“BUBBLING” OF THE PRESCRIBED CURVATURE FLOW ON THE TORUS

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Abstract. By a classical result of Kazdan-Warner, for any smooth sign-changing function f with negative mean on the torus (M, g_b) there exists a conformal metric $g = e^{2u}g_b$ with Gauss curvature $K_g = f$, which can be obtained from a minimizer u of Dirichlet's integral in a suitably chosen class of functions. As shown by Galimberti, these minimizers exhibit “bubbling” in a certain limit regime. Here we sharpen Galimberti's result by showing that all resulting “bubbles” are spherical. Moreover, we prove that analogous “bubbling” occurs in the prescribed curvature flow.

1. Background and results

1.1. **The Kazdan-Warner result.** Let (M, g_b) be a closed surface of genus zero. A classical result of Kazdan-Warner [18] characterizes those smooth functions f on M for which there exists a conformal metric $g = e^{2u}g_b$ on M with Gauss curvature $K_g = f$. By the uniformization theorem, with no loss of generality we may assume that the background metric g_b is flat with $K_{g_b} = 0$ and has volume $vol(M, g_b) = 1$. In view of the Gauss equation

$$K_g = e^{-2u}(-\Delta_{g_b} u + K_{g_b}) = -e^{-2u}\Delta_{g_b} u \tag{1.1}$$

we are then led to study the equation

$$-\Delta_{g_b} u = fe^{2u} \quad \text{on } M. \tag{1.2}$$

Theorem 1.1. *(Kazdan-Warner [17]) There exists a solution u of (1.2) if and only if either $f \equiv 0$, or if the function f changes sign and satisfies*

$$\int_M f d\mu_{g_b} < 0. \tag{1.3}$$

Leaving aside the trivial case when $f \equiv 0$ with corresponding solution $u \equiv 0$ of (1.2), in the case when f changes sign and satisfies (1.3) a solution u to (1.2) may be obtained by minimizing the Liouville energy (or Dirichlet integral)

$$E(u) = \frac{1}{2}\int_M |\nabla u|^2_{g_b}\, d\mu_{g_b}$$

in the class of functions

$$\mathcal{C} = \mathcal{C}_f = \{u \in H^1(M, g_b)\,;\ \int_M fe^{2u}\, d\mu_{g_b} = 0\}.$$

Observe that the constraint

$$\int_M f\, d\mu_g = \int_M f\, e^{2u}\, d\mu_{g_b} = 0, \tag{1.4}$$

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where $g = e^{2u} g_b$, is natural in view of (1.2) or the Gauss-Bonnet theorem.

Since both the energy $E(u)$ and the constraint (1.4) are left unchanged if we replace a function $u \in \mathcal{C}$ with $u + c$ for any $c \in \mathbb{R}$, in order to show existence of a minimizer for E in $\mathcal{C}$ traditionally one restricts attention to comparison functions $u \in \mathcal{C}$ with vanishing mean; see for instance Chang [8], pp. 2-4. However, normalizing the volume

$$vol(M, g) = \int_M d\mu_g = \int_M e^{2u}\, d\mu_{g_b} = 1 = vol(M, g_b) \tag{1.5}$$

of comparison functions will work equally well.

1.2. **"Bubbling" metrics.** Following the argument of [3] for surfaces of genus larger than one, Galimberti [15] showed "bubbling" of the Kazdan-Warner metrics in a certain limit regime. To describe his results, let f_0 be a smooth, non-constant function with $\max_{p \in M} f_0(p) = 0$, and for any $\lambda \in \mathbb{R}$ let $f_\lambda = f_0 + \lambda$. Then for any sufficiently small $\lambda > 0$ the function f_λ changes sign and satisfies (1.3). By Theorem 1.1 therefore there exists a solution $\hat{u}_\lambda$ of (1.2) which can be obtained as $\hat{u}_\lambda = u_\lambda + c_\lambda$ from a minimizer u_λ of E in the set $\mathcal{C}_\lambda = \mathcal{C}_{f_\lambda}$ satisfying (1.5).

Setting

$$\beta_\lambda := E(u_\lambda) = \min\{E(u);\ u \in \mathcal{C}_\lambda\},$$

then it follows from Theorem 1.1 that $\beta_\lambda \to \infty$ as $\lambda \downarrow 0$; see also (4.3) below. Moreover, with a delicate argument Galimberti is able to show that β_λ is non-increasing as a function of λ, which as in [3] then allows to control the total curvature of the metrics $\hat{g}_\lambda = e^{2\hat{u}_\lambda} g_b$ for suitable $\lambda \downarrow 0$ and to show that after rescaling the metrics suitably near local maximum points $p_\lambda^{(i)}$ of $\hat{u}_\lambda$ one or more "bubbles" may be extracted from $\hat{g}_\lambda$; see [15], Theorem 1.1.

One of our goals in the present paper is to better understand this "bubbling" behavior and to obtain the following more precise characterization.

Theorem 1.2. *Let $f_0 \le 0$ be a smooth, non-constant function with $\max f_0 = 0$ having only non-degenerate maxima p_0 where $f_0(p_0) = 0$. Then for suitable $\lambda_k \downarrow 0$, for $u_k = u_{\lambda_k}$ as above and suitable $i_0 \in \mathbb{N}$, $r_k^{(i)} \downarrow 0$, $p_k^{(i)} \to p_\infty^{(i)} \in M$ with $f_0(p_\infty^{(i)}) = 0$, $1 \le i \le i_0$, as $k \to \infty$ the following holds.*

i) We have $u_k \to -\infty$ locally uniformly on $M_\infty = M \setminus \{p_\infty^{(i)};\ 1 \le i \le i_0\}$.

ii) For each $1 \le i \le i_0$ there holds $r_k^{(i)}/\sqrt{\lambda_k} \to 0$, and in local Euclidean coordinates x around $p_k^{(i)} = 0$ with constants $c_k^{(i)} \to \infty$ we have

$$w_k^{(i)}(x) := u_k(r_k^{(i)} x) - c_k^{(i)} \to w_\infty(x) = \log\Big(\frac{2}{1 + |x|^2}\Big),$$

in H^2_{loc} on $\mathbb{R}^2$, where w_∞ induces the standard spherical metric $g_\infty = e^{2w_\infty} g_{\mathbb{R}^2}$ on $\mathbb{R}^2$ of curvature $K_{g_\infty} \equiv 1$, and $1 \le i_0 \le 2$.

Theorem 1.2, which is a special case of our Theorem 4.4 below, improves Galimberti's [15] Theorem 1.1, in particular, by ruling out the "slow blow-up" allowed in Galimberti's theorem. This is achieved with the help of the following Liouville-type result.

Theorem 1.3. *Suppose A is a negative definite and symmetric 2×2-matrix. Then there is no solution $w \in C^\infty(\mathbb{R}^2)$ of the equation*

$$-\Delta w = (1 + (Ax, x))e^{2w} \ \text{ on } \mathbb{R}^2$$

with $w \le C$ *and such that the induced metric* $h = e^{2w} g_{\mathbb{R}^2}$ *has finite volume and integrated curvature*

$$\int_{\mathbb{R}^2} e^{2w}\, dx < \infty, \quad \int_{\mathbb{R}^2} (1 + (Ax, x)) e^{2w}\, dx \in \mathbb{R}.$$

Section 4 of this paper, where we give the proof of these results, also contains a simplified proof of the crucial monotonicity property of β_λ.

In the same way as it improves Galimberti's result, Theorem 1.3 gives rise to an improvement of our earlier results in [3] on "bubbling" metrics of prescribed curvature on surfaces of genus > 1 by showing that also in the higher genus case all blow-ups must be spherical. The construction of "bubbling" solutions for the latter problem by del Pino-Roman [12], prompted by [3], thus might give a complete description of the set of all prescribed curvature metrics in that case. It would be nice to see if "bubbling" metrics of prescribed curvature could be constructed in analogous fashion via matched asymptotic expansion also in the genus zero case.

Our method for proving Theorem 1.2 is so robust that it may be applied also in the presence of perturbations. In particular, we are able to show analogous "bubbling" as $\lambda \downarrow 0$ also for a family of prescribed curvature flows for f_λ with suitably chosen initial data in $\mathcal{C}_\lambda$. This is our second goal in this paper.

