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Persistence, Magnitude and Blurred Homology

Master Thesis

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Abstract

We develop the necessary category theory background to define magnitude, an invariant of enriched categories akin to cardinality of sets, as well as persistence modules - viewed as $(\mathbb{R}, \leq)$ indexed diagrams in **Top**. We then re-develop Hepworth and Willerton's notion of magnitude homology and show that it can be used to categorify magnitude in the case of finite metric spaces. We show it can also detect useful topological properties of the underlying space. Finally we describe Nina Otter's blurred magnitdue homology and its relation to magnitude.

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Chapter 1

Introduction

Magnitude is a numerical invariant of metric spaces, first introduced by Leinster [15] in 2011. It arises as an extension of a concept from category theory - the Euler characteristic of a category - to a class of categories which encompasses metric spaces. Surprisingly, the notion of magnitude was first come upon not in mathematics, but in biology. Indeed in 1994, Solow and Polasky [24] introduced a quantity to represent the "effective number of species" in an ecosysytem. This quantity is precisely our magnitude, and justifies thinking of magnitude as the "effective number of points" in a (finite) metric space.

Shockingly for a purely categorical notion wholly independant of measure or integration, magnitude has much to say about various geometric aspects of metric spaces. It is able to detect properties of compact subsets of Euclidean space, like their Minkowski dimension (see [20]) or their volume (see [5]).

On its own, magnitude is just a real number, and thus is limited in reach. In order to capture more features of metric spaces, it is often fruitful to attempt to link such invariants with algebraic objects. Indeed magnitude has a categorification; that is to say, it can be viewed as the Euler characteristic of an aptly chosen chain complex of abelian groups. This chain complex, called magnitude homology, was first introduced by Hepworth-Willerton [13] and Leinster-Shulman [18]. It is itself an invariant of metric spaces, and encapsulates more information about the space than magnitude alone does. Magnitude is thus to magnitude homology what the Euler characteristic is to regular homology.

More recently, Otter [21] introduced a variant of magnitude homology, called blurred magnitude homology, connecting magnitude homology to ordinary homology.

Blurred magnitude homology is an example of a persistence module. Persistence modules are algebraic structures which first appeared in the context of the study of persistent homology. Intuitively speaking, the idea of persistent homology is to generate from a data set an increasing sequence of topological spaces Y_r, where Y_r is interpreted to represent the shape of the data for a certain parameter r. The homology $H_*(Y_r)$ together with the maps between these complexes induced by inclusions $Y_r \hookrightarrow Y_{r'}$ for $r \leq r'$ constitutes a persistence module.

Persistence modules, under certain assumptions, admit nice decompositions into simpler interval modules. The main result of the paper this thesis deals with relates the decomposition into interval modules of blurred magnitude homology of a finite metric space to its magnitude.

Our goal in this paper is to provide and centralise all necessary knowledge to understand the statement of Theorem 4.9 from [10]. Thus, we start with a generous background section into category theory, re-tracing the steps of Tom Leinster and leading to the definition of the magnitude of metric spaces. From there, we define regular and blurred magnitude homology, and state and prove the major results linking both with magnitude.

Chapter 2

Category Theory Background

Category theory is a general theory of mathematical structures and of the relations between them. It provides a general framework which allows for the description of a wide variety of other mathematical objects, unifying and conveniently expressing common constructions such as quotients, direct products, or duals. We provide in this section a rudimentary introduction to the (vast) topic, containing sufficient bases to properly understand and define concepts from the literature. This section mainly follows from "Categorification of Persistent Homology" by Bubenik and Scott [6].

2.1 Basics of Category Theory

Definition 1 (Category)**.** *A **category** is the data of*

- *a class of **objects**, denoted* $\mathrm{Ob}(\mathbf{C})$*;*
- *for any pair of objects* $X, Y \in \mathrm{Ob}(\mathbf{C})$*, a set of **morphisms** or **arrows**, denoted by* $\mathrm{Hom}(X, Y)$*. We frequently write* $f \colon X \to Y$ *as shorthand for* $f \in \mathrm{Hom}(X, Y)$*;*
- *For any object triple* X, Y, Z*, an associative map of sets*

$$\circ\colon \begin{cases} \mathrm{Hom}(Y, Z) \times \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z) \\ g, f \mapsto g \circ f, \end{cases}$$

  *called **composition** and denoted with the symbol* $\circ$*.*
- *For any object* X*, an **identity morphism*** Id_X*, which satisfies*

$$Id_X \circ f = f,$$
$$g \circ Id_X = g,$$

  for all W, Y *and* $f \in \mathrm{Hom}(W, X)$*,* $g \in \mathrm{Hom}(X, Y)$*.*

*A category is called **small** if the class* $\mathrm{Ob}(\mathbf{C})$ *is a set. We often describe categories by solely providing their object class and morphism set, if the composition and identity maps are obvious from context.*

Example 1. The class of all sets, together with morphisms the functions between them, forms a category called **Set**. Composition is the usual composition of functions, and identity maps are as one would expect.
In the same manner, one can define the categories **Vect** of all vector spaces, with morphisms linear maps between them, **Ab** the category with objects all abelian groups, and with morphisms the group homomorphisms between them, and **Top** the category of topological spaces with arrows being continuous functions.

Remark. The language of category theory is rooted in that of algebra. Indeed many algebraic objects can very easily be interpreted as categories, and those examples will justify the use of terms like *morphism*.
However, note that morphisms are not *required* to be functions and can theoretically be any element of any set. In the same manner, although we call the associative map $\circ$ ”composition”, it does not always correspond to our intuition of composition of functions. See the following examples, which highlight these subtleties.

Example 2. Any group $(G, *)$ can be viewed as a category **CG** with a single object $\bullet$ and with a morphism $g \in \mathrm{Hom}(\bullet, \bullet)$ for each element g in G. Composition is then given by the group operation on G: $g \circ h = (g * h)$ for all $g, h \in G$.

Example 3. Let $(P, \leq)$ be a partially ordered set, that is to say, $\leq$ is a reflexive, antisymmetric and transitive binary relation on P. We can then identify P with a small category, usually denoted $(P, \leq)$ itself. It has as objects the elements of P and morphisms are given by

$$\mathrm{Hom}(x, y) = \begin{cases} \{f_{xy}\} & \text{if } x \leq y; \\ \varnothing & \text{otherwise,} \end{cases}$$

for all $x, y \in P$.
One can think of the quantity $\mathrm{Card}(\mathrm{Hom}(x, y))$ as a binary variable: it has value 1 if $x \leq y$ is true, and 0 if the expression is false. Such categories will play an important role in the following, namely for $P = \mathbb{R}$ or $P = [0, \infty]$ equipped with the standard order.

Definition 2 (Isomorphic Objects)**.** *Two objects X, Y of a category* $\mathbf{C}$ *are called **isomorphic** if there exist morphisms $f\colon X \to Y$ and $g\colon Y \to X$ such that $f \circ g = \mathrm{Id}_Y$ and $g \circ f = \mathrm{Id}_X$.*

Remark. In categories such as **Vect** and **Ab**, this notion corresponds to standard definitions of isomorphisms of vector spaces and of abelian groups.

Definition 3 (Functors). *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be categories. A* ***functor*** *from* $\mathbf{C}$ *to* $\mathbf{D}$*, written* $F\colon \mathbf{C} \to \mathbf{D}$*, is the data of:*

- *A map* $F\colon \mathrm{Ob}(\mathbf{C}) \to \mathrm{Ob}(\mathbf{D})$*;*
- *For all* $X, Y \in \mathrm{Ob}(\mathbf{C})$*, a map* $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(F(X), F(Y))$*.*

These maps must be compatible with the composition and identity: if $g\colon Y \to Z$ *and* $f\colon X \to Y$*, then* $F(g \circ f) = F(g) \circ F(f)$ *and* $F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$*.*

Remark. The notion of a functor expresses relationships between categories, in much the same way as morphisms let us describe relationships between objects in a category.

Example 4. Any category $\mathbf{C}$ can be equipped with a so-called *identity functor*. This is simply the trivial functor $\mathrm{Id}_{\mathbf{C}}\colon \mathbf{C} \to \mathbf{C}$ which acts in the following manner:

$$\begin{cases} \mathrm{Id}_{\mathbf{C}}(X) = X & \forall X \in \mathrm{Ob}(\mathbf{C}); \\ \mathrm{Id}_{\mathbf{C}}(f) = f & \forall f\colon C_1 \to C_2 \text{ morphism in } \mathbf{C}. \end{cases}$$

Example 5 (Homology and Homotopy). Denote by $H_k(-)$ degree k singular homology with integer coefficients. Then for a given topological space $X \in \mathrm{Ob}(\mathbf{Top})$, $H_k(-)$ maps X to an (abelian) group.
Furthermore, if we have other topological spaces Y, Z and continuous maps $g\colon Y \to Z$ and $f\colon X \to Y$, then we obtain new group homomorphisms $H_k(f)\colon H_k(X) \to H_k(Y)$ and $H_k(g)\colon H_k(Y) \to H_k(Z)$ such that $H_k(g \circ f) = H_k(g) \circ H_k(f)$.
H_k thus defines a functor $\mathbf{Top} \to \mathbf{Grp}$. If instead we had chosen to work with coefficients in some field $\mathbb{F}$, we would have obtained a functor $\mathbf{Top} \to \mathbf{Vect}$.
In much the same way, given a pointed topological space (X, x), the k-th homotopy π_k also defines a functor $\mathbf{Top} \to \mathbf{Grp}$. It indeed matches pointed topological spaces to groups, and continuous functions between pointed spaces to group homomorphisms.

Definition 4 (Composition of Functors). *Suppose we have* 2 *functors* $F\colon \mathbf{C} \to \mathbf{D}$ *and* $G\colon \mathbf{D} \to \mathbf{E}$*. Then we can define a* ***composite functor****, denoted* $GF\colon \mathbf{C} \to \mathbf{E}$*, by:*

$$\begin{cases} GF(X) = G(FX) & \forall X \in \mathrm{Ob}(\mathbf{C}); \\ GF(f) = G(Ff) & \forall f\colon C_1 \to C_2 \textit{ morphism in } \mathbf{C}. \end{cases}$$

In the same way that functors allow us to express relationships between categories, natural transformations allow us to express relationships between functors.

Definition 5. *Let* $\mathbf{C}$*,* $\mathbf{D}$ *be categories and* $F, G\colon \mathbf{C} \to \mathbf{D}$*. A* ***natural transformation*** η *between* F *and* G *is the data of morphisms in* $\mathbf{D}$*:* $\eta_X\colon F(X) \to G(X)$ *for* *all $X \in \mathrm{Ob}(\mathbf{C})$ such that for all $Y \in \mathrm{Ob}(\mathbf{C})$ and $f \in \mathrm{Hom}(X, Y)$, the following diagram commutes:*

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ {\scriptstyle F(f)}\big\downarrow & & \big\downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\eta_Y} & G(Y) \end{array}$$

In other words: $G(f) \circ \eta_X = \eta_Y \circ F(f)$ for all X, Y, f.
Such a natural transformation is called a ***natural isomorphism*** *if η_X is an isomorphism for all X. We write this relationship with the notation $F \cong G$.*

Remark. Informally, a natural transformation lets us transform a functor into another consistently over a category.

Example 6 (Hurewicz Homomorphism)**.** Recall from the previous that for any integer $k \geq 1$, when working with integer coefficients, k-th homotopy and k-th homology are both functors **Top** $\to$ **Grp**.

Suppose that we have a pointed topological space (X, x). Then the Hurewicz theorem guarantees the existence of a group morphism

$$h_X : \pi_1(X, x) \to H_1(X)$$

which to the class $[\gamma]$ of a loop at x associates the class $\{\gamma\}$ in $H_1(X)$, viewing γ as a singular 1-cycle. We claim that $h := (h_X)_{X \in \mathrm{Ob}(\mathbf{Top})}$ is a natural transformation between π_1 and H_1.
To show this, suppose that we have another pointed space (Y, y) as well as a continuous function $f \colon (X, x) \to (Y, y)$. From f we can recover two group homomorphisms: $\pi_1 f : \pi_1(X, x) \to \pi_1(Y, y)$ and $f_* \colon H_1(X) \to H_1(Y)$. All of these fit into the following (not necessarily commutative) diagram:

$$\begin{array}{ccc} \pi_1(X, x) & \xrightarrow{h_X} & H_1(X) \\ {\scriptstyle \pi_1 f}\big\downarrow & & \big\downarrow{\scriptstyle f_*} \\ \pi_1(Y, y) & \xrightarrow[h_Y]{} & H_1(Y) \end{array}$$

To prove h is a natural transformation, it suffices to show that this diagram commutes. For this purpose, we consider $[\gamma] \in \pi_1(X, x)$ and compute its image under both compositions:

$$\begin{aligned} h_Y \circ \pi_1 f([\gamma]) &= h_X[f \circ \gamma] \\ &= \{f \circ \gamma\}. \end{aligned}$$

Through the other path:

$$\begin{aligned} f_* \circ h_X([\gamma]) &= f_*\{\gamma\} \\ &= \{f \circ \gamma\}. \end{aligned}$$

Thus the diagram commutes, and h is proven to be a natural transformation from π_1 to H_1.

Functors and natural transformations give us the right tools to be able to define an isomorphism-like relation for categories.

Definition 6 (Equivalence of Categories)**.** *2 categories* $\mathbf{C}, \mathbf{D}$ *are said to be* ***equivalent*** *if there exist functors* $F\colon \mathbf{C} \to \mathbf{D}$ *and* $G\colon \mathbf{D} \to \mathbf{C}$ *such that*

- $GF \cong \mathrm{Id}_{\mathbf{C}}$*;*
- $FG \cong \mathrm{Id}_{\mathbf{D}}$.

2.2 Monoidal and Enriched Categories

Monoidal categories are a special type of category equipped with an operation, which we can apply to pairs of objects to obtain a new object. This section is taken from [19].

Definition 7. *A* ***monoidal category*** *is a category* $\mathbf{C}$*, equipped with an associative binary operation* $\otimes$ *and a unit object* $\mathbb{1}_{\mathbf{C}}$*.*

Remark. In rigorous detail, the product $\otimes$ is a *bifunctor* $\mathbf{C} \times \mathbf{C} \to \mathbf{C}$. This definition would require us to define the notion of a *product category*; for our purposes, all that matters is that we have a way of obtaining, from any pair of objects $X, Y \in \mathrm{Ob}(\mathbf{C})$, a third object in an associative way.

Example 7. Set - The prototypical example of a monoidal category is **Set** equipped with the cartesian product $\times$, and with a one-point set as identity element $\{*\}$.

FDVect - The category of finite-dimensional vector spaces over some field $\mathbb{F}$ is a monoidal category with product given by the tensor product $\otimes$ and with identity the base field $\mathbb{F}$.

Posets in $\mathbb{R}$ - Recall that $([0, \infty], \geq)$ can be viewed as a category with object set $[0, \infty]$ and with a single arrow $x \to y$ if $x \geq y$. This category is also monoidal: the product is given by standard addition $+$ with identity 0.

Definition 8 (Enriched Categories)**.** *Let* $(\mathbf{V}, \otimes, \mathbb{1}_{\mathbf{V}})$ *be a monoidal category. Then a* ***category enriched in*** $\mathbf{V}$ *or* $\mathbf{V}$*-category* $\mathbf{C}$ *is the data of:*

- *a collection of objects* $\mathrm{Ob}(\mathbf{C})$*;*
- *for each pair of objects* $X, Y \in \mathrm{Ob}(\mathbf{C})$*, an object of* $\mathbf{V}$ *written* $\mathrm{Hom}(X, Y)$*;*
- *identity maps* $\mathbb{1}_{\mathbf{V}} \to \mathrm{Hom}(X, X)$ *for all* $X \in \mathrm{Ob}(\mathbf{C})$*;*
- *a composition operation.*

The composition consists of maps

$$\mathrm{Hom}(X, Y) \otimes \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$$

for all triples $X, Y, Z \in \mathrm{Ob}(\mathbf{C})$*.*

Remark. Intuitively speaking, an enriched category is a category in which the Hom sets are no longer required to be sets, but are taken to be objects in some monoidal category. For standard categories, recall that composition of morphisms was a binary operation $\mathrm{Hom}(-,-) \times \mathrm{Hom}(-,-) \to \mathrm{Hom}(-,-)$. Notably, it relied on the set structure of $\mathrm{Hom}(-,-)$ to construct a cartesian product. This cartesian product is no longer available to us in this more general setting, and it is replaced by the monoidal product.
Observe also that compositions are simply morphisms in $\mathbf{V}$.

