3 Randomization Tests

3.1 Introductory Example

a Hail prevention: ("Grossversuch IV" in central Switz. 1978-83)
Does spreading of siver iodide into potential hail clouds diminish the total hail energy?
(Simple ideas, only simple combinatorics and prob. needed.)

Target variable: Hail energy, measured for n clouds.
Two groups: treated versus control.

$$Y_i \quad : \quad \text{Hail energy for cloud } i$$
$$G_i = \begin{cases} 1 & \text{if cloud } i \text{ is treated,} \\ 0 & \text{otherwise.} \end{cases}$$

We expect Y_i to be usually smaller for $G_i = 1$ than for $G_i = 0$.

b Observed:

$Y_i = y_i^*$	16672	25	855	0	152	0	46	1219
$G_i = g_i^*$	1	1	0	0	0	1	1	0

g_i^*: random choice of the clouds that are treated.

(In reality there were 216 clouds of which 94 were treated.)

Statistical test! H_0 : no effect.

($\longrightarrow$ Proof by contradiction!)

t test for independent samples?

We do not like to assume any particular distr. for the Y_is!!

3.2 Statistical idea

a Null hypothesis = Probability model

usually is a distribution of the Y_i. $G_i = g_i^*$ assumed to be given.

Randomization tests: G_i random, $Y_i = y_i^*$ considered as fixed!

(Analysis "conditional, fiven the y_i^*s".)

If the treatment has no influence on hail energy,

the same observations y_i^* would result,

if the treatment had been given by $\underline{g}^{(1)} = [0, 1, 0, 0, 1, 1, 0, 1]$

or according to any other choice.

Random choice:

Each choice of $n/2 = 4$ elements from $n = 8$

has the same probability

$$p = \binom{8}{4}^{-1} = \frac{1}{70}$$

This determines the null hypothesis.

b **Test statistic:** designed to assume extreme values when the alternative is true.

Alternative: y_i^* with $g_i^* = 1$ are generally smaller.

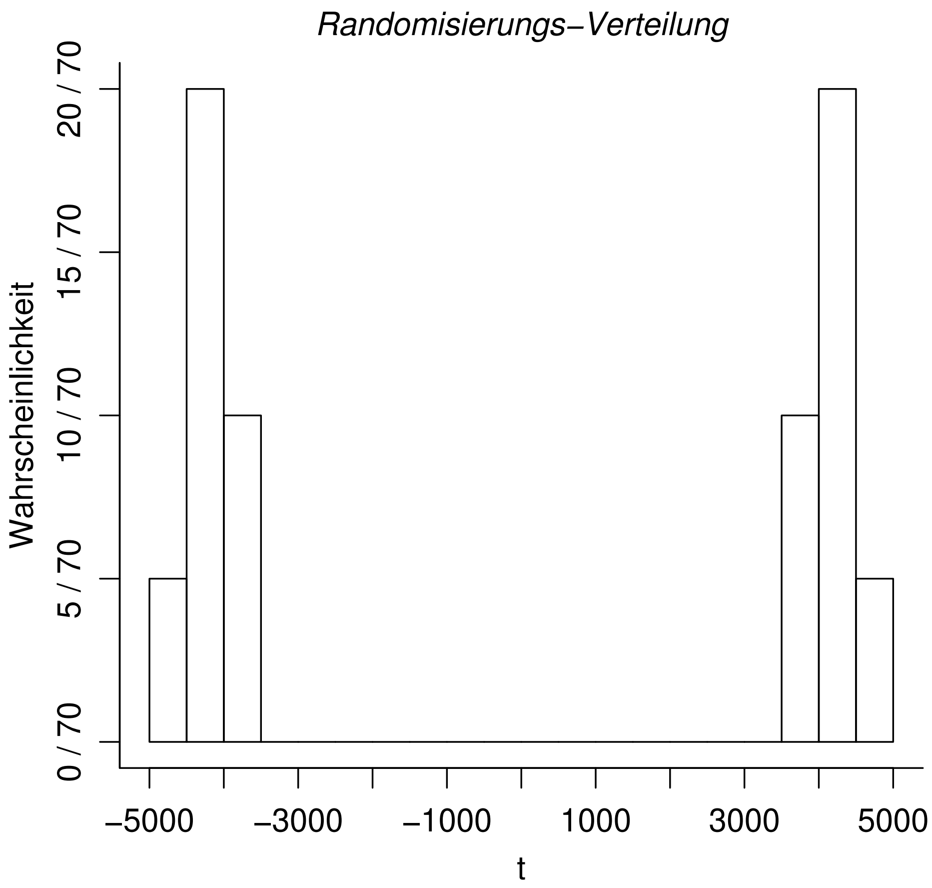
$$T\langle \underline{g}, \underline{y}^* \rangle = \frac{1}{n/2} \sum_{i:g_i=0} y_i^* - \frac{1}{n/2} \sum_{i:g_i=1} y_i^* = \frac{2}{n} \sum_i y_i^* (1 - 2g_i).$$