1.3. **Prescribed curvature flow.** Given a function f satisfying the assumptions of Theorem 1.1, for any $u_0 \in \mathcal{C} = \mathcal{C}_f$ we study the equation

$$u_t = \alpha f - K \quad \text{on } M \times [0, \infty[\tag{1.6}$$

with initial data $u\big|_{t=0} = u_0$, where $K = K_g$ with $g = g(t) = e^{2u(t)} g_b$ at any time $t \ge 0$ and where the function $\alpha = \alpha(t)$ is determined so that $u(t) \in \mathcal{C}$ for all $t \ge 0$; that is, we require the condition

$$\frac{1}{2}\frac{d}{dt}\Big(\int_M f\, d\mu_g\Big) = \int_M u_t f\, d\mu_g = \int_M (\alpha f - K) f\, d\mu_g = 0. \tag{1.7}$$

Solving for α then we find

$$\alpha = \int_M fK\, d\mu_g \Big/ \int_M f^2\, d\mu_g. \tag{1.8}$$

The flow (1.6)-(1.7) is the negative L^2-gradient flow for E on $\mathcal{C}$ with respect to the evolving metrics. Indeed, for a sufficiently smooth solution $u = u(t)$ of (1.6)-(1.7) on an interval $[0, T]$ by (1.1) there holds

$$\frac{d}{dt} E(u(t)) = -\int_M u_t \Delta_{g_b} u\, d\mu_{g_b} = \int_M (\alpha f - K) K\, d\mu_g = -\int_M |u_t|^2\, d\mu_g \le 0. \tag{1.9}$$

Hence we also have the uniform a-priori bound

$$E(u(T)) + \int_0^T \int_M |u_t|^2\, d\mu_g\, dt \le E(u_0) \tag{1.10}$$

for any $T > 0$. Also note that by (1.4) and the Gauss-Bonnet theorem for a solution u of (1.6)-(1.7) the volume is preserved with

$$\frac{1}{2}\frac{d}{dt} vol(M, g(t)) = \int_M u_t\, d\mu_g = \alpha \int_M f\, d\mu_g - \int_M K\, d\mu_g = 0. \tag{1.11}$$

Normalizing the initial data $g_0 = e^{2u_0} g_b$ to satisfy $vol(M, g_0) = 1$, then we have

$$vol(M,g) = \int_M d\mu_g = \int_M e^{2u}\, d\mu_{g_b} = 1 \tag{1.12}$$

for all $t > 0$, and we see that (1.6)-(1.7) induces a flow in the space

$$\mathcal{C}^* = \mathcal{C}^*_f = \{u \in H^1(M, g_b)\,;\ \int_M f e^{2u}\, d\mu_{g_b} = 0,\ \int_M e^{2u}\, d\mu_{g_b} = 1\}.$$

We then have the following result, which is parallel to yet unpublished, independent results of Ngô and Xu [21].

Theorem 1.4. *Suppose that f is smooth, changes sign, and satisfies* (1.3). *Then for any smooth $u_0 \in \mathcal{C}^*$ there exists a unique, global smooth solution u of* (1.6)-(1.7) *with initial data $u|_{t=0} = u_0$ and satisfying $u(t) \in \mathcal{C}^*$ as well as the energy bound $E(u(t)) \le E(u_0)$ for all t. Moreover, we have $u(t) \to u_\infty$ in $H^2(M, g_b)$ (and smoothly) as $t \to \infty$ suitably, where $u_\infty + c_\infty$ is a smooth solution of* (1.2) *for some $c_\infty \in \mathbb{R}$.*

While Theorem 1.4 shows that for *fixed* f as in Theorem 1.1 there is a global, smooth solution of the flow (1.6)-(1.7) for any given smooth $u_0 \in \mathcal{C}^* = \mathcal{C}^*_f$, in Section 5 below we observe that, similar to the time-independent case described in Theorem 1.2, for suitable $\lambda_k \downarrow 0$ and suitable initial data $u_{0k} \in \mathcal{C}^*_{f_{\lambda_k}}$ the *family* of flows (1.6)-(1.7) with prescribed curvature functions f_{λ_k} exhibits "bubbling" as $k \to \infty$ when t is large in the following sense.

Theorem 1.5. *Let f_0 be as in Theorem 1.2 above. Then for suitable $\lambda_k \downarrow 0$, suitable $u_{0k} \in \mathcal{C}^*_{f_{\lambda_k}}$ with $E(u_{0k}) - \beta_{\lambda_k} \le \delta_k^2 \downarrow 0$, and sufficiently large $t_k \ge T_k \to \infty$ as $k \to \infty$ the conclusions of Theorem 1.2 hold for $u_k = u^{\delta_k}_{\lambda_k}(t_k)$, where $u^{\delta_k}_{\lambda_k}(t)$ is the solution to* (1.6)-(1.7) *for f_{λ_k} with data $u^{\delta_k}_{\lambda_k}(0) = u_{0k}$, $k \in \mathbb{N}$.*

1.4. **Related work, open problems.** Our interest in the study of (1.6)-(1.7) is inspired by a recent preprint of Ngô and Xu [22] on a flow approach to the results of Escobar-Schoen [14] on the Kazdan-Warner [18] problem of prescribed scalar curvature on a closed Riemannian manifold of dimension $n \ge 3$ with vanishing Yamabe invariant.

Also in $n = 2$ dimensions many open questions remain. For instance, in Section 6, we speculate about possible consequences of "bubbling" for the nature of convergence of the flow (1.6)-(1.7). To put this question into perspective, recall that in the case when $f \equiv 0$ the flow (1.6) corresponds to the 2-dimensional Ricci flow on (M, g_b) for which Hamilton [16] established global existence and exponentially fast convergence. [1] Also in the setting of Theorem 1.4 our Theorem 3.2 below shows that the convergence is exponentially fast when u_∞ is a strict relative minimizer of E in $\mathcal{C}^*$, which nicely complements the result of Hamilton for $f \equiv 0$ (where for a flat background metric we have $u_\infty = 0$). Not surprisingly, we also have unconditional convergence of the flow when the solution of (1.2) is unique; see our Theorem 3.3 below. Moreover, with the help of the Łojasiewicz-Simon inequality Ngô and Xu [21] showed unconditional convergence of the prescribed curvature flow on the torus for any given analytic sign-changing function f with negative average.

[1] In fact, Hamilton [16] shows global existence and exponentially fast convergence of the normalized Ricci flow on any closed surface (M, g_0). For the sphere his work was completed by Chow [11]; see also [26] for a simpler proof of exponentially fast convergence in this case.

In contrast, the nature of convergence is not clear if f is only assumed to be smooth (or even less regular), and spiralling "bubbles" might give rise to infinitely many different subsequential limits of the prescribed curvature flow for f_λ as $t \to \infty$ for suitable f_0 having degenerate maxima and sufficiently small $\lambda > 0$. As a first step towards a better understanding of this case, in Theorem 6.1 we apply our methods from Section 4 to analyze "bubbling" in such a degenerate case.

Moreover, it would be interesting to investigate "bubbling" of prescribed curvature metrics and of the prescribed curvature flow also in the case when f consists of a regular part and a sum of Dirac masses, where solutions of (1.2) would correspond to conical metrics of prescribed curvature with prescribed opening angles, as in the work of Troyanov [30], Bartolucci-De Marchis-Malchiodi [1], Malchiodi-Ruiz [19], and Carlotto-Malchiodi [7]; or in the case of Chern-Simons vortices, as in the work of Bartolucci and Tarantello [2], [28]. Likewise, it would be of interest to investigate "bubbling" of metrics of prescribed Q-curvature in arbitrary even dimensions $n \geq 4$, or in the corresponding, higher order prescribed Q-curvature flow introduced by Brendle [4].

1.5. **Outline.** In the following Sections 2-3 we first complete the analysis of the flow (1.6)-(1.7) and give the proof of Theorem 1.4 as well as of the convergence results alluded to above. The main Section 4 then contains the details of our "bubble" analysis, and in Section 5 we show that these results may be applied to yield corresponding results on "bubbling" of the flow (1.6), (1.7). Finally, Section 6 contains a first study of the degenerate case.

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2. Global existence of the prescribed curvature flow

Given a function f as in Theorem 1.4 and smooth initial data $u_0 \in \mathcal{C}^*$ we now show that we can find a unique smooth solution u of (1.6)-(1.7) defined for all time.

Note that for any $u \in \mathcal{C}^*$ Jensen's inequality gives the bound

$$2\bar{u} := 2\int_M u\, d\mu_{g_b} \leq \log\Big(\int_M e^{2u}\, d\mu_{g_b}\Big) = 0 \tag{2.1}$$

for the average of u.

Next recall the following result of Trudinger [31], sharpened by Moser [20].

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^2$ be bounded. For any $\beta \leq 4\pi$ there holds*

$$\sup\{\int_\Omega e^{u^2}\, dx;\ u \in H_0^1(\Omega), \|\nabla u\|^2_{L^2} \leq \beta\} < \infty,$$

and the constant $\beta_0 = 4\pi$ is best possible.

In fact, for any bounded domain $\Omega \subset \mathbb{R}^2$, any $u \in H_0^1(\Omega)$ with $\|\nabla u\|^2_{L^2} \leq \beta < 4\pi$ it is not hard to see that

$$\frac{1}{\mu(\Omega)}\int_\Omega e^{u^2}\, dx \leq \frac{4\pi}{4\pi - \beta}.$$

Using a partition of unity one can obtain a similar result on any closed surface; see the lecture notes of Chang [8], or [6], Theorem 4.4, for reference.