Example 8. Category enriched in Set - These are simply regular categories: the Hom morphisms classes are sets, the composition is given by functions $\mathrm{Hom}(-,-) \times \mathrm{Hom}(-,-) \to \mathrm{Hom}(-,-)$, and identity morphisms are arrows $\{*\} \to \mathrm{Hom}(X, X)$ - which can be identified with elements in $\mathrm{Hom}(X, X)$.

Category enriched in FDVect - When $\mathbf{V} = \mathbf{FDVect}$, $\mathrm{Hom}(-,-)$ has the structure of a vector space. We call these *linear categories*.

Category enriched in posets in $\mathbb{R}$ - Taking $\mathbf{V} = ([0, \infty], \geq)$, we obtain a *generalised metric space*. Such a category consists of a collection X of objects or points. Every pair $x, y \in X$ is assigned a real number $\mathrm{Hom}(x, y) =: d(x, y) \in [0, \infty]$, which we can interpret as a distance function.
The composition axioms mean that for all triple $x, y, z \in X$, there exists a morphism in $([0, \infty], \geq)$:

$$\mathrm{Hom}(x, y) + \mathrm{Hom}(y, z) \to \mathrm{Hom}(x, z)$$

which translates to the triangle inequality, since morphisms in the category $([0, \infty], \geq)$ encode order relationships:

$$d(x, y) + d(y, z) \geq d(x, z).$$

Note that these spaces are less restricted than standard metric spaces; the metric is not presumed to be symmetric, the separation axiom does not hold, and ∞ is a valid distance.

2.3 Introduction of Persistence Modules

A persistence module is a structure that aims to capture the persistence of topological features of an object across a range of parameters. Most often, they consist of a collection of abelian groups or vector fields, each corresponding to a certain filtration of a topological set. A natural parameter to consider is time; as such, persistence module are a suitable tool to analyse the evolution of topological data over a time period.
Persistence modules have been a staple tool of topological data analysis ever since they were introduced into the field in 2005 and remain one of the primary structures studied in the field.

In this section, we look at persistence modules and define them from the point of view of category theory. After a detour through abelian categories to establish important properties, we will later state and give an idea of the proof of a key decomposition theorem linking persistence modules and graded modules over polynomial rings. This is a fruitful algebraic approach which lets us use results in commutative algebra to learn more about persistence.

Definition 9. *A* ***persistence module*** *is a functor* $M\colon (\mathbb{R}, \leq) \to \mathbf{Vect}$.
A ***morphism of persistence modules*** *is a natural transformation of such functors. Persistence modules and their morphisms form a category, denoted* $\mathbf{Vect}^{(\mathbb{R},\leq)}$. *This category is also called "diagrams in* $\mathbf{Vect}$ *indexed in* $\mathbb{R}$*".*

What exactly does a persistence module M look like? To see this, consider an ordered triple $x \leq y \leq z \in \mathbb{R}$. Then:

- $M(x)$, $M(y)$ and $M(z)$ are $\mathbb{F}$-vector fields for some field $\mathbb{F}$;
- $M(x \leq y), M(y \leq z)$ are linear maps $M(x) \to M(y)$, $M(y) \to M(z)$ respectively such that $M((y \leq z) \circ (x \leq y)) = M(y \leq z) \circ M(x \leq y)$.

Let $\eta\colon M \to N$ be a natural transformation of persistence modules. Then η is the data of linear maps $\eta(x)\colon M(x) \to N(x)$ for all $x \in \mathbb{R}$ such that if $x \leq y \in \mathbb{R}$ then

$$\eta(y) \circ M(x \leq y) = N(x \leq y) \circ \eta(x),$$

i.e the following diagram commutes:

$$\begin{tikzcd} M(x) \arrow[r, "\eta(x)"] \arrow[d, "M(x\leq y)"'] & N(x) \arrow[d, "N(x\leq y)"] \\ M(y) \arrow[r, "\eta(y)"'] & N(y) \end{tikzcd}$$

Remark. The term persistence module is often used to refer to slightly different functors. For instance, it is frequent for diagrams in the categories $\mathbf{Vect}^{(\mathbb{N},\leq)}$ or $\mathbf{Vect}^{([0,\infty],\leq)}$ to be called persistence modules.

2.3.1 Key Examples

Persistence modules naturally appear in Algebraic Topology when studying data sets. In the following, we discuss classic recipes used to define persistence modules, and introduce some ubiquitous examples, like the Vietoris-Rips complex. This section follows from [7].

In the general story, one considers a space X and converts it into a simplicial complex Y. The complex is then decomposed into a system of subsets $Y_{r\in K}$, where the index set K is traditionally one of $\mathbb{R}, \mathbb{N}$ or a finite subset of integers $[\![0, n]\!]$. The values r correspond to the parameter depending on which we are studying the evolution of X.
The subsets $Y_{r\in K}$ are required to verify the following properties:

- $Y_r \subset Y_s \;\; \forall r \leq s$;
- $\bigcup\limits_{r\in K} Y_r = Y$.

In the following, we assume $K = [0, \infty]$.

The assignment $r \mapsto Y_r$ lets us define a functor $F\colon ([0,\infty], \leq) \to \mathbf{SCpx}$ into the category of simplicial complexes. We define F by:

- $F(r) = Y_r$ for $r \in [0, \infty]$;
- $F(r \leq s) = i_{rs}\colon Y_r \hookrightarrow Y_s$ the inclusion.

Letting H_k denote the functor $\mathbf{Top} \to \mathbf{Vect}$ which to a simplicial complex S associates the vector field $H_k(S)$, we then obtain a chain complex of persistence modules $(H_k F)_{k\in\mathbb{N}} : ([0,\infty], \leq) \to \mathbf{Vect}$.

Remark. Note that in light of this recipe, it is sufficient to describe the assignment $r \mapsto Y_r$ to obtain a persistence module.

We will now describe here two important filtrations, which can be used to define persistence modules as per the recipe: the Čech and Vietoris-Rips filtrations.

Definition 10 (Čech Filtration)**.** *Suppose* (X, d) *is a metric space. The* ***Čech complex*** $\mathcal{C}(X)$ *associated to* X *is the simplicial complex defined in the following way:*

- 0*-simplices are all points in* X*;*
- *for* $r \geq 0$ *and a* k*-simplex* $\sigma = \{x_0 \dots x_k\}$*, we have:*

$$\sigma \in \mathcal{C}_r(X) \iff \bigcap_{i=0}^{k} B(x_i, r) \neq \emptyset.$$

$B(x_i, r)$ *is the ball centered at* x_i *with radius* r*.*

Remark. This construction is of particular interest, because in the case where X is a compact, Riemannian manifold, a theorem guarantees that $\mathcal{C}_r(X)$ is homotopy equivalent to X for small enough values of r.

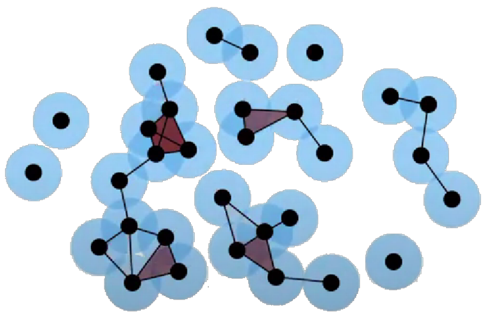


Figure 2.1: The Čech filtration of a set for parameter r the radius of a blue ball.

Remark. The Čech complex is somewhat cumbersome to work with due to the computationally expensive work involved in checking which simplices belong to the complex. The following variant of the Čech complex aims to simplify this aspect of the construction by only requiring pairwise distances to be known in order to compute the complex.

Definition 11 (Vietoris-Rips Filtration)**.** *Suppose* (X, d) *is a metric space. The* ***Vietoris-Rips complex*** $\mathcal{R}(X)$ *associated to* X *is the simplicial complex defined in the following way:*

- 0*-simplices are all points in* X*;*
- *for* $r \geq 0$ *and a* k*-simplex* $\sigma = \{x_0 \dots x_k\}$*, we have:*

$$\sigma \in \mathcal{R}_r(X) \iff d(x_i, x_j) \leq r, \ \forall i, j \colon 0 \leq i, j \leq k.$$

Remark. Note that the 1-skeleton of the Vietoris-Rips complex already encodes the entirety of the data of the pairwise distances between elements in X. As such, the entire simplicial complex can be recovered from the 1-skeleton.

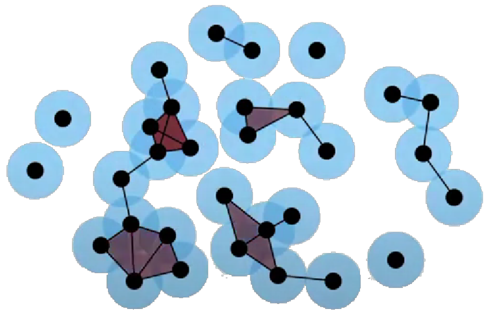


Figure 2.2: The Vietoris-Rips complex of the same underlying set for parameter $2r$.

2.3.2 Abelian Categories

We now turn our attention back to some more abstract category theory in the form of abelian categories. Abelian categories encompass many categories which behave in certain ways like **Ab**, the category of abelian groups. The category $\mathbf{Vect}^{(\mathbb{R},\leq)}$ is one such category, and we will derive from that knowledge useful constructions, like a notion of direct sum of persistence modules. These notions will play a key role in stating the barcode decomposition theorem many further results rely on. Throughout this section, let $\mathbf{C}$ be a category, with object class $\mathrm{Ob}(\mathbf{C})$.

Kernels, Cokernels

Definition 12 (Initial, Terminal Objects)**.** *We say that an object $\varnothing \in \mathrm{Ob}(\mathbf{C})$ is **initial** if, for all $X \in \mathrm{Ob}(\mathbf{C})$, there exists a unique morphism $\varnothing \to X$.*
*Dually, we say that an object $* \in \mathrm{Ob}(\mathbf{C})$ is **terminal** if, for all $X \in \mathrm{Ob}(\mathbf{C})$, there exists a unique morphism $X \to *$.*
If an object is both initial and terminal, it is called a 0-object. Copying abelian groups, it is then written 0.

Definition 13 (The 0-morphism)**.** *If a category $\mathbf{C}$ has a 0-object, then we can define a 0-morphism between any two objects $X, Y \in \mathrm{Ob}(\mathbf{C})$. Consider the following diagram:*

$$X \xrightarrow{\alpha} 0 \xrightarrow{\beta} Y$$

*where α, β are the unique morphisms given by terminality (resp. initiality). Then the 0-**morphism** is given by $0_{XY} = \beta \circ \alpha$.*

Lemma 1. *Initial (resp. terminal) objects are unique up to isomorphism.*

Proof. We prove uniqueness of initial objects. The proof of uniqueness for terminal objects is symmetric.
Assume $X, Y \in \mathrm{Ob}(\mathbf{C})$ are both initial objects. Then there exist unique morphisms $f \in \mathrm{Hom}(X, Y)$ and $g \in \mathrm{Hom}(Y, X)$. Thus $f \circ g \in \mathrm{Hom}(Y, Y)$. By initiality of Y, $\mathrm{Hom}(Y, Y) = \{\mathrm{Id}_Y\}$. Hence $f \circ g = \mathrm{Id}_Y$. Similarly, it holds that $g \circ f = \mathrm{Id}_X$. Hence X, Y are isomorphic.

$$\begin{tikzcd} X \arrow[r, bend right, "f"'] \arrow[loop below, "\mathrm{Id}_X"] & Y \arrow[l, bend right, "g"'] \arrow[loop below, "\mathrm{Id}_Y"] \end{tikzcd}$$

□

Lemma 2. *Let $W, X, Y, Z \in \mathrm{Ob}(\mathbf{C})$, $f \in \mathrm{Hom}(Y, Z)$, $g \in \mathrm{Hom}(W, X)$. Let 0_{XY} be the 0-morphism from X to Y. Then:*

- $f \circ 0_{XY} = 0_{XZ}$;
- $0_{XY} \circ g = 0_{WY}$.

Proof. We show the first equality.
Write α, β, γ the unique morphisms $X \to 0$, $0 \to Z$ and $0 \to Y$. We have $0_{XY} = \gamma \circ \alpha$, $0_{XZ} = \beta \circ \alpha$:

$$\begin{tikzcd} & & Z \\ X \arrow[r, "\alpha"] & 0 \arrow[ur, "\beta"] \arrow[dr, "\gamma"'] & \\ & & Y \arrow[uu, "f"'] \end{tikzcd}$$

Note that $f \circ \gamma \in \mathrm{Hom}(0, Z) = \{\beta\}$. Hence $f \circ \gamma = \beta$, and, composing on the right with α:

$$f \circ \underbrace{\gamma \circ \alpha}_{0_{XY}} = \underbrace{\beta \circ \alpha}_{0_{XZ}}$$

The second equality follows in the same way. □

Definition 14 (Monomorphism, Epimorphism). *We say that a morphism $f \in \mathrm{Hom}(X,Y)$ is a **monomorphism** if $\forall\, W \in \mathrm{Ob}(\mathbf{C})$, $\forall\, g,h \in Hom(W,X)$:*

$$f \circ g = f \circ h \Rightarrow g = h.$$

*Dually, f is an **epimorphism** if $\forall\, Z \in \mathrm{Ob}(\mathbf{C})$, $\forall\, g,h \in Hom(Y,Z)$:*

$$g \circ f = h \circ f \Rightarrow g = h.$$

Monomorphisms and epimorphisms extend the notion of injectivity and surjectivity to morphisms.

Remark. Recall again that in general categories, elements of Hom sets are not necessarily functions, hence injectivity and surjectivity cannot be defined by usual properties.

Definition 15 (Kernels). *Let $\mathbf{C}$ be a category with a 0-object, $X, Y \in \mathrm{Ob}(\mathbf{C})$ and $f \in \mathrm{Hom}(X,Y)$. The **kernel** of f is the equalizer of f and 0_{XY}.*
In other words, the kernel of f is an object in $\mathbf{C}$, usually denoted $\ker f$, together with a morphism $j\colon \ker f \to X$ which satisfies the following properties:

- $f \circ j = 0_{\ker f Y}$
- *For all $W \in \mathrm{Ob}(\mathbf{C})$ and $g \in \mathrm{Hom}(W,X)$ such that $f \circ g = 0_{WY}$, there exists a unique morphism $\tilde{g}\colon W \to \ker f$ such that $g = j \circ \tilde{g}$.*

This information is synthesized in the following commutative diagram:

$$\begin{tikzcd}
\ker f \arrow[r, "j"] & X \arrow[r, "f"] & Y \\
W \arrow[u, "\tilde{g}"] \arrow[ur, "g"] \arrow[urr, "0_{WY}"'] & &
\end{tikzcd}$$

Lemma 3. *The map j is a monomorphism.*

Proof. Let $\alpha, \beta \in \mathrm{Hom}(W, \ker f)$ such that $j \circ \alpha = j \circ \beta$.
We have $f \circ (j \circ \alpha) = (f \circ j) \circ \alpha = 0 \circ \alpha = 0$. Hence by universal property of j, there exists a unique morphism γ such that

$$j \circ \gamma = j \circ \alpha.$$

α and β are both valid choices for γ, so by unicity we have $\alpha = \beta$, and hence j is a monomorphism. □

Lemma 4. *If (W, g) and (V, h) are two kernels of f, then there exists a unique isomorphism $k\colon W \to V$ such that $h \circ k = g$.*

Proof. We have by definition of the kernel $f \circ g = f \circ h = 0$, hence both maps factor through each other: there exist unique maps $\alpha\colon W \to V$ and $\beta\colon V \to W$ such that $h \circ \alpha = g$ and $g \circ \beta = h$.