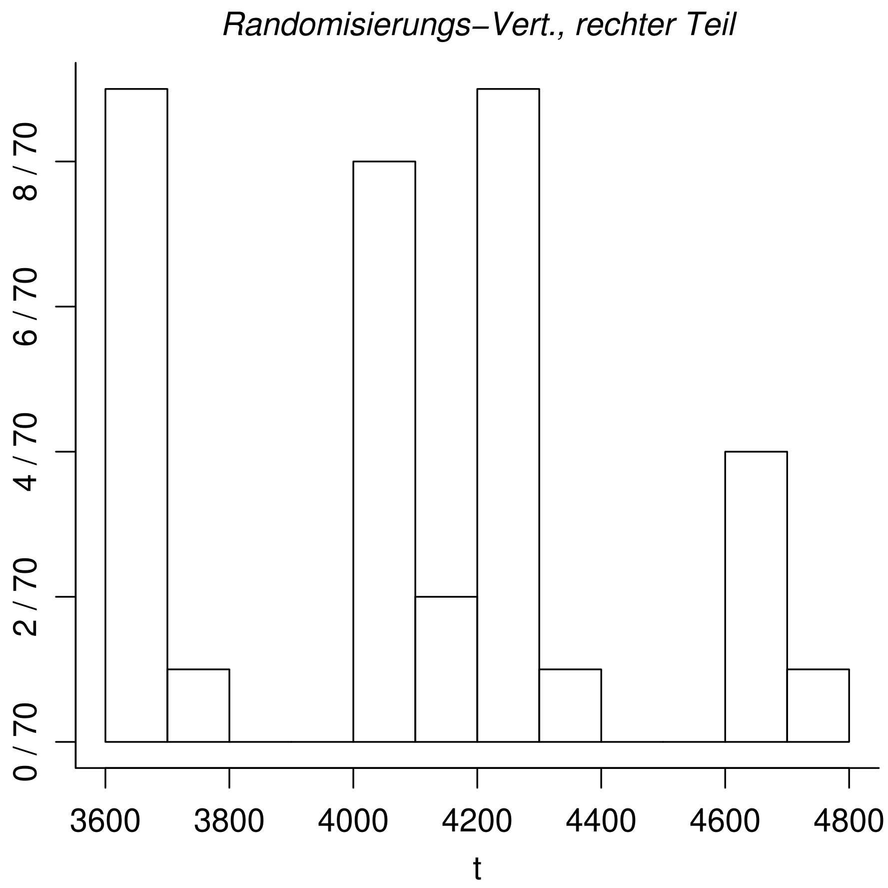
c What is the distribution of T under H_0?

$y_1^*, \ldots, y_n^*$ given $\longrightarrow \leq \binom{n}{n/2}$ possible values for T.

$$P\langle T\langle \underline{G}, \underline{y}^* \rangle = t \rangle \;=\; \frac{\#\{\underline{g} \mid T\langle \underline{g}, \underline{y}^* \rangle = t\}}{\binom{n}{n/2}}$$

„randomization distribution"


Randomisierungs−Verteilung
Wahrscheinlichkeit
0 / 70
5 / 70
10 / 70
15 / 70
20 / 70
−5000
−3000
−1000
1000
3000
5000
t


Randomisierungs−Vert., rechter Teil
0 / 70
2 / 70
4 / 70
6 / 70
8 / 70
3600
3800
4000
4200
4400
4600
4800
t

d **Rejection region**: $\alpha = 5\%$ most extreme values (as precisely as possible).

Example: $\{t \mid t \geq 4643.25\}$ (one sided).

e Experiment:

$$T\langle \underline{g}^*, \underline{y}^*\rangle = \tfrac{1}{4}(855 + 0 + 152 + 1219) - \tfrac{1}{4}(16672 + 25 + 0 + 46) = -3629.25$$

An effect in the wrong direction was observed!

