



Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Magnitude Homology

Master Thesis

Nadja Häusermann

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Advisor: Dr. Sara Kališnik Hintz

Department of Mathematics, ETH Zürich

Abstract

The magnitude of finite metric spaces is a cardinality-like invariant counting the 'effective number of points' of a metric space. By equipping graphs with the shortest path metric, we can view them as metric spaces and thus consider their magnitude. We give several examples of magnitude of finite metric spaces and graphs, we prove how the magnitude behaves with respect to the cartesian product of metric spaces. Since the shortest-path metric is integer-valued, we can also define magnitude of graphs as a formal power series. The magnitude of graphs satisfies cardinality-like properties such as an inclusion-exclusion principle and additivity with respect to disjoint unions. These properties can also be proven using the magnitude homology of graphs, which categorifies the magnitude. We prove several properties of magnitude homology, including a Mayer-Vietoris type theorem, which implies the inclusion-exclusion principle for magnitude. Furthermore, we study a type of graphs called diagonal graphs, for which the coefficients of the magnitude alternate in sign and prove that joins are diagonal. We provide several computer calculated examples of magnitude homology using our code written for SageMath.

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Chapter 1

Introduction

The magnitude of finite metric spaces is a cardinality-like invariant introduced in [5] by Leinster. The magnitude can be interpreted as counting the ‘effective number of points’ of the space, which is illustrated by the following example. Consider a metric space (A, d) with 3 points and distances according to Figure 1.1 below. We calculate the magnitude of A with the help of the so-called symmetry matrix:

$$Z_A = \begin{pmatrix} 1 & e^{-10} & e^{-10} \\ e^{-10} & 1 & e^{-0.1} \\ e^{-10} & e^{-0.1} & 1 \end{pmatrix}$$

The entry of the symmetry matrix corresponding to $x, y \in A$ is given by $e^{-d(x,y)}$. The matrix Z_A is invertible and in this special case the magnitude of (A, d) is given by the sum over all entries of the inverse Z_A^{-1} . This yields

$$|A| = \sum_{i=1}^3 \sum_{j=1}^3 (Z_A^{-1})(i, j) = 2.05$$

up to 2 decimals. We can also scale the space (A, d) by a factor $t > 0$ and compute the magnitude of every scaled version. This procedure produces a function in t , which we have plotted in Figure 1.2. We call this the magnitude function of the metric space A . For small t , meaning the space is viewed from far away, all three points are close together and therefore the magnitude

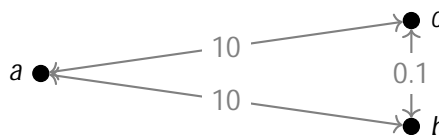


Figure 1.1: Schematic figure of the metric space (A, d) , indicating the distances between the points.

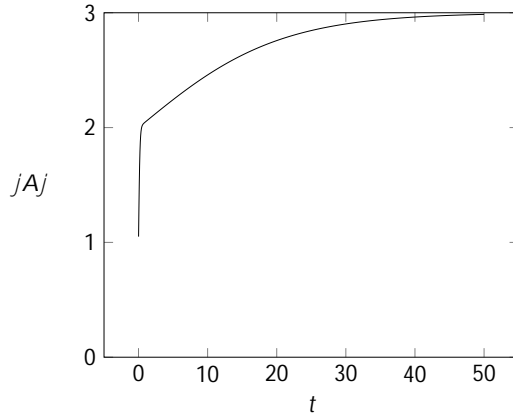


Figure 1.2: Magnitude function of the metric space (A, d) .

is close to 1, as there is only 1 ‘effective point’. After zooming in a bit, the point a can be seen separately from the other two points, thus the magnitude is around 2. When t is getting larger, all three points can be seen individually and the magnitude approaches 3. Indeed, for every finite metric space, the magnitude approaches the number of points of that space as t goes to infinity.

Magnitude first appeared in [9] by Solow and Polasky in 1994. They studied biological diversity and called what we define as magnitude the ‘effective number of species’. The definition of magnitude of a metric space comes from a more general definition of magnitude of enriched categories, which is explained in [5], but is beyond the scope of this thesis.

Instead, we define the magnitude of a matrix and directly introduce magnitude for finite metric spaces using symmetry matrices. In [6], Leinster generalized the notion of magnitude to the setting of graphs. By viewing graphs as metric spaces on their vertex set with the shortest path distance, one can apply the definition of magnitude for finite metric spaces. The additional property that the shortest path distance is integer-valued can be used to define the magnitude of a graph as a power series in a formal variable q , or equivalently, as a rational function in the formal variable q . We denote this magnitude of a graph G by jGj_q . The magnitude for graphs also satisfies cardinality-like properties, Leinster proved in [6] that the magnitude of graphs is additive with respect to the disjoint union of graphs. Furthermore, let G and H be two graphs, then we can take their cartesian product $G \times H$, in analogy to the cardinality of the cartesian product of sets, the magnitude satisfies

$$jG \times Hj_q = jGj_q \cdot jHj_q, \quad (1.1)$$

and with some additional assumptions on the graphs, the following inclusion-exclusion principle for the magnitude holds:

$$jG \sqcup Hj_q = jGj_q + jHj_q - jG \cap Hj_q.$$

Leinster also gave an example of two connected graphs that share the same magnitude, but have different Tutte polynomials, showing that the magnitude is not derived by this previously known graph invariant but contains other information.

Hepworth and Willerton introduced the magnitude homology of graphs in [3]. The magnitude homology of a graph G is a bigraded homology theory denoted by $MH_{k,l}(G)$. It categorifies the magnitude in a sense that the graded Euler characteristic of the magnitude homology equals the magnitude:

$$\sum_{k,l \geq 0} (-1)^k \text{rank}(MH_{k,l}(G)) q^l = |G|_q. \quad (1.2)$$

By applying this relation to properties of the magnitude homology, Hepworth and Willerton were able to prove the above properties of the magnitude found by Leinster. In particular, they showed that

$$|G \sqcup H|_q = |G|_q + |H|_q,$$

which implies the additivity of magnitude with respect to disjoint unions. Equation (1.1) about the magnitude of the cartesian product of graphs can be deduced by a Künneth type theorem for magnitude homology. Furthermore, Hepworth and Willerton proved a Mayer-Vietoris type theorem for magnitude homology that relates the magnitude homology of a union of graphs G and H (with some further assumptions) to the magnitude homology of $G \sqcup H$, and the intersection $G \cap H$. Concretely, they showed that there is a split short exact sequence

$$0 \rightarrow MH_{k,l}(G \cap H) \rightarrow MH_{k,l}(G) \oplus MH_{k,l}(H) \rightarrow MH_{k,l}(G \sqcup H) \rightarrow 0,$$

which recovers the inclusion-exclusion principle for the magnitude. By considering Equation (1.2), one can deduce that for a graph G where all magnitude homology groups $MH_{k,l}(G)$ are trivial for $k \neq l$, the coefficients of the magnitude alternate in sign. This can be seen for example in the magnitude of the triangle, which is

$$|K_3|_q = 3 - 6q + 12q^2 - 24q^3 + 48q^4 - 96q^5 + \dots$$

Such graphs are called diagonal graphs and Hepworth and Willerton proved that the join of two non-empty graphs is diagonal. This is true, for example, of the complete graphs, which are the iterated joins of one-vertex graphs.

The main portion of this thesis is dedicated to the paper 'Categorifying The Magnitude of a Graph' by Hepworth and Willerton [3] and proving its statements in detail. Furthermore, we have written our own code to compute magnitude homology of graphs using SageMath.

In Chapter 2 of this thesis, we introduce magnitude for matrices and finite metric spaces. Using the shortest path metric on graphs, we study the magnitude of graphs. In Chapter 3, we define magnitude homology of graphs and show how it categorifies the magnitude. Furthermore, we compute the magnitude homology of disjoint unions and prove a Mayer-Vietoris type theorem. Finally, we study diagonal graphs and prove that all joins of non-empty graphs belong to this type of graphs.

Chapter 2

Magnitude

The goal of this chapter is to define magnitude of graphs, so that we can later establish a connection between magnitude and the magnitude homology. We do this by first defining magnitude of matrices and finite metric spaces. It is also possible to approach magnitude via enriched categories. This is explained, for example, in [7, Section 5.2.3.]

Notation: To avoid confusion with notation, we denote throughout the thesis the cardinality of a finite set A by $\#A$, as the standard notation $|A|$ denotes the magnitude.

2.1 The Magnitude of a Matrix

Let k be a commutative semiring (this is a ring with a multiplicative unity, but here we do not require every element to have an additive inverse) and let A be a finite set. A matrix $Z \in k^{A \times A}$ is to be understood as an $\#A \times \#A$ square matrix over k indexed by the elements of the set A . For any two elements $a, b \in A$, we write $Z(a, b)$ for the entry of Z indexed by (a, b) . Furthermore, we denote by $e \in k^A$ the column vector $e = (1, \dots, 1)^T$ where each entry is 1. We mostly follow [7, Section 5.2.1.] for this section.

Definition 2.1. A weighting on a matrix $Z \in k^{A \times A}$ is a column vector $w \in k^A$ such that $Zw = e$. Furthermore, we say a weighting is positive, if all its entries are positive. Analogously we define negative and non-negative weightings.

Definition 2.2. A coweighting on a matrix $Z \in k^{A \times A}$ is a row vector $v \in k^A$ such that $vZ = e^T$.

By the definition of matrix multiplication, a column vector $w \in k^A$ is a weighting on a matrix $Z \in k^{A \times A}$ if

$$\forall a \in A \quad \sum_{b \in A} Z(a, b)w(b) = 1.$$

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Analogously, a row vector $v \in \mathbb{R}^A$ is a coweighting on a matrix $Z \in \mathbb{R}^{A \times A}$ if

$$\sum_{a \in A} v(a)Z(a, b) = 1.$$

Furthermore, we observe that for any matrix $Z \in \mathbb{R}^{A \times A}$, and w and v a weighting and coweighting on Z respectively, then

$$\sum_{a \in A} w(a) = e^T w = vZw = ve = \sum_{a \in A} v(a).$$

That is, if w is a weighting on a matrix Z , and v is a coweighting on the same matrix, then the sum of the entries of w is equal to the sum of the entries of v . This allows us to make the following definition.

Definition 2.3. For a matrix $Z \in \mathbb{R}^{A \times A}$ that admits both a weighting and a coweighting, let w be any weighting on Z and v be any coweighting on Z . The magnitude $|Z|$ of Z is the common quantity

$$|Z| = \sum_{a \in A} w(a) = \sum_{a \in A} v(a).$$

Remark 2.4. When the matrix $Z \in \mathbb{R}^{A \times A}$ is invertible, it has a unique weighting $w = Z^{-1}e$ and a unique coweighting $v = e^T Z^{-1}$. In this case, the magnitude is the sum of the entries of the inverse Z^{-1} .

$$|Z| = \sum_{a \in A} w(a) = \sum_{a \in A} (Z^{-1}e)(a) = \sum_{a \in A} \sum_{b \in A} Z^{-1}(a, b).$$

Not every matrix has a weighting and thus a magnitude, consider for example the matrix that has 0 as every entry. However, the following proposition gives us a class of matrices that do possess magnitude, namely the positive definite matrices.

Proposition 2.5 ([5, Proposition 2.4.3]). For a positive definite matrix $Z \in \mathbb{R}^{A \times A}$, the magnitude $|Z|$ is defined and equal to

$$|Z| = \sup_{0 \neq x \in \mathbb{R}^A} \frac{(\sum_{a \in A} x(a))^2}{x^T Z x},$$

and the supremum is attained exactly when x is a non-zero scalar multiple of the unique weighting on Z .

Proof. Since Z is positive definite, we can use the Cauchy-Schwarz inequality, which states that for any two row vectors $x, w \in \mathbb{R}^A$

$$(x^T Z x)(w^T Z w) \geq (x^T Z w)^2,$$

2.2. The Magnitude of a Finite Metric Space

with equality if and only if x and w are linearly dependent, that is, if one is a non-zero scalar multiple of the other. Any positive definite matrix is invertible, so by Remark 2.4 we can take $w \in \mathbb{R}^A$ to be the unique weighting on Z to rewrite the left-hand-side of the Cauchy-Schwarz inequality as

$$(x^T Z x)(w^T Z w) = (x^T Z x)(w^T e) = (x^T Z x) \sum_{a \in A} w(a).$$

The right-hand-side of the Cauchy-Schwarz equation is

$$(x^T Z w)^2 = (x^T e)^2 = \left(\sum_{a \in A} x(a) \right)^2.$$

Combining these two equalities and using that for all $x \in \mathbb{R}^A \setminus \{0\}$ the term $x^T Z x > 0$ because Z is positive definite, we conclude

$$|Zx| = \sum_{a \in A} w(a) \frac{\left(\sum_{a \in A} x(a) \right)^2}{(x^T Z x)}$$

with equality if and only if x is a scalar multiple of the weighting w . □

2.2 The Magnitude of a Finite Metric Space

In this section, we define the magnitude of finite metric spaces. We also provide results that support the interpretation of the magnitude of a finite metric space as a 'measure of the effective number of points' of that space, as well as some introductory examples of magnitude and general properties. We consider metric spaces in which the metric is allowed to take the value ∞ , because the metric we work with in the following chapters has this property. We follow [7, Section 5.2.4.].

Definition 2.6. The magnitude $|A|$ of a finite metric space (A, d) is the magnitude of the matrix $Z = Z_A \in \mathbb{R}^{A \times A}$ whose entry corresponding to $(a, b) \in A \times A$ is given by

$$Z_A(a, b) = e^{-d(a,b)},$$

if Z_A has a magnitude. The matrix Z_A is called the symmetry matrix.

Definition 2.7. A column vector $w \in \mathbb{R}^A$ is a weighting for the finite metric space (A, d) if w is a weighting on its symmetry matrix, so if $Z_A w = e$.

Note that the matrix Z_A of a finite metric space (A, d) is symmetric. Thus, if $w \in \mathbb{R}^A$ is a weighting for (A, d) , then

$$w^T Z_A = w^T Z_A^T = (Z_A w)^T = e^T,$$

so the transpose w^T is a coweighting on Z_A . Hence, if a weighting for the metric space (A, d) exists, then Z_A admits both a weighting and a coweighting and the following is well defined.

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Definition 2.8. For a finite metric space (A, d) that admits a weighting, the magnitude of (A, d) is

$$|A| = |Z_A| = \sum_{a \in A} w(a)$$

for any weighting w of A .

There are metric spaces that do not possess a weighting and hence do not have a magnitude, see Example 2.25, which we will discuss later.

Example 2.9 ([5, Examples 2.1.1.]) Let us look at some simple metric spaces and their magnitude.

- i) The magnitude of the empty space is 0 because its symmetry matrix is the empty matrix. The one point space has magnitude 1, its symmetry matrix is (1) .
- ii) Let (A, d) be the metric space consisting of precisely two points a_1 and a_2 that are at distance $x > 0$ apart from each other. To calculate the magnitude of A , we have to find its symmetry matrix Z_A , which is

$$Z_A = \begin{pmatrix} 1 & e^{-x} \\ e^{-x} & 1 \end{pmatrix}.$$

This matrix is invertible with inverse

$$Z_A^{-1} = \frac{1}{1 - e^{-2x}} \begin{pmatrix} 1 & e^{-x} \\ e^{-x} & 1 \end{pmatrix}.$$

By Remark 2.4, the magnitude of A is

$$|A| = |Z_A| = \sum_{1 \leq i, j \leq 2} Z_A^{-1}(i, j) = \frac{2}{1 - e^{-2x}} = 1 + \tanh \frac{x}{2}.$$

In Figure 2.1 we plot the magnitude as a function of the distance between the two points. We can see that for large x , the magnitude is close to 2, which is the number of points in A . If x is small, the magnitude is closer to 1 because in that case the two points are not far apart from each other, so they do not appear as separate points but more as one 'blurry' point. In this sense, magnitude captures the effective number of points of a finite metric space.

The next results provide us with some metric spaces that do have magnitude. The first proposition states that if we scale up a space big enough, it will have a magnitude.

Proposition 2.10 ([8, Theorem 2]). Let (A, d) be a finite metric space with n points and suppose that for all $a, b \in A$ with $a \neq b$ the distance

$$d(a, b) > \log(n - 1).$$

Then A possesses a weighting and therefore its magnitude is defined.

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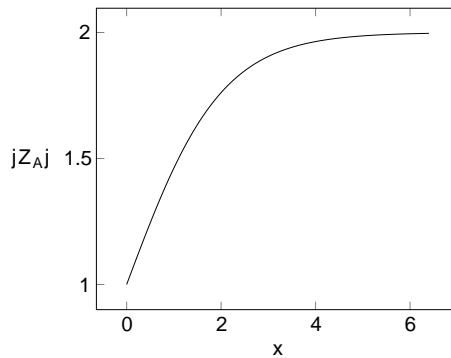


Figure 2.1: The magnitude of the two point space.

Proof. First, note that the statement is true for $n = 0$ and $n = 1$, as we have seen in Example 2.9 above. Next, consider $n \geq 2$. The diagonal entries of the symmetry matrix Z_A are all 1 and for any $1 \leq i \neq j \leq n$ we use the condition from the statement to deduce

$$Z_A(i, j) = e^{-d(i,j)} < e^{-\log(n-1)} = \frac{1}{n-1}.$$

Consider any symmetric matrix $Z \in \mathbb{R}^{n \times n}$ satisfying these conditions, meaning for all $i = 1, \dots, n$ the entry $Z(i, i) = 1$ and for any $1 \leq i \neq j \leq n$ the matrix satisfies $0 < Z(i, j) < \frac{1}{n-1}$. By Proposition 2.5, it is enough to show that Z is positive definite. Let $x \in \mathbb{R}^n$ be any column vector, then

$$\begin{aligned} x^T Z x &= \sum_{i=1}^n x(i) Z(i, i) x(i) + \sum_{1 \leq i \neq j \leq n} x(i) Z(i, j) x(j) \\ &= \sum_{i=1}^n x(i)^2 + \sum_{1 \leq i \neq j \leq n} x(i) Z(i, j) x(j). \end{aligned} \quad (2.1)$$

We need to show that $x^T Z x \geq 0$ with equality if and only if $x = 0$, so let us now bound the second sum in (2.1) from below. The negative summands can be written as $x(i) Z(i, j) x(j) = -|Z(i, j)| |x(i)| |x(j)|$ because all entries of Z are non-negative by assumption, thus we obtain the estimate

$$x(i) Z(i, j) x(j) > -\frac{1}{n-1} |x(i)| |x(j)| \quad (2.2)$$

for the negative summands. Note that the positive summands satisfy the lower bound in (2.2) as well. Using this estimation and some calculations, we

obtain for all $x \in \mathbb{R}^n$

$$\begin{aligned} x^T Z x &= \sum_{i=1}^n x(i)^2 - \frac{1}{n-1} \sum_{1 \leq i < j \leq n} |x(i) - x(j)|^2 \\ &= \frac{1}{2(n-1)} \sum_{1 \leq i < j \leq n} (|x(i) - x(j)|^2 + |x(i) + x(j)|^2) - \frac{1}{n-1} \sum_{1 \leq i < j \leq n} |x(i) - x(j)|^2 \\ &= \frac{1}{2(n-1)} \sum_{1 \leq i < j \leq n} (|x(i) - x(j)| + |x(i) + x(j)|)^2 \geq 0. \end{aligned}$$

It is left to show that equality holds if and only if $x = 0$. The last inequality in the calculation above is an equality if and only if $|x(i) - x(j)| = |x(i) + x(j)| = a$ for some $a \in \mathbb{R}_0$. The first lower bound is an equality if and only if there are no positive terms that we bounded from below and, because $Z(i, j) < \frac{1}{n-1}$ is a strict inequality, if and only if there are no negative terms. This leaves us with $a = 0$ as the only possibility, which shows that Z is positive definite. \square

2.2.1 Homogeneous Metric Spaces

In this subsection we introduce a condition that ensures a metric space to have magnitude. Concretely, we look at homogeneous finite metric spaces.

Definition 2.11. A metric space (A, d) is homogeneous if its isometry group acts transitively on the points of A .

Proposition 2.12 ([5, Proposition 2.1.5]). A finite homogeneous metric space (A, d) with n points, and any fixed point $a_0 \in A$ has a positive weighting and its magnitude is given by

$$|A| = \frac{n}{\sum_{a \in A} e^{-d(a, a_0)}} = \frac{n^2}{\sum_{a, b \in A} e^{-d(a, b)}}.$$

Proof. We will give a concrete weighting for A using the following claim.

Claim: For any $x \in A$, the sum $S(x) = \sum_{a \in A} Z_A(x, a)$ is independent of $x \in A$.

Proof of Claim: Let $x, y \in A$ be any two points. Because A is homogeneous, there exists an isometry $f: A \rightarrow A$ such that $f(x) = y$. Thus,

$$S(x) = \sum_{a \in A} e^{-d(x, a)} = \sum_{a \in A} e^{-d(f(x), f(a))} = \sum_{a \in A} e^{-d(y, f(a))} = \sum_{b \in A} e^{-d(y, b)} = S(y).$$

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The claim allows us to write $S = \sum_{a \in A} e^{-d(x,a)}$ for any $x \in A$ instead of $S(x)$. Consider the column vector $w \in \mathbb{R}^A$ given by $w(b) = 1/S$ for all $b \in A$. We show that w is a weighting for A . Indeed, for any $a \in A$ we have

$$\sum_{b \in A} Z_A(a,b)w(b) = \sum_{b \in A} e^{-d(a,b)} \frac{1}{S} = \frac{1}{S} S = 1.$$

Using the weighting w to calculate the magnitude of A yields

$$|A| = \sum_{a \in A} w(a) = \frac{n}{S} = \frac{n}{\sum_{a \in A} e^{-d(a,a_0)}},$$

which proves the first equality. Note also that $w(a) = \frac{1}{S} > 0$ for every $a \in A$, so the weighting is positive, and

$$\frac{n}{S} = \frac{n^2}{\sum_{a \in A} S} = \frac{n^2}{\sum_{a \in A} \sum_{b \in A} e^{-d(a,b)}}.$$

□

2.2.2 Cartesian Product of Metric Spaces

The cardinality of sets is multiplicative with respect to the Cartesian product. Since magnitude is a cardinality-like invariant, we would expect that it behaves similarly. Indeed, in this short subsection we prove such a relation.

Definition 2.13. Let (A, d_A) and (B, d_B) be two metric spaces. We denote by $A \times B$ the metric space consisting of the set $A \times B$ equipped with the metric given by

$$d((a,b), (a',b')) = d_A(a, a') + d_B(b, b').$$

The map d in the definition above is indeed a metric for $A \times B$, which can immediately be verified using that d_A and d_B are metrics for A and B respectively.

Proposition 2.14. Suppose that (A, d_A) and (B, d_B) are finite metric spaces with weightings $w \in \mathbb{R}^A$ and $v \in \mathbb{R}^B$ respectively. A weighting $\alpha \in \mathbb{R}^{A \times B}$ for the space $(A \times B, d)$ is given by

$$\alpha(a,b) = w(a) v(b)$$

and the magnitude is $|A \times B| = |A| |B|$.

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Proof. We verify by direct calculations that $x \in \mathbb{R}^{A \times B}$ from the statement is a weighting. Let $(a, b) \in A \times B$ be any point, then

$$\begin{aligned} \sum_{(c,d) \in A \times B} Z_{A \times B}((a,b), (c,d)) x(c,d) &= \sum_{(c,d) \in A \times B} e^{-d((a,b),(c,d))} w(c)v(d) \\ &= \sum_{(c,d) \in A \times B} e^{-d_A(a,c) - d_B(b,d)} w(c)v(d) \\ &= \sum_{c \in A} e^{-d_A(a,c)} w(c) \sum_{d \in B} e^{-d_B(b,d)} v(d) = 1. \end{aligned}$$

Since all sums are finite there is no problem exchanging the order. The last equality holds because w and v are weightings for A and B respectively. Lastly, we can calculate the magnitude

$$\sum_{(a,b) \in A \times B} x(a,b) = \sum_{(a,b) \in A \times B} w(a)v(b) = \sum_{a \in A} w(a) \sum_{b \in B} v(b).$$

□

2.2.3 Expansion of a Finite Metric Space

This short subsection explores the relation of the magnitude of two finite metric spaces that are expansions of each other.

Definition 2.15. A metric space (A, d_A) is an expansion of a metric space (B, d_B) if there exists a distance decreasing surjection $f: A \rightarrow B$, that is a surjective map $f: A \rightarrow B$ such that for any $a, b \in A$ the distance $d_A(a, b) \leq d_B(f(a), f(b))$.

The following example of an expansion of a metric space will appear again later in Section 2.4.

Example 2.16. Let (A, d) be any finite metric space. Consider any factor $\alpha < 1$, the metric space $(A, \alpha d)$ can be thought of as the space (A, d) zoomed in. This space $(A, \alpha d)$ is an expansion of (A, d) since the identity $\text{id}: A \rightarrow A$ is a distance decreasing surjection.

Lemma 2.17 ([5, Lemma 2.2.5]). Let (A, d_A) and (B, d_B) be finite metric spaces, each admitting a non-negative weighting. If A is an expansion of B , then $|A| \geq |B|$.

Proof. Suppose A is an expansion of B and let $f: A \rightarrow B$ be a distance decreasing surjection. Choose a right inverse $g: B \rightarrow A$ of f . We observe that $\forall a \in A, b \in B$ the distance

$$d_A(a, g(b)) \leq d_B(f(a), f(g(b))) = d_B(f(a), b),$$

which implies

$$\sum_{b \in B} d_B(f(a), b) \leq \sum_{b \in B} d_A(a, g(b)) = \sum_{b \in B} Z_A(a, g(b)). \quad (2.3)$$

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Let w and v be non-negative weightings for A and B respectively, using the defining property of weightings and the estimation (2.3) we obtain

$$\begin{aligned}
 |A| &= \sum_{a \in A} w(a) \sum_{b \in B} Z_B(f(a), b) v(b) = \sum_{(a,b) \in A \times B} w(a) Z_B(f(a), b) v(b) \\
 &= \sum_{(a,b) \in A \times B} w(a) Z_A(a, g(b)) v(b) = \sum_{b \in B} v(b) \sum_{a \in A} w(a) Z_A(a, g(b)) \\
 &= |B|.
 \end{aligned}$$

Note that we need both w and v to be non-negative weightings for the estimation. □

2.3 The Magnitude of a Graph

In this section we explain how we view graphs as metric spaces in order to use the previous section to study their magnitudes. We also discuss a way to deal with the implicit choice of the base in the symmetry matrix and apply it to an example.

