Asymptotic limits of a Ginzburg-Landau type functional

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Dedicated to Stefan Hildebrandt
on the occasion of his 60th birthday

Abstract

We consider a Ginzburg-Landau type of functional involving a section of a line bundle over a Riemann surface and a connection on this bundle. We select a scaling parameter ε in such a way that self-duality is preserved and that the infimum of the functional stays bounded as ε tends to 0, and we perform a corresponding limit analysis.

1 Introduction

Let Ω be an open domain on some Riemann surface Σ, with – possibly empty – smooth boundary $\partial\Omega$. Let L be a complex line bundle over $\bar{\Omega}$, equipped with a Hermitian metric $< \cdot, \cdot >$. For a section u of L, we write

$$|u(x)| =< u(x), u(x) >^{\frac{1}{2}} .$$

Ginzburg-Landau functionals are defined for a section u of L and a unitary connection A on L. That A is unitary means that

$$d < u, v >= \langle \nabla_A u, v \rangle + \langle u, \nabla_A v \rangle \tag{1.1}$$

for sections u, v of L, where d is the exterior derivative and ∇_A is the covariant derivative defined by A. We employ the physical convention

$$\nabla_A u := (d - iA)u,$$

i.e. we let A be real valued. In local coordinates (x^1, x^2) on Σ, we write

$$\nabla_A^k := \nabla_A \left(\frac{\partial}{\partial x^k} \right) =: \partial_k - iA^k \quad (k = 1, 2).$$

The curvature of A is

$$F := dA,$$

i.e.

$$F^{kj} = \partial_k A^j - \partial_j A^k = i\left(\nabla_A^k \nabla_A^j - \nabla_A^j \nabla_A^k\right). \tag{1.2}$$

Again, this is the physical convention. The prototype of a Ginzburg-Landau functional is the Yang-Mill-Higgs functional

$$E(u, A, \Omega) = \int_\Omega \left\{ |dA|^2 + |\nabla_A u|^2 + \frac{1}{4}\left(1 - |u|^2\right)^2 \right\} \mathrm{dvol}(x), \tag{1.3}$$

where dvol is the volume form of some fixed Kähler metric on Σ. The Euler-Lagrange equations for E are

$$\Delta_A u = -\frac{1}{2}\left(1 - |u|^2\right) u \tag{1.4}$$

$$\partial_k F^{kj} = -\mathrm{Im}\left\langle \left(\partial_j - iA^j\right) u, u\right\rangle, \tag{1.5}$$

where

$$\Delta_A = \nabla_A^k \nabla_A^k \tag{1.6}$$

is the Laplacian defined by A, with the analysts' sign convention, and where we employ the usual summation convention.

In more invariant form, (1.5) may be rewritten as

$$- *dh = Re\,\langle iu, \nabla_A u\rangle \tag{1.7}$$

with

$$h := *dA,$$

$*$ being the usual star operator defined by the conformal structure. An important feature of E is its gauge invariance, i.e. its invariance under the substitution

$$(u, A) \to (u \exp(i\psi), A + d\psi) \tag{1.8}$$

for a real valued function ψ.

Another important feature of E is the self-duality. Namely, decomposing ∇_A into its $(1,0)$ and $(0,1)$ parts,

$$\nabla_A = \partial_A + \bar\partial_A,$$

in case $\Omega = \mathbb{R}^2$, and if $|u(x)| \to 1$, $\nabla_A u(x) \to 0$ sufficiently fast as $|x| \to \infty$, then E can be rewritten as

$$E(u, A) = \int_{\mathbb{R}^2} \left\{ 2\left|\bar\partial_A u\right|^2 + \left| *F - \frac{1}{2}(1 - |u|^2)\right|^2 \right\} dx + 2\pi d, \tag{1.9}$$

for some integer d, the so-called vortex number; see [18], p. 54. Thus, we see that the infimum of E, namely $2\pi d$, is attained iff the vortex equations

$$\bar{\partial}_A u = 0 \tag{1.10}$$

and

$$* F = \frac{1}{2}(1 - |u|^2) \tag{1.11}$$

are satisfied. Of course, since E is nonnegative, this is possible only if $d \geq 0$. (If $d < 0$, one should consider antiholomorphic sections instead of holomorphic ones.) Taubes ([28]) showed that for any collection of d points $x_j \in \mathbb{R}^2$, possibly with multiplicities, there exists a solution, unique up to gauge equivalence, of the vortex equations with

$$u(x_j) = 0 \quad j = 1, \dots, d. \tag{1.12}$$

Likewise, if Ω is a compact Riemann surface Σ, E can be rewritten as

$$E(u, A, \Sigma) = \int_\Sigma \left\{ 2 \left| \bar{\partial}_A u \right|^2 + \left| \Lambda F - \frac{1}{2}(1 - |u|^2) \right|^2 \right\} \mathrm{dvol}(x) + 2\pi \deg L, \tag{1.13}$$

where $\deg L$ is of course the degree of L and Λ denotes contraction with the Kähler form of Σ. Thus, again the infimum $2\pi \deg L$ of E is realized by the solutions of the vortex equations

$$\bar{\partial}_A u = 0 \tag{1.14}$$

$$\Lambda F = \frac{1}{2}(1 - |u|^2). \tag{1.15}$$

Integrating (1.15) over Σ, one sees that a solution can only exist if

$$2\pi \deg L < \frac{1}{2} \mathrm{Vol}\Sigma. \tag{1.16}$$

This obstruction, however, is easily circumvented by replacing the term $(1 - |u|^2)$ by $(\tau - |u|^2)$, for $\tau \in \mathbb{R}$ satisfying

$$2\pi \deg L < \frac{\tau}{2} \mathrm{Vol}\Sigma. \tag{1.17}$$

The resulting equations have been solved and studied by Bradlow and García-Prada, see e.g. [5], [6], [12], [13], [14].

The Ginzburg-Landau functional originated in the theory of superconductivity, where Ω represents the cross section of a wire and $u(x)$ is a complex order parameter. $|u(x)| = 1$ corresponds to a superconducting phase, and therefore, one wishes to constrain u to have absolute value 1. Of course, there are topological obstructions for that; namely if d in (1.9) or $d := \deg L$ in (1.13) is not

0, then any section u will have at least $|d|$ zeroes (counted with multiplicity), the so-called vortices. For this reason, the family of functionals

$$\tilde{E}_\varepsilon(u, A, \Omega) := \int_\Omega \left\{ |dA|^2 + |\nabla_A u|^2 + \frac{1}{4\varepsilon^2}\left(1 - |u|^2\right)^2 \right\} \mathrm{dvol}(x), \qquad (1.18)$$

depending on a real parameter $\varepsilon > 0$, has been studied, and the limiting behavior as $\varepsilon \to 0$ has been investigated. The first mathematical treatment is due to Berger-Chen [1]. They performed the limit analysis in the class of rotationally symmetric solutions $u_\varepsilon, A_\varepsilon$ on $\mathbb{R}^2$. Their results are quite explicit, and as $\varepsilon \to 0$, they found a "nonlinear desingularization"; namely, $h_\varepsilon := *dA_\varepsilon$ tends to a limit h that satisfies the London equation

$$\Delta h - h = -2\pi d \delta(x) \text{ and } h(x) \to 0 \quad \text{for } |x| \to \infty,$$

where $\delta(x)$ is the Dirac delta function, and d is the vortex number. Of course, since only rotationally symmetric solutions are considered, in the limit, one obtains a singularity of multiplicity d at the origin.