Null hypothesis is not rejected; no effect is demonstrated.

f* Assumption of the test: Independence

$\longrightarrow$ Randomization among 76 “potential hail days”

Among these, 33 have been assigned to the treatment.

Number of treated days is random.

$\longrightarrow$ Analysis conditional on the number of hail days with treatm.

g $\binom{76}{33} = 36 \cdot 10^{20}$ possible choices

$\longrightarrow$ Simulation of the randomization distribution.

3.3 Tests for the Two Sample Problem

a Randomization tests are adequate even if the experimental procedure does not contain any randomization.

Assumptions in this case:

- The observations must be equally distributed under H_0 and
- independent

Then, the presupposed probability α of error of the first kind holds precisely.
The randomization tests are in this sense the the "gold standard" of statistical tests.

(* Weaker assumption: "Exchangeability".)

b If the observations are random:

Sample $[Y_1, ..., Y_n] \longrightarrow$ ordered sample $Y_{[1]}, ..., Y_{[n]}$
or empirical distribution function $\widehat{F}_n$ (s. Bootstrap)

Distribution of the test statistic, conditional on $\widehat{F}_n$,
is the randomization distribution.

Cond. prob. of an error of the first kind, given $\widehat{F}_n$, is α

c Arbitrary test statistic.

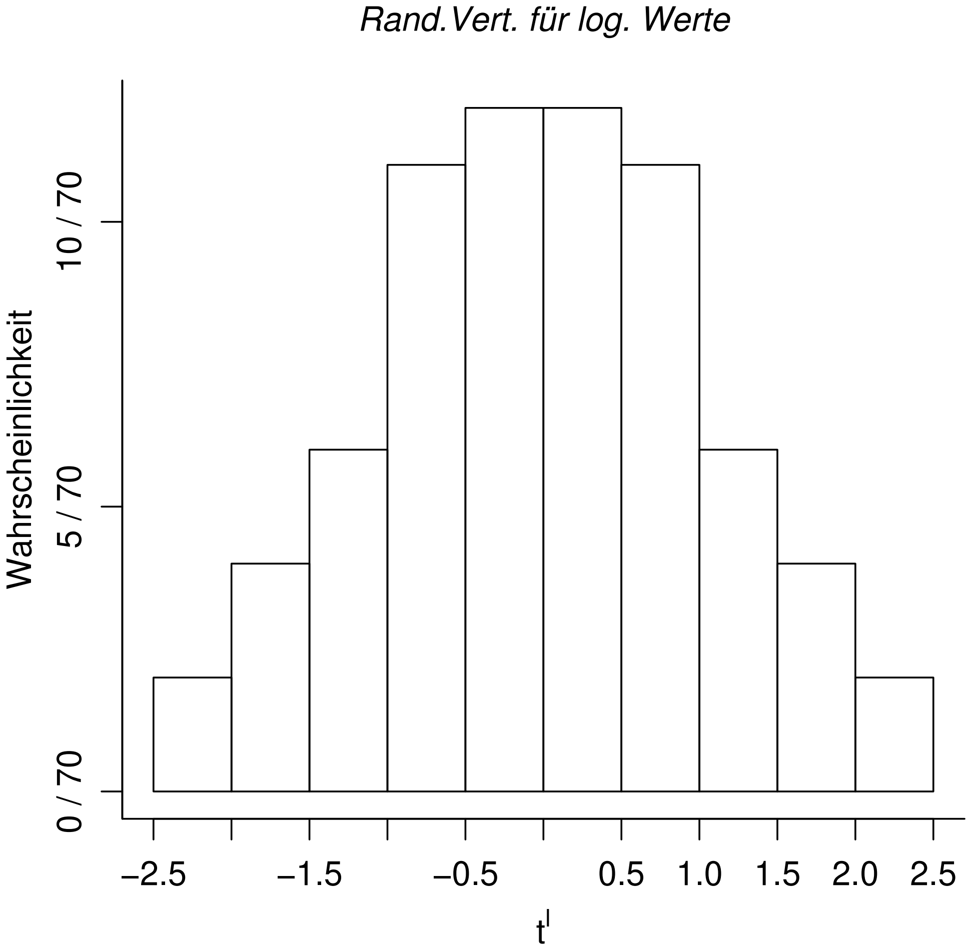
The difference of means is not robust ...