Theorem 2.2. *Let (M, g_0) be closed and orientable. Then for any $\beta < 4\pi$ there holds*

$$C_{TM}(\beta) = \sup\{\int_M e^{u^2}\, d\mu_{g_0};\ u \in H^1(M, g_0),\ \|\nabla u\|^2_{L^2} \le \beta,\ \bar{u} = 0\} < \infty.$$

Applying the bounds in Theorem 2.2 to a function $u \in H^1(M, g_0)$, and estimating

$$2|p(u - \bar{u})| \le 2\pi(u - \bar{u})^2 / \|\nabla u\|^2_{L^2(M,g_0)} + \frac{p^2}{2\pi}\|\nabla u\|^2_{L^2(M,g_0)}$$

via Young's inequality, for any $p \in \mathbb{R}$ we then find

$$\int_M e^{2p(u-\bar{u})}\, d\mu_{g_0} \le C_{TM}(2\pi) e^{\frac{p^2}{2\pi}\|\nabla u\|^2_{L^2(M,g_0)}}. \tag{2.2}$$

In particular, on the torus (M, g_b) for any $u \in H^1(M, g_b)$ satisfying (1.12) we can bound

$$1 = \int_M d\mu_g = \int_M e^{2u}\, d\mu_{g_b} = e^{2\bar{u}} \int_M e^{2(u-\bar{u})}\, d\mu_{g_b} \le C e^{2\bar{u}}$$

with a constant $C = C(E(u))$, and we conclude the uniform lower bound

$$\bar{u} \ge -m_0 \tag{2.3}$$

for the average of u with some $m_0 = m_0(E(u)) > 0$. Moreover, for any $p \in \mathbb{R}$ the bounds (2.1) and (2.3) together with (2.2) give

$$\int_M e^{2pu}\, d\mu_{g_b} = e^{2p\bar{u}} \int_\Omega e^{2p(u-\bar{u})}\, d\mu_{g_b} \le C(p, E(u)). \tag{2.4}$$

Throughout the remainder of this section suppose that $u = u(t)$ solves (1.6)-(1.7) with initial data $u\big|_{t=0} = u_0 \in \mathcal{C}^* = \mathcal{C}^*_f$ for some smooth, sign-changing function f on M satisfying (1.3). Note that by (1.10) the bounds (2.3), (2.4) then hold with uniform constants $m_0 = m_0(E(u_0)) > 0$, $C(p, E(u_0))$, respectively.

Lemma 2.3. *There exists a constant $m_1 > 0$ such that for all $t > 0$ there holds $\int_M f^2\, d\mu_g \ge m_1$.*

Proof. By Hölder's inequality, assumption (1.3), and (2.4) we have

$$0 < \big|\int_M f\, d\mu_{g_b}\big|^2 \le \int_M e^{-2u}\, d\mu_{g_b} \int_M f^2 e^{2u}\, d\mu_{g_b} \le C \int_M f^2\, d\mu_g,$$

uniformly in $t > 0$. □

As an immediate consequence we obtain the following result.

Lemma 2.4. *There exists a constant $\alpha_0 > 0$ such that for all $t > 0$ there holds $|\alpha| \le \alpha_0$.*

Proof. By (1.1) and (1.10) we can estimate

$$\begin{aligned}\big|\int_M fK\, d\mu_g\big| &= \big|\int_M f(-\Delta_{g_b} u)\, d\mu_{g_b}\big| \le \int_M |\nabla f|_{g_b} |\nabla u|_{g_b}\, d\mu_{g_b} \\ &\le C\|f\|_{C^1} E^{1/2}(u) \le C E^{1/2}(u_0) =: C_0.\end{aligned}$$

From (1.8) and Lemma 2.3 it then follows that

$$|\alpha| \le C_0 m_1^{-1} =: \alpha_0,$$

as claimed. □

The above bounds suffice to show existence of a unique solution u to (1.6), (1.7) for all time. Indeed, using (1.1) to write equation (1.6) in the form

$$u_t = \alpha f - K = \alpha f + e^{-2u}\Delta_{g_b} u = \alpha f + \Delta_g u, \tag{2.5}$$

from Lemma 2.4 we conclude that

$$|u_t - \Delta_g u| = |\alpha f| \le |\alpha_0| \, \|f\|_{L^\infty} =: C_1,$$

uniformly in $t > 0$, and by the maximum principle, applied to the function $u \pm C_1 t$, it follows that

$$\sup_M |u(t)| \le \sup_M |u_0| + C_1 t \tag{2.6}$$

for all $t > 0$. In particular, equation (1.7) is uniformly parabolic on any finite time interval; therefore, the unique local solution u to (1.6), (1.7) that we can construct with the help of a standard fixed point argument (as explained in detail for instance in [6], Proposition 6.3) may be extended globally.

In fact, with the help of (1.10) we can turn (2.6) into a uniform estimate for all time. For convenience, let

$$F = F(t) = \int_M |K - \alpha f|^2 \, d\mu_g, \; G = G(t) = \int_M |\nabla(K - \alpha f)|_g^2 \, d\mu_g.$$

Then we have the following result.

Lemma 2.5. *There holds* $\sup_{t>0} \|u(t)\|_{L^\infty} < \infty$.

Proof. By (1.10) we have

$$\int_0^\infty F(t)\, dt \le E(u_0) < \infty.$$

Hence for any $T > 0$ we can find $t_T \in [T, T+1]$ such that

$$F(t_T) = \inf_{T<t<T+1} F(t) \le E(u_0).$$

But then in view of the uniform bound $\|e^u\|_{L^6(M,g_b)} \le C(3, E(u_0))$ from (2.4), and writing (1.1) in the form

$$-\Delta_{g_b} u = K e^{2u} = (K - \alpha f)e^u \cdot e^u + \alpha f e^{2u},$$

at time t_T, with uniform constants $C > 0$ by Hölder's inequality we have

$$\begin{aligned} \|\Delta_{g_b} u\|_{L^{3/2}(M,g_b)} &\le \|(K - \alpha f)e^u\|_{L^2(M,g_b)} \|e^u\|_{L^6(M,g_b)} + C \\ &\le C F(t_T)^{1/2} + C \le C. \end{aligned}$$

Elliptic regularity then yields

$$\|u - \bar{u}\|_{L^\infty} \le C\|\nabla^2 u\|_{L^{3/2}(M,g_b)} \le C\|\Delta_{g_b} u\|_{L^{3/2}(M,g_b)} \le C,$$

and in view of (2.1) and (2.3) we conclude the uniform bound $\|u(t_T)\|_{L^\infty} \le C$. Upon shifting time by t_T, from (2.6) we now find that

$$\sup_{T+1 \le s \le T+2} \|u(t)\|_{L^\infty} \le \sup_{t_T \le s \le T+2} \|u(t)\|_{L^\infty} \le \|u(t_T)\|_{L^\infty} + 2C_1 \le C. \tag{2.7}$$

Since $T > 0$ is arbitrary, the claim follows. □

Thus, in particular, the metrics $g(t)$ will be uniformly equivalent to the background metric g_b for all $t > 0$.

3. Convergence

In order to study convergence of the flow we consider the evolution of curvature. From (1.1) and (1.6) it follows that

$$K_t = \frac{d}{dt}(-e^{-2u}\Delta_{g_b}u) = -2u_tK - \Delta_g u_t = 2K(K-\alpha f) + \Delta_g(K-\alpha f). \tag{3.1}$$

Thus, and again using (1.6), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_M |K-\alpha f|^2\,d\mu_g &= \int_M \big((K_t-\alpha_t f)(K-\alpha f) - (K-\alpha f)^3\big)\,d\mu_g\\ &= -\int_M |\nabla(K-\alpha f)|_g^2\,d\mu_g + 2\int_M K(K-\alpha f)^2\,d\mu_g - \int_M (K-\alpha f)^3\,d\mu_g\\ &= -\int_M |\nabla(K-\alpha f)|_g^2\,d\mu_g + 2\alpha\int_M f(K-\alpha f)^2\,d\mu_g + \int_M (K-\alpha f)^3\,d\mu_g,\end{aligned} \tag{3.2}$$

where we observe that the term involving α_t vanishes due to (1.7). Using (1.6) to write $\alpha f - K = u_t$ for brevity, by Hölder's inequality we can bound the last term

$$\|u_t\|^3_{L^3(M,g)} \le \|u_t\|_{L^2(M,g)}\|u_t\|^2_{L^4(M,g)}.$$

In view of Lemma 2.5, by the Gagliardo-Nirenberg inequality then with uniform constants $C > 0$ for any $t > 0$ we have

$$\begin{aligned}\|u_t\|^2_{L^4(M,g)} &\le C\|u_t\|^2_{L^4(M,g_b)} \le C\|u_t\|_{L^2(M,g_b)}\|u_t\|_{H^1(M,g_b)}\\ &\le C\|u_t\|_{L^2(M,g)}\|u_t\|_{H^1(M,g)}.\end{aligned}$$

Recalling the notation

$$F(t) = \int_M |u_t|^2\,d\mu_g, \quad G(t) = \int_M |\nabla u_t|_g^2\,d\mu_g = \int_M |\nabla u_t|^2_{g_b}\,d\mu_{g_b},$$

by Young's inequality finally we can bound

$$\|u_t\|^3_{L^3(M,g)} \le C\|u_t\|^2_{L^2(M,g)}\|u_t\|_{H^1(M,g)} \le CF^2 + \frac{1}{2}(F+G). \tag{3.3}$$

Together with the uniform bound $|\alpha f| \le C_1$ from Lemma 2.4, equation (3.2) then gives the differential inequality

$$\frac{dF}{dt} + G \le C_2F + C_3F^2 \quad \text{on } [0,\infty[\tag{3.4}$$

with uniform constants $C_2 = 4C_1 + 1, C_3 > 0$.

