank him, as well as Vin de Silva, for all of their support and mentorship. Funding from a Melvin and Joan Lane Stanford Graduate Fellowship helped me to devote myself to my work.

While at Stanford I was fortunate to be surrounded by many supportive students. I learned a lot from my Ph.D. 'siblings,' Jacek Skryzalin, Pawin Vongmasa, Henry Adams and Michael Lesnick, during our spirited meetings. My friends, the one and only Evita Nestoridi (Tuffi), Peter Hintz (Piddilein), Jeremy Booher, Alexandr Zamorzaev (Sanicka), Francois Greer (Bro') and Iurie Boreico made graduate school year a lot of fun (even in the first year, when we were studying for quals). Special thanks also goes to Davorin Lešnik, who patiently read my dissertation and gave me suggestions on how to improve it.

Finally, I would like to thank my husband Peter, my parents Zlata and Toni, my brother Jure, my grandmother Lojzka and my parents-in-law Tanja and Joe for their unwavering support over the years.

Contents

List of Figures

Introduction

Classical topologists developed homology in order to 'measure' shape. In simplest terms, homology counts the occurrences of patterns, such as the number of connected components, loops and voids. The adaptation of homology to the study of point cloud data sets is called persistent homology [22, 19, 10].

The idea is that the union of discs with radius r centered around points from the data set recovers the underlying shape of the point cloud. We do not know *a priori* how to choose the radius. Persistent homology computes and keeps track of the changes in the homology of the Čech complex of a point cloud over a range of radii parameters r. The output is a barcode, i.e. a collection of intervals. Each interval corresponds to a topological feature which appears at the value of a parameter given by the left hand endpoint of the interval and disappears at the value given by the right hand endpoint.

Barcodes have been useful for understanding the topology and geometry of individual data sets in [9],[11], [2], etc. However, they can also be used in situations where the data points themselves are equipped with geometric structure. For example, databases of chemical compounds and of images have this property. In this situation, by assigning barcodes to the data points, we obtain a database of barcodes. Geometric structures on the collection of barcodes are important for analyzing the database, using, for example, methods from machine learning. One such structure is the bottleneck distance [15], which is a metric imposed on the set of all persistence barcodes, complex vectors [20, 21] and persistence landspaces [6] have also been used. One might also attempt to find a coordinate system on the set of barcodes.

Adcock et al. [1] identified polynomials that can serve this purpose. The disadvantage of using such coordinates is that they are not stable with respect to the standard distance functions (bottleneck distance, Wasserstein distance) used on the barcode space. They interpreted the space of persistence barcodes as embedded in the geometric points of an affine scheme over $\mathbb{R}$. As a result, some important functions, notably max and min functions, were not included. This suggested to us that it would be valuable to carry out a parallel analysis where tropical functions on the barcodes are studied. This is what I attempt to do in my thesis.

Tropical geometry has been developed over the last two decades to understand a wide variety of problems. Tropical polynomial problems are interpretable as linear problems, with inequalities. This makes its methods useful in various kinds of optimization problems. Tropical geometry can also be used to approximate, in an appropriate sense, ordinary algebraic geometric problems, and therefore are useful in their solution. In addition, it solves a number of enumerative geometric problems [26]. Although much work has been done, there is not yet a complete translation of the methods of algebraic geometry to the tropical situation. One of the main objects of study in algebraic geometry, invariant theory, has to our knowledge not been studied in the tropical setting. In my thesis I initiate the translation of invariant theory by studying some special cases. Although some ideas from ordinary invariant theory translate, there are interesting differences.

Tropical algebra is based on the study of the tropical semiring $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$. In this semiring

$$a \oplus b := \min{(a, b)} \quad \text{and} \quad a \odot b := a + b.$$

A min-plus monomial expression is any product of variables $x_1, \ldots, x_n$ that represent elements in the tropical semiring, where repetition is allowed. A tropical monomial expression allows taking quotients in addition to products. Max-plus monomial expressions are defined analogously, this time taking elements from the arctic semiring. Min-plus, max-plus and tropical polynomial expressions are finite linear combinations of appropriate monomial expressions (tropical polynomials, the way they are defined by tropical algebraists, are really Laurent polynomials, because negative exponents are allowed) and they all form semirings. I define the degree of a polynomial expression following the ordinary polynomial case. I also define rational tropical expressions as quotients of tropical polynomial expressions. The set of equivalence classes is a semiring of rational tropical functions.

The passage from expressions to functions is not one-to-one. Because I am primarily interested in studying functions on the barcode space, I identify expressions that define the same functions. The quotient we obtain in the case of min-plus expressions is the semiring of min-plus polynomials. I define the semirings of tropical and max-plus polynomials analogously. Note that this is not the standard 'polynomial semiring' construction well known to algebraists. Of course, whenever we deal with equivalence classes, we might ask ourselves what the canonical representative of a class could be. I define the 'minimal representation' of a polynomial which arises from the graph and show that it exists and is unique. Another related question is: 'Given a polynomial expression, when is a monomial appearing in the expression redundant?' I give a sufficient and necessary condition that involves convex combinations of coefficients appearing in monomials. From here we can define the degree of a polynomial (because we show that all expressions representing it have the same degree).

I call a tropical polynomial in n variables symmetric if it is invariant under the action of S_n that permutes the variables. The fundamental theorem of ordinary symmetric polynomials has an equivalent in min-plus, tropical and max-plus semirings. This is also generalized to rational functions.

Furthermore, I consider the case when the tropical polynomial semiring has nr variables that come in n blocks of r variables each and are permuted by the symmetric group S_n. I call a tropical polynomial in nr variables r-symmetric if it is invariant under the action of S_n that permutes the blocks. I define elementary r-symmetric polynomials and show that they separate orbits. As opposed to the ordinary polynomial case, the semiring of r-symmetric tropical polynomials is not finitely generated, but the elementary r-symmetric polynomials do generate r-symmetric rational functions.

Once all the algebra is in place, I move to the question of max-plus and tropical rational functions on the barcode space. Aside from being symmetric, the functions should take the same value if we adjoin zero length intervals. I first identify appropriate functions on spaces of barcodes with a fixed number of bars and then assemble them to get functions on the entire barcode space. Assembling them turns out to be somewhat problematic, because the additive unit in the tropical semiring is ∞ and in the arctic semiring $-\infty$. To fix this I represent each interval by a pair (x, d), where x is the left endpoint and d the length of the interval. The length is always nonnegative and I additionally assume that $x \geq 0$. This allows me to work within $([0, \infty), \max, +)$, where 0 is the unit with respect to max. When assigning simplicial complexes to data sets, the parameter is the radius, which is nonnegative, so this is not severe a restriction.

The only max-plus polynomials that satisfy these conditions are linear combinations of lengths. I prove that these are stable, but the drawback is that there are not enough of them to separate the barcodes. For this reason I turn my attention to tropical rational functions. There seems to be little hope of trying to figure out what the relationship between coefficients should be for a function to stay the same after adjoining a zero length vector. Therefore I focused primarily on finding functions that would work even if the entire ring of such functions is hard to describe. I identified a subsemiring of such functions, proved that they separate points and that they are stable.

In the last part of the thesis I tested my coordinate functions on the MNIST dataset of handwritten digits and compared the results with those obtained by Adcock et. al. [1]. The results were slightly better, but not by much. It turned out that functions that worked best are linear combinations of lengths, which fits in with the intuition about barcodes, i.e.

the longer the length, the greater the importance of a feature.

Overview:

In Chapter 1 I discuss different ways of assigning simplical complexes to point cloud data and explain how persistent homology works. I rely on papers by Gunnar Carlsson [8, 7], Robert Ghrist [23] and Frédéric Chazal et al. [12].

In Chapter 2 I mention different methods for assigning vectors to barcodes, with an emphasis on work by Adcock et. al. [1]. I also formally define the barcode space and different metrics on it [1, 15]. I demostrate with an example that these coordinates are not stable.

Chapter 3 focuses on the necessary definitions from tropical algebra. Aside from the definitions of the tropical and arctic semirings themselves and tropical polynomials, everything is my work.

In Chapter 4 I introduce symmetric min-plus, tropical and max-plus polynomials and study their properties. Among other things I prove that the fundamental theorem of ordinary symmetric polynomials has an equivalent in min-plus, tropical and max-plus semirings. This is also generalized to rational functions.

Chapter 5 continues discussion from Chapter 5, but with r-symmetric min-plus, tropical and max-plus polynomials. The semiring of r-symmetric tropical polynomials is not finitely generated, but quite surprisingly the elementary r-symmetric polynomials do generate r-symmetric rational functions.

Chapter 6 is devoted to functions on the barcode space. We identify appropriate max-plus polynomials and prove that they are stable. Because they do not separate points, we take into account tropical rational functions as well. We show that no finite set of such functions separates barcodes with a fixed number of bars (and therefore the whole barcode space). I find a countably infinite set of functions that separate the barcodes and are stable.

Chapter 7 presents the results of using coordinate functions identified in Chapter 6 on the MNIST dataset.

Chapter 1

Persistent Homology

The rapid development of information technology in the last few decades has produced data at an unprecedented rate. For example, businesses collect large amounts of information on current and potential customers, biologists keep track of how predator/prey populations change over time, medical doctors gather gene expression data from cancer patients, etc. Even though specialists are interested in answers to concrete questions, like how to divide customers into groups for marketing activities or how to confirm a circular model for predator-prey populations, often what they are actually trying to understand is the shape of the data.

Topology is the branch of mathematics which deals with shape. It does that in two distinct ways–by building compressed combinatorial representations of shapes (triangulation) and by 'measuring' aspects of shape (homotopy groups, homology groups). In simplest terms, homology counts the occurrences of patterns, such as the number of connected components, loops and voids. The adaptation of this technique to the study of point cloud data is called persistent homology. We always take homology groups with coefficients in a field $\mathbf{k}$.

The concept emerged independently in the work of Frosini, Ferri, and collaborators in Bologna [22], Italy, of Robins at Boulder, Colorado [27], and of Edelsbrunner, Letscher and

Zomorodian at Duke, North Carolina [19]. Zomorodian and Carlsson then gave this idea a firm theoretical footing [10].

There are a number of excellent introductory papers written on the topic. To write this section, I drew inspiration from Topology and Data and Topological Pattern Recognition for Point Cloud Data by Gunnar Carlsson [8, 7], Barcodes: The persistent topology of data by Robert Ghrist [23] and the Structure and Stability of Persistence Modules by Frédéric Chazal et al. [12].

A Software package for computing persistent homology and the accompanying JPlex Matlab Tutorial are available at http://appliedtopology.github.io/javaplex/ [29].

1.1 Clouds of Data

Large datasets are often given in the form of very long vectors or arrays (for instance, DNA sequences or pixel arrays) that reside in a space of potentially high dimension. Following the common usage, we therefore call any finite collection of points in $\mathbb{R}^n$ a *point cloud*.

Consider the example in Figure 1.1. If asked about the shape of this data set, most people would say that it is a circle or that the dots lie on a circle. For point clouds residing in a low-dimensional ambient space, there are numerous approaches for inferring features based on planar projections. But what if a point cloud lies in a very high-dimensional space? How would we analyze it? A statistician might try to fit a Gaussian, an exponential or some other standard descriptor to this collection of points. However, this is an example where most standard methods of data analysis fail.

With continuous objects, like circles, mathematicians have no trouble determining the presence of loops and higher dimensional voids. Homology is a standard tool. We can compute homology groups by hand using a variety of techniques (exact sequence of a pair, Mayer-Vietoris sequence, excision theorem), but direct computation from the definition is not feasible for general spaces. For this reason we work with spaces equipped with

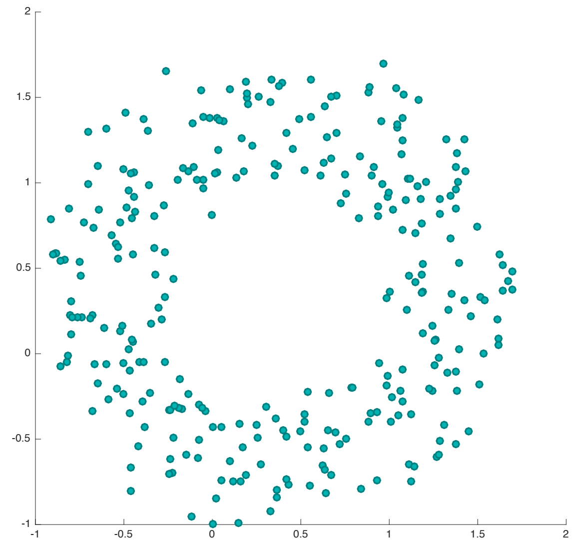

Figure 1.1: A collection of points sampled from a noisy circle.

particularly nice structures, namely simplicial complexes.

1.2 From Point Clouds to Simplicial Complexes

Formally, an *abstract simplicial complex* K on a finite set of points V is a family of non-empty subsets Σ of V such that $\sigma \in \Sigma$ and $\tau \subseteq \sigma$ implies that $\tau \in \Sigma$. We call V the set of vertices. A $\tau \in K$ of size $k+1$ is a *k-simplex*. A 0-simplex is a point or a vertex, a 1-simplex is an edge, a 2-simplex is a triangle and a 3-simplex is a tetrahedron (see Figure 1.2 for examples). Intuitively, a simplicial complex structure on a space is an expression of

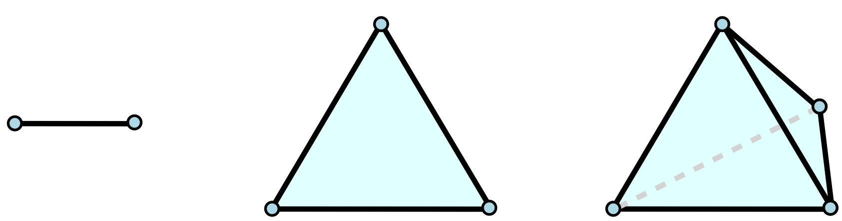

Figure 1.2: Examples of simplicial complexes.

the space as a union of points, intervals, triangles, and higher dimensional analogues glued together along faces.

Given a topological space X, there are a number of simplicial complexes which can be constructed from X. One such is the nerve complex, which is extremely useful in homotopy theory.

Definition 1.1. *Let X be a topological space and let $U = \{U_\alpha\}_{\alpha \in A}$ be any covering of X. The nerve of U, denoted by $\mathscr{N}U$, is the abstract simplicial complex with vertex set A, and where a family $\{\alpha_0, \ldots, \alpha_k\}$ spans a k-simplex if and only if $U_{\alpha_0} \cap \ldots \cap U_{\alpha_k} \neq \emptyset$.*

We use the nerve because of the following theorem, which provides criteria that guarantee that $\mathscr{N}U$ is homotopy equivalent to the underlying space X [4].

Theorem 1.2. *Suppose that X and U are as in Definition 1.1, and suppose that the covering consists of open sets and is numerable. Suppose further that for all $\emptyset \neq T \subseteq A$, we have that $\bigcap_{t \in T} U_t$ is either contractible or empty. Then $\mathscr{N}U$ is homotopy equivalent to X.*

With this theorem we can confidently approximate unions of certain open sets with simplicial complexes. What remains is to choose a method for generating 'good' coverings. If X is a metric space, we can take a covering consisting of balls of radius ϵ: $\{B_\epsilon(x)\}_{x \in X}$. The nerve associated to this covering is called the *Čech complex* and denoted by $\check{\mathrm{C}}(X, \epsilon)$.

Theorem 1.3. *Let M be a compact Riemannian manifold. Then there is a positive number e so that $\check{\mathrm{C}}(M, \epsilon)$ is homotopy equivalent to M whenever $\epsilon \leq e$. Moreover, for every $\epsilon \leq e$, there is a finite subset $V \subseteq M$ so that the subcomplex of $\check{\mathrm{C}}(V, \epsilon)$ is also homotopy equivalent to M.*

This theorem shows that Čech complex is a topologically faithful simplicial model when X is a Riemannian manifold as long as the radius of balls we choose for covering is small enough.

Given a point cloud, we take a union of balls with radius ϵ centered at the points of the point cloud. If the centers lie on a submanifold M of $\mathbb{R}^n$, and the point cloud is sufficiently dense in M, then this complex is the Čech complex attached to a covering of M by balls. If we additionally assume that ϵ is small enough, then by Theorem 1.3 the complex computes the homology of M correctly. For this reason we use the Čech complex to approximate the homology of M. The only drawback is that the Čech complex is hard to compute as it relies on precise distances between the vertices. Since it also requires the storage of the entire boundary operator, the construction is computationally expensive.

Vietoris–Rips complex, often shortened to Rips complex, is a computationally less expensive alternative. Given a metric space X equipped with distance function d, the Rips complex for X with parameter value ϵ has a vertex set X and $\{x_0, x_1, \ldots, x_k\}$ spans a k-simplex if and only if $d(x_i, x_j) \leq \epsilon$ for all $0 \leq i, j \leq k$. We denote it by $VR(X, \epsilon)$. The Rips complex is a special combinatorial simplicial complex known as a flag complex and it is maximal among all simplicial complexes with the given 1-skeleton. Because the combinatorics of the 1-skeleton completely determines the complex, the complex can be stored as a graph. This is what makes it less expensive than the Čech complex.

Choosing Rips complex over Čech is not without cost. The penalty for this simplicity is that it is not immediately clear what is encoded in the homotopy type of the complex. Example in Figure 1.3 shows that the Rips complex does not necessarily give the homology of the union of balls.

If X is a finite subset of $\mathbb{R}^n$, with the standard metric, then

$$\check{\mathrm{C}}(X, \epsilon) \subseteq VR(X, 2\epsilon) \subseteq \check{\mathrm{C}}(X, 2\epsilon).$$

Intuitively, the Rips complex of X is nested between two Čech complexes for X. For this reason we can take it to be a fairly good approximation.

For larger datasets, if we include every data point as a vertex, as in the Rips construction,

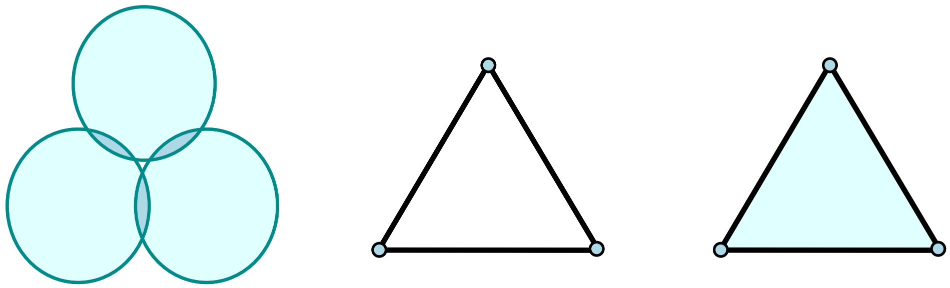

Figure 1.3: Observe a metric space X with three pairwise equidistant vertices. The Čech and Rips complexes of the covering on the left each have three edges. The Čech complex does not have any 2-simplices since the sets do not have a common intersection. The Rips complex, however, includes the triangle since all points have pairwise distance less than twice the radius of the balls. The Rips complex does not give the homology of the union of balls since $\mathrm{H}_1(\check{\mathrm{C}}(X,\mathrm{r})) = \mathbf{k}$, while $\mathrm{H}_1(VR(X,\mathrm{r})) = 0$.

we will again have to deal with too many simplices for efficient computation. The witness complex and the lazy witness complex address this problem. In both cases we select a subset of the metric space X, called *landmark points*, and only these points will be vertices in the complex we build. We denote the set of landmark points by L.

Let $\epsilon > 0$. For every $x \in X$ let m_x be the distance from x to the set L. The *strong witness complex* attached to this data is the complex $W_s(X, L, \epsilon)$ whose vertex set is L, and a collection $\{l_0, \ldots, l_k\}$ spans a k-simplex if and only if there is such a point $x \in X$, the *witness*, that $d(x, l_i) \leq m_x + \epsilon$ for all i. In analogy to the Rips complex, we also consider a complex in which the 1-simplices are identical to those of $W^s(X, L, \epsilon)$ but where $\{l_0, \ldots, l_k\}$ spans a k-simplex if and only if all the pairs (l_i, l_j) are 1-simplices. This complex we denote by $W^s_{VR}(X, L, \epsilon)$ and call the *lazy witness complex*. The adjective lazy refers to the fact that the lazy witness complex is a flag complex.

We have not exhausted all the possibilities. One can define weak witness complex, lazy weak witness complex, alpha complexes, etc. [18, 8].

1.3 Persistence

In the previous section I describe ways of representing a point cloud by a simplicial complex. In all cases the construction depends on a certain parameter ϵ. For small ϵ, the complex is a discrete set; for large ϵ, it is a high-dimensional simplex. What is the optimal choice for ϵ?

Consider the point cloud data X and Rips complexes as illustrated in Figure 1.4. We observe Betti numbers in dimensions 0 and 1. When ϵ is small, there are no loops, so $\beta_1 = 0$, while β_0 is the cardinality of X. For slightly bigger ϵ, $\beta_1 = 2$ and $\beta_0 = |X| - 4$. As we increase ϵ even more, the two loops get filled in and another bigger forms, yielding $\beta_1 = 1$ and $\beta_0 = 1$. Finally, for large ϵ, $\beta_1 = 0$ and $\beta_0 = 1$.

Given a variety of complexes attached to different parameters, it seems that it is a mistake to ask which value of ϵ is optimal and rather than just counting the loops and connected components, we need a means of declaring which holes are essential and which can be safely ignored. In the case of X in our example, we wish to say that the two smaller loops are due to noise and the bigger one represents of true feature of the point cloud. However, homology allows no such explanation: a hole is a hole no matter how fragile or how fine.

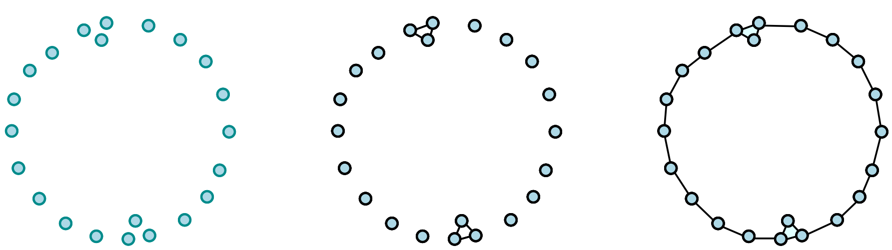

Figure 1.4: A sampling X from a circle with noise. If we choose the parameter ϵ to be very small, the associated simplicial complex has a set of vertices that consists of points from X. There are no higher dimensional simplices. As we increase ϵ two loops appear. As we increase it more, these loops get filled in by 2-dimensional simplices and a new cycle appears. For big enough ϵ the associated Rips complex is a single high dimensional simplex.

Homology does have an extra property that we have not yet taken into account, namely functoriality. First observe that we have an inclusion $\check{\mathrm{C}}(X,\epsilon) \hookrightarrow \check{\mathrm{C}}(X,\epsilon')$ for every $\epsilon \leq \epsilon'$. In Figure 1.4 the complex in the left includes into the one on the right. The two small cycles in the complex on the left vanish in the one on the right, since they are filled in by 2-simplices. In the complex on the right we have a larger cycle not present in the complex on the left. As we increase the parameter, this larger cycle persists for quite a while until it is finally filled in. We declare the loops in the complex on the left to be noise, because they do not persist over a significant parameter range. The larger loop on the other hand, captures a real feature of the data, because it is long-lived.

In the following section, we formalize this observation and present a tool that captures the desired summary of the behavior of homology under all choices of values for the scale parameter ϵ.

1.4 Persistent Homology

Definition 1.4. *Let $\mathbf{k}$ be a field and $\mathscr{P}$ be a partially ordered set. A persistence $\mathscr{P}$-vector space over $\mathbf{k}$ is an indexed family of $\mathbf{k}$-vector spaces $\{V_r\}_{r\in\mathscr{P}}$ and a doubly-indexed family of linear maps*

$$(L_V(r,r')\colon V_r \to V_{r'} \,|\, r \leq r'),$$

which satisfy the composition law

$$L_V(r',r'') \circ L_V(r,r') = L_V(r,r'')$$

for all $r \leq r' \leq r''$. For all r $L_V(r,r)$ is the identity map on V_r.

A linear transformation F of $\mathscr{P}$-persistence vector spaces over $\mathbf{k}$ from $\{V_r\}$ to $\{W_r\}$ is such a family of linear transformations $f_r\colon V_r \to W_r$ that for all $r \leq r'$, the diagrams

$$\begin{tikzcd} V_r \arrow[r, "L_V(r,r')"] \arrow[d, "f_r"'] & V_{r'} \arrow[d, "f_{r'}"] \\ W_r \arrow[r, "L_W(r,r')"'] & W_{r'} \end{tikzcd}$$

commute in the sense that

$$f_{r'} \circ L_V(r, r') = L_W(r, r') \circ f_r.$$

For a point cloud X and a homology functor H with coefficients in a field $\mathbf{k}$, the family $\{\mathrm{H}(\check{\mathrm{C}}(X, r))\}_{r\in[0,\infty)}$ is an $\mathbb{R}$-persistence vector space. The same holds if we use Rips or witness constructions.