Optimal test statistic? $\longrightarrow$ optimize power for alternative(s)!

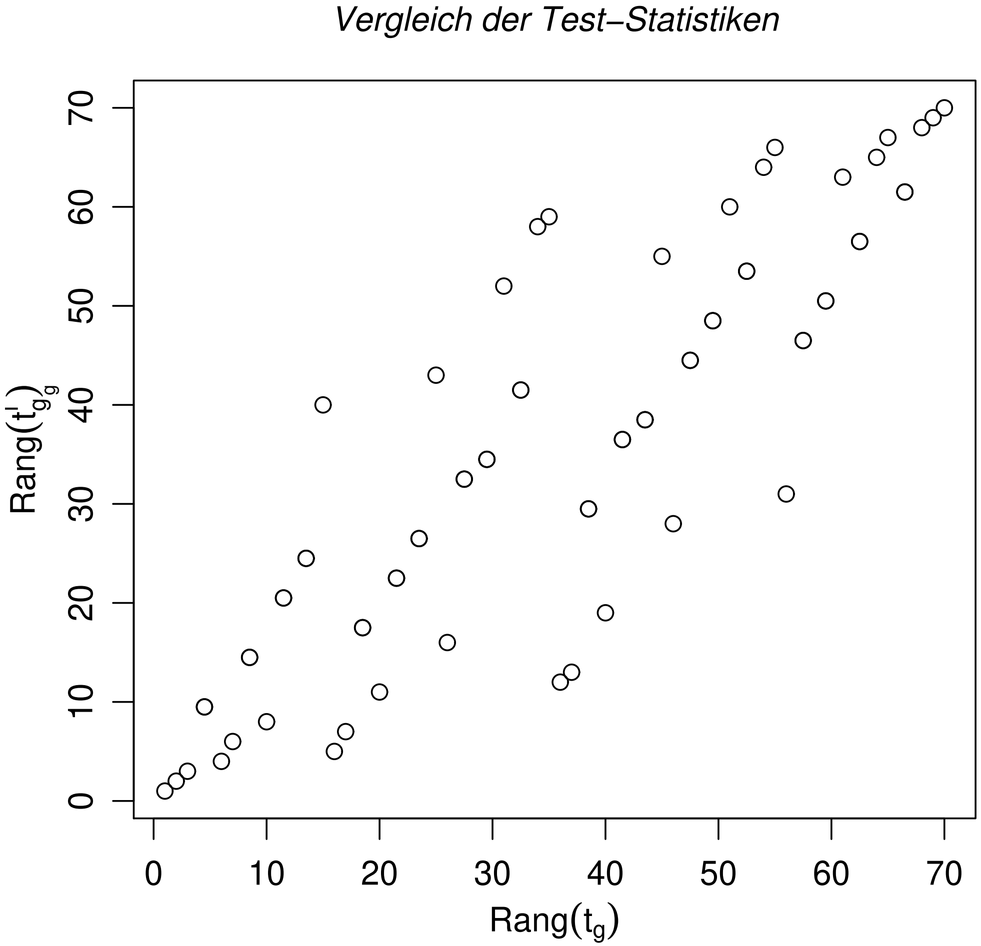
Needs fixed (family of) distribution(s)

$\longrightarrow$ optimal test statistic (e.g., likelihood ratio test)

d Example: Log transformation, then difference of means (preferably robustized)



Rand.Vert. für log. Werte
Wahrscheinlichkeit
0 / 70
5 / 70
10 / 70
−2.5
−1.5
−0.5
0.5
1.0
1.5
2.0
2.5
t'



Vergleich der Test−Statistiken
Rang(t'g)
Rang(tg)
0
10
20
30
40
50
60
70

e **Robustness.** Why should we use a robust test statistic,
if the test keeps the level without this "preventive measure"?

f **Rank sum test** of Wilcoxon, Mann and Whitney (U-Test),

$$T\langle \underline{g}, \underline{y} \rangle = \sum\nolimits_{g_i=1} R_i = \sum\nolimits_i g_i R_i \ ,$$

Quite robust $\longrightarrow$ First choice for the 2 sample problem

Distribution of the test statistic under H_0 as before.

g* Hail experiment: Complicated test statistic, two-dimensional
$\longrightarrow$ two dimenstional rejection region.

3.4 One Sample and Matched Pairs

a **Example Tranquilizer.**

Target variable: „Hamilton depression scale factor IV".

