ETH zürich

D-MATH

Magnitude Homology

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Magnitude Workshop

Outline

1. The Magnitude of a Graph

2. The Definition of Magnitude Homology

3. Induced Maps

4. Disjoint Unions

5. The Mayer-Vietoris Sequence

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The Magnitude of a Graph

Definition

A **graph** G is a pair $(\mathrm{V}(G), \mathrm{E}(G))$ consisting of a finite set $\mathrm{V}(G)$ representing the **vertices** of G and a set $\mathrm{E}(G)$ of unordered pairs of distinct vertices. The set $\mathrm{E}(G)$ is the set of **edges** of G.

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Definition:

The **shortest path metric** d_G (or just d if there is no risk of confusion) is the map $d_G \colon \mathrm{V}(G) \times \mathrm{V}(G) \to [0, \infty]$ which sends any two vertices $x, y \in \mathrm{V}(G)$ to the length of a shortest path from x to y in G if such a path exists. If no such path exists, then $d_G(x, y) = \infty$.

The Magnitude of a graph

Definition:
For any graph G, let $|G|_q$ be the magnitude of the symmetry matrix $Z_G = Z_G(q)$ given by

$$Z_G(x, y) = q^{d(x,y)}$$

for $x, y \in G$. The matrix $Z_G(q)$ is viewed as an element in $\mathbb{Q}(q)$ or $\mathbb{Z}[[q]]$.

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Example (The complete bipartite graph $K_{3,2}$)
The complete bipartite graph $K_{3,2}$ for integers has magnitude

$$|K_{3,2}|_q = \frac{5 - 7q}{(1 + q)(1 - 2q^2)}.$$

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The Definition of Magnitude Homology

Definition:
Let G be a graph. The **length** ℓ of a tuple $(x_0, \ldots, x_k) \in G^{k+1}$ of vertices of G is the sum

$$\ell(x_0, \ldots, x_k) = \sum_{i=0}^{k-1} d(x_i, x_{i+1}).$$

Note that for $k = 0$ the length of a tuple (x_0) is $\ell(x_0) = 0$.

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Definition:
The **magnitude chain group** $\mathrm{MC}_{k,l}(G)$ of a graph G in bidegree (k, l) for $k, l \geq 0$ is the free abelian group generated by $(k+1)$-tuples $(x_0, ..., x_k)$ of vertices of G satisfying $x_0 \neq x_1 \neq \ldots \neq x_k$ and $\ell(x_0, \ldots x_k) = l$.

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Definition:
For a graph G and integers $l \geq 0, k \geq 1$ we define the differential

$$\partial \colon \mathrm{MC}_{k,l}(G) \to \mathrm{MC}_{k-1,l}(G)$$

by the alternating sum $\partial = \sum_{i=1}^{k-1} (-1)^i \partial_i$, where $\partial_i \colon \mathrm{MC}_{k,l}(G) \to \mathrm{MC}_{k-1,l}(G)$ is defined on the generators by

$$\partial_i(x_0, \ldots, x_k) = \begin{cases} (x_0, \ldots, \widehat{x_i}, \ldots, x_k) & \text{if } \ell(x_0, \ldots, \widehat{x_i}, \ldots, x_k) = l \\ 0 & \text{else} \end{cases}$$

and then linearly extended to the whole group $\mathrm{MC}_{k,l}(G)$.

The Definition of Magnitude Homology

Lemma:

For any graph G and integers $l \geq 0$ and $k \geq 2$, the composition

$$\mathrm{MC}_{k,l}(G) \xrightarrow{\partial} \mathrm{MC}_{k-1,l}(G) \xrightarrow{\partial} \mathrm{MC}_{k-2,l}(G)$$

is equal to the zero map.

The Definition of Magnitude Homology

Definition:

The **magnitude chain complex** $\mathrm{MC}_{*,*}(G)$ of a graph G is the direct sum

$$\bigoplus_{l \geq 0} \mathrm{MC}_{*,l}(G)$$

of chain complexes.