Definition 2.18. A graph G is a pair $(V(G), E(G))$ consisting of a finite set $V(G)$ representing the vertices of G and a set $E(G)$ of unordered pairs of distinct vertices. The set $E(G)$ is the set of edges of G .

This means the graphs that we consider in this thesis are finite, undirected graphs without loops and multiple edges. To denote a vertex x of a graph G we also write $x \in G$ instead of the lengthier $x \in V(G)$. Given any such graph G , we can view it as a metric space on the set of vertices $V(G)$ with the following metric.

Definition 2.19. The shortest path metric d_G (or just d if there is no risk of confusion) is the map $d_G: V(G) \times V(G) \rightarrow [0, \infty]$ which sends any two vertices $x, y \in V(G)$ to the length of a shortest path from x to y in G if such a path exists. If no such path exists, then $d_G(x, y) = \infty$.

With this definition we obtain a metric space (G, d_G) , if we allow the metric to take the value ∞ , and hence we can look at its magnitude using the results from Section 2.2.

Example 2.20 (Complete bipartite graph $K_{3,2}$). Let us study the magnitude of the complete bipartite graph $K_{3,2}$ with the vertices labelled as in Figure 2.2 above. The symmetry matrix $Z = Z_{K_{3,2}}$ is

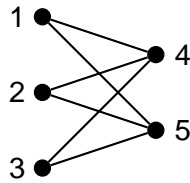


Figure 2.2: Complete bipartite graph $K_{2,3}$.

$$Z = \begin{pmatrix} 0 & 1 & e^{-2} & e^{-2} & e^{-1} & e^{-1} \\ e^{-2} & 1 & e^{-2} & e^{-1} & e^{-1} & e^{-1} \\ e^{-2} & e^{-2} & 1 & e^{-1} & e^{-1} & e^{-1} \\ e^{-1} & e^{-1} & e^{-1} & 1 & e^{-2} & e^{-2} \\ e^{-1} & e^{-1} & e^{-1} & e^{-2} & 1 & 1 \end{pmatrix}.$$

For the weighting w on Z denote $N = (1 + e^{-1})(1 - 2e^{-2})$. For $i = 1, 2, 3$ we define $w(i) = \frac{1 - e^{-1}}{N}$ and for $i = 4, 5$ we define $w(i) = \frac{1 - 2e^{-1}}{N}$. A direct calculation yields $Zw = e$, hence the magnitude is

$$|Z| = \sum_{i=1}^5 w(i) = \frac{5 - 7e^{-1}}{(1 + e^{-1})(1 - 2e^{-2})} = 2.43,$$

up to two decimals.

In the following, we address the implicit choice we made in the definition of magnitude of a finite metric space. The entries of the symmetry matrix are defined to have base e^{-1} , but we could have also chosen any other base. The approach to deal with this implicit choice we discuss here is from [6, Section 2], in Section 2.4 we explain an alternative way. Here, we make use of the fact that the shortest path distance of a graph is integer valued. Let q be a formal variable, which we use to replace e^{-1} . That is, for a graph G viewed as a metric space with the shortest path metric d we define the symmetry matrix $Z_G = Z_G(q)$ whose entry corresponding to $(x, y) \in G^2$ is given by

$$Z_G(x, y) = q^{d(x,y)},$$

where we use the convention that $q^{\infty} = 0$. We consider $Z_G = Z_G(q)$ as a matrix over the ring $\mathbb{Q}(q)$ of rational functions in the formal variable q . By considering this approach, we get the following advantage. The matrix $Z_G(0)$ is the identity, therefore the determinant $\det(Z_G(q))$ has constant term 1 and is invertible in the field $\mathbb{Q}(q)$. It follows that $Z_G(q)$ is invertible over $\mathbb{Q}(q)$ and thus G has a well defined magnitude, which is a rational function in q . Alternatively, we can view $Z_G(q)$ as a matrix over the ring $\mathbb{Z}[[q]]$ of power series in q and because $\det(Z_G(q))$ is also invertible over $\mathbb{Z}[[q]]$ by the same reasoning as above, we find that $Z_G(q)$ is an invertible matrix over $\mathbb{Z}[[q]]$ and

hence we can also view the magnitude as a formal power series in q . Viewing the magnitude as a rational function over q or as a formal power series in q gives the same result because they are equal when viewed as elements in the ring $\mathbb{Q}((q))$ of formal Laurent series in q , of which both $\mathbb{Q}(q)$ and $\mathbb{Z}[[q]]$ are subrings. In conclusion, we have the following alternative definition of magnitude for a graph.

Definition 2.21. For any graph G , let $|jGj_q$ be the magnitude of the symmetry matrix $Z_G = Z_G(q)$ given by

$$Z_G(x, y) = q^{d(x,y)}$$

for $x, y \in G$. The matrix $Z_G(q)$ is viewed as an element of $\mathbb{Q}(q)$ or $\mathbb{Z}[[q]]$.

The next result generalizes the previous example of the complete bipartite graph $K_{2,3}$ and uses the new approach of viewing the magnitude as a rational function in q .

Proposition 2.22 ([6, Example 3.4]). The complete bipartite graph $K_{m,n}$ for integers $m, n \geq 1$ has magnitude

$$|jK_{m,n}j_q = \frac{(m+n)(2mn - m - n)q}{(1+q)(1 - (m-1)(n-1)q^2)}.$$

Proof. Let the bipartition of $K_{m,n}$ be given by A and B where $\#A = m$ and $\#B = n$. The symmetry matrix $Z = Z_{K_{m,n}}(q)$ is

$$Z(x, y) = \begin{cases} 1 & \text{if } x = y \\ q^2 & \text{if } x \notin y \text{ and } (x, y \in A \text{ or } x, y \in B) \\ q & \text{if } x \in A, y \in B \text{ or } x \in B, y \in A. \end{cases}$$

Let us denote $N = (1+q)(1 - (m-1)(n-1)q^2)$, we define a column vector $w \in \mathbb{Q}(q)^{m+n}$ by

$$w(a) = \frac{1 + (1 - n)q}{N} \quad \forall a \in A$$

and

$$w(b) = \frac{1 + (1 - m)q}{N} \quad \forall b \in B.$$

Direct calculation shows that for $a \in A$ we have

$$\begin{aligned} \sum_{x \in A \cup B} Z(a, x)w(x) &= \\ &= \frac{1 + (1 - n)q}{N} + (m-1)q^2 \frac{1 + (1 - n)q}{N} + nq \frac{1 + (1 - m)q}{N} = 1, \end{aligned}$$

and for $b \in B$

$$\begin{aligned} \sum_{x \in A \cup B} Z(b, x)w(x) &= \\ &= mq \frac{1 + (1-n)q}{N} + \frac{1 + (1-m)q}{N} + (n-1)q^2 \frac{1 + (1-m)q}{N} = 1. \end{aligned}$$

This proves that w is a weighting on Z and thus the magnitude is

$$\begin{aligned} |K_{m,n}|_q &= \sum_{i=1}^{m+n} w(i) = m \frac{1 + (1-n)q}{N} + n \frac{1 + (1-m)q}{N} \\ &= \frac{(m+n)(1 + (1-m)q)}{(1+q)(1 + (1-m)q)(1 + (1-n)q)}. \end{aligned}$$

□

2.4 The Magnitude Function

In the following section, we introduce a second way to deal with the implicit choice we made in the definition of magnitude of a finite metric space. The idea is to not only look at the magnitude as one number like we did at the start, but consider the so-called magnitude function. To every finite metric space (A, d) , we consider the family of metric spaces $\{tA\}_{t>0}$ where tA denotes the metric space (A, td) , which is just a scaled version of (A, d) . If we look at the magnitude of each scaled version we get a function in t that captures how the scaling of the space changes the magnitude. This scaling gets rid of the implicit choice of the base because we add an additional factor $t > 0$ in the exponent of the entries in the symmetry matrix.

Definition 2.23. The magnitude function of a finite metric space (A, d) is the (partially defined) function $t \mapsto |tA|$ for $t \in \mathbb{R}_{>0}$.

Examples of the magnitude function already appeared in the introduction and in item ii) of Example 2.9. The next proposition that proves several properties about the magnitude function further supports the interpretation of magnitude as the effective number of points. But we will see in the example afterwards that the magnitude function does not always behave 'nicely'. To prove the proposition, we use Subsection 2.2.3.

Proposition 2.24 ([5, Proposition 2.2.6]). Let (A, d) be a finite metric space with n points.

- i) The magnitude function $|tA|$ is defined for all but finitely many $t > 0$.
- ii) For sufficiently large $t > 0$, the magnitude function is increasing in t .
- iii) The limit $\lim_{t \rightarrow \infty} |tA| = n$.

2.4. The Magnitude Function

Proof. Let $\mathbb{R}^{A \times A}$ denote the space of real $A \times A$ matrices, the subset $GL(A)$ of invertible matrices is an open subset of $\mathbb{R}^{A \times A}$. By entry-wise convergence, we find the following limit in $\mathbb{R}^{A \times A}$

$$Z_{tA} = \sum_{i=0}^{\infty} \frac{t^i}{i!} 1_A \in GL(A), \quad (2.4)$$

where 1_A denotes the identity matrix of $\mathbb{R}^{A \times A}$. Recall that for any matrix $Z \in GL(A)$, we have an explicit formula for the unique weighting w_Z on Z given by

$$w_Z(a) = \sum_{b \in A} Z^{-1}(a, b) = \sum_{b \in A} \frac{\text{adj}(Z)(a, b)}{\det(Z)} \quad \forall a \in A, \quad (2.5)$$

where $\text{adj}(Z)$ denotes the adjugate of Z .

i) The limit $\lim_{t \rightarrow \infty} Z_{tA} = 1_A$ lies in the open set $GL(A)$, hence there exists $T > 0$ such that $Z_{tA} \in GL(A)$ for all $t > T$, or in other words, the matrices Z_{tA} are invertible for t sufficiently large. We can define the matrix $Z_{tA} \in \mathbb{C}^{A \times A}$ given by $(Z_{tA})(a, b) = e^{t \delta(a, b)}$ for any $t \in \mathbb{C}$. The determinant $\det(Z_{tA})$ is a polynomial in the entries of Z_{tA} and thus a holomorphic function in t . By our first observation, we know that $\det(Z_{tA}) \neq 0$ for $t > T$, and thus by the holomorphicity of the determinant, there can only be finitely many zeros of $\det(Z_{tA})$ in $(0, \infty)$. It follows that $|tA|$ is defined for all but finitely many $t > 0$ by Remark 2.4.

ii) For this proof we use Lemma 2.17 about the magnitude of an expansion of a metric space, so we need to show two things. First, we show that for sufficiently large $t > 0$, the space tA has a positive weighting. For every $a \in A$, the function $Z^{-1} w_Z(a)$ is continuous on $GL(A)$ by the explicit formula (2.5). Let $a \in A$ be any point, since $w_{1_A}(a) = 1$, the continuity implies that there exists an open neighbourhood $U_a \subset GL(A)$ of 1_A such that for every $Z \in U_a$ the weighting $w_Z(a) > 0$. Taking the intersection $\bigcap_{a \in A} U_a$, we get an open neighbourhood $U \subset GL(A)$ of 1_A such that

$$\forall a \in A \quad \forall Z \in U \quad w_Z(a) > 0.$$

By the convergence in (2.4) it follows that $Z_{tA} \in U$ for sufficiently large t . Second, for any $t_1 > t_2$ large enough such that both t_1A and t_2A have a positive weighting and well defined magnitude, the space t_1A is an expansion of t_2A as we saw in Example 2.16. Finally, we can apply Lemma 2.17 and conclude that $|t_1A| \geq |t_2A|$ for sufficiently large $t_1 > t_2$.

iii) The explicit formula (2.5) for the unique weighting of an invertible matrix Z implies that the function which sends an invertible matrix

2.4. The Magnitude Function

$Z \in GL(A)$ to its magnitude $|Z|$ is continuous. For t large enough, the matrix Z_{tA} is invertible because of (2.4), hence we can calculate the limit

$$\lim_{t \rightarrow \infty} |Z_{tA}| = \lim_{t \rightarrow \infty} |Z_{tA}| = \lim_{t \rightarrow \infty} |Z_{tA}| = |1_A| = n.$$

□

Example 2.25 (Complete bipartite graph $K_{3,2}$). Let us study the magnitude function of the complete bipartite graph $K_{3,2}$. We can use Proposition 2.22. By replacing q with e^t we obtain the magnitude function

$$|K_{2,3}| = \sum_{i=1}^5 w(i) = \frac{5 - 7e^{-t}}{(1 + e^{-t})(1 - 2e^{-2t})},$$

which we have plotted in Figure 2.3. We see that for $\log(\sqrt{2})$, the magnitude

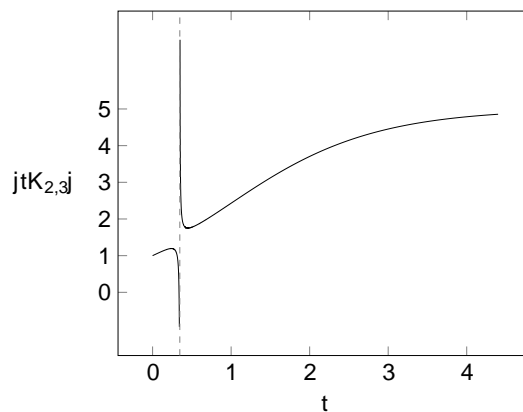


Figure 2.3: Magnitude function $|K_{2,3}|$.

function is not defined and hence the metric space $(\sqrt{2})K_{3,2}$ does not possess magnitude. Furthermore, all the properties of the magnitude function we have proven in Proposition 2.24 can be observed, but there are also some unexpected behaviours. There are intervals on which the magnitude function is decreasing, and it even takes on negative values for certain t . At some point $t > 0$, the magnitude is larger than 5, the number of points in $K_{3,2}$. Looking at the subspace $K_{3,1} \subset K_{3,2}$, we find another unexpected behaviour. By Proposition 2.22 the magnitude function of $K_{3,1}$

$$|K_{3,1}| = \frac{4 - 2e^{-t}}{1 + e^{-t}}.$$

For $t = 0.3$ we have $|K_{3,2}| = 1.09$ and $|K_{3,1}| = 1.45$ rounded up to two decimal points, so there is a strict subspace $K_{3,1}$ that has larger magnitude for some $t > 0$. To summarise, we have seen an example for a finite metric space that satisfies the following 'strange' properties

2.4. The Magnitude Function

- $\exists t > 0$ such that $\|j_t A_j\|$ is undetermined;
- $\exists t > 0$ such that $\|j_t A_j\|$ is decreasing at t ;
- $\exists t > 0$ such that $\|j_t A_j\| < 0$;
- $\exists B \subset A$ a strict subspace such that there is t_0 with $\|j_{t_0} B_j\| > \|j_{t_0} A_j\|$.

Example 2.26. In this example we calculate the magnitude function of the complete graph and cyclic graph using Proposition 2.12 about the magnitude of a homogeneous space. The complete graph K_n and the cyclic graph C_n are both homogeneous because of their symmetry. Thus, we find

$$\|j_t K_n\| = \frac{n^2}{n(n-1)e^{-t} + n} = \frac{n}{(n-1)e^{-t} + 1}.$$

For the cyclic graphs we look at the cases even and odd separately. If n is odd and we fix any vertex $a_0 \in C_n$, then for each $i = 1, \dots, \frac{n-1}{2}$ there are precisely two vertices that are distance i away from a_0 and thus

$$\|j_t C_n\| = \frac{n}{1 + 2 \sum_{i=1}^{\frac{n-1}{2}} e^{-ti}} = \frac{n(e^t - 1)}{2e^{\frac{t(n+1)}{2}} - e^t - 1}.$$

When n is even and $a_0 \in C_n$ is any fixed vertex, then there is precisely one vertex at distance $n/2$ from a_0 , and for every $i = 1, \dots, \frac{n}{2} - 1$ there are precisely two vertices at distance i from a_0 . Hence the magnitude function is

$$\|j_t C_n\| = \frac{n}{1 + 2 \sum_{i=1}^{\frac{n}{2}-1} e^{-ti} + e^{-\frac{tn}{2}}} = \frac{n(e^t - 1)}{(e^{\frac{tn}{2}} - 1)(e^t + 1)}.$$

Chapter 3

Magnitude Homology

In this chapter, we define the magnitude homology of a graph and see how it categorifies the magnitude from the previous chapter. We will compute the magnitude homology of several examples and also include some computer calculations from our own code that can be found in Appendix A.1. Furthermore, we describe the magnitude homology of a disjoint union of graphs and prove a Mayer-Vietoris type theorem for magnitude homology. Finally, there is a type of graph that has zero magnitude homology groups everywhere except on the diagonal. These graphs are the topic of the last section in this chapter. This section is based on [3]. We also need some homological algebra, as a reference we used [10].

3.1 The Definition of Magnitude Homology

We start by defining the magnitude homology of a graph and apply the definition to several examples. We deduce some basic properties and prove how the magnitude homology relates to the magnitude of a graph. There are also some computer calculated tables to illustrate the magnitude homology of some examples. Let us start by defining the setting we need throughout the chapter.

Definition 3.1. Let G be a graph. The length ℓ of a tuple $(x_0, \dots, x_k) \in G^{k+1}$ of vertices of G is the sum

$$\ell(x_0, \dots, x_k) = \sum_{i=0}^{k-1} d(x_i, x_{i+1}).$$

Note that for $k=0$ the length of a tuple (x_0) is $\ell(x_0) = 0$.

We use the notation $(x_0, \dots, \hat{x}_i, \dots, x_k)$ to denote the tuple we obtain by removing the entry x_i from the tuple (x_0, \dots, x_k) .

3.1. The Definition of Magnitude Homology

Remark 3.2. For every graph G and tuple (x_0, \dots, x_k) of vertices of G , the triangle inequality implies that

$$d(x_0, \dots, x_k) \leq d(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_k) + d(x_{i-1}, x_i) + d(x_i, x_{i+1}, \dots, x_k),$$

which we refer to as the triangle inequality of

Definition 3.3. The magnitude chain group $MC_{k,l}(G)$ of a graph G in bidegree (k, l) for $k, l \geq 0$ is the free abelian group generated by $(k+1)$ -tuples (x_0, \dots, x_k) of vertices of G satisfying $x_0 \in x_1 \in \dots \in x_k$ and $d(x_0, \dots, x_k) = l$.

We do allow a tuple $(x_0, \dots, x_k) \in MC_{k,l}(G)$ to have two equal entries, as long as they are not next to each other, see also Example 3.10 below.

Definition 3.4. For a graph G and integers $l \geq 0, k \geq 1$ we define the differential

$$\partial_l: MC_{k,l}(G) \rightarrow MC_{k-1,l}(G)$$

by the alternating sum $\partial_l = \sum_{i=1}^k (-1)^i \partial_{l,i}$, where $\partial_{l,i}: MC_{k,l}(G) \rightarrow MC_{k-1,l}(G)$ is defined on the generators by

$$\partial_{l,i}(x_0, \dots, x_k) = \begin{cases} (x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_k) & \text{if } d(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_k) = l \\ 0 & \text{else} \end{cases} \quad (3.1)$$

and then linearly extended to the whole group $MC_{k,l}(G)$.

The condition $d(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_k) = l$ in (3.1) says that the length l of the tuple (x_0, \dots, x_k) must be preserved if we remove the i -th entry of the tuple. This condition is equivalent to

$$d(x_{i-1}, x_i) + d(x_i, x_{i+1}) = d(x_{i-1}, x_{i+1}) \quad (3.2)$$

because these are the only summands that differ in $d(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_k)$ and $d(x_0, \dots, x_k)$. This also shows why the differential ∂_l is well defined. We know by definition that $x_{i-1} \in x_i$ and $x_i \in x_{i+1}$, so the distances $d(x_{i-1}, x_i)$ and $d(x_i, x_{i+1})$ are both greater than or equal to 1 and thus the right-hand-side is also greater than or equal to 1, which shows $x_{i-1} \in x_{i+1}$. Since we want to define a homology, we need a chain complex. The next lemma shows that $MC_{\cdot,l}(G)$ is a chain complex for any graph G and a fixed integer $l \geq 0$.

Lemma 3.5. For any graph G and integers $l \geq 0$ and $k \geq 2$, the composition

$$MC_{k,l}(G) \xrightarrow{\partial_l} MC_{k-1,l}(G) \xrightarrow{\partial_l} MC_{k-2,l}(G)$$

is equal to the zero map.

3.1. The Definition of Magnitude Homology

Proof. Let (x_0, \dots, x_k) be any generator of $MC_{k,l}(G)$, we start by showing that for all $0 \leq i < j \leq k$ the relation

$$\partial_i \partial_j(x_0, \dots, x_k) = \partial_{j-1} \partial_i(x_0, \dots, x_k) \quad (3.3)$$

holds. By definition and by using that $i < j$, the left-hand-side equals to $(x_0, \dots, \mathbf{x}_i, \dots, \mathbf{x}_j, \dots, x_k)$ if

$$\partial(x_0, \dots, \mathbf{x}_i, \dots, x_k) = 1 \text{ and } \partial(x_0, \dots, \mathbf{x}_j, \dots, \mathbf{x}_i, \dots, x_k) = 1 \quad (3.4)$$

and to 0 otherwise. The triangle inequality for ∂ from Remark 3.2 implies

$$\partial(x_0, \dots, \mathbf{x}_i, \dots, \mathbf{x}_j, \dots, x_k) - \partial(x_0, \dots, \mathbf{x}_j, \dots, \mathbf{x}_i, \dots, x_k) - \partial(x_0, \dots, x_k) = 1,$$

so the condition we have found in (3.4) is equivalent to the condition $\partial(x_0, \dots, \mathbf{x}_i, \dots, \mathbf{x}_j, \dots, x_k) = 1$. Analogously, the right-hand-side of (3.3) is equal to $(x_0, \dots, \mathbf{x}_i, \dots, \mathbf{x}_j, \dots, x_k)$ if

$$\partial(x_0, \dots, \mathbf{x}_i, \dots, \mathbf{x}_j, \dots, x_k) = 1 \text{ and } \partial(x_0, \dots, \mathbf{x}_j, \dots, x_k) = 1 \quad (3.5)$$

and to 0 otherwise. In this case one needs to pay attention because $i < j$ and by removing x_i first, the indices afterwards shift, so the $(j-1)$ th entry of the tuple $(x_0, \dots, \mathbf{x}_i, \dots, x_k)$ is not x_{j-1} but x_j . With the same argument as before, using the triangle inequality for ∂ , we see that the condition from (3.5) is equivalent to $\partial(x_0, \dots, \mathbf{x}_i, \dots, \mathbf{x}_j, \dots, x_k) = 1$ as well, which proves that the relation (3.3) does indeed hold. To conclude, we calculate the composition of the differential map with itself. First, we split the sum up:

$$\begin{aligned} \partial \partial &= \sum_{i=1}^{k-2} \binom{k-1}{i} \partial_i \sum_{j=1}^{k-1} \binom{k-1}{j} \partial_j = \\ &= \sum_{i=1}^{k-2} \sum_{j=1}^{i-1} \binom{k-1}{i} \binom{k-1}{j} (\partial_i \partial_j) + \sum_{i=1}^{k-2} \binom{k-1}{i} \binom{k-1}{i} (\partial_i \partial_i) + \sum_{i=1}^{k-2} \sum_{j=i+1}^{k-1} \binom{k-1}{i} \binom{k-1}{j} (\partial_i \partial_j) \end{aligned} \quad (3.6)$$

We can apply the relation from (3.3) to the last sum in this expression and reindex to obtain

$$\begin{aligned} \sum_{i=1}^{k-2} \sum_{j=i+1}^{k-1} \binom{k-1}{i} \binom{k-1}{j} (\partial_i \partial_j) &= \sum_{i=1}^{k-2} \sum_{j=i+1}^{k-1} \binom{k-1}{i} \binom{k-1}{j-1} (\partial_{j-1} \partial_i) \\ &= \sum_{i=1}^{k-2} \sum_{j=i}^{k-2} \binom{k-1}{i} \binom{k-1}{j} (\partial_j \partial_i). \end{aligned}$$

By adding the middle sum from (3.6), the terms with $i = j$ vanish:

$$\sum_{i=1}^{k-2} \binom{k-1}{i} \binom{k-1}{i} (\partial_i \partial_i) + \sum_{i=1}^{k-2} \sum_{j=i}^{k-2} \binom{k-1}{i} \binom{k-1}{j} (\partial_j \partial_i) = \sum_{i=1}^{k-2} \sum_{j=i+1}^{k-1} \binom{k-1}{i} \binom{k-1}{j} (\partial_j \partial_i)$$

3.1. The Definition of Magnitude Homology

In the first sum of (3.6) we exchange the order of summation to get

$$\sum_{i=1}^{k-2} \sum_{j=1}^{i-1} (-1)^{j+i} (\mathbb{1}_i - \mathbb{1}_j) = \sum_{j=1}^{k-3} \sum_{i=j+1}^{k-2} (-1)^{j+i} (\mathbb{1}_i - \mathbb{1}_j).$$

The last two results sum up to 0, so the composition of the differential with itself vanishes. □

Definition 3.6. The magnitude chain complex $MC_{\cdot, l}(G)$ of a graph G is the direct sum

$$\bigoplus_{l=0}^M MC_{\cdot, l}(G)$$

of chain complexes.

Remark 3.7. We mostly consider one component at a time of the magnitude chain complex of a graph G . Hence, for simplicity we also call the chain complex $MC_{\cdot, l}(G)$ for a fixed $l \geq 0$ the magnitude chain complex of G .

Definition 3.8. The magnitude homology $MH_{\cdot, l}(G)$ of a graph G is the bigraded abelian group defined by the homology groups

$$MH_{k, l}(G) = H_k(MC_{\cdot, l}(G))$$

for $k, l \geq 0$.

Before we compute the magnitude homology of some examples, we prove an immediate property which shows that some of the magnitude homology groups are always trivial.

Proposition 3.9. If $k > l$, then the magnitude homology $MH_{k, l}(G) = 0$ for any graph G .

Proof. Let $k > l \geq 0$ be arbitrary. For any two distinct vertices $x, y \in G$, their distance $d(x, y) \geq 1$ because we have defined to be the shortest path metric. Thus, any tuple (x_0, \dots, x_k) of vertices of G satisfying $x_0 \neq x_1 \neq \dots \neq x_k$ has length

$$l(x_0, \dots, x_k) = d(x_0, x_1) + \dots + d(x_{k-1}, x_k) \geq k > l.$$

Hence, there are no generators of $MC_{k, l}(G)$ and consequently also the magnitude homology $MH_{k, l}(G)$ vanishes. □

Example 3.10 (Four-cycle). Let us calculate the magnitude homology groups for the four-cycle C_4 for $0 \leq l \leq 2$. To do this, we look at a fixed and find the chain complex $MC_{\cdot, l}(C_4)$ and its homology groups. We denote the vertices of the graph by a_1, a_2, a_3, a_4 according to Figure 3.1.