9 patients, before and after taking the tranquilizer

before ($X_i^{(1)}$)	1.83	0.50	1.62	2.48	1.68	1.88	1.55	3.06	1.30
after ($X_i^{(2)}$)	0.878	0.647	0.598	2.05	1.06	1.29	1.06	3.14	1.29
Difference ($-Y_i$)	0.952	-0.147	1.022	0.43	0.62	0.59	0.49	-0.08	0.01

b Matched Pairs.

Differences $Y_i = X_i^{(2)} - X_i^{(1)}$

distributed symmetrically around 0?

H_0: For each Y_i, $+$ and – sign is equally probable

G_i = sign, $|Y_i|$ = „Y_i" of a two sample problem.

For each configuration $\underline{g}^{(\ell)} = [g_1^{(\ell)}, ..., g_n^{(\ell)}]$

the probability is $= 1/2^n$.

c Fix a test statistic $T\langle \underline{g}, \underline{z} \rangle$

$g_i = +1$ or $= -1$, $z_i > 0$.

Randomization distr. $P\langle T\langle \underline{G}, \underline{z} \rangle = t \rangle = \#\{\underline{g} \mid T\langle \underline{g}, \underline{z} \rangle = t\} /2^n$

- $T\langle \underline{g}, \underline{z} \rangle = (1/n) \sum_i g_i z_i = \text{ave}_i \langle y_i \rangle$
  corresponds to the t test for matched pairs.

- $T\langle \underline{g}, \underline{z} \rangle = \#\{i : g_i = 1\}$: sign test.

- $T\langle \underline{g}, \underline{z} \rangle = \sum_{i:g_i=1} R_i$, R_i: rank of z_i:
  signed rank test of Wilcoxon

e Example:

```
> wilcox.test(d.tranquilizer[,1], d.tranquilizer[,2],
  paired=TRUE)
        Wilcoxon signed rank test
data:  d.tranquilizer[, 1] and d.tranquilizer[, 2]
V = 40, p-value = 0.03906
alternative hypothesis: true mu is not equal to 0
```

barely significant.

Beware of before-after comparisons!
Adequate: Comparison with control or cross over experiment

3.5 Estimators and Confidence Intervals

a Model: Testing problem was: Is the distr. symmetric around 0?
More general: ... symmetric around μ
$\Leftrightarrow \quad Y_i - \mu$ symmetric around 0.

Test: Test statistic $T\langle \underline{g}, \underline{y} - \mu\underline{1}\rangle$.
Large values indicate deviation from $H_0 : \mu$.

b This yields an estimator:

$$\widehat{\mu} = \text{arg min}_{\mu}\langle T\langle \underline{g}, \underline{y} - \mu\underline{1}\rangle\rangle$$

c Signed rank Test $\longrightarrow$ Hodges-Lehmann estimator.

Form Walsh averages $(X_h + X_i)/2$.

$$\widehat{\mu} = \text{med}_{h \le i} \langle (X_h + X_i)/2 \rangle \; .$$

Example Tranquilizer: 45 Walsh averages

-0.1470, -0.1135, -0.0800, -0.0685, -0.0350, 0.0100, ..., 1.022

Median $\widehat{\mu} = 0.46$

d* Derivation: $X_{[k]}$ kth smallest value.

$X_{[k]} > 0,\ Z_{hk} = (X_{[h]} + X_{[k]})/2,\ h < k$

$Z_{hk} < 0$, if $|X_{[h]}| > |X_{[k]}|$.

$$\#\{Z_{hk} < 0\} = \#\{h \mid |X_{[h]}| < |X_{[k]}|\} = R_{[k]} - 1$$

$$R_{[k]} = \#\{h \mid Z_{hk} > 0,\ h \leq k\}.$$

$X_{[k]} < 0 \Longrightarrow Z_{hk} < 0$, if $h < k$.

$$T\langle \underline{g}, \underline{z} \rangle = \sum_{i:g_i=1} R_i = \#\{[h,k] \mid Z_{hk} > 0, h \leq k\}$$

Null hypothesis $\mu = \mu_0$:

$T\langle \underline{g}, \underline{z} \rangle = \sum_{i:g_i=1} R_i = \#\{[h,k] \mid Z_{hk} > \mu_0, h \leq k\}$

Test is least significant if $= \frac{n(n+1)}{2}$

$\longrightarrow\ \widehat{\mu} = \text{median}\langle Z_{hk} \mid h \leq k \rangle$.

f Confidence intervall for the signed rank test:

Limits of the acceptance interval of T:

c and $c' = n(n+1)/2 + 1 - c$

Confidence limits $=$ cth and c'th Walsh average.