In the nonsymmetric case, Chen [7] also obtained an existence result for solutions on bounded domains $\Omega \subset \mathbb{R}^2$ with prescribed d for any fixed positive ε.

More recently, Bethuel-Brézis-Hélein ([2], [3]) simplified the functional $\tilde{E}_\varepsilon$ by dropping the term $|dA|^2$ and succeeded in performing the limit analysis without the symmetry assumption. They found, in particular, that the singularities of minimizers $u_\varepsilon, A_\varepsilon$ decouple in the limit, i.e. that all singularities of the limit have multiplicity ± 1. Thus, in particular, for $|d| \neq 1$, the rotationally symmetric solutions of Berger-Chen cannot be minimizing (for the modified functional) for sufficiently small ε. The results in [2], [3] were obtained only for star-shaped domains in $\mathbb{R}^2$. This restriction was removed, and some of the arguments were considerably simplified by Struwe ([25], [26]).

Related results were obtained by Hardt-Lin [16], Chen-Lin [9], Lin [19], and, more recently, by del Pino-Felmer [11].

Bethuel-Riviére [4], and more recently Orlandi [22] and J. Qing [23], studied the original functionals $\tilde{E}_\varepsilon$ and obtained results analogous to those of Bethuel-Brézis-Hélein and Struwe.

A characteristic feature of the functionals $\tilde{E}_\varepsilon$ is that self-duality is lost for $\varepsilon \neq 1$, and that

$$\liminf_{\varepsilon \to 0} \left(\inf \tilde{E}_\varepsilon \right) \qquad (1.19)$$

is infinite under fixed nontrivial natural boundary conditions. In order to restore self-duality (and to thus make the results better applicable in Riemann surface theory) and to get finite limits for the infima of the functionals, we consider here instead the functionals

$$E_\varepsilon(u, A, \Omega) := \int_\Omega \left\{ \varepsilon^2 |dA|^2 + |\nabla_A u|^2 + \frac{1}{4\varepsilon^2}\left(1 - |u|^2\right)^2 \right\} \mathrm{dvol}(x). \qquad (1.20)$$

The Euler-Lagrange equations for (1.20) are

$$\Delta_A u = -\frac{1}{2\varepsilon^2}\left(1-|u|^2\right)u \tag{1.21}$$

$$\varepsilon^2 \partial_k F^{kj} = -\mathrm{Im}\left\langle \left(\partial_j - iA_j\right)u, u\right\rangle, \tag{1.22}$$

or, with $h := *dA$, equivalently

$$-\varepsilon^2 * dh = \mathrm{Re}\left\langle iu, \nabla_A u\right\rangle. \tag{1.23}$$

Again, the latter are satisfied by solutions of the vortex equations on $\mathbb{R}^2$,

$$\overline{\partial}_A u = 0 \tag{1.24}$$

$$-i\varepsilon^2 * F = \frac{1}{2}\left(1-|u|^2\right). \tag{1.25}$$

On a general Riemann surface Σ, the second equation becomes

$$\varepsilon^2 \Lambda F = \frac{1}{2}\left(1-|u|^2\right). \tag{1.26}$$

The compatibility condition (1.16) now becomes

$$2\pi\varepsilon^2 \deg L < \frac{1}{2}\mathrm{Vol}(\Sigma), \tag{1.27}$$

and this is obviously satisfied for sufficiently small $\varepsilon > 0$.

In fact, if $\Omega \subset \mathbb{R}^2$ is a flat domain, E_ε can be obtained from $E = E_1$ by a simple rescaling of the domain. Namely,

$$E_\varepsilon\left(u_\varepsilon, A_\varepsilon, \Omega\right) = E\left(v_\varepsilon, B_\varepsilon, \Omega_\varepsilon\right), \tag{1.28}$$

if the various quantities are related via

$$\begin{aligned} v_\varepsilon(x) &= u_\varepsilon(x_0+\varepsilon x), \\ B_\varepsilon(x) &= A(x_0+\varepsilon x) = \varepsilon A_1(x_0+\varepsilon x)dx_1 + \varepsilon A_2(x_0+\varepsilon x)dx_2, \end{aligned}$$

and $\Omega = \{x_0 + \varepsilon x : x \in \Omega_\varepsilon\}$ for some fixed $x_0 \in \Omega$; that is,

$$\Omega_\varepsilon = \left\{\frac{1}{\varepsilon}(y-x_0) : y \in \Omega\right\}.$$

Of course, solutions of the corresponding equations are likewise related by rescaling. This rescaling with parameter ε will therefore be often applied in the present paper. In particular, the limiting analysis as $\varepsilon \to 0$ becomes equivalent to an analysis of solutions of (1.4), (1.5) near vortices, i.e. where $u(x) = 0$. Since these equations admit solutions having vortices of arbitrary degree, in contrast to [3], no decoupling of the singularities will occur.

If Ω happens to have a nonempty boundary, one wishes to impose boundary conditions on u and A. Because of the gauge invariance (1.8), it is not meaningful to impose Dirichlet boundary conditions for both u and A. Instead, natural boundary conditions are

$$|u| = 1 \quad \text{on } \partial\Omega \tag{1.29}$$

and

$$\langle iu, \nabla_A(\tau)u\rangle = g \quad \text{on } \partial\Omega \tag{1.30}$$

where τ denotes a unit tangent vector field along $\partial\Omega$, and g is given.

In view of the scaling properties and of (1.23), one might also replace (1.30) by the condition

$$\langle iu, \nabla_A(\tau)u\rangle = \varepsilon g \quad \text{on } \partial\Omega. \tag{1.31}$$

This would lead to a nicer behavior of the curvature dA for minimizers of E_ε in the limit $\varepsilon \to 0$.

If Ω is diffeomorphic to the disc, we may consider a section u as a complex valued function with prescribed degree $\deg(u, \partial\Omega) = d$. In the general case, the topology of the bundle L also imposes a global condition on sections.

In this paper we will focus on the latter topological conditions and we consider the asymptotic behavior as $\varepsilon \to 0$ of sections of a line bundle L over a compact Riemann surface Σ without boundary which minimize the scaled, self-dual Ginzburg-Landau energy E_ε.

In this self-dual case, the minimizers of E_ε are precisely the solutions of the first-order vortex equations (1.24) and (1.25), respectively (1.26). For these solutions, we have the following result.