3.1. The Definition of Magnitude Homology

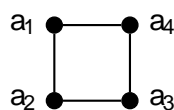


Figure 3.1: Four-cycle.

- Let $l = 0$.
 - $MC_{0,0}(C_4)$ is the free abelian group generated by the tuples of vertices in C_4 such that $\sum(x_0) = 0$. Every 1-tuple has length zero, so the generators for $MC_{0,0}(C_4)$ are the vertices of C

$$(a_1), (a_2), (a_3), (a_4)$$

- For $k > 0$, the groups $MC_{k,0}(C_4) = 0$ by Proposition 3.9 above.

Thus, the magnitude chain complex $MC_{\cdot,0}(C_4)$ is

$$\cdots \rightarrow 0 \rightarrow 0 \rightarrow MC_{0,0}(C_4) \rightarrow 0 \rightarrow \cdots$$

all its differentials are zero and the magnitude homology groups are

$$MH_{k,0}(C_4) = \begin{cases} MC_{0,0}(C_4) & \text{if } k = 0 \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} \mathbb{Z}^4 & \text{if } k = 0 \\ 0 & \text{else} \end{cases}$$

- Let $l = 1$.
 - $MC_{0,1}(C_4)$ is generated by tuples (x_0) of vertices in C_4 with length $\sum(x_0) = 1$, but tuples with exactly one entry have length 0, so the magnitude chain $MC_{0,1}(C_4) = 0$.
 - $MC_{1,1}(C_4)$ is generated by tuples (x_0, x_1) of two distinct vertices in C_4 with $\sum(x_0, x_1) = 1$, which means that x_0 and x_1 are neighbours in C_4 . So, the generators for $MC_{1,1}(C_4)$ are

$$(a_1, a_2), (a_1, a_4), (a_2, a_3), (a_2, a_1), (a_3, a_4), (a_3, a_2), (a_4, a_1), (a_4, a_3)$$

$$\text{and hence } MC_{1,1}(C_4) = \mathbb{Z}^8.$$

- As above, for $k > 1$ we have $MC_{k,1}(C_4) = 0$ by Proposition 3.9.

We get the magnitude chain complex $MC_{\cdot,1}(C_4)$:

$$\cdots \rightarrow 0 \rightarrow MC_{1,1}(C_4) \rightarrow 0 \rightarrow \cdots$$

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Again, all the differentials are zero and the magnitude homology groups are

$$\begin{aligned} \text{MH}_{k,1}(C_4) &= \begin{cases} \text{MC}_{1,1}(C_4) & \text{if } k = 1 \\ 0 & \text{else} \end{cases} \\ &= \begin{cases} \mathbb{Z}^8 & \text{if } k = 1 \\ 0 & \text{else} \end{cases} \end{aligned}$$

• Let $l = 2$.

- $\text{MC}_{0,2}(C_4) = 0$ as before.
- $\text{MC}_{1,2}(C_4)$ is generated by tuple (x_0, x_1) of two distinct vertices of C_4 such that $d(x_0, x_1) = 2$, that is, the shortest path between x_0 and x_1 has length 2. This yields the four generators

$$(a_1, a_3), (a_2, a_4), (a_3, a_1), (a_4, a_2),$$

$$\text{so } \text{MC}_{1,2}(C_4) = \mathbb{Z}^4.$$

- $\text{MC}_{2,2}(C_4)$ is generated by tuple (x_0, x_1, x_2) of vertices of C_4 such that $x_0 \neq x_1, x_1 \neq x_2$, and $d(x_0, x_1, x_2) = 2$. Because the neighbouring vertices are distinct, the distance between them is greater than or equal to 1, implying that for every generator of $\text{MC}_{2,2}(C_4)$ the distances $d(x_0, x_1) = 1$ and $d(x_1, x_2) = 1$. To calculate the number of generators of $\text{MC}_{2,2}(C_4)$, note that in the cyclic graph C_4 every vertex has exactly two neighbours. For x_0 we can take any of the four vertices and the vertices x_1 and x_2 each have to be one of the two neighbours of the previous entry. It follows that there are precisely $4 \cdot 2 \cdot 2 = 16$ generators, concretely they are given by

$$\begin{aligned} &(a_1, a_2, a_3), (a_1, a_2, a_1), (a_1, a_4, a_1), (a_1, a_4, a_3), \\ &(a_2, a_3, a_4), (a_2, a_3, a_2), (a_2, a_1, a_2), (a_2, a_1, a_4), \\ &(a_3, a_4, a_1), (a_3, a_4, a_3), (a_3, a_2, a_3), (a_3, a_2, a_1), \\ &(a_4, a_1, a_2), (a_4, a_1, a_4), (a_4, a_3, a_4), (a_4, a_3, a_2). \end{aligned}$$

- As before, for $k \geq 2$ we have $\text{MC}_{k,2}(C_4) = 0$ by Proposition 3.9.

The chain complex $\text{MC}_{\cdot,2}(C_4)$ is

$$0 \rightarrow \text{MC}_{2,2}(C_4) \xrightarrow{0} \text{MC}_{1,2}(C_4) \rightarrow 0,$$

so all the differentials except maybe $d_2: \text{MC}_{2,2}(C_4) \rightarrow \text{MC}_{1,2}(C_4)$ are zero and the magnitude homology groups $\text{MH}_{k,2}(C_4)$ all vanish except for maybe $k = 1$ or $k = 2$. To find the remaining magnitude homology groups, we need to

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determine the differential map. Let us calculate the image of all generators of $MC_{2,2}(C_4)$:

$$\begin{aligned}
 \mathbb{F}(a_1, a_2, a_3) &= (a_1, a_3) & \mathbb{F}(a_1, a_2, a_1) &= 0 \\
 \mathbb{F}(a_1, a_4, a_1) &= 0 & \mathbb{F}(a_1, a_4, a_3) &= (a_1, a_3) \\
 \mathbb{F}(a_2, a_3, a_4) &= (a_2, a_4) & \mathbb{F}(a_2, a_3, a_2) &= 0 \\
 \mathbb{F}(a_2, a_1, a_2) &= 0 & \mathbb{F}(a_2, a_1, a_4) &= (a_2, a_4) \\
 \mathbb{F}(a_3, a_4, a_1) &= (a_3, a_1) & \mathbb{F}(a_3, a_4, a_3) &= 0 \\
 \mathbb{F}(a_3, a_2, a_3) &= 0 & \mathbb{F}(a_3, a_2, a_1) &= (a_3, a_1) \\
 \mathbb{F}(a_4, a_1, a_2) &= (a_4, a_2) & \mathbb{F}(a_4, a_1, a_4) &= 0 \\
 \mathbb{F}(a_4, a_3, a_4) &= 0 & \mathbb{F}(a_4, a_3, a_2) &= (a_4, a_2)
 \end{aligned}$$

From this calculation we see that every generator of $MC_{1,2}(C_4)$ is in the image of \mathbb{F} , thus the magnitude homology $MH_{1,2}(C_4) = 0$. The remaining homology group $MH_{2,2}(C_4) = \ker(\mathbb{F}) \cap MC_{2,2}(C_4)$ because the image of $\mathbb{F}: MC_{3,2}(C_4) \rightarrow MC_{2,2}(C_4)$ is zero. Using the calculations of the differential above, we find that the kernel is the free abelian group generated by the twelve generators

$$\begin{aligned}
 &(a_1, a_2, a_1), (a_1, a_4, a_1), (a_2, a_3, a_2), (a_2, a_1, a_2), \\
 &(a_3, a_4, a_3), (a_3, a_2, a_3), (a_4, a_1, a_4), (a_4, a_3, a_4), \\
 &(a_1, a_2, a_3), (a_1, a_4, a_3), (a_2, a_3, a_4), (a_2, a_1, a_4), \\
 &(a_3, a_4, a_1), (a_3, a_2, a_1), (a_4, a_1, a_2), (a_4, a_3, a_2).
 \end{aligned}$$

Hence, the magnitude homology $MH_{2,2}(C_4) = \mathbb{Z}^{12}$.

These results could also be verified by computer calculations we did in SageMath. The code to the program can be found in Appendix A.1. Table 3.1 shows the ranks of the magnitude homology groups of the four-cycle computed with this program. It appears that only the diagonal entries are non-zero and the rank $\text{rank}(MC_{l,l}(C_4)) = 4(l + 1)$ for $l \geq 0$. Indeed, we prove later in Section 3.5 that the only non-zero homology groups of the four-cycle are on the diagonal. Furthermore, one approach to compute the ranks of the diagonal entries is by using a Künneth Theorem for magnitude homology [3, Chapter 5].

In the next example we describe the magnitude homology of the discrete graph E_n for $n \geq 1$, this is the graph on n vertices without any edges.

Example 3.11 (Discrete graph E_n). We need the following observations to understand the magnitude chain groups of the discrete graph for $n \geq 1$. Any two distinct vertices $x, y \in E_n$ have distance $d(x, y) = \infty$. A generator (x_0, \dots, x_k) of $MC_{k,l}(E_n)$ for $k, l \geq 0$ has finite length and the length of a tuple is always 0. Therefore, the only non-trivial magnitude chain group of E_n is $MC_{0,0}(E_n)$ and it is

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$l \setminus k$	0	1	2	3	4	5	6	7	8	9	10
0	4	0	0	0	0	0	0	0	0	0	0
1	0	8	0	0	0	0	0	0	0	0	0
2	0	0	12	0	0	0	0	0	0	0	0
3	0	0	0	16	0	0	0	0	0	0	0
4	0	0	0	0	20	0	0	0	0	0	0
5	0	0	0	0	0	24	0	0	0	0	0
6	0	0	0	0	0	0	28	0	0	0	0
7	0	0	0	0	0	0	0	32	0	0	0
8	0	0	0	0	0	0	0	0	36	0	0
9	0	0	0	0	0	0	0	0	0	40	0
10	0	0	0	0	0	0	0	0	0	0	44

Table 3.1: Ranks of the magnitude homology of C_4 computed with SageMath.

the free abelian group generated by tuples (x_0) for $x_0 \in E_n$, this is the free abelian group generated by the vertices of E_n . All differential maps of the magnitude chain complex must be zero, so the magnitude homology groups are

$$MH_{k,l}(E_n) = MC_{k,l}(E_n) = \begin{cases} \mathbb{Z}^n & \text{if } k = l = 0 \\ 0 & \text{else} \end{cases}$$

Example 3.12 (Complete graph K_n). In this example, we calculate the homology groups for the complete graph K_n for any $n \geq 2$. First, we need to understand the magnitude chain complex of K_n . In the complete graph, the shortest path distance between any two distinct vertices is 1. Thus, any tuple (x_0, \dots, x_k) of vertices of K_n such that $x_0 \neq \dots \neq x_k$ has length

$$l(x_0, \dots, x_k) = \sum_{i=0}^{k-1} d(x_i, x_{i+1}) = k.$$

It follows that for any $l \geq 0$ the only non-zero magnitude chain group in the chain complex $MC_{\cdot,l}(K_n)$ is $MC_{l,l}(K_n)$ and the generators for this group are all tuples (x_0, \dots, x_l) of vertices of K_n such that $x_0 \neq \dots \neq x_l$. There are precisely $(n-1)!$ generators. Because all differentials in the magnitude chain complex are zero, the magnitude homology groups of the complete graph are

$$MH_{k,l}(K_n) = MC_{k,l}(K_n) = \begin{cases} \mathbb{Z}^{n(n-1)!} & \text{if } k = l \\ 0 & \text{else} \end{cases}$$

In all of the examples above, the magnitude homology groups in bidegree $(0, 0)$ and $(1, 1)$ were free abelian and generated by the vertices and the oriented edges respectively, the next proposition shows that this is true in general. Before stating it, let us introduce some notation. First, we write $\mathbb{Z}A$ for

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the free abelian group on the set A . Second, for a graph G we denote by $\overset{\#}{E}(G)$ the set of oriented edges of G , that is $\overset{\#}{E}(G) = \{f(x, y) \mid f(x, y) \in E(G)\}$. So, for any edge $f(x, y) \in E(G)$ there are two oriented edges $(x, y), (y, x) \in \overset{\#}{E}(G)$.

Proposition 3.13. For any graph G , the magnitude homology groups satisfy:

- i) $MH_{0,0}(G) = \mathbb{Z} V(G)$;
- ii) $MH_{1,1}(G) = \mathbb{Z} \overset{\#}{E}(G)$.

Proof. i) The magnitude chain group $MC_{0,0}(G)$ is generated by tuples (x_0) of vertices in G such that $\ell(x_0) = 0$. Every 1-tuple has length 0, so $MC_{0,0}(G)$ is the free abelian group on the vertices of G . By Proposition 3.9, all other groups in the chain complex $MC_{\cdot,0}(G)$ are zero, so all the differential maps are zero and $MH_{0,0}(G) = MC_{0,0}(G) = \mathbb{Z} V(G)$.

ii) The magnitude chain group $MC_{0,1}(G) = 0$ because, as stated above, all the 1-tuples have length 0. By Proposition 3.9, also the groups $MC_{k,1}(G)$ for $k > 1$ vanish. Hence, all the differentials in $MC_{\cdot,1}(G)$ are zero and the magnitude homology group $MH_{1,1}(G) = MC_{1,1}(G)$. The group $MC_{1,1}(G)$ is generated by tuples (x_0, x_1) of distinct vertices in G with length $\ell(x_0, x_1) = 1$, that is if $(x_0, x_1) \in \overset{\#}{E}(G)$ is an oriented edge of the graph G .

□

Remark 3.14. For every graph G , Proposition 3.13 shows that

- i) $\text{rank}(MH_{0,0}(G)) = \#V(G)$;
- ii) $\text{rank}(MH_{1,1}(G)) = 2 \#E(G)$.

The next theorem we prove is justifying the name magnitude homology by establishing a connection between the magnitude homology and the magnitude from Chapter 2. First, we need a lemma that describes another way to compute the magnitude of a graph.

Lemma 3.15 ([6, Prop. 3.9.]). For any graph G

$$|G|_q = \sum_{k=0}^{\infty} (-1)^k \sum_{\substack{x_0, \dots, x_k \in G \\ x_0 \neq \dots \neq x_k}} q^{\ell(x_0, \dots, x_k)}.$$

Proof. We calculate the magnitude using a weighting w_G . For any vertex $x \in G$ we define

$$w_G(x) = \sum_{k=0}^{\infty} (-1)^k \sum_{\substack{x_1, \dots, x_k \in G \\ x \neq x_1 \neq \dots \neq x_k}} q^{\ell(x, x_1, \dots, x_k)}.$$

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Let $y \in G$ be any vertex. To show that w_G is a weighting we need to verify that

$$\sum_{x \in G} q^{d(y,x)} w_G(x) = 1.$$

We can start rewriting the left-hand-side by splitting up the sum and using the definition of w_G :

$$\sum_{x \in G} q^{d(y,x)} w_G(x) = w_G(y) + \sum_{x \in G \setminus \{y\}} q^{d(y,x)} \sum_{k=0}^{\infty} \binom{-1}{k} q^k \sum_{\substack{x_1, \dots, x_k \in G \\ x \in \{x_1, \dots, x_k\}}} q^{d(x, x_1, \dots, x_k)}$$

After rearranging the sums and plugging in the definition for $w_G(y)$ we obtain

$$\begin{aligned} & \sum_{k=0}^{\infty} \binom{-1}{k} q^k \sum_{\substack{x_1, \dots, x_k \in G \\ y \in \{x_1, \dots, x_k\}}} q^{d(y, x_1, \dots, x_k)} + \sum_{k=0}^{\infty} \binom{-1}{k} q^k \sum_{\substack{x, x_1, \dots, x_k \in G \\ y \in \{x, x_1, \dots, x_k\}}} q^{d(y, x, x_1, \dots, x_k)} \\ &= \sum_{k=0}^{\infty} \binom{-1}{k} q^k \sum_{\substack{x_1, \dots, x_k \in G \\ y \in \{x_1, \dots, x_k\}}} q^{d(y, x_1, \dots, x_k)} + \sum_{k=0}^{\infty} \binom{-1}{k} q^k \sum_{\substack{x, x_1, \dots, x_k \in G \\ y \in \{x, x_1, \dots, x_k\}}} q^{d(y, x, x_1, \dots, x_k)} = 1, \end{aligned}$$

where the last equation holds because it is a telescoping sum. Therefore, w_G is a weighting and we can calculate the magnitude

$$\begin{aligned} |G|_q &= \sum_{x \in G} w_G(x) = \sum_{x \in G} \sum_{k=0}^{\infty} \binom{-1}{k} q^k \sum_{\substack{x_1, \dots, x_k \in G \\ x \in \{x_1, \dots, x_k\}}} q^{d(x, x_1, \dots, x_k)} \\ &= \sum_{k=0}^{\infty} \binom{-1}{k} q^k \sum_{\substack{x_0, \dots, x_k \in G \\ x_0 \in \{x_0, \dots, x_k\}}} q^{d(x_0, \dots, x_k)}. \end{aligned}$$

□

Theorem 3.16. Let G be a graph, then

$$\sum_{k,l=0}^{\infty} \binom{-1}{k} \text{rank}(\text{MH}_{k,l}(G)) q^l = |G|_q.$$

Proof. For any graded abelian group $C = \bigoplus_{i \in \mathbb{Z}} C_i$ the Euler characteristic $c(C)$ is defined to be the alternating sum $c(C) = \sum_{i \in \mathbb{Z}} (-1)^i \text{rank}(C_i)$. By a known result, which can be found in [2, Theorem 2.44], the Euler characteristic satisfies

$$c(\text{MC}_{\bullet, \bullet}(G)) = c(\text{MH}_{\bullet, \bullet}(G)),$$

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for all $l \geq 0$. Using this relation, we calculate

$$\begin{aligned} \sum_{k,l \geq 0} (-1)^k \text{rank}(\text{MH}_{k,l}(G)) q^l &= \sum_{l \geq 0} \text{c}(\text{MH}_{\cdot,l}(G)) q^l \\ &= \sum_{l \geq 0} \text{c}(\text{MC}_{\cdot,l}(G)) q^l \\ &= \sum_{k,l \geq 0} (-1)^k \text{rank}(\text{MC}_{k,l}(G)) q^l. \end{aligned}$$

The magnitude chain group $\text{MC}_{k,l}(G)$ is a free abelian group, so its rank is just the number of generators:

$$\text{rank}(\text{MC}_{k,l}(G)) = \#\{(x_0, \dots, x_k) \in G^{k+1} \mid x_0 \in \dots \in x_k \wedge (x_0, \dots, x_k) = l\}$$

Consider the sum

$$\sum_{l \geq 0} \#\{(x_0, \dots, x_k) \in G^{k+1} \mid x_0 \in \dots \in x_k \wedge (x_0, \dots, x_k) = l\} q^l,$$

if we sum over tuples instead of summing over $l \geq 0$, we see that the above sum is equal to

$$\sum_{\substack{(x_0, \dots, x_k) \in G^{k+1} \\ x_0 \in \dots \in x_k}} q^{(x_0, \dots, x_k)}.$$

Combining all these observations, we conclude that

$$\sum_{k,l \geq 0} (-1)^k \text{rank}(\text{MH}_{k,l}(G)) q^l = \sum_{k \geq 0} (-1)^k \sum_{\substack{(x_0, \dots, x_k) \in G^{k+1} \\ x_0 \in \dots \in x_k}} q^{(x_0, \dots, x_k)}$$

and by Lemma 3.15 above, the statement follows. □

In other words, this theorem tells us that taking the graded Euler characteristic of the magnitude homology of a graph returns the magnitude as a power series. Let us illustrate this relation on some examples.

Example 3.17 (Complete graph). We have calculated both the magnitude and the magnitude homology for the complete graph. From Example 3.12, we know the ranks of the magnitude homology are

$$\text{rank}(\text{MH}_{k,l}(K_n)) = \begin{cases} n(n-1)^l & \text{if } k = l \\ 0 & \text{else} \end{cases}$$

By the theorem above, the magnitude of the complete graph is equal to the power series

$$|K_n|_q = \sum_{k,l \geq 0} (-1)^k \text{rank}(\text{MH}_{k,l}(K_n)) q^l = \sum_{l \geq 0} (-1)^l n(n-1)^l q^l.$$

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This is a geometric series and is equal to the rational function

$$\sum_{l=0}^{\infty} (n-1)^l (n-1)^l q^l = n \frac{1}{1 - ((n-1)q)} = \frac{n}{(n-1)q + 1},$$

which is indeed what we first calculated in Example 2.26.

Example 3.18 (Five-cycle). With our code for SageMath, we could obtain the following table of the ranks of the magnitude homology groups $\text{MH}_{k,l}(C_5)$ of the five-cycle fork, $l \leq 10$. This table can be used to find the first coefficients of the

$l \backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	5	0	0	0	0	0	0	0	0	0	0
1	0	10	0	0	0	0	0	0	0	0	0
2	0	0	10	0	0	0	0	0	0	0	0
3	0	0	10	10	0	0	0	0	0	0	0
4	0	0	0	30	10	0	0	0	0	0	0
5	0	0	0	0	50	10	0	0	0	0	0
6	0	0	0	0	20	70	10	0	0	0	0
7	0	0	0	0	0	80	90	10	0	0	0
8	0	0	0	0	0	0	180	110	10	0	0
9	0	0	0	0	0	0	40	320	130	10	0
10	0	0	0	0	0	0	0	200	500	150	10

Table 3.2: Ranks of the magnitude homology of C_5 computed with SageMath.

magnitude of the five-cycle by applying Theorem 3.16. We have to take the alternating sum of each row, which results in

$$\begin{aligned} |jC_5|_q &= 5 - 10q + 10q^2 + (10 - 10)q^3 + (30 + 10)q^4 + (50 - 10)q^5 \\ &\quad + (20 - 70 + 10)q^6 + (80 + 90 - 10)q^7 + (180 - 110 + 10)q^8 \\ &\quad + (40 - 320 + 130 - 10)q^9 + (200 + 500 - 150 + 10)q^{10} + \dots \\ &= 5 - 10q + 10q^2 - 20q^4 + 40q^5 - 40q^6 + 80q^8 - 160q^9 + 160q^{10} + \dots \end{aligned}$$

In Example 2.26, we saw that the magnitude of the five-cycle is the rational function

$$|jC_5|_q = \frac{5(q-1)}{2q^3 - q - 1},$$

which is indeed equal to the power series starting with the same terms as the one we found above.

Notice that in Table 3.2, the non-zero ranks of the magnitude homology are all in the lower triangle, as we have proved in Proposition 3.9. But furthermore, one observes that the non-zero ranks are still somewhat concentrated

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around the diagonal. The next result will give some explanation for that phenomenon. Recall that the diameter d of a graph G is the maximum length of a shortest path between any two vertices in the graph, formally

$$d = \max_{v \in V(G)} \left(\max_{u \in V(G)} d(v, u) \right).$$

Proposition 3.19. Let G be a graph and suppose that for some $0 < l < k$ the magnitude homology $MH_{k,l}(G) \neq 0$. If G has finite diameter $d > 0$, then $\frac{l}{d} < k$ and moreover, if $d > 1$ and $l > 0$, then $\frac{l}{d} < k$.

Proof. If the magnitude homology $MH_{k,l}(G) \neq 0$, then also the magnitude chain group $MC_{k,l}(G)$ is non-trivial and hence there exists a generator $(x_0, \dots, x_k) \in MC_{k,l}(G)$. In particular, the tuple (x_0, \dots, x_k) satisfies $x_0 \neq \dots \neq x_k$ and $\sum_{i=0}^k d(x_i, x_{i+1}) = l$. By definition of the diameter d , every pair of vertices $x, y \in G$ has distance $d(x, y) \leq d$. Thus,

$$l = \sum_{i=0}^k d(x_i, x_{i+1}) \leq (k+1)d, \quad (3.7)$$

which implies $\frac{l}{d} \leq k+1$.

Assume now that $d > 1$ and $l > 0$. We prove the statement by contradiction, so suppose that $\frac{l}{d} = k+1$. As above, let $(x_0, \dots, x_k) \in MC_{k,l}(G)$ be a generator, then

$$\sum_{i=0}^k d(x_i, x_{i+1}) = l = (k+1)d.$$

From Equation (3.7), we see that this is only possible if for all $i = 0, \dots, k-1$ the distance $d(x_i, x_{i+1}) = d$. The diameter is greater than 1 by assumption, hence in particular $d(x_0, x_1) = d > 1$. Thus, there exists an internal vertex $y \in G$ on a shortest path between x_0 and x_1 , this vertex divides the shortest path from x_0 to x_1 in a shortest path from x_0 to y and a shortest path from y to x_1 , therefore

$$d(x_0, y) + d(y, x_1) = d(x_0, x_1).$$

It follows that $\sum_{i=0}^k d(x_i, x_{i+1}) = (k+1)d$. To finish the proof, we show that the generator (x_0, \dots, x_k) is the image of $(x_0, y, x_1, \dots, x_k)$ under the differential ∂ , which implies that every generator of $MC_{k,l}(G)$ vanishes in homology, contradicting the assumption that $MH_{k,l}(G) \neq 0$. So, we need to calculate $\sum_{i=0}^k d(x_i, x_{i+1})$ for $i = 2, \dots, k-1$. Note that because $y \neq x_1$ the sum of the distances

$$d(y, x_1) + d(x_1, x_2) > d(x_1, x_2),$$

which shows that $\sum_{i=2}^k d(x_i, x_{i+1}) = 0$. For $i = 3, \dots, k-1$ we have

$$d(x_{i-1}, x_i) + d(x_i, x_{i+1}) = 2d > d(x_i, x_{i+2}),$$

implying $\sum_{i=3}^k d(x_i, x_{i+1}) = 0$. Altogether, we find

$$\partial((x_0, y, x_1, \dots, x_k)) = (x_0, \dots, x_k),$$

which concludes the proof. □

Example 3.20. Let us look again at the ve-cycle and Table 3.2. The diameter of the ve-cycle is 2. Thus, by the result above, the non-zero homology groups $H_k(C_5)$ satisfy

$$\frac{1}{2} \leq k \leq 1,$$

so they are not too far off the diagonal.

3.2 Induced Maps

The goal of this chapter is to categorify the magnitude. Concretely, note that the magnitude itself can be viewed as an element in the set of power series in one formal variable with integer coefficients. In this section, however, we will see that there is a functor between the category of graphs and the category of graded abelian groups, that associates to a graph its magnitude homology. This categorification allows us to prove results about magnitude that could already be shown in [6] from a different viewpoint. We use basic notions from category theory in this section, see [4] as a reference. Let us first define what we mean by the category of graphs, in particular we need to define the maps between graphs.