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Remark:
For simplicity, we also call the chain complex $\mathrm{MC}_{*,l}(G)$ for a fixed $l \geq 0$ the magnitude chain complex of G.

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Definition:
The **magnitude homology** $\mathrm{MH}_{*,*}(G)$ of a graph G is the bigraded abelian group defined by the homology groups

$$\mathrm{MH}_{k,l}(G) = \mathrm{H}_k(\mathrm{MC}_{*,l}(G))$$

for $k, l \geq 0$.

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Proposition:

If $k > l$, then the magnitude homology $\mathrm{MH}_{k,l}(G) = 0$ for any graph G.

Example: (The four-cycle)

- Let $l = 0$.

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$$(a_1), (a_2), (a_3), (a_4).$$
- Let $l = 1$.
  - $\mathrm{MC}_{0,1}(C_4) = 0$.
  - $\mathrm{MC}_{1,1}(C_4)$ is generated by the eight tuples
$$(a_1, a_2), (a_1, a_4), (a_2, a_3), (a_2, a_1), (a_3, a_4), (a_3, a_2), (a_4, a_1), (a_4, a_3).$$

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  - $\mathrm{MC}_{2,2}(C_4)$ is generated by the 16 tuples

$$\begin{aligned}
&(a_1, a_2, a_3), (a_1, a_2, a_1), (a_1, a_4, a_1), (a_1, a_4, a_3),\\
&(a_2, a_3, a_4), (a_2, a_3, a_2), (a_2, a_1, a_2), (a_2, a_1, a_4),\\
&(a_3, a_4, a_1), (a_3, a_4, a_3), (a_3, a_2, a_3), (a_3, a_2, a_1),\\
&(a_4, a_1, a_2), (a_4, a_1, a_4), (a_4, a_3, a_4), (a_4, a_3, a_2).
\end{aligned}$$

Example: (The four-cycle)

The images of the generators under the differential $\partial : \mathrm{MC}_{2,2}(C_4) \to \mathrm{MC}_{1,2}(C_4)$ are

$$
\begin{aligned}
\partial(a_1,a_2,a_3) &= -(a_1,a_3) & \partial(a_1,a_2,a_1) &= 0\\
\partial(a_1,a_4,a_1) &= 0 & \partial(a_1,a_4,a_3) &= -(a_1,a_3)\\
\partial(a_2,a_3,a_4) &= -(a_2,a_4) & \partial(a_2,a_3,a_2) &= 0\\
\partial(a_2,a_1,a_2) &= 0 & \partial(a_2,a_1,a_4) &= -(a_2,a_4)\\
\partial(a_3,a_4,a_1) &= -(a_3,a_1) & \partial(a_3,a_4,a_3) &= 0\\
\partial(a_3,a_2,a_3) &= 0 & \partial(a_3,a_2,a_1) &= -(a_3,a_1)\\
\partial(a_4,a_1,a_2) &= -(a_4,a_2) & \partial(a_4,a_1,a_4) &= 0\\
\partial(a_4,a_3,a_4) &= 0 & \partial(a_4,a_3,a_2) &= -(a_4,a_2).
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\partial(a_1,a_4,a_1) &= 0 & \partial(a_1,a_4,a_3) &= -(a_1,a_3)\\
\partial(a_2,a_3,a_4) &= -(a_2,a_4) & \partial(a_2,a_3,a_2) &= 0\\
\partial(a_2,a_1,a_2) &= 0 & \partial(a_2,a_1,a_4) &= -(a_2,a_4)\\
\partial(a_3,a_4,a_1) &= -(a_3,a_1) & \partial(a_3,a_4,a_3) &= 0\\
\partial(a_3,a_2,a_3) &= 0 & \partial(a_3,a_2,a_1) &= -(a_3,a_1)\\
\partial(a_4,a_1,a_2) &= -(a_4,a_2) & \partial(a_4,a_1,a_4) &= 0\\
\partial(a_4,a_3,a_4) &= 0 & \partial(a_4,a_3,a_2) &= -(a_4,a_2).
\end{aligned}$$