Consider the composition $g \circ \beta \circ \alpha$. We have:

$$\underbrace{(g \circ \beta)}_{\text{h}} \circ \alpha = h \circ \alpha = g \circ \mathrm{Id}_W \,.$$

Since g is a monomorphism, this implies that $\beta \circ \alpha = \mathrm{Id}_W$. Considering the composition $h \circ \alpha \circ \beta$ symetrically yields $\alpha \circ \beta = \mathrm{Id}_V$.
Hence α is an isomorphism and it verifies $h \circ \alpha = g$.

$$\begin{tikzcd}
& X & \\
W \arrow[ur, "g"] \arrow[rr, bend right, "\alpha"'] & & V \arrow[ul, "h"'] \arrow[ll, bend right, "\beta"']
\end{tikzcd}$$

□

Remark. There can be ambiguity as to what "the kernel of f" can mean, since it is rigorously a pair (object, morphism). Without further detail, it most often refers to the object in the pair.

Definition 16 (Cokernels)**.** *Dually, we define the* ***cokernel*** *of f to be the pair $(\operatorname{Coker} f \in \mathrm{Ob}(\mathbf{C}),\ q\colon Y \to \operatorname{Coker} f)$ such that:*

- $q \circ f = 0$
- *For all $Z \in \mathrm{Ob}(\mathbf{C})$ and $h \in \mathrm{Hom}(Y, Z)$ such that $h \circ f = 0_{XZ}$, there exists a unique morphism $\tilde{h}\colon Y \to \operatorname{Coker} f$ such that $h = \tilde{h} \circ j$.*

Products, Coproducts

Definition 17 (Products)**.** *Let $X, Y \in \mathrm{Ob}(\mathbf{C})$. The* ***product*** *of X and Y, if it exists, is an object $X \times Y$, together with "projection morphisms" $p_X\colon X \times Y \to X$; $p_Y\colon X \times Y \to Y$ that satisfy the following property:*
$\forall\ W \in \mathrm{Ob}(\mathbf{C})$, $f_X\colon W \to X$, $f_Y\colon W \to Y$, there exists a unique morphism $f\colon W \to X \times Y$ such that

$$\begin{cases} f_X = p_X \circ f; \\ f_Y = p_Y \circ f. \end{cases}$$

In other words, f makes the following diagram commutes:

$$\begin{tikzcd}
 & X \\
W \arrow[ur, "f_X"] \arrow[r, dashed, "f"] \arrow[dr, "f_Y"'] & X \times Y \arrow[u, "p_X"'] \arrow[d, "p_Y"] \\
 & Y
\end{tikzcd}$$

Definition 18 (Coproducts)**.** *Dually, we define the **product** of X and Y, if it exists, to be an object $X \oplus Y$, together with "injection morphisms" $i_X \colon X \to X \oplus Y$; $i_Y \colon Y \to X \oplus Y$ that satisfy the following property:*
$\forall\ Z \in \mathrm{Ob}(\mathbf{C})$, $g_X \colon X \to Z$, $g_Y \colon Y \to Z$, there exists a unique morphism $g \colon X \oplus Y \to Z$ such that

$$\begin{cases} g_X = i_X \circ g; \\ g_Y = i_Y \circ g. \end{cases}$$

In other words, g makes the following diagram commutes:

$$\begin{tikzcd}
X \arrow[d, "\iota_X"'] \arrow[dr, "g_X"] & \\
X \oplus Y \arrow[r, dashed, "g"] & Z \\
Y \arrow[u, "\iota_Y"] \arrow[ur, "g_Y"'] &
\end{tikzcd}$$

Definition of an Abelian Category

We now have all the tools needed to define an abelian category.

Definition 19 (Abelian Categories)**.** *A category $\mathbf{C}$ is called an **abelian category** if:*

- $\mathbf{C}$ *contains a* 0*-object;*
- $\mathbf{C}$ *contains all possible products and coproducts;*
- *every morphism in* $\mathbf{C}$ *has a kernel and a cokernel;*
- *every monomorphism is a kernel;*
- *every epimorphism is a cokernel.*

Example 9. The prototypical example of an abelian category is **Ab** the category of abelian groups.

0**-object:** The 0-object is the trivial group $(\{0\}, +)$.

Products, Coproducts: Products and coproducts are given by the product $\times$ and the direct sum $\oplus$ respectively; the universal properties defining these operations for abelian groups coincide with the categorical definitions of products and coproducts.

Kernels, Cokernels: Consider abelian groups X, Y and $f\colon X \to Y$ a morphism in **Ab**. Let $\iota\colon \ker(f) \hookrightarrow X$ be the inclusion. We claim that the pair $(\ker(f), \iota)$ is a kernel of f, in the sense of abelian catgories. Indeed, $f \circ \iota = 0$, and for some abelian group W and morphism $g\colon W \to X$, $f \circ g = 0$ implies that $\operatorname{Im}(g) \subset \ker(f)$, in which case we can define the co-restriction of g to $\ker(f)$:

$$\tilde{g}\colon \begin{cases} W \to \ker(f) \\ w \mapsto g(w) \end{cases}$$

which clearly verifies $g = \iota \circ \tilde{g}$.
Dually, letting $\pi\colon Y \to \operatorname{Coker}(f) = B/\operatorname{Im}(f)$ be the quotient projection map, we claim that $(\operatorname{Coker}(f), \pi)$ is a cokernel for f. Again it is clear that $\pi \circ f = 0$, and if we are given an abelian group Z and a homomorphism $h\colon Y \to Z$ such that $h \circ f = 0$, then h descends to $\tilde{h}$ on $\operatorname{Coker}(f)$ and we have $h = \tilde{h} \circ \pi$.

Monomorphisms are kernels: Suppose now that $f\colon X \to Y$ is a monomorphism; we will show that f is the kernel of a well-chosen morphism. First, we show that f is injective. By the previous, f has a kernel $(\ker(f),\ k\colon \ker(f) \to X)$. We have $f \circ k = f \circ 0 = 0$, which implies $k = 0$ since f is a monomorphism. However, we proved that k itself is also a monomorphism, and we have $k \circ \mathrm{Id}_{\ker f} = k \circ 0$, hence $\mathrm{Id}_{\ker f} = 0$, and thus $\ker(f) = \{0\}$. Therefore, f is injective.
Now define the projection map $\Pi\colon Y \to Y/\operatorname{Im}(f)$. It is clear that $\Pi \circ f = 0$. Furthermore, if $g\colon W \to Y$ is a morphism which verifies $\Pi \circ g = 0$, then $\operatorname{Im}(g) \subset \ker(\Pi) = \operatorname{Im}(f)$. Therefore, the composition map $\tilde{g} = f^{-1} \circ g$ is well-defined, and it is the unique map verifying $g = f \circ \tilde{g}$.

Epimorphisms are cokernels: The proof is the dual of the previous.

In much the same way, the category **Vect** of vector spaces is an abelian category. We will make extensive use of this fact in the following.

Theorem 1. *If* **C** *is an abelian category, then in particular* **C** *is pre-additive:*

- *morphism sets have the structure of an abelian group;*
- *composition is bilinear with respect to the abelian group operation;*
- *any finite product is also a coproduct, and vice-versa.*

Lemma 5. *If* **C** *is a category and* **D** *is an abelian category, then the category* $\mathbf{D}^{\mathbf{C}}$ *of functors* $F\colon \mathbf{C} \to \mathbf{D}$ *inherits the structure of an abelian category. The required structures are constructed objectwise.*
In particular, the category of persistence modules $\mathbf{Vect}^{(\mathbb{R},\leq)}$ *inherits the structure of an abelian category from* **Vect***. We can thus construct direct sums of persistence modules, as well as add natural transformations together.*

Remark. Given two persistence modules M, N, what does the product $M \oplus N$ look like?

Construct a new persistence module $P\colon (\mathbb{R}, \leq) \to \mathbf{Vect}$ by setting

$$\begin{cases} P(x) = M(x) \oplus N(x) \ \forall x \in \mathbb{R} \\ P(x \leq y) = M(x \leq y) + N(x \leq y) \end{cases}$$

It is easy to check that this module is $M \oplus N$. In the same manner, persistence modules have kernels and cokernels; these objects are themselves persistence modules, and they are comuted coordinate-wise.

2.3.3 Finite Type Diagrams

To be able to extract any information from persistence modules, it is important that we understand their structure. To do so, we will interest ourselves in a specific class of persistence modules, which decompose nicely into sums of simpler and more easily understood modules.

Definition 20 (Finite type diagrams)**.** *Let $\mathcal{I}$ be an interval in $\mathbb{R}$ and $\mathbb{F}$ a field. We define the **interval module** $\chi_{\mathcal{I}}\colon (\mathbb{R}, \leq) \to \mathbf{Vect}$ by:*

$$\chi_{\mathcal{I}}(x) = \begin{cases} \mathbb{F} \text{ if } x \in \mathcal{I}; \\ 0 \text{ otherwise.} \end{cases}$$

$$\chi_{\mathcal{I}}(x \leq y) = \begin{cases} \mathrm{Id}_{\mathbb{F}} \text{ if } x, y \in \mathcal{I}; \\ 0 \text{ otherwise.} \end{cases}$$

*A persistence module F is said to be of **finite type** if it is a sum of interval modules. F is said to be **indecomposable** if $F = M \oplus N \Rightarrow M = 0$ or $N = 0$.*

Lemma 6. *For all $\mathcal{I} \subset \mathbb{R}$, the diagram $\chi_{\mathcal{I}}$ is indecomposable.*

Proof. Suppose we have an interval $\mathcal{I}$ such that $\chi = \chi_{\mathcal{I}} = P \oplus Q$. If there exists some $c \notin I$, then $\chi(c) = 0 = P(c) \oplus Q(c)$ and hence $P(c) = Q(c) = 0$.

Now let $a \in \mathcal{I}$. Then $\chi(a) = \mathbb{F} = P(a) \oplus Q(a)$ and thus either $P(a) = 0$ and $Q(a) = \mathbb{F}$ or $P(a) = \mathbb{F}$ and $Q(a) = 0$. Assume the latter.
Pick $b \in \mathcal{I}$ such that $a \leq b$. Since $Q(a) = 0$, then necessarily $Q(a \leq b)$ is the 0-morphism. But $\chi(a \leq b) = \mathrm{Id}_{\mathbb{F}}$, so we must have $P(a \leq b) = \mathrm{Id}_{\mathbb{F}}$.
From $P(b) \oplus Q(b) = \chi(b) = \mathbb{F}$ and $P(a \leq b) = \mathrm{Id}_{\mathbb{F}}$ we obtain $P(b) = \mathbb{F}$ and $Q(b) = 0$.
Repeating this reasoning for the $z \in \mathcal{I}$ such that $z \leq a$ similarly yields

$P(z) = \mathbb{F}$, $Q(z) = 0$, and $P(z \leq a) = \mathrm{Id}_{\mathbb{F}}$, $Q(z \leq a) = 0$.
Hence $P = \chi_{\mathcal{I}}$ and $Q = 0$. $\square$

Theorem 2. *Suppose $F\colon (\mathbb{R}, \leq) \to$ **Vect** is a persistence module such that $F(x)$ is finite dimensional for all x. Then F has a decomposition into interval modules. This decomposition is unique up to re-ordering the intervals.*

Remark. This decomposition theorem gives us access to a (very large) class of finite-type persistence modules, which we will make extensive use of in the following.
The proof of this theorem in the most general of cases is outside the scope of this paper. See [9] for the proof of existence of the decomposition. See [4] for the proof of the uniqueness. We provide intuition for the result in the simplified case of modules $(\mathbb{N}, \leq) \to$ **Vect**, following [7], [26] and [8].

Proof idea. Suppose we have such a persistence module $M\colon (\mathbb{N}, \leq) \to$ **Vect**. Recall that M is the data of $\mathbb{F}$-vector spaces M_n as well as a collection of maps $M(n-1 \leq n)\colon M_{n-1} \to M_n$. To simplify notation we write these maps $M_{n-1\to n}$. We construct from them a (graded) $\mathbb{F}[t]$-module $\theta(M)$:

$$\theta(M) = \bigoplus_{n\in\mathbb{N}} M_n.$$

Suppose $v = (v_0, v_1, \ldots) \in \theta(M)$. Then the module action is given by:

$$t \cdot v = (0,\, M_{0\to 1}(v_0),\, M_{1\to 2}(v_1), \ldots)$$

Now, if we have a map between persistence modules $f\colon M \to N$, we can define the map $\theta(f)$ by letting it act pointwise:

$$\theta(f)\colon \begin{cases} \bigoplus_{n\in\mathbb{N}} M_n \to \bigoplus_{n\in\mathbb{N}} N_n \\ (v_0, v_1, \ldots) \mapsto (f(v_0), f(v_1), \ldots) \end{cases}$$

In this manner we have defined a functor θ from the category of $\mathbb{N}$-indexed persistence modules into the category **GrMod** of $\mathbb{N}$-graded $\mathbb{F}[t]$-modules.
Going the other direction, we can also define a functor $\zeta\colon$ **GrMod** $\to$ $\mathbf{Vect}^{(\mathbb{N},\leq)}$.
Given an $\mathbb{N}$-graded $\mathbb{F}[t]$-module O, we may from it define a persistence module ζO by:

- $\zeta O(n) = O_n$;
- $\zeta O(n \leq m) = t^{m-n}$,

and again, morphisms are mapped pointwise.
The functors θ and ζ establish an equivalence of categories between **GrMod** and $\mathbf{Vect}^{(\mathbb{N},\leq)}$. This correspondance lets us apply classification theorems

of (finitely-generated) graded $\mathbb{F}[t]$-modules to persistence modules, which yields the desired result.

$\square$

Whenever a persistence module $F\colon (\mathbb{R}, \leq) \to \mathbf{Vect}$ has a decomposition into interval modules, we can represent it using a **persistence barcode**. This is the multiset of all intervals that occur in the definition - that is to say, pairs (a, b) of startpoints and endpoints of intervals in the decomposition.
The multiset of these pairs is called the **persistence diagram** associated with F. This notion can sometimes be extended to persistence modules which do not admit an interval decomposition.

Chapter 3

Magnitude of Metric Spaces

Magnitude is an isometric invariant of metric spaces that encodes much geometric data - volume ([5]), Minkowski dimension... ([20]) It arises when viewing metric spaces as specific types of categories, and adapting to them a construction from homotopy theory.
The conceptual ancestor of magnitude is the Euler characteristic, itself an invariant of topological spaces. The notion of Euler characteristic for categories was first introduced in work by Leinster [14]. In further work [15], he generalised the notion of Euler characteristic of categories to enriched categories, and renamed it "magnitude" for a certain class of categories corresponding to generalised metric spaces.
In the following, we will re-trace the steps of these definitions for our own needs.

This section follows mostly from [17], [16] and [13].

3.1 Magnitude of a matrix

Definition 21 (Weighting, Coweighting, Matrix Magnitude)**.** *Let R be a ring with multiplicative identity 1_R, A a finite set, and let $Z \in R^{A\times A}$ be a matrix with coefficients in the ring R and indexed by the elements of A.*
A ***weighting*** *on Z is a column vector $w \in R^A$ satisfying*

$$Zw = e,$$

where e is the column vector with all entries equal to 1_R. In other words:

$$\sum_{b\in A} Z(a,b)w_b = 1_R \qquad \text{(for all } a \in A.)$$

Dually, we define a ***coweighting*** *on Z to be a row vector $v \in R^A$ satisfying*

$$vZ = e^T,$$

that is to say:

$$\sum_{a\in A} v_a Z(a,b) = 1_R \qquad \text{(for all } b \in A.\text{)}$$

Suppose Z admits both a weighting w and a coweighting v. Then we have

$$\sum_{w\in A} w_a = e^T w = vZw = ve = \sum_{a\in A} v_a$$

*and hence the value $\sum_{a\in A} w_a = \sum_{a\in A} v_a$ is independant of the choice of weighting (or coweighting). We call this quantity the **magnitude** of Z and denote it by $|Z|$.*

Lemma 7. *Suppose Z has an inverse matrix Z^{-1}. Then it has a unique weighting and a unique coweighting, and its magnitude is the sum of the entries of Z^{-1}:*

$$|Z| = \sum_{a,b\in A} Z^{-1}(a,b).$$

Proof. We first prove that the magnitude of Z is indeed given by this expression. Let $w \in R^A$ be given by $w_a = \sum_{b\in A} Z^{-1}(a,b)$: the a-th entry is obtained by summing all values in row a of Z^{-1}. We verify that this w is a weighting for Z:

$$\begin{aligned}
(Zw)_a &= \sum_{b\in A} Z(a,b) w_b \\
&= \sum_{b\in A} Z(a,b) \left(\sum_{c\in A} Z^{-1}(b,c) \right) \\
&= \sum_{c\in A} \sum_{b\in A} Z(a,b) Z^{-1}(b,c) \\
&= \sum_{c\in A} \mathrm{Id}(a,c) \\
&= 1_R.
\end{aligned}$$

In the same manner one can verify that v: $v_a = \sum_{a\in A} Z^{-1}(a,b)$ is a valid coweighting, and so we get

$$|Z| = \sum_{a\in A} w_a = \sum_{a,b\in A} Z^{-1}(a,b)$$

This weighting is unique because note that if w' is another weighting for Z, then $Zw' = e \implies w' = Z^{-1}e = w$. □

Theorem 3 (Magnitude of Invertible Matrix)**.** *Let $X, Z_X \in M_{n,n}(\mathbb{R})$ be as above. Then if the matrix Z_X is invertible, it admits a unique weighting, and we have*

$$|X| = \sum_{i,j=1}^{n} Z_X^{-1}(i,j).$$

3.2 The Euler Characteristic of a Finite Category

With the definition of magnitude of a matrix in tow, we can now start with the definition of the Euler characteristic of a category.