Definition 3.21. For graphs G and H we define a map of graphs or morphism of graphs $f : G \rightarrow H$ to be a map $f : V(G) \rightarrow V(H)$ on the vertex sets such that

$$\forall x, y \in E(G) \quad f(x), f(y) \in E(H) \text{ or } f(x) = f(y).$$

In other words, a map of graphs $f : G \rightarrow H$ sends an edge in G either to an edge in H or contracts the edge to a single vertex in H . Thus, f sends a path in G to a path in H , which might have fewer edges. This leads us to the following observation.

Remark 3.22. Alternatively, we can define a map of graphs $f : G \rightarrow H$ to be a map on the vertex sets $f : V(G) \rightarrow V(H)$ such that

$$\forall x, y \in V(G) \quad d_H(f(x), f(y)) \leq d_G(x, y).$$

With this second definition we can easily see that for any map $f : G \rightarrow H$ between graphs, the length of any tuple $(x_0, \dots, x_k) \in G^{k+1}$ satisfies

$$d(f(x_0), \dots, f(x_k)) \leq d(x_0, \dots, x_k) \tag{3.8}$$

The set of graphs together with the maps of graphs form the category of graphs, this is indeed a category, which can be verified in a straightforward way. As mentioned above, we define a functor from the category of graphs to the category of bigraded abelian groups.

Definition 3.23. If $f: G \rightarrow H$ is a map of graphs, the induced chain map $f_{\#}: MC_{\bullet}(G) \rightarrow MC_{\bullet}(H)$ is defined on generators by

$$f_{\#}(x_0, \dots, x_k) = \begin{cases} (f(x_0), \dots, f(x_k)) & \text{if } \langle f(x_0), \dots, f(x_k) \rangle = \langle x_0, \dots, x_k \rangle \\ 0 & \text{else} \end{cases}$$

Proposition 3.24. Let $f: G \rightarrow H$ be a map of graphs. The above defined induced map $f_{\#}$ is indeed a chain map, that is, it commutes with the differential

$$f_{\#} \partial = \partial f_{\#}.$$

Proof. It is enough to verify the commutativity on the generators. Let $k, l \geq 0$ and let $(x_0, \dots, x_k) \in MC_{k,l}(G)$ be a generator. Similarly to the proof of Lemma 3.5, we show that for any $i \in \{1, \dots, k-1\}$ the equation

$$(f_{\#} \partial_i)(x_0, \dots, x_k) = (\partial_i f_{\#})(x_0, \dots, x_k) \tag{3.9}$$

holds. The left-hand-side is equal to $(f(x_0), \dots, \hat{x}_i, \dots, f(x_k))$ if

$$\langle x_0, \dots, \hat{x}_i, \dots, x_k \rangle = l \text{ and } \langle f(x_0), \dots, \hat{f}(x_i), \dots, f(x_k) \rangle = l$$

and to 0 otherwise. To simplify the condition, note that by the triangle inequality for $\langle \cdot \rangle$ from Remark 3.2 and the inequality (3.8) we have

$$\langle f(x_0), \dots, \hat{f}(x_i), \dots, f(x_k) \rangle - \langle x_0, \dots, \hat{x}_i, \dots, x_k \rangle - \langle x_0, \dots, x_k \rangle = l.$$

Thus, the left-hand-side of Equation (3.9) is non-zero if and only if

$$\langle f(x_0), \dots, \hat{f}(x_i), \dots, f(x_k) \rangle = l.$$

The right-hand-side of (3.9) is equal to $(f(x_0), \dots, \hat{f}(x_i), \dots, f(x_k))$ if

$$\langle f(x_0), \dots, f(x_k) \rangle = l \text{ and } \langle f(x_0), \dots, \hat{f}(x_i), \dots, f(x_k) \rangle = l$$

and to zero otherwise. Continuing as we did for the left-hand-side, we find the inequality

$$\langle f(x_0), \dots, \hat{f}(x_i), \dots, f(x_k) \rangle - \langle f(x_0), \dots, f(x_k) \rangle - \langle x_0, \dots, x_k \rangle = l,$$

from which we can conclude that both sides of (3.9) are equal. The differential is the alternating sum $\partial = \sum_{i=1}^k (-1)^i \partial_i$, so the result follows. \square

Definition 3.25. Let $f: G \rightarrow H$ be a map of graphs. The induced map in homology is the map

$$f : MH_{\bullet}(G) \rightarrow MH_{\bullet}(H)$$

induced by the chain map $f_{\#}$.

Proposition 3.26. The assignment sending any graph G to its magnitude homology $MH_{\bullet, \bullet}(G)$, and any map of graphs to its induced map $f_{\#}$ is a functor from the category of graphs to the category of bigraded abelian groups.

Proof. We need to verify that

- i) for any graph G , the induced map of the identity $(1_G)_{\#} = 1_{MH_{\bullet, \bullet}(G)}$;
- ii) for any two maps of graphs $f: G \rightarrow H, g: H \rightarrow K$, the composition satisfies $(g \circ f)_{\#} = g_{\#} \circ f_{\#}$.

The first item follows immediately from the definitions; the chain map induced by the identity 1_G is the identity on the chains $MC_{\bullet, \bullet}(G)$ and thus the induced map in homology is the identity on $MH_{\bullet, \bullet}(G)$.

For the second item, let $f: G \rightarrow H, g: H \rightarrow K$ be two maps of graphs. We proceed similarly as in Lemma 3.5 and in the previous proposition. Consider first the chains. So, let $k, l \geq 0$, and let $(x_0, \dots, x_k) \in MC_{k, l}(G)$ be a generator. On the one hand, from the definition it follows that $(g \circ f)_{\#}(x_0, \dots, x_k)$ equals to $((g \circ f)(x_0), \dots, (g \circ f)(x_k))$ if

$$\langle (g \circ f)(x_0), \dots, (g \circ f)(x_k) \rangle = 1$$

and to 0 otherwise. On the other hand, the image $(g_{\#} \circ f_{\#})(x_0, \dots, x_k)$ equals to $(g(f(x_0)), \dots, g(f(x_k)))$ if

$$\langle f(x_0), \dots, f(x_k) \rangle = 1 \text{ and } \langle g(f(x_0)), \dots, g(f(x_k)) \rangle = 1$$

and to 0 otherwise. By using the inequality (3.8) we obtain

$$\langle (g \circ f)(x_0), \dots, (g \circ f)(x_k) \rangle \leq \langle f(x_0), \dots, f(x_k) \rangle \leq \langle x_0, \dots, x_k \rangle = 1,$$

so the composition $(g_{\#} \circ f_{\#})(x_0, \dots, x_k)$ is non-zero if and only if

$$\langle g(f(x_0)), \dots, g(f(x_k)) \rangle = 1.$$

We can conclude that $(g_{\#} \circ f_{\#}) = (g \circ f)_{\#}$ and therefore also in homology $(g \circ f)_{\#} = (g_{\#} \circ f_{\#})$. \square

The induced maps in bidegrees $(0, 0)$ and $(1, 1)$ can be described concretely. Recall that by Proposition 3.13, for any graph G

- the magnitude homology $MH_{0,0}(G) = MC_{0,0}(G)$ is the free abelian group generated by the vertices of G ;
- the magnitude homology $MH_{1,1}(G) = MC_{1,1}(G)$ is the free abelian group generated by the oriented edges of G .

This is why we speak of a vertex and an oriented edge in $MH_{0,0}(G)$ and $MH_{1,1}(G)$ respectively, and not of their equivalence classes.

Proposition 3.27. Let $f: G \rightarrow H$ be a map of graphs, then

- i) the induced map $f_{\#}: MH_{0,0}(G) \rightarrow MH_{0,0}(H)$ sends a vertex $x \in V(G)$ to the vertex $f(x) \in V(H)$;
- ii) the induced map $f_{\#}: MH_{1,1}(G) \rightarrow MH_{1,1}(H)$ sends $(x, y) \in \overset{\#}{E}(G)$ to the oriented edge $(f(x), f(y)) \in \overset{\#}{E}(H)$ if that is an edge, and to 0 otherwise.

Proof. i) The induced map $f_{\#}$ sends a vertex $(x_0) \in MC_{0,0}(G)$ to $(f(x_0))$ if $\text{length}(f(x_0)) = 0$ by definition. But the length of a 1-tuple is always 0, so for any $(x_0) \in MC_{0,0}(G)$ we have

$$f_{\#}((x_0)) = (f(x_0)).$$

- ii) The induced chain map is defined on a generator $(x_0, x_1) \in MC_{1,1}(G)$ by

$$(x_0, x_1) \mapsto \begin{cases} (f(x_0), f(x_1)) & \text{if } (f(x_0), f(x_1)) \in \overset{\#}{E}(H) \\ 0 & \text{else.} \end{cases}$$

Because f is a map of graphs, we have

$$d_H(f(x_0), f(x_1)) = d_G(x_0, x_1) = 1.$$

Hence, either $d_H(f(x_0), f(x_1)) = 1$ which means $(f(x_0), f(x_1)) \in \overset{\#}{E}(H)$ and $f_{\#}((x_0, x_1)) = (f(x_0), f(x_1))$, or $d_H(f(x_0), f(x_1)) = 0$ implying that $f_{\#}((x_0, x_1)) = 0$.

□

Definition 3.28. A map of graphs $f: G \rightarrow H$ is an isomorphism of graphs if there exists a map of graphs $g: H \rightarrow G$ which is an inverse to f , that is, such that $f \circ g = 1_H$ and $g \circ f = 1_G$.

Remark 3.29. We can characterise an isomorphism of graphs in the following way. A map $f: V(G) \rightarrow V(H)$ on the vertex sets of two graphs is an isomorphism of graphs if it is bijective and if

$$(x, y) \in E(G) \iff (f(x), f(y)) \in E(H). \tag{3.10}$$

Indeed, such a map is a map of graphs and because of the property (3.10) its inverse f^{-1} is also a map of graphs. For the other direction let $f: G \rightarrow H$ be an isomorphism of graphs as in the Definition 3.28 above and let $g: H \rightarrow G$ be its mutual inverse. Being mutually inverse to each other, the maps f and g must be bijections on the vertex sets of the graphs. Together with the fact that they are also maps of graphs, we obtain the implication $(x, y) \in E(G) \implies (f(x), f(y)) \in E(H)$ from f and the implication $(f(x), f(y)) \in E(H) \implies (g(f(x)), g(f(y))) \in E(G) = f^{-1} \circ (x, y) \in E(G)$ from g .

Corollary 3.30. Let $f : G \rightarrow H$ be a map of graphs. If the induced map in homology $f : MH_*(G) \rightarrow MH_*(H)$ is an isomorphism, that is, f is a group isomorphism in each bidegree, then f is an isomorphism of graphs.

Proof. If $f : MH_*(G) \rightarrow MH_*(H)$ is an isomorphism, then in particular $f : MH_{0,0}(G) \rightarrow MH_{0,0}(H)$ is an isomorphism. By Proposition 3.27, the map $f : V(G) \rightarrow V(H)$ on the vertices is bijective. Let us now check the Condition (3.10). The map $f : MH_{1,1}(G) \rightarrow MH_{1,1}(H)$ is an isomorphism as well, which implies that the image of any oriented edge $(x_0, x_1) \in \vec{E}(G)$ under f is non-zero, hence $f(x_0), f(x_1) \in E(H)$ is an edge in H by Proposition 3.27. Conversely, any edge $f(y_0), f(y_1) \in E(H)$ is in the image of f , so $f^{-1}(y_0), f^{-1}(y_1) \in E(G)$ is an edge in G . \square

3.3 Disjoint Unions

This section shows how the magnitude homology behaves with respect to disjoint unions. As a consequence, we obtain that the magnitude of two disjoint graphs is the sum of the magnitudes of each of the graphs.

Proposition 3.31. Let G and H be two graphs, we denote $G \sqcup H$ their disjoint union. Let $i : G \rightarrow G \sqcup H$ and $j : H \rightarrow G \sqcup H$ be the inclusion maps. The induced map on the direct sum

$$i \oplus j : MH_*(G) \oplus MH_*(H) \rightarrow MH_*(G \sqcup H)$$

is an isomorphism.

Proof. We first consider the chain level, let $k, l \geq 0$. Let (x_0, \dots, x_k) be a generator of the chain group $MC_{k,l}(G \sqcup H)$, so the length $\ell(x_0, \dots, x_k) = l < \infty$ and in particular for all $i \in \{0, \dots, k-1\}$ the distance

$$d_{G \sqcup H}(x_i, x_{i+1}) \leq \ell(x_0, \dots, x_k) < \infty.$$

It follows that the vertices x_0, \dots, x_k all lie either in G or all lie in H . With this observation, we define the map $h : MC_{k,l}(G \sqcup H) \rightarrow MC_{k,l}(G) \oplus MC_{k,l}(H)$ defined on the generators by

$$h(x_0, \dots, x_k) = \begin{cases} ((x_0, \dots, x_k), 0) & \text{if } x_0, \dots, x_k \in V(G) \\ (0, (x_0, \dots, x_k)) & \text{if } x_0, \dots, x_k \in V(H) \end{cases}$$

and extend linearly. By considering the images of the generators, we see that h is an inverse to $i \oplus j$, hence also the induced map in homology $i \oplus j$ is an isomorphism. \square

Corollary 3.32. The magnitude of the disjoint union of two graphs G and H satisfies

$$|G \sqcup H|_q = |G|_q + |H|_q.$$

Proof. The rank of the direct sum of two abelian groups A and B satisfies $\text{rank}(A \oplus B) = \text{rank}(A) + \text{rank}(B)$. Thus, the Euler characteristic of a direct sum of two graded abelian groups C and D is $c(C \oplus D) = c(C) + c(D)$. Using this together with Proposition 3.31 above and Theorem 3.16 about the relation of magnitude and magnitude homology, we obtain

$$\begin{aligned} |G \oplus H|_q &= \sum_{l=0}^{\infty} c(\text{MH}_{l,1}(G \oplus H)) q^l \\ &= \sum_{l=0}^{\infty} c(\text{MH}_{l,1}(G)) q^l + \sum_{l=0}^{\infty} c(\text{MH}_{l,1}(H)) q^l \\ &= |G|_q + |H|_q. \end{aligned}$$

□

3.4 The Mayer-Vietoris Sequence

In this section, we prove a Mayer-Vietoris type theorem for the magnitude homology, in fact, we obtain a split short exact Mayer-Vietoris sequence. Again, we can use the result about magnitude homology to deduce the so-called inclusion-exclusion principle for magnitude. We finish the section by using the Mayer-Vietoris theorem to compute the magnitude homology of trees and wedge sums of graphs. In this section we consider a connected graph X . The connectivity simplifies some definitions but is not a real constraint because with the result from last section we can describe the magnitude of a disconnected graph by looking at its connected components. We start by giving some definitions that are in analogy to a convex subset in \mathbb{R}^n , in which for any two points, the shortest path connecting them lies also in the convex set.

Definition 3.33. A subgraph $U \subseteq X$ is called convex if

$$\forall u, v \in U \quad d_U(u, v) = d_X(u, v).$$

Definition 3.34. Let $U \subseteq X$ be a convex subgraph. We say X projects to U if for every vertex $x \in X$ there is a vertex $p(x) \in U$ such that

$$\forall u \in U \quad d(x, u) = d(x, p(x)) + d(p(x), u). \quad (3.11)$$

If the graph X projects to a subgraph U , then for any vertex $x \in X$ there exists precisely one vertex $p(x) \in U$ satisfying property (3.11). Indeed, assume there are two vertices $p_1, p_2 \in U$ with this property. By taking $p(x) = p_1$ we obtain

$$d(x, p_2) = d(x, p_1) + d(p_1, p_2)$$

and with $p(x) = p_2$ we have

$$d(x, p_1) = d(x, p_2) + d(p_2, p_1).$$

These equations imply that $d(p_1, p_2) = 0$, so any two points satisfying (3.11) are equal. Furthermore, the unique vertex $p(x)$ is the vertex of U closest to the vertex x , since for every other vertex $v \in U \setminus \{u\}$ the distance

$$d(x, v) = d(x, u) + d(u, v) > d(x, u).$$

This is also analogous to convex subsets in \mathbb{R}^n ; every point in \mathbb{R}^n has a closest point in any convex subset.

Remark 3.35. Because of the uniqueness we just discussed, we find that if a graph X projects to a subgraph U , then there is a well-defined map: $X \rightarrow U, x \mapsto p(x)$.

Remark 3.36. We have seen that X projects to a subgraph U , then the projection $p(x)$ of any vertex $x \in X$ is the vertex in U that is closest to x . But it is not true that X projects to a subgraph U if each vertex of X has a closest vertex in U . Take for example X to be the vertex cycle with its subgraph U consisting of two adjacent edges, see Figure 3.2a below. Both vertices a_3 and a_5 have a closest vertex in the red subgraph, but there is no projection. Thus, projecting to X is stronger than the property that each point of X has a closest point in U .

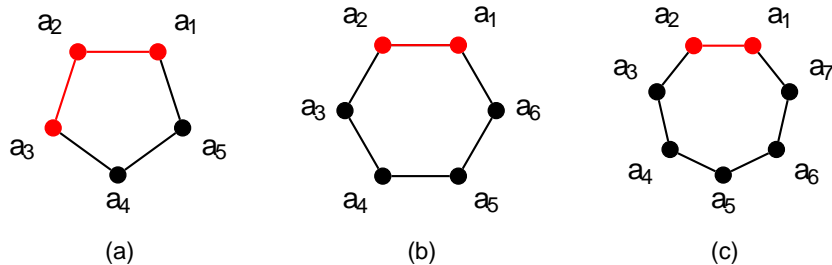


Figure 3.2: The vertex cycle on the left does not project to the red subgraph. The even cycle in the middle projects to the red subgraph, the projection is $p(a_3) = p(a_4) = a_2$ and $p(a_5) = p(a_6) = a_1$. The odd cycle on the right does not project to its red subgraph because the vertex a_5 can not be projected to the red edge.

Example 3.37. As illustrated in Figure 3.2b and 3.2c, we consider the following two examples.

- i) Every even cyclic graph projects to any of its edges.
- ii) No odd cyclic graph projects to any of its edges.

In this section, we work with the union of two graphs. This is to be understood as the union $G \sqcup H$ of labelled graphs G and H , with vertex set $V(G \sqcup H) = V(G) \sqcup V(H)$ and $E(G \sqcup H) = E(G) \sqcup E(H)$. It is not the disjoint union of the graphs, see also the example in Figure 3.3.

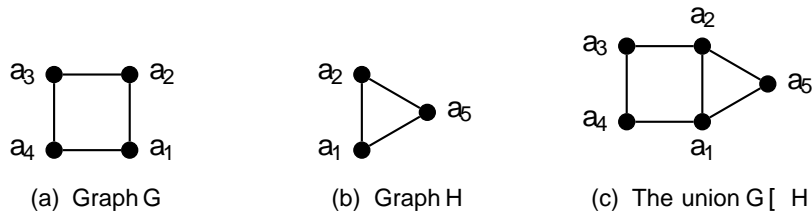


Figure 3.3: Example of a labelled union of graphs.

Definition 3.38. A projecting decomposition is a triple $(X; G, H)$ consisting of a graph X and subgraphs $G \subseteq X$ such that

- i) $X = G [H$;
- ii) $G \setminus H$ is convex in X ;
- iii) H projects to $G \setminus H$.

Given a projecting decomposition $(X; G, H)$ we write

$$\begin{aligned} i^G: G &\rightarrow X, & j^G: G \setminus H &\rightarrow G, \\ i^H: H &\rightarrow X, & j^H: G \setminus H &\rightarrow H \end{aligned}$$

for the inclusions.

Definition 3.39. A decomposition map $f: (X; G, H) \rightarrow (X^0; G^0, H^0)$ is a map of graphs $f: X \rightarrow X^0$ such that $f(G) \subseteq G^0$ and $f(H) \subseteq H^0$.

Definition 3.40. A decomposition map $f: (X; G, H) \rightarrow (X^0; G^0, H^0)$ is projecting if $H = f^{-1}(H^0)$ and if for every vertex $x \in H$ the equality $f(p(h)) = p(f(h))$ holds.

Lemma 3.41 ([6, Lemma 4.4]). Let X be a graph with subgraphs $G, H \subseteq X$ such that $X = G [H$. Every path in X from a vertex in G to a vertex in H contains a vertex in the intersection $G \cap H$.

Proof. Let $x_0 \in G$ and $x_n \in H$ be two vertices connected by a path P in X given by x_0, x_1, \dots, x_n . Let $i \in \{0, \dots, n\}$ be the largest index such that the vertex $x_i \in G$. We show that $x_i \in H$ as well, which will give us the desired vertex in the intersection. This is clear if $i = n$ by assumption. If $i < n$, then $x_{i+1} \notin G$, so the edge $\{x_i, x_{i+1}\}$ is not in G either. But because $X = G [H$, the edge $\{x_i, x_{i+1}\}$ must be in $E(H)$, implying that $x_i \in H$. \square

Lemma 3.42. In a projecting decomposition $(X; G, H)$, all the subgraphs

$$G \setminus H \subseteq X, \quad G \setminus H \subseteq G, \quad G \setminus H \subseteq H, \quad G \subseteq X, \quad H \subseteq X$$

are convex.

Proof. The subgraphs $G \setminus H \cap X$ and $G \setminus H \cap H$ are convex by definition of a projecting decomposition. For the rest, note first that in any graph B with subgraph $A \subseteq B$ and vertices $x, y \in A$, we have $d_B(x, y) = d_A(x, y)$ because every path in A is also a path in B . Let us now show that $G \setminus H$ is convex in G . Consider vertices $u, v \in G \setminus H$, by the observation we just made we obtain the inequalities

$$d_X(u, v) \leq d_G(u, v) \leq d_{G \setminus H}(u, v). \quad (3.12)$$

But we know that $d_{G \setminus H}(u, v) = d_X(u, v)$, because $G \setminus H \cap X$ is convex, so all the inequalities in (3.12) are in fact equalities.

To show that $G \setminus X$ is convex we follow the proof given in [6, Lemma 4.3]. Let $u, v \in G$ be arbitrary vertices. We assume that X is connected, so let $n = d_X(u, v) < \infty$ and take a shortest path P in X from u to v with the additional assumption that it has the greatest possible number of vertices in G of all those shortest paths. Let us say P is the path $u = x_0, x_1, \dots, x_n = v$, we show by contradiction that P must already be a path in G . Suppose there exists an index $j \in \{0, \dots, n\}$ such that $x_j \notin G$. By Lemma 3.41 above, there exist indices $i, k \geq 0$ such that $0 \leq i < j < k \leq n$ and $x_i, x_k \in G \setminus H$. Consider the path x_i, \dots, x_k . It is a shortest path in X from x_i to x_k because it is part of a shortest path from u to v , thus $d_X(x_i, x_k) = k - i$. The subgraph $G \setminus H \cap X$ is convex by definition, so $d_{G \setminus H}(x_i, x_k) = k - i$ as well. Hence, there exists a path $x_i = u_i, \dots, u_k = x_k$ in $G \setminus H$, but then the path

$$u = x_0, x_1, \dots, x_i = u_i, \dots, u_k = x_k, x_{k+1}, \dots, x_n = v$$

is another shortest path in X that contains strictly more vertices in G than P . This is a contradiction and we conclude that the path P already lies in G and therefore $d_X(u, v) = d_G(u, v)$. By the argument at the beginning of the proof, we have the other inequality $d_X(u, v) \leq d_G(u, v)$. The proof that $H \cap X$ is convex is completely analogous. \square

This lemma is important for the proof of the Mayer-Vietoris sequence, as it ensures that the distance of two vertices in any subgraph of the decomposition is the same even when the vertices are viewed in a larger subgraph of the decomposition. Recall that a chain subcomplex B of a chain complex C of abelian groups is a chain complex such that $B_n \subseteq C_n$ is a subgroup for every $n \in \mathbb{Z}$ and the differential of B is the restriction of the differential on C , that is, if the inclusion $i: B \hookrightarrow C$ is a chain map.

Definition 3.43. Given a projecting decomposition $(X; G, H)$, let $MC_{\bullet}(G, H)$ denote the chain subcomplex $MC_{\bullet}(G \sqcup H)$ spanned by those tuples (x_0, \dots, x_k) with entries all in G or all in H .

Theorem 3.44 (Excision for magnitude chains). Let $(X; G, H)$ be a projecting decomposition. For every $l \geq 0$, the inclusion

$$MC_{\bullet, l}(G, H) \hookrightarrow MC_{\bullet, l}(G \sqcup H)$$

is a quasi-isomorphism, that is, it induces an isomorphism in homology.

The proof of excision for magnitude homology is rather lengthy, so we split it up into several lemmas. The general idea is to show that the quotient complex $MC_{\cdot,l}(G \sqcup H)/MC_{\cdot,l}(G, H)$ is acyclic by showing that some auxiliary chain subcomplexes are also acyclic. Let $(X; G, H)$ denote a projecting decomposition throughout the proof.

Definition 3.45. Let $l \geq 0$ and let $a, b \in X$ be a pair of vertices such that either $a \in G \cap H$ and $b \in H \cap G$, or $a \in H \cap G$ and $b \in G \cap H$; that is, a and b are not both contained in G and not both contained in H . Define $A_{\cdot,l}(a, b)$ to be the chain subcomplex of $MC_{\cdot,l}(G \sqcup H)$ spanned by the tuple (x_0, \dots, x_k) for which $x_0 = a, x_k = b$ and $x_1, \dots, x_{k-1} \in G \setminus H$.

Lemma 3.46. Let $l \geq 0$ and $a, b \in X$ be a pair of vertices not both contained in G and not both contained in H . The chain complex $A_{\cdot,l}(a, b)$ is acyclic, that is, for every $k \geq 0$ the homology group $H_k(A_{\cdot,l}(a, b)) = 0$.