The image $\mathrm{im}(\partial) = \mathrm{MC}_{1,2}(C_4)$ and the kernel $\ker(\partial)$ is generated by the twelve generators

$$\begin{gathered}
(a_1,a_2,a_1),(a_1,a_4,a_1),(a_2,a_3,a_2),(a_2,a_1,a_2),\\
(a_3,a_4,a_3),(a_3,a_2,a_3),(a_4,a_1,a_4),(a_4,a_3,a_4),\\
(a_1,a_2,a_3)-(a_1,a_4,a_3),(a_2,a_3,a_4)-(a_2,a_1,a_4),\\
(a_3,a_4,a_1)-(a_3,a_2,a_1),(a_4,a_1,a_2)-(a_4,a_3,a_2).
\end{gathered}$$

Example: (The four-cycle)

$l \diagdown k$	0	1	2	3	4	5	6	7	8	9	10
0	4	0	0	0	0	0	0	0	0	0	0
1	0	8	0	0	0	0	0	0	0	0	0
2	0	0	12	0	0	0	0	0	0	0	0
3	0	0	0	16	0	0	0	0	0	0	0
4	0	0	0	0	20	0	0	0	0	0	0
5	0	0	0	0	0	24	0	0	0	0	0
6	0	0	0	0	0	0	28	0	0	0	0
7	0	0	0	0	0	0	0	32	0	0	0
8	0	0	0	0	0	0	0	0	36	0	0
9	0	0	0	0	0	0	0	0	0	40	0
10	0	0	0	0	0	0	0	0	0	0	44

Table: Ranks of the magnitude homology of C_4 computed with SageMath.

Ranks of $\mathrm{MH}_{0,0}(G)$ and $\mathrm{MH}_{1,1}(G)$

Proposition:

For any graph G, the magnitude homology groups satisfy:

i) $\mathrm{MH}_{0,0}(G) \cong \mathbb{Z}\,\mathrm{V}(G)$;

ii) $\mathrm{MH}_{1,1}(G) \cong \mathbb{Z}\,\vec{\mathrm{E}}(G)$.

Remark:

For every graph G, the ranks of $\mathrm{MH}_{0,1}(G)$ and $\mathrm{MH}_{1,1}(G)$ are

i) $\mathrm{rank}(\mathrm{MH}_{0,0}(G)) = \#\,\mathrm{V}(G)$;

ii) $\mathrm{rank}(\mathrm{MH}_{1,1}(G)) = 2 \cdot \#\,\mathrm{E}(G)$.

Magnitude and Magnitude Homology

Theorem:

Let G be a graph, then

$$|G|_q = \sum_{k,l \geq 0} (-1)^k \operatorname{rank}(\mathrm{MH}_{k,l}(G)) \cdot q^l = \sum_{l \geq 0} \chi(\mathrm{MH}_{*,l}(G)) \cdot q^l .$$

Example: (The five-cycle)

$$|G|_q = \sum_{k,l\geq 0} (-1)^k \operatorname{rank}(\mathrm{MH}_{k,l}(G)) \cdot q^l = \sum_{l\geq 0} \chi(\mathrm{MH}_{*,l}(G)) \cdot q^l.$$

$l \backslash k$	0	1	2	3	4	5	6	7	8	9
0	5	0	0	0	0	0	0	0	0	0
1	0	10	0	0	0	0	0	0	0	0
2	0	0	10	0	0	0	0	0	0	0
3	0	0	10	10	0	0	0	0	0	0
4	0	0	0	30	10	0	0	0	0	0
5	0	0	0	0	50	10	0	0	0	0
6	0	0	0	0	20	70	10	0	0	0
7	0	0	0	0	0	80	90	10	0	0
8	0	0	0	0	0	0	180	110	10	0
9	0	0	0	0	0	0	40	320	130	10

Table: Ranks of the magnitude homology of C_5 computed with SageMath.