Definition 22 (Euler Characteristic for Finite Categories)**.** *Let* $\mathbf{C}$ *be a finite category - that is to say, a category which has only finitely many objects, and in which the* Hom *sets are all finite. Define the matrix* $Z_{\mathbf{C}} \in \mathbb{Z}^{\mathrm{Ob}(\mathbf{C})\times\mathrm{Ob}(\mathbf{C})}$ *by*

$$Z_C(a, b) = \mathrm{Card}(\mathrm{Hom}(a, b)),$$

for all $a, b \in \mathrm{Ob}(\mathbf{C})$*. We call the quantity* $|Z_C|$ *the* ***Euler characteristic*** *of the category* $\mathbf{C}$*, if it exists.*

Remark. One can assign to a category a topological space that encodes its data, called a **classifying space**. The name "Euler characteristic" is justified by the following theorem.

Theorem 4. *Let* $\mathbf{C}$ *be a finite category. Under appropriate circumstances, the Euler characteristic of* $\mathbf{C}$ *is equal to the Euler characteristic of its classifying space.*

3.3 Magnitude of an Enriched Category

In the following, let $(\mathbf{V}, \otimes, \mathbb{1}_{\mathbf{V}})$ denote a monoidal category, R a ring, and $m\colon \mathrm{Ob}(\mathbf{V}) \to R$ a "size" map verifying that $m(X) = m(Y)$ whenever $X \cong Y$, and satisfying multiplicativity axioms: $m(X \otimes Y) = m(X)m(Y)$ and $m(\mathbb{1}_{\mathbf{V}}) = 1_R$.

Definition 23 (Magnitude for Enriched Categories)**.** *Let* $\mathbf{C}$ *be a category enriched in* $\mathbf{V}$ *with finitely many objects.*

- *The* ***similarity matrix*** *of* $\mathbf{C}$ *is the matrix* $Z_{\mathbf{C}} \in R^{\mathrm{Ob}(\mathbf{C})\times\mathrm{Ob}(\mathbf{C})}$ *defined by* $Z_{\mathbf{C}}(x, y) = m(\mathrm{Hom}(x, y))$ *for all* $x, y \in \mathrm{Ob}(\mathbf{C})$*.*
- *The* ***magnitude*** *of* $\mathbf{C}$ *is the magnitude of* $Z_{\mathbf{C}}$ *if it is defined:* $|\mathbf{C}| = |Z_{\mathbf{C}}|$*.*

Example 10. Categories enriched in Set - Recall that $(\mathbf{Set}, \times, \{*\})$ is a monoidal category, and categories enriched in **Set** correspond to the standard definition of categories. In this case we can take $R = \mathbb{R}$ and the map m to be Card. This yields a notion of magnitude for small categories, which is exactly the same as the Euler characteristic discussed in the previous section.

Categories enriched in FDVect - For categories enriched over (finite dimensional) vector spaces, one possible choice of m is the dimension. This notion is related to that of the Euler form of an algebra.

Categories enriched in posets in $\mathbb{R}$ - Recall that categories enriched in subsets of $\mathbb{R}$ such as $([0,\infty],\geq)$ are so-called generalised metric spaces. Taking $R = \mathbb{R}$, we are left with the choice of m. The multiplicativity condition forces m to be of the form $m\colon x \mapsto e^{\lambda x}$ for some arbitrary coefficient λ. Choose for instance $\lambda = -1$. This yields a notion of magnitude for finite metric spaces, which we expand on in the following.

3.4 Magnitude of Finite Metric Spaces

Definition 24. *Let $X = \{x_1 \ldots x_n\}$ be a finite metric space equipped with a metric d, and define the* ***magnitude matrix*** *$Z_X \in M_{n,n}(\mathbb{R})$ by*

$$Z_X(i,j) = e^{-d(x_i,x_j)}.$$

A ***weighting*** *on Z_X is a colum vector $w = (w_i)_{1\leq i\leq n} \in M_{n,1}(\mathbb{R})$ such that*

$$Z_X w = e_n,$$

where e_n is the column vector of 1*s.*

If Z_X admits a weighting, then we define the ***magnitude of*** *Z_X to be*

$$|Z_X| = \sum_{i=1}^{n} w_i.$$

By extension we define the magnitude of X to be the magnitude of Z_X if it is defined.

Remark. This definition of the magnitude of a finite metric space is an immediate application of the definition of the magnitude of categories enriched in $([0,\infty],\geq)$. Note that there is a choice of scale, implicit in the definition of the magnitude; here we picked the size function to be $m\colon x \mapsto e^{-x}$, but other choices are possible. We will define a new object to capture the information encoded in magnitude regardless of the choice of scale.

Remark. Note that this definition is indeed valid for *generalised* metric spaces, and not just metric spaces. In particular, we could work with spaces in which the metric is not symmetric and without the separation axiom.

Remark. Not every metric space has a well-defined magnitude. However, there are certain large classes of metric spaces which do have an easily-expressed magnitude, as we will see in the next few theorems.

Definition 25 (Homogeneous Metric Spaces)**.** *A metric space (X,d) is called* ***homogeneous*** *if its isometry group acts transitively on the points of X. That is to say, for all $x,y \in X$, there is an isometry $f : X \to X$ such that $f(x) = y$.*

Theorem 5 (Magnitude of homogeneous spaces)**.** *Let (X, d) be a homogeneous, finite metric space and x_0 be a fixed point in X. Then X has well-defined magnitude:*

$$|X| = \frac{\mathrm{Card}(X)}{\sum_{y \in X} e^{-d(x_0,y)}}.$$

Proof. To begin with, note that the quantity $\sum_{y \in X} e^{-d(x_0,y)}$ is independant of the choice of x_0. Indeed, let $x_1 \in X$ be a point distinct from x_0. Then by homogeneity, there exists an isometry $f : X \to X$ mapping x_0 to x_1. Hence:

$$\begin{aligned}\sum_{y \in X} e^{-d(x_0,y)} &= \sum_{y \in X} e^{-d(f(x_0),f(y))} \\ &= \sum_{z \in X} e^{-d(x_1,z)}.\end{aligned}$$

Call this shared quantity q and define $w \in \mathbb{R}^X$ by $w(x) = \frac{1}{q}$ for all $x \in X$. Letting Z_X be the magnitude matrix for X we now have, for z a point in X:

$$(Z_X w)(z) = \frac{1}{q} \sum_{x \in X} e^{-d(z,x)} = \frac{1}{q} q = 1.$$

In the same manner it is clear that w^T is a coweighting. Hence, magnitude is well defined and we have:

$$|X| = \sum_{x \in X} \frac{1}{q} = \frac{\mathrm{Card}(X)}{q} = \frac{\mathrm{Card}(X)}{\sum_{y \in X} e^{-d(x_0,y)}}.$$

□

Remark. Many interesting spaces, in particular many graphs, are homogeneous spaces. In the following, all considered graphs are equipped with the shortest path metric. Descriptions of automorphism groups of common graphs in further detail can be found in [22].

Complete Graphs - All complete graphs K_n are homogeneous spaces. Indeed, identifying the vertices with $[\![1, n]\!]$, it is clear that any graph automorphism of K_n can be identified with a permutation in S_n and vice-versa. Note also that all permutations are isometries of K_n since $d(x, y) = d(\sigma(x), \sigma(y)) = 1$ for all vertices x, y of K_n and permutation σ.
In particular, S_n acts transitively on $[\![1, n]\!]$, and so K_n is a homogeneous space.

Cycle Graphs - It is known that the automorphism group of the cyclic graph on n vertices C_n is isomorphic to D_n. In particular it contains a copy of $\mathbb{Z}/n\mathbb{Z}$, which provides a transitive action on the vertices.

Definition 26. *Let X be a finite metric space as above. Denote tX the metric space (X, td) for $t \geq 0$, $t \in \mathbb{R}$. The partially defined function $t \to |tX|$ is called the **magnitude function** of X.*

Remark. The magnitude of tX is equal to the magnitude of the similarity matrix $Z_{tX}(i,j) = e^{-td(x_i,x_j)}$. Note that this is equivalent to working in X with a size function $m\colon x \mapsto e^{-tx}$; hence the magnitude function resolves the issue of the arbitrary choice of scale.

Theorem 6 (Basic Facts about the Magnitude Function)**.** *Let (X,d) be a finite metric space. Then the following hold:*

1. *the quantity $|tX|$ is defined for all but finitely many t;*
2. *the magnitude function is increasing for sufficiently large t;*
3. $\lim_{t\to\infty} |tX| = \mathrm{Card}(X)$.

Proof. For fact 1, note that for $a, b \in X$, we have $Z_{tX}(a,a) = e^{-t0} = 1$ for all t and $\lim_{t\to\infty} Z_{tX}(a,b) = 0$. Hence Z_{tX} converges to the identity matrix I_n in absolute norm. The determinant function is continuous, so for sufficiently large t, $\det(Z_{tX}) > 0$, and so the similarity is invertible and has well-defined magnitude. Fact 3 is a consequence of the same convergence. Observe that the matrix function $A \in \mathrm{GL}_n(\mathbb{R}) \mapsto |A|$ is continuous, since $A \mapsto A^{-1}$ and $A \mapsto \sum_{i,j} A_{i,j}$ are continuous. So $\lim_{t\to\infty} |Z_{tX}| = |I_n| = n = \mathrm{Card}(X)$. □

Example 11. The 2-point space - Let (X,d) be a finite metric space consisting of 2 points x, y with distance $d(x,y) = d$. Then the magnitude matrix of tX is given by:

$$Z_{tX} = \begin{pmatrix} 1 & e^{-td} \\ e^{-td} & 1 \end{pmatrix}$$

This matrix is invertible with inverse

$$Z_{tX}^{-1} = \frac{1}{1-e^{-2td}} \begin{pmatrix} 1 & -e^{-td} \\ -e^{-td} & 1 \end{pmatrix}$$

and hence we obtain the magnitude function:

$$|tX| = \frac{1}{1-e^{-2td}}(2 - 2e^{-td}) = \frac{2}{1+e^{-td}}.$$

The graph of this function is the following:

Recall that magnitude is often said to encode the "effective cardinality" of the space. Morally, as the distance between the points approaches 0, then they become indistinguishable and the magnitude approaches 1. As the distance betwwen the points goes to ∞, they separate and the magnitude converges to 2.

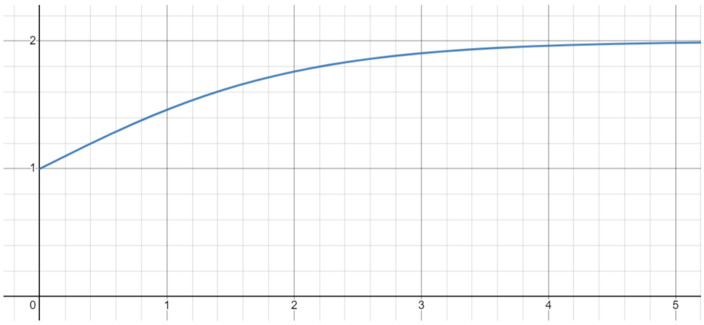


Figure 3.1: Magnitude of the two point space.

Complete Graphs - In the preceding, we noted that K_n has an easily expressed magnitude as a homogeneous space. Letting x_0 be a set vertex:

$$|tK_n| = \frac{\mathrm{Card}(K_n)}{\sum_{y \in K_n} e^{-d(x_0,y)}} = \frac{n}{1+(n-1)e^{-t}}.$$

In the case $n = 5$ we obtain the following:

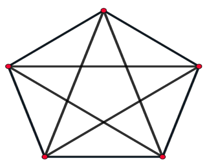


Figure 3.2: The K_5 graph.

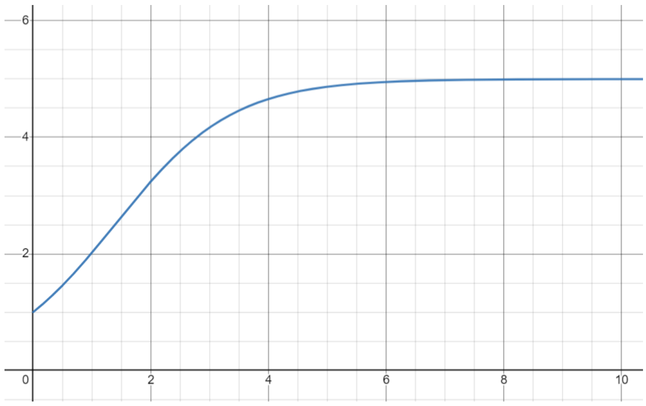


Figure 3.3: Magnitude $|tK_5|$

Cycle Graphs - We also saw in the previous that cycle graphs are homogeneous spaces, and therefore their magnitude function is easy to express as well:

$$|tC_n| = \frac{\mathrm{Card}(K_n)}{\sum_{y \in K_n} e^{-d(x_0,y)}} = \begin{cases} \dfrac{n}{1+2\sum_{j=1}^{\frac{n-1}{2}} e^{-jt}} & \text{if } n \text{ is odd;} \\ \dfrac{n}{1+2\sum_{j=1}^{\frac{n-2}{2}} e^{-jt} + e^{-\frac{n}{2}t}} & \text{if } n \text{ is even.} \end{cases}$$

We can for example look at the magnitude function of the cycle graph C_5, which is given by the above formula:

$$|tC_5| = \frac{5}{1+2e^{-t}+2e^{-2t}},$$

with graph:

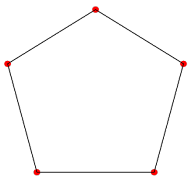


Figure 3.4: The C_5 graph.

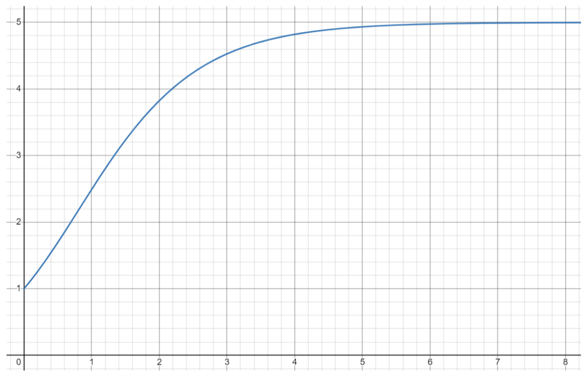
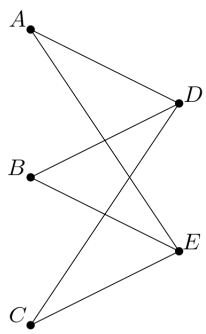


Figure 3.5: Magnitude $|tC_5|$.