Theorem 1.1 *Let $(u_\varepsilon, A_\varepsilon)$ be solutions of (1.24), (1.26) on some compact Riemann surface Σ, with fixed $d = \deg L \geq 0$. Then for some sequence $\varepsilon_n \to 0$, there exist points x_j, $j = 1, \dots, l \leq d$, such that*

$$|u_\varepsilon| \to 1, \ \nabla_{A_\varepsilon} u_\varepsilon \to 0, \ dA_\varepsilon \to 0$$

uniformly on compact subsets of $\Sigma \setminus \{x_1, \dots, x_l\}$. Moreover, for $h_\varepsilon := *dA_\varepsilon$, we have

$$h_\varepsilon \to 2\pi \sum_{j=1}^{l} \delta(x_j)$$

in the sense of measures, where the delta functions have to be counted with multiplicity.

An analogous result holds on $\mathbb{R}^2$; however, one has to deal with dichotomy, that is, vortices moving off to infinity.

Theorem 1.1 yields a method for degenerating a line bundle L on Σ of degree d into a flat line bundle with $|d|$ singular points (counted with multiplicity) and a covariantly constant section.

The proof of Theorem 1.1 gives similar but slightly weaker results on the asymtotic behaviour in the interior of minimizers of E_ε in the general case.

If u were real instead of complex valued (and A trivial), one would obtain a functional of the type studied by Modica and Mortola ([21], [20]) and many other authors. Such functionals are of great physical importance and mathematical interest. For real valued u, in the limit $\varepsilon \to 0$, a phase transition takes place along a real hypersurface, whereas in our context of a complex valued u, a different phase is realized only on a set of real codimension 2. One might expect such a behavior also for similar functionals on domains of higher dimension.

Most of the research represented in the present paper was carried out during the summer 1994 when the three authors met in Zürich. Since then, a very interesting independent development has taken place that relates to our work. Namely, the above self-duality equations are the two-dimensional analogue of the Seiberg-Witten equations that Taubes [30], [31] used to relate the Seiberg-Witten and Gromov invariants in four-dimensional geometry through a similar change of scale.

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2 Preliminary estimates

Most of the results in this section are well-known from previous work cited above.

Since the space of solutions of our equations is invariant under gauge transformations, one cannot have regularity results without fixing the gauge. The most common gauge is the Coulomb gauge where one requires

$$d^* A = 0. \tag{2.1}$$

This may be supplemented by the boundary condition

$$A(\nu) = 0 \quad \text{on } \partial\Omega, \tag{2.2}$$

where ν denotes a unit normal vector field. Another gauge that is more useful for rotationally symmetric solutions was discovered by Cronström [10]. If we use standard polar coordinates (r, θ) on $\mathbb{R}^2$ and write

$$A = S(r,\theta)d\theta + T(r,\theta)dr, \tag{2.3}$$

then Cronström's gauge condition is simply

$$T \equiv 0, \tag{2.4}$$

that is,

$$A = S(r,\theta)d\theta. \tag{2.5}$$

If such an A is radially symmetric (which holds if in addition (2.1)is satisfied), then

$$A = S(r)d\theta. \tag{2.6}$$

Returning to the Coulomb gauge, we note that (2.1) and (1.7) yield

$$-\Delta A = (d^*d + dd^*)A = \langle iu, \nabla_A u\rangle\,, \tag{2.7}$$

and (1.4) and (2.7) together constitute an elliptic system; similarly (1.21) and (1.22) together with (2.1). Therefore, whenever we fix the Coulomb gauge, we may apply elliptic regularity theory. In particular, smoothness of solutions is automatic.

Lemma 2.1 *Any solution $(u_\varepsilon, A_\varepsilon)$ of the Ginzburg-Landau equations (1.21), (1.22) satisfies*

$$|u_\varepsilon| \le 1 \quad \text{in} \;\; \Omega. \tag{2.8}$$

Proof. As noted, in the Coulomb gauge, u_ε is smooth. We compute

$$\frac{1}{2}\Delta |u_\varepsilon|^2 = -\frac{1}{2\varepsilon^2}|u_\varepsilon|^2\left(1-|u|^2\right) + |\nabla_{A_\varepsilon} u_\varepsilon|^2\,, \tag{2.9}$$

and (2.8) follows from the maximum principle. □

For small radii $r > 0$, $x_0 \in \Sigma$ let $B_r(x_0;\Sigma)$ denote the geodesic ball of radius r around x_0 in Σ and let $B_r(x_0) = B_r(x_0;\mathbb{R}^2)$ for brevity. Also denote $B = B_1(0;\mathbb{R}^2)$ the unit disc.

Lemma 2.2 *For any $E_0 > 0$ there exists a constant $C = C(\Sigma, E_0)$ such that for $0 < \epsilon < 1$ any solution (u_ϵ, A_ϵ) of the Ginzburg-Landau equation (1.21), (1.22) on a ball $B_\epsilon(x_0;\Sigma)$ with*

$$E_\epsilon\big(u_\epsilon, A_\epsilon; B_\epsilon(x_0;\Sigma)\big) \le E_0$$

satisfies

$$|\nabla |u_\epsilon||\,(x_0) \le C\epsilon^{-1}.$$

(We may assume $C \ge 1$.)

Proof. Rescaling $v(x) = u_\epsilon(x_0 + \epsilon x)$, etc., we may assume $\epsilon = 1$, $B_\epsilon(x_0;\Sigma) = B \subset \mathbb{R}^2$, and we denote $u_\epsilon = u$, $A_\epsilon = A$ for convenience. For simplicity, we assume that on B we have the standard Euclidean metric.

Specifying the Coulomb gauge

$$d^*A = 0 \qquad \text{on } B$$

with boundary condition

$$A(\nu) = 0 \qquad \text{on } \partial B,$$

by a result of Uhlenbeck [32], p. 35 f., we can estimate

$$\int_{\partial B} |A|^2 \, do + \int_B |\nabla A|^2 \, dx = \int_B |dA|^2 \, dx \le E_1(u, A; B) \le E_0.$$

Moreover, equation (1.22) – or rather (1.23) – becomes

$$-\Delta A = (d^* d + dd^*) A = \mathrm{Im}\langle u, \nabla_A u\rangle \qquad \text{in } B,$$

and we conclude that $A \in H^{2,2}_{loc}(B)$ with

$$\|A\|_{H^{2,2}(B')} \le C \left(\|A\|_{L^2(B)} + \|\Delta A\|_{L^2(B)}\right) \le C E_1(u, A; B) \le C E_0$$

for any domain $B' \subset\subset B$, where $C = C(B')$. In particular, $A \in L^\infty(B') \cap W^{1,p}(B')$ for any $p < \infty$, any $B' \subset\subset B$, with bounds depending only on E_0, B', and p.

Now rewrite equation (1.21) in the form

$$-\Delta u = \frac{1}{2} u(1 - |u|^2) + (\Delta_A u - \Delta u)$$

and observe that

$$|\Delta_A u - \Delta u| \le C\big(\, |A| \, |\nabla_A u| + (|\nabla A| + |A|^2) \, |u| \,\big)$$

by the above is locally bounded in L^2.