Our goal is to find invariants whose description is finite in size and that do not depend on the underlying field of computation. We call such invariants discrete. There is no classification theorem for $\mathbb{R}$-persistence vector spaces, but there is one for $\mathbb{N}$-persistence vector spaces, which is based on the Structure Theorem for PID's.

To any $\mathbb{N}$-persistence vector spaces $\{V_n\}$ we assign a graded module $\theta(\{V_n\})$ over the graded polynomial ring $\mathbf{k}[x]$, where x is assigned degree 1, as follows:

$$\theta(\{V_n\}) = \bigoplus_{n\geq 0} V_n,$$

where the n-th graded part is the vector space V_n. The action of the polynomial generator x is given by

$$x \cdot \{v_n\} = \{v'_n\}, \text{ where } v'_n = L_v(n-1, n)v_{n-1} \text{ and } v'_0 = 0.$$

The map θ is a functor from the category of $\mathbb{N}$-persistent vector spaces to the category of graded $\mathbf{k}[x]$-modules. In fact it is an equivalence of categories, since an inverse functor can be given by $V_* \to \{V_n\}$, where the morphisms $L_V(m,n)$ are given by multiplication by x^{n-m}.

Theorem 1.5. *Let V^* denote any finitely generated non-negatively graded $\mathbf{k}[x]$-module. Then there are integers $\{i_1,\ldots,i_m\}$, $\{j_1,\ldots,j_n\}$, $\{l_1,\ldots,l_n\}$, and an isomorphism*

$$V_* \simeq \bigoplus_{s=1}^{m} x^{i_s} \cdot \mathbf{k}[x] \oplus \bigoplus_{t=1}^{n} x^{j_t} \cdot (\mathbf{k}[x]/(x^{l_t}))$$

The decomposition is unique up to permutation of factors.

There is a restriction in Theorem 1.5. Namely, the $\mathbf{k}[x]$-modules must be finitely generated. The following proposition shows which $\mathbb{N}$-persistent $\mathbf{k}$-vector spaces correspond under θ to finitely generated non-negatively generated $\mathbf{k}[x]$-modules.

Proposition 1.6. *Let $\{V_n\}$ be a $\mathbb{N}$-persistent $\mathbf{k}$-vector space. Then $\theta(\{V_n\})$ is a finitely generated $\mathbf{k}[x]$-module if and only if every vector space V_n is finite dimensional and*

$$L_V(n,n+1)\colon V_n \to V_{n+1}$$

is an isomorphism for sufficiently large n. We call such $\{V_n\}$ a tame *$\mathbb{N}$-persistent $\mathbf{k}$-vector space.*

The classification theorem has a natural interpretation. The free portions are in bijective correspondence with those homology generators which come into existence at parameter i_s and which persist for all future parameter values. The torsion elements correspond to those homology generators which appear at parameter j_t and disappear at parameter $j_t + l_t$.

This implies that the isomorphism classes of tame persistence vector spaces arising from point cloud data are in one to one correspondence with finite subsets (with multiplicity) of intervals. Such sets are represented visually in two ways, one as families of intervals on the non-negative real lines, and the other as a collection of points in the half-plane

$$\mathscr{H} = \{(x, y) \mid -\infty < x < y \leq \infty\}.$$

The first representation is called a *barcode*, and the second a *persistence diagram*. We use both expressions interchangeably.

The methodology we use to study the homology of point clouds is as follows:

- Choose any partial order preserving map $f \colon \mathbb{N} \to \mathbb{R}$.

  One way of choosing this map would be as follows. Given any finite point cloud X there are only finitely many real values that the distance function takes on X. This implies that there are only finitely many real values at which there are transitions in the complex no matter which construction we use. Let these transition values be $\{t_0, t_1, \ldots, t_N\}$, $t_0 \leq t_1 \leq \ldots \leq t_N$, and define $f \colon \mathbb{N} \to \mathbb{R}$ as follows

  $$f(n) = t_n \text{ for } n \leq N, \text{ and } f(n) = t_N \text{ for } n \geq N.$$

- Construct a family of simplicial complexes $\{C(X, f(n))\}$ for parameter values in $\{f(n)\}$. We can use any method, Čech, Rips or lazy complex.

- Apply homology functor H with coefficients in a field $\mathbf{k}$. This yields a tame $\mathbb{N}$-persistence vector space $\{\mathrm{H}C(X, f(n))\}$. Tameness follows from the finiteness hypothesis on X and the nature of the constructions.

- Compute the barcodes associated to $\{\mathrm{H}C(X, f(n))\}$.

In interpreting the output, a long interval in the barcode indicates the presence of an essential homology class, which persists over a long range of parameter values. Short intervals indicate cycles which are short-lived. They correspond either to noise or inadequate sampling. Of course, what is short, depends on a data set, and we do not always want to ignore short bars. Sometimes the whole multiscale version of the space is of interest.

Example 1.7. *Observe once again a point cloud sampled from a noisy circle. Figure 1.5 depicts persistent homology barcodes in dimensions 0 and 1. In the 0-dimensional barcode a lot of bars persist over small values of the parameter. As we increase the parameter, they merge and only one remains that persists over a significant range of parameters, implying that the underlying shape we are sampling from has 1 connected component. Similarly, we see many cycles present for small values of parameter, but only one that persists over a considerable range of parameter values.*

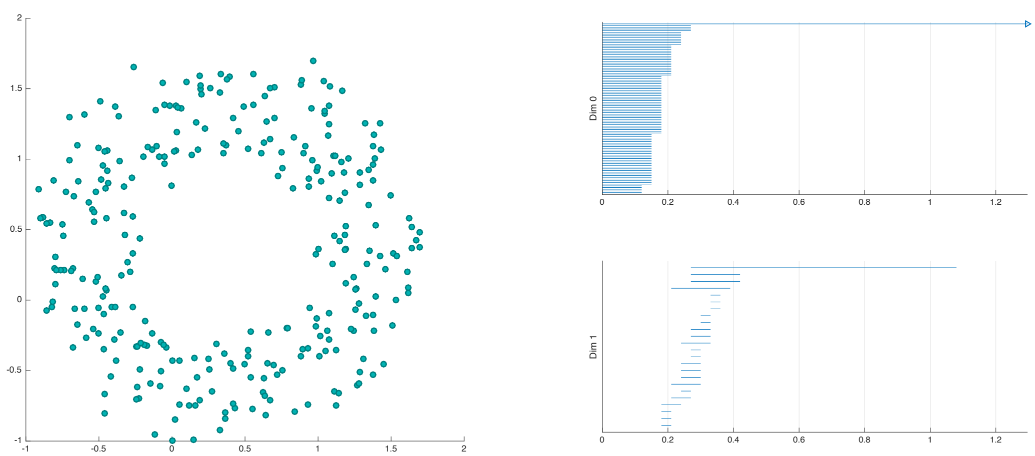

Figure 1.5: A collection of points sampled from a circle.

In some cases the data we get is already a finite simplicial complex K equipped with a real-valued function f on the vertex set of K. For example, Adcock et al. [3] analyzed a dataset of liver lesions by building simplicial complexes on each image and then used grayscale values of pixels as a filter.

We construct a sequence of simplicial subcomplexes K_i of K by including any simplex

$\sigma \in K$ with the property that for every vertex $v \in \sigma$, $f(v) \leq i$:

$$K_i = \{\sigma \in K \mid \forall v \in \sigma, f(v) \leq i\}.$$

Intuitively, $f(v)$ represents the vertex at which v enters the filtration and $\max_{v\in\sigma} f(v)$ determines the point at which a simplex $\sigma \in K$ enters the filtration. Now we have the filtered complex

$$K_{f_1} \subseteq K_{f_2} \subseteq \ldots \subseteq K_{f_m} = K,$$

where f_1, denotes the minimum value obtained by f on the vertex se of K, f_2 the second smallest value, etc., and finally, and f_m is the maximum value obtained by f. We call this the *sublevelset filtration* of (X, f).

To this sequence of simplicial complexes and inclusions between them, we apply the homology functor H in a field **k** and obtain a tame persistence vector space for which we can compute a barcode.

Chapter 2

Coordinatizing the Barcode Space using Polynomials

In the previous chapter we introduced persistent homology as a tool that identifies structure in point clouds. The output of this invariant is a collection of intervals rather than a vector, which makes persistent homology hard to combine with standard methods in machine learning, which accept vectors as an input.

Many researchers have tried to remedy this by attaching numerical quantities to barcodes. For example, di Fabio, Ferri [20] and Landi [21] make use of complex vectors, Bubenik [6] defines *persistence landspaces*, etc.

At Stanford Adcock et al. [1] identified an algebra of polynomials on the barcode space that can be used as coordinates. In this section we formally define the barcode space and the metrics commonly used and show what the disadvantages of using polynomial coordinates are. These shortcomings prompt us to search for other types of coordinates on the barcode space.

2.1 The Barcode Space

Before delving into a discussion about functions, we must formally define what we mean by a *barcode space*. We can represent a barcode with exactly n intervals by a vector $(x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$, where x_i denotes the left endpoint of the i-th interval and y_i the right endpoint. Since the ordering of the intervals does not matter, we take the orbit space of the action of the symmetric group on n letters on the product $(\mathbb{R} \times \mathbb{R})^n$ given by permuting the coordinates. We denote this set by B_n.

To ignore intervals of length 0, we define an equivalence relation on $\coprod_n B_n$, which we denote by $\sim$. It is generated by equivalences of the form

$$((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)) \sim ((x_1, y_1), (x_2, y_2), \ldots, (x_{n-1}, y_{n-1})),$$

whenever $x_n = y_n$. The barcode set B is the quotient

$$\coprod_n B_n/\sim.$$

2.2 Metrics on the Barcode Space

To define distance between two barcodes, we need to specify the distance between any pair of intervals, as well as the distance between any interval and the set of zero length intervals $\Delta = \{(x, x) \mid -\infty < x < \infty\}$. Set

$$\mathrm{d}_\infty((x_1, y_1), (x_2, y_2)) = \max(|x_1 - x_2|, |y_1 - y_2|).$$

The distance between an interval and the set Δ is

$$\mathrm{d}_\infty((x, y), \Delta) = \frac{y - x}{2}.$$

Let $\mathscr{B}_1 = \{I_\alpha\}_{\alpha\in A}$ and $\mathscr{B}_2 = \{J_\beta\}_{\beta\in B}$ be barcodes. For finite sets A and B, and any bijection θ from a subset $A' \subseteq A$ to $B' \subseteq B$, the penalty of θ, $P_\infty(\theta)$, is

$$P_\infty(\theta) = \max(\max_{a\in A'}(\mathrm{d}_\infty(I_a, J_{\theta(a)})), \max_{a\in A\setminus A'} \mathrm{d}_\infty(I_a, \Delta), \max_{b\in B\setminus B'} \mathrm{d}_\infty(I_b, \Delta)).$$

The *bottleneck distance* [15] is

$$\mathrm{d}_\infty(\mathscr{B}_1, \mathscr{B}_2) = \min_\theta P_\infty(\theta),$$

where the minimum is over all possible bijections from subsets of A to subsets of B.

There are other metrics also commonly used for barcode spaces. Setting the penalty for θ as above to

$$P_p(\theta) = \sum_{a\in A'}(\mathrm{d}_\infty(I_a, J_{\theta(a)})^p + \sum_{a\in A\setminus A'} \mathrm{d}_\infty(I_a, \Delta)^p + \sum_{b\in B\setminus B'} \mathrm{d}_\infty(I_b, \Delta)^p,$$

yields the pth-Wasserstein distance ($p \geq 1$) between $\mathscr{B}_1$, $\mathscr{B}_2$:

$$\mathrm{d}_p(\mathscr{B}_1, \mathscr{B}_2) = (\min_\theta P_p(\theta))^{\frac{1}{p}}.$$

Example 2.1. *Consider two barcodes in dimensions 0 and 1 depicted in Figure 2.1. First*

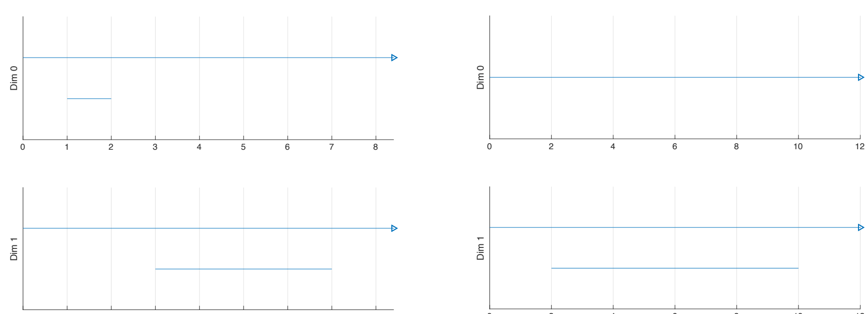

Figure 2.1: Barcode $\mathscr{B}_1$ in dimensions 0 and 1 on the left and barcode $\mathscr{B}_2$ on the right.

we observe what happens in dimension 0. Barcode $\mathscr{B}_1$ contains two bars, while $\mathscr{B}_2$ only one. We align the longest bars and match $[1,2]$ of length one with a bar whose length is 0. The penalty for this partial bijection is $\frac{1}{2}$. If we do not align the infinite bars, the penalty increases. So the Bottleneck distance in dimension 0 between the $\mathscr{B}_1$ and $\mathscr{B}_2$ is $\frac{1}{2}$. In dimension 1 $\mathscr{B}_1$ and $\mathscr{B}_2$ the optimal bijection aligns the two infinite length bars and the remaining two with the penalty of 3.

We are interested in finding 'stable' functions on the barcode space, ie. functions Lipschitz with respect to distances on the barcode space.

2.3 Functions on the Barcode Space

We can assemble all B_n into a directed system

$$B_1 \xrightarrow{i_1} B_2 \xrightarrow{i_2} B_3 \xrightarrow{i_3} B_4 \xrightarrow{i_4} \cdots$$

where the maps $i_n \colon B_n \to B_{n+1}$ are given by

$$i_n(((x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n))) = (((x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n),(0,0))).$$

The direct limit of this system is B_∞. There is a natural isomorphism $B_\infty \to B$. We first observe functions on all of B_∞ and then keep the ones that respect the quotient B. Each function F on B_∞ can be identified with an infinite vector $(f_1, f_2, f_3, \ldots)$ of functions $f_n \colon B_n \to \mathbb{R}$ satisfying the compatibility condition

$$f_{n+1} \circ i_n = f_n.$$

The set of all such vectors of functions forms a semiring $\mathscr{R}$ under coordinatewise addition and multiplication. We are interested in a subsemiring of $\mathscr{R}$, $\mathscr{R}'$, for which intervals of length zero are considered equal, i.e. where

$$F(((x_1, y_1), \ldots, (x_{n-1}, y_{n-1}), (x, x))) = F(((x_1, y_1), \ldots, (x_{n-1}, y_{n-1}), (x', x')))$$

for all x, $x' \in \mathbb{R}$ and a function F on B_∞. The reason for this is that small perturbations to the input data to the persistence algorithms are reflected in small perturbations in lengths of intervals and appearance of short intervals. We do not wish the values of coordinate functions evaluated on these barcodes to be far away.

The set of these functions is too large to deal with effectively, so Aaron Adcock et al. [1] focus on a subset of polynomials.

2.4 Polynomial Coordinates

To identify polynomials on the barcode space, we need some facts about multisymmetric polynomials that appear in Dalbec's Multisymmetric functions [17] . Let

$$\Lambda_{n,r} = \mathbb{R}[x_{1,1}, x_{1,2}, \ldots, x_{n,r}]^{S_n}$$

denote the ring of S_n invariants, where the symmetric group permutes blocks of r-numbers each. Let

$$i_n^m \colon \mathbb{R}[x_{1,1}, x_{1,2}, \ldots, x_{n,r}] \to \mathbb{R}[x_{1,1}, x_{1,2}, \ldots, x_{m,r}], m \leq n$$

be such that $x_{i,j} = 0$ if $i > m$ and $1 \leq j \leq r$. The map i_n^m is S_m-equivariant. Here $S_m \subseteq S_n$ is the subgroup of permutations of the first m elements of the set $\{1, \ldots, n\}$. We have the composites

$$\Lambda_{n,r} = \mathbb{R}[x_{1,1}, x_{1,2}, \ldots, x_{n,r}]^{S_n} \hookrightarrow \mathbb{R}[x_{1,1}, x_{1,2}, \ldots, x_{n,r}]^{S_m} \xrightarrow{i_n^m} \mathbb{R}[x_{1,1}, x_{1,2}, \ldots, x_{m,r}]^{S_m} = \Lambda_{m,r}$$

which we denote by $\rho_{n,m}$, and therefore the inverse system

$$\dots \xrightarrow{\rho_{n+1,n}} \Lambda_{n,r} \xrightarrow{\rho_{n,n-1}} \Lambda_{n-1,r} \xrightarrow{\rho_{n-1,n-2}} \dots \xrightarrow{\rho_{2,1}} \Lambda_{1,r}.$$

The graded inverse limit Λ_r is known as the ring of *r-multisymmetric functions.* Its grading is induced by the grading on $\mathbb{R}[x_{1,1}, x_{1,2}, \dots, x_{n,r}]$.

To easier describe $\Lambda_{n,r}$, we take a look at a few sets of generators. Given a monomial $\mathbf{m}$ in $\mathbb{R}[x_{1,1}, x_{1,2}, \dots, x_{n,r}]$, we may construct its symmetrization $\operatorname{Sym}\mathbf{m}$ by summing over its orbit under the group action. It turns out that as an algebra $\Lambda_{n,r}$ is generated by the symmetrizations of monomials involving $\{x_{1,1}, x_{1,2}, \dots, x_{1,r}\}$. They are given by the formulas

$$p_{\mathbf{a}} = \mathbf{m}_{\mathbf{a}} = \sum_i x_{i,1}^{a_1} \cdots x_{i,r}^{a_r},$$

where $(a_1, \dots, a_r) \in \mathbb{N}^r$ and are called the *multisymmetric power sums.* While there are relations among the power sums in finitely many variables, they freely generate the inverse limit Λ_r, making it a polynomial algebra.

In the case of barcodes r equals 2. We write $x_{i,1} = x_i + y_i$ and $x_{i,2} = x_i - y_i$, where x_i denotes the left endpoint of the i-th interval and y_i the right endpoint. The subalgebra $\mathscr{D}$ of polynomials on barcodes can be described as follows [1] .

Theorem 2.2. *As a subalgebra of Λ_2, $\mathscr{D}$ is freely generated by the set of elements of the form $p_{a,b}$ where $b \geq 1$.*

2.5 Lack of Stability

For coordinate functions to be useful in applications, stability is the key property. There are stability theorems for barcodes arising from filtered complexes [13, 14] and we wish to have similar theorems for the coordinate functions that we are using. This would guarantee that if two barcodes $\mathscr{B}_1$ and $\mathscr{B}_2$ are close in the bottleneck distance, the coordinates $F(\mathscr{B}_1)$

and $F(\mathscr{B}_2)$ are close, ie.

$$|F(\mathscr{B}_1) - F(\mathscr{B}_2)| \leq C\mathrm{d}_\infty(\mathscr{B}_1, \mathscr{B}_2)$$

for some constant C.

This unfortunately does not happen for $p_{a,b}$ identified in Theorem 2.2. Consider a sequence of barcodes given by $\mathscr{B}_n = \{(1,2),(3,4),\ldots,(2n+1,2n+2)\}$ for $n \geq 0$. The first few barcodes of the sequence are depicted in Figure 2.2. The botttleneck distance between

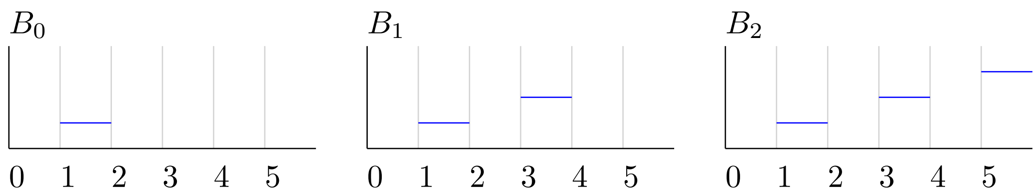

Figure 2.2: Barcodes $\mathscr{B}_0$, $\mathscr{B}_1$ and $\mathscr{B}_2$.

$\mathscr{B}_n$ and $\mathscr{B}_0$ is

$$\mathrm{d}_\infty(\mathscr{B}_0, \mathscr{B}_n) = \frac{1}{2}.$$

for all $n \geq 1$. We evaluate $p_{a,b}$ for $b \geq 1$ on $\mathscr{B}_n$

$$p_{a,b}(\mathscr{B}_n) = \sum_{i=0}^{n}(4i+3)^a \cdot 1^b = \sum_{i=0}^{n}(4i+3)^a$$

and compute

$$|p_{a,b}(\mathscr{B}_n) - p_{a,b}(\mathscr{B}_0)| = \sum_{i=1}^{n}(4i+3)^a.$$

As n tends to infinity, $|p_{a,b}(\mathscr{B}_n) - p_{a,b}(\mathscr{B}_0)|$ tends to infinity and therefore cannot be bounded by a constant.

We can also come up with sequences of barcodes that prove that not all $p_{a,b}$ are stable with respect to the p-th Wasserstein distance.

For $\mathscr{B}_n = \{(1,2), (\frac{1}{\sqrt[a+b]{2}}, \frac{2}{\sqrt[a+b]{2}}), \ldots, (\frac{1}{\sqrt[a+b]{n}}, \frac{2}{\sqrt[a+b]{n}})\}$,

$$\mathrm{d}_p(\mathscr{B}_1, \mathscr{B}_n)^p = \sum_{i=2}^{n} (\frac{1}{2\sqrt[a+b]{i}})^p = \frac{1}{2^p} \sum_{i=2}^{n} \frac{1}{\sqrt[a+b]{i^p}}$$

and

$$|p_{a,b}(\mathscr{B}_n) - p_{a,b}(\mathscr{B}_0)| = \sum_{i=2}^{n} (\frac{3}{\sqrt[a+b]{i}})^a \cdot (\frac{1}{\sqrt[a+b]{i}})^b = 3^a \sum_{i=2}^{n} \frac{1}{i}.$$

If $p > a+b$, $\mathrm{d}_p(\mathscr{B}_1, \mathscr{B}_n)$ converges, whereas $|p_{a,b}(\mathscr{B}_n) - p_{a,b}(\mathscr{B}_0)|$ is divergent.

For $\mathscr{B}_n = \{(1,2), (\frac{2^{(2(a+b))^2}}{\sqrt[p^2]{2^{2(a+b)}}}, \frac{2^{(2(a+b))^2}+1}{\sqrt[p^2]{2^{2(a+b)}}}), \ldots, (\frac{n^{(2(a+b))^2}}{\sqrt[p^2]{n^{2(a+b)}}}, \frac{n^{(2(a+b))^2}+1}{\sqrt[p^2]{n^{2(a+b)}}})\}$,

$$\mathrm{d}_p(\mathscr{B}_1, \mathscr{B}_n)^p = \sum_{i=2}^{n} (\frac{1}{2\sqrt[p^2]{i^{2(a+b)}}})^p = \frac{1}{2^p} \sum_{i=2}^{n} \frac{1}{\sqrt[p]{i^{2(a+b)}}}$$

and

$$|p_{a,b}(\mathscr{B}_n) - p_{a,b}(\mathscr{B}_0)| = \sum_{i=2}^{n} (\frac{2i^{(2(a+b))^2}+1}{\sqrt[p^2]{i^{2(a+b)}}})^a \cdot (\frac{1}{\sqrt[p^2]{i^{2(a+b)}}}))^b = \sum_{i=2}^{n} \frac{(2i^{(2(a+b))^2}+1)^a}{\sqrt[p^2]{i^{(2(a+b))^2}}}.$$

If $p < 2(a+b)$, $\mathrm{d}_p(\mathscr{B}_1, \mathscr{B}_n)$ converges, whereas $|p_{a,b}(\mathscr{B}_n) - p_{a,b}(\mathscr{B}_0)|$ is divergent by the divergence test.

This example shows what the problem with the polynomial coordinates is. Namely, if $a > 0$, then the coordinate of a potentially short bar (x_i, y_i) might be big because the bar appeared late in the filtration and not because it is particularly long.

In the next few chapters we aim to identify stable coordinates. Since max appears in the bottleneck and Wasserstein distances, we came to the idea that 'polynomials semirings' over max-plus and min-plus (tropical) semirings might be better suited. We start by establishing results in the tropical setting analogous to ones proved by Dalbec in the case of ordinary polynomials.

Chapter 3

Tropical and Max-Plus Arithmetics

Tropical algebra is based on the study of the tropical semiring $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$. In this semiring, addition and multiplication are defined as follows:

$$a \oplus b := \min(a, b) \quad \text{and} \quad a \odot b := a + b.$$

Both are commutative and associative. The times operator $\odot$ takes precedence when plus $\oplus$ and times $\odot$ occur in the same expression. The distributive law holds:

$$a \odot (b \oplus c) = a \odot b \oplus a \odot c.$$

Moreover, Freshman's Dream holds for all powers of n in tropical arithmetic:

$$(a \oplus b)^n = a^n \oplus b^n. \tag{3.1}$$

Expression b^{-1} is the inverse of b with respect to $\odot$ and equals $-b$ in ordinary arithmetic.