Example 12 (A counterexample)**.** So far it may seem that all graphs that arise from magnitude functions are remarkably well-behaved. That is not the case in general, and many degeneracies may occur. A graph exhibiting such issues is the bipartite graph $K_{3,2}$:



The magnitude matrix of $K_{3,2}$ is given by:

$$Z_{K_{3,2}} = \begin{array}{cc} \begin{matrix} A & B & C & D & E \end{matrix} & \\ \begin{pmatrix} 1 & e^{-2t} & e^{-2t} & e^{-t} & e^{-t} \\ e^{-2t} & 1 & e^{-2t} & e^{-t} & e^{-t} \\ e^{-2t} & e^{-2t} & 1 & e^{-t} & e^{-t} \\ e^{-t} & e^{-t} & e^{-t} & 1 & e^{-2t} \\ e^{-t} & e^{-t} & e^{-t} & e^{-2t} & 1 \end{pmatrix} & \begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix} \end{array},$$

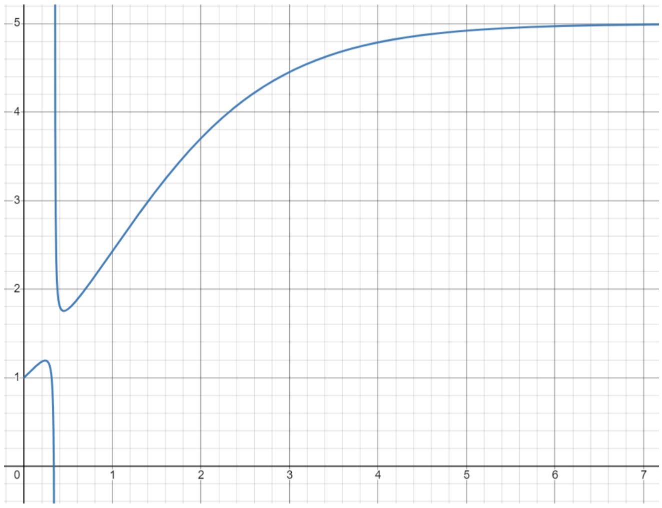
for which one can check that the following column vector is a valid weighting:

$$w = \frac{1}{p}\begin{pmatrix} 1-e^{-t} \\ 1-e^{-t} \\ 1-e^{-t} \\ 1-2e^{-t} \\ 1-2e^{-t} \end{pmatrix},$$

where $p = (1+e^{-t})(1-2e^{-2t})$. This yields the following magnitude function, summing over all the entries of w:

$$|tK_{3,2}| = \frac{5-7e^{-t}}{(1+e^{-t})(1-2e^{-2t})},$$

and this function has graph:



This function is not convex, is negative for some values of t, is undefined at $t = \frac{\ln(2)}{2}$, it is strictly decreasing on certain intervals, and strictly above the number of points on others.

3.5 Magnitude of Compact Metric Spaces

Magnitude can also be defined for certain classes of infinite metric spaces.

Definition 27 (Compact Definite-Positive Metric Spaces)**.** *A compact metric space X is called **definite-positive** if for all finite sub-metric spaces $F \subset X$, the magnitude matrix Z_F is positive-definite.*

Remark. Note that in this case, Z_F is similar to a diagonal matrix with strictly positive diagonal entries; thus $\det(Z_F) > 0$ and Z_F is invertible, and thus the magnitude of F is well-defined.

Definition 28. *Let (X, d) be a compact positive-definite metric space. Then we can define a magnitude for X in the following way:*

$$|X| = \sup(\{|F|\,,\ F \subset X \textit{ finite metric space}\}).$$

As for finite metric spaces, we call the function $t \in \mathbb{R}^+ \mapsto |tX|$ the magnitude function of X. It is not necessarily defined for all t.

Chapter 4

Magnitude Homology

Our goal is now to introduce an algebraic structure which encodes the magnitude function. To this end, we define an appropriate homology theory.

4.1 The Magnitude Chain Complex

In this section, we introduce the notion of magnitude homology - a chain complex for which the Euler characteristic (that is to say, the alternating sum of the ranks of the homology groups) corresponds to the magnitude function, in the case of finite spaces. This approach echoes other efforts to express key invariants as Euler characteristics. Notably, Khovanov homology categorifies the Jones polynomial, while Knot Floer homology categorifies the Alexander polynomial.

The idea of the magnitude chain complex was first introduced by Hepworth and Willerton in *Categorifying the magnitude of a graph* [13]; it was then extended to much more general categories by Leinster and Shulman in [18]. This section also follows from [10].

Definition 29 (Length of a tuple)**.** *Given a metric space X and $(x_0 \dots x_k)$ points in X, we define the **length** of the tuple to be:*

$$l(x_0 \dots x_k) = \begin{cases} 0 & \textit{if } k = 0; \\ \sum_{i=0}^{k-1} d(x_i, x_{i+1}) = d(x_0, x_1) + \dots + d(x_{k-1}, x_k) & \textit{otherwise.} \end{cases}$$

Definition 30 (Magnitude Chain Complex)**.** *Let X be a metric space. The **magnitude chain complex** $\mathrm{MC}_{*,\ell}(X)$ is the data of the following abelian groups for $k \in \mathbb{N}, \ell \geq 0$:*

$$\mathrm{MC}_{k,\ell}(X) = \left\langle (x_0 \dots x_k) \in X^{k+1} \mid x_i \neq x_{i+1} \; \forall i \in [\![0, k-1]\!],\; l((x_0 \dots x_k)) = \ell \right\rangle$$

In other words, $\mathrm{MC}_{k,\ell}(X)$ *is the free abelian group with basis tuples* $(x_0 \dots x_k)$ *in* X *such that two consecutive elements are distinct, and such that the length of the tuple is* ℓ*.*

The boundary operators $\partial_{k,\ell} \colon \mathrm{MC}_{k,\ell}(X) \to \mathrm{MC}_{k-1,\ell}(X)$ *are given by*

$$\partial_{k,\ell}(x_0 \dots x_k) = \sum_{i=0}^{k} (-1)^i d^i_{k,\ell}(x_0 \dots x_k)$$

where the auxilliary maps $d^i_{k,\ell}$ *are themselves given by:*

$$d^i_{k,\ell}(x_0 \dots x_k) = \begin{cases} 0 & \text{if } i = 0, k; \\ (x_0, x_1 \dots x_{i-1}, x_{i+1}, \dots x_k) & \text{if } d(x_{i-1}, x_{i+1}) = d(x_{i-1}, x_i) + d(x_i, x_{i+1}); \\ 0 & \text{otherwise.} \end{cases}$$

It can be readily checked that for a given $\ell \geq 0$, $\partial_{k,\ell} \circ \partial_{k-1,\ell} = 0$*, and thus these abelian groups indeed form a chain complex. We denote its homology groups, called the* ***magnitude homology groups****, by* $\mathrm{MH}_{*,\ell}(X)$*.*

Remark. Computing magnitude homology groups in the general case is rather difficult. Nevertheless, we present in the following some general results to gain intuition about them in some specific contexts.

Lemma 8 ([13])**.** *Let* X *be a finite metric space. Then for a fixed* $\ell \geq 0$*, the groups* $\mathrm{MH}_{k,X}(\ell)$ *vanish for large enough* k*.*

Proof. Denote by δ the minimum distance between 2 distinct points in X. Then $\forall (x_0 \dots x_k) \in X^{k+1}$, it is clear that $l(x_0 \dots x_k) \geq k\delta$. In particular, if k is large enough so that $k\delta > \ell$, then the length of any $k+1$-tuple $(x_0 \dots x_k)$ is strictly greater than ℓ, and so $\mathrm{MC}_{k,\ell}(X) = \{0\}$. Thus, for any given ℓ, $\mathrm{MC}_{k,\ell}(X) = \{0\}$ for big enough k.

□

Theorem 7 ([13])**.** *Let* G *be a graph, equipped with the shortest path metric. Then the following holds:*

- $\mathrm{MH}_{0,0}(G)$ *is the free abelian group generated by all vertices of* G*;*
- $\mathrm{MH}_{1,1}(G)$ *is the free abelian group generated by directed edges in* G*.*

Proof. We prove the second statement; the first can be proven with a very similar argument. Recall that we have the following sequence of abelian groups:

$$\ldots \longrightarrow \mathrm{MC}_{2,1}(G) \xrightarrow{\partial_2} \mathrm{MC}_{1,1}(G) \xrightarrow{\partial_1} \mathrm{MC}_{0,1}(G) \longrightarrow \{0\}$$

and that $\mathrm{MH}_{1,1}(G)$ is given by the quotient $\mathrm{MH}_{1,1}(G) = \ker(\partial_1)/\operatorname{Im}(\partial_2)$. Let us look at the other groups in the sequence.

- $\mathrm{MC}_{2,1}(G)$ is generated by tuples (x_0, x_1, x_2) in G such that $x_0 \neq x_1$ and $x_1 \neq x_2$ and such that $d(x_0, x_1) + d(x_1, x_2) = 1$. However the distinctness condition implies that $d(x_0, x_1) + d(x_1, x_2) \geq 2$, and thus $\mathrm{MC}_{2,1}(G) = \{0\}$.
- $\mathrm{MC}_{0,1}(G)$ is generated by elements x_0 in G such that $l(x_0) = 1$, which is the empty set since $l(x) = 0$ for all $x \in G$. Hence $\mathrm{MC}_{0,1}(G) = \{0\}$.

Thus $\mathrm{MH}_{1,1}(G) = \mathrm{MC}_{1,1}(G)$, and $\mathrm{MC}_{1,1}(G)$ is generated by all ordered tuples (x_0, x_1) in G such that $d(x_0, x_1) = 1$, that is to say all directed edges in G. □

Example 13 (Magnitude Homology of Complete Graphs)**.** Let K_n be the complete graph on K_n. These graphs are nicely behaved enough that we can obtain a full description of their magnitude homology groups:

$$\mathrm{MH}_{k,\ell}(K_n) = \begin{cases} \left\langle (x_0 \ldots x_k) \in K_n^{k+1} \mid x_i \neq x_{i+1},\ l((x_0 \ldots x_k)) = \ell \right\rangle & \text{if } k = \ell; \\ 0 & \text{if } k \neq \ell. \end{cases}$$

Again, this is derived in the same way as the above computations, by observing that non-diagonal groups are $\{0\}$.

4.2 Link Between Magnitude and the MH complex

This section presents the main result linking magnitude and magnitude homology, Theorem 8, which justifies why MH is considered to be a categorification of magnitude. This theorem was first stated and proven for graphs in [13] by Hepworth and Willerton, and generalised by Leinster and Shulman in [18].

Theorem 8. *Let X be a finite metric space. Then its magnitude is characterised by the magnitude chain complex in the following way:*

$$|tX| = \sum_{\ell \geq 0} \sum_{k=0}^{\infty} (-1)^k \operatorname{rk}(\mathrm{MH}_{k,\ell}(X)) e^{-\ell t}$$

for sufficiently large t.

Remark. Note that it is not obvious that this double-sum is well defined. As argued in the previous remark, for each value of ℓ, there are only a finite number of k such that $\mathrm{MH}_{k,\ell}(X)$ is non-zero. Hence, for each value of ℓ in the outer sum, the inner sum is finite. However, the outer sum may very well be infinite. The proof of the validity of the formula will include a proof of convergence.

To prove this result, we first state two lemmas that will be needed in the proof.

Lemma 9. *Let (C_*, ∂) be a chain complex with positive indices that is bounded above, that is to say there exists $n \in \mathbb{N}$ such that $C_N = 0$ for all $N > n$. Let H_* denote its homology complex. Then we have:*

$$\chi(C_*) = \chi(H_*)$$

or in other words:

$$\sum_{i=0}^{\infty}(-1)^i \operatorname{rk}(C_i) = \sum_{i=0}^{\infty}(-1)^i \operatorname{rk}(H_i)$$

Proof. As is standard, we write Z_i, B_i the cycle and boundary groups of index i, that is to say:

$$\begin{aligned} Z_i &= \ker(\partial_i), \quad \partial_i \colon C_i \to C_{i-1}; \\ B_i &= \operatorname{Im}(\partial_{i+1}), \quad \partial_{i+1} \colon C_{i+1} \to C_i. \end{aligned}$$

Consider the short exact sequence (SES) of abelian groups

$$0 \longrightarrow Z_i \hookrightarrow C_i \xrightarrow{\partial_i} B_{i-1} \longrightarrow 0$$

where the map $Z_i \hookrightarrow C_i$ is inclusion. Hence we have

$$\operatorname{rk}(C_i) = \operatorname{rk}(Z_i) + \operatorname{rk}(B_{i-1}). \tag{4.1}$$

Now consider the sequence

$$0 \longrightarrow B_i \hookrightarrow Z_i \xrightarrow{\pi} H_i \longrightarrow 0$$

where again the map $B_i \hookrightarrow Z_i$ is inclusion, and the map π is the canonical projection onto the quotient space. This sequence is also exact and thus we have

$$\operatorname{rk}(H_i) = \operatorname{rk}(Z_i) - \operatorname{rk}(B_i). \tag{4.2}$$

Now we may sum over i:

$$\begin{aligned}
\sum_{i\geq 0}(-1)^i \operatorname{rk}(H_i) &= \sum_{i\geq 0}(-1)^i \operatorname{rk}(Z_i) - \sum_{i\geq 0}(-1)^i \operatorname{rk}(B_i) \\
&= \sum_{i\geq 0}(-1)^i \operatorname{rk}(Z_i) - \sum_{j\geq 1}(-1)^{j-1} \operatorname{rk}(B_{j-1}) \quad j = i+1 \text{ in the right sum} \\
&= \sum_{i\geq 0}(-1)^i \operatorname{rk}(Z_i) - \sum_{j\geq 0}(-1)^{j-1} \operatorname{rk}(B_{j-1}) \quad \text{since} \quad B_{-1} = 0 \\
&= \sum_{i\geq 0}(-1)^i \operatorname{rk}(Z_i) + \sum_{j\geq 0}(-1)^{j} \operatorname{rk}(B_{j-1}) \\
&= \sum_{i\geq 0}(-1)^i \operatorname{rk}(C_i) \quad \text{by (1)}
\end{aligned}$$

□

Lemma 10. *Let (X, d) be a finite metric space, and write $q = e^{-t}$ for some $t \geq 0$. Then*

$$|tX| = \sum_{k\geq 0}(-1)^k \sum_{\substack{(x_0\ldots x_k)\in X \\ x_i\neq x_{i+1}}} q^{l(x_0\ldots x_k)}$$

for sufficiently large t.

Proof. We prove this result by finding a weighting for X. Recall that the magnitude matrix Z_{tX} is defined by

$$Z_{tX}(x, y) = q^{d(x,y)},$$

for all $x, y \in X$.

Consider the following candidate w_X for a weighting:

$$w_X(x) = \sum_{k\geq 0}(-1)^k \sum_{\substack{(x_0\ldots x_k)\in X \\ x_0=x, x_i\neq x_{i+1}}} q^{l(x_0\ldots x_k)}.$$

We first need to verify that this quantity is finite. To this end we denote δ the minimum distance between any 2 distinct points in X, and n the cardinality of X. Now, observe that the following inequalities hold for all x:

$$\begin{aligned}
|w_X(x)| &\leq \sum_{k\geq 0} \sum_{\substack{(x_0\ldots x_k)\in X \\ x_0=x, x_i\neq x_{i+1}}} q^{l(x_0\ldots x_k)} \\
&\leq \sum_{k\geq 0} n^k q^{k\delta} \\
&\leq \sum_{k\geq 0} (ne^{-\delta t})^k.
\end{aligned}$$

This is a geometric series with ratio $ne^{-\delta t}$. Hence it converges if t is large enough so that $ne^{-\delta t} < 1$. In the following, we suppose that this assumption is satisfied.