Hence, using also Lemma 2.1, in a first step we obtain that $u \in H^{2,2}_{loc}(B) \hookrightarrow W^{1,p}_{loc}(B)$ with uniform local bounds for any $p < \infty$. Thus, also the error term $|\Delta_A u - \Delta u|$ is locally bounded in L^p for any $p < \infty$. In particular, if we fix some $p > 2$ we find that $u \in W^{2,p}_{loc}(B) \hookrightarrow C^1(B)$ together with the (gauge-invariant) bound $|\nabla |u|| (0) \le C = C(E_0)$. □

Lemma 2.3 *There exists a constant $\epsilon_1 > 0$ with the following property: If $0 < \epsilon \le \rho < 1$ and (u_ϵ, A_ϵ) solves (1.21), (1.22) on a ball $B_\rho(x_0) \subset \Sigma$ with $E_\epsilon\big(u_\epsilon, A_\epsilon; B_\rho(x_0)\big) < \epsilon_1$, then $|u(x_0)| \ge \frac{1}{2}$.*

Proof. Suppose $|u(x_0)| \le \frac{1}{2}$. We may assume $E_\epsilon\big(u_\epsilon, A_\epsilon; B_\rho(x_0)\big) \le 1$. Then by Lemma 2.2 for all $x \in \Sigma$ such that

$$|x - x_0| \le C^{-1}\epsilon \le \rho, \quad C = C(\Sigma),$$

we have $|u(x)|^2 \le \frac{1}{2}$ and hence

$$E\big(u_\epsilon, A_\epsilon; B_\rho(x_0)\big) \ge \int_{B_{C^{-1}\epsilon}(x_0)} \frac{(1 - |u|^2)^2}{\epsilon^2} \, dx \ge \frac{\pi}{4C^2}.$$

Thus, if we let $\epsilon_1 = \frac{\pi}{4C^2} < 1$, the claim follows. □

3 A Bochner type formula and consequences

The proof of Theorem 1.1 will be a consequence of a Bochner-type formula and an ε_0-regularity estimate for equations (1.4), (1.5). For a Ginzburg-Landau type system without magnetic field in a different context these tools were developed by Chen-Struwe [8]. They were first applied in the present context (still without a magnetic field and in the case $d = 0$) by Chen-Lin [9]. Here we extend this method to problems with magnetic field; moreover, we allow vortex behaviour, that is, $d > 0$. Moreover, we scale $\epsilon = 1$.

For reference, we recall equations (1.4), (1.5), that is

$$\partial_k F^{kj} = -\mathrm{Im}\left\langle \left(\partial_j - iA^j\right) u, u\right\rangle \tag{3.1}$$

$$\Delta_A u = -\frac{1}{2}\left(1 - |u|^2\right) u. \tag{3.2}$$

Our Laplacian on functions is

$$\Delta := \partial_j \partial_j \quad (= -d^* d) \quad (\text{"analysts' Laplacian"})$$

For a section u of L, and function s, we have the product rule

$$\nabla_A(su) = s\nabla_A u + (ds)u.$$

As always, A is a unitary connection on L. In geodesic normal coordinates at the point under consideration and denoting the curvature tensor of the metric of Σ by

$$R_{jk} = R\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right),$$

we derive the following identities

$$\frac{1}{2}\Delta\left|\nabla_A u\right|^2 = \left\langle \left(\nabla_A\right)^2 u, \left(\nabla_A\right)^2 u\right\rangle + \mathrm{Re}\langle \Delta_A \nabla_A u, \nabla_A u\rangle, \tag{3.3}$$

$$\begin{aligned}\mathrm{Re}\ \left\langle \Delta_A \nabla_A u, \nabla_A u\right\rangle &= \mathrm{Re}\langle \nabla_A^k \nabla_A^k \nabla_A^j u, \nabla_A^j u\rangle \\ &= \mathrm{Re}\langle \nabla_A^k \nabla_A^j \nabla_A^k u, \nabla_A^j u\rangle + \mathrm{Re}\langle \nabla_A^k \left(-iF^{kj} u\right), \nabla_A^j u\rangle \\ &= \mathrm{Re}\langle \nabla_A^j \Delta_A u, \nabla_A^j u\rangle + 2\mathrm{Re}\langle -iF^{kj} \nabla_A^k u, \nabla_A^j u\rangle \\ &\quad + \mathrm{Re}\langle -i\left(\partial_k F^{kj}\right) u, \nabla_A^j u\rangle + \mathrm{Re}\langle R_{kj} \nabla_A^k u, \nabla_A^j u\rangle.\end{aligned} \tag{3.4}$$

Moreover, by (3.1), we have

$$\begin{aligned}\mathrm{Re}\langle -i\left(\partial_k F^{kj}\right) u, \nabla_A^j u\rangle &= -\mathrm{Im}\langle \nabla_A^j u, u\rangle \mathrm{Re}\big(-i\langle u, \nabla_A^j u\rangle\big) \\ &= \big(\mathrm{Im}\langle \nabla_A u, u\rangle\big)^2,\end{aligned} \tag{3.5}$$

and by (3.2)

$$\mathrm{Re}\,\langle \nabla_A \Delta_A u, \nabla_A u\rangle = -\frac{1}{2}\left(1-|u|^2\right)|\nabla_A u|^2 + \frac{1}{4}\left|d\,|u|^2\right|^2. \tag{3.6}$$

(Observe that $d\,|u|^2 = 2\mathrm{Re}\,\langle u, \nabla_A u\rangle$). From (3.1)–(3.4) we conclude

$$\frac{1}{2}\Delta\,|\nabla_A u|^2 \geq -\frac{1}{2}\left(1-|u|^2\right)|\nabla_A u|^2 - 2\,|F|\,|\nabla_A u|^2 - 2\,|R|\,|\nabla_A u|^2. \tag{3.7}$$

Next we consider the term

$$\frac{1}{2}\Delta\,|F|^2 = |\nabla F|^2 + F\Delta F. \tag{3.8}$$

Note that in order to be consistent with the convention $F = dA$ and $|dA|^2 = (\partial_1 A_2 - \partial_2 A_1)^2$, we have $|F|^2 = \left|F^{12}\right|^2$. By (3.3) there holds

$$\begin{aligned} \Delta F^{12} &= \partial_1\partial_1 F^{12} + \partial_2\partial_2 F^{12} = \partial_1\partial_1 F^{12} - \partial_2\partial_2 F^{21} \\ &= -\mathrm{Im}\left\langle \nabla_A^1 \nabla_A^2 u, u\right\rangle - \mathrm{Im}\left\langle \nabla_A^2 u, \nabla_A^1 u\right\rangle \\ &\quad +\mathrm{Im}\left\langle \nabla_A^2 \nabla_A^1 u, u\right\rangle + \mathrm{Im}\left\langle \nabla_A^1 u, \nabla_A^2 u\right\rangle. \\ &= F^{12}\,|u|^2 + 2\mathrm{Im}\left\langle \nabla_A^1 u, \nabla_A^2 u\right\rangle \end{aligned} \tag{3.9}$$