Example: (The five-cycle)

$l \diagdown k$	0	1	2	3	4	5	6	7	8	9
0	5	0	0	0	0	0	0	0	0	0
1	0	10	0	0	0	0	0	0	0	0
2	0	0	10	0	0	0	0	0	0	0
3	0	0	10	10	0	0	0	0	0	0
4	0	0	0	30	10	0	0	0	0	0
5	0	0	0	0	50	10	0	0	0	0
6	0	0	0	0	20	70	10	0	0	0
7	0	0	0	0	0	80	90	10	0	0
8	0	0	0	0	0	0	180	110	10	0
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0	5	0	0	0	0	0	0	0	0	0
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2	0	0	10	0	0	0	0	0	0	0
3	0	0	10	10	0	0	0	0	0	0
4	0	0	0	30	10	0	0	0	0	0
5	0	0	0	0	50	10	0	0	0	0
6	0	0	0	0	20	70	10	0	0	0
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Proposition:

Let G be a graph and suppose that for some $k, l \geq 0$ the magnitude homology $\mathrm{MH}_{k,l}(G) \neq 0$. If G has finite diameter $\delta > 0$, then $\frac{l}{\delta} \leq k$ and moreover, if $\delta > 1$ and $l > 0$, then $\frac{l}{\delta} < k$.

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The Category of Graphs

Definition:

For graphs G and H we define a **map of graphs** or **morphism of graphs** $f\colon G \to H$ to be a map $f\colon \mathrm{V}(G) \to \mathrm{V}(H)$ on the vertex sets such that

$$\forall \{x,y\} \in \mathrm{E}(G) \qquad \{f(x), f(y)\} \in \mathrm{E}(H) \text{ or } f(x) = f(y).$$

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Definition:
If $f\colon G \to H$ is a map of graphs, the **induced chain map** $f_{\#}\colon \mathrm{MC}_{*,*}(G) \to \mathrm{MC}_{*,*}(H)$ is defined on generators by

$$f_{\#}(x_0, \dots, x_k) = \begin{cases} (f(x_0), \dots, f(x_k)) & \text{if } \ell(f(x_0), \dots, f(x_k)) = \ell(x_0, \dots, x_k) \\ 0 & \text{else.} \end{cases}$$

The Category of Graphs

Proposition:

Let $f: G \to H$ be a map of graphs. The above defined induced map $f_\#$ is indeed a chain map, that is, it commutes with the differential ∂:

$$f_\# \circ \partial = \partial \circ f_\#.$$

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Definition:

Let $f\colon G \to H$ be a map of graphs. The **induced map in homology** is the map

$$f_*\colon \mathrm{MH}_{*,*}(G) \to \mathrm{MH}_{*,*}(H)$$

induced by the chain map $f_\#$.

The Category of Graphs

Proposition:

The assignment sending any graph G to its magnitude homology $\mathrm{MH}_{*,*}(G)$, and any map of graphs f to its induced map f_* is a functor from the category of graphs to the category of bigraded abelian groups.

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The Disjoint Union of Graphs

Proposition:

Let G and H be two graphs, we denote by $G \sqcup H$ their disjoint union. Let $i\colon G \to G \sqcup H$ and $j\colon H \to G \sqcup H$ be the inclusion maps. The induced map on the direct sum

$$i_* \oplus j_* \colon \mathrm{MH}_{*,*}(G) \oplus \mathrm{MH}_{*,*}(H) \to \mathrm{MH}_{*,*}(G \sqcup H)$$

is an isomorphism.

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Corollary:

The magnitude of the disjoint union of two graphs G and H satisfies

$$|G \sqcup H|_q = |G|_q + |H|_q.$$

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Definition:
Let $U \subset X$ be a convex subgraph. We say that X **projects** to U if for every vertex $x \in X$ there is a vertex $\pi(x) \in U$ such that

$$\forall u \in U \qquad d(x, u) = d(x, \pi(x)) + d(\pi(x), u).$$

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Remark:
If a graph X projects to a subgraph U, there is a well-defined map $\pi \colon X \to U$, where $x \mapsto \pi(x)$.