We prove that w_X is a valid weighting by computing $Z_{tX}w_X$. For $x \in X$:

$$\begin{aligned}
\left[Z_{tX}w_X\right](x) &= \sum_{y\in X} Z_{tX}(x,y)w_X(y) \\
&= \sum_{y\in X} q^{d(x,y)}w_X(y) \\
&= \underbrace{q^{d(x,x)}}_{=1} w_X(x) + \sum_{\substack{y\in X\\ y\neq x}} q^{d(x,y)}w_X(y) \qquad \text{(separating the } x \text{ term)} \\
&= w_X(x) + \sum_{\substack{y\in X\\ y\neq x}} \sum_{k\geq 0} (-1)^k \sum_{\substack{(y_0,\dots,y_k)\in X\\ y_0=y, y_i\neq y_{i+1}}} q^{d(x,y)}q^{l(y_0,\dots,y_k)} \\
&= w_X(x) + \sum_{\substack{y\in X\\ y\neq x}} \sum_{k\geq 0} (-1)^k \sum_{\substack{(y_0,\dots,y_k)\in X\\ y_0=y, y_i\neq y_{i+1}}} q^{l(x,y_0,\dots,y_k)} \\
&= \sum_{k\geq 0} (-1)^k \sum_{\substack{(x_0\dots x_k)\in X\\ x_0=x, x_i\neq x_{i+1}}} q^{l(x_0\dots x_k)} + \sum_{k\geq 0} (-1)^k \sum_{\substack{(y_0,\dots,y_k)\in X\\ y_0\neq x, y_i\neq y_{i+1}}} q^{l(x,y_0,\dots,y_k)}
\end{aligned}$$

The last equality is obtained by exchanging the sum on $\{y \in X \colon y \neq x\}$ and the sum on $\{k \geq 0\}$. This is legal, because the triple sum is convergent, as we are summing a finite quantity (the innermost double sum, which converges because of our assumption that $ne^{-\delta t} < 1$) a finite number of times (since X is a finite metric space).
After this switch, we are left with something of the form:

$$\sum_{k\geq 0} \sum_{\substack{y\in X\\ y\neq x}} \sum_{\substack{(y_0,\dots,y_k)\in X\\ y_0=y, y_i\neq y_{i+1}}} \cdots .$$

We obtain the final equation by noticing that the two innermost sums can be put together simply by dropping the $y_0 = y$ condition in the innermost sum.

In the final equation, every term that appears in the sum on the right thus also appears in the sum on the left, with an opposite sign. Thus after cancelling out we are only left with the single term from the left-hand sum corresponding to $k = 0$:

$$\left[Z_{tX}w_X\right](x) = (-1)^0 q^{l(x_0)} = q^0 = 1.$$

Hence w_X is a valid weighting for X and thus we have

$$|tX| = \sum_{x \in X} w_X(x) = \sum_{k \geq 0} (-1)^k \sum_{\substack{(x_0 \ldots x_k) \in X \\ x_i \neq x_{i+1}}} q^{l(x_0 \ldots x_k)}.$$

□

With these lemmas we can now prove the theorem very easily. Recall its statement:

Theorem 9 (Euler Characteristic of Magnitude Complex)**.** *Let (X, d) be a finite metric space. Then the following equality holds for t sufficiently large:*

$$|tX| = \sum_{\ell \geq 0} \sum_{k=0}^{\infty} (-1)^k \operatorname{rk}(\mathrm{MH}_{k,\ell}(X)) q^\ell$$

Proof.

$$\begin{aligned}
\sum_{\ell \geq 0} \sum_{k=0}^{\infty} (-1)^k \operatorname{rk}(\mathrm{MH}_{k,\ell}(X)) q^\ell &= \sum_{\ell \geq 0} \chi(\mathrm{MH}_{*,\ell}(X)) q^\ell \\
&= \sum_{\ell \geq 0} \chi(\mathrm{MC}_{*,\ell}(X)) q^\ell \qquad \text{(by the first lemma)} \\
&= \sum_{\ell \geq 0} \sum_{k=0}^{\infty} (-1)^k \operatorname{rk}(\mathrm{MC}_{k,\ell}(X)) q^\ell. \qquad \text{(A)}
\end{aligned}$$

Let us now count the generators of $\mathrm{MC}_{k,\ell}(G)$. There is one generator for each $k+1$-tuple $(x_0 \ldots x_k)$ such that $x_i \neq x_{i+1}$ for all $0 \leq i \leq k-1$, and such that $l(x_0 \ldots x_k) = \ell$. Any such tuple also corresponds to a generator. Thus, we have:

$$\operatorname{rk}(\mathrm{MC}_{k,\ell}(X)) = \sum_{\substack{(x_0 \ldots x_k) \in X \\ l(x_0 \ldots x_k) = \ell,\ x_i \neq x_{i+1}}} 1.$$

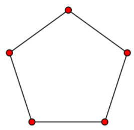
We plug this into equation (A) to obtain:

$$
\begin{aligned}
\sum_{\ell \geq 0} \sum_{k=0}^{\infty} (-1)^k \operatorname{rk}(\mathrm{MH}_{k,\ell}(X)) q^\ell &= \sum_{\ell \geq 0} \sum_{k=0}^{\infty} (-1)^k \left[\sum_{\substack{(x_0 \ldots x_k) \in X \\ l(x_0 \ldots x_k) = \ell,\ x_i \neq x_{i+1}}} q^\ell \right] \\
&= \sum_{k=0}^{\infty} (-1)^k \sum_{\ell \geq 0} \sum_{\substack{(x_0 \ldots x_k) \in X \\ l(x_0 \ldots x_k) = \ell,\ x_i \neq x_{i+1}}} q^\ell \\
&= \sum_{k=0}^{\infty} (-1)^k \sum_{\substack{(x_0 \ldots x_k) \in X \\ x_i \neq x_{i+1}}} q^\ell \\
&= |tX| \,. \qquad \text{(by the second lemma)}
\end{aligned}
$$

□

Remark. Note that in the proof, we managed to find a weighting for *any* finite metric space that is valid for sufficiently large t. In particular, we have also shown that the magnitude of a metric space can be seen as a formal power series in q.

Example 14 (5-cycle graph). For certain metric spaces, for which the magnitude function is already known, we can now verify that this formula indeed computes magnitude appropriately. We do so for the graph C_5, for which the ranks of the groups $\mathrm{MH}_{k,l}(C_5)$ are given by the following:



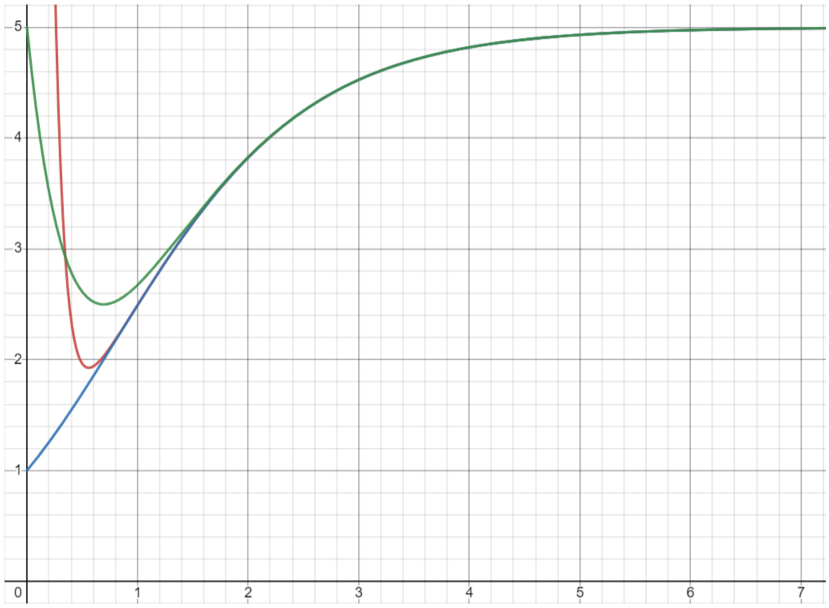
l \ k	0	1	2	3	4	5	6	7	8	9	10	11
0	5											
1		10										
2			10									
3			10	10								
4				30	10							
5					50	10						
6					20	70	10					
7						80	90	10				
8							180	110	10			
9							40	320	130	10		
10								200	500	150	10	
11									560	720	170	10

(a) Ranks of $\mathrm{MH}_{k,l}(C_5)$. Taken from [13].

This gives us the following approximation of the magnitude function:

$$|tC_5| \approx 5 - 10e^{-t} + 10e^{-2t} - 20e^{-4t} + 40e^{-5t} - 40e^{-6t} + 80e^{-8t} - 160e^{-9t} + 160e^{-10t}$$

On the following graph, we show the approximations of $|tC_5|$ obtained from this formula at orders 2 and 11:



(a) Approximations of $|tC_5|$ (in blue) at orders 2 (in green) and 11 (in red).

Note that even the order 2 approximation is very close to the actual magnitude function for $t > 2$. Indeed, recall that for $-1 < x < 1$, we have $\frac{1}{1+x} = 1 - x + x^2 + O(x^3)$; applying this to $|tC_5|$ yields

$$\begin{aligned} |tC_5| &= \frac{5}{1 + 2e^{-t} + 2e^{-2t}} \\ &= 5(1 - 2e^{-t} + 2e^{-2t} + O(e^{-3t})) \\ &= 5 - 10e^{-t} + 10e^{-2t} + O(e^{-3t}), \end{aligned}$$

for t large enough so that $2e^{-t} + 2e^{-2t} < 1$, which is equivalent to $t > \ln(\frac{4}{2\sqrt{3}-2}) \approx 1$. Note that this condition is similar to, but still distinct from, the condition that t is sufficiently large in the theorem, which requires that $5e^{-t} < 1$ or $t > \ln(5) \approx 1.6$.

Remark. Magnitude homology strictly refines magnitude. In particular, recent work has found graphs which have same magnitude, but different magnitude homology [11].

4.3 Geometric Interpretations of Magnitude Homology

In this section, we expand on some of the results which were given above in the case of graphs to build intuition for magnitude homology. We also relate *algebraic* properties of the magnitude complex to *geometric properties* of spaces; this also justifies our interest in the magnitude complex, since our purpose in defining it was to obtain a richer invariant, encoding more information about the space, than the magnitude function. The main source for this section is the text *Magnitude homology of enriched categories and metric spaces*, from Leinster and Shulman ([18]).

To start we recall the definition of the magnitude complex in degrees k, ℓ:

$$\mathrm{MC}_{k,\ell}(X) = \Big\langle (x_0 \dots x_k) \in X^{k+1} \mid x_i \neq x_{i+1}\ \forall i \in [\![0, k-1]\!],\ l((x_0 \dots x_k)) = \ell \Big\rangle$$

Observe that in particular, if $k = 0$, then:

- If $\ell > 0$, $\mathrm{MC}_{0,\ell}(X)$ is the free abelian group with empty basis. By convention, we thus have $\mathrm{MC}_{0,\ell}(X) = \{0\}$.
- If $\ell = 0$, then $\mathrm{MC}_{0,0}(X)$ is the free abelian group with a basis element corresponding to every point in X. In particular it coincides with the degree 0 singular chains.

Furthermore, observe that $d^0_{k,\ell} = d^k_{k,\ell} = 0$ implies that $\partial_{1,\ell} = d^0_{1,\ell} - d^1_{1,\ell} = 0$. So we always have $\mathrm{Im}(\partial_{1,\ell}) = 0$.

These observations lead to the following theorem:

Theorem 10. *Let X be a finite metric space. Then 0-th degree magnitude homology is given by:*

$$\mathrm{MH}_{0,\ell}(X) = \begin{cases} 0 & \text{if } \ell > 0; \\ \langle x \in X \rangle & \text{otherwise.} \end{cases}$$

Remark. In the context of graphs, we had also been able to relate the degree 1 magnitude homology $\mathrm{MH}_{0,*}(G)$ with directed edges. This identification is no longer possible in the context of more general finite metric spaces. Nevertheless, the structure of $\mathrm{MH}_{1,*}(X)$ still corresponds to a geometric property of X.

Definition 31 (Adjacence, Menger Convexity). *Let X be a metric space and x, y, z points in X, with $x \neq z$. We say that y is **between** x and z if $d(x,y) + d(y,z) = d(x,z)$. If furthermore $y \neq x$ and $y \neq z$, then we say that y is **strictly between** x and z. The points x and z are **adjacent** if there is no point strictly between them.*

The space X is called ***Menger convex*** *if any two distinct points are non-adjacent; that it to say, $\forall x \neq z \in X$, there exists a point y such that y is stricly between x and z.*

We use the symbol $\prec$ to denote the between relationship. Hence $x \prec y \prec z$ reads "y is strictly between x and z". Dually, if y is not strictly between them, we write $x \not\prec y \not\prec z$.

Theorem 11. *Let X be a finite metric space. Then 0-th degree magnitude homology is given by:*

$$\mathrm{MH}_{1,\ell}(X) = \begin{cases} 0 & \textit{if } \ell = 0; \\ \langle (x_0, x_1) \colon x_0 \neq x_1,\ d(x_0, x_1) = \ell,\ \forall y \in X,\ x_0 \not\prec y \not\prec x_1 \rangle & \textit{otherwise.} \end{cases}$$

Remark. Applying this theorem to graphs yields back the same result as in the previous, since adjacent points exactly correspond to directed edges.

Proof. Recall that for all $\ell \geq 0$ we have the following exact sequence:

$$\ldots \longrightarrow \mathrm{MC}_{2,\ell}(X) \xrightarrow{\partial_2} \mathrm{MC}_{1,\ell}(X) \xrightarrow{\partial_1} \mathrm{MC}_{0,\ell}(X) \longrightarrow \{0\}$$

and that

$$\mathrm{MH}_{1,\ell}(X) = \ker(\partial_1) / \operatorname{Im}(\partial_2).$$

In the degenerate case $\ell = 0$, $\mathrm{MC}_{1,\ell}(X) = \{0\}$, since no distinct points x_0, x_1 in X can verify $d(x_0, x_1) = 0$, which implies our result.
In the general case, recall that we have $\partial_1 = 0$, and thus $\ker(\partial_1) = \mathrm{MC}_{1,\ell}(X)$.
We turn our attention to $\operatorname{Im}(\partial_2)$. Consider a triple $(x_0, x_1, x_2) \in X^3$ that is a generator of $\mathrm{MC}_{2,\ell}(X)$; that is to say, we have $x_0 \neq x_1$, $x_1 \neq x_2$ and $l(x_0, x_1, x_2) = \ell$. Then we have:

$$\begin{aligned} \partial_2(x_0, x_1, x_2) &= d^0(x_0, x_1, x_2) - d^1(x_0, x_1, x_2) + d^2(x_0, x_1, x_2) \\ &= -d^1(x_0, x_1, x_2). \qquad\qquad (d^0,\ d^2 = 0) \end{aligned}$$

Using the definition of d^1 now, we obtain:

$$\partial_2(x_0, x_1, x_2) = \begin{cases} -(x_0, x_2) & \text{if } d(x_0, x_2) = \ell; \\ 0 & \text{otherwise.} \end{cases}$$

which yields that $\operatorname{Im}(\partial_2) = \big\langle (x_0, x_2) \in X^2 \colon \text{there exists } x_1 \in X \text{ so that } x_0 \prec x_1 \prec x_2 \big\rangle$.
Thus the only generators of $\mathrm{MC}_{1,\ell}(X)$ which do not vanish in the quotient are precisely those for which there exists no element in-between them, which coincides with the desciption of $\mathrm{MH}_{1,\ell}(X)$ in the theorem. $\square$

4.4 Additional Algebraic Results on Magnitude Homology

This section presents (but does not prove) two results mirroring properties of standard homology, both initially stated by Hepworth and Willerton in [13]. These results were proven and are valid for the specific case of graphs.

4.4.1 A Künneth Theorem for Magnitude Homology

Definition 32 (Cartesian Porduct of Graphs)**.** *The cartesian product of graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ is the graph $G \otimes H$ with vertex set $V(G) \times V(H)$ and which has an edge from (g_1, h_1) to (g_2, h_2) if $g_1 = g_2$ and $(h_1, h_2) \in E(H)$, or if $h_1 = h_2$ and $(g_1, g_2) \in E(G)$.*
The metric on $G \otimes H$ is given by:

$$d((g_1, h_1), (g_2, h_2)) = d_G(g_1, g_2) + d_H(h_1, h_2)$$

for any vertices (g_1, h_1) and (g_2, h_2) in $G \otimes H$.

Remark. In other words, for each vertex $g \in G$, there is a copy of the graph H, and points in each copy which correspond to the same vertex of H are connected by an edge.

Remark. The cartesian product is the natural tensor product from the view of enriched categories, hence our notation.