From (3.7) we derive

$$\begin{aligned} F\Delta F &= F^{12}\Delta F^{12} \\ &= (F^{12})^2\,|u|^2 + 2F^{12}\mathrm{Im}\left\langle \nabla_A^1 u, \nabla_A^2 u\right\rangle \\ &= |F|^2\,|u|^2 + F^{kj}\mathrm{Im}\big\langle \nabla_A^k u, \nabla_A^j u\big\rangle. \end{aligned} \tag{3.10}$$

Note incidentally that the term in (3.2) also equals

$$2\mathrm{Re}\big\langle -iF^{kj}\nabla_A^k u, \nabla_A^j u\big\rangle = 2F^{kj}\mathrm{Im}\big\langle \nabla_A^k u, \nabla_A^j u\big\rangle.$$

(3.8) and (3.10) imply

$$\frac{1}{2}\Delta\,|F|^2 \geq |F|^2\,|u|^2 - 2\,|F|\,|\nabla_A u|^2. \tag{3.11}$$

Combining (3.7) and (3.11), we find

$$\begin{aligned} &\frac{1}{2}\,\Delta\left(|\nabla_A u|^2 + |F|^2\right) \\ &\geq\ |F|^2\,|u|^2 - 6\,|F|\,|\nabla_A u|^2 - \frac{1}{2}\left(1-|u|^2\right)|\nabla_A u|^2 - 2\,|R|\,|\nabla_A u|^2. \end{aligned} \tag{3.12}$$

It may also be useful to note the equation (which also follows from (3.3)–(3.6), (3.8), (3.10))

$$\begin{aligned} \frac{1}{2}\,\Delta\left(|\nabla_A u|^2+|F|^2\right) & \qquad\qquad (3.13)\\ = \; & \left|(\nabla_A)^2\,u\right|^2+\langle\nabla_A u,u\rangle^2+|\nabla F|^2-3F^{kj}\mathrm{Im}\left\langle\nabla_A^k u,\nabla_A^j u\right\rangle\\ & +|F|^2\,|u|^2-\frac{1}{2}\left(1-|u|^2\right)|\nabla_A u|^2+\mathrm{Re}\left\langle R_{kj}\nabla_A^k u,\nabla_A^j u\right\rangle. \end{aligned}$$

Finally, we consider the term

$$\frac{1}{2}\Delta\left(1-|u|^2\right)^2=\left|\nabla\,|u|^2\right|^2+\left(|u|^2-1\right)\Delta\,|u|^2\,. \qquad (3.14)$$

From (3.2) we derive

$$\Delta\,|u|^2=-\,|u|^2\left(1-|u|^2\right)+2\,|\nabla_A u|^2\,. \qquad (3.15)$$

Equations (3.14), (3.15) yield

$$\frac{1}{2}\Delta\left(1-|u|^2\right)^2=\left|\nabla\,|u|^2\right|^2+2\left(|u|^2-1\right)|\nabla_A u|^2+|u|^2\left(1-|u|^2\right)^2. \qquad (3.16)$$

Finally, denoting

$$e(u,A):=|\nabla_A u|^2+|F|^2+\frac{1}{4}\left(1-|u|^2\right)^2,$$

from (3.12) and (3.16) we obtain the differential inequality

$$\begin{aligned} \frac{1}{2}\Delta e(u,A) \;\; \geq \;\; & |u|^2\left(|F|^2+\frac{1}{4}\left(1-|u|^2\right)^2\right) \qquad (3.17)\\ & -|\nabla_A u|^2\left(6\,|F|+\left(1-|u|^2\right)+2\,|R|\right). \qquad (3.18) \end{aligned}$$

Suppose $|u|\geq\frac{1}{2}$. Then we can estimate

$$\begin{aligned} 6\,|\nabla_A u|^2\,|F| & \leq |u|^2\,|F|^2+36\,|\nabla_A u|^4\\ |\nabla_A u|^2\,(1-|u|^2) & \leq \frac{1}{4}\,|u|^2\,(1-|u|^2)^2+4\,|\nabla_A u|^4\,; \end{aligned}$$

that is, we have proved:

Proposition 3.1 *Let (u,A) be a solution of (3.3), (3.4) with $|u|\geq\frac{1}{2}$ on Ω. Then there holds*

$$\Delta e(u,A)\geq -Ce(u,A)^2-4\,|R|\,e(u,A), \qquad (3.19)$$

for some absolute constant C.

Observe that estimate (3.18) remains true for any $\epsilon \in]0,1]$ with the same constant C and the same bound for the curvature of Σ, as is easily seen by scaling.

As a consequence of Proposition 3.1 we derive

Theorem 3.2 (ε_0-regularity estimate) *Let (u, A) be a solution of equations (3.1) and (3.2) on B_{2R} for $\varepsilon = 1$ with $|u| \geq \frac{1}{2}$. There exists $\varepsilon_0 > 0$ and a constant $C_0 > 0$ with the following property: If $E(u, A; B_{2R}) < \varepsilon_0$, then*

$$\sup_{B_{R/2}} e(u, A) \leq C_0 R^{-2} E(u, A; B_R). \tag{3.20}$$

Proof. As in [17] and [24], we choose $r_0 < R$ such that

$$(R - r_0)^2 \sup_{B_{r_0}} e(u, A) = \max_{0 \leq r \leq R} \left\{ (R - r)^2 \sup_{B_r} e(u, A) \right\}$$

and let $x_0 \in \bar{B}_{r_0}$ be determined such that

$$e_0 := (e(u, A))\,(x_0) = \sup_{B_{r_0}} e(u, A).$$

Now we are going to prove that $e_0 \leq 4((R - r_0)^{-2})$. Assume by contradiction that $\varrho_0 = e_0^{-\frac{1}{2}} \leq \frac{R - r_0}{2}$. We rescale

$$v(x) = u(x_0 + \varrho_0 x),\ B(x) = \varrho_0 A(x_0 + \varrho_0 x),\ \nabla_B v = \varrho_0 \nabla_A u,\ dB = \varrho_0^2 dA.$$

Let

$$e_{\varrho_0}(v, B) = |\nabla_B v|^2 + \varrho_0^{-2} |dB|^2 + \frac{\varrho_0^2}{4} \left(1 - |v|^2\right)^2 = \varrho_0^2 e(u, A).$$

Then we have

$$1 = e_{\varrho_0}(v, B)(0)$$

while

$$\sup_{B_1} e_{\varrho_0}(v, B) = \varrho_0^2 \sup_{B_{\varrho_0}(x_0)} e(u, A) \leq \varrho_0^2 \sup_{B_{\frac{R + r_0}{2}}} e(u, A) \leq 4\varrho_0^2 \sup_{B_{r_0}} e(u, A) \leq 4.$$

From Proposition 3.1, we get

$$\Delta e_{\varrho_0} \geq -C e_{\varrho_0}$$

where C is a constant. Then using Moser's sup-estimate, we have

$$\begin{aligned} 1 = (e_{\varrho_0}(v, B))\,(0) &\leq C \int_{B_1} e_{\varrho_0}(v, B) dx \\ &= C \int_{B_{\varrho_0}} e(u, A) dx \\ &\leq C \int_{B_R} e(u, A) dx < 1, \end{aligned} \tag{3.21}$$

if $\varepsilon_0 = C^{-1}$. This is a contradiction. Hence

$$e_0 \left(\frac{R - r_0}{2} \right)^2 \leq 1,$$

and we get

$$\left(\frac{R}{2} \right)^2 \sup_{B_{R/2}} e(u, A) \leq (R - r_0)^2 e_0 \leq 4;$$

that is,

$$\sup_{B_{R/2}} e(v, A) \leq 16 R^{-2}.$$

Scaling with R instead of ϱ_0, the desired conclusion then follows from (3.20). □

We may assume that $\epsilon_0 < \epsilon_1$, where $\epsilon_1 > 0$ is the constant determined in Lemma 2.3. As a corollary we obtain a simple proof for the following result of Taubes [29].