Lemma 3.1. *We have $F(t) \to 0$ uniformly as $t \to \infty$.*

Proof. Recalling the bound

$$\int_0^\infty F(t)\,dt \le E(u_0) \tag{3.5}$$

from (1.10), we have $\liminf_{t\to\infty} F(t) = 0$. Hence there exist $t_i \to \infty$ with $F(t_i) \to 0$ as $i \to \infty$. Upon integrating (3.4) over any interval $[t_i, t] \subset [t_i, T]$ we then find

$$\sup_{t_i<t<T} F(t) \le F(t_i) + (C_2 + C_3 \sup_{t_i<t<T} F(t))\int_{t_i}^\infty F(t)\,dt.$$

But by (3.5) we also have $\int_{t_i}^{\infty} F(t)\,dt \to 0$ as $i \to \infty$. Hence, for sufficiently large $i \in \mathbb{N}$ we can absorb the last term on the right on the left hand side of this inequality and let $T \to \infty$ to find

$$\sup_{t>t_i} F(t) \le 2F(t_i) + 2C_2 \int_{t_i}^{\infty} F(t)\,dt \to 0 \quad \text{as } i \to \infty,$$

proving our claim. □

Proof of Theorem 1.4. In view of (1.10) and Lemma 2.4 for suitable $t_i \to \infty$ we have $u_i = u(t_i) \to u_\infty$ weakly in $H^1(M, g_b)$ and strongly in $L^2(M, g_b)$, as well as $\alpha_i = \alpha(t_i) \to \alpha_\infty$ $(i \to \infty)$. Moreover, in view of Lemmas 2.5 and 3.1 we then also have convergence $e^{2u_i} \to e^{2u_\infty}$ in $L^p(M, g_b)$ for any $p < \infty$, as well as $e^{2u_i} u_t(t_i) \to 0$ in $L^2(M, g_b)$. Thus, passing to the limit $i \to \infty$ in the equation

$$e^{2u} u_t - \Delta_{g_b} u = \alpha f e^{2u}$$

equivalent to (1.6) at $t = t_i$, we find the identity

$$-\Delta_{g_b} u_\infty = \alpha_\infty f e^{2u_\infty} \quad \text{on } M.$$

In fact, since at $t = t_i$ by Lemmas 2.5 and 3.1 with $L^2 = L^2(M, g_b)$, etc., we can estimate

$$\begin{aligned} \|\Delta_{g_b}(u_i - u_\infty)\|_{L^2} &\le \|e^{u_i}\|_{L^\infty} F(t_i)^{1/2} + \|\alpha_i f e^{2u_i} - \alpha_\infty f e^{2u_\infty}\|_{L^2} \\ &\le \|e^{u_i}\|_{L^\infty} F(t_i)^{1/2} + |\alpha_i - \alpha_\infty| \|f e^{2u_i}\|_{L^\infty} \\ &\quad + \|\alpha_\infty f\|_{L^\infty} \|e^{2u_i} - e^{2u_\infty}\|_{L^2} \to 0 \quad (i \to \infty), \end{aligned} \tag{3.6}$$

we even have strong convergence $u_i \to u_\infty$ in $H^2(M, g_b)$ and uniformly.

Note that $\alpha_\infty \ne 0$; else u_∞ would be constant, and from (1.3), (1.4) we obtain the contradiction

$$0 = \lim_{i \to \infty} \int_M f e^{2u_i}\,d\mu_{g_b} = \int_M f e^{2u_\infty}\,d\mu_{g_b} = e^{2u_\infty} \int_M f\,d\mu_{g_b} < 0.$$

In fact, $\alpha_\infty > 0$. Otherwise, if we assume that $\alpha_\infty < 0$, using (1.3) and computing

$$\begin{aligned} 0 &\le 2 \int_M |\nabla_{g_b} u_\infty|^2 e^{-2u_\infty}\,d\mu_{g_b} \\ &= \int_M \Delta_{g_b} u_\infty e^{-2u_\infty}\,d\mu_{g_b} = -\alpha_\infty \int_M f\,d\mu_{g_b} < 0 \end{aligned}$$

we again find a contradiction. We thus may let $c_\infty = \frac{1}{2}\log \alpha_\infty \in \mathbb{R}$. Setting $\hat{u}_\infty = u_\infty + c_\infty$ then the metric $\hat{g}_\infty = e^{2\hat{u}_\infty} g_b$ has curvature $K_{\hat{g}_\infty} = f$.

This completes the proof of Theorem 1.4. □

In general we cannot say whether we have uniform convergence $u(t) \to u_\infty$ as $t \to \infty$. However, we have the following result.

Theorem 3.2. *Suppose that u_∞ is a strict relative minimizer of E in $\mathcal{C}^*$ in the sense that with a constant $c_0 > 0$ there holds*

$$d^2 L_{u_\infty}(h, h) = \int_M \left(|\nabla_{g_b} h|^2 - 2\alpha_\infty f h^2 e^{2u_\infty}\right) d\mu_{g_b} \ge 2c_0 ||h||_{H^1}^2 \tag{3.7}$$

for all $h \in T_{u_\infty}\mathcal{C}^*$, where for $u \in \mathcal{C}^*$ we let

$$T_u \mathcal{C}^* = \{h \in H^1(M, g_b);\ \int_M f h e^{2u}\,d\mu_{g_b} = 0 = \int_M h e^{2u}\,d\mu_{g_b}\},$$

and where for $u = u(t)$ or $u = u_\infty$ with $\alpha = \alpha(t)$ or $\alpha = \alpha_\infty$ we let

$$L_u(v) = E(v) - \frac{\alpha}{2}\int_M fe^{2v}\, d\mu_{g_b}$$

be the Lagrange functional associated with u. Then $u(t) \to u_\infty$, $\alpha(t) \to \alpha_\infty$ exponentially fast as $t \to \infty$; that is, with constants $C > 0$ there holds

$$F(t) = \|\alpha f - K\|^2_{L^2(g)} \le Ce^{-2c_0 t},$$

and hence

$$|\alpha(t) - \alpha_\infty| + \|u(t) - u_\infty\|_{H^2(M,g_b)} \le Ce^{-c_0 t}.$$

Proof. First observe that by (3.2) and a variant of (3.3) on account of Lemma 3.1 with error $o(1) \to 0$ as $t \to \infty$ there holds

$$\begin{aligned}\frac{1}{2}\frac{dF}{dt} &\le -(1+o(1))\int_M |\nabla(K-\alpha f)|^2_g\, d\mu_g + 2\alpha\int_M f(K-\alpha f)^2\, d\mu_g + o(1)F \\ &= -(1+o(1))d^2L_u(u_t,u_t) + o(1)F,\end{aligned}$$

where

$$d^2L_u(u_t,u_t) = \int_M \big(|\nabla u_t|^2_{g_b} - 2\alpha f u_t^2 e^{2u}\big)\, d\mu_{g_b}.$$

Moreover, by (1.7) and (1.11) we have $u_t = \alpha f - K \in T_u\mathcal{C}^*$ for any $t > 0$.