Definition 33 (Exterior Product)**.** *The exterior product is the map*

$$\square\colon \ \mathrm{MC}_{*,*}(G) \otimes \mathrm{MC}_{*,*}(H) \to \mathrm{MC}_{*,*}(G \otimes H)$$

with components in degrees k_1, k_2 and ℓ_1, ℓ_2 given by:

$$\square : \begin{cases} \mathrm{MC}_{k_1,\ell_1}(G) \otimes \mathrm{MC}_{k_2,\ell_2}(H) \to \mathrm{MC}_{k_1+k_2,\ell_1+\ell_2}(G \otimes H) \\ \\ (g_0, \dots, g_{k_1}) \otimes (h_0, \dots, h_{k_2}) \mapsto \sum\limits_{\sigma} \varepsilon(\sigma) \cdot ((g_{i_0}, h_{j_0}), \dots, (g_{i_{k_1+k_2}}, h_{j_{k_1+k_2}})) \end{cases}$$

here the sum ranges over all sequences $\sigma = ((i_0, j_0), \dots, (i_{k_1+k_2}, j_{k_1+k_2}))$ for which $i_0 = j_0 = 0$, $0 \leq i_r \leq k_1$ and $0 \leq j_r \leq k_2$ for all r, and such that the term (i_{r+1}, j_{r+1}) is obtained from the previous (i_r, j_r) by adding 1 to exactly one of the two components. The quantity $\varepsilon(\sigma)$ is $(-1)^n$, where n is the value $\sum_{r=0}^{k_1+k_2} j_r$.
The exterior product is a chain map, and so induces a map in homology that is denoted by the same symbol.

Remark. Viewed another way, each sequence σ corresponds to a path from $(0, 0)$ to (k_1, k_2) in $\mathbb{Z}^2$, moving along the lattice by one unit upwards or rightwards at each step. The quantity $\varepsilon(\sigma)$ then counts the number of grid squares below the chosen path.

Remark. This construction closely echoes that of the simplicial cross product; see [12] for details.

Theorem 12 (Künneth Theorem for Magnitude Homology)**.** *The exterior product in homology fits into a short exact sequence of chain complexes:*

$$\begin{array}{ccc} 0 \to \mathrm{MH}_{*,*}(G) \otimes \mathrm{MH}_{*,*}(H) & \xrightarrow{\quad\square\quad} & \mathrm{MH}_{*,*}(G \square H) \\ & & \downarrow \\ & & \mathrm{Tor}(\mathrm{MH}_{*-1,*}(G), \mathrm{MH}_{*,*}(H)) \to 0 \end{array}$$

This sequence is split, but not naturally. In particular, $\square$ *is an isomorphism if either* $\mathrm{MH}_{*,*}(G)$ *or* $\mathrm{MH}_{*,*}(H)$ *is torsion-free.*

Example 15 (Magnitude Homology of C_4)**.** Letting K_2 be the complete graph on 2 vertices, observe that we have $C_4 = K_2 \square K_2$. Recall that in the previous, in example 13, we computed magnitude homology of complete graphs:

$$\mathrm{MH}_{k,\ell}(K_n) = \begin{cases} \left\langle (x_0 \ldots x_k) \in K_n^{k+1} \mid x_i \neq x_{i+1},\ l((x_0 \ldots x_k)) = \ell \right\rangle & \text{if } k = \ell; \\ 0 & \text{if } k \neq \ell. \end{cases}$$

In the case of K_2, labelling its vertices v_1 and v_2, note that for any $k \in \mathbb{N}$, the condition $x_i \neq x_{i+1}$ enforces that there are only 2 tuples $(x_0 \ldots x_k) \in K_2^{k+1}$ verifying $l(x_0 \ldots x_k) = k$: one starting at v_1 and alternating between vertices, and the other starting at v_2 and alternating vertices. Thus we have:

$$\mathrm{MH}_{k,\ell}(K_2) = \begin{cases} \mathbb{Z}^2 & \text{if } k = \ell; \\ 0 & \text{if } k \neq \ell. \end{cases}$$

for all $k \in \mathbb{N}$.
These groups contain no torsion, hence $\square$ is an isomorphism, and thus:

$$\mathrm{MH}_{k,\ell}(C_4) \cong \bigoplus_{\substack{k_1,k_2:\ k_1+k_2=k \\ \ell_1,\ell_2:\ \ell_1+\ell_2=\ell}} \mathrm{MH}_{k_1,\ell_1}(K_2) \otimes \mathrm{MH}_{k_2,\ell_2}(K_2)$$

Observe that if $k \neq \ell$, then one of the two groups is always $\{0\}$, and so the sum is $\{0\}$. Otherwise, if $k = \ell$, there are $k+1$ factors in the sum, corresponding to $k_1 = 0, \ldots, k_1 = k$. Each factor is $\mathbb{Z}^2 \otimes \mathbb{Z}^2 = \mathbb{Z}^4$, which finally lets us deduce:

$$\mathrm{MH}_{k,\ell}(C_4) = \begin{cases} \mathbb{Z}^{4k+4} & \text{if } k = \ell; \\ 0 & \text{if } k \neq \ell. \end{cases}$$

4.4.2 The Mayer-Vietoris Sequence for Magnitude Homology

Definition 34. *Let G be a graph. A subgraph $U \subset G$ is called **convex** if $d_U(u,v) = d_G(u,v)$ for all u,v in U.*

Remark. This definition echoes the property of convex subsets in $\mathbb{R}^n$ that each pair of points is connexted by a line, which also lies in the subset.

Definition 35. *Let $U \subset G$ be a convex subgraph. G is said to **project onto** U if for all $g \in G$ that can be connected by an edge path to a vertex in U, there exists some $\pi(x) \in U$ such that*

$$d(x,u) = d(x,\pi(x)) + d(\pi(x),u)$$

*for all $u \in U$. We call π the **projection map**.*

Remark. This definition aims to generalise the property of convex subsets of $\mathbb{R}^n$ that there is a notion of nearest point to any point not in the convex subset.

Example 16. Complete Graphs - Complete graphs do not project onto any proper subgraph, since for any $U \subset K_n$ and $x \notin U$, it holds that $d(x,\pi(x)) + d(\pi(x),u) = 2 > d(x,u)$ for any $u \in U$.

Cyclic Graphs - Even cycles project onto any edge, that is to say onto any subgraph $U = (\{v_1, v_2\}, \{v_1 v_2\})$. Starting at any $x \notin U$, U can reached by 2 paths, one of which is shorter than the other. For odd cycles, the two paths have the same length, and so there is no "closest point to x in U".

Definition 36 (Projecting Decomposition). *A **projecting decomposition** is a triple $(G;U,V)$ consisting of a graph G and subgraphs U,V such that the following properties hold:*

- *$G = U \cup V$;*
- *$U \cap V$ is convex in G;*
- *V projects onto $U \cap V$.*

We denote inclusion maps with double indices; for instance $\iota^V_{U\cap V}$ is the inclusion $U \cap V \hookrightarrow V$.

Definition 37 (Notation). *If we have $(G;U,V)$ a projective decomposition, we denote by $\mathrm{MC}_{*,*}(U,V)$ the subcomplex of $\mathrm{MC}_{*,*}(G)$ spanned by tuples with entries lying entirely in U or entirely in V.*

Theorem 13 (Excision). *Let $(G;U,V)$ be a projecting decomposition. Then for all $\ell \geq 0$, the inclusion $\mathrm{MC}_{*,\ell}(U,V) \hookrightarrow \mathrm{MC}_{*,\ell}(G)$ induces isomorphisms in homology in all degrees.*

Remark. This result is an analogue to the standard excision result that holds for standard homology. Just like in the standard case, it leads to a Mayer-Vietoris sequence in homology.

Theorem 14 (Mayer-Vietoris for Magnitude Homology)**.** *Let $(G; U, V)$ be a projecting decomposition. Then there is a split short exact sequence in magnitude homology:*

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathrm{MH}_{*,*}(U \cap V) & \longrightarrow & \mathrm{MH}_{*,*}(U) \oplus \mathrm{MH}_{*,*}(V) \\ & & & & \downarrow \\ & & & & \mathrm{MH}_{*,*}(G) \longrightarrow 0 \end{array}$$

Remark. Note that this result is slightly more general than the standard Mayer-Vietoris; it gives a short exact sequence for any homological degree, while standard Mayer-Vietoris gives a short exact sequence on the level of chains, and only one long exact sequence on the level of homology.

Theorem 15 (Inclusion-Exclusion)**.** *If $(G; U, V)$ is a projecting decomposition, then for large enough t we have:*

$$|tG| = |tU| + |tV| - |tU \cap V|\,.$$

Proof. Let $\ell \in \mathbb{N}$ Additivity of the Euler characteristic applied to the Mayer-Vietoris sequence at ℓ yields:

$$\chi(\mathrm{MH}_{*,\ell}(U \cap V)) + \chi(\mathrm{MH}_{*,\ell}(G)) - \chi(\mathrm{MH}_{*,\ell}(U) \oplus \mathrm{MH}_{*,\ell}(V)) = 0$$

The Euler characteristic is additive with respect to direct sums, so we obtain:

$$\chi(\mathrm{MH}_{*,\ell}(G)) = \chi(\mathrm{MH}_{*,\ell}(U)) + \chi(\mathrm{MH}_{*,\ell}(V)) - \chi(\mathrm{MH}_{*,\ell}(U \cap V))$$

Multiplying by $e^{-\ell t}$ and summing over all ℓ, we obtain the summands involved in Theorem 9. Applying it yields the results. $\square$

Chapter 5

Blurred Magnitude Homology

Blurred magnitude homology is a variant of Magnitude Homology introduced by Nina Otter in recent work [21]. It provides an appropriate setting to answer open questions that were raised by Leinster and Shulman. In particular, it establishes a connection between magnitude homology and persistent homology. Our main goal throughout this section is to state and prove Theorem 16, the main result from *Persistent Magnitude* ([10]).

5.1 The Blurred Magnitude Complex

Definition 38. *The* ***blurred magnitude chain complex*** *of a metric space X is the chain complex of persistence modules* $\mathrm{BMC}_*(X)$ *defined by*

$$\mathrm{BMC}_k(X)(\ell) = \left\langle (x_0 \dots x_k) \in X^{k+1} \mid x_i \neq x_{i+1}, 0 \leq i \leq k-1,\ l(x_0 \dots x_k) \leq \ell \right\rangle$$

for $k \in \mathbb{N}$ *and* $\ell \in [0, \infty]$.
For a given ℓ, *the sequence of abelian groups* $(\mathrm{BMC}_k(X)(\ell))_{k \in \mathbb{N}}$ *defines a chain complex, where the boundary map* $\partial_{k,\ell}$ *is given by:*

$$\partial_{k,\ell}(x_0 \dots x_k) = \sum_{i=0}^{k} (-1)^i (x_0, \dots, \widehat{x_i}, \dots, x_k).$$

The ***blurred magnitude homology*** $\mathrm{BMH}_*(X)$ *is defined to be the homology of this chain complex.*

Remark. In both Nina Otter's paper and in Dejan Govc and Richard Hepworth's, the blurred magnitude complex is defined as a chain complex of abelian groups. However, to connect it with persistence modules, it will

be useful to work with coefficients in a field. To this end, we introduce a slightly different definition:
Let $\mathbb{F}$ be a base field and X a metric space. Then the **blurred magnitude chain complex with coefficients in** $\mathbb{F}$ associated with X is defined by

$$\mathrm{BMC}_{k,\mathbb{F}}(X)(\ell) = \bigoplus_{\substack{x_i \neq x_{i+1} \\ l(x_0 \dots x_k) \leq \ell}} \mathrm{Span}(x_0 \dots x_k)$$

where $\mathrm{Span}(x_0 \dots x_k)$ denotes the 1-dimensional $\mathbb{F}$ vector space with basis $(x_0 \dots x_k)$.
Note that this new definition makes $\mathrm{BMC}_{k,\mathbb{F}}(X)$ and $\mathrm{BMH}_{k,\mathbb{F}}(X)$ into persistence modules. To each $\ell \in [0, \infty]$ they associate a vector space, and they can be extended to the morphisms $\ell \leq \ell'$ of $([0, \infty], \leq)$ by inclusion.

Viewing these chain complexes as persistence modules allows us to state the main result:

Theorem 16. *Let X be a finite metric space, and let $\mathrm{BMH}_{*,\mathbb{F}}(X)$ denote its blurred magnitude homology with coefficients in $\mathbb{F}$. Suppose that $\mathrm{BMH}_{k,\mathbb{F}}(X)$ has a barcode decomposition for all k:*

$$\mathrm{BMH}_{k,\mathbb{F}}(X) = \bigoplus_{i=0}^{\varphi(k)} \chi_{[a_{ki},\, b_{ki}[}$$

Then the magnitude of X is given by the following formula for t sufficiently large:

$$|tX| = \sum_{k=0}^{\infty} \sum_{i=0}^{\varphi(k)} \left(e^{-a_{ki}t} - e^{-b_{ki}t} \right)$$

The proof of this theorem will require several lemmas that we will first state and prove.

Definition 39 (l-values)**.** *We call **l-values** and denote $\mathcal{L}$ the subset of $\mathbb{R}$ that is realised as lengths of tuples in X:*

$$\mathcal{L} = \left\{ l(x_0 \dots x_k) \mid (x_0 \dots x_k) \in X^{k+1},\ k \in \mathbb{N} \right\}.$$

Lemma 11. *The set $\mathcal{L}$ of l-values is countable and can be totally ordered:*

$$\mathcal{L} = l_0 < l_1 < \dots < l_j < \dots$$

l_0 will always be taken to be 0 in the following, as the length of any single element in X.

Proof. Denote δ the minimal distance between any two points in X. δ is strictly positive since X is a finite metric space. For any k we thus have that

$$\begin{aligned} l(x_0 \dots x_k) &= \sum_{i=0}^{k-1} d(x_i, x_{i+1}) \\ &\geq \sum_{i=0}^{k-1} \delta \\ &\geq k\delta. \end{aligned}$$

Hence in particular, if $N \geq 0$, then there exists an integer k_N such that $k \geq k_N \implies l(x_0 \dots x_k) > N$; for instance pick $k_N = \lceil \frac{N}{\delta} \rceil + 1$.
Therefore the set $\mathcal{L}_N := \mathcal{L} \cap [0, N]$ is comprised of tuples of length at most k_N, of which there are a finite amount, and thus $\mathcal{L}_N$ is finite itself.

$\mathcal{L}$ is countable as the countable union of finite sets $\mathcal{L} = \bigcup_{N \in \mathbb{N}} \mathcal{L}_N$ and thus can be totally ordered, as desired. $\square$

Lemma 12. *For all k, $l_j \in \mathcal{L}$ with $j \geq 1$ there is a short exact sequence of abelian groups:*

$$0 \longrightarrow \mathrm{BMC}_k(X)(l_{j-1}) \xrightarrow{\iota_k} \mathrm{BMC}_k(X)(l_j) \xrightarrow{\pi_k} \mathrm{MC}_{k,l_j}(X) \longrightarrow 0$$

This short exact sequence induces a long exact sequence in homology:

$$\dots \longrightarrow \mathrm{BMH}_k(X)(l_{j-1}) \xrightarrow{\iota_{k,*}} \mathrm{BMH}_k(X)(l_j) \xrightarrow{\pi_{k,*}} \mathrm{MH}_{k,l_j}(X) \xrightarrow{\partial *} \mathrm{BMH}_{k-1}(X)(l_{j-1}) \xrightarrow{\iota_{k-1,*}} \dots$$

This statement remains true with coefficients in a field. Furthermore, the long exact sequence terminates on both sides.

Proof. We first define the maps in the short exact sequence.
ι_k is simply the canonical inclusion: since $l_{j-1} < l_j$, every generator $(x_0 \dots x_k)$ of $\mathrm{BMC}_k(X)(l_{j-1})$ verifies $l(x_0 \dots x_k) \leq l_j$ and is therefore a generator of $\mathrm{BMC}_k(X)(l_j)$ as well. This map is clearly injective.

Define π_k on the generators of $\mathrm{BMC}_k(X)(l_j)$ in the following way:

$$\pi_k(x_0 \dots x_k) = \begin{cases} (x_0 \dots x_k) & \text{if } l(x_0 \dots x_k) = l_j; \\ 0 & \text{if } l(x_0 \dots x_k) < l_j. \end{cases}$$

and extend it by linearity. Note that if $(x_0 \dots x_k) \in \mathrm{BMC}_k(X)(l_j)$ verifies $l(x_0 \dots x_k) = l_j$ then in particular $(x_0 \dots x_k)$ is a generator of $\mathrm{MC}_{k,l_j}(X)$, and hence π_k has image contained in $\mathrm{MC}_{k,l_j}(X)$. Dually, if $(x_0 \dots x_k)$ is in $\mathrm{MC}_{k,l_j}(X)$, then it is also in $\mathrm{BMC}_k(X)(l_j)$, which implies that π_k is surjective.