Definitions

Definition:
A **projecting decomposition** is a triple $(X; G, H)$ consisting of a graph X and subgraphs $G, H \subset X$ such that

i) $X = G \cup H$;
ii) $G \cap H$ is convex in X;
iii) H projects to $G \cap H$.

Given a projecting decomposition $(X; G, H)$ we write

$$i^G : G \to X, \quad j^G : G \cap H \to G,$$
$$i^H : H \to X, \quad j^H : G \cap H \to H$$

for the inclusions.

Excision

Definition:
Given a projecting decomposition $(X; G, H)$, let $\mathrm{MC}_{*,*}(G, H)$ denote the chain subcomplex of $\mathrm{MC}_{*,*}(G \cup H)$ spanned by those tuples $(x_0, \ldots, x_k)$ with entries all in G or all in H.

Theorem: (Excision for magnitude chains)

Let $(X; G, H)$ be a projecting decomposition. For every $l \geq 0$, the inclusion

$$\mathrm{MC}_{*,l}(G, H) \to \mathrm{MC}_{*,l}(G \cup H)$$

is a quasi-isomorphism, that is, it induces an isomorphism in homology.

The Mayer-Vietoris Sequence

Theorem: (Mayer-Vietoris for magnitude homology)

Let $(X; G, H)$ be a projecting decomposition. There exists a split short exact sequence

$$0 \to \mathrm{MH}_{*,*}(G \cap H) \xrightarrow{(j_*^G, -j_*^H)} \mathrm{MH}_{*,*}(G) \oplus \mathrm{MH}_{*,*}(H) \xrightarrow{i_*^G \oplus i_*^H} \mathrm{MH}_{*,*}(G \cup H) \to 0.$$

The sequence is natural with respect to decomposition maps, and the splitting is natural with respect to projecting decomposition maps.

The Mayer-Vietoris Sequence

Corollary: (Inclusion-Exclusion principle)

For a projecting decomposition $(X; G, H)$, the magnitudes satisfy

$$|X|_q = |G|_q + |H|_q - |G \cap H|_q.$$

The Mayer-Vietoris Sequence

Definition:
Let G and H be graphs with chosen base vertices. The **wedge sum** $G \vee H$ of G and H is the graph we get by identifying the two base vertices to a single vertex.

Corollary:

Let G and H be graphs with fixed base vertices and denote the vertex of their wedge sum corresponding to the base vertices by P. The inclusion maps $a\colon G \to G \vee H$ and $b\colon H \to G \vee H$ induce isomorphisms

$$a_* \oplus b_* : \mathrm{MH}_{k,l}(G) \oplus \mathrm{MH}_{k,l}(H) \xrightarrow{\cong} \mathrm{MH}_{k,l}(G \vee H),$$

if $k > 0$ or $l > 0$, and an isomorphism

$$a_* \oplus b_* : (\mathrm{MH}_{0,0}(G) \oplus \mathrm{MH}_{0,0}(H)) / \operatorname{im}(j_*^G, -j_*^H) \xrightarrow{\cong} \mathrm{MH}_{0,0}(G \vee H),$$

where $j^G : P \to G$ and $j^H : P \to H$ are the inclusions.

Example: (Magnitude Homology of Trees)

The magnitude homology of a tree T is given by

$$\mathrm{MH}_{k,l}(T) \cong \begin{cases} \mathbb{Z}\,\mathrm{V}(T) & \text{if } k = l = 0 \\ \mathbb{Z}\,\vec{\mathrm{E}}(T) & \text{if } k = l > 0 \\ 0 & \text{else.} \end{cases}$$

Reference: Hepworth, R., & Willerton, S. (2017). Categorifying the magnitude of a graph. Homology, Homotopy and Applications, 19(2), 31-60. https://doi.org/10.4310/HHA.2017.v19.n2.a3

Github: https://github.com/nadjahae/Magnitude-Homology

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