Proof. We show that the identity map $\text{id}: A_{\cdot,l}(a, b) \rightarrow A_{\cdot,l}(a, b)$ is chain homotopic to the zero map, that is, that there exists a chain homotopy $s: A_{\cdot,l}(a, b) \rightarrow A_{\cdot+1,l}(a, b)$ satisfying $\partial s + s \partial = \text{id}$. This proves the lemma because two chain maps that are chain homotopic induce the same map in homology and if the identity id is the trivial map, then all homology groups of $A_{\cdot,l}(a, b)$ are zero. We begin by considering the case $a \in H \cap G$ and $b \in G \cap H$, let us define the map $s: A_{\cdot,l}(a, b) \rightarrow A_{\cdot+1,l}(a, b)$ defined on the generators $(x_0, \dots, x_k) \in A_{k,l}(a, b)$ by

$$s(x_0, \dots, x_k) = \begin{cases} (-1)^k(x_0, \dots, x_{k-1}, p(x_k), x_k) & \text{if } p(x_k) \in x_{k-1} \\ 0 & \text{else,} \end{cases}$$

Where p denotes the projection map from H to $G \setminus H$. Note that this map is well-defined; the fact that H projects to $G \setminus H$ ensures that the length is preserved, because $d(x_{k-1}, p(x_k)) + d(p(x_k), x_k) = d(x_{k-1}, x_k)$. To show that s is the required chain homotopy, we verify that for any $k \geq 0$, any generator $(x_0, \dots, x_k) \in A_{k,l}(a, b)$ satisfies

$$\sum_{i=1}^k (-1)^i (\partial_i s)(x_0, \dots, x_k) + \sum_{i=1}^{k-1} (-1)^i (s \partial_i)(x_0, \dots, x_k) = (x_0, \dots, x_k).$$

Note that for $i = 1, \dots, k-2$ we have

$$(\partial_i s)(x_0, \dots, x_k) = (-1)^k(x_0, \dots, x_i, \dots, x_{k-1}, p(x_k), x_k),$$

if $p(x_k) \in x_{k-1}$ and $d(x_{i-1}, x_{i+1}) = d(x_{i-1}, x_i) + d(x_i, x_{i+1})$, and otherwise, $(\partial_i s)(x_0, \dots, x_k) = 0$. Furthermore,

$$(s \partial_i)(x_0, \dots, x_k) = (-1)^{k-1}(x_0, \dots, x_i, \dots, x_{k-1}, p(x_k), x_k)$$

if $p(x_k) \notin x_{k-1}$ and $d(x_{i-1}, x_{i+1}) = d(x_{i-1}, x_i) + d(x_i, x_{i+1})$. Otherwise $(s \circ \mathbb{1}_i)(x_0, \dots, x_k) = 0$. Therefore, the sum

$$(\mathbb{1}_i \circ s)(x_0, \dots, x_k) + (s \circ \mathbb{1}_i)(x_0, \dots, x_k) = 0$$

for $i = 1, \dots, k-2$. All that is left to show is that

$$\underbrace{(-1)^{k-1} (\mathbb{1}_{k-1} \circ s)(x_0, \dots, x_k)}_{\textcircled{1}} + \underbrace{(-1)^k (\mathbb{1}_k \circ s)(x_0, \dots, x_k)}_{\textcircled{2}} + \underbrace{(-1)^k (s \circ \mathbb{1}_{k-1})(x_0, \dots, x_k)}_{\textcircled{3}} = (x_0, \dots, x_k).$$

We verify this by dealing with the three possible cases separately. For each case we use that H projects to $G \setminus H$ and in particular we need the following two equations

$$\begin{aligned} d(x_k, x_{k-2}) &= d(x_k, p(x_k)) + d(p(x_k), x_{k-2}), \\ d(x_k, x_{k-1}) &= d(x_k, p(x_k)) + d(p(x_k), x_{k-1}). \end{aligned} \quad (3.13)$$

- If $x_{k-1} = p(x_k)$, then $s(x_0, \dots, x_k) = 0$ and hence $\textcircled{1} = 0$ and $\textcircled{2} = 0$. The first equation of (3.13) implies $\mathbb{1}_{k-1}(x_0, \dots, x_k) = (x_0, \dots, x_{k-1}, x_k)$. From $x_{k-2} \notin x_{k-1} = p(x_k)$ it follows that

$$\begin{aligned} s(\mathbb{1}_{k-1}(x_0, \dots, x_k)) &= \\ (-1)^{k-1} (x_0, \dots, x_{k-2}, p(x_k), x_k) &= (-1)^{k-1} (x_0, \dots, x_k), \end{aligned}$$

hence $\textcircled{3} = (x_0, \dots, x_k)$.

- If $x_{k-1} \notin p(x_k)$ and $d(x_{k-2}, x_{k-1}) + d(x_{k-1}, x_k) > d(x_{k-2}, x_k)$, then $s(x_0, \dots, x_k) = (-1)^k (x_0, \dots, x_{k-1}, p(x_k), x_k)$, and $\mathbb{1}_{k-1}(x_0, \dots, x_k) = 0$, so we already have $\textcircled{3} = 0$. The second equation from (3.13) shows that $\mathbb{1}_k(s(x_0, \dots, x_k)) = (-1)^k (x_0, \dots, x_{k-1}, x_k)$ and thus the second summand $\textcircled{2} = (x_0, \dots, x_k)$. By both Equations (3.13) and the second assumption of this case it follows that

$$d(x_{k-2}, x_{k-1}) + d(x_{k-1}, p(x_k)) > d(x_{k-2}, p(x_k)),$$

so the differential $\mathbb{1}_{k-1}(s(x_0, \dots, x_k)) = 0$ and $\textcircled{1} = 0$.

- If $x_{k-1} \notin p(x_k)$ and $d(x_{k-2}, x_{k-1}) + d(x_{k-1}, x_k) = d(x_{k-2}, x_k)$, then $s(x_0, \dots, x_k) = (-1)^k (x_0, \dots, x_{k-1}, p(x_k), x_k)$ and thus $\textcircled{2} = (x_0, \dots, x_k)$. The assumptions also imply $\mathbb{1}_{k-1}(x_0, \dots, x_k) = (x_0, \dots, x_{k-1}, x_k)$ and with Equations (3.13) we obtain

$$d(x_{k-2}, x_{k-1}) + d(x_{k-1}, p(x_k)) = d(x_{k-2}, p(x_k)),$$

which shows that $p(x_k) \notin x_{k-2}$ because the left-hand side is non-zero. Altogether, we can calculate $\textcircled{1} = (x_0, \dots, x_{k-2}, p(x_k), x_k)$ and $\textcircled{3} = (x_0, \dots, x_{k-2}, p(x_k), x_k)$.

Indeed, in all three cases $\textcircled{1} + \textcircled{2} + \textcircled{3} = (x_0, \dots, x_k)$.

The proof of the case $b \in G \cap H$, $a \in H \cap G$ has the same structure as before, but we can not just exchange G and H because it is only assumed that H projects to the intersection $G \cap H$ and not G . We need to define a new map $s^0: A_{k+1}(a, b) \rightarrow A_k(x_0, \dots, x_k)$, let s^0 be defined on the generators by

$$s^0(x_0, \dots, x_k) = \begin{cases} (x_0, p(x_0), x_1, \dots, x_k) & \text{if } p(x_0) \in x_1 \\ 0 & \text{else.} \end{cases}$$

Again, this map is well-defined because H projects to $G \cap H$ and we need to show that $\partial \circ s^0 + s^0 \circ \partial = \text{id}$, or in terms of generators $(x_0, \dots, x_k) \in A_{k+1}(a, b)$:

$$\sum_{i=1}^k (-1)^i (\partial_i \circ s^0)(x_0, \dots, x_k) + \sum_{i=1}^{k-1} (-1)^i (s^0 \circ \partial_i)(x_0, \dots, x_k) = (x_0, \dots, x_k)$$

For $i = 3, \dots, k$ we have

$$\begin{aligned} \partial_i (s^0(x_0, \dots, x_k)) &= \\ &= (x_0, p(x_0), x_1, \dots, x_{i-1}, \dots, x_k) \quad \text{if } p(x_0) \in x_1 \text{ and} \\ & \quad d(x_{i-2}, x_i) = d(x_{i-2}, x_{i-1}) + d(x_{i-1}, x_i) \\ &= 0 \quad \text{else} \\ &= (s^0 \circ \partial_{i-1})(x_0, \dots, x_k), \end{aligned}$$

hence the difference $(\partial_i \circ s^0 - s^0 \circ \partial_{i-1})(x_0, \dots, x_k) = 0$ and it is only left to show that

$$\begin{aligned} & \left| \underbrace{(\partial_1 \circ s^0)(x_0, \dots, x_k)}_{\textcircled{1}} \right| + \left| \underbrace{(\partial_2 \circ s^0)(x_0, \dots, x_k)}_{\textcircled{2}} \right| + \\ & \quad \left| \underbrace{(s^0 \circ \partial_1)(x_0, \dots, x_k)}_{\textcircled{3}} \right| = (x_0, \dots, x_k). \end{aligned}$$

Similarly to before, we do a case by case analysis and use that H projects to $G \cap H$. In particular, we will refer to the equations

$$\begin{aligned} d(x_0, x_2) &= d(x_0, p(x_0)) + d(p(x_0), x_2), \\ d(x_0, x_1) &= d(x_0, p(x_0)) + d(p(x_0), x_1). \end{aligned} \tag{3.14}$$

- If $p(x_0) = x_1$, then $\textcircled{1} = \textcircled{2} = 0$. The first equation of (3.14) implies $\partial_1(x_0, \dots, x_k) = (x_0, x_2, \dots, x_k)$. Together with $x_2 \in x_1 = p(x_0)$ it follows that $(s^0 \circ \partial_1)(x_0, \dots, x_k) = (x_0, p(x_0), x_2, \dots, x_k) = (x_0, \dots, x_k)$ and hence $\textcircled{3} = (x_0, \dots, x_k)$.

- If $p(x_0) \notin x_1$ and $d(x_0, x_2) < d(x_0, x_1) + d(x_1, x_2)$, then we have $\mathbb{1}_1(x_0, \dots, x_k) = 0$ and it follows that $\textcircled{3} = 0$. The assumptions also imply $s^0(x_0, \dots, x_k) = (x_0, p(x_0), x_1, \dots, x_k)$, so $\textcircled{1} = (x_0, x_1, \dots, x_k)$ and with Equations (3.14) we find $d(p(x_0), x_2) < d(p(x_0), x_1) + d(x_1, x_2)$, which shows $\textcircled{2} = 0$.
- If $p(x_0) \notin x_1$ and $d(x_0, x_2) = d(x_0, x_1) + d(x_1, x_2)$, then the image $s^0(x_0, \dots, x_k) = (x_0, p(x_0), x_1, \dots, x_k)$, so the composition with $\mathbb{1}_1$ yields $(\mathbb{1}_1 \circ s^0)(x_0, \dots, x_k) = (x_0, x_1, \dots, x_k)$ and thus $\textcircled{1} = (x_0, \dots, x_k)$. By Equations (3.14) and the second assumption of this case we obtain

$$d(p(x_0), x_2) = d(p(x_0), x_1) + d(x_1, x_2), \quad (3.15)$$

which implies $\textcircled{2} = (\mathbb{1}_2 \circ s^0)(x_0, \dots, x_k) = (x_0, p(x_0), x_2, \dots, x_k)$. The right-hand-side of Equation (3.15) is non-zero, thus $p(x_0) \notin x_2$ and with the second assumption of this case it follows that the composition $(s^0 \circ \mathbb{1}_1)(x_0, \dots, x_k) = (x_0, p(x_0), x_2, \dots, x_k)$, so the last summand $\textcircled{3} = (x_0, p(x_0), x_2, \dots, x_k)$.

In all these three cases we find that indeed $\textcircled{1} + \textcircled{2} + \textcircled{3} = (x_0, \dots, x_k)$ and therefore we showed that the chain complex $A_{\bullet}(a, b)$ is acyclic. \square

Let us recall a definition and result from homological algebra, which we need for the next lemma.

Definition 3.47. A filtration on a chain complex C is an ordered family of chain subcomplexes

$$\dots \subset F_{p-1} \subset F_p \subset F_{p+1} \subset \dots$$

of C .

Lemma 3.48. Let $F_0 \subset F_1 \subset \dots \subset F_l = C$ be a filtration on a chain complex C . If for every index $i = 1, \dots, l$ the quotient complex F_i / F_{i-1} is acyclic, then the chain complex F/F_0 is also acyclic.

Proof. For every $i = 1, \dots, l$, the inclusion and projection map fit into the short exact sequence

$$0 \rightarrow F_{i-1} \rightarrow F_i \rightarrow F_i / F_{i-1} \rightarrow 0,$$

which induces the following long exact sequence in homology:

$$\dots \rightarrow H_{i+1}(F_i / F_{i-1}) \rightarrow H_i(F_{i-1}) \rightarrow H_i(F_i) \rightarrow H_i(F_i / F_{i-1}) \rightarrow \dots$$

The assumption that the quotient complex F_i / F_{i-1} is acyclic implies that the inclusion $F_{i-1} \rightarrow F_i$ induces an isomorphism in homology. Thus, the

inclusion $i: F_0 \rightarrow F_1$ is a quasi isomorphism as well because it is equal to the composition of the inclusions $F_0 \rightarrow F_1 \rightarrow \dots \rightarrow F_1$. The short exact sequence

$$0 \rightarrow F_0 \xrightarrow{i} F_1 \xrightarrow{p} F_1/F_0 \rightarrow 0,$$

where p is the projection map, induces the long exact sequence

$$\begin{aligned} \dots \rightarrow H_{i+1}(F_1/F_0) \xrightarrow{d} H_i(F_0) \xrightarrow{i} H_i(F_1) \xrightarrow{p} H_i(F_1/F_0) \\ \rightarrow H_{i-1}(F_0) \xrightarrow{i} H_{i-1}(F_1) \xrightarrow{p} H_{i-1}(F_1/F_0) \rightarrow \dots \end{aligned}$$

in homology with connecting homomorphism d . Exactness of the sequence implies that both p and d are trivial because $\ker(p) = \text{im}(i) = H(F_1)$ and $\text{im}(d) = \ker(i) = 0$. It follows that $0 = \text{im}(p) = \ker(d) = H(F_1/F_0)$. \square

Definition 3.49. For an integer $j \geq 0$, the j -th suspension $S^j C$ of a chain complex C is the chain complex with $(S^j C)_i = C_{i-j}$ for all $i \in \mathbb{Z}$.

Definition 3.50. Fix $l \geq 0$ and let $b \in (G \sqcup H) \setminus (G \setminus H)$ be a vertex. Let $B_{\cdot,l}(b) \subseteq H \otimes G$ denote the chain complex $B_{\cdot,l}(b)$ to be the subcomplex of $C_{\cdot,l}(G \sqcup H)$ spanned by tuples of the form (x_0, \dots, x_k) with $x_k = b$ and $x_0, \dots, x_{k-1} \in G$. We further denote by $\bar{B}_{\cdot,l}(b)$ the chain subcomplex of $B_{\cdot,l}(b)$ spanned by tuples (x_0, \dots, x_k) for which $x_0, \dots, x_{k-1} \in G \setminus H$. If instead $b \in G \cap H$, then we interchange G and H in the definition of $B_{\cdot,l}(b)$ and $\bar{B}_{\cdot,l}(b)$.

Lemma 3.51. The quotient complex $B_{\cdot,l}(b)/\bar{B}_{\cdot,l}(b)$ is acyclic for any $l \geq 0$ and any $b \in (G \sqcup H) \setminus (G \setminus H)$.

Proof. The idea of the proof is to use Lemma 3.48 above, so we start by defining a filtration. Let us first suppose that $b \in H \otimes G$, for $i = 0, \dots, l$ let $F_i \subseteq B_{\cdot,l}(b)$ be the chain subcomplex for which the group in degree $k > i$ is spanned by the tuples $(x_0, \dots, x_k) \in B_{k,l}(b)$ satisfying $x_i, \dots, x_{k-1} \in G \setminus H$, and for the groups in degree $k \leq i$ we impose no condition. Note that the restriction of the differential to these subgroups is well defined, so the F_i 's are indeed chain subcomplexes. We obtain a filtration

$$\bar{B}_{\cdot,l}(b) = F_0 \subseteq F_1 \subseteq \dots \subseteq F_l = B_{\cdot,l}(b)$$

of $B_{\cdot,l}(b)$ and by the previous lemma it is enough to show that for every $i = 1, \dots, l$ the quotient complex (F_i/F_{i-1}) is acyclic. So, let $i = 1, \dots, l$ be any fixed index, the generators of $(F_i/F_{i-1})_k$ are the generators of F_i that are not also generators of F_{i-1} because both F_i and F_{i-1} are free abelian groups. Concretely, for $k \leq i$ the generators of $(F_i/F_{i-1})_k$ are the tuples $(x_0, \dots, x_k) \in B_{k,l}(b)$ for which $x_{i-1} \in G \cap H$ and $x_i, \dots, x_{k-1} \in G \setminus H$, and

for $k \geq i - 1$ we have $(F_i)_k = (F_{i-1})_k = B_{k,i}(b)$ so the quotient complex $(F_i/F_{i-1}) = 0$. Let us define a map

$$x: \bigoplus_{(x_0, \dots, x_{i-1})}^M S^{i-1} A_{\cdot, l}{}^0(x_{i-1}, b) \rightarrow F_i/F_{i-1}$$

where the direct sum is taken over all tuples $(x_0, \dots, x_{i-1}) \in G^i$ with $x_{i-1} \in G \setminus H$ and $l^0 = \langle x_0, \dots, x_{i-1} \rangle$. Define x on a generator $(x_{i-1}, y_1, \dots, y_k)$ of $(S^{i-1} A_{\cdot, l}{}^0(x_{i-1}, b))_k = A_{k-i+1, l}{}^0(x_{i-1}, b)$ from the summand corresponding to (x_0, \dots, x_{i-1}) by

$$x((x_{i-1}, y_1, \dots, y_k)) = (-1)^{(i-1)k} (x_0, \dots, x_{i-1}, y_1, \dots, y_k).$$

Before continuing, we verify that this is indeed a generator of the quotient $(F_i/F_{i-1})_k$. The length satisfies

$$\langle x_0, \dots, x_{i-1}, y_1, \dots, y_k \rangle = \langle x_0, \dots, x_{i-1} \rangle + \langle x_{i-1}, y_1, \dots, y_k \rangle = l^0 + (l - l^0) = l,$$

as needed, the vertex $x_{i-1} \in G \setminus H$ by assumption, and by the definition of $A_{\cdot, l}{}^0(x_{i-1}, b)$ it follows that $y_k = b$ and $y_1, \dots, y_{k-1} \in G \setminus H$, so x is well-defined. Next, we show that x is an isomorphism of chain complexes. It is immediate from the definition that x is injective on the generators and because any generator $(x_0, \dots, x_k) \in (F_i/F_{i-1})_k$ is the image of $(x_{i-1}, \dots, x_k) \in (S^{i-1} A_{\cdot, l}{}^0(x_{i-1}, b))_k$ corresponding to the summand (x_0, \dots, x_{i-1}) , the map x is a bijection on the generators in each degree and consequently an isomorphism of groups. It is left to show that x is a chain map, for this we need to check that the following diagram commutes for every $k \geq 0$.

$$\begin{array}{ccc} (S^{i-1} A_{\cdot, l}{}^0(x_{i-1}, b))_k & \xrightarrow{x} & (F_i/F_{i-1})_k \\ \downarrow \mathbb{1} & & \downarrow \mathbb{1} \\ (S^{i-1} A_{\cdot, l}{}^0(x_{i-1}, b))_{k-1} & \xrightarrow{x} & (F_i/F_{i-1})_{k-1} \end{array}$$

First, note that the differential on F_i/F_{i-1} is induced by the differential $\mathbb{1}$ on $MC_{\cdot, l}(G \sqcup H)$, which is given in degree k by the alternating sum $\sum_{j=1}^k (-1)^j \mathbb{1}_j$. Let us look at the maps $\mathbb{1}_j$ and consider first $j = 1, \dots, i - 1$ and a generator $(x_0, \dots, x_k) \in (F_i/F_{i-1})_k$. If we label the vertices in the tuple $(x_0, \dots, x_j, \dots, x_k)$ by (y_0, \dots, y_{k-1}) , then we see that for the indices $j^0 > j$, the vertex x_{j^0} corresponds to y_{j^0-1} . Therefore, the vertices y_{i-1}, \dots, y_{k-2} are the vertices $x_i, \dots, x_{k-1} \in G \setminus H$ and it follows that $(x_0, \dots, x_j, \dots, x_k) \in (F_{i-1})_{k-1}$ if the length is preserved after removing x_j and hence $\mathbb{1}_j$ is the trivial map on F_i/F_{i-1} . For $j = i, \dots, k - 1$, the map $\mathbb{1}_j$ still removes the j -th entry of a

generator if that preserves the length, and otherwise sends it to 0. Now, we verify commutativity of the diagram, let (x_0, \dots, x_{i-1}) be a tuple of vertices in G with $x_{i-1} \in G \cap H$ and set $I^0 = I(x_0, \dots, x_{i-1})$. Let $(x_{i-1}, y_1, \dots, y_k) \in (S^{i-1}A_{j \in I^0}(x_{i-1}, b))_k$ be a generator. Viewed as an element in the direct sum $(S^{i-1}A_{j \in I^0}(x_{i-1}, b))_k$, the image of this generator under x is $(-1)^{(i-1)k}(x_0, \dots, x_{i-1}, y_1, \dots, y_k)$. With the considerations above, we apply the differential and obtain

$$\begin{aligned} \mathbb{1}((- 1)^{(i-1)k}(x_0, \dots, x_{i-1}, y_1, \dots, y_k)) = \\ \mathring{a} \sum_{j=i}^{k-1} (- 1)^{j+(i-1)k} \mathbb{1}_j(x_0, \dots, x_{i-1}, y_1, \dots, y_k). \end{aligned}$$

By first applying the differential and then the map x to the generator $(x_{i-1}, y_1, \dots, y_k)$, we get the sum

$$\mathring{a} \sum_{j=1}^{k-i} (- 1)^j x(\mathbb{1}_j(x_{i-1}, y_1, \dots, y_k)). \tag{3.16}$$

For any $j = 1, \dots, k-1$, the j -th summand in (3.16) can be further expressed as

$$(- 1)^j (- 1)^{(i-1)(k-1)}(x_0, \dots, x_{i-1}, y_1, \dots, y_{i+j-1}, \dots, y_k)$$

if $d(y_{i+j-2}, y_{i+j-1}) = d(y_{i+j-2}, y_{i+j-1}) + d(y_{i+j-1}, y_{i+j})$, and is 0 otherwise. After substituting $j+i-1 = j^0$ in (3.16), we obtain

$$\mathring{a} \sum_{j^0=i}^{k-1} (- 1)^{j^0+(k-2)(i-1)} \mathbb{1}_{j^0}(x_0, \dots, x_{i-1}, y_1, \dots, y_k),$$

and since $(- 1)^{2(i-1)} = 1$, the diagram commutes. By interchanging the roles of G and H , we obtain the proof if instead $b \in G \cap H$. \square

We are now able to prove the excision theorem for magnitude homology, let us recall the following fact from homological algebra.

Remark 3.52. Let D be a chain complex with a chain subcomplex $C \subseteq D$. If the quotient complex D/C is acyclic, then the inclusion $C \hookrightarrow D$ is a quasi-isomorphism. This statement follows from the induced long exact sequence in homology of the short exact sequence

$$0 \rightarrow C \xrightarrow{i} D \xrightarrow{p} D/C \rightarrow 0,$$

where p denotes the projection.

Proof of Theorem 3.44. The idea of the proof is to use again a filtration to show with Lemma 3.48 that the quotient complex $MC_{\cdot,l}(G[H])/MC_{\cdot,l}(G,H)$ is acyclic, which then implies with the previous Remark 3.52 that the inclusion map $MC_{\cdot,l}(G,H) \rightarrow MC_{\cdot,l}(G[H])$ is a quasi-isomorphism.

For $i = 0, \dots, l$ let F_i be the chain subcomplex of $MC_{\cdot,l}(G[H])$ spanned in degree $k \geq i$ by the tuples (x_0, \dots, x_k) such that x_0, \dots, x_{k-i} either all lie in G or all in H . When $k < i$ we impose no condition, this produces a filtration

$$MC_{\cdot,l}(G,H) = F_0 \supseteq F_1 \supseteq \dots \supseteq F_l = MC_{\cdot,l}(G[H]).$$

We claim that there is a chain map isomorphism

$$\bigoplus_{(x_{k-i+1}, \dots, x_k)} S^{i-1}(B_{\cdot,l}^{i-1}(x_{k-i+1}) / \bar{B}_{\cdot,l}^{i-1}(x_{k-i+1})) \rightarrow F_i / F_{i-1},$$

where the direct sum is taken over all tuples $(x_{k-i+1}, \dots, x_k) \in (G[H])^i$ with $x_{k-i+1} \in (G[H]) \setminus (G \setminus H)$ and we denote $I^0 = \langle (x_{k-i+1}, \dots, x_k) \rangle$. For constructing the isomorphism, let us look at one summand, say corresponding to (x_{k-i+1}, \dots, x_k) as described above. Consider the map

$$b: S^{i-1} B_{\cdot,l}^{i-1}(x_{k-i+1}) \rightarrow F_i / F_{i-1}$$

defined on generators by $b(x_0, \dots, x_{k-i+1}) = (x_0, \dots, x_k)$. Let us quickly verify that this is well defined. By assumption, $x_{k-i+1} \in (G[H]) \setminus (G \setminus H)$, so $B_{\cdot,l}^{i-1}(x_{k-i+1})$ is defined and a generator $(x_0, \dots, x_{k-i+1}) \in B_{k-i+1,l}^{i-1}(x_{k-i+1})$ satisfies $\langle (x_0, \dots, x_{k-i+1}) \rangle = I^0$ and

$$(x_0, \dots, x_{k-i}) \in \begin{cases} G & \text{if } x_{k-i+1} \in H \cap G \\ H & \text{if } x_{k-i+1} \in G \cap H \end{cases}$$

Then, the tuple (x_0, \dots, x_k) has length

$$\langle (x_0, \dots, x_k) \rangle = \langle (x_0, \dots, x_{k-i+1}) \rangle + \langle (x_{k-i+1}, \dots, x_k) \rangle = I^0 + I^0 = I^0$$

and x_0, \dots, x_{k-i} all lie either in G or in H , hence $(x_0, \dots, x_k) \in F_i$ and we can project it to $(x_0, \dots, x_k) \in F_i / F_{i-1}$.

Claim: The kernel $\ker(b) = S^{i-1} \bar{B}_{\cdot,l}^{i-1}(x_{k-i+1})$.