We now prove that $\ker(\pi_k) = \mathrm{Im}(\iota_k)$. By the definition of the map π_k we obtain that

$$\begin{aligned}
\ker(\pi_k) &= \Big\langle (x_0 \dots x_k) \in X^{k+1} \mid x_i \neq x_{i+1},\ l(x_0 \dots x_k) < l_j \Big\rangle \\
&= \Big\langle (x_0 \dots x_k) \in X^{k+1} \mid x_i \neq x_{i+1},\ l(x_0 \dots x_k) \leq l_{j-1} \Big\rangle \\
&= \mathrm{BMC}_k(X)(l_{j-1}) \\
&= \mathrm{Im}(\iota_k).
\end{aligned}$$

The long exact sequence clearly terminates on the right side, since all the abelian groups are 0 for negative k.
To see that it terminates on the left side, recall from the definition of the magnitude chain complex that for large enough k and fixed ℓ, $\mathrm{MC}_{k,\ell}(X) = 0$, because the length of a tuple $(x_0 \dots x_k)$ is bounded below by $k\delta$. This argument also yields $\mathrm{BMC}_k(X)(\ell) = 0$ for big enough k, and thus the long sequence terminates on the left. □

Lemma 13. *Suppose A_*, B_*, C_* are chain complexes of abelian groups that fit into a short exact sequence:*

$$0 \longrightarrow A_* \longrightarrow B_* \longrightarrow C_* \longrightarrow 0$$

Suppose also that the complexes terminate in both directions. Then we have:

$$\mathrm{rk}(B_*) = \mathrm{rk}(A_*) + \mathrm{rk}(C_*).$$

Proof. The short exact sequence of chain complexes yields short exact sequences

$$0 \longrightarrow A_n \longrightarrow B_n \longrightarrow C_n \longrightarrow 0$$

for all indices n. By additivity of the rank over short exact sequences of abelian groups, we thus have for all n:

$$\mathrm{rk}(B_n) = \mathrm{rk}(A_n) + \mathrm{rk}(C_n).$$

Since all sequences terminate we can multiply by $(-1)^n$ and sum over all n:

$$\sum_{n=-\infty}^{+\infty}(-1)^n \operatorname{rk}(B_n) = \sum_{n=-\infty}^{+\infty}(-1)^n \operatorname{rk}(A_n) + \sum_{n=-\infty}^{+\infty}(-1)^n \operatorname{rk}(C_n)$$
$$\implies \operatorname{rk}(B_*) = \operatorname{rk}(A_*) + \operatorname{rk}(C_*).$$

□

Lemma 14. *Let X be a finite metric space, and let $\mathrm{BMH}_{*,\mathbb{F}}(X)$ denote its blurred magnitude homology with coefficients in $\mathbb{F}$. Suppose that $\mathrm{BMH}_{k,\mathbb{F}}(X)$ has a barcode decomposition for all k:*

$$\mathrm{BMH}_{k,\mathbb{F}}(X) = \bigoplus_{i=0}^{\varphi(k)} \chi_{[a_{ki},\, b_{ki}[}$$

Then the numbers a_{ki}, b_{ki} appearing in the decomposition are l-values. In other words, for each interval boundary $\alpha = a_{ki}$ or $\alpha = b_{ki}$ there exists a tuple $(x_0 \dots x_k) \in X^{k+1}$ such that $l(x_0 \dots x_k) = \alpha$.

Proof. The proof will rely on the following observation: suppose X has l-values $0 = l_0 < l_1 < \dots < l_{j-1} < l_j < \dots$. Then for all $x, y \in]l_{j-1}, l_j[$, we have:

$$\mathrm{BMH}_{k,\mathbb{F}}(X)(x) = \mathrm{BMH}_{k,\mathbb{F}}(X)(y) = \mathrm{BMH}_{k,\mathbb{F}}(X)(l_{j-1}) \tag{A}$$

Let $u \geq 0$ be the left endpoint of an interval in the decomposition, and suppose the decomposition can be written as:

$$\mathrm{BMH}_{k,\mathbb{F}}(X) = \left(\bigoplus_{i=1}^{\alpha_k} \chi_{[a_i,\, b_i[}\right) \bigoplus \left(\bigoplus_{j=1}^{\beta_k} \chi_{[u,\, v_j[}\right) \tag{B}$$

where $a_i \neq u$ for all i. In other words, we separate the β_k intervals which have u as their left endpoint from those which do not.
Notice that we have $u \neq b_i$ for all i. Indeed, if it was the case that $u = b_i$ for some i, then we could obtain another valid decomposition of $\mathrm{BMH}_{k,\mathbb{F}}(X)$ by removing the interval modules $\chi_{[a,\, b_i=u[}$ and $\chi_{[u,\, v_j[}$ and replacing them by the interval $\chi_{[a_i,\, v_j[}$. This is a contradiction, because the decomposition is unique up to reordering the intervals, and so two decompositions cannot have different numbers of intervals.

Suppose further that u is not an l-value and let l_u be the l-value verifying $l_u = \max_{l \in \mathcal{L}}(l \,:\, l < u)$, the largest l-value inferior to u. Define

$$\varepsilon = \frac{1}{2}\min_{i,j,n}(\{|u - a_i|,\ |u - b_j|,\ |u - v_n|,\ |u - l_u|,\ |u - l_{u+1}|,\ \})$$

This quantity is well-defined and strictly positive since u is not equal to any a_i, b_j, l_u or v_n.

Let $x \in]u-\varepsilon, u[$ and $y \in]u, u+\varepsilon[$. Observe the following facts:

1. We have $\varepsilon < |u - l_u|$ and $\varepsilon < |u - l_{u+1}|$. So in particular $l_u < x < u < y < l_{u+1}$.
2. The condition $\varepsilon < |u - v_j|$ implies that y belongs to every interval of the form $[u,\ v_j[$
3. The conditions $\varepsilon < |u - a_i|$ and $\varepsilon < |u - b_i|$ imply that x and y belong to the same set of intervals of the form $[a_i,\ b_i[$

We obtain the following set of equations:

$$\begin{cases} \mathrm{BMH}_{k,\mathbb{F}}(X)(x) = \mathrm{BMH}_{k,\mathbb{F}}(X)(y) & \text{by (A) and observation 1} \\ \mathrm{BMH}_{k,\mathbb{F}}(X)(y) = \mathrm{BMH}_{k,\mathbb{F}}(X)(y) \bigoplus \mathbb{F}^{\beta_k} & \text{by (B) and observations 2 and 3} \end{cases}$$

which is a contradiction. Thus, u is an l-value. The proof that the right endpoints of intervals are l-values is symmetric. □

With these results in tow we are finally able to prove the theorem. We will first describe the steps of calculation, then justify them. Recall the statement of the theorem:

Theorem 17. *Let X be a finite metric space, and let $\mathrm{BMH}_{*,\mathbb{F}}(X)$ denote its blurred magnitude homology with coefficients in $\mathbb{F}$. Suppose that $\mathrm{BMH}_{k,\mathbb{F}}(X)$ has a barcode decomposition for all k:*

$$\mathrm{BMH}_{k,\mathbb{F}}(\mathrm{X}) = \bigoplus_{i=0}^{\varphi(k)} \chi_{[a_{ki},\ b_{ki}[}$$

Then the magnitude of X is given by the following formula for t sufficiently large:

$$|tX| = \sum_{k=0}^{\infty} \sum_{i=0}^{\varphi(k)} \left(e^{-a_{ki}t} - e^{-b_{ki}t} \right)$$

Proof. Throughout this proof, let:

- δ be the minimum non-zero distance between two elements in X;
- n be the cardinality of X;
- $\mathcal{L} = l_0 < l_1 < \ldots < l_j < \ldots$ be the set of l-values of X;

- $D_{i,k}(j)$ be the indicator function of $[a_{ki}, b_{ki}[$ applied at l_j: $D_{i,k}(j)$ is 1 if l_j is in the interval $[a_{ki}, b_{ki}[$, and 0 otherwise.

We now have the following computations:

$$\begin{aligned}
|tX| &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k \operatorname{rk}(\mathrm{MH}_{k,l_j}(X))e^{-l_j t} && (1)\\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k \left[\operatorname{rk}(\mathrm{BMH}_k(X)(l_j)) - \operatorname{rk}(\mathrm{BMH}_k(X)(l_{j-1}))\right] e^{-l_j t} && (2)\\
&= \sum_{k=0}^{\infty}(-1)^k \sum_{j=0}^{\infty}\left[\operatorname{rk}(\mathrm{BMH}_k(X)(l_j)) - \operatorname{rk}(\mathrm{BMH}_k(X)(l_{j-1}))\right] e^{-l_j t} && (3)\\
&= \sum_{k=0}^{\infty}(-1)^k \sum_{j=0}^{\infty}\operatorname{rk}(\mathrm{BMH}_k(X)(l_j))\left(e^{-l_j t} - e^{-l_{j+1} t}\right) && (4)\\
&= \sum_{k=0}^{\infty}(-1)^k \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} D_{i,k}(j)\left(e^{-l_j t} - e^{-l_{j+1} t}\right) && (5)\\
&= \sum_{k=0}^{\infty}(-1)^k \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} D_{i,k}(j)\left(e^{-l_j t} - e^{-l_{j+1} t}\right) && (6)\\
&= \sum_{k=0}^{\infty}(-1)^k \sum_{i=0}^{\infty}\left(e^{-a_{ki} t} - e^{-b_{k,i} t}\right) && (7)
\end{aligned}$$

Equation (1) is the result of the previous theorem. Equation (2) is obtained by applying Lemma 11 to the short exact sequence from Lemma 10.
Equation (3) is a switching of the order of summation. To show that this manipulation is valid, we show that the series - in its form from (1) - is absolutely convergent.
Let $J \in \mathbb{N}$. We show that the partial sum

$$\sum_{j=0}^{J}\sum_{k=0}^{\infty}(-1)^k \operatorname{rk}(\mathrm{MH}_{k,l_j}(X))e^{-l_j t}$$

is bounded above by a quantity that is independant of J.
First, recall that the $\mathrm{MH}_{k,l_j}(X)$ vanish for k large enough, so the inner sum

is finite. This allows us to swap the order of summation:

$$\begin{aligned}\sum_{j=0}^{J}\sum_{k=0}^{\infty}(-1)^k \operatorname{rk}(\mathrm{MH}_{k,l_j}(X))e^{-l_j t} &= \sum_{k=0}^{\infty}\sum_{j=0}^{J}(-1)^k \operatorname{rk}(\mathrm{MH}_{k,l_j}(X))e^{-l_j t} \\ &\leq \sum_{k=0}^{\infty}\sum_{\substack{(x_0\ldots x_k)\in X \\ x_i\neq x_{i+1}}} e^{-l(x_0\ldots x_k)t} \\ &\leq \sum_{k=0}^{\infty} n^{k+1}e^{-k\delta t}\end{aligned}$$

The second line follows from observing that a tuple $(x_0 \ldots x_k)$ such that $x_i \neq x_{i+1}$ contributes at most 1 to the inner sum for the term $l_j = l(x_0 \ldots x_k)$ and summing over j. The third line follows from the fact that there are at most $\mathrm{Card}(X)^{k+1}$ such tuples, and from the inequality $l(x_0 \ldots x_k) \geq k\delta$. Applying D'Alembert's rule to the final line, we obtain $ne^{-\delta t} < 1$ as a criterion for absolute convergence, which corresponds to the standing assumption that t is large enough. Since the bound is independant of J, it holds in particular for $J = +\infty$, and so we have absolute convergence of the original series.

Equation (5) is obtained from equation (4) by doing the following substitution:

$$\begin{aligned}\operatorname{rk}(\mathrm{BMH}_k(X)(l_j)) &= \operatorname{rk}\left(\bigoplus_{i=0}^{\varphi(k)} \chi_{[a_{ki},\, b_{ki}[}(l_j)\right) \\ &= \operatorname{rk}\left(\bigoplus_{i=0}^{\varphi(k)} \mathbb{F}^{D_{ik}(j)}\right) \\ &= \sum_{i=0}^{\varphi(k)} D_{i,k}(j).\end{aligned}$$

Finally, equation (7) comes from telescoping the sum. Indeed, fix k and i, and consider the values $D_{i,k}(j)$ for varying j. We have

$$D_{ik}(j) = \begin{cases} 0 & \text{if } l_j < a_{ki}; \\ 1 & \text{if } a_{ki} \leq l_j < b_{ki}; \\ 0 & \text{if } l_j \geq b_{ki}. \end{cases}$$

Hence the innermost sum in equation (6) can be rewritten as

$$\sum_{j=0}^{\infty} D_{i,k}(j)\left(e^{-l_j t} - e^{-l_{j+1}t}\right) = \sum_{a_{ki}\leq l_j < b_{ki}}\left(e^{-l_j t} - e^{-l_{j+1}t}\right)$$

This is a telescopic sum and thus only the first and last terms remain. Letting $l_\alpha = \min_j(l_j \mid l_j \geq a_{ki})$ and $l_\beta = \min_j(l_j \mid l_j \geq b_{ki})$ we thus have

$$\sum_{j=0}^{\infty} D_{i,k}(j) \left(e^{-l_j t} - e^{-l_{j+1} t}\right) = \left(e^{-l_\alpha t} - e^{-l_\beta t}\right)$$

However recall from the previous lemma that a_{ki} and b_{ki} are l-values, so $l_\alpha = a_{ki}$ and $l_\beta = b_{ki}$ and thus

$$\sum_{j=0}^{\infty} D_{i,k}(j) \left(e^{-l_j t} - e^{-l_{j+1} t}\right) = \left(e^{-a_{ki} t} - e^{-b_{ki} t}\right)$$

Replacing the innermost sum by this expression in (6) completes the proof.

$\square$

Chapter 6

Future Directions

In this section, we discuss more recent work having to do with magnitude and magnitude homology. Much of what we discuss here are ideas which were presented at the conference on magnitude held in Osaka, between December 4th and December 8th, 2023. For greater detail, many of the slides describing these new advances are available on the conference website [1], or in Tom Leinster's blog posts about the conference [2].

- Giuliamaria Menara introduced *Eulerian magnitude homology*, a variant of magnitude homology which replaces the condition $x_i \neq x_{i+1}$ from the definitions of the magnitude complex with $x_i \neq x_j$ for $i \neq j$. This allows magnitude homology to capture more geometric data, specifically when it is applied to graphs; it then notably encodes the number of triangle subgraphs in a given graph.
- Emily Roff introduced *iterated* magnitude homology [23], which generalises magnitude homology by making it applicable to categories with second order enrichment - categories enriched in some categroy of enriched categories.
- Recent work of Asao [3] conclusively demonstrated that magnitude homology and path homology of graphs, both previously unrelated, are in fact just two aspects of a much larger object, the magnitude-path spectral sequence. Active research continues in this area, from Hepworth and Roff.
- Leinster and Adrián Doña Mateo have worked on a theorem describing when two closed subspaces of Euclidean space have same magnitude homology. This condition turns out to be equivalent to geometric properties of the subspaces.
- Yu Tajima and Masahiko Yoshinaga [25] introduced magnitude homotopy type of metric spaces, an aptly constructed CW-complex which

has reduced homology group isomorphic to magnitude homology of the underlying space. Applying Morse theory to this new complex, they obtained a new proof of the Mayer-Vietoris type theorem.

In [10], Hepworth and Govc continue on to *reverse* the result of Theorem 16. Starting with a finitely presented persistence module M with barcode decomposition

$$M = \bigoplus_{i=1}^{k} \mathcal{X}_{[a_i, b_i[},$$

they define the *persistent magnitude* of M to be the quantity

$$|M| = \sum_{i=1}^{k} e^{-a_i} - e^{-b_i}$$

by analogy with the theorem result. They then apply this computation to several key examples of finitely presented persistence modules, such as the Rips complex, to obtain the notion of *Rips magnitude*.

Appendix A

Dummy Appendix

You can defer lengthy calculations that would otherwise only interrupt the flow of your thesis to an appendix.

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