Suppose that for some $0 < \rho < 1$ and some $t > 0$ we have

$$|\alpha(t) - \alpha_\infty| + \|u(t) - u_\infty\|_{H^2(M,g_b)} \le 2\rho. \tag{3.8}$$

Then there exist constants $\gamma, \delta \in \mathbb{R}$ such that $u_t + \gamma + \delta f \in T_{u_\infty}\mathcal{C}^*$, and with $g_\infty = e^{2u_\infty}g_b$ the equations

$$\begin{aligned}0 &= \int_M (u_t + \gamma + \delta f)\, d\mu_{g_\infty} = \int_M u_t\big(e^{2(u_\infty - u)} - 1\big)\, d\mu_g + \gamma, \\ 0 &= \int_M (u_t + \gamma + \delta f) f\, d\mu_{g_\infty} = \int_M u_t f\big(e^{2(u_\infty - u)} - 1\big)\, d\mu_g + \delta\int_M f^2\, d\mu_{g_\infty},\end{aligned}$$

respectively, give the bounds

$$|\gamma| + |\delta| \le C\|u(t) - u_\infty\|_{L^\infty}F^{1/2}(t) \le C\rho F^{1/2}(t).$$

Here we also use equations (1.7), (1.11), as well as the identities

$$\int_M d\mu_{g_\infty} = 1,\ \int_M f\, d\mu_{g_\infty} = 0,$$

and Lemma 2.3. In consequence, setting $h_0 = u_t + \gamma + \delta f$ for brevity, we have

$$\begin{aligned}d^2L_u(u_t,u_t) &= \int_M \big(|\nabla u_t|^2_{g_b} - 2\alpha f u_t^2 e^{2u}\big)\, d\mu_{g_b} \\ &= d^2L_{u_\infty}(h_0,h_0) + I + II \ge 2c_0||h_0||^2_{H^1} + I + II\end{aligned}$$

with error terms

$$\begin{aligned}I &= \int_M \big(|\nabla u_t|^2_{g_b} - |\nabla h_0|^2_{g_b}\big)\, d\mu_{g_b} = -\delta\int_M \nabla f\cdot\nabla(2u_t + \delta f)\, d\mu_{g_b} \\ &= 2\delta\int_M u_t\Delta_{g_b} f\, d\mu_{g_b} - \delta^2\int_M |\nabla f|^2_{g_b}\, d\mu_{g_b} = O(\rho F),\end{aligned}$$

where $a \cdot b = (a,b)_{g_b}$, and

$$\begin{aligned} II &= 2\int_M \left(\alpha_\infty f h_0^2 e^{2u_\infty} - \alpha f u_t^2 e^{2u}\right) d\mu_{g_b} \\ &= 2\alpha_\infty \int_M f\left(h_0^2 e^{2u_\infty} - u_t^2 e^{2u}\right) d\mu_{g_b} + O(\rho F) \\ &= 2\alpha_\infty \int_M f\left((h_0^2 - u_t^2) + u_t^2(1 - e^{2(u-u_\infty)})\right) d\mu_{g_\infty} + O(\rho F) = O(\rho F). \end{aligned}$$

Moreover, similar computations give

$$||h_0||^2_{L^2} = ||u_t||^2_{L^2} + O(\rho F),\ ||\nabla h_0||^2_{L^2} = ||\nabla u_t||^2_{L^2} - I = ||\nabla u_t||^2_{L^2} + O(\rho F).$$

Hence for sufficiently small $\rho > 0$ and all sufficiently large $t > 0$ satisfying (3.8) we have

$$\frac{1}{2}\frac{dF}{dt} \le -(2c_0 + o(1))||u_t||^2_{H^1} + O(\rho F) + o(1)F \le -c_0 F. \tag{3.9}$$

Having fixed such $\rho > 0$, now assume that for some $t_0 > 0$ there holds

$$|\alpha(t_0) - \alpha_\infty| + \|u(t_0) - u_\infty\|_{H^2(M,g_b)} \le \rho. \tag{3.10}$$

Then, if $t_0 > 0$ is sufficiently large, from (3.9) we find

$$F(t) \le F(t_0)e^{-2c_0(t-t_0)} \tag{3.11}$$

for all $t > t_0$, as long as there holds (3.8). We claim that if $t_0 > 0$ is sufficiently large, the bound (3.8) and hence also (3.11) will, in fact, hold true for all $t > t_0$.

Indeed, from (1.8) and (3.1) we have the equation

$$\begin{aligned} \alpha_t \int_M f^2\, d\mu_g &= \int_M f(K_t + 2Ku_t)\, d\mu_g - 2\alpha \int_M f^2 u_t\, d\mu_g \\ &= -\int_M f\Delta_g u_t\, d\mu_g - 2\alpha\int_M f^2 u_t\, d\mu_g. \end{aligned}$$

Integrating by parts, we have

$$\Big|\int_M f\Delta_g u_t\, d\mu_g\Big| = \Big|\int_M \Delta_g f u_t\, d\mu_g\Big| \le C\|f\|_{C^2}\|u_t\|_{L^2(M,g)} \le CF(t)^{1/2}.$$

Moreover, by Hölder's inequality we can bound

$$\Big|\alpha\int_M f^2 u_t\, d\mu_g\Big| \le C\|u_t\|_{L^2(M,g)} \le CF(t)^{1/2}.$$

Thus, from Lemma 2.3 and (3.11) we have

$$|\alpha_t| \le CF(t)^{1/2} \le CF(t_0)^{1/2}e^{-c_0(t-t_0)},$$

and for any $t > t_0$ we obtain

$$|\alpha(t) - \alpha(t_0)| \le \int_{t_0}^t |\alpha_t(s)|\, ds \le CF(t_0)^{1/2} \tag{3.12}$$

as long as there holds (3.8). Similarly, also using Lemma 2.5, with a constant $C = C(E(u_0))$ we can bound

$$\|u(t) - u(t_0)\|_{L^2(M,g_b)} \le \int_{t_0}^t \|u_t(s)\|_{L^2(M,g_b)}\, ds \le CF(t_0)^{1/2}. \tag{3.13}$$

From a computation analogous to (3.6) we then obtain

$$\|\Delta_{g_b}(u(t)-u(t_0))\|_{L^2(M,g_b)} \le C|\alpha(t)-\alpha(t_0)| + C\|e^{2u(t)}-e^{2u(t_0)}\|_{L^2(M,g_b)} + CF(t)^{1/2} + CF(t_0)^{1/2} \le CF(t_0)^{1/2},$$

and together with (3.12), (3.13) we conclude that

$$|\alpha(t)-\alpha(t_0)| + \|u(t)-u(t_0)\|_{H^2(M,g_b)} \le CF(t_0)^{1/2}. \tag{3.14}$$

But the latter will be strictly less than ρ if $t_0 > 0$ is sufficiently large. Together with (3.10) this shows that the barrier 2ρ in (3.8) will never be achieved for any $t > t_0$, if $t_0 > 0$ is sufficiently large, and for such $t_0 > 0$ the bound (3.8) and hence (3.11) hold for all $t > t_0$, as claimed.

Finally, passing to the limit $t = t_i \to \infty$ in (3.14) we see that we have

$$|\alpha_\infty - \alpha(t_0)| + \|u_\infty - u(t_0)\|_{H^2(M,g_b)} \le CF(t_0)^{1/2}$$

for all sufficiently large $t_0 > 0$. Renaming t_0 as t, and choosing $t_0 > 0$ such that (3.11) holds for all $t > t_0$, we then obtain the claim. □

Moreover, we have uniform convergence of the flow (1.6), (1.7) whenever the solution to (1.2) is unique.

Theorem 3.3. *Suppose that* (1.2) *has a unique solution. Then* $u(t) \to u_\infty$ *in* $H^2(M,g_b)$ *as* $t \to \infty$.

Proof. Suppose by contradiction that there exists $\rho > 0$ and a sequence $t_i \to \infty$ such that

$$\|u(t_i) - u_\infty\|_{H^2(M,g_b)} \ge \rho \text{ for all } i \in \mathbb{N}.$$

Then, as in the proof of Theorem 1.4, by (1.10) as well as Lemmas 2.4 and 2.5 there exists a subsequence $\Lambda \subset \mathbb{N}$ and a function v_∞ such that $v_i = u(t_i) \to v_\infty$ weakly in $H^1(M,g_b)$ with $e^{2v_i} \to e^{2v_\infty}$ in $L^p(M,g_b)$ for any $p < \infty$ as $i \to \infty$, $i \in \Lambda$, and such that, in addition $\beta_i = \alpha(t_i) \to \beta_\infty$. Still following the proof of Theorem 1.4, with the help of Lemma 3.1 we then conclude strong convergence $v_i \to v_\infty$ in $H^2(M,g_b)$, and $v_\infty \in \mathcal{C}^*$ solves the equation

$$-\Delta_{g_b} v_\infty = \beta_\infty f e^{2v_\infty} \text{ on } M. \tag{3.15}$$

Again, (1.3) and (1.4) imply that $\beta_\infty > 0$ and, with $d_\infty = \frac{1}{2}\log\beta_\infty \in \mathbb{R}$, the function $\hat{v} = v_\infty + d_\infty$ solves (1.2).

But by assumption the function $\hat{u}$ constructed in the proof of Theorem 1.4 is the unique solution of equation (1.2). It follows that $\hat{v} = \hat{u} = u_\infty + c_\infty$, and $v_\infty = u_\infty + c_\infty - d_\infty$. But then from (1.12) we obtain

$$1 = \int_M e^{2v_\infty}\, d\mu_{g_b} = e^{2(c_\infty - d_\infty)} \int_M e^{2u_\infty}\, d\mu_{g_b} = e^{2(c_\infty - d_\infty)}.$$

Hence $c_\infty = d_\infty$; that is, $v_\infty = u_\infty$, and $v_i \to u_\infty$ in $H^2(M,g_b)$ contrary to our initial assumption. □

Our next goal is to analyze the "shape" of the metrics $g(t)$ and to establish "bubbling" of the prescribed curvature flow analogous to [27] in a certain limit regime. We first consider the static (time-independent) case.