Proof of Claim: Let $k \geq 0$, because the image $b(x_0, \dots, x_{k-i+1}) = (x_0, \dots, x_k)$ of a generator $(x_0, \dots, x_{k-i+1}) \in (S^{i-1} \bar{B}_{\cdot,l}^{i-1}(x_{k-i+1}))_k$ satisfies that the vertices x_0, \dots, x_{k-i} are in the intersection $G \setminus H$, the vertices x_0, \dots, x_{k-i+1} all lie either in G or all in H . Therefore, the tuple $(x_0, \dots, x_k) \in F_{i-1}$ and

$$S^{i-1} \bar{B}_{\cdot,l}^{i-1}(x_{k-i+1}) \subseteq \ker(b).$$

For the other inclusion, let B be the set of generators of $(S^{i-1} B_{\cdot,l}^{i-1}(x_{k-i+1}))_k$ and let

$$g = \sum_{t \in B} a_t \cdot t \in \ker(b),$$

for some coefficients $n_t \in \mathbb{Z}$ be an arbitrary element in the kernel, meaning $b(g) \in F_{i-1}$. It follows that for every $(x_0, \dots, x_{k-i+1}) = t \in B$ such that $n_t \neq 0$, already $t \in F_{i-1}$ because the map sending a generator $(x_0^0, \dots, x_{k-i+1}^0) \in B$ to $(x_0^0, \dots, x_k^0) \in F_i$ is injective. By definition, this means the vertices x_0, \dots, x_{k-i+1} all either lie in G or all in H . Additionally, we know that $x_{k-i+1} \in (G \setminus H) \cap (G \setminus H)$ and $t \in (S^{i-1} B_{\setminus I^0(x_{k-i+1})})_k$, so

$$(x_0, \dots, x_{k-i}) \in \begin{cases} G & \text{if } x_{k-i+1} \in H \cap G \\ H & \text{if } x_{k-i+1} \in G \cap H \end{cases}$$

which implies that $(x_0, \dots, x_{k-i}) \in G \setminus H$ and thus $t \in S^{i-1} \bar{B}_{\setminus I^0(x_{k-i+1})}$ and also $g \in S^{i-1} \bar{B}_{\setminus I^0(x_{k-i+1})}$.

It follows that b factors through $S^{i-1} \bar{B}_{\setminus I^0(x_{k-i+1})}$, and we get the injective map

$$\bar{b}: S^{i-1}(B_{\setminus I^0(x_{k-i+1})} / \bar{B}_{\setminus I^0(x_{k-i+1})}) \rightarrow F_i / F_{i-1}$$

induced by b . Taking the direct sum as described earlier, we obtain the injective map

$$x: \bigoplus_{(x_{k-i+1}, \dots, x_k)}^M S^{i-1}(B_{\setminus I^0(x_{k-i+1})} / \bar{B}_{\setminus I^0(x_{k-i+1})}) \rightarrow F_i / F_{i-1}.$$

Note that a generator of the quotient chain complex F_i / F_{i-1} in degree k is a tuple $(x_0, \dots, x_k) \in MC_{k,i}(G \setminus H)$ such that x_0, \dots, x_{k-i} either all lie in G or all in H and x_{k-i+1} does not lie in the same subgraph as x_0, \dots, x_{k-i} . Hence, (x_{k-i+1}, \dots, x_k) is a tuple of vertices in $G \setminus H$ and $x_{k-i+1} \in (G \setminus H) \cap (G \setminus H)$. If we denote $\setminus(x_{k-i+1}, \dots, x_k) = I^0$, then the generator (x_0, \dots, x_k) is the image of $(x_0, \dots, x_{k-i+1}) \in S^{i-1} B_{\setminus I^0(x_{k-i+1})}$ under \bar{b} corresponding to the summand (x_{k-i+1}, \dots, x_k) . This shows that x is also surjective and thus it is an isomorphism in each degree $k \geq 0$. It is left to check that x is a chain map, to do this we prove that \bar{b} is a chain map for each summand. That is, for every tuple $(x_{k-i+1}, \dots, x_k) \in (G \setminus H)^i$ with $x_{k-i+1} \in (G \setminus H) \cap (G \setminus H)$, the diagram

$$\begin{array}{ccc} (S^{i-1}(B_{\setminus I^0(x_{k-i+1})} / \bar{B}_{\setminus I^0(x_{k-i+1})}))_k & \xrightarrow{\bar{b}} & (F_i / F_{i-1})_k \\ \downarrow \mathbb{1} & & \downarrow \mathbb{1} \\ (S^{i-1}(B_{\setminus I^0(x_{k-i+1})} / \bar{B}_{\setminus I^0(x_{k-i+1})}))_{k-1} & \xrightarrow{\bar{b}} & (F_i / F_{i-1})_{k-1} \end{array}$$

commutes for every $k \geq 0$. The differential $\mathbb{1}: (F_i / F_{i-1})_k \rightarrow (F_i / F_{i-1})_{k-1}$ is induced by the alternating sum $\mathbb{1} = \sum_{j=1}^k (-1)^j \mathbb{1}_j$, we show that for every $j = k-i+1, \dots, k-1$ the map $\mathbb{1}_j$ is trivial. A generator $(x_0, \dots, x_k) \in F_i$ satisfies that the vertices x_0, \dots, x_{k-i} all either lie in H or in G . In the tuple

3.4. The Mayer-Vietoris Sequence

$(x_0, \dots, x_j, \dots, x_k)$, where the entry x_j is removed, the entries x_0, \dots, x_{k-i} stay the same. Therefore, if the tuple $(x_0, \dots, x_j, \dots, x_k)$ still has length l , then it is a generator of the chain complex F_{l-1} in degree $k-1$, otherwise $\mathbb{1}_j$ maps it to 0. Thus, $\mathbb{1}_j(x_0, \dots, x_k) = 0 \in (F_l / F_{l-1})$ and the sum of the differential simplices to $\mathbb{1} = \sum_{j=1}^k (-1)^j \mathbb{1}_j$. Let $(x_0, \dots, x_{k-i+1}) \in S^{i-1} B_{l-1} \cap \bar{B}_{l-1} \cap (x_{i-1+k}) / \bar{B}_{l-1} \cap (x_{i-1+k})$ be a generator, its images under each of the two compositions in the diagram are

$$(x_0, \dots, x_{k-i+1}) \xrightarrow{\bar{\mathbb{1}}} (x_0, \dots, x_k) \xrightarrow{\mathbb{1}} \sum_{j=1}^{k-i} (-1)^j \mathbb{1}_j(x_0, \dots, x_k)$$

and

$$(x_0, \dots, x_{k-i+1}) \xrightarrow{\mathbb{1}} \sum_{j=1}^{k-i} (-1)^j \mathbb{1}_j(x_0, \dots, x_{k-i+1}) \xrightarrow{\bar{\mathbb{1}}} \sum_{j=1}^{k-i} (-1)^j \mathbb{1}_j(x_0, \dots, x_k),$$

so the diagram commutes and x is a chain map. The previous Lemma 3.51 states that $S^{i-1}(B_{l-1} \cap (x_{k-i+1}) / \bar{B}_{l-1} \cap (x_{k-i+1}))$ is acyclic, it follows that the chain complex (F_l / F_{l-1}) is acyclic too for every $l \geq 1, \dots, lg$. \square

With the excision theorem, we can deduce the Mayer-Vietoris sequence for magnitude homology, which relates the magnitude homologies of X , G , H , and $G \setminus H$. Note that unlike for singular homology, the Mayer-Vietoris sequence we obtain here splits into short exact sequences.

Theorem 3.53 (Mayer-Vietoris for magnitude homology). Let $(X; G, H)$ be a projecting decomposition. There exists a split short exact sequence

$$0 \rightarrow MH_{\bullet}(G \setminus H) \xrightarrow{(j^G, j^H)} MH_{\bullet}(G) \oplus MH_{\bullet}(H) \xrightarrow{j^G \oplus j^H} MH_{\bullet}(G \sqcup H) \rightarrow 0.$$

The sequence is natural with respect to decomposition maps, and the splitting is natural with respect to projecting decomposition maps.

Proof. Let $l \geq 0$ and consider the sequence

$$0 \rightarrow MC_{\bullet,l}(G \setminus H) \xrightarrow{(j^G_{\#}, j^H_{\#})} MC_{\bullet,l}(G) \oplus MC_{\bullet,l}(H) \xrightarrow{i} MC_{\bullet,l}(G, H) \rightarrow 0,$$

where the map i is induced by the inclusions $MC_{\bullet,l}(G) \rightarrow MC_{\bullet,l}(G, H)$ and $MC_{\bullet,l}(H) \rightarrow MC_{\bullet,l}(G, H)$.

Claim: This sequence is exact.

Proof of Claim: Lemma 3.42 implies that for any tuple (x_0, \dots, x_k) of vertices in X that can also be viewed as a tuple of vertices in any of the subgraphs G , H , or $G \setminus H$, the length does not change depending on which subgraph we

consider the vertices to be in. This ensures that the chain maps induced by inclusions send a tuple (x_0, \dots, x_k) to itself viewed with vertices in the larger graph instead of sending it to 0. It follows that the map $(j_{\#}^G, j_{\#}^H)$ is injective. To see that $\ker(i) = \text{im}(j_{\#}^G, j_{\#}^H)$, let $k \geq 0$ be arbitrary and note that for every generator $(x_0, \dots, x_k) \in MC_{k,l}(G \setminus H)$, the image $i \cdot (j_{\#}^G, j_{\#}^H)(x_0, \dots, x_k) = 0$, hence $\text{im}(j_{\#}^G, j_{\#}^H) \subseteq \ker(i)$. For the other inclusion, let $a \in \ker(i_{\#}^G - i_{\#}^H)$ be arbitrary. We can write $a = (\sum_{i=1}^k n_i g_i, \sum_{i_0=1}^{k_0} m_{i_0} h_{i_0})$ for some generators $g_i \in MC_{k,l}(G)$ and $h_{i_0} \in MC_{k,l}(H)$, and some coefficients $n_i, m_{i_0} \in \mathbb{Z}$. We obtain the equation

$$\sum_{i=1}^k n_i g_i + \sum_{i_0=1}^{k_0} m_{i_0} h_{i_0} = 0 \in MC_{k,l}(G, H),$$

and it follows that the coefficients are non-zero only if the vertices in their corresponding generator all lie in the intersection $G \setminus H$ and the indices of the non-zero coefficients come in pairs $1 \leq i_0, i_0^0 \leq k$ such that the generators $g_{i_0} = h_{i_0^0}$ and $n_{i_0} = -m_{i_0^0}$. This implies

$$a = (j_{\#}^G, j_{\#}^H) \left(\sum_{j=1}^l n_j g_j \right),$$

where we sum over all non-zero coefficients. We conclude that also the inclusion $\ker(i) \subseteq \text{im}(j_{\#}^G, j_{\#}^H)$ holds. The map i is surjective by definition of the chain complex $MC_{\cdot,l}(G, H)$.

In the induced long exact sequence in homology, we can use the isomorphism $H_n(MC_{\cdot,l}(G, H)) \cong H_n(MH_{\cdot,l}(G \sqcup H))$ from the Excision Theorem 3.44 to obtain the following long exact sequence:

$$\begin{aligned} \dots \rightarrow MH_{n,l}(G \setminus H) \xrightarrow{(j_{\#}^G, j_{\#}^H)} MH_{n,l}(G) \oplus MH_{n,l}(H) \\ \xrightarrow{(i_{\#}^G - i_{\#}^H)} MH_{n,l}(G \sqcup H) \rightarrow MH_{n-1,l}(G \setminus H) \rightarrow \dots \end{aligned} \quad (3.17)$$

Next, we will show that this long exact sequence splits into many short exact sequences by giving a left inverse to $(j_{\#}^G, j_{\#}^H)$. Consider the composition Y given by

$$MH_{n,l}(G) \oplus MH_{n,l}(H) \rightarrow MH_{n,l}(H) \xrightarrow{p} MH_{n,l}(G \setminus H),$$

where the first map projects to the second component of the direct sum and p is the map from Remark 3.35 that exists because H projects to the intersection $G \setminus H$. For any $k \geq 0$ and any generator $(x_0, \dots, x_k) \in MC_{k,l}(G \setminus H)$, we have

$$\begin{aligned} Y \cdot (j_{\#}^G, j_{\#}^H)(x_0, \dots, x_k) &= Y((x_0, \dots, x_k), (x_0, \dots, x_k)) \\ &= p \cdot ((x_0, \dots, x_k)) = (x_0, \dots, x_k), \end{aligned}$$

because the vertices x_0, \dots, x_k already lie in $G \setminus H$, so their projection is $p(x_i) = x_i$ for all $i = 0, \dots, k$. Therefore, the long exact sequence (3.17) splits into the split short exact sequences

$$0 \rightarrow MH_{k,l}(G \setminus H) \xrightarrow{(j^G, j^H)} MH_{k,l}(G) \oplus MH_{k,l}(H) \xrightarrow{(i^G, i^H)} MH_{k,l}(G \sqcup H) \rightarrow 0$$

for all $k, l \geq 0$. It is only left to check the naturality claims. First, let $f : (X; G, H) \rightarrow (X^0; G^0, H^0)$ be a decomposition map and consider the following diagram for $k, l \geq 0$:

$$\begin{array}{ccccc} MH_{k,l}(G \setminus H) & \xrightarrow{(j^G, j^H)} & MH_{k,l}(G) \oplus MH_{k,l}(H) & \xrightarrow{(i^G, i^H)} & MH_{k,l}(G \sqcup H) \\ \downarrow (f_{j_{G \setminus H}}) & \textcircled{1} & \downarrow ((f_{j_G}, f_{j_H})) & \textcircled{2} & \downarrow f \\ MH_{k,l}(G^0 \setminus H^0) & \xrightarrow{(j^{G^0}, j^{H^0})} & MH_{k,l}(G^0) \oplus MH_{k,l}(H^0) & \xrightarrow{(i^{G^0}, i^{H^0})} & MH_{k,l}(G^0 \sqcup H^0) \end{array}$$

At the level of maps of graphs, it is clear that the following diagrams commute for $Y \subseteq G, H$.

$$\begin{array}{ccc} G \setminus H & \xrightarrow{j^Y} & Y \\ \downarrow f_{j_{G \setminus H}} & & \downarrow f_{j_Y} \\ G^0 \setminus H^0 & \xrightarrow{j^{Y^0}} & Y^0 \end{array} \quad \begin{array}{ccc} Y & \xrightarrow{i^Y} & G \sqcup H \\ \downarrow f_{j_Y} & & \downarrow f \\ Y^0 & \xrightarrow{i^{Y^0}} & G^0 \sqcup H^0 \end{array} \quad (3.18)$$

By using the functoriality of the magnitude homology, it follows that the squares $\textcircled{1}$ and $\textcircled{2}$ commute. Therefore, the Mayer-Vietoris short exact sequence is natural with respect to decomposition maps.

Second, let $f : (X; G, H) \rightarrow (X^0; G^0, H^0)$ be any projecting decomposition map and consider the diagram

$$\begin{array}{ccccc} MH_{k,l}(G) \oplus MH_{k,l}(H) & \xrightarrow{F_Y} & MH_{k,l}(G \setminus H) \oplus MH_{k,l}(G \sqcup H) \\ \downarrow ((f_{j_G}, f_{j_H})) & \textcircled{3} & \downarrow ((f_{j_{G \setminus H}}, f)) \\ MH_{k,l}(G^0) \oplus MH_{k,l}(H^0) & \xrightarrow{F_{Y^0}} & MH_{k,l}(G^0 \setminus H^0) \oplus MH_{k,l}(G^0 \sqcup H^0) \end{array}$$

where F_Y is the splitting isomorphism induced by the left inverse γ_Y we constructed in the proof. Concretely, for every $x \in MH_{k,l}(G) \oplus MH_{k,l}(H)$ the map is defined by

$$F_Y(x) = (\gamma_Y(x), (i^G, i^H)(x))$$

and analogous for F_{Y^0} with the maps from the projecting decomposition $(X^0; G^0, H^0)$. First, for any generator $(g_0, \dots, g_k) \in MC_{k,l}(G)$, the two compositions in the diagram map the corresponding generator $((g_0, \dots, g_k), 0)$ of

the direct sum $MC_{k,l}(G) \oplus MC_{k,l}(H)$ to

$$((g_0, \dots, g_k), 0) \xrightarrow{F^Y} (0, i_{\#}^G(g_0, \dots, g_k)) \xrightarrow{((f_{j_{G \setminus H}})_{\#}, f_{\#})} (0, f_{\#}(i_{\#}^G(g_0, \dots, g_k)))$$

and

$$((g_0, \dots, g_k), 0) \xrightarrow{((f_{j_G})_{\#}, (f_{j_H})_{\#})} ((f_{j_G})_{\#}(g_0, \dots, g_k), 0) \xrightarrow{F^{Y_0}} (0, i_{\#}^G((f_{j_G})_{\#}(g_0, \dots, g_k))),$$

which are equal in homology by functoriality and the second diagram in (3.18). Second, let $(h_0, \dots, h_k) \in MC_{k,l}(H)$ be a generator, the images of the corresponding generator $(0, (h_0, \dots, h_k)) \in MC_{k,l}(G) \oplus MC_{k,l}(H)$ under the two compositions of the diagram are

$$(0, (h_0, \dots, h_k)) \xrightarrow{F^Y} (p_{\#}(h_0, \dots, h_k), i_{\#}^H(h_0, \dots, h_k)) \xrightarrow{((f_{j_{G \setminus H}})_{\#}, f_{\#})} ((f_{j_{G \setminus H}})_{\#}(p_{\#}(h_0, \dots, h_k)), f_{\#}(i_{\#}^H(h_0, \dots, h_k)))$$

and

$$(0, (h_0, \dots, h_k)) \xrightarrow{((f_{j_G})_{\#}, (f_{j_H})_{\#})} (0, (f_{j_H})_{\#}(h_0, \dots, h_k)) \xrightarrow{F^{Y_0}} (p_{\#}((f_{j_H})_{\#}(h_0, \dots, h_k)), i_{\#}^H((f_{j_H})_{\#}(h_0, \dots, h_k))).$$

We know that $(p \circ f)(h) = (f \circ p)(h)$ for every vertex $h \in H$ by definition of a projecting decomposition map and thus by functoriality and the second commutative diagram in (3.18) we conclude that ③ also commutes. Therefore, the splitting is natural with respect to projecting decomposition maps. \square

As we have seen before in the chapter about disjoint unions, the result about magnitude homology can be used to deduce a property about the magnitude itself.

Corollary 3.54 (Inclusion-Exclusion). For a projecting decomposition $(X; G, H)$, the magnitudes satisfy

$$|X|_q = |G|_q + |H|_q - |G \setminus H|_q.$$

Proof. From the Mayer-Vietoris split short exact sequence of Theorem 3.53 we know that for any $k, l \geq 0$, there is an isomorphism

$$MH_{k,l}(G) \oplus MH_{k,l}(H) = MH_{k,l}(G \setminus H) \oplus MH_{k,l}(G \sqcup H)$$

and by using the additivity of the rank with respect to the direct sum we obtain

$$\text{rank}(MH_{k,l}(G)) + \text{rank}(MH_{k,l}(H)) = \text{rank}(MH_{k,l}(G \setminus H)) + \text{rank}(MH_{k,l}(G \sqcup H)).$$

Taking the alternating sum over $k \geq 0$ of the equation above, yields

$$c(MH_{\cdot,l}(G)) + c(MH_{\cdot,l}(H)) = c(MH_{\cdot,l}(G \setminus H)) + c(MH_{\cdot,l}(G \sqcup H))$$

for all $l \geq 0$. After rearranging and multiplying both sides by q^l and summing over $l \geq 0$ we get

$$\sum_{l=0}^{\infty} c(MH_{\cdot,l}(G \sqcup H))q^l = \sum_{l=0}^{\infty} c(MH_{\cdot,l}(G))q^l + \sum_{l=0}^{\infty} c(MH_{\cdot,l}(H))q^l - \sum_{l=0}^{\infty} c(MH_{\cdot,l}(G \setminus H))q^l.$$

The statement follows from applying Theorem 3.16, which relates the magnitude to the magnitude homology. \square

The Mayer-Vietoris short exact sequence can be used to compute the magnitude homology of certain graphs as the following corollary shows.

Definition 3.55. Let G and H be graphs with chosen base vertices. The **wedge sum** $G _ H$ of G and H is the graph we get by identifying the two base vertices to a single vertex.

Corollary 3.56. Let G and H be graphs with fixed base vertices and denote the vertex of their wedge sum corresponding to the base vertices by p . The inclusion maps $a: G \rightarrow G _ H$ and $b: H \rightarrow G _ H$ induce isomorphisms

$$a_* : MH_{k,l}(G) \oplus MH_{k,l}(H) \xrightarrow{\cong} MH_{k,l}(G _ H),$$

if $k > 0$ or $l > 0$, and an isomorphism

$$a_* : (MH_{0,0}(G) \oplus MH_{0,0}(H)) / \text{im}(j^G, j^H)_* \xrightarrow{\cong} MH_{0,0}(G _ H),$$

where $j^G: P \rightarrow G$ and $j^H: P \rightarrow H$ are the inclusions.

Proof. After labelling the base vertices in G and H by P , the wedge sum $G _ H$ is the same as the labelled union $G \sqcup H$. The intersection $G \setminus H = P$ is just one vertex, so the triple $(G _ H; G, H)$ is a projecting decomposition and by the Mayer-Vietoris Theorem we obtain the short exact sequence

$$0 \rightarrow MH_{\cdot,l}(P) \xrightarrow{(j^G, j^H)_*} MH_{\cdot,l}(G) \oplus MH_{\cdot,l}(H) \xrightarrow{(a, b)_*} MH_{\cdot,l}(G _ H) \rightarrow 0.$$

The magnitude homology of a point is

$$MH_{k,l}(P) = \begin{cases} \mathbb{Z} & \text{if } k = l = 0 \\ 0 & \text{else,} \end{cases}$$

so it follows by exactness that for $k > 0$ or $l > 0$, the map

$$a \quad b : MH_{k,l}(G) \rightarrow MH_{k,l}(H) \oplus MH_{k,l}(G \setminus H)$$

is an isomorphism. If $k = l = 0$, then by exactness, the map $a \quad b$ is surjective and with the homomorphism theorem we obtain the isomorphism

$$(MH_{0,0}(G) \oplus MH_{0,0}(H)) / \ker(a \quad b) \cong MH_{0,0}(G \setminus H)$$

induced by $a \quad b$. Using the exactness of the Mayer-Vietoris sequence we see that $\ker(a \quad b) = \text{im}(j^G, \quad j^H)$ and the result follows. \square

Remark 3.57. The condition that we need a projecting decomposition for the Mayer-Vietoris sequence is necessary as the following example shows. Consider the diamond graph X as in Figure 3.4 below with subgraphs G and H induced by the vertices a_1, a_2, a_3 and a_1, a_3, a_4 . Clearly the union $G \cup H$ is X and the intersection $G \cap H$ is convex in X . The subgraphs G and H are both the triangle K_3 , so their

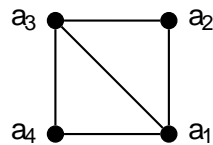


Figure 3.4: The diamond graph.

magnitude homology is the free abelian group with rank

$$\text{rank}(MH_{k,l}(K_3)) = \begin{cases} 3 - 2^l & \text{if } k = l \\ 0 & \text{else} \end{cases}$$

as we have seen in Example 3.12. The intersection $G \cap H$ can be viewed as K_2 , hence the rank of its magnitude homology is $\text{rank}(MH_{k,l}(G \cap H)) = 2$ if $k = l$ and 0 otherwise. Neither G nor H project to the intersection. If the Mayer-Vietoris theorem would hold in this example, then the rank of the magnitude homology would equal

$$\text{rank}(MH_{k,l}(X)) = 2 \text{rank}(MH_{k,l}(K_3)) + \text{rank}(MH_{k,l}(G \cap H)). \quad (3.19)$$

Computer calculations show that the ranks of the magnitude homology are as in Table 3.3. So, for example in $\text{bideg}(2, 2)$ the formula (3.19) does not hold, showing that the projection assumption is necessary.

We finish this section by describing the magnitude homology of trees using the Mayer-Vietoris sequence.

3.4. The Mayer-Vietoris Sequence

0	0	1	2	3	4	5	6	7	8
0	4	0	0	0	0	0	0	0	0
1	0	10	0	0	0	0	0	0	0
2	0	0	24	0	0	0	0	0	0
3	0	0	0	58	0	0	0	0	0
4	0	0	0	0	140	0	0	0	0
5	0	0	0	0	0	338	0	0	0
6	0	0	0	0	0	0	816	0	0
7	0	0	0	0	0	0	0	1970	0
8	0	0	0	0	0	0	0	0	4756

Table 3.3: Ranks of the magnitude homology of the diamond graph.

Example 3.58 (Magnitude homology of trees). In this example, we show that the magnitude homology of a tree T is

$$MH_{k,l}(T) = \begin{cases} \mathbb{Z} \langle V(T) \rangle & \text{if } k = l = 0 \\ \mathbb{Z} \langle \vec{E}(T) \rangle & \text{if } k = l > 0 \\ 0 & \text{else} \end{cases} \quad (3.20)$$

For an easier readability, let us introduce a functor from the category of trees to the category of bigraded abelian groups representing the right-hand-side of the isomorphism. That is, for fixed $k, l \geq 0$, let $F_{k,l}(T)$ of a tree T be defined as

$$F_{k,l}(T) = \begin{cases} \mathbb{Z} \langle V(T) \rangle & \text{if } k = l = 0 \\ \mathbb{Z} \langle \vec{E}(T) \rangle & \text{if } k = l > 0 \\ 0 & \text{else} \end{cases}$$

For a map of graphs $f: T \rightarrow S$ between trees S and T , let f also denote the induced map by the functor $F_{k,l}$, it is defined as follows. For $k = l = 0$, the map $f: \mathbb{Z} \langle V(T) \rangle \rightarrow \mathbb{Z} \langle V(S) \rangle$ maps a vertex $x \in V(T)$ to $f(x) \in V(S)$ and is extended linearly. For $k = l > 0$, the map $f: \mathbb{Z} \langle \vec{E}(T) \rangle \rightarrow \mathbb{Z} \langle \vec{E}(S) \rangle$ is defined on generators by sending an oriented edge $(x, y) \in \vec{E}(T)$ to the oriented edge $(f(x), f(y)) \in \vec{E}(S)$ if $f(x) \neq f(y)$ and to 0 otherwise. In all the other cases, the induced map has to be the zero map and it is clear from the definitions that this is indeed a functor. With this setup, we further claim that the isomorphism (3.20) is natural with respect to maps of trees, that is, for every map of graphs $f: T \rightarrow S$ between two trees, the following diagram with the above claimed isomorphism commutes.

$$\begin{array}{ccc} MH_{k,l}(T) & \xrightarrow{\cong} & F_{k,l}(T) \\ \downarrow f & & \downarrow f \\ MH_{k,l}(S) & \xrightarrow{\cong} & F_{k,l}(S) \end{array}$$

Let $Q: F_{k,l} \rightarrow MH_{k,l}$ be a transformation defined for any tree T by the morphism q_T of graded abelian groups mapping a vertex $x \in V(T)$ to $(x) \in MH_{0,0}(T)$ for $k = l = 0$. For $k = l > 0$, the map q_T sends a generator $(x, y) \in F_{k,l}(T)$, which is an oriented edge $(x, y) \in E(T)$, to the homology class of the $(k+1)$ -tuple (x, y, x, y, \dots) , where the entries alternate between the vertices x and y . We show that this is a natural transformation, that is, the following diagram commutes for each pair $k, l \geq 0$ and any map of trees: $f: T \rightarrow S$.

$$\begin{array}{ccc} F_{k,l}(T) & \xrightarrow{Q_T} & MH_{k,l}(T) \\ \downarrow f & & \downarrow f \\ F_{k,l}(S) & \xrightarrow{Q_S} & MH_{k,l}(S) \end{array}$$

For $k \neq l$, there is nothing to prove because both maps Q_S and Q_T are trivial. If $k = l = 0$, the diagram commutes because for any generator $(x) \in F_{0,0}(T) = \mathbb{Z} V(T)$ there is equality

$$f(Q_T(x)) = (f(x)) = Q_S(f(x)).$$

For $k = l > 0$, let $(x, y) \in F_{k,l}(T) = \mathbb{Z} E(T)$ be a generator. The images under the compositions are

$$f(Q_T(x, y)) = \begin{cases} \frac{\delta}{k+1} (f(x), f(y), f(x), f(y), \dots) & \text{if } d(f(x), f(y)) = 1 \\ 0 & \text{else} \end{cases}$$

and

$$Q_S(f(x, y)) = \begin{cases} \frac{\delta}{k+1} (f(x), f(y), f(x), f(y), \dots) & \text{if } f(x) \neq f(y) \\ 0 & \text{else} \end{cases}$$

In the first composition, the length $(f(x), f(y), f(x), f(y), \dots) = l = k$ if and only if the distance $d(f(x), f(y)) = 1$, which is the case if and only if (x, y) is an edge in T . Because f is a map of graphs, this is true precisely if $f(x) \neq f(y)$, so the two images (x, y) above are equal and the diagram commutes. So, the transformation Q is natural and we now prove that Q_T is already our desired isomorphism by induction on the number of edges in the tree T and using the Mayer-Vietoris short exact sequence in the induction step. For the basis of the induction let us consider the cases $e=0$ and $e=1$.