We also work with the max-plus semiring $(\mathbb{R} \cup \{-\infty\}, \boxplus, \odot)$, where multiplication of two elements is defined as before, but adding means taking their maximum instead of the minimum:

$$a \boxplus b := \max(a, b) \quad \text{and} \quad a \odot b := a + b.$$

Its operations are associative, commutative and distributive as in tropical algebra.

3.1 Min-Plus and Tropical Polynomial Expressions

Let $x_1, x_2, \ldots, x_n$ be variables representing elements in the tropical semiring. A *min-plus monomial expression* is any product of these variables, where repetition is allowed. A *tropical monomial expression* allows taking quotients in addition to products. By commutativity, we can sort the product and write monomial expressions with the variables raised to exponents.

A *min-plus polynomial expression* is a finite linear combination of min-plus monomial expressions:

$$p(x_1, x_2, \ldots, x_n) = a_1 \odot x_1^{a_1^1} x_2^{a_2^1} \ldots x_n^{a_n^1} \oplus a_2 \odot x_1^{a_1^2} x_2^{a_2^2} \ldots x_n^{a_n^2} \oplus \ldots \oplus a_m \odot x_1^{a_1^m} x_2^{a_2^m} \ldots x_n^{a_n^m},$$

Here the coefficients $a_1, a_2, \ldots a_m$ are real numbers and the exponents a_j^i for $1 \leq j \leq n$ and $1 \leq i \leq m$ are nonnegative integers. Similarly, a *tropical polynomial expression* is a finite linear combination of tropical monomial expressions:

$$p(x_1, x_2, \ldots, x_n) = a_1 \odot x_1^{a_1^1} x_2^{a_2^1} \ldots x_n^{a_n^1} \oplus a_2 \odot x_1^{a_1^2} x_2^{a_2^2} \ldots x_n^{a_n^2} \oplus \ldots \oplus a_m \odot x_1^{a_1^m} x_2^{a_2^m} \ldots x_n^{a_n^m},$$

where the coefficients $a_1, a_2, \ldots a_m$ are real numbers and the exponents a_j^i for $1 \leq j \leq n$ and $1 \leq i \leq m$ are integers. Tropical expressions are called tropical polynomials in other sources (see *Tropical Mathematics* [28] or *Introduction to Tropical Geometry* [25]).

The *total degree of an expression* $p(x_1, x_2, \dots, x_n)$ is

$$\deg p = \max_{1 \le i \le m} (a_1^i + a_2^i + \dots + a_n^i).$$

Each tropical polynomial expression represents a concave piece-wise linear function from $(\mathbb{R} \cup \{\infty\})^n$ to $\mathbb{R} \cup \infty$. Tropical polynomial expressions whose image is contained in $\mathbb{R}$ are $\mathbb{R}$-tropical polynomial expressions.

Example 3.1. *Let $n = 3$. A tropical monomial expression*

$$x_2 \odot x_1 \odot x_3 \odot x_2 \odot x_2 \odot x_1 = x_1^2 \odot x_2^3 \odot x_3 = x_1^2 x_2^3 x_3$$

represents the linear function

$$(x_1, x_2, x_3) \mapsto x_2 + x_1 + x_3 + x_2 + x_2 + x_1 = 2x_1 + 3x_2 + x_3.$$

The passage from tropical polynomial expressions to functions is not one-to-one. For example,

$$x_1^2 \oplus x_2^2 = x_1^2 \oplus x_2^2 \oplus x_1 x_2.$$

We say that the monomial $c_0 \odot x_1^{c_1} \odot x_2^{c_2} \odot \dots \odot x_n^{c_n}$ in

$$p(x_1, x_2, \dots, x_n) = \bigoplus_{i=1}^{k} a_0^i \odot x_1^{a_1^i} \odot x_2^{a_2^i} \odot \dots \odot x_n^{a_n^i} \oplus c_0 \odot x_1^{c_1} \odot x_2^{c_2} \odot \dots \odot x_n^{c_n}$$

is *redundant* if for all $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$

$$c_0 + c_1 x_1 + c_2 x_2 + \dots + c_n x_n \ge \min_{1 \le i \le k} (a_0^i + a_1^i x_1 + \dots + a_n^i x_n).$$

3.2 Redundancy in Tropical Polynomial Expressions

In this section we identify the conditions a monomial in a tropical polynomial expression must satisfy in order to be redundant.

The following theorem [5] will come handy in proving the statements of this type:

Theorem 3.2 (Strong Separating Hyperplane Theorem)**.** *Let K and C be disjoint nonempty convex subsets of $\mathbb{R}^n$. Suppose K is compact and C is closed. Then there exists a nonzero hyperplane that strongly separates K and C.*

For a finite set A, $\mathrm{Conv}(A)$ denotes the convex hull of the elements from A.

Proposition 3.3 (Redundancy, special example)**.** *Let*

$$p(x_1, x_2, \ldots, x_n) = \bigoplus_{i=1}^{k} x_1^{a_1^i} \odot x_2^{a_2^i} \odot \ldots \odot x_n^{a_n^i} \oplus x_1^{c_1} \odot x_2^{c_2} \odot \ldots \odot x_n^{c_n}$$

Then the monomial $x_1^{c_1} \odot x_2^{c_2} \odot \ldots \odot x_n^{c_n}$ is redundant if and only if $(c_1, \ldots, c_n)$ is contained in the convex hull of points $\{(a_1^1, \ldots, a_n^1), \ldots, (a_1^k, \ldots, a_n^k)\}$.

Proof. ($\Leftarrow$) Since $(c_1, c_2, \ldots, c_n)$ is contained in the convex hull of points

$$\{(a_1^1, \ldots, a_n^1), \ldots, (a_1^k, \ldots, a_n^k)\},$$

such real numbers $t_1, t_2, \ldots, t_k$ exist that $t_i \geq 0$, $t_1 + t_2 + \ldots + t_k = 1$ and

$$(c_1, c_2, \ldots, c_n) = \sum_{i=1}^{k} t_i (a_1^i, \ldots, a_n^i).$$

Let $x_1, x_2, \ldots, x_n$ be fixed real numbers. Then

$$\begin{aligned} x_1^{c_1} \odot x_2^{c_2} \odot \ldots \odot x_n^{c_n} &= c_1x_1 + c_2x_2 + \ldots + c_nx_n \\ &= (\sum_{i=1}^{k} t_i a_1^i)x_1 + (\sum_{i=1}^{k} t_i a_2^i)x_2 + \ldots + (\sum_{i=1}^{k} t_i a_n^i)x_n \\ &= t_1 \sum_{j=1}^{n} a_j^1 x_j + t_2 \sum_{j=1}^{n} a_j^2 x_j + \ldots + t_k \sum_{j=1}^{n} a_j^k x_j \\ &\geq \min_{i=1,\ldots,n} \sum_{j=1}^{n} a_j^i x_j \\ &= \bigoplus_{i=1}^{k} x_1^{a_1^i} \odot x_2^{a_2^i} \odot \ldots \odot x_n^{a_n^i} \end{aligned}$$

This proves that $x_1^{c_1} \odot x_2^{c_2} \odot \ldots \odot x_n^{c_n}$ is redundant.

($\Rightarrow$) We prove this part by contradiction. Suppose $(c_1, \ldots, c_n)$ is not contained in the convex hull C of points $\{(a_1^1, \ldots, a_n^1), \ldots, (a_1^k, \ldots, a_n^k)\}$. Since $C \subset \mathbb{R}^n$ and $(c_1, \ldots, c_n)$ are both convex, compact subsets of $\mathbb{R}^n$, there exists a nonzero hyperplane H that separates them. Let $\vec{t}$ be a point in H. Hyperplane $H - \vec{t}$ goes through the origin and separates $C - \vec{t}$ from $\vec{c} - \vec{t}$, where $\vec{c} = (c_1, \ldots, c_n)$. Let $\vec{n} = (x_1, \ldots, x_n)$ be the unit normal vector to $H - \vec{t}$ that points in the direction of the half space, which contains $C - \vec{t}$. Then

$$((a_1^i, \ldots, a_n^i) - \vec{t}) \cdot \vec{n} > 0$$

for all i, and

$$(\vec{c} - \vec{t}) \cdot \vec{n} < 0.$$

It follows from here that

$$c_1x_1 + \ldots c_nx_n < a_1^i x_1 + \ldots a_n^i x_n$$

for all i. Consequently $x_1^{c_1} \odot x_2^{c_2} \odot \ldots \odot x_n^{c_n}$ cannot be a redundant monomial. □

In Proposition 3.3 we observed a minimum of linear functions, whose graphs go through the origin and found a condition that makes a linear function of that type redundant. Now we prove a similar statement concerning general linear functions.

Proposition 3.4 (Redundancy, general case)**.** *Let*

$$p(x_1, x_2, \ldots, x_n) = \bigoplus_{i=1}^{k} a_0^i \odot x_1^{a_1^i} \odot x_2^{a_2^i} \odot \ldots \odot x_n^{a_n^i} \oplus c_0 \odot x_1^{c_1} \odot x_2^{c_2} \odot \ldots \odot x_n^{c_n}$$

Then the monomial $c_0 \odot x_1^{c_1} \odot x_2^{c_2} \odot \ldots \odot x_n^{c_n}$ *is redundant if and only if* $(c_0, c_1, \ldots, c_n)$ *is contained in*

$$C = \bigcup_{M=1}^{\infty} \mathrm{Conv}(\bigcup_{i=1}^{k} \{(a_0^i, a_1^i, \ldots, a_n^i)\} \cup \bigcup_{i=1}^{k} \{(a_0^i + M, a_1^i, \ldots, a_n^i)\}).$$

Proof. ($\Leftarrow$) For $(c_0, c_1, c_2, \ldots, c_n)$ contained in C such real numbers $t_0, t_1, \ldots, t_k, t'_0, t'_1, \ldots, t'_k$ exist that $t_i, t'_i \geq 0$,

$$t_0 + t_1 + \ldots + t_k + t'_0 + t'_1 + \ldots + t'_k = 1$$

and

$$(c_0, c_1, c_2, \ldots, c_n) = \sum_{i=0}^{k} t_i(a_0^i, a_1^i, \ldots, a_n^i) + \sum_{i=0}^{k} t'_i(a_0^i + M, a_1^i, \ldots, a_n^i).$$

Let $x_0 = 1$ and let $x_1, x_2, \ldots, x_n$ be fixed real numbers. Then

$$\begin{aligned}
c_0 \odot x_1^{c_1} \odot x_2^{c_2} \odot \ldots \odot x_n^{c_n} &= c_0 + c_1x_1 + c_2x_2 + \ldots + c_nx_n \\
&= M \sum_{i=0}^{k} t'_i + \sum_{j=0}^{n} \sum_{i=0}^{k} (t_i + t'_i) a_j^i x_j \\
&= M \sum_{i=0}^{k} t'_i + \sum_{i=0}^{k} (t_i + t'_i) (\sum_{j=0}^{n} a_j^i x_j) \\
&\geq \sum_{i=0}^{k} (t_i + t'_i) (\sum_{j=0}^{n} a_j^i x_j) \\
&\geq \min_{i=0,\ldots,n} \sum_{j=1}^{n} a_j^i x_j \\
&= \bigoplus_{i=0}^{k} a_0^i \odot x_1^{a_1^i} \odot x_2^{a_2^i} \odot \ldots \odot x_n^{a_n^i}
\end{aligned}$$

This proves that $c_0 \odot x_1^{c_1} \odot x_2^{c_2} \odot \ldots \odot x_n^{c_n}$ is redundant.

($\Rightarrow$) We prove this part by contradiction. Suppose $\{(c_0, c_1, \ldots, c_n)\}$ is not contained in

C. The set $C \subset \mathbb{R}^n$ is convex since it is a union of nested convex sets. It is also closed. Since $\{(c_0, c_1, \ldots, c_n)\}$ is convex and compact, there exists a nonzero hyperplane H that separates them. Let $\vec{t}$ be a point in H. Hyperplane $H - \vec{t}$ goes through the origin and separates $C - \vec{t}$ from $\vec{c} - \vec{t}$, where $\vec{c} = (c_0, c_1, \ldots, c_n)$. Let $\vec{n} = (x_0, x_1, \ldots, x_n)$ be the unit normal vector to $H - \vec{t}$ that points in the direction of the half space, which contains $C - \vec{t}$. Then

$$((a_0^i + M, a_1^i, \ldots, a_n^i) - \vec{t}) \cdot \vec{n} > 0$$

for all i, any positive integer M, and

$$(\vec{c} - \vec{t}) \cdot \vec{n} < 0.$$

It follows from here that

$$c_0x_0 + c_1x_1 + \ldots c_nx_n < x_0M + a_0^ix_0 + a_1^ix_1 + \ldots a_n^ix_n. \tag{3.2}$$

If $x_0 = 0$, we are done. Let $0 < x_0 \leq 1$ and choose $M = 0$. If we divide the above equation by x_0, we get

$$c_0 + c_1(\frac{x_1}{x_0}) + \ldots c_n(\frac{x_n}{x_0}) < a_0^i + a_1^i(\frac{x_1}{x_0}) + \ldots a_n^i(\frac{x_n}{x_0})$$

for all i. This proves $c_0 \odot x_1^{c_1} \odot x_2^{c_2} \odot \ldots \odot x_n^{c_n}$ cannot be a redundant monomial.

Finally, suppose $-1 \leq x_0 < 0$. We rearrange terms in Equation 3.2:

$$c_0 + c_1x_1 + \ldots c_nx_n < c_0(1 - x_0) + (x_0M + a_0^ix_0 - a_0^i) + a_0^i + a_1^ix_1 + \ldots a_n^ix_n.$$

We observe that

$$c_0(1 - x_0) + (Mx_0 + a_0^ix_0 - a_0^i) \leq 2c_0 + Mx_0 + a_0^i + 1$$

By choosing M sufficiently big, we can make the right hand side negative, which implies

$$c_0 + c_1x_1 + \dots c_nx_n < a_0^i + a_1^ix_1 + \dots a_n^ix_n.$$

This finishes the proof. □

Let us take a look at one example.

Example 3.5. *Let $p(x) = \min\{1+x, 2x, c_0+c_1x\}$. Expression c_0+c_1x is redundant if and only if $(c_0, c_1) \in C$, where C is depicted in Figure 3.1a) From the Figure 3.1a) we see that*

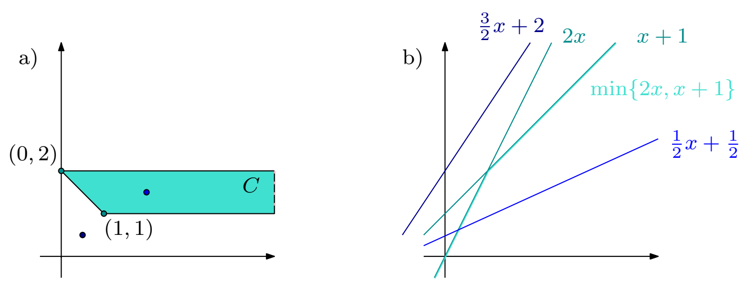

Figure 3.1: a) Region C. b) Graphs of linear functions and redundancy.

setting $c_0 = 2$, $c_1 = \frac{3}{2}$ makes expression $c_0 + c_1x$ redundant in $p(x)$. That is not the case if $c_0 = \frac{1}{2}$, $c_1 = \frac{1}{2}$.

3.3 Min-Plus and Tropical Polynomials

Let p and q be tropical polynomial expressions. If

$$p(x_1, x_2, \dots, x_n) = q(x_1, x_2, \dots, x_n)$$

for all $(x_1, x_2, \dots, x_n) \in (\mathbb{R} \cup \infty)^n$, then p and q are *functionally equivalent.*

Functional equivalence $\sim$ is an equivalence relation on the set of all tropical polynomial expressions.

Definition 3.6. *Tropical polynomials are the semiring of equivalence classes of tropical polynomial expressions with respect to* $\sim$. *In the case of* n *variables we denote it by* $\texttt{Trop}[x_1, x_2, \ldots, x_n]$. *Min-plus polynomials are the semiring of equivalence classes of min-plus polynomial expressions with respect to* $\sim$. *In the case of* n *variables we denote it by* $\texttt{MinPlus}[x_1, x_2, \ldots, x_n]$.

Remark 3.7. *Note that our tropical polynomial semiring is not obtained using the standard 'polynomial semiring' construction. That construction yields the semiring of tropical polynomial expressions.*

Tropical polynomial expressions in the same equivalence class determine the same graph in $\mathbb{R}^{n+1}$ and are piece-wise linear, concave functions. This means that each tropical polynomial p determines a graph. Associated with each piece-wise linear function is a finite family $\mathcal{Q} = \{Q_1, \ldots, Q_m\}$ of maximal closed domains (a closed domain is a closure of a nonempty open set) such that $\mathbb{R}^n = \bigcup \mathcal{Q}$ and p is linear on every domain in $\mathcal{Q}$. We call the linear function g_i on $\mathbb{R}^n$ which coincides with p on $Q_i \in \mathcal{Q}$ a *component* of p. Domains being maximal means that p does not agree with g_i on a set strictly containing Q_i. We call Q_i the *domain component* of p.

The minimal representation of a tropical polynomial p is such a tropical expression

$$a_1 \odot x_1^{a_1^1} x_2^{a_2^1} \ldots x_n^{a_n^1} \oplus a_2 \odot x_1^{a_1^2} x_2^{a_2^2} \ldots x_n^{a_n^2} \oplus \ldots \oplus a_m \odot x_1^{a_1^m} x_2^{i_2^m} \ldots x_n^{a_n^m}$$

functionally equivalent to p that contains no redundant monomials.

Theorem 3.8. *Every tropical polynomial has a unique minimal representation. The subset, where the minimum in the minimal representation is attained at exactly one index, is open and dense in the domain of the polynomial.*

Proof. Let p be any tropical polynomial, $\mathcal{Q} = \{Q_1, \ldots, Q_m\}$ the set of domain components and let $g_1, \ldots, g_m$ be the components of p.

We claim that $\min\limits_{i\in\mathbb{N}_{\leq m}} g_i$ is the minimal representation of p.

Let $(x_1,\ldots,x_n)\in\mathbb{R}^n$. Since $\mathbb{R}^n$ is a union of $\mathcal{Q}$, i exists such that $(x_1,\ldots,x_n)\in Q_i$. By construction $p(x_1,\ldots,x_n)=g_i(x_1,\ldots,x_n)\geq \min\limits_{j\in\mathbb{N}_{\leq m}} g_j(x_1,\ldots,x_n)$. We must show that

$$g_i(x_1,\ldots,x_n)\leq g_j(x_1,\ldots,x_n)$$

for all j. Take $(y_1,\ldots,y_n)\in\mathring{Q}_j$. Since $(y_1,\ldots,y_n)$ lies in the interior of Q_j we can choose such $t_1,t_2\in(0,1)$ with $t_1+t_2=1$, that $t_1(x_1,\ldots,x_n)+t_2(y_1,\ldots,y_n)\in Q_j$. Since p is concave

$$t_1p(x_1,\ldots,x_n)+t_2p(y_1,\ldots,y_n)\leq p(t_1(x_1,\ldots,x_n)+t_2(y_1,\ldots,y_n)).$$

We can transform both sides of the equations as follows

$$t_1g_i(x_1,\ldots,x_n)+t_2g_j(y_1,\ldots,y_n)\leq g_j(t_1(x_1,\ldots,x_n)+t_2(y_1,\ldots,y_n)).$$

Since g_j is linear

$$g_j(t_1(x_1,\ldots,x_n)+t_2(y_1,\ldots,y_n))=t_1g_j(x_1,\ldots,x_n)+t_2g_j(y_1,\ldots,y_n).$$

So

$$t_1g_i(x_1,\ldots,x_n)+t_2g_j(y_1,\ldots,y_n)\leq t_1g_j(x_1,\ldots,x_n)+t_2g_j(y_1,\ldots,y_n)$$

implying

$$g_i(x_1,\ldots,x_n)\leq g_j(x_1,\ldots,x_n)$$

since $t_1>0$. This proves that $p(x_1,\ldots,x_n)=\min\limits_{j\in\mathbb{N}_{\leq m}} g_j(x_1,\ldots,x_n)$.

Before proving that this representation is minimal, we prove that

$$\mathring{Q}_i = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, \forall j \neq i,\, g_i(x_1, \ldots, x_n) < g_j(x_1, \ldots, x_n)\}.$$

We denote $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, \forall j \neq i,\, g_i(x_1, \ldots, x_n) < g_j(x_1, \ldots, x_n)\}$ by D_i. Clearly, $D_i \subseteq Q_i$. To show that $D_i \subseteq \mathring{Q}_i$, it suffices to show that D_i is open. Let $G_i \colon \mathbb{R}^n \to \mathbb{R}^{m-1}$ be such that $G_i = (g_1 - g_i, \ldots, g_{i-1} - g_i, g_{i+1} - g_i, \ldots, g_n - g_i)$. G_i is continuous and $D_i = G_i^{-1}((0,\infty)^{m-1})$ and therefore open.

Let $(x_1, \ldots, x_n) \in \mathring{Q}_i$ and $j \neq i$. Suppose that $g_i(x_1, \ldots, x_n) = g_j(x_1, \ldots, x_n)$. Since closed domains are maximal this implies that $(x_1, \ldots, x_n) \in Q_j$. This would imply that $\mathring{Q}_i$ intersects $\mathring{Q}_j$ and that on this intersection g_i and g_j match since they both equal p. Since they are both linear, $g_i = g_j$, a contradiction.

We are now in position to prove that the subset, where the minimum in the minimal representation is attained at exactly one index, i.e. $\bigcup_{j \in \mathbb{N}_{\leq m}} \mathring{Q}_i$, is open and dense in the domain of the polynomial. It is open since it is a union of open sets. It is dense since

$$\overline{\bigcup_{j \in \mathbb{N}_{\leq m}} \mathring{Q}_i} = \bigcup_{j \in \mathbb{N}_{\leq m}} \overline{\mathring{Q}_i} = \bigcup_{j \in \mathbb{N}_{\leq m}} Q_i = \mathbb{R}^n.$$

To see that $\min\limits_{j \in \mathbb{N}_{\leq m}} g_i(x_1, \ldots, x_n)$ is a minimal representation, take $(x_1, \ldots, x_n) \in \mathring{Q}_i$. Since $\mathring{Q}_i = D_i$, $g_i(x_1, \ldots, x_n) < g_j(x_1, \ldots, x_n)$ for all $j \neq i$. This proves that the representation is minimal.

Suppose we have two minimal representations $\min\limits_{j \in \mathbb{N}_{\leq m_1}} g_i$ and $\min\limits_{j \in \mathbb{N}_{\leq m_2}} h_j$. It suffices to show that every g_i equals some h_j. Then the statement will follow by symmetry. The set D_i is nonempty and open. On this set g_i equals $\min\limits_{j \in \mathbb{N}_{\leq m_2}} h_j$. Since g_i is linear and different h_j appearing in $\min\limits_{j \in \mathbb{N}_{\leq m_2}} h_j$ represent different linear functions, j exists such that $g_i = h_j$ on this nonempty open set and therefore for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$. $\square$

Before defining the degree of a tropical polynomial, we prove the following proposition.

Proposition 3.9. *Let p be a tropical polynomial expression and p' the minimal representation of p. Then*

$$\deg\, p := \deg\, p'.$$

Proof. It is always true that $\deg p \geq \deg p'$. We must show that $\deg p \leq \deg p'$. Let

$$p(x_1, x_2, \ldots, x_n) = \bigoplus_{i=1}^{k} a_0^i \odot x_1^{a_1^i} \odot x_2^{a_2^i} \odot \ldots \odot x_n^{a_n^i} \oplus c_0 \odot x_1^{c_1} \odot x_2^{c_2} \odot \ldots \odot x_n^{c_n}$$

and suppose

$$p'(x_1, x_2, \ldots, x_n) = \bigoplus_{i=1}^{k} a_0^i \odot x_1^{a_1^i} \odot x_2^{a_2^i} \odot \ldots \odot x_n^{a_n^i}.$$

According to Proposition 3.4 $(c_1, c_2, \ldots, c_n)$ is contained in $\mathrm{conv}\{(a_1^1, \ldots, a_n^1), \ldots, (a_1^k, \ldots, a_n^k)\}$. This implies that

$$\begin{aligned} c_1 + c_2 + \ldots + c_n &= \left(\sum_{i=1}^{k} t_i a_1^i\right) + \left(\sum_{i=1}^{k} t_i a_2^i\right) + \ldots + \left(\sum_{i=1}^{k} t_i a_n^i\right) \\ &= t_1 \sum_{j=1}^{n} a_j^1 + t_2 \sum_{j=1}^{n} a_j^2 + \ldots + t_k \sum_{j=1}^{n} a_j^k \\ &\leq \max_{i=1,\ldots,n} \sum_{j=1}^{n} a_j^i \end{aligned}$$

Therefore $\max_{i=1,\ldots,n} \sum_{j=1}^{n} a_j^i = \max\{\max_{i=1,\ldots,n} \{\sum_{j=1}^{n} a_j^i\}, \sum_{j=1}^{n} c_j\}$ and in turn that $\deg p \leq \deg p'$. The same argument works for a representation with more than one redundant monomial. □

This implies that all functionally equivalent tropical expressions have the same degree, so we can define the *degree of a tropical polynomial* p as follows:

$$\mathrm{Deg} p = \deg q,$$

where q is any tropical expression representing p.