4. "Bubble" analysis in the static case

4.1. **Bounds for total curvature.** Let f_0 be a smooth, non-constant function with $\max_{p\in M} f_0(p) = 0$, and for any $\lambda \in \mathbb{R}$ let $f_\lambda = f_0 + \lambda$ as in [13], [3], or [15]; also set $\mathcal{C}_\lambda = \mathcal{C}_{f_\lambda}$, $\mathcal{C}^*_\lambda = \mathcal{C}^*_{f_\lambda}$.

Fix some sufficiently small $\lambda_0 > 0$ so that f_{λ_0} changes sign and satisfies (1.3). For any $0 < \lambda < \lambda_0$ then by Theorem 1.1 there exists a solution $\hat{u}_\lambda = u_\lambda + c_\lambda$ of (1.2), where u_λ minimizes E in the set $\mathcal{C}^*_\lambda$ and thus satisfies the equation

$$-\Delta_{g_b} u_\lambda = \alpha_\lambda f_\lambda e^{2u_\lambda} \tag{4.1}$$

with a constant $\alpha_\lambda > 0$, and where $c_\lambda = \frac{1}{2}\log \alpha_\lambda$. Moreover, setting $\hat{g}_\lambda = e^{2\hat{u}_\lambda} g_b$ we have

$$vol(M, \hat{g}_\lambda) = \int_M e^{2(u_\lambda + c_\lambda)}\, d\mu_{g_b} = e^{2c_\lambda} = \alpha_\lambda. \tag{4.2}$$

Note that by Theorem 1.1 we must have

$$\beta_\lambda := E(u_\lambda) = \min\{E(u);\ u \in \mathcal{C}^*_\lambda\} \to \infty \ \text{ as } \lambda \downarrow 0. \tag{4.3}$$

Indeed, if we assume $\beta_\lambda \le C_1$ and express $|f_0| = -f_0 = \lambda - f_\lambda$, condition (1.3) and the Gauss-Bonnet identity (1.4) together with (1.12) and the bound (2.4) give

$$\begin{aligned} 0 < \Big(\int_M |f_0| d\mu_{g_b}\Big)^2 &\le \int_M |f_0| e^{-2u_\lambda}\, d\mu_{g_b} \cdot \int_M |f_0| e^{2u_\lambda}\, d\mu_{g_b} \\ &\le C\|f_0\|_{L^\infty} \int_M (\lambda - f_\lambda) e^{2u_\lambda}\, d\mu_{g_b} = C\lambda \|f_0\|_{L^\infty}, \end{aligned}$$

with a constant $C > 0$ depending only on C_1, which is only possible if $\lambda \ge \lambda_1$ for some $\lambda_1 > 0$.

On the other hand, as shown by Galimberti [15] with the help of the method introduced in [3] for surfaces of genus larger than one, for suitable $\lambda \downarrow 0$ the total curvature of the metrics $\hat{g}_\lambda = e^{2\hat{u}_\lambda} g_b$ is uniformly bounded. The crucial ingredient in the derivation of this bound is the following monotonicity property established by Galimberti [15], Proposition 3.7 and Corollary 3.8.

Lemma 4.1. *The function $\lambda \mapsto \beta_\lambda$ is non-increasing in λ for $0 < \lambda < \lambda_0$, and for every such λ we have the bound*

$$\limsup_{\mu \downarrow \lambda} \frac{\beta_\lambda - \beta_\mu}{\mu - \lambda} \ge \frac{\alpha_\lambda}{2}.$$

The argument by Galimberti [15] is rather long and technical. For our later convenience we therefore include the following short proof.

Proof. Fix $0 < \lambda < \lambda_0$, and let $u_\lambda \in \mathcal{C}^*_\lambda$ be a minimizer of E as above. Then by (1.4) for $\delta < 0$ close to zero with suitable numbers $\delta', \delta'' \in]\delta, 0[$ we have

$$\int_M f_\lambda e^{2(u_\lambda + \delta f_\lambda)}\, d\mu_{g_b} = 2\delta \int_M f_\lambda^2 e^{2(u_\lambda + \delta' f_\lambda)}\, d\mu_{g_b} = 2\delta \int_M f_\lambda^2 e^{2u_\lambda}\, d\mu_{g_b} + O(\delta^2),$$

while also using (1.12) we find

$$\int_M e^{2(u_\lambda + \delta f_\lambda)}\, d\mu_{g_b} = 1 + 2\delta \int_M f_\lambda e^{2(u_\lambda + \delta'' f_\lambda)}\, d\mu_{g_b} = 1 + O(\delta^2).$$

Thus, if $0 < |\delta| \ll 1$ is sufficiently small it follows that

$$u_\lambda + \delta f_\lambda \in \mathcal{C}_\mu$$

with

$$\mu = \lambda - 2\delta \int_M f_\lambda^2 e^{2u_\lambda}\, d\mu_{g_b} + O(\delta^2). \tag{4.4}$$

In particular, $\mu > \lambda$ for $\delta < 0$ sufficiently close to zero, and $\delta = O(\mu - \lambda)$.

On the other hand, we have

$$E(u_\lambda + \delta f_\lambda) = E(u_\lambda) + \delta \int_M \nabla u_\lambda \cdot \nabla f_\lambda\, d\mu_{g_b} + O(\delta^2),$$

where

$$\int_M \nabla u_\lambda \cdot \nabla f_\lambda\, d\mu_{g_b} = \int_M (-\Delta_{g_b} u_\lambda) f_\lambda\, d\mu_{g_b} = \alpha_\lambda \int_M f_\lambda^2 e^{2u_\lambda}\, d\mu_{g_b}$$

in view of (4.1), and by (4.4) there holds

$$\begin{aligned}\beta_\mu \le E(u_\lambda + \delta f_\lambda) &\le E(u_\lambda) + \delta\alpha_\lambda \int_M f_\lambda^2 e^{2u_\lambda}\, d\mu_{g_b} + O(\delta^2)\\ &= \beta_\lambda - \frac{\alpha_\lambda}{2}(\mu - \lambda) + O((\mu-\lambda)^2) < \beta_\lambda\end{aligned}$$

for $\delta < 0$ sufficiently close to zero. Hence, the map $\lambda \mapsto \beta_\lambda$ is non-increasing, and

$$\limsup_{\mu\downarrow\lambda} \frac{\beta_\lambda - \beta_\mu}{\mu - \lambda} \ge \frac{\alpha_\lambda}{2},$$

as claimed. □

Moreover, using the same comparison function as in [3], Lemma 3.1, also used by Galimberti [15], we find the following bound on β_λ.

Lemma 4.2. *There holds*

$$\limsup_{\lambda\downarrow 0} \frac{\beta_\lambda}{\log(1/\lambda)} \le 4\pi.$$

Galimberti [15], Proposition 3.3, obtains a similar bound with an unspecified constant instead of 4π.

Proof. Let $p_0 \in M$ be such that $f_0(p_0) = 0$. We may assume that we have local Euclidean coordinates x near $p_0 = 0$. Letting $A = \frac{1}{2} Hess_{f_0}(p_0)$, for a suitable constant $L > 0$ we have

$$f_0(x) = (Ax, x) + O(|x|^3) \ge -\lambda/2 \text{ on } B_{\sqrt{\lambda}/L}(0),$$

and $f_\lambda \ge \lambda/2$ on $B_{\sqrt{\lambda}/L}(0)$. As in [3] we then set $w_\lambda(x) = z_\lambda(Lx/\sqrt{\lambda})$, where $z_\lambda \in H_0^1(B_1(0))$ is given by $z_\lambda(x) = \log(1/|x|)$ for $\lambda \le |x| \le 1$ and $z_\lambda(x) = \log(1/\lambda)$ for $|x| \le \lambda$, satisfying

$$\|\nabla w_\lambda\|_{L^2}^2 = \|\nabla z_\lambda\|_{L^2}^2 = 2\pi \log(1/\lambda).$$

Extending $w_\lambda(x) = 0$ outside $B_{\sqrt{\lambda}/L}(0)$, for sufficiently small $\lambda > 0$ and any $s > 0$ we obtain

$$\begin{aligned}\int_M f_\lambda e^{2sw_\lambda} d\mu_{g_b} &= \int_M f_\lambda d\mu_{g_b} + \int_{B_{\sqrt{\lambda}/L}(0)} f_\lambda (e^{2sw_\lambda} - 1) dx\\ &= \int_M f_0\, d\mu_{g_b} + \lambda + \int_{B_{\sqrt{\lambda}/L}(0)} f_\lambda (e^{2sw_\lambda} - 1) dx.\end{aligned}$$