If $e=0$, the tree T is a discrete graph, so from the previous Example 3.11 it follows that the magnitude homology $MH_{0,0}(T)$ is the free abelian group generated by the vertices of T , and for $k > 0$ or $l > 0$ the magnitude homology $MH_{k,l}(T) = 0$. The same holds for the group $F_{k,l}(T)$ and by definition of Q_T it is clear that the map is an isomorphism.

In the case $e=1$, let us denote the only edge in T by $e_0 = (v_0, v_1)$ and

and the magnitude homology of T . We have seen before in Proposition 3.13 that the only non-zero magnitude homology groups for $k \neq 0$ and $l = 1$ are $MH_{0,0}(T) = Z \langle V(T) \rangle$ and $MH_{1,1}(T) = Z \langle E(T) \rangle$. For any $l > 1$ and $k \geq 0$, observe that a tuple (x_0, \dots, x_k) of vertices in T^{k+1} with $x_0 \neq x_1 \neq \dots \neq x_k$ of finite length must be either $(v_0, v_1, v_0, v_1, \dots)$ or $(v_1, v_0, v_1, v_0, \dots)$ since v_0 and v_1 are the only two connected vertices in T . The length of these $(k+1)$ -tuples is $\ell(v_0, v_1, v_0, v_1, \dots) = \ell(v_1, v_0, v_1, v_0, \dots) = k$. Therefore, the only non-trivial group in the magnitude chain complex $MC_{\cdot, \cdot}(T)$ is $MC_{l,l}(T)$, which is the free abelian group generated by those $(k+1)$ -tuples $(v_0, v_1, v_0, v_1, \dots)$ and $(v_1, v_0, v_1, v_0, \dots)$. Consequently, the same holds for the magnitude homology and with the definition of $F_{k,l}(T)$ and Q_T it immediately follows that Q_T is an isomorphism.

Suppose that the number of edges ≥ 2 and assume that Q_T is an isomorphism if the number of edges is strictly smaller than n . The idea is to show the induction step by writing the tree T as a union of two subtrees T_1, T_2 with strictly less edges than T and such that $(T; T_1, T_2)$ is a projecting decomposition, so that we can apply the Mayer-Vietoris Theorem for magnitude homology. Since T is a tree with at least two vertices, it has a leaf $= \{x_0, x_1\} \in E(T)$, where x_0 denotes the vertex that has no other neighbours except for x_1 . Let T_2 be the subtree of T induced by the vertices x_0 and x_1 , so its only edge is the leaf $\{x_0, x_1\}$ and let T_1 be the subtree of T induced by the vertex set $V(T) \setminus \{x_0\}$. Then, the number of edges $\#E(T_1) < \#E(T)$ and $\#E(T_2) < \#E(T)$, and the tree T is the union $T_1 \cup T_2$. Furthermore, the intersection $T_1 \cap T_2$ consists only of the vertex x_1 , thus it is convex in T , and T_2 projects to $T_1 \cap T_2$. Hence, $(T; T_1, T_2)$ is a projecting decomposition and we can apply the Mayer-Vietoris Theorem for magnitude homology to obtain the split short exact sequence

$$0 \rightarrow MH_{k,l}(T_1 \setminus T_2) \rightarrow MH_{k,l}(T_1) \oplus MH_{k,l}(T_2) \rightarrow MH_{k,l}(T) \rightarrow 0,$$

for any $k, l \geq 0$.

Claim: There is an analogous short exact sequence but with $F_{k,l}$ instead of $MH_{k,l}$. In particular, we want to show that

$$0 \rightarrow F_{k,l}(T_1 \setminus T_2) \xrightarrow{(j^{T_1}, j^{T_2})} F_{k,l}(T_1) \oplus F_{k,l}(T_2) \xrightarrow{(i^{T_1}, i^{T_2})} F_{k,l}(T) \rightarrow 0$$

is a short exact sequence for all $k \geq 0$.

Proof of Claim: For $k \neq l$ all the involved groups are trivial, so there is nothing to show. If $k = l = 0$, then the sequence is concretely given by

$$0 \rightarrow Z \langle \{x_1\} \rangle \xrightarrow{(j^{T_1}, j^{T_2})} Z \langle V(T_1) \rangle \oplus Z \langle V(T_2) \rangle \xrightarrow{(i^{T_1}, i^{T_2})} Z \langle V(T) \rangle \rightarrow 0.$$

The map (j^{T_1}, j^{T_2}) is injective because it maps the generator $\{x_1\}$ to $(x_1, x_1) \in Z \langle V(T_1) \rangle \oplus Z \langle V(T_2) \rangle$, which is non-zero. Every vertex $x \in T \setminus \{x_0\}$

is a vertex in T_1 , so it is the image of $(x, 0) \in Z \times V(T_1) \times Z \times V(T_2)$ under the map $(i^{T_1} \ i^{T_2})$, the vertex $x_0 \in T$ is the image of $(0, x_0) \in Z \times V(T_1) \times Z \times V(T_2)$. Hence, the map $(i^{T_1} \ i^{T_2})$ is surjective. The only thing left to show for the claim is that $\text{im}((j^{T_1}, \ j^{T_2})) = \ker(i^{T_1} \ i^{T_2})$. From

$$x_1 \xrightarrow{(j^{T_1}, \ j^{T_2})} (x_1, \ x_1) \xrightarrow{(i^{T_1} \ i^{T_2})} 0,$$

the inclusion $\text{im}((j^{T_1}, \ j^{T_2})) \subseteq \ker(i^{T_1} \ i^{T_2})$ follows. For the other inclusion, let us denote the rest of the vertices in T by x_1, \dots, x_n and let

$$y = \left(\sum_{i=1}^n a_i x_i, \ b_0 x_0 + b_1 x_1 \right) \in \ker(i^{T_1} \ i^{T_2})$$

for some $a_1, \dots, a_n, b_0, b_1 \in Z$ be an arbitrary element from the kernel. That is,

$$\sum_{i=1}^n a_i x_i + b_0 x_0 + b_1 x_1 = 0 \in Z \times V(T)$$

and it follows that the coefficients $b_0 = 0$ and $a_i = 0$ for every $i > 1$, furthermore $b_1 = -a_1$. We conclude that $(j^{T_1}, \ j^{T_2})(a_1 x_1) = (a_1 x_1, \ -a_1 x_1) = y$ and therefore the other inclusion $\ker(i^{T_1} \ i^{T_2}) \subseteq \text{im}((j^{T_1}, \ j^{T_2}))$ holds.

If $k = l > 0$, then the sequence is

$$0 \rightarrow (j^{T_1}, \ j^{T_2}) \times Z \times \#E(T_1) \times Z \times \#E(T_2) \xrightarrow{(i^{T_1} \ i^{T_2})} Z \times \#E(T) \rightarrow 0.$$

Thus, we only need to show that $(i^{T_1} \ i^{T_2})$ is an isomorphism. The set of oriented edges $\#E(T)$ is the disjoint union

$$\#E(T) = \#E(T_2) \sqcup \#E(T_1),$$

so $(i^{T_1} \ i^{T_2})$ is an isomorphism on the generators and the exactness follows.

Because Q is a natural transformation, we obtain the following commutative diagram, where both rows are exact:

$$\begin{array}{ccccccc} 0 & \longrightarrow & MH_{k,l}(T_1 \setminus T_2) & \longrightarrow & MH_{k,l}(T_1) & \oplus & MH_{k,l}(T_2) & \longrightarrow & MH_{k,l}(T) & \longrightarrow & 0 \\ & & \uparrow Q_{T_1 \setminus T_2} & & \uparrow (Q_{T_1}, Q_{T_2}) & & \uparrow Q_T & & & & \\ 0 & \longrightarrow & F_{k,l}(T_1 \setminus T_2) & \longrightarrow & F_{k,l}(T_1) & \oplus & F_{k,l}(T_2) & \longrightarrow & F_{k,l}(T) & \longrightarrow & 0 \end{array}$$

The maps $Q_{T_1 \setminus T_2}$, Q_{T_1} , and Q_{T_2} are isomorphism by our induction hypothesis, so with the five-lemma we can conclude that Q_T is an isomorphism as well.

3.5 Diagonal Graphs

We have observed before with Table 3.1 that the magnitude homology of the four-cycle appears to be non-zero only on the diagonal. In this chapter, we show that this is indeed the case for the four-cycle and a whole class of graphs called joins. We will also see some further examples of this property, let us first define the notion of a diagonal graph.

Definition 3.59. A graph G is diagonal if for all $k \in \mathbb{I}$ the magnitude homology $MH_{k,l}(G) = 0$, that is, if its magnitude homology is non-zero only on the diagonal.

The diagonality of a graph has an immediate effect on the magnitude, as the following result demonstrates.

Proposition 3.60. If a graph G is diagonal, then the coefficients of its magnitude $|G|_q$ alternate in sign and the magnitude determines the magnitude homology up to isomorphism.

Proof. Let G be a diagonal graph. Using Theorem 3.16, which relates the magnitude and the magnitude homology, we obtain

$$|G|_q = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k \text{rank}(MH_{k,l}(G)) q^l = \sum_{l=0}^{\infty} (-1)^l \text{rank}(MH_{l,l}(G)) q^l.$$

This immediately shows that the coefficients of $|G|_q$ alternate in sign. By Proposition 3.9 we know that for any $l \geq 0$ the group $MC_{l+1,l}(G) = 0$, therefore the magnitude homology $MH_{l,l}(G)$ is free abelian and thus determined up to isomorphism by its rank, which is the absolute value of the corresponding coefficient in $|G|_q$. \square

Using the previous section, we can prove the following case of diagonality.

Proposition 3.61. A graph X that admits a projecting decomposition into diagonal graphs, that is, there is a projecting decomposition $(X; G, H)$ such that G and H are diagonal, is diagonal itself.

Proof. We can apply the Mayer-Vietoris Theorem 3.53 for magnitude homology to the projecting decomposition $(X; G, H)$ and obtain the short exact sequence

$$0 \rightarrow MH_{k,l}(G \setminus H) \rightarrow MH_{k,l}(G) \rightarrow MH_{k,l}(H) \rightarrow MH_{k,l}(G \sqcup H) \rightarrow 0.$$

For $k \in \mathbb{I}$, this sequence simplifies to

$$0 \rightarrow MH_{k,l}(G \setminus H) \rightarrow MH_{k,l}(G \sqcup H) \rightarrow 0$$

because G and H are diagonal. By exactness, it follows that the magnitude homology $MH_{k,l}(G \sqcup H) = 0$.

\square

Let us introduce a new operation on two graphs and study its properties, Figure 3.5 demonstrates the operation on an example.

Definition 3.62. The join $G \ast H$ of two graphs G and H is the graph obtained from the disjoint union $G \sqcup H$ by adding the edge xy for all vertices $x \in G$ and all $y \in H$.

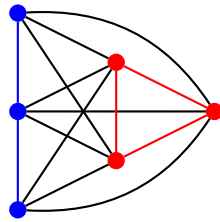


Figure 3.5: Example of the join between the blue graph and the red graph.

Lemma 3.63. Let G and H be two non-empty graphs and consider two vertices $a, b \in G \ast H$. Their distance $d(a, b) \in \{0, 1, 2\}$ and $d(a, b) = 2$ implies that a and b both lie in G or both in H .

Proof. There are two cases. First, if the vertex $a \in G$ and $b \in H$ or vice versa, then by definition of the join their distance is $d(a, b) = 1$. Second, both of them are in G or both in H , without loss of generality, say $a, b \in G$. If $a = b$, then the distance $d(a, b) = 0$. Else $a \neq b$ and we differentiate whether the edge ab is in G . If $ab \notin E(G)$, we use that H is not empty, so there exists a vertex $c \in H$, which is a neighbour of a and b by the definition of the join, hence $d(a, b) = 2$. If $ab \in E(G)$, then the distance $d(a, b) = 1$. \square

Theorem 3.64. The join of two non-empty graphs G and H is diagonal.

The proof of this theorem is quite long, so we use several definitions and lemmas to prove it. To simplify their statements, let us fix two non-empty graphs G and H and an integer $l \geq 0$ until we have proven the theorem above.

Definition 3.65. For any $i \in \{0, \dots, l-1\}$ let F^i denote the chain subcomplex of the chains $MC_{\bullet}(G \ast H)$ spanned by those generators (x_0, \dots, x_k) such that there exists an index $j \leq i$ for which $d(x_j, x_{j+1}) = 2$.

We denote by F^{-1} the zero chain complex to shorten some of the following statements. Before we continue, let us check that this definition produces indeed a chain subcomplex of $MC_{\bullet}(G \ast H)$.

Remark 3.66. Let $i \in \{0, \dots, l-1\}$ and $k \geq 0$ be arbitrary and let (x_0, \dots, x_k) be any generator of F_k^i . We need to check that the restriction of the differential $\partial = \sum_{j=0}^{k-1} (-1)^{j^0} \partial_{j^0}$ is well-defined on F^i . If $i+1 < j^0 \leq k-1$, then $\partial_{j^0}(x_0, \dots, x_k)$ either is zero or the entries x_0, \dots, x_{i+1} do not change, in any case the image $\partial_{j^0}(x_0, \dots, x_k) \in F_{k-1}^i$. If $1 \leq j^0 \leq i+1$, then by definition

$$\partial_{j^0}(x_0, \dots, x_k) = \begin{cases} (x_0, \dots, x_{j^0-1}, x_{j^0+1}, \dots, x_k) & \text{if } d(x_{j^0-1}, x_{j^0+1}) = d(x_{j^0-1}, x_{j^0}) + d(x_{j^0}, x_{j^0+1}) \\ 0 & \text{else} \end{cases}$$

We need to show that $(x_0, \dots, x_{j^0-1}, x_{j^0+1}, \dots, x_k) \in F_{k-1}^i$ if we are in the first case. By Lemma 3.63 above, the distance $d(x_{j^0-1}, x_{j^0+1}) \in \{0, 1, 2\}$. The condition $x_{j^0-1} \in X_{j^0} \in X_{j^0+1}$ implies that $d(x_{j^0-1}, x_{j^0}) \in \{1, 2\}$ and $d(x_{j^0}, x_{j^0+1}) \in \{1, 2\}$. Thus, the image $\partial_{j^0}(x_0, \dots, x_k)$ is non-zero only if the distance $d(x_{j^0-1}, x_{j^0+1}) = 2$ and $d(x_{j^0-1}, x_{j^0}) = d(x_{j^0}, x_{j^0+1}) = 1$. It follows that $\partial_{j^0}(x_0, \dots, x_k) \in F_{k-1}^i$ because $j^0 - 1 \leq i$.

We obtain a filtration

$$F^0 \subset F^1 \subset \dots \subset F^{l-1} \subset MC_{\cdot, l}(G \wr H)$$

of the chain complex $MC_{\cdot, l}(G \wr H)$.

Definition 3.67. Let $x \in G \wr H$ be any vertex, we define $\mathcal{A}_{\cdot, l}(x)$ to be the chain subcomplex of $MC_{\cdot, l}(G \wr H)$ that is generated by tuples of the form (x, x_1, \dots, x_k) with $d(x, x_1) = 2$. Furthermore, we define $\mathcal{B}_{\cdot, l}(x)$ to be the chain subcomplex of $MC_{\cdot, l}(G \wr H)$ generated by tuples of the form (x, x_1, \dots, x_k) .

By the same argument as we have seen in Remark 3.66 above, it is clear that these are indeed chain subcomplexes.

Lemma 3.68. For every $i \in \{0, \dots, l-1\}$, there is a commutative diagram

$$\begin{array}{ccc} \bigoplus^L S^i \mathcal{A}_{\cdot, l-i}(x_i) & \hookrightarrow & \bigoplus^L S^i \mathcal{B}_{\cdot, l-i}(x_i) \\ \downarrow a & & \downarrow b \\ F^i / F^{i-1} & \hookrightarrow & MC_{\cdot, l}(G \wr H) / F^{i-1} \end{array}$$

where the direct sums are taken over tuples $(x_0, \dots, x_i) \in (G \wr H)^{i+1}$ with distance $d(x_j, x_{j+1}) = 1$ for every $j \in \{0, \dots, i-1\}$, and the upper map is the direct sum of the inclusion maps.

Proof. We first let $i = 0, \dots, l - 1$ be arbitrary and define the map \bar{a} on the summand corresponding to $(x_0, \dots, x_i) \in (G \otimes H)^{i+1}$ as described above by

$$\bar{a}: S^i A_{\cdot, l-i}(x_i) \rightarrow F^i / F^{i-1} \\ (x_i, \dots, x_k) \mapsto (-1)^{ik} (x_0, \dots, x_i, \dots, x_k)$$

for any generator $(x_i, \dots, x_k) \in (S^i A_{\cdot, l-i}(x_i))_k$. First, let us verify that this map is well-defined. The generator $(x_i, \dots, x_k) \in A_{k-i, l-i}(x_i)$ satisfies $\ell(x_i, \dots, x_k) = l - i$, which implies that the tuple (x_0, \dots, x_k) has length

$$\ell(x_0, \dots, x_k) = \sum_{j=0}^{i-1} d(x_j, x_{j+1}) + \ell(x_i, \dots, x_k) = i + l - i = l,$$

and $(x_0, \dots, x_k) \in F_k^i$ because $d(x_i, x_{i+1}) = 2$. Next, we check that \bar{a} is a chain map, that is, that the following diagram commutes for every $k \geq 0$.

$$\begin{array}{ccc} (S^i A_{\cdot, l-i}(x_i))_k & \xrightarrow{\bar{a}} & (F^i / F^{i-1})_k \\ \downarrow \mathbb{1} & & \downarrow \mathbb{1} \\ (S^i A_{\cdot, l-i}(x_i))_{k-1} & \xrightarrow{\bar{a}} & (F^i / F^{i-1})_{k-1} \end{array}$$

The differential $\mathbb{1}: (S^i A_{\cdot, l-i}(x_i))_k \rightarrow (S^i A_{\cdot, l-i}(x_i))_{k-1}$ is the alternating sum $\mathbb{1} = \sum_{j=1}^k (-1)^j \mathbb{1}_j$ and the differential on (F^i / F^{i-1}) is induced by the alternating sum $\mathbb{1} = \sum_{j=1}^k (-1)^j \mathbb{1}_j$. Let $j = 1, \dots, i$ and consider a generator $(x_0, \dots, x_k) \in F_k^i$. With the argument from Remark 3.66 we find that $\mathbb{1}_j(x_0, \dots, x_k) \in F_{k-1}^{i-1}$ and thus the map $\mathbb{1}_j$ is trivial on (F^i / F^{i-1}) . The differential on (F^i / F^{i-1}) therefore simplifies to

$$\mathbb{1} = \sum_{j=i+1}^k (-1)^j \mathbb{1}_j.$$

Note that for any index $j = 1, \dots, k - i - 1$ and an arbitrary generator $(x_i, \dots, x_k) \in (S^i A_{\cdot, l-i}(x_i))_k$, on one hand

$$(-1)^i \mathbb{1}_{j+i} \bar{a}(x_i, \dots, x_k) = (-1)^{ik+i} \mathbb{1}_{j+i}(x_0, \dots, x_k)$$

and on the other hand, because $\mathbb{1}_j(x_i, \dots, x_k) = (-1)^j (x_i, \dots, x_{i+j}, \dots, x_k)$ if the length is preserved and 0 otherwise, we have

$$\bar{a} \mathbb{1}_j(x_i, \dots, x_k) = (-1)^{i(k-1)} \mathbb{1}_{i+j}(x_0, \dots, x_k).$$

It follows that $\bar{a} \mathbb{1}_j = (-1)^i \mathbb{1}_{j+i} \bar{a}$ for $j = 1, \dots, k - i - 1$ and we obtain

$$\bar{a} \mathbb{1} = \sum_{j=1}^{k-i-1} (-1)^j \bar{a} \mathbb{1}_j = \sum_{j=1}^{k-i-1} (-1)^{j+i} \mathbb{1}_{j+i} \bar{a} = \sum_{j^0=i+1}^{k-1} (-1)^{j^0} \mathbb{1}_{j^0} \bar{a} = \mathbb{1} \bar{a},$$

which proves that \bar{a} is a chain map and thus so is a . The generators of $(F^i / F^{i-1})_k$ are the tuples (x_0, \dots, x_k) with distance $d(x_i, x_{i+1}) = 2$ and $d(x_0, x_1) = \dots = d(x_{i-1}, x_i) = 1$ because these are the generators of F^i that do not lie in F^{i-1} . Any such generator corresponds to a tuple (x_0, \dots, x_i) of vertices in $(G \times H)^{i+1}$ with $d(x_0, x_1) = \dots = d(x_{i-1}, x_i) = 1$ and is the image of $(x_i, \dots, x_k) \in (S^i A_{i-1}(x_i))_k$ under the map \bar{a} corresponding to the summand (x_0, \dots, x_i) . The injectivity of a on the generators follows from the definition, so we conclude that a is an isomorphism. Analogous to a , let us define the map $b: (S^i B_{i-1}(x_i)) \rightarrow MC_{i-1}(G \times H) / F^{i-1}$ on the summand corresponding to the tuple $(x_0, \dots, x_i) \in (G \times H)^{i+1}$ with the conditions above by

$$\bar{b}: (S^i B_{i-1}(x_i)) \rightarrow MC_{i-1}(G \times H) / F^{i-1} \\ (x_i, \dots, x_k) \mapsto (-1)^{ik} (x_0, \dots, x_i, \dots, x_k)$$

for any generator $(x_i, \dots, x_k) \in (S^i B_{i-1}(x_i))_k$. With this definition, it immediately follows that the diagram in the statement commutes. We also need to check that \bar{b} is a chain map, that is, that the diagram

$$\begin{array}{ccc} (S^i B_{i-1}(x_i))_k & \xrightarrow{\bar{b}} & (MC_{i-1}(G \times H) / F^{i-1})_k \\ \downarrow \bar{a} & & \downarrow \bar{a} \\ (S^i B_{i-1}(x_i))_{k-1} & \xrightarrow{\bar{b}} & (MC_{i-1}(G \times H) / F^{i-1})_{k-1} \end{array}$$

commutes for any $k \geq 0$. The differential on $(S^i B_{i-1}(x_i))$ is the alternating sum $\bar{a} = \sum_{j=1}^k (-1)^j \bar{a}_j$ and with the same argument as before, it follows that the differential on $MC_{i-1}(G \times H) / F^{i-1}$ simplifies to $\bar{a} = \sum_{j=i+1}^k (-1)^j \bar{a}_j$. We continue analogous to \bar{a} and find that for every $j = 1, \dots, k-i-1$ the equation $\bar{b} \bar{a}_j = (-1)^j \bar{a}_{i+j} \bar{b}$ holds. Indeed, both the right and left-hand-side map a generator $(x_i, \dots, x_k) \in (S^i B_{i-1}(x_i))_k$ to $(-1)^{ik+i} \bar{a}_{i+j}(x_0, \dots, x_k)$. We conclude that \bar{b} is a chain map as well. \square

For the next lemma, we consider the following setting. Let $x \in G \times H$ be any vertex and without loss of generality assume that $x \in G$, otherwise exchange the roles of G and H . Furthermore, x an arbitrary vertex $y \in H$.

Lemma 3.69. The inclusion $A_{i-1}(x) \hookrightarrow B_{i-1}(x)$ induces the trivial map in homology.

Let us introduce the following definition for the proof of Lemma 3.69.

Definition 3.70. The height of a generator $(x, x_1, \dots, x_k) \in A_{i-1}(x)$ is the largest integer h such that for all odd indices $i \leq h$ the distance $d(x, x_i) = 2$ and for all even indices $i \leq h$ the distance $d(x, x_i) = 1$.

Note that the height of any generator $(x, x_1, \dots, x_k) \in A_{i,l}(x)$ is at most k , and at least 1 because $d(x, x_1) = 2$ by definition of $A_{i,l}(x)$. In the proof, we will need the following observations regarding the height of a generator.

Remark 3.71. Let $(x, x_1, \dots, x_k) \in A_{i,l}(x)$ be a generator of height h . By Lemma 3.63, we deduce that all the vertices x_j for $1 \leq j \leq h$ and j even lie in H , and the vertices x_j for $1 \leq j \leq h$ and j odd lie in G . Furthermore, we can conclude that $d(x_1, x_2) = d(x_2, x_3) = d(x_3, x_4) = \dots = d(x_{h-1}, x_h) = 1$.