3.4 Max-Plus Polynomials

Most of the definitions concerning max-plus polynomials are analogous to the definitions concerning tropical polynomials with $\oplus$ substituted by $\boxplus$.

Let $x_1, x_2, \ldots, x_n$ be variables representing elements in the max-plus semiring. A *max-plus polynomial expression* is a finite linear combination of tropical monomial expressions:

$$p(x_1, x_2, \ldots, x_n) = a_1 \odot x_1^{a_1^1} x_2^{i_2^1} \ldots x_n^{a_n^1} \boxplus a_2 \odot x_1^{a_1^2} x_2^{a_2^2} \ldots x_n^{a_n^2} \boxplus \ldots \boxplus a_m \odot x_1^{a_1^m} x_2^{a_2^m} \ldots x_n^{a_n^m}.$$

Here the coefficients $a_1, a_2, \ldots a_m$ are real numbers and the exponents a_j^i for $1 \leq j \leq n$ and $1 \leq i \leq m$ are nonnegative integers.

We say that $c_0 \odot x_1^{c_1} \odot x_2^{c_2} \odot \ldots \odot x_n^{c_n}$ in

$$p(x_1, x_2, \ldots, x_n) = \boxplus_{i=1}^{k} a_0^i \odot x_1^{a_1^i} \odot x_2^{a_2^i} \odot \ldots \odot x_n^{a_n^i} \boxplus c_0 \odot x_1^{c_1} \odot x_2^{c_2} \odot \ldots \odot x_n^{c_n}$$

is *redundant* if for all $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$

$$c_0 + c_1 x_1 \odot c_2 x_2 + \ldots + c_n x_n \leq \max_{1 \leq i \leq k} (a_0^i + a_1^i x_1 + \ldots + a_n^i x_n).$$

Proposition 3.3 and Proposition 3.4 extend directly to the max-plus setting since

$$-\max_{1 \leq i \leq k} a_i = \min_{1 \leq i \leq k} -a_i$$

for any real numbers $a_1, \ldots, a_k$.

Proposition 3.10 (Redundancy, special example)**.** *Let*

$$p(x_1, x_2, \ldots, x_n) = \boxplus_{i=1}^{k} x_1^{a_1^i} \odot x_2^{a_2^i} \odot \ldots \odot x_n^{a_n^i} \boxplus x_1^{c_1} \odot x_2^{c_2} \odot \ldots \odot x_n^{c_n}$$

Then the monomial $x_1^{c_1} \odot x_2^{c_2} \odot \ldots \odot x_n^{c_n}$ is redundant if and only if $(c_1, \ldots, c_n)$ is contained *in the convex hull of points* $\{(a_1^1, \ldots, a_n^1), \ldots, (a_1^k, \ldots, a_n^k)\}$.

Proposition 3.11 (Redundancy, general case)**.** *Let*

$$p(x_1, x_2, \ldots, x_n) = \boxplus_{i=1}^k a_0^i \odot x_1^{a_1^i} \odot x_2^{a_2^i} \odot \ldots \odot x_n^{a_n^i} \boxplus c_0 \odot x_1^{c_1} \odot x_2^{c_2} \odot \ldots \odot x_n^{c_n}$$

Then the monomial $c_0 \odot x_1^{c_1} \odot x_2^{c_2} \odot \ldots \odot x_n^{c_n}$ *is redundant if and only if* $(c_0, c_1, \ldots, c_n)$ *is contained in*

$$C = \bigcup_{M=1}^{\infty} \mathrm{Conv}(\bigcup_{i=1}^{k} \{(a_0^i, a_1^i, \ldots, a_n^i)\} \cup \bigcup_{i=1}^{k} \{(a_0^i - M, a_1^i, \ldots, a_n^i)\}).$$

Definition 3.12. *Max-plus polynomials are the semiring of equivalence classes of max-plus polynomial expressions with respect to functional equivalence relation* $\sim$*. In the case of* n *variables we denote it by* $\texttt{MaxPlus}[x_1, x_2, \ldots, x_n]$.

An equivalent of Proposition 3.9 holds in the max-plus setting allowing us to define the *degree of a max-plus polynomial* p represented by

$$q(x_1, x_2, \ldots, x_n) = a_1 \odot x_1^{a_1^1} x_2^{a_2^1} \ldots x_n^{a_n^1} \oplus a_2 \odot x_1^{a_1^2} x_2^{a_2^2} \ldots x_n^{a_n^2} \oplus \ldots \oplus a_m \odot x_1^{a_1^m} x_2^{a_2^m} \ldots x_n^{a_n^m},$$

as

$$\deg p = \max_{1 \leq i \leq m} (a_1^i + a_2^i + \ldots + a_n^i).$$

3.5 Tropical Rational Functions

In the semirings of tropical polynomials and max-plus polynomials the operation $\odot$ is not invertible. We have more flexibility to manipulate expressions if we allow inverses with respect to $\odot$.

Definition 3.13. *A tropical rational expression r is a quotient*

$$r(x_1, \ldots, x_n) = p(x_1, \ldots, x_n) \odot q(x_1, \ldots, x_n)^{-1},$$

where p is a tropical polynomial expression and q is an $\mathbb{R}$-tropical polynomial expression.

Remark 3.14. *Tropical rational expressions are the localization of the semiring of tropical polynomial expressions with respect to the multiplicatively closed set of $\mathbb{R}$-tropical polynomial expressions.*

We say that tropical rational expressions r and s are *functionally equivalent* and write $r \sim s$ if

$$r(x_1, x_2, \ldots, x_n) = s(x_1, x_2, \ldots, x_n)$$

for all $(x_1, x_2, \ldots, x_n) \in (\mathbb{R} \cup \infty)^n$.

Since

$$-\min(a, b) = \max(-a, -b),$$

tropical rational expressions are composed of taking the maxima and minima of linear functions, i.e. the set of tropical rational expressions is the smallest subset of functions $\mathbb{R}^n \to \mathbb{R}$ containing all constant maps, projections and closed under $+$, min and max.

Conversely, any function from this set can be represented by an expression of the form $p \odot q^{-1}$, where p and q are tropical polynomial expressions. The algorithm to produce p and q is best demonstrated by an example.

Example 3.15. *Let* $r(x_1, x_2) = x_1^{-1} \odot x_2 \oplus (x_2)^{-1} \oplus (x_2 \odot x_1 \oplus x_1)^{-1}$. *We can write*

$$\begin{aligned}
r(x_1, x_2) &= \min(-x_1 + x_2, -x_2, -\min(x_2 + x_1, x_1)) \\
&= \min(-x_1 + x_2 + \min(x_2 + x_1, x_1), -x_2 + \min(x_2 + x_1, x_1), 0) - \min(x_2 + x_1, x_1) \\
&= \min(\min(2x_2, x_2), \min(x_1, -x_2), 0) - \min(x_2 + x_1, x_1) \\
&= \min(2x_2, x_2, x_1, -x_2, 0) - \min(x_2 + x_1, x_1) \\
&= \min(3x_2, 2x_2, x_1 + x_2, 0, x_2) - \min(2x_2 + x_1, x_1 + x_2) \\
&= (x_2^2 \oplus x_2^2 \oplus x_1 x_2 \oplus 0 \oplus x_2) \odot (x_2^2 x_1 \oplus x_1 x_2)^{-1}.
\end{aligned}$$

This is similar to adding fractions with different denominators.

Definition 3.16. *The semiring of equivalence classes of tropical rational expressions with respect to the functions equivalence relation is* `RTrop`$[x_1, x_2, \ldots, x_n]$ *and is called the* semiring of rational tropical functions.

Chapter 4

Symmetric Tropical Polynomials, Min-Plus Polynomials and Tropical Rational Functions

4.1 Symmetric Tropical Polynomials

Definition 4.1. *A tropical polynomial* $p \in \mathtt{Trop}[x_1, x_2, \ldots, x_n]$ *is symmetric if*

$$p(x_1, \ldots, x_n) = p(x_{\pi(1)}, \ldots, x_{\pi(n)})$$

for every permutation $\pi \in S_n$.

We denote the semiring of symmetric tropical polynomials by $\mathtt{Trop}[x_1, x_2, \ldots, x_n]^{S_n}$. We work with a fixed n throughout this section.

Example 4.2. *Let* $n = 3$. *The tropical polynomials* $x_1^2 \oplus x_2^2 \oplus x_3^2$ *and* $x_1 \odot x_2 \odot x_3$ *are symmetric.*

We define a symmetrization operator:

$$\begin{aligned}\mathrm{Sym}\colon\ \texttt{Trop}[x_1, x_2, \ldots, x_n] &\to \texttt{Trop}[x_1, x_2, \ldots, x_n]^{S_n}\\ p(x_1, x_2, \ldots, x_n) &\mapsto \textstyle\bigoplus_{\pi \in S_n} p(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)}).\end{aligned}$$

Proposition 4.3. *Let $p, q \in \texttt{Trop}[x_1, x_2, \ldots, x_n]$. Then:*

1. $\mathrm{Sym}(p \oplus q) = \mathrm{Sym}(p) \oplus \mathrm{Sym}(q)$,
2. $a \odot \mathrm{Sym}(p) = \mathrm{Sym}(a \odot p)$.

Proof. We leave the proof to the reader. □

Proposition 4.4. *Let $p \in \texttt{Trop}[x_1, \ldots, x_n]$. Then*

$$p \textit{ is symmetric} \Leftrightarrow \mathrm{Sym}(p) = p.$$

Proof. ($\Rightarrow$) Suppose p is symmetric. Then $p(x_1, x_2, \ldots, x_n) = p(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})$ for all $\pi \in S_n$ and $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Since $\oplus$ is idempotent,

$$p(x_1, x_2, \ldots, x_n) = \bigoplus_{\pi \in S_n} p(x_1, x_2, \ldots, x_n) = \bigoplus_{\pi \in S_n} p(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})$$

for all $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Consequently, $\mathrm{Sym}(p) = p$.

($\Leftarrow$) By definition $\mathrm{Sym}(p)$ is symmetric and since $p = \mathrm{Sym}(p)$ so is p. □

The following symmetric tropical polynomials will play an important role in our discussion.

Definition 4.5. *Given variables $x_1, \ldots, x_n$, we define the elementary symmetric tropical* *polynomials* $e_1, \ldots, e_n \in \mathtt{Trop}[x_1, x_2, \ldots, x_n]$ *by the formulas*

$$\begin{aligned} e_1 &= x_1 \oplus \ldots \oplus x_n, \\ &\vdots \\ e_k &= \mathrm{Sym}(x_1 \odot \ldots \odot x_k), \\ &\vdots \\ e_n &= x_1 \odot x_2 \odot \ldots \odot x_n. \end{aligned}$$

The total degree of expression e_k is k. Elementary symmetric tropical polynomials give coordinates on $\mathbb{R}^n/S_n$. In other words, they separate orbits.

Proposition 4.6. *Let* $[x_1, \ldots, x_n]$ *and* $[y_1, \ldots, y_n]$ *be two orbits under the* S_n*-action on* $\mathbb{R}^n$*. If*

$$e_i([x_1, \ldots, x_n]) = e_i([y_1, \ldots, y_n])$$

for all i*, then* $[x_1, \ldots, x_n] = [y_1, \ldots, y_n]$.

Proof. Suppose $[x_1, \ldots, x_n]$ and $[y_1, \ldots, y_n]$ are orbits for which

$$e_i([x_1, \ldots, x_n]) = e_i([y_1, \ldots, y_n])$$

for all i. We assume without loss of generality that $x_1 \leq x_2 \leq \ldots \leq x_n$ and $y_1 \leq y_2 \leq \ldots \leq y_n$. Since

$$e_1([x_1, \ldots, x_n]) = e_1([y_1, \ldots, y_n]),$$

it follows that $x_1 = y_1 = e_1([x_1, \ldots, x_n])$. Next note that

$$x_1 + x_2 = e_2([x_1, \ldots, x_n]) = e_2([y_1, \ldots, y_n]) = y_1 + y_2.$$

Since $x_1 = y_1$, this implies $x_2 = y_2$. We repeat these steps until we get $x_i = y_i$ for $i \leq n-1$.

Lastly,

$$x_1 + x_2 + \ldots + x_n = e_n([x_1, \ldots, x_n]) = e_n([y_1, \ldots, y_n]) = y_1 + y_2 + \ldots + y_n.$$

Since $x_i = y_i$ for $i \leq n-1$, it follows from this last equation that $x_n = y_n$ and we are done. $\square$

The goal of the remainder of this section is to prove the following theorem, which states that elementary symmetric polynomials generate symmetric tropical polynomials.

Theorem 4.7. *Every symmetric tropical polynomial in* $\mathtt{Trop}[x_1, x_2, \ldots, x_n]$ *can be written as a tropical polynomial in the elementary symmetric tropical polynomials* $e_1, \ldots, e_n$ *and* e_n^{-1}.

A referee from the journal I submitted the paper on this topic to suggested that a version of this theorem might hold for polynomial semirings over any semiring and that this more general statement could be deduced using the 'transfer principle' [30]. A relevant counterexample is the semiring tropical polynomial expressions. Tropical expression $x^2 \oplus y^2$ is not a polynomial in e_1,e_2 on the level of expressions, but equals e_1^2 in Trop$[x, y]$. The transfer principle fails because we are not making a statement about specific equations but rather about generating sets. The problem is that an element in the ring might be expressed by multiplying, adding and *subtracting* polynomials, whereas in the semiring the subtraction step cannot be carried out.

It is actually quite surprising that one can give an argument in the tropical case that avoids subtraction.

Lemma 4.8. *Let us suppose that*

$$p(x_1, x_2, \ldots, x_n) = \bigoplus_{1 \leq s \leq m} a_s \odot x_1^{i_1^s} \odot \ldots \odot x_n^{i_n^s}$$

is a symmetric tropical polynomial. Then

$$p(x_1, x_2, \ldots, x_n) = \bigoplus_{1 \le s \le m} \mathrm{Sym}(a_s \odot x_1^{i_1^s} \odot \ldots \odot x_n^{i_n^s}).$$

Proof. Follows from Propositions 4.4 and 4.3. □

Lemma 4.9. *Suppose $i_{j_1}, i_{j_2}, \ldots, i_{j_k}$, $k \le n$, are positive integers and $a = \min(i_{j_1}, i_{j_2}, \ldots, i_{j_k})$. Then*

$$e_k^a \odot \mathrm{Sym}(x_1^{i_{j_1}-a} \odot \ldots \odot x_k^{i_{j_k}-a}) = \mathrm{Sym}(x_1^{i_{j_1}} \odot \ldots \odot x_k^{i_{j_k}}).$$

Proof of Lemma 4.9. Since the Freshman's Dream holds in tropical arithmetic, the expression on the left equals

$$(\bigoplus_{\rho \in S_n} x_{\rho(1)}^a \odot \ldots \odot x_{\rho(k)}^a) \odot (\bigoplus_{\pi \in S_n} x_{\pi(1)}^{i_{j_1}-a} \odot \ldots \odot x_{\pi(k)}^{i_{j_k}-a}).$$

By distributivity and commutativity, we can rewrite it as

$$\bigoplus_{\rho \in S_n} \bigoplus_{\pi \in S_n} x_{\pi(1)}^{i_{j_1}-a} \odot \ldots \odot x_{\pi(k)}^{i_{j_k}-a} \odot x_{\rho(1)}^a \odot \ldots \odot x_{\rho(k)}^a.$$

We must show that

$$\bigoplus_{\rho \in S_n} \bigoplus_{\pi \in S_n} x_{\pi(1)}^{i_{j_1}-a} \odot \ldots \odot x_{\pi(k)}^{i_{j_k}-a} \odot x_{\rho(1)}^a \odot \ldots \odot x_{\rho(k)}^a = \bigoplus_{\sigma \in S_n} x_{\sigma(1)}^{i_{j_1}} \odot \ldots \odot x_{\sigma(k)}^{i_{j_k}}.$$

The right hand side is bigger than the left hand side since the minimum is taken over a smaller set. We must show that

$$x_{\pi(1)}^{i_{j_1}-a} \odot \ldots \odot x_{\pi(k)}^{i_{j_k}-a} \odot x_{\rho(1)}^a \odot \ldots \odot x_{\rho(k)}^a \ge \bigoplus_{\sigma \in S_n} x_{\sigma(1)}^{i_{j_1}} \odot \ldots \odot x_{\sigma(k)}^{i_{j_k}}$$

for any $\pi, \rho \in S_n$ and the claim will follow. Let

$$M = \{m \in \{1, 2, \ldots, k\} \,|\, j_m \in \{1, 2, \ldots, k\} \text{ exists such that } \pi(m) = \rho(j_m)\}$$

and let

$$J = \{j_m \in \{1, 2, \ldots, k\} \,|\, m \in \{1, 2, \ldots, k\} \text{ exists such that } \pi(m) = \rho(j_m)\}.$$

We denote the elements of M by $m_1, \ldots, m_l$, the elements of $\{1, 2, \ldots, k\} \setminus M$ by $s_1, \ldots, s_{k-l}$ and the elements of $\{1, 2, \ldots, k\} \setminus J$ by $q_1, \ldots, q_{k-l}$. We simplify the expression

$$x_{\pi(1)}^{i_{j_1}-a} \odot \ldots \odot x_{\pi(k)}^{i_{j_k}-a} \odot x_{\rho(1)}^{a} \odot \ldots \odot x_{\rho(k)}^{a}$$

to

$$\bigodot_{r=1}^{l} x_{\pi(m_r)}^{i_{j_{m_r}}} \odot \bigodot_{r=1}^{k-l} x_{\pi(s_r)}^{i_{j_{s_r}}-a} \odot \bigodot_{r=1}^{k-l} x_{\rho(q_r)}^{a}.$$

For all $r = 1, \ldots, k-l$

$$x_{\pi(s_r)}^{i_{j_{s_r}}-a} \odot x_{\rho(q_r)}^{a} \geq x_{\pi(s_r)}^{i_{j_{s_r}}} \oplus x_{\rho(q_r)}^{i_{j_{s_r}}}.$$

Tropically multiplying (adding) these inequalities for applicable r yields

$$\begin{aligned} \bigodot_{r=1}^{l} x_{\pi(m_r)}^{i_{j_{m_r}}} \odot \bigodot_{r=1}^{k-l} (x_{\pi(s_r)}^{i_{j_{s_r}}-a} \odot x_{\rho(q_r)}^{a}) \quad &\geq \quad \bigodot_{r=1}^{l} x_{\pi(m_r)}^{i_{j_{m_r}}} \odot \bigodot_{r=1}^{k-l} (x_{\pi(s_r)}^{i_{j_{s_r}}} \oplus x_{\rho(q_r)}^{i_{j_{s_r}}}) \\ &\geq \quad \bigoplus_{\sigma \in S_n} x_{\sigma(1)}^{i_{j_1}} \odot \ldots \odot x_{\sigma(k)}^{i_{j_k}}. \end{aligned}$$

In the last step we use distributivity to expand $\bigodot_{r=1}^{l} x_{\pi(m_r)}^{i_{j_{m_r}}} \odot \bigodot_{r=1}^{k-l} (x_{\pi(s_r)}^{i_{j_{s_r}}} \oplus x_{\rho(q_r)}^{i_{j_{s_r}}})$ and then take the minimum over a bigger set.

□

Lemma 4.10. *We can express the symmetrization of any min-plus monomial as a min-plus polynomial in the elementary symmetric polynomials* $e_1, \ldots, e_n$.

Proof. We prove the statement by induction on $\operatorname{Deg} p$.

If $\operatorname{Deg} p = 0$, then $p \equiv a = a(e_1, \ldots, e_n)$.

Suppose now that we can express all symmetric min-plus polynomials of the required form of total degree less than m as min-plus polynomials in $e_1, \ldots, e_n$.

Let

$$p(x_1, \ldots, x_n) = \operatorname{Sym}(a \odot x_1^{i_1} \odot \ldots \odot x_n^{i_n}) = a \odot \operatorname{Sym}(x_1^{i_1} \odot \ldots \odot x_n^{i_n}),$$

where $\operatorname{Deg} p = \deg \operatorname{Sym}(a \odot x_1^{i_1} \odot \ldots \odot x_n^{i_n}) = m$ and $i_1, \ldots, i_n$ are nonnegative.

Suppose exactly $i_{j_1}, i_{j_2}, \ldots, i_{j_k}$ are nonzero. By Lemma 4.9 we can write

$$p(x_1, \ldots, x_n) = a \odot e_k^b \odot \operatorname{Sym}(x_1^{i_{j_1}-b} \odot \ldots \odot x_k^{i_{j_k}-b})$$

where $b = \min(i_{j_1}, \ldots, i_{j_k}) > 0$.

Since

$$\operatorname{Deg} \operatorname{Sym}(x_1^{i_{j_1}-b} \odot \ldots \odot x_k^{i_{j_k}-b}) < m,$$

the claim follows by induction for polynomials of the specified form. $\square$

Proof of Theorem 4.7. Let p be a symmetric min-plus polynomial. By Lemma 4.8 we can write it as

$$p(x_1, \ldots, x_n) = \bigoplus_{1 \le s \le m} \operatorname{Sym}(a_s \odot x_1^{i_1^s} \odot \ldots \odot x_n^{i_n^s}).$$

By Lemma 4.10 each $\operatorname{Sym}(a_s \odot x_1^{i_1^s} \odot \ldots \odot x_n^{i_n^s})$ can be written as a min-plus polynomial in $e_1, \ldots, e_n$. Therefore so can p.

Let q be any symmetric tropical polynomial. We can write it as $q = \frac{qe_n^j}{e_n^j}$, where j is such an integer that qe_n^j is a symmetric min-plus polynomial. $\square$

A symmetric tropical polynomial p can be written in terms of elementary symmetric tropical polynomials in many ways. Therefore the uniqueness statement of the Fundamental Theorem of Symmetric Polynomials does not hold in the tropical setting. However, if we work with a particular tropical expression we can make an analogue claim.

Corollary 4.11 (Uniqueness)**.** *If we apply the algorithm used to prove Theorem 4.7 to the minimal representation of a symmetric tropical polynomial expression, then the polynomial expression in* $e_1, \ldots, e_n$ *and* e_n^{-1} *is also minimal.*

Proof. Let

$$p = \bigoplus_j \mathrm{Sym}(a_j \odot x_n^{i_n^j+\ldots+i_1^j-k} x_{n-1}^{i_n^j+\ldots+i_2^j-k} \odot \ldots \odot x_2^{i_n^j+i_{n-1}^j-k} \odot x_1^{i_n^j-k}),$$

be a minimal tropical polynomial expression in n variables, where $i_1^j, \ldots, i_n^j$ are all positive integers. Applying the algorithm to p produces

$$\bigoplus_j a_j \odot e_n^{i_n^j-k} \odot e_{n-1}^{i_{n-1}^j} \odot \ldots \odot e_1^{i_1^j}.$$

Assume now that this expression in $e_1, \ldots, e_n$ is not minimal. This means that a j_0 exists and for any $(x_1, \ldots, x_n)$ an l such that

$$(a_{j_0} \odot e_n^{i_n^{j_0}-k} \odot e_{n-1}^{i_{n-1}^{j_0}} \odot \ldots \odot e_1^{i_1^{j_0}})(x_1, \ldots, x_n) \geq (a_l \odot e_n^{i_n^l-k} \odot e_{n-1}^{i_{n-1}^l} \odot \ldots \odot e_1^{i_1^l})(x_1, \ldots, x_n).$$

Without loss of generality we may assume $x_1 \leq x_2 \leq \ldots \leq x_n$. This implies that a j_0 exists and for any $(x_1, \dots, x_n)$ an l so that

$$\begin{aligned} a_l \odot x_1^{i_n^l + i_{n-1}^l + \dots + i_1^l - k} \odot \dots \odot x_n^{i_n^l - k} &= a_l \odot (x_1 \odot \dots \odot x_n)^{i_n^l - k} \odot \dots \odot x_1^{i_1^l} \\ &= (a_l \odot e_n^{i_n^l - k} \odot e_{n-1}^{i_{n-1}^l} \odot \dots \odot e_1^{i_1^l})(x_1, \dots, x_n) \\ &\leq (a_{j_0} \odot e_n^{i_n^{j_0} - k} \odot e_{n-1}^{i_{n-1}^{j_0}} \odot \dots \odot e_1^{i_1^{j_0}})(x_1, \dots, x_n) \\ &= a_{j_0} \odot (x_1 \odot \dots \odot x_n)^{i_n^{j_0} - k} \odot \dots \odot x_1^{i_1^{j_0}} \\ &= a_{j_0} \odot x_1^{i_n^{j_0} + i_{n-1}^{j_0} + \dots + i_1^{j_0} - k} \odot \dots \odot x_n^{i_n^{j_0} - k}. \end{aligned}$$

It follows that the term $\mathrm{Sym}(a_{j_0} \odot x_n^{i_n^{j_0} + \dots + i_1^{j_0} - k} x_{n-1}^{i_n^{j_0} + \dots + i_2^{j_0} - k} \odot \dots \odot x_2^{i_n^{j_0} + i_{n-1}^{j_0} - k} \odot x_1^{i_n^{j_0} - k})$ in the original expression must have been redundant. This is a contradiction. $\square$

The following corollary is a tropical polynomial analogoue of the Fundamental Theorem of Symmetric Polynomials.