Note that after substituting $y = Lx/\sqrt{\lambda}$ for $s > 1$ we have

$$\lambda \int_{B_{\sqrt{\lambda}/L}(0)} e^{2sw_\lambda} dx = \frac{\lambda^2}{L^2} \int_{B_1(0)} e^{2sz_\lambda} dy = \frac{\pi \lambda^{4-2s}}{L^2} + \frac{2\pi\lambda^2}{L^2} \int_\lambda^1 r^{1-2s}\, dr$$
$$= \frac{\pi\lambda^{4-2s}}{L^2} + \frac{\pi\lambda^{4-2s}}{(s-1)L^2} - \frac{\pi\lambda^2}{(s-1)L^2}.$$

Since we have $\lambda/2 \le f_\lambda \le \lambda$ on $B_{\sqrt{\lambda}/L}(0)$, for $3/2 \le s \le 3$ we can estimate

$$\frac{3\pi}{4L^2}\lambda^{4-2s} - \frac{3\pi}{L^2}\lambda^2 \le \int_{B_{\sqrt{\lambda}/L}(0)} f_\lambda(\mathrm{e}^{2sw_\lambda} - 1)dx \le \frac{3\pi}{L^2}\lambda^{4-2s} - \frac{3\pi}{4L^2}\lambda^2.$$

Hence for $s = 2 + O(1/\log(1/\lambda))$ we can achieve that $\int_M f_\lambda \mathrm{e}^{2sw_\lambda} d\mu_{g_b} = 0$; that is, $sw_\lambda \in \mathcal{C}_\lambda$ with

$$\|\nabla(sw_\lambda)\|^2_{L^2} = s^2\|\nabla w_\lambda\|^2_{L^2} = 8\pi\log(1/\lambda) + O(1).$$

Thus, for any $K > 4\pi$ and sufficiently small $\lambda > 0$ there results

$$\beta_\lambda \le K\log(1/\lambda),$$

as desired. □

Applying the monotonicity trick from [23], [24] as in [3] or [15], we observe that the monotone function β_λ is almost everywhere differentiable and then use the bounds from Lemmas 4.1, 4.2 to obtain that

$$\liminf_{\lambda\downarrow 0}(\lambda\alpha_\lambda) \le 2\liminf_{\lambda\downarrow 0}\big(\lambda\big|\frac{d\beta_\lambda}{d\lambda}\big|\big) \le 8\pi. \tag{4.5}$$

Indeed, if we assume that for some $0 < \lambda_1 < \lambda_0$ and some $c_0 > 4\pi$ for almost all $0 < \lambda < \lambda_1$ the absolutely continuous part of the differential of β_λ satisfies

$$|\beta'_\lambda| = \big|\frac{d\beta_\lambda}{d\lambda}\big| \ge \frac{c_0}{\lambda},$$

then for $K = 2\pi + c_0/2 > 4\pi$ and any sufficiently small $0 < \lambda < \lambda_1$ we obtain

$$\beta_\lambda - \beta_{\lambda_1} \ge \int_\lambda^{\lambda_1} |\beta'_s|\, ds \ge c_0 \int_\lambda^{\lambda_1} \frac{ds}{s} \ge c_0\log(1/\lambda) + C > K\log(1/\lambda),$$

contradicting the bound in Lemma 4.2.

Estimating

$$|K_{\hat{g}_\lambda}| = |f_\lambda| \le \lambda - f_0 = 2\lambda - f_\lambda, \tag{4.6}$$

from (4.5) and (4.2) together with (1.4) we then conclude the bound

$$\liminf_{\lambda\downarrow 0}\int_M |K_{\hat{g}_\lambda}|\, d\mu_{\hat{g}_\lambda} \le 2\liminf_{\lambda\downarrow 0}(\lambda\alpha_\lambda) \le 16\pi \tag{4.7}$$

for the total curvature of the metrics $\hat{g}_\lambda$. For the normalized metrics $g_\lambda = e^{2u_\lambda}g_b$ we have $K_{g_\lambda} = \alpha_\lambda K_{\hat{g}_\lambda}$ and $\int_M |K_{g_\lambda}|\, d\mu_{g_\lambda} = \int_M |K_{\hat{g}_\lambda}|\, d\mu_{\hat{g}_\lambda}$.

4.2. **Concentration of curvature.** Motivated by (4.7) in the following for a sequence $\lambda_k \downarrow 0$ we consider functions $v_k \in \mathcal{C}^*_{\lambda_k}$ with corresponding metrics $g_k = e^{2v_k} g_b$ such that

$$-\Delta_{g_b} v_k = K_{g_k} e^{2v_k} = \alpha_k f_{\lambda_k} e^{2v_k} + h_k e^{2v_k} \tag{4.8}$$

with α_k satisfying $\limsup_{k\to\infty}(\lambda_k\alpha_k) \le 8\pi$, and with functions h_k on M such that $\|h_k e^{2v_k}\|_{L^2(M,g_b)} =: \varepsilon_k \to 0$ as $k \to \infty$. In particular, for suitable $\lambda_k \downarrow 0$ in view of (4.5) we may choose $v_k = u_{\lambda_k} \in \mathcal{C}^*_{\lambda_k}$, satisfying (4.8) with $h_k = 0$, $k \in \mathbb{N}$. However, by allowing the "error term" h_k we later will be able to apply the results below also in the flow context, where $v_k = u(t_k)$ for a solution $u = u(t)$ to (1.6), (1.7) and $h_k = u_t(t_k)$ for a sequence of times $t_k \to \infty$.

Set $s^{\pm} = \pm \max\{\pm s, 0\}$ for any $s \in \mathbb{R}$. Note that similar to (4.6), upon writing $|K_{g_k}| = -K_{g_k} + 2K^+_{g_k}$ and estimating $K^+_{g_k} \le \alpha_k\lambda_k + |h_k|$, from the Gauss-Bonnet identity (or by integrating (4.8)) we obtain the bound

$$\begin{aligned} \limsup_{k\to\infty} \int_M |K_{g_k}|\, d\mu_{g_k} &= 2\limsup_{k\to\infty} \int_M K^+_{g_k} e^{2v_k}\, d\mu_{g_b} \\ &\le 2\limsup_{k\to\infty}(\alpha_k\lambda_k + \|h_k e^{2v_k}\|_{L^2(M,g_b)}) \le 16\pi \end{aligned} \tag{4.9}$$

similar to (4.7) for the total curvature of the metrics g_k, $k \in \mathbb{N}$.

Lemma 4.3. *Given (v_k) as above we have $\alpha_k \to \infty$ as $k \to \infty$. Moreover, there is a sequence of radii $R_k \to 0$ such that with error $o(1) \to 0$ as $k \to \infty$ there holds*

$$\sup_{p_0\in M} \int_{B_{R_k}(p_0;g_b)} K^+_{g_k}\, d\mu_{g_k} = \pi/2 + o(1). \tag{4.10}$$

Proof. Suppose that there exists a sequence $\Lambda \subset \mathbb{N}$ and a number $R > 0$ such that for all $k \in \Lambda$ there holds

$$\sup_{p_0\in M} \int_{B_R(p_0;g_b)} K^+_{g_k}\, d\mu_{g_k} < \pi. \tag{4.11}$$

In particular, by (4.9) condition (4.11) will be satisfied if $\liminf_{k\to\infty} \alpha_k < \infty$.

We will show that (4.11) leads to a contradiction. Hence there exist radii $R_k \to 0$ with (4.10); moreover, $\alpha_k \to \infty$ $(k \to \infty)$, as claimed.