Example Tranquilizer: $c = 6$, $c' = 40$,

Confidence interval $[0.01, 0.786]$.

h For a general test statistic $T\langle \underline{G}, \underline{z}^*; \mu\rangle$: Let

$$Q\langle\beta\rangle = P\langle T\langle \underline{G}, \underline{z}^*; \mu\rangle > T\langle \underline{g}^*, \underline{z}^*; \mu\rangle\rangle - \beta$$

Estimator = solution of $Q\langle\beta = 0.5\rangle = 0$.

Confidence limits = solution of

$Q\langle\beta = 0.025\rangle = 0$ and $Q\langle\beta = 0.975\rangle = 0$.

Not difficult!

3.6 More than 2 Samples

a Simple Analysis of Variance

Randomization = assignment of observations to groups.
Nummber of observations in each group is fixed.

Rank the y_is among all observations $\longrightarrow R_i$
Average the ranks over groups
$\bar{R}_h = \text{ave}_{g_i=h} R_i$. $\quad \mathcal{E}\langle \bar{R}_h \rangle = (n+1)/2$.
Form weighted mean of squares of deviations

$$T\langle \underline{g}, \underline{y} \rangle = \frac{12}{n(n+1)} \sum\nolimits_h n_h \left(\bar{R}_h - \frac{n+1}{2} \right)^2$$

Kruskal-Wallis test. $\qquad$ 2 samples $\longrightarrow$ U-Test.