Corollary 4.12 (Fundamental Theorem of Symmetric Min-Plus Polynomials)**.** *Every symmetric min-plus polynomial in* $\mathtt{MinPlus}[x_1, x_2, \dots, x_n]$ *can be written as a min-plus polynomial in the elementary symmetric tropical polynomials* $e_1, \dots, e_n$*. If we apply the algorithm used to prove Theorem 4.7 to the minimal representation of a min-plus polynomial, then the expression in* $e_1, \dots, e_n$ *is also minimal.*

4.2 Symmetric Max-Plus Polynomials

Definition 4.13. *A max-plus polynomial* $p \in \mathtt{MaxPlus}[x_1, x_2, \dots, x_n]$ *is symmetric if*

$$p(x_1, \dots, x_n) = p(x_{\pi(1)}, \dots, x_{\pi(n)})$$

for every permutation $\pi \in S_n$*.*

We define symmetrization operator $\boxplus\mathrm{Sym}$ on $\mathtt{MaxPlus}[x_1, x_2, \dots, x_n]$ and elementary symmetric max-plus polynomials analogously to how it is done in Subsection 4.1, but with

$\oplus$ replaced by $\boxplus$:

$$\begin{aligned}\sigma_1 &= x_1 \boxplus \ldots \boxplus x_n,\\ &\vdots\\ \sigma_k &= {}_{\boxplus}\mathrm{Sym}(x_1 \odot \ldots \odot x_k),\\ &\vdots\\ \sigma_n &= x_1 \odot x_2 \odot \ldots \odot x_n.\end{aligned}$$

The proof of following theorem can be proved similarly as in the previous section.

Theorem 4.14 (Fundamental Theorem of Symmetric Max-Plus Polynomials)**.** *Every symmetric max-plus polynomial in* $\texttt{MaxPlus}[x_1, x_2, \ldots, x_n]$ *can be written as a max-plus polynomial in the elementary symmetric max-plus polynomials* $\sigma_1, \ldots, \sigma_n$*. If we apply the algorithm used to prove Theorem 4.7 to the minimal representation of a max-plus polynomial, then the expression in* $\sigma_1, \ldots, \sigma_n$ *is also minimal.*

4.3 Symmetric Rational Tropical Functions

Definition 4.15. *A rational tropical function* $r \in \texttt{RTrop}[x_1, x_2, \ldots, x_n]$ *is symmetric if*

$$r(x_1, \ldots, x_n) = r(x_{\pi(1)}, \ldots, x_{\pi(n)})$$

for all permutations $\pi \in S_n$*.*

We denote the algebra of symmetric rational tropical functions by $\texttt{RTrop}[x_1, x_2, \ldots, x_n]^{S_n}$. We can extend Sym to $\texttt{RTrop}[x_1, x_2, \ldots, x_n]$:

$$\begin{aligned}\mathrm{Sym}\colon\ \texttt{RTrop}[x_1, \ldots, x_n] &\to \texttt{RTrop}[x_1, \ldots, x_n]^{S_n}\\ r(x_1, \ldots, x_n) &\mapsto \bigoplus_{\pi \in S_n} r(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)}).\end{aligned}$$

The operator Sym is well-defined, additive, and commutes with tropical multiplication. A rational tropical function r is symmetric if and only if $\mathrm{Sym}(r) = r$.

Theorem 4.16. *Every symmetric rational tropical function function in* $\texttt{RTrop}[x_1, x_2, \ldots, x_n]$ *can be written as a rational tropical function in the elementary symmetric tropical polynomials* $e_1, \ldots, e_n$.

Proof. Any rational tropical function r may be written as

$$r = p \odot q^{-1},$$

where p and q are in $\texttt{Trop}[x_1, x_2, \ldots, x_n]$ and whose monomials all have nonnegative powers.

Let $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Since r is symmetric,

$$p(x_{\pi(1)}, \ldots, x_{\pi(n)}) \odot q(x_1, \ldots, x_n) = p(x_1, \ldots, x_n) \odot q(x_{\pi(1)}, \ldots, x_{\pi(n)})$$

for all $\pi \in S_n$. Tropically summing over $\pi \in S_n$ gives

$$\bigoplus_{\pi \in S_n} (p(x_{\pi(1)}, \ldots, x_{\pi(n)}) \odot q(x_1, \ldots, x_n)) = \bigoplus_{\pi \in S_n} (p(x_1, \ldots, x_n) \odot q(x_{\pi(1)}, \ldots, x_{\pi(n)})).$$

By distributivity,

$$(\bigoplus_{\pi \in S_n} p(x_{\pi(1)}, \ldots, x_{\pi(n)})) \odot q(x_1, \ldots, x_n) = p(x_1, \ldots, x_n) \odot (\bigoplus_{\pi \in S_n} q(x_{\pi(1)}, \ldots, x_{\pi(n)})).$$

Since this holds for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$\mathrm{Sym}(p) \odot q = p \odot \mathrm{Sym}(q),$$

and consequently

$$p \odot q^{-1} = \mathrm{Sym}(p) \odot \mathrm{Sym}(q)^{-1}.$$

By Theorem 4.7 $\mathrm{Sym}(q)$ and $\mathrm{Sym}(p)$ are tropical polynomials in $e_1, \ldots, e_n$. Consequently

r is a rational tropical function in $e_1, \ldots, e_n$. $\square$

Chapter 5

r-Symmetric Tropical Polynomials, Min-Plus Polynomials and Rational Functions

5.1 r-Symmetric Tropical Polynomials

A tropical polynomial in n variables is symmetric if it is invariant under the action of S_n that permutes the variables. We can generalize this definition as follows: a tropical polynomial in nr variables, divided into n blocks of r variables each, is r-symmetric if it is invariant under the action of S_n that permutes the blocks while preserving the order of the variables within each block.

We state the relevant results for the case when $r = 2$, but by induction we can prove similar statements for a general r (with $r = 2$ as the base case). We focus on $r = 2$ because persistence barcodes, persistence analogoues of Betti numbers, are collections of intervals. Each interval is given as a point (x, y) and represents a feature which is 'born' at x and which 'dies' at y. Since the order of intervals does not matter, we must identify functions symmetric with respect to the action of S_n on $(\mathbb{R}^2)^n$ that permutes pairs.

Fix n. Let the symmetric group S_n act on the matrix of indeterminates

$$X = \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \\ \vdots & \vdots \\ x_{n,1} & x_{n,2} \end{pmatrix}$$

by left multiplication. We want to find a generating set for the subset of $\mathtt{Trop}[x_{1,1}, x_{1,2}, \ldots, x_{n,2}]$ that is invariant under the action of S_n described above.

Definition 5.1. *A tropical polynomial $p \in \mathtt{Trop}[x_{1,1}, x_{1,2}, \ldots, x_{n,2}]$ is 2-symmetric if*

$$p(x_{1,1}, x_{1,2}, \ldots, x_{n,1}, x_{n,2}) = p(x_{\pi(1),1}, x_{\pi(1),2}, \ldots, x_{\pi(n),1}, x_{\pi(n),2})$$

for every permutation $\pi \in S_n$.

Definition 5.2. *A min-plus polynomial $p \in \mathtt{MinPlus}[x_{1,1}, x_{1,2}, \ldots, x_{n,2}]$ is 2-symmetric if*

$$p(x_{1,1}, x_{1,2}, \ldots, x_{n,1}, x_{n,2}) = p(x_{\pi(1),1}, x_{\pi(1),2}, \ldots, x_{\pi(n),1}, x_{\pi(n),2})$$

for every permutation $\pi \in S_n$.

Given a tropical monomial in the variables $x_{1,1}, x_{1,2}, \ldots, x_{n,2}$, we construct its *exponent matrix* from the matrix X by replacing each variable by its exponent.

Example 5.3. *Let $n = 2$. The exponent matrix of $x_{1,1} \odot x_{2,2}$ is*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We define the *symmetrization* map with respect to the row permutation action of S_n:

$$\begin{aligned}\texttt{Trop}[x_{1,1}, x_{1,2}, \ldots, x_{n,2}] &\to \texttt{Trop}[x_{1,1}, x_{1,2}, \ldots, x_{n,2}]^{S_n} \\ p(x_{1,1}, x_{1,2}, \ldots, x_{n,1}, x_{n,2}) &\mapsto \bigoplus\nolimits_{\pi \in S_n} p(x_{\pi(1),1}, x_{\pi(1),2}, \ldots, x_{\pi(n),1}, x_{\pi(n),2}).\end{aligned}$$

We denote this map by Sym_2.

Example 5.4. *Let $n = 2$. The symmetrization of $x_{1,1} \odot x_{2,2}$ is*

$$\mathrm{Sym}_2(x_{1,1} \odot x_{2,2}) = x_{1,1} \odot x_{2,2} \oplus x_{2,1} \odot x_{1,2}.$$

Proposition 5.5. *Let $p(x_{1,1}, \ldots, x_{n,2}), q(x_{1,1}, \ldots, x_{n,2}) \in \texttt{Trop}[x_{1,1}, \ldots, x_{n,2}]$. Then:*

1. $\mathrm{Sym}_2(p \oplus q)(x_{1,1}, \ldots, x_{n,2}) = \mathrm{Sym}_2(p)(x_{1,1}, \ldots, x_{n,2}) \oplus \mathrm{Sym}_2(q)(x_{1,1}, \ldots, x_{n,2})$,
2. $a \odot \mathrm{Sym}_2(p)(x_{1,1}, \ldots, x_{n,2}) = \mathrm{Sym}_2(a \odot p)(x_{1,1}, \ldots, x_{n,2})$.
3. *p is 2-symmetric* $\Leftrightarrow \mathrm{Sym}_2(p) = p$.

Proof. This proof is similar to the proofs of Propositions 4.3 and 4.4. $\square$

We want to identify an equivalent of elementary symmetric tropical polynomials in this setting. Let

$$\mathscr{E}_n = \left\{ \begin{pmatrix} e_{1,1} & e_{1,2} \\ e_{2,1} & e_{2,2} \\ \vdots & \vdots \\ e_{n,1} & e_{n,2} \end{pmatrix} \neq [0]_n^2 \;\middle|\; e_{i,j} \in \{0,1\} \text{ for } i = 1, 2, \ldots, n, \text{ and } j = 1, 2 \right\}.$$

Each matrix $E \in \mathscr{E}_n$ determines a tropical monomial $P(E)$.

Example 5.6. *Let $n = 3$. If*

$$E = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

then $P(E) = x_{1,1} \odot x_{2,1} \odot x_{3,2}$.

We denote the set of orbits under the row permutation action on $\mathscr{E}_n$ by $\mathscr{E}_n/S_n$. Each orbit $\{E_1, E_2, \ldots E_m\}$ determines a 2-symmetric tropical polynomial

$$P(E_1) \oplus P(E_2) \oplus \ldots \oplus P(E_m).$$

Definition 5.7. *We call 2-symmetric tropical polynomials that arise from orbits $\mathscr{E}_n/S_n$ elementary. We let $e_{(e_{1,1},e_{1,2}),\ldots,(e_{n,1},e_{n,2})}$ denote the tropical polynomial that arises from the orbit*

$$\left[\begin{pmatrix} e_{1,1} & e_{1,2} \\ e_{2,1} & e_{2,2} \\ \vdots & \vdots \\ e_{n,1} & e_{n,2} \end{pmatrix}\right].$$

Example 5.8. *Let $n = 2$. The set of orbits under the S_2 action is*

$$\mathscr{E}_2/S_2 = \left\{ \begin{array}{c} \left[\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right], \left[\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\right], \left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right], \left[\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\right], \\ \\ \left[\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\right] \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right], \left[\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\right], \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right], \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] \end{array} \right\}.$$

A few examples of elementary 2-symmetric tropical polynomials are:

$$\begin{aligned}
e_{[(1,1)(1,1)]} &= x_{1,1} \odot x_{1,2} \odot x_{2,2} \odot x_{2,1}, \\
e_{[(1,0)(1,1)]} &= x_{1,1} \odot x_{2,1} \odot x_{2,2} \oplus x_{1,1} \odot x_{1,2} \odot x_{2,1}, \\
e_{[(1,1)(0,1)]} &= x_{1,1} \odot x_{1,2} \odot x_{2,2} \oplus x_{1,2} \odot x_{2,1} \odot x_{2,2}, \\
e_{[(1,0)(0,0)]} &= x_{1,1} \oplus x_{2,1}.
\end{aligned}$$

For simplicity we write $e_{[(1,0)]}$ *instead* $e_{[(1,0)(0,0)]}$ *when* n *is clear from the context. Similarly* $e_{[(1,1)^2]}$ *represents* $e_{[(1,1)(1,1)]}$.

Now we show that elementary 2-symmetric tropical polynomials give coordinates on $\mathbb{R}^{2n}/S_n$.

Theorem 5.9. *Let* $[(x_1, y_1), \ldots, (x_n, y_n)]$ *and* $[(x'_1, y'_1), \ldots, (x'_n, y'_n)]$ *be two orbits under the row permutation action on* $\mathbb{R}^{2n}$. *If*

$$e([(x_1, y_1), \ldots, (x_n, y_n)]) = e([(x'_1, y'_1), \ldots, (x'_n, y'_n)])$$

for all elementary 2-symmetric tropical polynomials e, *then*

$$[(x_1, y_1), \ldots, (x_n, y_n)] = [(x'_1, y'_1), \ldots, (x'_n, y'_n)].$$

Proof. Suppose that $x_1 \leq x_2 \leq \ldots \leq x_n$, $x'_1 \leq x'_2 \leq \ldots \leq x'_n$, $y_1 \leq y_2 \leq \ldots \leq y_n$ and $y'_1 \leq y'_2 \leq \ldots \leq y'_n$.

Let $[(x_1, y_{\pi(1)}), \ldots, (x_n, y_{\pi(n)})]$ and $[(x'_1, y'_{\rho(1)}), \ldots, (x'_n, y'_{\rho(n)})]$ be two orbits under the row permutation action on $\mathbb{R}^{2n}$ that satisfy

$$e([(x_1, y_{\pi(1)}), \ldots, (x_n, y_{\pi(n)})]) = e([(x'_1, y'_{\rho(1)}), \ldots, (x'_n, y'_{\rho(n)})])$$

for all elementary 2-symmetric tropical polynomials.

Applying $e_{[(1,0)]}$, we get $x_1 = x_1'$. Applying $e_{[(1,0),(1,0)]}$, we get

$$x_1 + x_2 = x_1' + x_2'$$

and from here $x_2 = x_2'$ and so on. Finally, applying $e_{[(1,0)^n]}$ yields $x_n = x_n'$.

We use a similar argument using $e_{[(0,1)]}, e_{[(0,1),(0,1)]}, \ldots, e_{[(0,1)^n]}$ to show that

$$y_1 = y_1', y_2 = y_2', \ldots, y_n = y_n'.$$

Given evaluations of elementary 2-symmetric tropical polynomials on $[(x_1, y_{\pi(1)}), \ldots, (x_n, y_{\pi(n)})]$, we must prove that the permutation ρ on the set of pairs $[(x_1, y_{\rho(1)}), \ldots, (x_n, y_{\rho(n)})]$ equals π. We prove this by induction.

First we show that $y_{\pi(1)} = y_{\rho(1)}$.

Let $I = \{y_k \in \{y_1, y_2, \ldots, y_n\} \mid y_k < y_{\pi(1)}\}$. We evaluate $e_{[(1,1)(0,1)^{|I|}]}$ where $|I|$ is the cardinality of I.

The following inequalities hold

$$\begin{aligned}
&x_1 + y_{\pi(1)} + \textstyle\sum_{y_k \in I} y_k \leq x_m + y_{\pi(m)} + (\sum_{y_k \in I} y_k - y_{\pi(m)} + y_{\pi(1)}) && \text{for } y_{\pi(m)} < y_{\pi(1)} \\
&x_1 + y_{\pi(1)} + \textstyle\sum_{y_k \in I} y_k \leq x_m + y_{\pi(m)} + (\sum_{y_k \in I} y_k) && \text{for } y_{\pi(1)} \leq y_{\pi(m)}
\end{aligned}$$

since $x_1 \leq x_2 \leq \ldots \leq x_n$.

It follows from here that

$$e_{[(1,1)(0,1)^{|I|}]}([(x_1, y_{\pi(1)}), \ldots, (x_n, y_{\pi(n)})]) = x_1 + y_{\pi(1)} + \sum_{y_k \in I} y_k.$$

Now we note that $e_{[(1,1)(0,1)^{|I|}]}([(x_1, y_{\rho(1)}), \ldots, (x_n, y_{\rho(n)})])$ equals

$$\min\Big(\min_{\{k \,|\, y_{\rho(1)} \in I\}} (x_k + \sum_{y_k \in I} y_k + y_{\pi(1)}), \min_{\{k \,|\, y_{\rho(1)} \notin I\}} (x_k + \sum_{y_k \in I} y_k + y_{\rho(1)})\Big).$$

For this minimum to equal $x_1 + y_{\pi(1)} + \sum_{y_k \in I} y_k$, we must have $y_{\rho(1)} \leq y_{\pi(1)}$.

A similar argument using $J = \{y_k \in \{y_1, y_2, \ldots, y_n\} \mid y_k < y_{\rho(1)}\}$ and evaluating $e_{[(1,1)(0,1)^{|J|}]}$ shows that $y_{\pi(1)} \leq y_{\rho(1)}$.

We conclude that $y_{\pi(1)} = y_{\rho(1)}$.

Now suppose that $y_{\pi(s)} = y_{\rho(s)}$ for all $s < m$. We want to show that $y_{\pi(m)} = y_{\rho(m)}$. Let us first suppose that $\pi(i) < \pi(m)$ for $i < m$. Let

$$I = \{y_k \in \{y_1, y_2, \ldots, y_n\} \mid y_k < y_{\pi(m)}\}.$$

We evaluate at $e_{[(1,1)^{m-1}(0,1)^{|I|-m+2}]}$ where $|I|$ is the cardinality of I and conclude that $y_{\rho(m)} \leq y_{\pi(m)}$.

Let $\{i_1, \ldots, i_{m-1}\}$ be a permutation of $\{\pi(1), \ldots, \pi(m-1)\}$ for which $i_1 \leq i_2 \leq \ldots \leq i_{m-1}$.

Now suppose that $i_{(m-2)} < \pi(m) < i_{(m-1)}$. Let

$$I = \{y_k \in \{y_1, y_2, \ldots, y_n\} \mid y_k < y_{\pi(m)}\}.$$

Let s be such that $i_{(m-2)} = \pi(s)$. We evaluate at $e_{[(1,0)(1,1)^{m-2}(0,1)^{|I|-m+3}]}$ where $|I|$ is the cardinality of I and conclude that then $y_{\rho(m)} \leq y_{\pi(m)}$.

We use a similar argument in the following cases

$$i_{(m-3)} < \pi(m) < i_{(m-2)}, i_{(m-4)} < \pi(m) < i_{(m-3)}, \ldots, i_1 < \pi(m) < i_2$$

and conclude that $y_{\rho(m)} \leq y_{\pi(m)}$ in each.

Finally, let $\pi(m) < i_1$ and let $I = \{y_k \in \{y_1, y_2, \ldots, y_n\} \mid y_k < y_{\pi(m)}\}$. Applying $e_{[(1,0)^{m-1}(1,1)(0,1)^{|I|}]}$ we deduce that $y_{\rho(m)} \leq y_{\pi(m)}$.

So in all cases $y_{\rho(m)} \leq y_{\pi(m)}$. A similar argument holds for the permutation ρ. It follows that $y_{\pi(m)} = y_{\rho(m)}$. $\square$

Unfortunately an equivalent of Theorem 4.7 does not hold for r-symmetric tropical polynomials. We show that when $r = 2$ and $n = 2$ no finite set of generators exists for the 2-symmetric tropical polynomials.

Proposition 5.10. $\texttt{Trop}[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}]^{S_2}$ *is not finitely generated.*

Proof. Suppose a finite set of monomials, $\{x_{1,1}^{i_{1,1}} \odot x_{1,2}^{i_{1,2}} \odot x_{2,1}^{i_{2,1}} \odot x_{2,2}^{i_{2,2}}\}_{i \in I}$, exists such that

$$\mathrm{Sym}_2(I) = \{\mathrm{Sym}_2(x_{1,1}^{i_{1,1}} \odot x_{1,2}^{i_{1,2}} \odot x_{2,1}^{i_{2,1}} \odot x_{2,2}^{i_{2,2}})\}_{i \in I}$$

generates $\texttt{Trop}[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}]^{S_2}$. Since I is finite, an integer $d \leq 1$ exists such that

$$0 \leq i_{1,1}, i_{1,2}, i_{2,1}, i_{2,2} \leq d - 1$$

for all $i \in I$.

We show by contradiction that $x_{1,1}^d \odot x_{1,2} \oplus x_{2,1}^d \odot x_{2,2}$ is not generated by

$$G = \{\mathrm{Sym}_2(x_{1,1}^{j_{1,1}} \odot x_{1,2}^{j_{1,2}} \odot x_{2,1}^{j_{2,1}} \odot x_{2,2}^{j_{2,2}})\}_{j_{1,1}, j_{1,2}, j_{2,1}, j_{2,2} \leq d-1},$$

and consequently not by $\mathrm{Sym}_2(I)$.

First note that this tropical polynomial expression representing the function is minimal. Suppose $x_{1,1}^d \odot x_{1,2} \oplus x_{2,1}^d \odot x_{2,2}$ can be generated by elements of G; that is, can be represented by a tropical expression P, which is a tropical sum of tropical products of the elements from G (we can expand any expression to one of this form using distributivity). The term $x_{1,1}^d \odot x_{1,2}$ only appears in P if P contains a tropical product of symmetrizations of monomials $S_1 S_2 \dots S_n$, $S_i \in G$, where for some a, $0 \leq a < d$,

$$S_1 = \mathrm{Sym}_2(x_{1,1}^a \odot x_{1,2}) = x_{1,1}^a \odot x_{1,2} \oplus x_{2,1}^a \odot x_{2,2}, \text{ and } S_j = \mathrm{Sym}_2(x_{1,1}^{a_j}) = x_{1,1}^{a_j} \oplus x_{2,1}^{a_j}$$

for other j such that $\sum_{j=2}^n a_j = d - a$ (this must hold because P is functionally equivalent to $x_{1,1}^d \odot x_{1,2} \oplus x_{2,1}^d \odot x_{2,2}$).

Expression P contains $x_{2,1}^d \odot x_{2,2}$ by symmetry. But it also contains, for example, $x_{1,1}^a \odot x_{1,2} \odot x_{2,1}^{d-a}$. By assumption P is functionally equivalent to $x_{1,1}^d \odot x_{1,2} \oplus x_{2,1}^d \odot x_{2,2}$, so $x_{1,1}^d \odot x_{1,2} \oplus x_{2,1}^d \odot x_{2,2}$ is the minimal expression for P as argued earlier. This implies that

$$x_{1,1}^a \odot x_{1,2} \odot x_{2,1}^{d-a} \geq x_{1,1}^d \odot x_{1,2} \oplus x_{2,1}^d \odot x_{2,2}$$

for all $x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2} \in \mathbb{R}$. This is equivalent to

$$0 \geq \min\{(d-a)(x_{1,1} - x_{2,1}), a(x_{2,1} - x_{1,1}) + x_{2,2} - x_{1,2}\}$$

at all points, which clearly does not hold. This contradicts our initial assumption and shows that the minimal form of P cannot be $x_{1,1}^d \odot x_{1,2} \oplus x_{2,1}^d \odot x_{2,2}$ and in turn implies that $x_{1,1}^d \odot x_{1,2} \oplus x_{2,1}^d \odot x_{2,2}$ cannot be generated by G. $\square$

5.2 r-Symmetric Max-Plus Polynomials

A max-plus polynomial in nr variables, divided into n blocks of r variables each, is r-symmetric if it is invariant under the action of S_n that permutes the blocks while preserving the order of the variables within each block. We define 2-symmetric max-plus polynomials in Section 5.1 with $\oplus$ replaced by $\boxplus$ and all results from that section hold in this setting.