Corollary 3.3 (Gap theorem) *Let (u, A) be a solution of equations (1.4) and (1.5) on $\mathbb{R}^2$. Then, if $E(u, A; \mathbb{R}^2) < \epsilon_0$, it follows that*

$$|u|^2 = 1, \ \nabla_A u = 0 \ \textit{and} \ dA = 0 \quad \textit{on} \ \mathbb{R}^2.$$

Proof. By Lemma 2.3 we have $|u| \geq \frac{1}{2}$ everywhere. Thus the claim follows if we let $R \to \infty$ in Theorem 3.2. □

4 Proof of Theorem 1.1

As in [25], [26], we obtain the concentration points x_j from the following Lemma.

For $0 < \varepsilon < \varepsilon_0$, $\varrho > 0$, and minimizers $(u_\varepsilon, A_\varepsilon)$ of E_ε, consider

$$\Sigma_\varepsilon = \{x \in \Omega : E_\varepsilon(u_\varepsilon, A_\varepsilon; B_\varrho(x)) \geq \varepsilon_0\}$$

and its cover $(B_\varrho(x))_{x \in \Sigma_\varepsilon}$. By Vitali's covering lemma there exists a finite collection of disjoint balls $B_i = B_\varrho(x_i)$, $x_i \in \Sigma_\varepsilon$, $1 \leq i \leq I = I(\varepsilon)$, such that

$$\bigcup_{x \in \Sigma_\varepsilon} B_\varrho(x) \subset \bigcup_i B_{5\varrho}(x_i).$$

Lemma 4.1 *Let ε_0 be given as in Theorem 3.2. There exists a number $J_0 = J_0(L)$ such that for any $\varrho > 0$ any disjoint collections of balls $B_\varrho(x_j)$, $x_j \in \Sigma$, $1 \leq j \leq J$, with $E(u_\varepsilon, A_\varepsilon; B_\varrho(x_j)) \geq \varepsilon_0$ there holds $J \leq J_0$.*

Proof. As the $(u_\varepsilon, A_\varepsilon)$ are solutions to the self-duality equations, there holds

$$E_\varepsilon(u_\varepsilon, A_\varepsilon) \le 2\pi \deg L =: C_1.$$

Hence

$$J \le \sum_i \frac{8E_\varepsilon\left(u_\varepsilon, A_\varepsilon; B_\varrho(x_i)\right)}{\varepsilon_0} \le 8C_1\varepsilon_0^{-1}.$$

□

As a consequence of Lemma 4.1, the number $I(\varepsilon)$ introduced above is bounded independently of ε and ϱ. In particular, we may choose $\varrho = \sqrt{\varepsilon}$. Then we may assume that $I(\varepsilon) = I_0$ is independent of ε and that $x_i = x_i^\varepsilon \to x_i^0$ as $\varepsilon \to 0$ for $1 \le i \le I_0$.

Let $\Sigma_0 = \{x_i^0; 1 \le i \le I_0\}$ be the set of limits of concentration points.

For $0 < \varepsilon < 1$ consider any point x_0 such that

$$\inf_i |x_0 - x_i^\varepsilon| \ge 5\sqrt{\varepsilon}.$$

If follows that $x_0 \notin \Sigma_\varepsilon$ and therefore

$$E_\varepsilon\big(u_\varepsilon, A_\varepsilon; B_{\sqrt{\varepsilon}}(x_0)\big) < \varepsilon_0.$$

Rescale $(u_\varepsilon, A_\varepsilon)$ around x_0. Note that we also have to rescale the Kähler metric and the rescaled metric converges smoothly locally to the standard metric on $\mathbb{R}^2$ as $\varepsilon \to 0$. The rescaled solutions $(v_\varepsilon, B_\varepsilon)$ are defined on the ball $B_{1/\sqrt{\varepsilon}}(0)$. Applying Theorem 3.2 with $R = \frac{1}{2\sqrt{\varepsilon}}$ we obtain

$$\frac{(1 - |u_\varepsilon|^2)}{4}(x_0) \le \sup_{B_{1/2\sqrt{\varepsilon}}(0)} e(v_\varepsilon, B_\varepsilon) \le C_0\varepsilon_0\varepsilon. \tag{4.1}$$

In particular, for any set $\Sigma' \subset\subset \Sigma \setminus \Sigma_0$ there holds

$$\sup_{\Sigma'} \left(1 - |u_\varepsilon|^2\right) \le C\varepsilon \to 0$$

as $\varepsilon \to 0$, and it follows that for small $\varepsilon > 0$ the total degree d of L, endowed with the connection A_ε, equals the sum of the local degrees d_i of (L, A_ε), restricted to a neighborhood of x_i^0. Finally, on a blown-up neighborhood $B_\varrho(x_i^0)$ of a point x_i^0, the rescaled equations (1.24), (1.26) approximate the vortex equations (1.10), (1.11) on $\mathbb{R}^2$. We thus expect the energy of $(u_\varepsilon, A_\varepsilon)$, restricted to any such ball $B_\varrho(x_i^0)$, to be bounded from below by the energy of a vortex configuartion on $\mathbb{R}^2$ with degree d_i.

In fact, we have

Lemma 4.2 *For any index i there holds*

$$\liminf_{\varepsilon \to 0} E_\varepsilon\big(u_\varepsilon, A_\varepsilon; B_{\sqrt{\varepsilon}}(x_i^0)\big) \ge 2\pi\, |d_i|\,.$$

Proof. Again denote by $(v_\varepsilon, B_\varepsilon)$ the rescaled solution, defined on a ball $B_{1/\sqrt{\varepsilon}}(0)$. Note that the difference from the scaled metric to the Euclidean metric vanishes as $\varepsilon \to 0$, uniformly on $B_{1/\sqrt{\varepsilon}}(0)$.

Hence we have

$$E_\varepsilon\big(u_\varepsilon, A_\varepsilon; B_{\sqrt{\varepsilon}}(x_i^0; \Sigma)\big) \geq E_1\big(v_\varepsilon, B_\varepsilon; B_{1/\sqrt{\varepsilon}}(0; \mathbb{R}^2)\big) + o(1)$$

where $o(1) \to 0$ as $\varepsilon \to 0$ and where we emphasize the use of the Kähler metric on Σ and the Euclidean metric on $\mathbb{R}^2$, respectively.