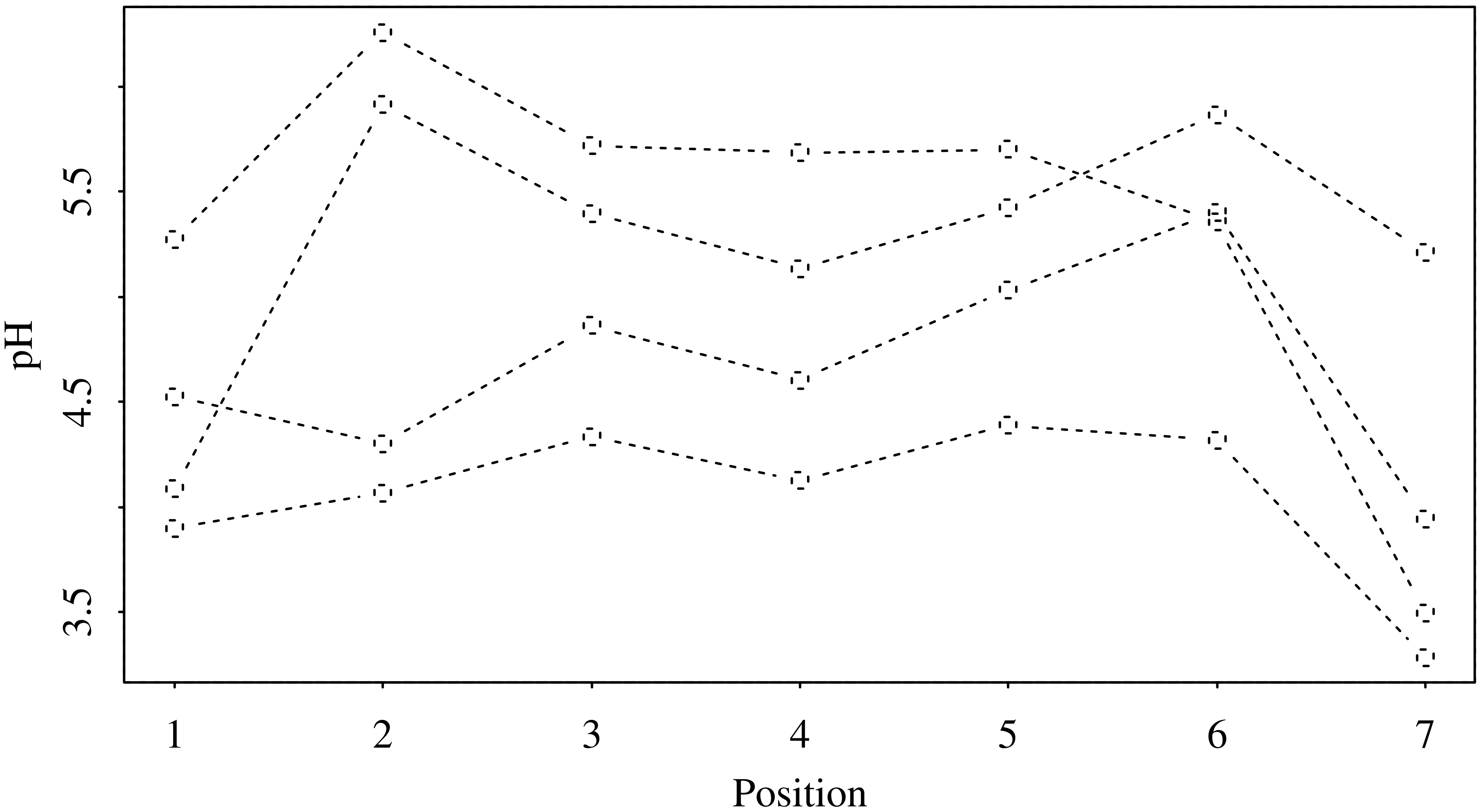
b **Matched Samples** = block design

n blocks, m treatments.

Randomization?

c **Example acidic soils**

	Position						
Block	1	2	3	4	5	6	7
1	4.09	5.91	5.40	5.13	5.43	5.87	5.21
2	3.90	4.07	4.34	4.13	4.39	4.32	3.29
3	5.27	6.26	5.72	5.69	5.70	5.36	3.50
4	4.53	4.30	4.86	4.61	5.03	5.40	3.95



pH
5.5
4.5
3.5
1
2
3
4
5
6
7
Position

Friedman test. R_{ij} = Rank of observation j in block i.
$\widetilde{R}_j = \text{ave}_i \langle R_{ij} \rangle$ average rank of sample j.

$$T = \frac{12n}{m(m+1)} \sum_{j=1}^{m} (\widetilde{R}_j - (m+1)/2)^2 .$$

d

	Position						
Block	1	2	3	4	5	6	7
1	1	7	4	2	5	6	3
2	2	3	6	4	7	5	1
3	2	7	6	4	5	3	1
4	3	2	5	4	6	7	1
Summe	8	19	21	14	23	21	6
Mittel	2	4.75	5.25	3.5	5.75	5.25	1.5

```
> friedman.test(t.dt)
        Friedman rank sum test
data:  t.dt
Friedman chi-squared = 14.8, df = 6,
p-value = 0.02199
```

e Analysis of variance for this example:

```
> summary(aov(pH~trans+pos,data=t.d))

            Df  Sum Sq Mean Sq F value Pr(>F)
trans        1    0.12    0.12    0.17   0.68
pos          1    0.18    0.18    0.27   0.61
Residuals   25   16.57    0.66
```

3.7 Correlation and Regression

a Correlation and simple regression.

$X_i,\ Y_i$ (X_i random or fixed)

Null hypothesis: "no relationship"

Randomization = matching = permutation of $\underline{Y}$.

Probability of each permutation = $1/n! = 1/(n(n-1)...2 \cdot 1)$.

Test statistic:

- simple (Pearson) correlation,
- rank correlation,
- robust estimator of the regression coefficient, ...

b **Multiple Regression:**

Permutation of $\underline{Y}$ for testing the null hypothesis that there is no relationship between all explanatory variables and the target variable.

c **Time Series**: Are the observations independent?

Randomization: Permutation.

Test statistic: e.g. first autocorrelation.

d **Multiple Regression: Single coefficient** (or several)

$\longrightarrow$ no proper randomization model.

e* Permutations and other randomizations.

Regression and correlation: permutations.

Two or more samples: subsets (“choices”).

there are many more permutations;

many of them lead to the same partition into groups

$\longrightarrow$ same randomization distribution.

Hail experiment: Number of potential hail days was random,

proportions of treated days also random

randomization distribution: Choices of 33 from 76 days

$\longrightarrow$ conditional test, given the number of

treated and control days.

- Randomization tests keep the level exactly,
  without any assumptions on the distribution.
  (Independence of observations is essentially assumed.)

- The test statistic may be arbitrarily complicated.
  Choose considering (informally) the power.
  Choose robust test statistic (e.g., based on ranks)!

- Confidence intervals can also be constructed.