Definition 5.11. *We call 2-symmetric max-plus polynomials that arise from orbits $\mathscr{E}_n/S_n$ elementary. We let $\sigma_{(e_{1,1},e_{1,2}),\ldots,(e_{n,1},e_{n,2})}$ denote the tropical polynomial that arises from the orbit*

$$\left[\begin{pmatrix} e_{1,1} & e_{1,2} \\ e_{2,1} & e_{2,2} \\ \vdots & \vdots \\ e_{n,1} & e_{n,2} \end{pmatrix}\right].$$

5.3 r-Symmetric Rational Tropical Functions

Definition 5.12. *A rational tropical function $r \in \texttt{RTrop}[x_{1,1}, x_{1,2}, \ldots, x_{n,1}, x_{n,2}]$ is 2-symmetric if*

$$r(x_{1,1}, x_{1,2}, \ldots, x_{n,1}, x_{n,2}) = r(x_{\pi(1),1}, x_{\pi(1),2}, \ldots, x_{\pi(n),1}, x_{\pi(n),2})$$

for all permutations $\pi \in S_n$.

We denote the algebra of 2-symmetric rational tropical functions by $\texttt{RTrop}[x_{1,1}, \ldots, x_{n,2}]^{S_n}$. We can extend Sym_2 to $\texttt{RTrop}[x_{1,1}, \ldots, x_{n,2}]$.

$$\begin{aligned} \mathrm{Sym}_2\colon\ \texttt{RTrop}[x_{1,1}, \ldots, x_{n,2}] &\to \texttt{RTrop}[x_{1,1}, \ldots, x_{n,2}]^{S_n} \\ r(x_{1,1}, x_{1,2}, \ldots, x_{n,1}, x_{n,2}) &\mapsto \bigoplus\nolimits_{\pi \in S_n} r(x_{\pi(1),1}, x_{\pi(1),2}, \ldots, x_{\pi(n),1}, x_{\pi(n),2}). \end{aligned}$$

Operator Sym_2 is well-defined, additive, and commutes with tropical multiplication. A rational tropical function r is 2-symmetric if and only if $\mathrm{Sym}_2(r) = r$.

Theorem 5.13. *Every symmetric rational tropical function in $\texttt{RTrop}[x_{1,1}, \ldots, x_{n,2}]$ can be written as a rational tropical function in the elementary 2-symmetric tropical polynomials.*

We prove this by induction with respect to a special order on min-plus monomials. Note that any such monomial $x_{1,1}^{j_{1,1}} \odot x_{1,2}^{j_{1,2}} \odot \ldots \odot x_{n,1}^{j_{n,1}} \odot x_{n,2}^{j_{n,2}}$ may be represented by a 2n-tuple $(j_{1,1}, j_{1,2}, \ldots, j_{n,1}, j_{n,2}) \in \mathbb{Z}_{\geq 0}^{2n}$. The number of nonzero entries in such a $2n$-tuple is a measure of how 'spread out' a monomial is:

$$S(x_{1,1}^{j_{1,1}} \odot x_{1,2}^{j_{1,2}} \odot \ldots \odot x_{n,1}^{j_{n,1}} \odot x_{n,2}^{j_{n,2}}) = |\,\{(i,k) \in \{1,2,\ldots,n\} \times \{1,2\} \,|\, j_{i,k} \neq 0\}\,|.$$

Let

$$m_1 = x_{1,1}^{j_{1,1}} \odot x_{1,2}^{j_{1,2}} \odot \ldots \odot x_{n,1}^{j_{n,1}} \odot x_{n,2}^{j_{n,2}}$$

and

$$m_2 = x_{1,1}^{s_{1,1}} \odot x_{1,2}^{s_{1,2}} \odot \ldots \odot x_{n,1}^{s_{n,1}} \odot x_{n,2}^{s_{n,2}}.$$

Notice that $\mathrm{Deg}(m_1) = \sum_{i=1}^{n}(j_{i,1} + j_{i,2})$ and $\mathrm{Deg}(m_2) = \sum_{i=1}^{n}(s_{i,1} + s_{i,2})$. Then we define order $>_S$ as follows:

$$\begin{aligned} m_1 >_S m_2 \quad \Leftrightarrow \quad & \mathrm{Deg}(m_1) > \mathrm{Deg}(m_2) \\ \text{or} \quad & \mathrm{Deg}(m_1) = \mathrm{Deg}(m_2), S(m_1) < S(m_2) \\ \text{or} \quad & \mathrm{Deg}(m_1) = \mathrm{Deg}(m_2), S(m_1) = S(m_2), m_1 >_{lex} m_2. \end{aligned}$$

Example 5.14. *Let*

$$m_1 = x_{1,1} \odot x_{1,2} \odot x_{2,1},$$

and

$$m_2 = x_{1,1} \odot x_{1,2}^2.$$

We have $\mathrm{Deg}(m_1) = \mathrm{Deg}(m_2) = 3$*,* $S(m_1) = 3$*,* $S(m_2) = 2$*. Since* $S(m_2) < S(m_1)$*, we have* $m_2 >_S m_1$*.*

Proposition 5.15. *Relation* $>_S$ *is a well-ordering on the set of min-plus monomials.*

Lemma 5.16. *The 2-symmetrization of any min-plus monomial can be written as a rational tropical function in the elementary symmetric polynomials.*

Proof. We prove the statement by induction on the set of min-plus monomials well-ordered by $>_S$. Monomial $x_{1,1}^0 \odot x_{2,1}^0 \odot \ldots \odot x_{1,n}^0 \odot x_{2,n}^0 = 0$ and as such is a rational function of 2-symmetric elementary polynomials. By definition the symmetrization of a monomial of degree 1 is a 2-symmetric elementary polynomial.

Let us suppose that this statement holds for all monomials $x_{1,1}^{s_{1,1}} \odot \ldots \odot x_{n,2}^{s_{n,2}}$ with

$$x_{1,1}^{s_{1,1}} \odot \ldots \odot x_{n,2}^{s_{n,2}} <_S x_{1,1}^{i_{1,1}} \odot \ldots \odot x_{n,2}^{i_{n,2}}.$$

We must show that it holds for $x_{1,1}^{i_{1,1}} \odot \ldots \odot x_{n,2}^{i_{n,2}}$ and then the statement will follow by induction. If $i_{1,1}, \ldots, i_{n,2} \leq 1$, then $\mathrm{Sym}_2(x_{1,1}^{i_{1,1}} \odot \ldots \odot x_{n,2}^{i_{n,2}})$ is an elementary symmetric polynomial by definition.

Otherwise, let us suppose that exactly $i_{j_1^1,j_2^1}, i_{j_1^2,j_2^2}, \ldots, i_{j_1^k,j_2^k}$ among $i_{1,1}, \ldots, i_{n,2}$ are positive. Let $a = \min(i_{j_1^1,j_2^1}, i_{j_1^2,j_2^2}, \ldots, i_{j_1^k,j_2^k})$ and

$$e = \mathrm{Sym}_2(x_{j_1^1,j_2^1} \odot \ldots \odot x_{j_1^k,j_2^k}).$$

We observe

$$e^a \odot \mathrm{Sym}_2(x_{j_1^1,j_2^1}^{i_{j_1^1,j_2^1}-a} \odot \ldots \odot x_{j_1^k,j_2^k}^{i_{j_1^k,j_2^k}-a}).$$

Since the Freshman's Dream holds in tropical arithmetic, the expression on the left equals

$$(\bigoplus_{\rho \in S_n} x^a_{\rho(j_1^1),j_2^1} \odot \ldots \odot x^a_{\rho(j_1^k),j_2^k}) \odot (\bigoplus_{\pi \in S_n} x_{\pi(j_1^1),j_2^1}^{i_{j_1^1,j_2^1}-a} \odot \ldots \odot x_{\pi(j_1^k),j_2^k}^{i_{j_1^k,j_2^k}-a}).$$

By distributivity, we can rewrite it as

$$\bigoplus_{\rho \in S_n} \bigoplus_{\pi \in S_n} x^a_{\rho(j_1^1),j_2^1} \odot \ldots \odot x^a_{\rho(j_1^k),j_2^k} \odot x_{\pi(j_1^1),j_2^1}^{i_{j_1^1,j_2^1}-a} \odot \ldots \odot x_{\pi(j_1^k),j_2^k}^{i_{j_1^1,j_2^1}-a}.$$

By commutativity and Freshman's Dream this equals

$$\mathrm{Sym}_2(x_{j_1^1,j_2^1}^{i_{j_1^1,j_2^1}} \odot \ldots \odot x_{j_1^k,j_2^k}^{i_{j_1^k,j_2^k}}) \oplus \bigoplus_{\rho \in S_n} \bigoplus_{\pi \in S_n^\rho} x^a_{\rho(j_1^1),j_2^1} \odot \ldots \odot x^a_{\rho(j_1^k),j_2^k} \odot x_{\pi(j_1^1),j_2^1}^{i_{j_1^1,j_2^1}-a} \odot \ldots \odot x_{\pi(j_1^k),j_2^k}^{i_{j_1^1,j_2^1}-a}.$$

Here $S_n^\rho = \{\pi \in S_n \mid r \in \{1, \ldots, k\}$ exists such that $(\pi(j_1^r), j_2^r) \notin \{(\rho(j_1^s), j_2^s)\}_{s=1}^k\}$.

We denote $x^a_{\rho(j_1^1),j_2^1} \odot \ldots \odot x^a_{\rho(j_1^k),j_2^k} \odot x_{\pi(j_1^1),j_2^1}^{i_{j_1^1,j_2^1}-a} \odot \ldots \odot x_{\pi(j_1^k),j_2^k}^{i_{j_1^1,j_2^1}-a}$ by $p_{\rho,\pi}$ and write:

$$\mathrm{Sym}_2(x_{j_1^1,j_2^1}^{i_{j_1^1,j_2^1}} \odot \ldots \odot x_{j_1^k,j_2^k}^{i_{j_1^k,j_2^k}}) = e^a \odot \mathrm{Sym}_2(x_{j_1^1,j_2^1}^{i_{j_1^1,j_2^1}-a} \odot \ldots \odot x_{j_1^k,j_2^k}^{i_{j_1^k,j_2^k}-a}) \odot (\bigoplus_{\rho \in S_n} \bigoplus_{\pi \in S_n^\rho} p_{\rho,\pi})^{-1}.$$

Since $\bigoplus_{\rho\in S_n}\bigoplus_{\pi\in S_n^\rho} p_{\rho,\pi}$ is symmetric and Sym_2 is additive,

$$\bigoplus_{\rho\in S_n}\bigoplus_{\pi\in S_n^\rho} p_{\rho,\pi} = \mathrm{Sym}_2(\bigoplus_{\rho\in S_n}\bigoplus_{\pi\in S_n^\rho} p_{\rho,\pi}) = \bigoplus_{\rho\in S_n}\bigoplus_{\pi\in S_n^\rho} \mathrm{Sym}_2(p_{\rho,\pi}).$$

Observe that

$$p_{\rho,\pi} <_S x_{j_1^1,j_2^1}^{i_{j_1^1,j_2^1}} \odot \ldots \odot x_{j_1^k,j_2^k}^{i_{j_1^k,j_2^k}}$$

for every $\pi \in S_n^\rho$ and $\rho \in S_n$. This holds since $\mathrm{Deg}\, p_{\rho,\pi} = \mathrm{Deg}\, x_{j_1^1,j_2^1}^{i_{j_1^1,j_2^1}} \odot \ldots \odot x_{j_1^k,j_2^k}^{i_{j_1^k,j_2^k}}$ and because by definition S_n^ρ contains π for which such r exists that $(\pi(j_1^r), j_2^r) \notin \{(\rho(j_1^s), j_2^s)\}_{s=1}^k$ and therefore $S(p_{\rho,\pi}) \geq S(x_{j_1^1,j_2^1}^{i_{j_1^1,j_2^1}} \odot \ldots \odot x_{j_1^k,j_2^k}^{i_{j_1^k,j_2^k}}) + 1$.

Moreover,

$$x_{j_1^1,j_2^1}^{i_{j_1^1,j_2^1}-a} \odot \ldots \odot x_{j_1^k,j_2^k}^{i_{j_1^k,j_2^k}-a} <_S x_{j_1^1,j_2^1}^{i_{j_1^1,j_2^1}} \odot \ldots \odot x_{j_1^k,j_2^k}^{i_{j_1^k,j_2^k}},$$

so by the inductive hypothesis $\mathrm{Sym}_2(x_{j_1^1,j_2^1}^{i_{j_1^1,j_2^1}-a} \odot \ldots \odot x_{j_1^k,j_2^k}^{i_{j_1^k,j_2^k}-a})$ can be written as a rational tropical function in 2-symmetric elementary tropical polynomials.

By induction $\mathrm{Sym}_2(p_{\rho,\pi})$ are rational tropical functions in 2-symmetric elementary tropical polynomials.

Since we can express $\mathrm{Sym}_2(x_{j_1^1,j_2^1}^{i_{j_1^1,j_2^1}} \odot \ldots \odot x_{j_1^k,j_2^k}^{i_{j_1^k,j_2^k}})$ as a rational tropical function in 2-symmetric elementary tropical polynomials, the proof is complete. □

Proof of Theorem 5.13. Any rational tropical function r may be written as

$$r(x_{1,1},\ldots,x_{n,2}) = p(x_{1,1},\ldots,x_{n,2}) \odot q(x_{1,1},\ldots,x_{n,2})^{-1},$$

where p and q are 2-symmetric tropical polynomials. It follows from Lemma 5.16 that 2-symmetrization of any min-plus polynomial can be written as a rational tropical function in the elementary symmetric polynomials (using additivity of Sym_2). Consequently, $\mathrm{Sym}_2(p(x_{1,1},\ldots,x_{n,2}))$ and $\mathrm{Sym}_2(q(x_{1,1},\ldots,x_{n,2}))$ are rational tropical functions in elementary 2-symmetric tropical polynomials as is their tropical quotient $r(x_{1,1},\ldots,x_{n,2})$. □

Chapter 6

Max-Plus, Min-Plus Type Coordinates

In this chapter we identify max-plus polynomials that we can use as coordinates on the barcode space and prove that they are stable with respect to the bottleneck and Wasserstein distances.

We work with a slightly modified definition of the barcode space. In Chapter 2 we encode a barcode with n intervals as a vector $(x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$, where x_i denotes the left endpoint of the i-th interval and y_i the right endpoint. Here we represent the same barcode as $(x_1, d_1, x_2, d_2, \ldots, x_n, d_n)$ where x_i is the left endpoint of the i-th interval and d_i its length. A space of barcodes with n intervals is the orbit space of S_n on the product $([0, \infty) \times [0, \infty))^n$ given by permuting the coordinates. We denote it by B'_n. The barcode space B' is the quotient

$$\coprod_n B'_n/\sim,$$

where $\sim$ is generated by equivalences of the form

$$[(x_1, d_1), (x_2, d_2), \ldots, (x_n, d_n)] \sim [(x_1, d_1), (x_2, d_2), \ldots, (x_{n-1}, d_{n-1})],$$

whenever $d_n = 0$.

6.1 Max-Plus Polynomials on the Barcode Space

In this section we find all max-plus polynomials that we can use as coordinates on the barcode space and prove that they are stable with respect to the bottleneck and Wasserstein distances.

The first step is to identify max-plus polynomials on the the image of $B'_n \to B'$, B_n. This is the set obtained from B'_n by taking the quotient of the following equivalence relation: two multisets of n intervals each,

$$I = \{(x_1, d_1), (x_2, d_2), \ldots, (x_n, d_n)\} \text{ and } J = \{(x_1, d_1), (x_2, d_2), \ldots, (x_n, d_n)\}$$

are equivalent if such subsets $A, B \subseteq \{1, \ldots, n\}$ exist, that for all $\alpha \in A$, $\beta \in B$ multisets of intervals $I \setminus \{(x_\alpha, 0)\}$ and $J \setminus \{(x_\beta, 0)\}$ are identical.

If $\mathcal{W}_i \subseteq ([0, \infty) \times [0, \infty))^n$ is the subset of n-tuples of pairs $(x_1, d_1, x_2, d_2, \ldots, x_n, d_n)$, which satisfy $d_i = 0$, then these functions are precisely the 2-symmetric max-plus polynomials whose restriction to $\mathcal{W}_i$ is independent of x_i for all i.

Lemma 6.1. *Let the minimal representation of a max-plus polynomial $p(x_1, d_1, \ldots, x_n, d_n)$ be*

$$\boxplus_{i=1\ldots,m} a_0^i \odot x_1^{a_1^i} \odot d_1^{b_1^i} \odot \ldots \odot x_n^{a_n^i} \odot d_n^{b_n^i}.$$

Then p restricted to $\mathcal{W}_j$ is independent of x_j if and only if $a_j^i = 0$ for all $i = 1, \ldots, m$.

Proof. The direction $(\Leftarrow)$ follows immediately. We must show $(\Rightarrow)$. Suppose not all a_j^i are 0. Let I be the set of indices i for which

$$|a_j^i| = \max\{|a_j^i|\,;\, i = 1, \ldots, m\}.$$

Let $i_0 \in I$. If $a_j^{i_0} > 0$, then $p(0,\dots,0,x_j,0,\dots,0) = a_0^{i_0} \odot x_j^{a_j^{i_0}}$ for x_j big. In this case $p(x_1,d_1,\dots,x_n,d_n)$ depends on x_i. By assumption $a_j^{i_0} \geq 0$. This implies that the only way the expression does not depend on x_i is if $a_j^{i_0} = 0$. $\square$

Corollary 6.2. *The subsemiring of max-plus polynomials whose restriction to $\mathcal{W}_i$ is independent of x_i for all i contains precisely the max-plus polynomials of the form*

$$\boxplus_{i=1\dots,m} a_0^i \odot d_1^{b_1^i} \odot \dots \odot d_n^{b_n^i}.$$

We denote this semiring by D_n.

Proposition 6.3. *Let $D_n^{S_n}$ denote the subring of elements of D_n which are invariant under the action of S_n. Then $\sigma_{[(0,1)]}, \sigma_{[(0,1)^2]}, \dots, \sigma_{[(0,1)^n]}$ generate D^{S_n}, in the sense that any element of D^{S_n} is of the form*

$$\boxplus_{i=1\dots,m} a_0^i \odot \sigma_{[(0,1)]}^{b_1^i} \odot \dots \odot \sigma_{[(0,1)^n]}^{b_n^i},$$

where $a_0^i \in \mathbb{R}$ and all b_j^i nonnegative integers.

Proof. The statement follows from Theorem 4.12. $\square$

Now that we have identified appropriate functions for each B_n separately, we must assemble them to get functions on the barcode space. When $n \geq m$, the natural inclusion

$$\begin{array}{rcl} B_m & \to & B_n \\ \{(x_1,d_1),\dots,(x_m,d_m)\} & \mapsto & \{(x_1,d_1),\dots,(x_m,d_m),(0,0),\dots,(0,0)\} \end{array}$$

induces $i_{n,m}\colon D_n \to D_m$, defined by

$$i_{n,m}(f)((x_1,d_1),\dots,(x_m,d_m)) = f((x_1,d_1),\dots,(x_m,d_m),(0,0),\dots,(0,0)),$$

The map $i_{n,m}$ is S_m-equivariant (S_m acts by permuting the first m pairs of variables). It follows that we may construct composites

$$i_n^m \colon D_n^{S_n} \hookrightarrow D_n^{S_m} \xrightarrow{i_{n,m}^{S_m}} D_m^{S_m}$$

and an inverse system

$$\ldots \xrightarrow{i_n^{n+1}} D_n^{S_n} \xrightarrow{i_{n-1}^n} D_{n-1}^{S_{n-1}} \xrightarrow{i_{n-2}^{n-1}} \ldots \xrightarrow{i_1^2} D_1^{S_1}.$$

Observe that $\sigma_{[(0,1)^k]}$ with $k \neq n$ is mapped to $\sigma_{[(0,1)^k]}$ with i_{n-1}^n and that $i_{n-1}^n(\sigma_{[(0,1)^n]}) = \sigma_{[(0,1)^{n-1}]})$. Therefore i_{n-1}^n are surjections for all positive integers n. We do not wish to include functions such as

$$\max_{i \in \mathbb{N}} x_i$$

and for this reason we take a filtered inverse limit of these objects instead of the inverse limit. The total degree is the filter we use. Recall that $\operatorname{Deg} p$ of a max-plus polynomial

$$p(x_1, x_2, \ldots, x_n) = a_1 \odot x_1^{i_1^1} x_2^{i_2^1} \ldots x_n^{i_n^1} \boxplus a_2 \odot x_1^{i_1^2} x_2^{i_2^2} \ldots x_n^{i_n^2} \boxplus \ldots \boxplus a_m \odot x_1^{i_1^m} x_2^{i_2^m} \ldots x_n^{i_n^m}$$

is $\max_{1 \leq j \leq m}(i_1^j + i_2^j + \ldots + i_n^j)$. Let

$${}_kD_n = \{p \in D_n \mid \operatorname{Deg} p \leq k\}$$

Map i_{n-1}^n induces ${}_ki_{n-1}^n \colon {}_kD_n^{S_n} \xrightarrow{{}_ki_{n-1}^n} {}_kD_{n-1}^{S_{n-1}}$. We denote the inverse limit of this system by $\mathscr{D}^k$. The space of max-plus polynomials on the barcode space, $\mathscr{D}$, is precisely $\cup_{k=1}^{\infty} \mathscr{D}^k$.

Definition 6.4. *A semiring* $(\mathscr{R}, +, \cdot)$ *is called* filtered *if there exists such a family of subsemirings* $\{\mathscr{R}_d\}_{d \in \mathbb{N}}$ *of* $(\mathscr{R}, +, \cdot)$ *for operation* $+$ *that*

- $\mathscr{R}_d \subset \mathscr{R}_{d'}$ *for* $d \leq d'$*,*

- $\mathscr{R} = \bigcup_d \mathscr{R}_d$,
- $\mathscr{R}_d \cdot \mathscr{R}_{d'} \subset \mathscr{R}_{d+d'}$ *for all* $d, d' \in \mathbb{N}$.

Theorem 6.5. *Max-plus polynomials on the barcode space, $\mathscr{D}$, have the structure a filtered semiring. They are generated by elements of the form $\sigma_{[(0,1)^n]}$, where n is a positive integer.*

6.2 Stability of Max-Plus Polynomials

In Chapter 2 we argue that stability is the key property that coordinate functions should satisfy. In this section we prove that the functions from $\mathscr{D}$ are stable with respect to the bottleneck and Wasserstein distances.

Theorem 6.6 (Bottleneck Stability of Max-Plus Polynomials)**.** *Let $\mathscr{D}$ be the filtered semiring of max-plus polynomials. If $F \in \mathscr{D}$, then a constant C exists such that*

$$|F(\mathscr{B}_1) - F(\mathscr{B}_2)| \leq C\, d_\infty(\mathscr{B}_1, \mathscr{B}_2)$$

for any pair of barcodes $\mathscr{B}_1$ and $\mathscr{B}_2$.

Lemma 6.7. *A constant C exists such that*

$$|\sigma_{[(0,1)^n]}(\mathscr{B}_1) - \sigma_{[(0,1)^n]}(\mathscr{B}_2)| \leq C\, d_\infty(\mathscr{B}_1, \mathscr{B}_2)$$

for any pair of barcodes $\mathscr{B}_1$ and $\mathscr{B}_2$ and any $n \in \mathbb{N}$.

Proof. Let $\mathscr{B}_1 = \{(x_1, d_1), \ldots, (x_{l_1}, d_{l_1})\}$ and $\mathscr{B}_2 = \{(x'_1, d'_1), \ldots, (x'_{l_2}, d'_{l_2})\}$ be such that $\mathscr{B}_1 \neq \mathscr{B}_2$ and $d_1 \geq d_2 \geq \ldots \geq d_{l_1} \geq 0$.

Without loss of generality assume that $\sigma_{[(0,1)^n]}(\mathscr{B}_1) \geq \sigma_{[(0,1)^n]}(\mathscr{B}_2)$. If $n > l_1$ or $n > l_2$, we add 0 length intervals to $\mathscr{B}_1$, $\mathscr{B}_2$ to achieve that their length is n.

Let θ be a bijection where the penalty is minimal, i.e. where $P_\infty(\theta) = \mathrm{d}_\infty(\mathscr{B}_1, \mathscr{B}_2)$. Assume that θ matches (x_1, d_1) with (x'_1, d'_1), (x_2, d_2) with (x'_2, d'_2), …, (x_n, d_n) with (x'_n, d'_n)

(some of these intervals might be 0 length intervals). Note that for all i in this matching,

$$|\frac{d_i - d'_i}{2}| \leq \max_{i=1,\ldots,m}(|x_i - x'_i|, |d_i - d'_i + x_i - x'_i|) \leq \mathrm{d}_\infty(\mathscr{B}_1, \mathscr{B}_2). \tag{6.1}$$

By the definition of minimal matching $\max_{i=1,\ldots,m}(|x_i - x'_i|, |d_i - d'_i + x_i - x'_i|) \leq \mathrm{d}_\infty(\mathscr{B}_1, \mathscr{B}_2)$. We must prove the first inequality. Notice that if $|\frac{d_i - d'_i}{2}| \leq |x_i - x'_i|$, this follows automatically. If $|\frac{d_i - d'_i}{2}| > |x_i - x'_i|$, then

$$|\frac{d_i - d'_i}{2}| \leq |d_i - d'_i + x_i - x'_i|,$$

proving Inequality 6.1.