Proof of Lemma 3.69. Denote by i the inclusion $A_{i,l}(x) \hookrightarrow B_{i,l}(x)$. By constructing a chain homotopy $s: A_{i,l}(x) \rightarrow B_{i+1,l}(x)$ satisfying $s \circ \partial + \partial \circ s = i$, we show that the inclusion i is chain homotopic to the zero map, which proves the lemma. Let us consider for any $i \geq 1$ the following map $s_i: A_{i,l}(x) \rightarrow B_{i+1,l}(x)$ defined on a generator $(x, x_1, \dots, x_k) \in A_{i,l}(x)$ with height h by

$$s_i(x, x_1, \dots, x_k) = \begin{cases} (x, y, \underbrace{x, y, \dots, x_i}_{i+1 \text{ entries}}, x_{i+1}, \dots, x_k) & \text{if } i = h \\ 0 & \text{if } i < h \end{cases}$$

To verify that s_i is well-defined, we check that the length of the $(k+2)$ -tuple $s_i(x, x_1, \dots, x_k) = (x, y, x, y, \dots, x_i, \dots, x_k)$ is l if $i = h$. If $i = h$ is even, then $x_i \in H$ by Remark 3.71 and thus the distance $d(x, x_i) = 1$ because $x \in G$. Analogously, if $i = h$ is odd, then $x_i \in G$ and $d(y, x_i) = 1$ and hence

$$\begin{aligned} \ell(x, y, \underbrace{x, y, \dots, x_i}_{i+1 \text{ entries}}, x_{i+1}, \dots, x_k) &= i \cdot d(x, y) + 1 + \ell(x_i, \dots, x_k) \\ &= i + 1 + \ell(x_i, \dots, x_k). \end{aligned}$$

The length of $\ell(x_i, \dots, x_k)$ can be calculated by using Remark 3.71 and the definition of $A_{i,l}(x)$, which imply

$$\begin{aligned} l = \ell(x, x_1, \dots, x_k) &= d(x, x_1) + \sum_{j=1}^{i-1} d(x_j, x_{j+1}) + \ell(x_i, \dots, x_k) \\ &= 2 + i - 1 + \ell(x_i, \dots, x_k). \end{aligned}$$

It follows that $\ell(x_i, \dots, x_k) = l - i + 1$ and thus $\ell(x, y, x, y, \dots, x_i, \dots, x_k) = l$. Next, we will verify the following relations between the maps s_i and ∂_i :

- i) $\partial_i \circ s_j = 0$ for $j < i$;
- ii) $\partial_i \circ s_{i+1} = s_{i+1} \circ \partial_{i+1}$;
- iii) $\partial_i \circ s_{i+2} = s_{i+2} \circ \partial_{i+2}$;
- iv) $\partial_i \circ s_i = i$;

v) $s_i = 0$ if $i > h$.

Let $(x, x_1, \dots, x_k) \in A_{k,l}(x)$ be a generator with height h .

i) Let $1 \leq j < i$, if we remove the j -th entry of $(x, y, x, y, \dots, x_i, \dots, x_k)$, then there are two consecutive entries of y or x in the resulting tuple, depending on whether j is even or odd. In any case, the length will not be preserved after removing this entry, so $s_j(x, x_1, \dots, x_k) = 0$.

ii) Let $i \leq h$. The left-hand-side is equal to

$$s_i(x, x_1, \dots, x_k) = \left| \left\{ \begin{array}{c} x, y, x, y, \dots, x_i, \dots, x_k \\ \hline \{z\} \\ \text{i+1 entries} \end{array} \right\} \right|$$

if $i \leq h$ and the length of the tuple $(x, y, x, y, \dots, x_i, \dots, x_k) = l$, and equal to 0 otherwise. The right-hand-side is

$$\begin{aligned} s_{i+1}(x, x_1, \dots, x_k) &= \left| \left\{ \begin{array}{c} x, y, x, y, \dots, x_{i+1}, \dots, x_k \\ \hline \{z\} \\ \text{i+2 entries} \end{array} \right\} \right| \\ &= \left| \left\{ \begin{array}{c} x, y, x, y, \dots, x_{i+1}, \dots, x_k \\ \hline \{z\} \\ \text{i+1 entries} \end{array} \right\} \right| \end{aligned}$$

if $i+1 \leq h$ and the length $(x, y, x, y, \dots, x_{i+1}, \dots, x_k) = l$, and is equal to 0 otherwise. It follows that the two sides are equal if $i \leq h-1$ or $i > h$. In the case $i = h$, the right-hand-side is 0. Let us check that the left-hand-side is also 0. First, assume that $i = h$ is odd. Hence, the distance $d(y, x_{i+1}) \in 2$ by definition of the height and the vertex x_i lies in G by Remark 3.71. Therefore,

$$\left| \left\{ \begin{array}{c} y, x_i \\ \hline \{z\} \\ \text{= 1} \end{array} \right\} \right| + \left| \left\{ \begin{array}{c} x_i, x_{i+1} \\ \hline \{z\} \\ \text{2f 1,2g} \end{array} \right\} \right| \in \left| \left\{ \begin{array}{c} y, x_{i+1} \\ \hline \{z\} \\ \text{2f 0,1g} \end{array} \right\} \right|$$

and it follows that

$$\left| \left\{ \begin{array}{c} x, y, x, y, \dots, y, x_i, x_{i+1}, \dots, x_k \\ \hline \{z\} \\ \text{i+1 entries} \end{array} \right\} \right| \in \left| \left\{ \begin{array}{c} x, y, x, y, \dots, y, x_i, \dots, x_k \\ \hline \{z\} \\ \text{i+1 entries} \end{array} \right\} \right|$$

which shows that the left-hand-side is also equal to 0. In a similar way, if $i = h$ is even, then $x_i \in H$ and $d(x, x_{i+1}) \in 2$. We find

$$\left| \left\{ \begin{array}{c} x_i, x_i \\ \hline \{z\} \\ \text{= 1} \end{array} \right\} \right| + \left| \left\{ \begin{array}{c} x_i, x_{i+1} \\ \hline \{z\} \\ \text{2f 1,2g} \end{array} \right\} \right| \in \left| \left\{ \begin{array}{c} x_i, x_{i+1} \\ \hline \{z\} \\ \text{2f 0,1g} \end{array} \right\} \right|$$

and also conclude that $s_{i+1}(x, x_1, \dots, x_k) = 0$.

iii) Let $i \geq 1, j \geq i + 2$, the left-hand-side is

$$\mathbb{1}_j \circ \mathbb{s}_i(x, x_1, \dots, x_k) = \left(\underbrace{x, y, x_2, y, \dots, x_i, \dots, x_{j-1}, \dots, x_k}_{i+1 \text{ entries}} \right)$$

if $i \leq h$ and the length $\ell(x, y, x, y, \dots, x_i, \dots, x_{j-1}, \dots, x_k) = l$, and is 0 otherwise. For the right-hand-side, recall

$$\mathbb{1}_j^{-1}(x, x_1, \dots, x_k) = \begin{cases} (x, x_1, \dots, x_{j-1}, \dots, x_k) & \text{if } \ell(x, x_1, \dots, x_{j-1}, \dots, x_k) = l \\ 0 & \text{else.} \end{cases}$$

Note that because $j - 1 \geq i + 1$, the first entries up to x_i in the tuples $(x, x_1, \dots, x_{j-2}, x_{j-1}, \dots, x_k)$ and (x, x_1, \dots, x_k) are the same. In particular, this implies that $i \leq h$, where h denotes the height of (x, x_1, \dots, x_k) , if and only if i is smaller than or equal to the height of $(x, x_1, \dots, x_{j-1}, \dots, x_k)$. Thus,

$$\mathbb{s}_i \circ \mathbb{1}_j^{-1}(x, x_1, \dots, x_k) = \left(\underbrace{x, y, x_2, y, \dots, x_i, \dots, x_{j-1}, \dots, x_k}_{i+1 \text{ entries}} \right)$$

if $i \leq h$ and the length is preserved, otherwise $\mathbb{s}_i \circ \mathbb{1}_j^{-1}(x, x_1, \dots, x_k) = 0$. So, both sides are equal.

iv) We have previously observed that $1 \leq h$, therefore

$$(x, x_1, \dots, x_k) \stackrel{\mathbb{1}_1}{\sim} (x, y, x_1, \dots, x_k) \stackrel{\mathbb{1}_1^{-1}}{\sim} (x, x_1, \dots, x_k),$$

which is just the inclusion.

v) Let $1 \leq j \leq i$, we begin by investigating $\mathbb{1}_j(x, x_1, \dots, x_k)$. Assume that $\mathbb{1}_j(x, x_1, \dots, x_k) \neq 0$, this can only happen if $d(x_{j-1}, x_{j+1}) = 2$ and $d(x_{j-1}, x_j) = d(x_j, x_{j+1}) = 1$, as we have seen in Remark 3.66. It follows that x_{j-1} and x_{j+1} both lie either in G or both lie in H by Lemma 3.63. Suppose further that $\mathbb{1}_j(x, x_1, \dots, x_k)$ has height $h^0 \leq j$. By Remark 3.71, it follows that if j is even, then $x_{j-1} \in G$ and $x_{j+1} \in H$ (note that we consider the tuple $(x, x_i, \dots, x_{j-1}, \dots, x_k)$), and if j is odd, then $x_{j-1} \in H$ and $x_{j+1} \in G$. This is a contradiction and thus $\mathbb{1}_j(x, x_1, \dots, x_k)$ is either 0 or a generator of height at most $j - 1$. In particular, $h^0 \leq j - 1 < i$ implies that $\mathbb{s}_i \circ \mathbb{1}_j(x, x_1, \dots, x_k) = 0$.

Let us define the map $\mathbb{s}: A_{j,l}(x) \rightarrow B_{j+1,l}(x)$ as the sum $s = \sum_{i=1}^l \mathbb{s}_i$. Note that because the height of a generator in $A_{k,l}(x)$ is smaller than or equal to k , the sum defining the map \mathbb{s} is finite for each $k \geq 0$, because for $i > k$ the map \mathbb{s}_i is trivial. With the properties above, we are able to show that \mathbb{s} is the

desired chain homotopy between \mathbb{f}_i and the zero-map. The composition $\mathbb{f}_i \circ s_1$ can be simplified to

$$\begin{aligned} \mathbb{f}_i \circ s_1 &= \sum_{i-1 \leq j=1}^k \mathbb{a}_i \mathbb{a}_j (1)^{i+j} \mathbb{f}_j \circ s_1 \stackrel{i)}{=} \sum_{i-1 \leq j=i}^k \mathbb{a}_i \mathbb{a}_j (1)^{i+j} \mathbb{f}_j \circ s_1 \\ &= \mathbb{a}_i (1)^{i+i} \mathbb{f}_i \circ s_1 + \sum_{i-1 \leq j=i+1}^k \mathbb{a}_i (1)^{i+i+1} \underbrace{\mathbb{f}_{i+1} \circ s_1}_{\stackrel{ii)}{=} \mathbb{f}_{i+1} \circ s_{+1}} + \sum_{i-1 \leq j=i+2}^k \mathbb{a}_i \mathbb{a}_j (1)^{i+j} \underbrace{\mathbb{f}_j \circ s_1}_{\stackrel{iii)}{=} s_1 \circ \mathbb{f}_{j-1}} \\ &= \mathbb{f}_i \circ s_1 + \sum_{i-1 \leq j=i+2}^k \mathbb{a}_i \mathbb{a}_j (1)^{i+j} s_1 \circ \mathbb{f}_{j-1} \\ &= \mathbb{f}_i \circ s_1 + \sum_{i-1 \leq j=i+1}^{k-1} \mathbb{a}_i \mathbb{a}_j (1)^{i+j} s_1 \circ \mathbb{f}_j. \end{aligned}$$

For the other composition, we calculate

$$s_1 \circ \mathbb{f}_i = \sum_{i-1 \leq j=1}^{k-1} \mathbb{a}_i \mathbb{a}_j (1)^{i+j} s_1 \circ \mathbb{f}_j \stackrel{v)}{=} \sum_{i-1 \leq j=i+1}^{k-1} \mathbb{a}_i \mathbb{a}_j (1)^{i+j} s_1 \circ \mathbb{f}_j$$

and therefore $s_1 \circ \mathbb{f}_i + \mathbb{f}_i \circ s_1 = \mathbb{f}_i \circ s_1$, which is the inclusion by the property iv) above. \square

Using the previous two lemmas, we can deduce the following.

Lemma 3.72. For $i = 0, \dots, l-1$, the inclusion $F^i / F^{i+1} \rightarrow MC_{i,l}(G \circlearrowleft H) / F^{i+1}$ induces the zero map in homology.

Proof. Lemma 3.69 states that the upper map in the commutative diagram of Lemma 3.68 induces the trivial map in homology. Because \mathbb{a} is an isomorphism, it follows that the lower map in the diagram, which is precisely the inclusion from the statement of this lemma, induces the trivial map in homology as well. \square

Finally, we are able to prove that joins of non-empty graphs are diagonal.

Proof 3.64. We first look at the chain subcomplex $F^{l-1} \rightarrow MC_{i,l}(G \circlearrowleft H)$. Recall that it is spanned by generators (x_0, \dots, x_k) such that $d(x_j, x_{j+1}) = 2$ for some $j \in \{0, \dots, k-1\}$. Let $k < l$ and consider any generator $(x_0, \dots, x_k) \in MC_{k,l}(G \circlearrowleft H)$, the inequality $\sum_{i=0}^k d(x_i, x_{i+1}) = l > k$ implies that there exists an index $j \in \{0, \dots, k-1\}$ such that $d(x_j, x_{j+1}) > 1$ and thus by Lemma 3.63, the distance $d(x_j, x_{j+1}) = 2$. Hence, if $k < l$, then $MC_{k,l}(G \circlearrowleft H) = F_k^{l-1}$. Note that for $k = l$, a generator $(x_0, \dots, x_k) \in MC_{k,l}(G \circlearrowleft H)$ satisfies $\sum_{i=0}^k d(x_i, x_{i+1}) = l$, which is only possible if for all $i \in \{0, \dots, k-1\}$ the distance $d(x_i, x_{i+1}) = 1$.

Therefore, the group F_k^{l-1} is trivial for $k = l$. Next, we prove by induction that for $i = 0, \dots, l-1$ the inclusion $F^i \hookrightarrow MC_{\cdot, l}(G \wr H)$ induces the zero map in homology. This finishes the proof because if the inclusion $F^{l-1} \hookrightarrow MC_{\cdot, l}(G \wr H)$, which is the identity for $k < l$ by the first observation, induces the zero map in homology, then the magnitude homology $MH_{k, l}(G \wr H) = 0$ for $k < l$. The base case $i = 0$ of the induction is part of Lemma 3.72. Fix any $i = 0, \dots, l-2$ and suppose that for $0 \leq i' \leq i$ the inclusion $F^{i'} \hookrightarrow MC_{\cdot, l}(G \wr H)$ does induce the zero map in homology, we show that this also holds for the inclusion $i: F^{i+1} \hookrightarrow MC_{\cdot, l}(G \wr H)$. The short exact sequence

$$0 \rightarrow F^i \hookrightarrow MC_{\cdot, l}(G \wr H) \xrightarrow{p} MC_{\cdot, l}(G \wr H)/F^i \rightarrow 0$$

with the inclusion and projection map p induces the following long exact sequence in homology:

$$\dots \rightarrow H_k(F^i), MH_{k, l}(G \wr H) \xrightarrow{p} H_k(MC_{\cdot, l}(G \wr H)/F^i) \rightarrow H_{k-1}(F^i) \rightarrow \dots$$

By the induction hypothesis, the map in homology induced by the inclusion is the zero map and thus exactness of the long exact sequence implies that p is injective. Consider the composition

$$F^{i+1} \hookrightarrow MC_{\cdot, l}(G \wr H) \xrightarrow{p} MC_{\cdot, l}(G \wr H)/F^i.$$

If we can show that this composition induces the zero map in homology, then we are done because the injectivity of p implies that the composition $(p \circ i) = p \circ i = 0$ if and only if i is the zero map. Let us rewrite $p \circ i$ by the composition

$$F^{i+1} \hookrightarrow F^{i+1}/F^i \hookrightarrow MC_{\cdot, l}(G \wr H)/F^i. \tag{3.21}$$

This is indeed the same map as $p \circ i$ because a generator $(x_0, \dots, x_k) \in F_k^{i+1}$ gets sent to $(x_0, \dots, x_k) + F_k^i$ under both compositions. By Lemma 3.72, the second map in the composition (3.21) induces the zero map in homology, and thus the composition $p \circ i$ induces the zero map in homology as well. \square

Let us look at some examples of graphs that can be written as joins and are therefore diagonal by the theorem we just proved.

Example 3.73 (Four-cycle). The four-cycle is the join $C_4 = E_2 \wr E_2$, as depicted in Figure 3.6. Hence, the four-cycle is indeed diagonal.

Example 3.74 (Complete multipartite graphs). The complete multipartite graph K_{n_1, \dots, n_k} with maximal independent subsets of size $n_1, \dots, n_k - 1$ is the iterated join $E_{n_1} \wr E_{n_2} \wr \dots \wr E_{n_k}$ of the discrete graphs E_{n_i} , and is therefore diagonal. In particular, the complete graph K_n can be viewed as the complete multipartite graph of n independent sets with precisely one vertex each, so K_n is diagonal. This was already shown in Example 3.12.

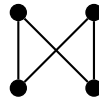


Figure 3.6: The four-cycle realized as the join $E_2 \vee E_2$.

Example 3.75 (1-skeleta of platonic solids). • The 1-skeleton of the tetrahedron is just the complete graph K_4 so it is diagonal.

- The 1-skeleton of the cube is the cartesian product $K_2 \times K_2$ and diagonal by [3, Proposition 7.3].
- The 1-skeleton of the octahedron is $E_2 \vee E_2$ and thus diagonal.
- Computer calculations with our code showed that the 1-skeleton of the dodecahedron is not diagonal, the ranks of the magnitude homology groups can be seen in Table 3.4.

$i \backslash k$	0	1	2	3	4	5	6	7	8
0	20	0	0	0	0	0	0	0	0
1	0	60	0	0	0	0	0	0	0
2	0	0	60	0	0	0	0	0	0
3	0	0	120	60	0	0	0	0	0
4	0	0	60	360	60	0	0	0	0
5	0	0	0	380	600	60	0	0	0
6	0	0	0	60	1320	840	60	0	0
7	0	0	0	0	1020	3240	1080	60	0
8	0	0	0	0	180	4620	6120	1320	60

Table 3.4: The ranks of the magnitude homology groups of the Dodecahedral graph.

- The 1-skeleton of the icosahedron is diagonal as shown by using algebraic morse theory.

Appendix A

Appendix

A.1 Code

The following code written in SageMath has been used to produce the tables containing ranks of the magnitude homology of different graphs. The code can also be accessed through <https://github.com/nadjahae/Magnitude-Homology>.

```
# input: list
# output: list without duplicates , same order
def removeduplicates (x):
    removed = []
    for i in x:
        if i in removed:
            pass
        else:
            removed.append(i)
    return removed

# input: k,l integers
# output: list with all unordered partitions of n
# with precisely k summands
def partitions (k,l):
    par = []
    if (k > l or k<0 or l<0) :
        return par
    elif (k == 0):
        if (l == 0):
            par.append([0])
            return par
        else:
```

```

        return par
    elif (l == 0):
        return par
    elif (k == 1):
        par.append([l])
        return par
    else:
        for i in range(1,l):
            prev = partitions(k-1, l-i)
            for j in prev:
                j.insert(0, i)
                par.append(j)
        par = removeduplicates(par)
        return par

#returns all the vertices in the graph G that
# have precisely distance dist from center
def graph_ball (G, center, dist):
    ball=[]
    for vertex in range(G.order()):
        if dist == G.distance(center, vertex):
            ball.append(vertex)
    return ball

#returns all generators of  $M_{\mathbb{Q},l}(G)$  that start
# with the vertex x0 and correspond to the given
# partition
def build_generator (G, x0, partition):
    generators = []
    length = len(partition)
    if length == 1:
        d = partition[0]
        if d == 0:
            generators.append(x0)
        else:
            x1vars = graph_ball(G, x0, d)
            for x1 in x1vars:
                generators.append([x0, x1])
    else:
        x1vars = graph_ball(G, x0, partition[0])
        parminuslead = partition[1:length]
        for x1 in x1vars:
            gen_i = build_generator(G, x1, parminuslead)
            for following in gen_i:

```

```

        following.insert(0, x0)
        generators.append(following)
    return generators

#returns all generators of  $M_{k,l}(G)$ 
def genMC (G,k,l):
    generators = []
    n = G.order()
    pars = partitions(k, l)

    for par in pars:
        for i in range(n):
            i_generators = build_generator(G,i,par)
            for j in i_generators:
                generators.append(j)
    return generators

#returns the rank of  $M_{k,l}(G)$ 
def rankMC (G, k, l):
    return len (genMC(G, k, l))

#returns the matrix corresponding to the
# differential map  $M_{k,l}(G) \rightarrow M_{k-1,l}(G)$ 
def differential (G, k, l):
    rows = rankMC(G, k-1, l)
    cols = rankMC(G, k, l)

    diff = matrix(ZZ, rows, cols, sparse=True)

    gen_domain = genMC(G, k, l)
    gen_codomain = genMC(G, k-1, l)

    for gen in gen_domain:
        col = gen_domain.index(gen)
        for i in range(1,k):
            copygen = gen.copy()
            copygen.pop(i)
            if copygen in gen_codomain:
                row = gen_codomain.index(copygen)
                diff[row, col] = ( 1) ** i
    return diff

#returns the chain complex  $M_{k,l}(G)$ 
def lchain_complex (G, l):

```

```

data = dict()
for k in range(l+2):
    data[k] = differential(G, k, l)
return ChainComplex(data, degree=-1, base_ring=ZZ)

#returns a table of the ranks of the magnitude homology
# groups  $MH_{k,l}(G)$  for  $k=0, \dots, k_{max}$  and  $l=0, \dots, l_{max}$ 
def table_hom_ranks (G, k_max, l_max):
    rows=[]
    row0 = [i for i in range(k_max+2)]
    row0.insert(0,0)
    rows.append(row0)
    for i in range(1,l_max+2):
        current_row = []
        Magnitude_chain_complex = lchain_complex(G, i-1)
        for j in range(k_max+1):
            current_row.append(Magnitude_chain_complex.betti(j))
        current_row.insert(0,i-1)
        rows.append(current_row)
    return table(rows = rows, header_row = True, header_column=True)

```

A.2 Tables

Further examples of ranks of magnitude homology groups calculated with SageMath and our own code given in Appendix A.1.

$l \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
0	5	0	0	0	0	0	0	0	0	0	0	0
1	0	10	0	0	0	0	0	0	0	0	0	0
2	0	0	10	0	0	0	0	0	0	0	0	0
3	0	0	10	10	0	0	0	0	0	0	0	0
4	0	0	0	30	10	0	0	0	0	0	0	0
5	0	0	0	0	50	10	0	0	0	0	0	0
6	0	0	0	0	20	70	10	0	0	0	0	0
7	0	0	0	0	0	80	90	10	0	0	0	0
8	0	0	0	0	0	0	180	110	10	0	0	0
9	0	0	0	0	0	0	40	320	130	10	0	0
10	0	0	0	0	0	0	0	200	500	150	10	0
11	0	0	0	0	0	0	0	0	560	720	170	10

Table A.1: The ranks of the magnitude homology groups of C_5

$l \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11
0	8	0	0	0	0	0	0	0	0	0	0	0
1	0	16	0	0	0	0	0	0	0	0	0	0
2	0	0	16	0	0	0	0	0	0	0	0	0
3	0	0	0	16	0	0	0	0	0	0	0	0
4	0	0	8	0	16	0	0	0	0	0	0	0
5	0	0	0	16	0	16	0	0	0	0	0	0
6	0	0	0	0	16	0	16	0	0	0	0	0
7	0	0	0	0	0	16	0	16	0	0	0	0
8	0	0	0	0	8	0	16	0	16	0	0	0
9	0	0	0	0	0	16	0	16	0	16	0	0
10	0	0	0	0	0	0	16	0	16	0	16	0
11	0	0	0	0	0	0	0	16	0	16	0	16

Table A.2: The ranks of the magnitude homology groups of the cyclic graph C_8 .

$l \setminus k$	0	1	2	3	4	5	6	7	8
0	10	0	0	0	0	0	0	0	0
1	0	30	0	0	0	0	0	0	0
2	0	0	30	0	0	0	0	0	0
3	0	0	120	30	0	0	0	0	0
4	0	0	0	480	30	0	0	0	0
5	0	0	0	0	840	30	0	0	0
6	0	0	0	0	1440	1200	30	0	0
7	0	0	0	0	0	7200	1560	30	0
8	0	0	0	0	0	0	17280	1920	30

Table A.3: The ranks of the magnitude homology groups of the Petersen Graph.

$l \setminus k$	0	1	2	3	4	5	6	7	8
0	20	0	0	0	0	0	0	0	0
1	0	60	0	0	0	0	0	0	0
2	0	0	60	0	0	0	0	0	0
3	0	0	120	60	0	0	0	0	0
4	0	0	60	360	60	0	0	0	0
5	0	0	0	380	600	60	0	0	0
6	0	0	0	60	1320	840	60	0	0
7	0	0	0	0	1020	3240	1080	60	0
8	0	0	0	0	180	4620	6120	1320	60

Table A.4: The ranks of the magnitude homology groups of the Dodecahedral graph.

l	k	0	1	2	3	4	5	6	7
0		12	0	0	0	0	0	0	0
1		0	60	0	0	0	0	0	0
2		0	0	240	0	0	0	0	0
3		0	0	0	912	0	0	0	0
4		0	0	0	0	3420	0	0	0
5		0	0	0	0	0	12780	0	0
6		0	0	0	0	0	0	47712	0
7		0	0	0	0	0	0	0	178080

Table A.5: The ranks of the magnitude homology groups of the Icosahedral graph

l	k	0	1	2	3	4	5	6	7
0		20	0	0	0	0	0	0	0
1		0	80	0	0	0	0	0	0
2		0	0	170	0	0	0	0	0
3		0	0	120	320	0	0	0	0
4		0	0	0	570	590	0	0	0
5		0	0	0	0	1560	1040	0	0
6		0	0	0	0	720	3900	1850	0
7		0	0	0	0	0	3960	8760	3200

Table A.6: The ranks of the magnitude homology groups of the Folkman graph.

l	k	0	1	2	3	4	5	6	7	8
0		4	0	0	0	0	0	0	0	0
1		0	10	0	0	0	0	0	0	0
2		0	0	24	0	0	0	0	0	0
3		0	0	0	58	0	0	0	0	0
4		0	0	0	0	140	0	0	0	0
5		0	0	0	0	0	338	0	0	0
6		0	0	0	0	0	0	816	0	0
7		0	0	0	0	0	0	0	1970	0
8		0	0	0	0	0	0	0	0	4756

Table A.7: The ranks of the magnitude homology groups of the Diamond graph.

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