Suppose that $d_i \geq 0$. Then, via an integration by parts as in [18], p. 54, we obtain

$$\begin{aligned} E_1\big(v_\varepsilon,\, B_\varepsilon; B_{1/\sqrt{\varepsilon}}(0)\big) = & \int_{B_{1/\sqrt{\varepsilon}}(0)} \Big\{ \big|\bar{\partial}_{B_\varepsilon} v_\varepsilon\big|^2 + \Big|F_\varepsilon + \frac{1}{2}\big(|v_\varepsilon|^2 - 1\big)\Big|^2 \Big\}\, dx \\ & + \int_{\partial B_{1/\sqrt{\varepsilon}}(0)} \operatorname{Im}\big\langle v_\varepsilon, \tau \cdot \nabla_{B_\varepsilon} v_\varepsilon \big\rangle\, do + \int_{B_{1/\sqrt{\varepsilon}}(0)} F_\varepsilon\, dx, \end{aligned}$$

where $F_\varepsilon = *dB_\varepsilon$ is the curvature of B_ε and τ is a (positively oriented) unit tangent vector field along $\partial B_{1/\sqrt{\varepsilon}}(0)$. Note that by (4.1) we can write

$$v_\varepsilon = \varrho_\varepsilon e^{i\psi_\varepsilon} \qquad \text{on } \partial B_{1/\sqrt{\varepsilon}}(0)$$

with $\varrho_\varepsilon^2 \geq 1 - C\varepsilon$. Moreover,

$$\operatorname{Im}\big\langle v_\varepsilon, \tau \cdot \nabla_{B_\varepsilon} v_\varepsilon \big\rangle = \varrho_\varepsilon^2 \big(d\psi - B_\varepsilon(\tau)\big).$$

Thus, and using (4.1), we deduce that

$$\int_{\partial B_{1/\sqrt{\varepsilon}}(0)} \operatorname{Im}\big\langle v_\varepsilon, \tau \cdot \nabla_{B_\varepsilon} v_\varepsilon \big\rangle\, do = \int_{\partial B_{1/\sqrt{\varepsilon}}(0)} \big(d\psi - B_\varepsilon(\tau)\big)\, do - \mu(\varepsilon),$$

where

$$\begin{aligned} \mu(\varepsilon) &= \int_{\partial B_{1/\sqrt{\varepsilon}}(0)} (1 - \varrho_\varepsilon^2) \frac{\operatorname{Im}\big\langle v_\varepsilon, \tau \cdot \nabla_{B_\varepsilon} v_\varepsilon \big\rangle}{\varrho_\varepsilon^2}\, do \\ &\leq \frac{\pi}{\sqrt{\varepsilon}} \sup_{\partial B_{1/\sqrt{\varepsilon}}(0)} \Big\{ \big|1 - \varrho_\varepsilon^2\big|^2 + \frac{1}{\varrho_\varepsilon^2} \left|\nabla_{B_\varepsilon} v_\varepsilon\right|^2 \Big\} \\ &\leq C\sqrt{\varepsilon} \to 0 \qquad \text{as } \varepsilon \to 0. \end{aligned}$$

But on the other hand, by Stokes' theorem

$$\int_{\partial B_{1/\sqrt{\varepsilon}}(0)} B_\varepsilon(\tau)\, do = \int_{B_{1/\sqrt{\varepsilon}}(0)} F_\varepsilon\, dx.$$

Hence we find that

$$E_1\big(v_\varepsilon, B_\varepsilon; B_{1/\sqrt{\varepsilon}}(0)\big) \geq \int_{\partial B_{1/\sqrt{\varepsilon}}(0)} d\psi\, do - C\sqrt{\varepsilon} = 2\pi d_i - C\sqrt{\varepsilon}.$$

Similary, if $d_i < 0$ we find the asymptotic lower bound $2\pi\,|d_i|$, concluding the proof. □

From the estimate

$$\begin{aligned} 2\pi d &= E_\varepsilon(u_\varepsilon, A_\varepsilon; \Sigma) \geq \sum_i E_\varepsilon\big(u_\varepsilon, A_\varepsilon; B_{\sqrt{\varepsilon}}(x_i^0)\big) \\ &\geq 2\pi \sum_i |d_i| - o(1) \end{aligned}$$

and using the fact that

$$\sum_i d_i = d$$

we then deduce that $I_0 \leq d$ and $d_i > 0$ for each i. Moreover, for any compact subset Σ' of $\Sigma \setminus \Sigma_0$, the energy

$$E'_\varepsilon\,(u_\varepsilon, A_\varepsilon; \Sigma')$$

becomes arbitrarily small for sufficiently small $\varepsilon > 0$. Using Theorem 3.2 (observing that $|u| \geq \frac{1}{2}$ by Lemma 2.3) and scaling back, we then obtain that

$$|\nabla_{A_\varepsilon} u_\varepsilon|^2 + \frac{1}{4\varepsilon^2}\left(1 - |u|^2\right)^2 + \varepsilon^2\,|dA_\varepsilon|^2 \to 0 \tag{4.2}$$

uniformly on Σ' as $\varepsilon \to 0$. This implies already

$$|u_\varepsilon|^2 \to 1,\ \nabla_{A_\varepsilon} u_\varepsilon \to 0$$

as $\varepsilon \to 0$.

In order to show that the curvature of A_ε tends to 0 as well, we use the equation (1.21) and (1.26) to show the following estimate.

For $x_0 \in \Sigma'$ and $0 < \varrho < \operatorname{dist}(x_0, \partial\Sigma')$ let

$$\phi_\varepsilon(\varrho) = ⨍_{B_\varrho(x_0)} \frac{1 - |u_\varepsilon|^2}{\varepsilon^2}\, dx,$$

where $⨍_{B_\varrho(x_0)} \dots$ denotes the average over $B_\varrho(x_0)$. We may assume that $\varepsilon \leq \sqrt{\varepsilon} \leq \varrho_0 = \frac{1}{2}\operatorname{dist}(x_0, \partial\Sigma')$ and that $|u_\varepsilon|^2 \geq \frac{1}{2}$ on Σ'.

Lemma 4.3 $\sup_{\varepsilon \leq \varrho \leq \varrho_0} \phi_\varepsilon(\varrho) \to 0$ *as* $\varepsilon \to 0$.

Proof. Multiplying (1.21) by $\bar{u}$, we obtain the equation

$$\frac{1}{2}\Delta\big(1-|u|^2\big) + |\nabla_A u|^2 = \frac{|u|^2}{2\epsilon^2}\big(1-|u|^2\big) \geq \frac{1-|u|^2}{4\epsilon^2} \tag{4.3}$$

Here and in the following we write u for u_ε etc.