Then

$$\begin{aligned} n\mathrm{d}_\infty(\mathscr{B}_1, \mathscr{B}_2) &\geq \textstyle\sum_{i=1}^n \frac{(d_i - d'_i)}{2} \\ &= \tfrac{1}{2}(\textstyle\sum_{i=1}^n d_i - \sum_{i=1}^n d'_i) \\ &= \tfrac{1}{2}(\sigma_{[(0,1)^n]}(\mathscr{B}_1) - \textstyle\sum_{i=1}^n d'_i) \\ &\geq \tfrac{1}{2}(\sigma_{[(0,1)^n]}(\mathscr{B}_1) - \sigma_{[(0,1)^n]}(\mathscr{B}_1)). \end{aligned}$$

The last inequality holds since $\sum_{i=1}^n d'_i \leq \sigma_{[(0,1)^n]}(\mathscr{B}_2)$. Also note that we chose $d_1, \ldots, d_n$ in a way that $\sigma_{[(0,1)^n]}(\mathscr{B}_1) = \sum_{i=1}^n d_i$.

We deduce that

$$|\sigma_{[(0,1)^n]}(\mathscr{B}_1) - \sigma_{[(0,1)^n]}(\mathscr{B}_1)| \leq 2n\mathrm{d}_\infty(\mathscr{B}_1, \mathscr{B}_2),$$

proving that $\sigma_{[(0,1)^n]}$ is Lipschitz with constant $2n$. □

Proof of Theorem 6.6. Suppose F_1 and F_2 are such that C_1 and C_2 exist such that

$$|F_1(\mathscr{B}_1) - F_1(\mathscr{B}_2)| \leq C_1\mathrm{d}_\infty(\mathscr{B}_1, \mathscr{B}_2)$$

and

$$|F_2(\mathscr{B}_1) - F_2(\mathscr{B}_2)| \leq C_2\mathrm{d}_\infty(\mathscr{B}_1, \mathscr{B}_2)$$

for any any pair of barcodes $\mathscr{B}_1$ and $\mathscr{B}_2$.

Let $H = F_1 + F_2$. Then

$$\begin{aligned} |H(\mathscr{B}_1) - H(\mathscr{B}_2)| &= |F_1(\mathscr{B}_1) + F_2(\mathscr{B}_1) - F_1(\mathscr{B}_2) - F_2(\mathscr{B}_2)| \\ &\leq |F_1(\mathscr{B}_1) - F_1(\mathscr{B}_2)| + |F_2(\mathscr{B}_1) - F_2(\mathscr{B}_2)| \\ &\leq C_1 \mathrm{d}_\infty(\mathscr{B}_1, \mathscr{B}_2) + C_2 \mathrm{d}_\infty(\mathscr{B}_1, \mathscr{B}_2) \\ &\leq (C_1 + C_2)\mathrm{d}_\infty(\mathscr{B}_1, \mathscr{B}_2). \end{aligned}$$

Let $H = \max(F_1, F_2)$. Then

$$F_1(\mathscr{B}_2) \leq F_1(\mathscr{B}_1) + |F_1(\mathscr{B}_2) - F_1(\mathscr{B}_1))| \leq H(\mathscr{B}_1) + |F_1(\mathscr{B}_2) - F_1(\mathscr{B}_1))|,$$

and similarly $F_2(\mathscr{B}_2) \leq H(\mathscr{B}_1) + |F_2(\mathscr{B}_2) - F_2(\mathscr{B}_1))|$. It follows that

$$H(\mathscr{B}_2) \leq H(\mathscr{B}_1) + \max(|F_1(\mathscr{B}_2) - F_1(\mathscr{B}_1))|, |F_2(\mathscr{B}_2) - F_2(\mathscr{B}_1))|),$$

and by symmetry we conclude that

$$|H(\mathscr{B}_1) - H(\mathscr{B}_2)| \leq \max(C_1, C_2)\mathrm{d}_\infty(\mathscr{B}_1, \mathscr{B}_2).$$

Any function F from the filtered semiring of max-plus polynomials is generated by taking maxima and sums of $\sigma_{[(0,1)^n]}$. Since stability is preserved under these two operations and since $\sigma_{[(0,1)^n]}$ are stable according to Lemma 6.7, F is also stable. □

Theorem 6.8 (Wasserstein Stability of Max-Plus Polynomials)**.** *Let $\mathscr{D}$ be the filtered semiring of max-plus polynomials. If $F \in \mathscr{D}$, then a constant C exists such that*

$$|F(\mathscr{B}_1) - F(\mathscr{B}_2)| \leq C\, d_q(\mathscr{B}_1, \mathscr{B}_2)$$

for any pair of barcodes $\mathscr{B}_1$ and $\mathscr{B}_2$.

Proof. Let $\mathscr{B}_1 = \{(x_1, d_1), \ldots, (x_{l_1}, d_{l_1})\}$ and $\mathscr{B}_2 = \{(x'_1, d'_1), \ldots, (x'_{l_2}, d'_{l_2})\}$ be such that $\mathscr{B}_1 \neq \mathscr{B}_2$ and $d_1 \geq d_2 \geq \ldots \geq d_{l_1} \geq 0$.

Without loss of generality assume that $\sigma_{[(0,1)^n]}(\mathscr{B}_1) \geq \sigma_{[(0,1)^n]}(\mathscr{B}_2)$. If $n > l_1$ or $n > l_2$, we add 0 length intervals to $\mathscr{B}_1$, $\mathscr{B}_2$ to achieve that their length is n.

Let θ be a bijection where the penalty is minimal, i.e. where $P_p(\theta) = \mathrm{d}_p(\mathscr{B}_1, \mathscr{B}_2)$. Assume that θ matches (x_1, d_1) with (x'_1, d'_1), (x_2, d_2) with (x'_2, d'_2), $\ldots$, (x_n, d_n) with (x'_n, d'_n) (some of these intervals might be 0 length intervals). For all i in this matching,

$$|\frac{d_i - d'_i}{2}|^q \leq \max_{i=1,\ldots,m}(|x_i - x'_i|, |d_i - d'_i + x_i - x'_i|)^q$$

since $x \mapsto x^q$ is increasing for $x > 0$. Then

$$\begin{array}{rcl}
(\sigma_{[(0,1)^n]}(\mathscr{B}_1) - \sigma_{[(0,1)^n]}(\mathscr{B}_2))^q & \leq & (\sigma_{[(0,1)^n]}(\mathscr{B}_1) - \sum_{i=1}^n d'_i)^q \\
& = & (\sum_{i=1}^n d_i - \sum_{i=1}^n d'_i)^q \\
& \leq & 2^q(\sum_{i=1}^n |\frac{d_i - d'_i}{2}|)^q \\
& \leq & 2^q(n)^{q-1}(\sum_{i=1}^n |\frac{d_i - d'_i}{2}|^q) \\
& \leq & 2^q(n)^{q-1}P_q(\theta) \\
& = & 2^q n^{q-1}\mathrm{d}_q(\mathscr{B}_1, \mathscr{B}_2).
\end{array}$$

The first inequality holds since $\sum_{i=1}^n d'_i \leq \sigma_{[(0,1)^n]}(\mathscr{B}_2)$. Also note that we chose $d_1, \ldots, d_n$ in a way that $\sigma_{[(0,1)^n]}(\mathscr{B}_1) = \sum_{i=1}^n d_i$. To bound $\sum_{i=1}^n |\frac{d_i - d'_i}{2}|^q$ we use Hőlder's inequality. $\square$

6.3 Tropical Rational Functions on the Barcode Space

While the the functions belonging to $\mathscr{D}$ are stable and can be used to assign vectors to barcodes, they do not separate points in the barcode space. Because there simply are not enough functions among max-plus polynomials to separate points, we expand the set of functions we observe to all tropical rational functions.

Let

$$((x_1, d_1), \ldots, (x_n, d_n)), ((x'_1, d'_1), \ldots, (x'_n, d'_n)) \in [0, \infty)^{2n}.$$

Without loss of generality we assume that they are lexicographically ordered.

The tropical rational functions that respect equivalence classes of B_n, R_n, must respect the following equivalence relation $\sim$ on $[0, \infty)^{2n}$:

$$((x_1, d_1), \ldots, (x_n, d_n)) \sim ((x'_1, d'_1), \ldots, (x'_n, d'_n)) \Leftrightarrow \forall i : \; d_i = d'_i \wedge (x_i = x'_i \vee d_i = 0).$$

Theorem 6.9. *No finite subset of R_n exists which separates nonequivalent points in B_n.*

Proof. Assume $\{f_1, \ldots, f_m\} \in R_n$ separates nonequivalent points in B_n. Let $\vec{x} = (x_1, d_1, \ldots, x_n, d_n)$ and $\vec{x}' = (x'_1, d'_1, \ldots, x'_n, d'_n)$

$$g(\vec{x}, \vec{x}') = \max\{|f_1(\vec{x}) - f_1(\vec{x}')|, \ldots, |f_m(\vec{x}) - f_m(\vec{x}')|\}.$$

The function g is the 1-distance between vectors $(f_1(\vec{x}), \ldots, f_n(\vec{x}))$ and $(f_1(\vec{x}'), \ldots, f_n(\vec{x}'))$. Thus $g(\vec{x}, \vec{x}') = 0$ if and only if $\vec{x}$ and $\vec{x}'$ are equivalent points.

Since $|x| = \max(x, -x)$, g is a tropical rational function, so we can write it as

$$\begin{aligned}\max_{i=1,\ldots,l_1} (\sum_{k=1}^{n} (a_{k,i} x_k + b_{k,i} d_k) + \sum_{k=1}^{n} (a'_{k,i} x'_k + b'_{k,i} d'_k) + c_i) - \\ \max_{j=1,\ldots,l_2} (\sum_{k=1}^{n} (s_{k,j} x_k + t_{k,j} d_k) + \sum_{k=1}^{n} (s'_{k,j} x'_k + t'_{k,j} d'_k) + u_j). \quad (6.2)\end{aligned}$$

For $\vec{x}_0 = (x, 0, \ldots, x, 0)$ and $\vec{y}_0 = (y, 0, \ldots, y, 0)$, $g(\vec{x}_0, \vec{y}_0) = 0$ and consequently

$$\max_{i=1,\ldots,l_1} (x \sum_{k=1}^{n} a_{k,i} + y \sum_{k=1}^{n} a'_{k,i} + c_i) = \max_{j=1,\ldots,l_2} (x \sum_{k=1}^{n} s_{k,j} + y \sum_{k=1}^{n} s'_{k,j} + u_j).$$

$\sum_{k=1}^{n} a_{k,i}$ cannot be 0 for all i unless $\sum_{k=1}^{n} s_{k,i}$ is 0 for all i (if they were not, we could set y to 0 and choose a large x to make the right hand side of the equation larger than the left hand side). The same holds for $\sum_{k=1}^n a'_{k,i}$ and $\sum_{k=1}^n s'_{k,i}$. If $\sum_{k=1}^n a_{k,i}$, $\sum_{k=1}^n a'_{k,i}$, $\sum_{k=1}^n s_{k,i}$ and $\sum_{k=1}^n s'_{k,i}$ are 0 for all i, then g is independent of $x_1, \ldots, x_n, x'_1, \ldots, x'_n$. This cannot happen since when $n = 1$ it would imply that

$$0 = g(1,1,1,1) = g(1,1,2,1) \neq 0.$$

We can construct a similar example for other n. It follows that $\sum_{k=1}^n a_{k,i}$, $\sum_{k=1}^n a'_{k,i}$, $\sum_{k=1}^n s_{k,i}$ and $\sum_{k=1}^n s'_{k,i}$ do not all equal 0 for all i. Choose

$$x, y > \frac{\max\limits_{i \in \mathbb{N}_{\leq l_1}} (u_i, c_i)}{\min(\max\limits_{i \in \mathbb{N}_{\leq l_1}} (\sum\limits_{k=1}^n a_{k,i} + \sum\limits_{k=1}^n a_{k,i}), \max\limits_{i \in \mathbb{N}_{\leq l_2}} (\sum\limits_{k=1}^n s_{k,i} + \sum\limits_{k=1}^n s'_{k,i}))}.$$

We denote the set of indices for which the maximum of $\max\limits_{j=1,\ldots,l_2} (x \sum\limits_{k=1}^n a_{k,i} + y \sum\limits_{k=1}^n a'_{k,i} + c_i)$ is attained at $x \sum_{k=1}^n a_{k,i} + y \sum_{k=1}^n a'_{k,i} + c_i$ by I. We define J to be the set of indices where $\max\limits_{j=1,\ldots,l_2} (x \sum\limits_{k=1}^n s_{k,j} + y \sum\limits_{k=1}^n s'_{k,j} + u_j)$ is attained at $x \sum_{k=1}^n s_{k,j} + y \sum_{k=1}^n s'_{k,j} + u_j$.

Let $i \in I$ and $j \in J$. Since

$$x \sum_{k=1}^n a_{k,i} + y \sum_{k=1}^n a'_{k,i} + c_i = x \sum_{k=1}^n s_{k,j} + y \sum_{k=1}^n s'_{k,j} + u_j$$

for all big enough x and y, $\sum_{k=1}^n a_{k,i} = \sum_{k=1}^n s_{k,j}$, $\sum_{k=1}^n a'_{k,i} = \sum_{k=1}^n s'_{k,j}$ and $c_i = u_j$.

For any index i let S_i denote the set of all such $(\vec{x}, \vec{x}') \in [0,\infty)^{2n} \times [0,\infty)^{2n}$ that

$$\max_{i=1,\ldots,l_1} (\sum_{k=1}^n (a_{k,i} x_k + b_{k,i} d_k) + \sum_{k=1}^n (a'_{k,i} x'_k + b'_{k,i} d'_k) + c_i)$$

is attained in i. We similarly define T_j to be the set of all such $(\vec{x}, \vec{x}')$ that

$$\max_{j=1,\ldots,l_2} (\sum_{k=1}^n (s_{k,i} x_k + t_{k,i} d_k) + \sum_{k=1}^n (s'_{k,i} x'_k + t'_{k,i} d'_k) + u_i)$$

is attained in j. The sets S_i and T_j are closed for all i and j. Let

$$U = [0,\infty)^{2n} \times [0,\infty)^{2n} \setminus \Big(\bigcup_{i \notin I} S_i \cup \bigcup_{j \notin J} T_j \Big)$$

Since $(\vec{x}_0, \vec{y}_0) \in U$ and U is open in $[0,\infty)^{2n} \times [0,\infty)^{2n}$, a positive ϵ exists such that $([x, x+\epsilon] \times [0,\epsilon])^n \times ([y, y+\epsilon] \times [0,\epsilon])^n \subseteq U$. There exist such $i_0 \in I$ and $j_0 \in J$ that

$$\begin{aligned}
0 &= g((x,\epsilon,\ldots,x,\epsilon),(x,\epsilon,\ldots,x,\epsilon)) \\
&= x\textstyle\sum_{k=1}^n a_{k,i_0} + x\sum_{k=1}^n a'_{k,i_0} + \epsilon\sum_{k=1}^n (b_{k,i_0} + b'_{k,i_0}) + c_{i_0} - \\
&\quad\ (x\textstyle\sum_{k=1}^n s_{k,j_0} + x\sum_{k=1}^n s'_{k,j_0} + \epsilon\sum_{k=1}^n (t_{k,j_0} + t'_{k,j_0}) + u_{j_0}) \\
&= (\epsilon\textstyle\sum_{k=1}^n (b_{k,i_0} + b'_{k,i_0}) + c_{i_0}) - (\epsilon\sum_{k=1}^n (t_{k,j_0} + t'_{k,j_0}) + u_{j_0}) \\
&= x\textstyle\sum_{k=1}^n a_{k,i_0} + y\sum_{k=1}^n a'_{k,i_0} + \epsilon\sum_{k=1}^n (b_{k,i_0} + b'_{k,i_0}) + c_{i_0} - \\
&\quad\ (x\textstyle\sum_{k=1}^n s_{k,j_0} + y\sum_{k=1}^n s'_{k,j_0} + \epsilon\sum_{k=1}^n (t_{k,j_0} + t'_{k,j_0}) + u_{j_0}) \\
&= g((x,\epsilon,\ldots,x,\epsilon),(y,\epsilon,\ldots,y,\epsilon)).
\end{aligned}$$

We choose such x and y that $x \neq y$. For this choice, $g((x,\epsilon,\ldots,x,\epsilon),(y,\epsilon,\ldots,y,\epsilon)) \neq 0$, which is a contradiction. $\square$

Theorem 6.9 states that no finite subset of symmetric min-plus, max-plus or tropical rational functions exists that separates barcodes. In this section we identify a countable set of tropical rational functions on the barcode space that does.

Theorem 6.10. *Let $\{\sigma_{(e_{1,1},e_{1,2}),\ldots,(e_{n,1},e_{n,2})}\}$ be the set of elementary 2-symmetric max-plus polynomials. Functions, defined by*

$$E_{m,(e_{1,1},e_{1,2}),\ldots,(e_{n,1},e_{n,2})}(x_1,d_1,\ldots,x_n,d_n) := \sigma_{(e_{1,1},e_{1,2}),\ldots,(e_{n,1},e_{n,2})}(x_1 \oplus d_1^m, d_1, \ldots, x_n \oplus d_n^m, d_n),$$

for $m \in \mathbb{N}$, $i \in \mathbb{N}_{\leq n}$ are contained in R_n. Furthermore, they separate nonequivalent points in B_n.

Proof. Restricted to $d_i = 0$, expressions $x_i \oplus d_i^m$ are 0 and therefore independent of x_i and consequently so are their post-compositions with $e_{(e_{1,1},e_{1,2}),\dots,(e_{n,1},e_{n,2})}$. This implies that $E_{m,(e_{1,1},e_{1,2}),\dots,(e_{n,1},e_{n,2})}(x_1, d_1, \dots, x_n, d_n)$ is contained in R_n.

We must show that if $(x_1, d_1, \dots, x_n, d_n)$ and $(x'_1, d'_1, \dots, x'_n, d'_n)$ are not equivalent in B_n, we can find such $E_{m,(e_{1,1},e_{1,2}),\dots,(e_{n,1},e_{n,2})}$ that

$$E_{m,(e_{1,1},e_{1,2}),\dots,(e_{n,1},e_{n,2})}(x_1, d_1, \dots, x_n, d_n) \neq E_{m,(e_{1,1},e_{1,2}),\dots,(e_{n,1},e_{n,2})}(x'_1, d'_1, \dots, x'_n, d'_n).$$

Let $(x_1, d_1, \dots, x_n, d_n)$ and $(x'_1, d'_1, \dots, x'_n, d'_n)$ be nonequivalent. Without loss of generality assume that $d_1 \leq \dots \leq d_n$ and $d'_1 \leq \dots \leq d'_n$.

Some of the d's, say $d_1, \dots, d_{k-1} = 0$ can be 0 (if $k = 1$ none of d's is 0). The point $(x_1, 0, \dots, x_{k-1}, 0, x_k, d_k, \dots x_n, d_n)$ is equivalent to $(0, 0, \dots, 0, 0, x_k, d_k, \dots x_n, d_n)$ and consequently

$$\begin{aligned} E_{m,(e_{1,1},e_{1,2}),\dots,(e_{n,1},e_{n,2})}&(x_1, 0, \dots, x_{k-1}, 0, x_k, d_k, \dots x_n, d_n) = \\ &E_{m,(e_{1,1},e_{1,2}),\dots,(e_{n,1},e_{n,2})}(0, 0, \dots, 0, 0, x_k, d_k, \dots x_n, d_n) \end{aligned} \tag{6.3}$$

for all m and $(e_{1,1}, e_{1,2}), \dots, (e_{n,1}, e_{n,2})$. Similarly, if $d'_1, \dots, d'_{l-1} = 0$, then

$$(x'_1, 0, \dots, x'_{l-1}, 0, x'_l, d'_l, \dots x'_n, d'_n) \sim (0, 0, \dots, 0, 0, x'_l, d'_l, \dots x'_n, d'_n)$$

and consequently

$$\begin{aligned} E_{m,(e_{1,1},e_{1,2}),\dots,(e_{n,1},e_{n,2})}&(x'_1, 0, \dots, x'_{l-1}, 0, x'_l, d'_l, \dots x'_n, d'_n) = \\ &E_{m,(e_{1,1},e_{1,2}),\dots,(e_{n,1},e_{n,2})}(0, 0, \dots, 0, 0, x'_l, d'_l, \dots x'_n, d'_n) \end{aligned} \tag{6.4}$$

for all m and $(e_{1,1}, e_{1,2}), \dots, (e_{n,1}, e_{n,2})$.

Choose $m \in \mathbb{N}$ such that

$$m > \max(\max_{k \leq i \leq n} \frac{x_i}{d_i}, \max_{l \leq i \leq n} \frac{x'_i}{d'_i}).$$

For this m,

$$(x_1 \oplus d_1^m, d_1, \ldots, x_n \oplus d_n^m, d_n) = (0, 0, \ldots, 0, 0, x_k, d_k, \ldots x_n, d_n)$$

and

$$(x'_1 \oplus d_1'^m, d'_1, \ldots, x'_n \oplus d_n'^m, d_n) = (0, 0, \ldots, 0, 0, x'_l, d'_l, \ldots x'_n, d'_n)$$

Theorem 5.9guarantees existence of such $e \in \{\sigma_{(e_{1,1}, e_{1,2}), \ldots, (e_{n,1}, e_{n,2})}\}$ that

$$e(0, 0, \ldots, 0, 0, x_k, d_k, \ldots x_n, d_n) \neq e(0, 0, \ldots, 0, 0, x'_l, d'_l, \ldots x'_n, d'_n).$$

If we now take $E_{m,e}$ for this choice of m and this e,

$$E_{m,e}(x_1, d_1, \ldots x_n, d_n) \neq E_{m,e}(x'_1, d'_1, \ldots, x'_n, d'_n)$$

and we are done. $\square$

It is hard to characterize *all* tropical rational functions on B_n, so we work with a subsemiring of functions obtained by taking maxima, adding and substracting functions from $\{E_{m,(e_{1,1}, e_{1,2}), \ldots, (e_{n,1}, e_{n,2})}\}$. We denote this subsemiring by G_n or $G_n^{S_n}$ when we wish to stress that all the functions contained in it are symmetric. We have restriction maps $i_{n,m} \colon G_n \to G_m$, when $n \geq m$, induced by

$$i_{n,m}(f)(x_1, d_1, \ldots, x_m, d_m, \ldots, x_n, d_n) = f(x_1, d_1, \ldots, x_m, d_m, 0, 0, \ldots, 0, 0),$$

The map $i_{n,m}$ is S_m-equivariant, where S_m acts by permuting the first m pairs of variables.

Maps $i_{n,n-1}$ transform the generators of G_n as follows:

$$E_{m,(0,0)^j(1,0)^k(0,1)^l(1,1)^p} \mapsto \begin{cases} E_{m,(0,0)^{j-1}(1,0)^k(0,1)^l(1,1)^p} & \text{if } j \neq 0 \\ E_{m,(1,1)^{n-1}} & \text{if } j=0, k=0, l=0 \\ E_{m,(0,1)^{l-1}(1,1)^p} & \text{if } j=0, k=0, l \geq 1 \\ E_{m,(1,0)^{k-1}(1,1)^p} & \text{if } j=0, k \geq, l=0 \\ E_{m,(1,0)^{k-1}(0,1)^l(1,1)^p} \boxplus E_{m,(1,0)^k(0,1)^{l-1}(1,1)^p} & \text{if } j=0, k \geq, l \geq 1 \end{cases}$$

Here $p = n - l - k - j$. Therefore we $i_{n,n-1}$ is a surjection from G_n to G_{n-1} and we may construct composites

$$i_n^{n-1} \colon G_n^{S_n} \hookrightarrow G_n^{S_{n-1}} \xrightarrow{i_{n,n-1}^{S_{n-1}}} G_{n-1}^{S_{n-1}}.$$

We cannot proceed as we did in the case of max-plus polynomials, since we cannot define a degree of a tropical rational expression. However we can write any $r \in G_n$ as

$$\max_{i=1,\dots,l_1}\Big(\sum_{k=1}^{n}(a_{k,i}x_k + b_{k,i}d_k) + \sum_{k=1}^{n}(a'_{k,i}x'_k + b'_{k,i}d'_k) + c_i\Big) - \max_{j=1,\dots,l_2}\Big(\sum_{k=1}^{n}(s_{k,j}x_k + t_{k,j}d_k) + \sum_{k=1}^{n}(s'_{k,j}x'_k + t'_{k,j}d'_k) + u_j\Big). \quad (6.5)$$

Now set

$${}_kG_n^{S_n} = \{r \in G_n \,|\, r \sim p \oplus q^{-1}, p, q \text{ are max-plus polynomials with } \deg p, \deg q \leq k\}$$

Map i_n^{n-1} induces ${}_ki_n^{n-1} \colon {}_kG_n^{S_n} \xrightarrow{{}_ki_n^{n-1}} {}_kG_{n-1}^{S_{n-1}}$. We denote the inverse limit of this system by $\mathscr{G}^k$. Let $\mathscr{G} = \cup_{k=1}^{\infty}\mathscr{G}^k$.