Let $\varphi \in C_0^\infty\big(B_2(0)\big)$ satisfy $0 \leq \varphi \leq 1$, $\varphi = 1$ on $B_1(0)$. For $\varepsilon \leq \varrho \leq \varrho_0$ we scale $\varphi_\varrho(x) = \varphi\left(\frac{x-x_0}{\varrho}\right)$.

Multiplying (4.3) by φ_ϱ and integrating by parts, we then obtain the estimate

$$\int_{B_\varrho} \frac{1-|u|^2}{\epsilon^2}\,dx \leq 4\int_{B_{2\varrho}(x_0)} |\nabla_A u|^2\,dx + C_0^2 \int_{B_{2\varrho}\setminus B_\varrho} \frac{1-|u|^2}{\varrho^2}\,dx$$

where $B_\varrho = B_\varrho(x_0)$, etc., and with a constant C_0 independent of ϱ and ε. Multiplying by $\frac{\varepsilon^2}{\varrho^2}$ and "filling the hole" on the right à la Widman [33] in a different context, we find

$$\begin{aligned}\big(1+C_0^2\frac{\epsilon^2}{\varrho^2}\big)\int_{B_\varrho}\frac{1-|u|^2}{\varrho^2}\,dx &\leq 16\pi\epsilon^2\sup_{\Sigma'}|\nabla_A u|^2 + C_0^2\frac{\epsilon^2}{\varrho^2}\int_{B_{2\varrho}}\frac{1-|u|^2}{\varrho^2}\,dx\\ &= \epsilon^2 o(1) + 4C_0^2\frac{\epsilon^2}{\varrho^2}\int_{B_{2\varrho}}\frac{1-|u|^2}{4\varrho^2}\,dx,\end{aligned}$$

where $o(1) \to 0$ as $\epsilon \to 0$ independent of ϱ.

For $\epsilon \leq \varrho \leq \varrho_0$ let

$$\psi_\epsilon(\varrho) = \int_{B_\varrho}\frac{1-|u|^2}{\varrho^2}\,dx.$$

Then we have

$$\psi_\varepsilon(\varrho) \leq \varepsilon^2 o(1) + \frac{4C_0^2\varepsilon^2}{\varrho^2 + C_0^2\varepsilon^2}\psi_\varepsilon(2\varrho).$$

Note that for $2C_0\varepsilon \leq \varrho$ there holds

$$\frac{4C_0^2\varepsilon^2/\varrho^2}{1+C_0^2\varepsilon^2/\varrho^2} \leq \frac{4}{5} = \theta$$

and hence

$$\psi_\varepsilon(\varrho) \leq \varepsilon^2 o(1) + \theta\psi_\varepsilon(2\varrho).$$

By iteration, for $2C_0\varepsilon \leq \varrho \leq \bar{\varrho} \leq \frac{\varrho_0}{2}$ therefore we obtain that

$$\psi_\varepsilon(\varrho) \leq \varepsilon^2 o(1) + \psi_\varepsilon(\bar{\varrho}). \tag{4.4}$$

Moreover, for $\varrho = \varrho_0$ we have

$$\begin{aligned}\psi_\varepsilon(\varrho_0) &\leq \int_{B_{\varrho_0}}\frac{1-|u|^2}{\varrho_0^2}\,dx = \pi\varepsilon\fint_{B_{\varrho_0}}\frac{1-|u|^2}{\varepsilon}\,dx\\ &\leq \pi\varepsilon\sup_{\Sigma'}\frac{1-|u|^2}{\varepsilon} \leq \varepsilon\,o(1) \leq \varrho_0^2 o(1).\end{aligned}$$

Hence for $\bar\varrho = \frac{1}{2}\varrho_0$ there holds

$$\psi_\varepsilon(\bar\varrho) \le \varepsilon^2 o(1) + 4C_0^2 \frac{\varepsilon^2}{\bar\varrho^2}\psi_\varepsilon(\varrho_0) \le \varepsilon^2 o(1). \tag{4.5}$$

Combining estimates (4.4) and (4.5), we find

$$\sup_{2C_0\varepsilon \le \varrho \le \varrho_0} \phi_\varepsilon(\varrho) \to 0 \qquad \text{as } \varepsilon \to 0.$$

Finally, observe that for $\varepsilon \le \varrho \le 2C_0\varepsilon$ there holds

$$\phi_\varepsilon(\varrho) \le 4C_0^2 \phi_\varepsilon(2C_0\varepsilon).$$

The claim follows. □

Lemma 4.4 $\sup_{\Sigma'} \frac{1-|u_\varepsilon|^2}{\varepsilon^2} \to 0$ *as* $\varepsilon \to 0$.

Proof. From (4.3) we have for $u = u_\varepsilon$, etc.,

$$-2\varepsilon^2 \Delta \left(\frac{1-|u|^2}{\varepsilon^2}\right) + \frac{1-|u|^2}{\varepsilon^2} \le 4\left|\nabla_A u\right|^2.$$

Fix $x_0 \in \Sigma'$ and suppose $\varepsilon \le \varrho_0 = \frac{1}{2}\operatorname{dist}(x_0, \partial\Sigma')$.

Let

$$g = \frac{1-|u|^2}{\varepsilon^2} \colon B_\varepsilon(x_0) \to \mathbb{R},$$

and for $\delta \ge \sup_{\Sigma'} \left|\nabla_A u\right|^2$ define

$$\bar g(x) = g(x_0 + \varepsilon x) + \delta\left|x\right|^2,$$

satisfying the equation

$$-2\Delta\bar g + \bar g \le 4\big(\left|\Delta_A u\right|^2 - \delta\big) \le 0 \quad \text{in } B_1(0).$$

That is, $\bar g > 0$ is a sub-solution for the operator $(-2\Delta + 1)$ on $B_1(0)$ and Moser's sup-estimate and Lemma 4.2 imply that

$$\begin{aligned}\left(\frac{1-|u|^2}{\varepsilon^2}\right)(x_0) = \bar g(0) &\le C\int_{B_1(0)} \bar g\, dx \\ &\le C \fint_{B_\varepsilon(x_0)} \frac{1-|u|^2}{\varepsilon^2}\, dx + C\delta \le o(1) + C\delta,\end{aligned}$$

where $o(1) \to 0$ as $\varepsilon \to 0$.

Since we may let $\delta \to 0$ as $\varepsilon \to 0$, the claim follows. □

Proof of Theorem 1.1 (completed): From Lemma 4.3 and equation (1.26) we now immediately deduce the asserted convergence

$$\sup_{\Sigma'} |\Lambda F_\varepsilon| \leq C \sup_{\Sigma'} \frac{1 - |u_\varepsilon|^2}{\varepsilon^2} \to 0$$

as $\varepsilon \to 0$.

Finally, since the total topological charge

$$\int_\Sigma \Lambda F_\varepsilon = 2\pi \deg L$$

is independent of ε, the preceding blow-up analysis and convergence result imply that

$$\Lambda F_\varepsilon \rightharpoonup 2\pi \sum_{j=1}^{l} d_j \delta(x_j)$$

in the sense of measure, as claimed.

This concludes the proof of Theorem 1.1. □

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