Theorem 6.11. *Tropical rational functions in $\mathscr{G}$ form a filtered semiring and they separate points in the barcode space. As a semiring $\mathscr{G}$ is generated by elements of the form $E_{m,(1,0)^k(0,1)^l(1,1)^p}$ where k, l, p are nonnegative integers and m is a positive integer.*

6.4 Stability of Tropical Rational Functions in $\mathscr{G}$

Theorem 6.12 (Bottleneck stability of functions in $\mathscr{G}$)**.** *If $F \in \mathscr{G}$, then a constant C exists such that*

$$|F(\mathscr{B}_1) - F(\mathscr{B}_2)| \leq C\, d_\infty(\mathscr{B}_1, \mathscr{B}_2)$$

for any pair of barcodes $\mathscr{B}_1$ and $\mathscr{B}_2$.

Lemma 6.13. *Let $m_i = \min\{x_i, md_i\}$ and $m_j = \min\{x_j, md_j\}$. Then*

$$|m_i - m_j| \leq 2m \max(|x_i - x_i'|, |d_i - d_i' + x_i - x_i'|).$$

Proof. If $x_i \leq md_i$ and $x_j \leq md_j$, then

$$|m_i - m_j| = |x_i - x_j|.$$

If $x_i \geq md_i$ and $x_j \geq md_j$, then

$$|m_i - m_j| = |md_i - md_j| = m|d_i - d_j|.$$

Let $x_i \leq md_i$ and $x_j > md_j$ (the case when $x_i > md_i$ and $x_j \leq md_j$ is analogous). Since $0 \leq x_i \leq md_i$,

$$-md_j \leq x_i - md_j \leq m(d_i - d_j).$$

On the other hand $-x_j < -md_j \leq 0$ and consequently

$$x_i - x_j < x_i - md_j \leq x_i.$$

It follows that

$$|x_i - md_j| \leq \max\{|x_i - x_i'|, m|d_i - d_i'||$$

and consequently

$$|m_i - m_j| \leq \max\{|x_i - x_i'|, m|d_i - d_i'|\} \leq m \max\{|x_i - x_i'|, |d_i - d_i'|\}.$$

By triangle inequality

$$|d_i - d_i'| \leq |d_i - d_i' + x_i - x_i'| + |x_i - x_i'| \leq 2\max(|x_i - x_i'|, |d_i - d_i' + x_i - x_i'|).$$

Finally these two inequalities imply

$$\max(|x_i - x_i'|, |d_i - d_i' + x_i - x_i'|) \leq 2m \max(|x_i - x_i'|, |d_i - d_i' + x_i - x_i'|)$$

$\square$

Proof of Theorem 6.12. Take $E = E_{m,(0,1)^l(1,0)^k(1,1)^p}$. Let $\mathscr{B}_1 = \{(x_1, d_1), \ldots, (x_{l_1}, d_{l_1})\}$ and $\mathscr{B}_2 = \{(x_1', d_1'), \ldots, (x_{l_2}', d_{l_2}')\}$ be such that $\mathscr{B}_1 \neq \mathscr{B}_2$. Without loss of generality assume that

$$E_{m,(0,1)^l(1,0)^k(1,1)^p}(\mathscr{B}_1) \geq E_{m,(0,1)^l(1,0)^k(1,1)^p}(\mathscr{B}_2)$$

and

$$E_{m,(0,1)^l(1,0)^k(1,1)^p}(\mathscr{B}_1) = \sum_{i=1}^{p}(m_i + d_i) + \sum_{i=p+1}^{p+k} m_i + \sum_{i=p+k+1}^{p+k+l} d_i.$$

If $l_1, l_2 < p + k + l$, we add 0 length intervals to both barcodes.

Let θ be a bijection where the penalty is minimal, i.e. where $P_\infty(\theta) = \mathrm{d}_\infty(\mathscr{B}_1, \mathscr{B}_2)$. Assume that θ matches (x_1, d_1) with (x_1', d_1'), (x_2, d_2) with (x_2', d_2'), $\ldots$, (x_{p+k+l}, d_{p+k+l}) with (x_{p+k+l}', d_{p+k+l}'). Recall that for all i in this matching,

$$\left|\frac{d_i - d_i'}{2}\right| \leq \max_{i=1,\ldots,m}(|x_i - x_i'|, |d_i - d_i' + x_i - x_i'|).$$

Then

$$\begin{aligned} E(\mathscr{B}_1) - E(\mathscr{B}_2) &= \textstyle\sum_{i=1}^{p}(m_i + d_i) + \sum_{i=p+1}^{p+k} m_i + \sum_{i=p+k+1}^{p+k+l} d_i - E(\mathscr{B}_2)) \\ &\leq \textstyle\sum_{i=1}^{p}(m_i - m'_i + d_i - d'_i) + \sum_{i=p+1}^{p+k}(m_i - m'_i) + \sum_{i=p+k+1}^{p+k+l}(d_i - d'_i) \\ &= 2|\textstyle\sum_{i=1}^{p} \frac{m_i - m'_i}{2} + \sum_{i=1}^{p} \frac{d_i - d'_i}{2} + \sum_{i=p+1}^{p+k} \frac{m_i - m'_i}{2} + \sum_{i=p+k+1}^{p+k+l} \frac{d_i - d'_i}{2}| \\ &\leq 2(\textstyle\sum_{i=1}^{p} |\frac{m_i - m'_i}{2}| + \sum_{i=1}^{p} |\frac{d_i - d'_i}{2}| + \sum_{i=p+1}^{p+k} |\frac{m_i - m'_i}{2}| + \sum_{i=p+k+1}^{p+k+l} |\frac{d_i - d'_i}{2}|) \\ &\leq 2(pP_\infty(\theta) + pP_\infty(\theta) + kP_\infty(\theta) + lP_\infty(\theta)) \\ &\leq 2(2p + k + l)\mathrm{d}_\infty(\mathscr{B}_1, \mathscr{B}_2). \end{aligned}$$

This proves that E is Lipschitz. In Proof of Theorem 6.6 we showed that stable functions on the barcode space are preserved under taking sums, maxima and minima. Since $E_{m,(e_{1,1},e_{1,2}),\ldots,(e_{n,1},e_{n,2})}$ are stable as any $F \in \mathscr{G}$ is composed of taking sums, maxima and minima of $E_{m,(e_{1,1},e_{1,2}),\ldots,(e_{n,1},e_{n,2})}$. □

Theorem 6.14 (Wasserstein stability of functions in $\mathscr{G}$)**.** *If $F \in \mathscr{G}$, then a constant C exists such that*

$$|F(\mathscr{B}_1) - F(\mathscr{B}_2)| \leq C\,\mathrm{d}_q(\mathscr{B}_1, \mathscr{B}_2)$$

for any pair of barcodes $\mathscr{B}_1$ and $\mathscr{B}_2$.

Proof. We denote the function $E_{m,(0,1)^l(1,0)^k(1,1)^p}$ by E. Let $\mathscr{B}_1 = \{(x_1, d_1), \ldots, (x_{l_1}, d_{l_1})\}$ and $\mathscr{B}_2 = \{(x'_1, d'_1), \ldots, (x'_{l_2}, d'_{l_2})\}$ be such that $\mathscr{B}_1 \neq \mathscr{B}_2$. Without loss of generality assume that

$$E_{m,(0,1)^l(1,0)^k(1,1)^p}(\mathscr{B}_1) \geq E_{m,(0,1)^l(1,0)^k(1,1)^p}(\mathscr{B}_2)$$

and

$$E_{m,(0,1)^l(1,0)^k(1,1)^p}(\mathscr{B}_1) = \sum_{i=1}^{p}(m_i + d_i) + \sum_{i=p+1}^{p+k} m_i + \sum_{i=p+k+1}^{p+k+l} d_i.$$

If $l_1, l_2 < p + k + l$, we add 0 length intervals to both barcodes.

Let θ be a bijection where the penalty is minimal, i.e. where $P_\infty(\theta) = \mathrm{d}_q(\mathscr{B}_1, \mathscr{B}_2)$.

Assume that θ matches (x_1, d_1) with (x'_1, d'_1), (x_2, d_2) with (x'_2, d'_2), $\ldots$, (x_{p+k+l}, d_{p+k+l}) with (x'_{p+k+l}, d'_{p+k+l}). Recall that for all i in this matching,

$$|\frac{d_i - d'_i}{2}|^q \leq \max_{i=1,\ldots,m}(|x_i - x'_i|, |d_i - d'_i + x_i - x'_i|)^q$$

since $x \mapsto x^q$ is increasing for $x > 0$. Then

$$\begin{aligned}
|E(\mathscr{B}_1) - E(\mathscr{B}_2)|^q &= (\textstyle\sum_{i=1}^{p}(m_i + d_i) + \sum_{i=p+1}^{p+k} m_i + \sum_{i=p+k+1}^{p+k+l} d_i - E(\mathscr{B}_2))^q \\
&\leq (\textstyle\sum_{i=1}^{p}(m_i - m'_i + d_i - d'_i) + \sum_{i=p+1}^{p+k}(m_i - m'_i) + \sum_{i=p+k+1}^{p+k+l}(d_i - d'_i))^q \\
&= 2^q|\textstyle\sum_{i=1}^{p} \frac{m_i - m'_i}{2} + \sum_{i=1}^{p} \frac{d_i - d'_i}{2} + \sum_{i=p+1}^{p+k} \frac{m_i - m'_i}{2} + \sum_{i=p+k+1}^{p+k+l} \frac{d_i - d'_i}{2}|^q \\
&\leq 2^q(\textstyle\sum_{i=1}^{p} |\frac{m_i - m'_i}{2}| + \sum_{i=1}^{p} |\frac{d_i - d'_i}{2}| + \sum_{i=p+1}^{p+k} |\frac{m_i - m'_i}{2}| + \sum_{i=p+k+1}^{p+k+l} |\frac{d_i - d'_i}{2}|)^q \\
&\leq 2^q(2p + k + l)^{q-1}(2p + k + l)P_q(\theta)^q \\
&= 2^q(2p + k + l)^q \mathrm{d}_p(\mathscr{B}_1, \mathscr{B}_2)^q.
\end{aligned}$$

The first inequality holds since $\sum_{i=1}^{p}(m'_i + d'_i) + \sum_{i=p+1}^{p+k} m'_i + \sum_{i=p+k+1}^{p+k+l} d'_i \leq E(\mathscr{B}_2)$. Second to last inequality uses Hőlder's inequality.

This proves that E is Lipschitz. In Proof of Theorem 6.6 we showed that stable functions on the barcode space are preserved under taking sums, maxima and minima. Since $E_{m,(e_{1,1},e_{1,2}),\ldots,(e_{n,1},e_{n,2})}$ are stable, $F \in \mathscr{G}$ is also stable as it is composed of taking sums, maxima and minima of $E_{m,(e_{1,1},e_{1,2}),\ldots,(e_{n,1},e_{n,2})}$. $\square$

Chapter 7

Classifying Digits

Adcock et al. [1] used polynomial coordinates to classify digits from the MNIST database [24] of handwritten digits. In this Chapter I compare classification results they obtained with mine, which were classified using tropical coordinates. Aaron Adcock provided the matlab code needed to convert digital images into filtrations.

While homology itself cannot distinguish between the digits - 1, 5, and 7 never have loops, 0, 6, 9 always have loops, 8 has two loops, while 2, 3, 4 might or might not have loops, depending on style - we can use persistent homology as a measurement of shape. Figure 7.1 shows the first 100 digits of the database. The original black and white images

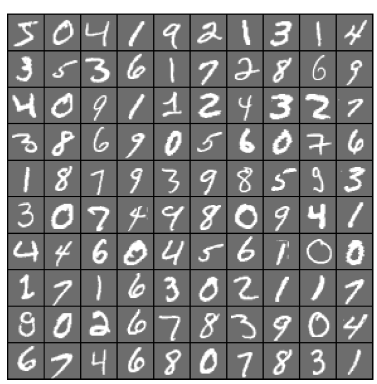

Figure 7.1: The first 100 images of the MNIST database.

were first normalized, scaled into a 20×20 pixel bounding box and anti-aliased, which introduced grayscale levels. Pixel values are 0 to 255, where 0 means background (white), 255 means foreground (black).

Following Collins et al. [16], we first threshold (setting pixel values greater than 100 to 1 and the rest to 0) to produce a binary image. We construct four filtrations as follows. For each pixel we add a vertex, for any pair of adjacent pixels (diagonals included) an edge and for any triple of adjacent pixels a 2-simplex. We sweep across the rows from the left and the right and across the columns from top to bottom and vice versa. This adds spatial information into what would otherwise be a purely topological measurement. We take both Betti 0 and Betti 1.

This extra spatial information reveals the location of various topological features. For example, though a '9' and '6' both have one connected component and one loop, the loop will appear at different locations in the 1-dimensional homology top-down sweep for the '9' and '6' (see Figure 7.3). In digits with no loops 0-dimensional homology right to left sweep distinguishes '3' from other digits (see Figure 7.2).

We can use different methods for turning barcodes into vectors. Adcock et al. selected four features,

$$\begin{aligned}
&\textstyle\sum_i x_i(y_i - x_i)\\
&\textstyle\sum_i (y_{\max} - y_i)(y_i - x_i)\\
&\textstyle\sum_i x_i^2(y_i - x_i)^4\\
&\textstyle\sum_i (y_{\max} - y_i)^2(y_i - x_i)^4
\end{aligned}$$

which when applied to the four sweeps, each with a 0-dimensional and 1- dimensional barcode, gives a feature vector of total size 32. I used command `fitcecoc` in matlab to get an error-correcting output codes (ECOC) multiclass model. This model was trained using support vector machine (SVM). I obtained the best results using the Gaussian kernel. As is typical when using a SVM, I scaled each coordinate such that the values were between 0 and 1. To measure the classification accuracy I used 100-fold cross-validation. See Table 7.1

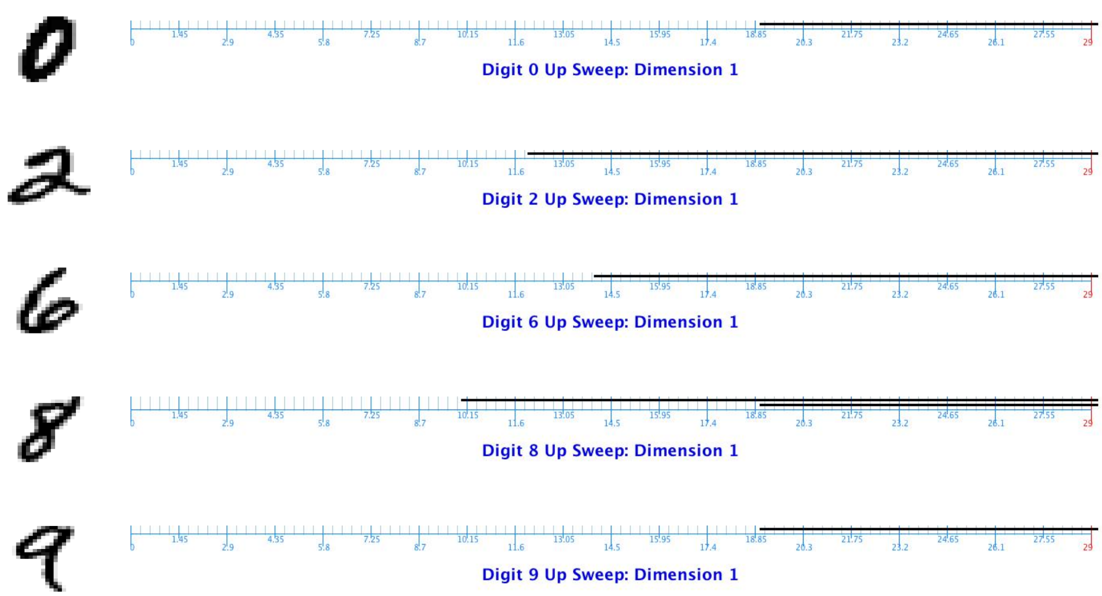

Figure 7.2: 1-dimensional homology bottom to top sweep for ‘0’, ‘2’, ‘6’, ‘8’ and ‘9’.

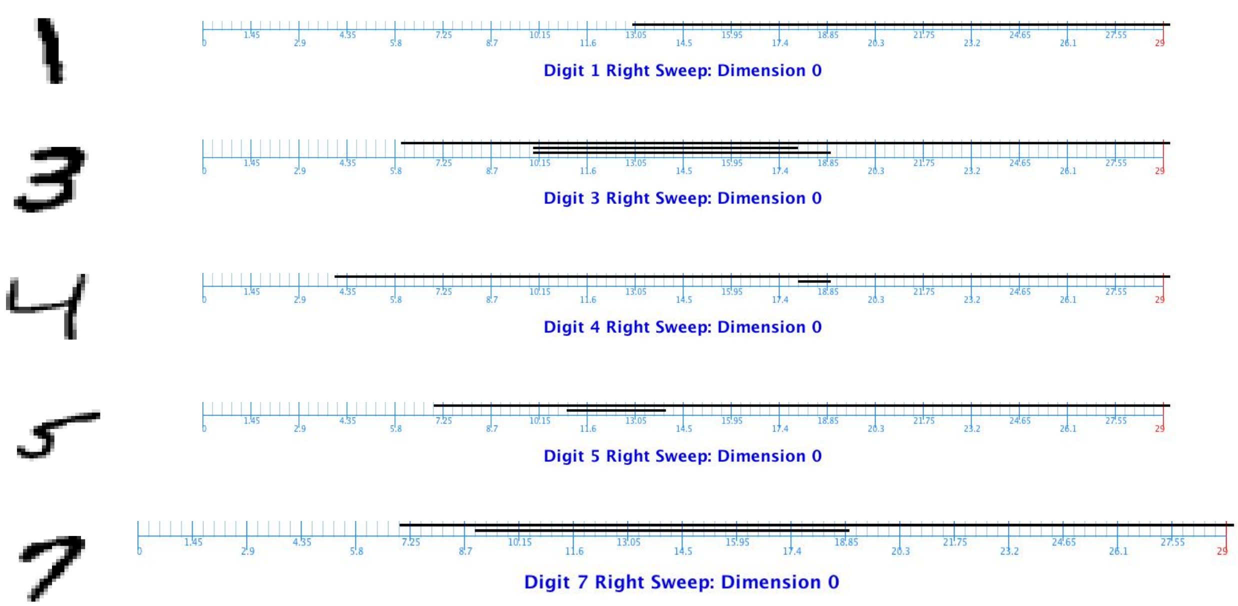

Figure 7.3: 0-dimensional homology right to left sweep for ‘1’, ‘3’, ‘4’, ‘5’ and ‘7’.

for results.

1000 digits	5000 digits	10000 digits
87.5%	90.04%	91.04%

Table 7.1: Classification accuracy using ordinary polynomial coordinates.

Using the following max-plus type coordinates

$\max_i d_i$ $\qquad$ $\max_{i<j}(d_i + d_j)$

$\max_{i<j<k}(d_i + d_j + d_k)$ $\qquad$ $\max_{i<j<k<l}(d_i + d_j + d_k + d_l)$

$\sum_i d_i$ $\qquad$ $\sum_i \min(28d_i, x_i)$

$\sum_i(\max_i(\min(28d_i, x_i) + d_i) - (\min(28d_i, x_i) + d_i))$.

yields slightly better results 7.2. Note that I used a many functions involving sums of lengths of intervals. These yielded the best results, which is perhaps not surprising since when using persistent homology and interpreting the barcode, we assign importance to features depending on over what range of parameters they persist.

1000 digits	5000 digits	10000 digits
87.70%	91.36%	92.41%

Table 7.2: Classification accuracy using max-plus type coordinates.

This method just demonstrates how one can use persistent homology with other machine learning algorithms and does not outperform existing classification algorithms. Figure 7.4 shows examples of digits that were not correctly The most common confusion is between a '5' and a '2' written with no loop. Other common confusions occur when topological changes occurred to the digit, for example when '8' is written with no loops, etc.

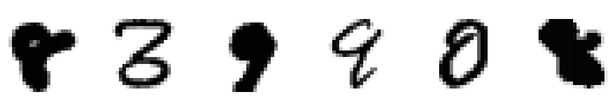

Figure 7.4: Common Misclassifications.

These examples also show the power of combining topology with geometry, and in particular demonstrate how coordinates can serve as a method for organizing the collection of all barcodes, and therefore any database whose members produce barcodes. They are also stable with respect to the bottleneck and Wasserstein distances.

Bibliography

[1] A. Aadcock, E. Carlsson, and G. Carlsson. “The Ring of Algebraic Functions on Persistence Bar Codes”. In: *ArXiv e-prints* (2013).

[2] H. Adams and G. Carlsson. “On the non-linear statistics of range image patches”. In: *SIAM J. Imaging Sci.* 2 (2008), pp. 110–117.

[3] A. Adcock, D. Rubin, and G. Carlsson. “Classification of Hepatic Lesions Using the Matching Metric”. In: *Computer Vision and Image Understanding* 121 (2014), pp. 36–42.

[4] A. Björner. “Topological methods”. In: *Handbook of Combinatorics* 1,2 (1995), pp. 71819–1872.

[5] K. C. Border. *Separating Hyperplane Theorems*. available at `http://people.hss.caltech.edu/~kcb/Notes/SeparatingHyperplane.pdf`. Notes, California Institute of Technology.

[6] Peter Bubenik. “Statistical Topological Data Analysis Using Persistence Landscapes”. In: *Journal of Machine Learning Research* 16.1 (2015), pp. 77–102.

[7] G. Carlsson. “Topological pattern recognition for point cloud data”. In: *Acta Numerica* 23 (2013), pp. 289–368.

[8] G. Carlsson. “Topology and Data”. In: *Bulletin of the American Mathematical Society* 46 (2009), pp. 255–308.

[9] G. Carlsson et al. "On the local behavior of spaces of natural images". In: *International Journal of Computer Vision* 76 (2008), pp. 1–12.

[10] Gunnar Carlsson and Afra J. Zomorodian. "Computing persistent homology". In: *Discrete and Computational Geometry* 33 (2005), pp. 249–274.

[11] J. M. Chan, R. Rabadan, and G. Carlsson. "Topology of viral evolution". In: *Proceedings of the National Academy of Sciences* (2013).

[12] Frédéric Chazal et al. "The structure and stability of persistence modules". In: *ArXiv e-prints* (2012).

[13] D. Cohen-Steiner, H. Edelsbrunner, and J. Harer. "Stability of persistence diagrams". In: *Discrete and Computational Geometry* 37 (2007), pp. 103–120.

[14] D. Cohen-Steiner et al. "Lipschitz functions have L^p-stable persistence". In: *Foundations of Computational Mathematics* 10 (2 2010), pp. 127–139.

[15] David Cohen-Steiner, Herbert Edelsbrunner, and John Harer. "Stability of Persistence Diagrams". In: *Discrete Comput. Geom.* 37.1 (2007), pp. 103–120.

[16] Anne Collins et al. "A Barcode Shape Descriptor for Curve Point Cloud Data". In: *Proceedings of the First Eurographics Conference on Point-Based Graphics*. SPBG'04. 2004, pp. 181–191.

[17] J. Dalbec. "Multisymmetric functions". In: *Beiträge zur Algebra und Geometrie* 40 (1 1999), pp. 27–51.

[18] Herbert Edelsbrunner and John Harer. *Computational Topology: An Introduction.* American Mathematical Society, 2010.

[19] Herbert Edelsbrunner, David Letscher, and Afra J. Zomorodian. "Topological persistence and simplification". In: *Discrete and Computational Geometry* 28 (2002), pp. 511–533.

[20] B. Di Fabio and M. Ferri. "Comparing persistence diagrams through complex vectors". 2015.

[21] M. Ferri and C. Landi. "Representing size functions by complex polynomials". In: *Proc. Math. Met. in Pattern Recognition* 9 (1999), pp. 16–19.

[22] P. Frosini and C. Landi. *Size Theory as a Topological Tool for Computer Vision*. Tech. rep.

[23] R. Ghrist. "Barcodes: The persistent topology of data". In: *Bulletin of the American Mathematical Society* 45 (2008), pp. 61–75.

[24] Y. LeCun and C. Cortes. "The MNIST Database". http://yann.lecun.com/exdb/mnist/. 2015.

[25] D. Maclagan and B. Sturmfels. *Introduction to tropical geometry*. Vol. 161. American Mathematical Soc., 2015.

[26] G. Mikhalkin. "Amoebas of algebraic varieties and tropical geometry". In: *Different Faces of Geometry* 3 (2014), pp. 257–300.

[27] V. Robins. "Computational topology at multiple resolutions." PhD thesis. University of Colorado, 2000.

[28] D. Speyer and B. Sturmfels. "Tropical mathematics". In: *Mathematics Magazine* 82 (2009), pp. 163–173.

[29] A. Tausz, M. Vejdemo-Johansson, and H. Adams. "JavaPlex: A research software package for persistent (co)homology". In: *Proceedings of ICMS 2014, Lecture Notes in Computer Science* 8592 (2014), pp. 129–136.

[30] "Théorie des systèmes linéaires dans les dioides". 1992.