After passing to a subsequence we may assume that (4.11) holds for any $k \in \mathbb{N}$; in addition, we may assume that $R > 0$ is so small that we can introduce Euclidean coordinates on $B_R(p_0) = B_R(p_0; g_b)$ for any $p_0 \in M$ and we henceforth omit g_b.

i) We claim that the functions v_k are uniformly bounded on M from above. Fix a point $p_0 \in M$. In Euclidean coordinates around $p_0 = 0$ we then obtain the equation

$$-\Delta v_k = K_{g_k} e^{2v_k} \text{ on } B = B_R(0).$$

Split $v_k = v_k^{(0)} + v_k^{(+)} + v_k^{(-)}$, where $\Delta v_k^{(0)} = 0$ in B with $v_k^{(0)} = v_k$ on ∂B, and where $v_k^{(\pm)} \in H^1_0(B)$ solve

$$-\Delta v_k^{(\pm)} = K^{\pm}_{g_k} e^{2v_k} \text{ on } B.$$

Then $\pm v_k^{(\pm)} \ge 0$ by the maximum principle. Moreover, in view of the uniform L^1-bound (4.9) we also have uniform bounds

$$\|v_k^{(\pm)}\|_{W^{1,p}(B)} \le C \text{ for any } 1 \le p < 2.$$

Hence a subsequence $(v_k^{(\pm)})$ converges weakly in $W^{1,3/2}(B)$ and pointwise almost everywhere. Finally, from (4.11) and [5], Theorem 1, we have the uniform bound

$$\|e^{v_k^{(+)}}\|_{L^p(B)} \le C \quad \text{for any } 1 \le p < 4. \tag{4.12}$$

Likewise, if we choose $\gamma = 1/20 > 0$ so that

$$\gamma \int_M |K_{g_k}| \, d\mu_{g_k} \le \pi$$

for all sufficiently large $k \in \mathbb{N}$, from [5], Theorem 1, we find

$$\|e^{-\gamma v_k^{(-)}}\|_{L^p(B)} \le C \quad \text{for any } 1 \le p < 4. \tag{4.13}$$

Thus, by Jensen's inequality and in view of $v_k^{(0)} \le v_k - v_k^{(-)}$, on any disc $D \subset B$ the average of $\gamma v_k^{(0)}$ can be bounded

$$\begin{aligned} exp\big(\fint_D \gamma v_k^{(0)} \, dx\big) &\le \fint_D e^{\gamma v_k^{(0)}} \, dx \le C\big(\int_B e^{2\gamma v_k} \, dx \cdot \int_B e^{-2\gamma v_k^{(-)}} \, dx\big)^{1/2} \\ &\le C\big(\int_M e^{2v_k} \, d\mu_{g_b}\big)^{\gamma/2} = C, \end{aligned}$$

where we used Hölder's inequality in the second and in the final estimate.

By the mean value property of harmonic functions there results a uniform bound $v_k^{(0)} \le C_1$ on $B_{2R/3}(0)$ for all $k \in \mathbb{N}$. Harnack's inequality, applied to the functions $C_1 - v_k^{(0)} \ge 0$, $k \in \mathbb{N}$, then shows that either for a subsequence we have $|v_k^{(0)}| \le C$ on $B_{R/2}(0)$ for all $k \in \mathbb{N}$, or $v_k^{(0)} \to -\infty$ uniformly on $B_{R/2}(0)$ as $k \to \infty$.

Covering M with finitely many balls $B_{R/2}(x_i)$, $1 \le i \le I$, we note that for each $B_i = B_R(x_i)$ we have

$$\int_{B_i} K_{g_k}^+ \, d\mu_{g_k} \le \pi, \ 1 \le i \le I,$$

and bounding $v_k \le v_k^{(+)} + v_k^{(0)} \le v_k^{(+)} + C$ on each ball, from (4.12) we conclude

$$\|e^{v_k}\|_{L^p(M,g_b)} \le C \quad \text{for any } 1 \le p < 4.$$

Recalling the equation

$$-\Delta v_k = K_{g_k} e^{2v_k} = \alpha_k f_{\lambda_k} e^{2v_k} + h_k e^{2v_k}$$

with $\alpha_k f_{\lambda_k} \le \alpha_k \lambda_k \le 8\pi + o(1)$ up to an error $o(1) \to 0$, and recalling that by assumption we have $\|h_k e^{2v_k}\|_{L^2} \to 0$ as $k \to \infty$, we then obtain a uniform bound $v_k^{(+)} \le C\|v_k^{(+)}\|_{W^{2,3/2}} \le C$ on $B_R(x_i)$, uniformly in $1 \le i \le I$, and hence $v_k \le C$ on M, uniformly in $k \in \mathbb{N}$.

In addition, we now conclude the uniform bound $|v_k^{(0)}| \le C$ on every ball $B_{R/2}(x_i)$, $1 \le i \le I$, for some C independent of i and $k \in \mathbb{N}$. Indeed, if we suppose that $v_k^{(0)} \to -\infty$ and hence $v_k \le v_k^{(+)} + v_k^{(0)} \le C + v_k^{(0)} \to -\infty$ uniformly as $k \to \infty$ on *some* ball $B_{R/2}(x_i)$, by considering the decompositions of v_k in the overlap regions of adjacent balls, we must have $v_k^{(0)} \to -\infty$ uniformly as $k \to \infty$ on *every* $B_{R/2}(x_i)$, $1 \le i \le I$. But then $v_k \to -\infty$ uniformly on M as $k \to \infty$, contradicting (1.12). Thus, $|v_k^{(0)}| \le C$ on every ball $B_{R/2}(x_i)$, and by harmonicity of $v_k^{(0)}$ a subsequence $(v_k^{(0)})$ smoothly converges locally on every ball $B_{R/2}(x_i)$, ensuring that $v_k \to v_\infty$ weakly in $W^{1,3/2}(M)$ and pointwise almost everywhere.

ii) It now follows that $\alpha_k \to \infty$ as $k \to \infty$. Indeed, suppose by contradiction that we have a uniform bound $\alpha_k \le C < \infty$. Then by i) with a constant $C > 0$ independent of k we have the uniform bound $v_k \le C$ on M, and together with the bound $se^s \ge -1$ for $s \le 0$ we find that $\alpha_k f_{\lambda_k} e^{v_k} v_k \le C$, uniformly on M. Moreover, we have $\|v_k\|_{L^2} \le C\|v_k\|_{W^{1,3/2}} \le C$. But then upon multiplying (4.8) with v_k we obtain the bound

$$\begin{aligned}\beta_{\lambda_k} \le \int_M |\nabla v_k|^2 \, d\mu_{g_b} &\le \int_M \alpha_k f_{\lambda_k} e^{2v_k} v_k \, d\mu_{g_b} + \int_M h_k e^{2v_k} v_k \, d\mu_{g_b} \\ &\le C + C\|h_k e^{2v_k}\|_{L^2} \|v_k\|_{L^2} \le C,\end{aligned}$$

contradicting (4.3).

iii) Finally, recall that a subsequence $v_k \to v_\infty$ weakly in $W^{1,3/2}(\Omega)$. Fatou's lemma then gives

$$\liminf_{k\to\infty} \int_M |f_{\lambda_k}| e^{2v_k} \, d\mu_{g_b} \ge \int_M |f_0| e^{2v_\infty} \, d\mu_{g_b} > 0,$$

and with ii) we obtain

$$\int_M \alpha_k |f_{\lambda_k}| e^{2v_k} \, d\mu_{g_b} \to \infty \quad (k \to \infty).$$

But estimating

$$\int_M \alpha_k |f_{\lambda_k}| e^{2v_k} \, d\mu_{g_b} \le \int_M |K_{g_k}| \, d\mu_{g_k} + \|h_k e^{2v_k}\|_{L^2} \le 16\pi + o(1)$$

with error $o(1) \to 0$ $(k \to \infty)$ we then arrive at the desired contradiction. □

In particular, by Lemma 4.3 and (4.9) with error $o(1) \to 0$ as $k \to \infty$ there holds

$$\pi/2 - o(1) \le \alpha_k \lambda_k \le 8\pi + o(1) \text{ for all } k \in \mathbb{N}. \tag{4.14}$$

Note that so far we did not need to assume that all maxima p_0 where $f_0(p_0) = 0$ are non-degenerate.

4.3. **Blow-up analysis.** In this section we partially complete Galimberti's analysis of the shape of the metrics g_k in the time-independent case and extend his result to the more general case above. Like Galimberti's work, our analysis follows the outline of our previous joint work with Borer [3].

Theorem 4.4. *Let $f_0 \le 0$ be a smooth, non-constant function with* $\max f_0 = 0$ *having only non-degenerate maxima p_0 where $f_0(p_0) = 0$. Then for any (v_k) as in* (4.8) *above for suitable $i_0 \in \mathbb{N}$, $r_k^{(i)} \downarrow 0$, $p_k^{(i)} \to p_\infty^{(i)} \in M$ with $f_0(p_\infty^{(i)}) = 0$, $1 \le i \le i_0$, for a subsequence $k \to \infty$ the following holds.*

i) We have $v_k \to -\infty$ locally uniformly on $M_\infty = M \setminus \{p_\infty^{(i)};\ 1 \le i \le i_0\}$.

ii) For each $1 \le i \le i_0$ there holds $r_k^{(i)}/\sqrt{\lambda_k} \to 0$, and in local Euclidean coordinates x around $p_k^{(i)} = 0$ with constants $c_k^{(i)} \to \infty$ we have

$$w_k^{(i)}(x) := v_k(r_k^{(i)} x) - c_k^{(i)} \to w_\infty(x) = \log\Big(\frac{2}{1+|x|^2}\Big) \text{ in } H^2_{loc}(\mathbb{R}^2),$$

where w_∞ induces the standard spherical metric $g_\infty = e^{2w_\infty} g_{\mathbb{R}^2}$ on $\mathbb{R}^2$ of curvature $K_{g_\infty} \equiv 1$, and $1 \le i_0 \le 2$.

Thus we can rule out one of the possible blow-up limits in Galimberti's [15] Theorem 1.1; moreover, we obtain the improved bound $i_0 \le 2$ for the number of "bubbles". However, like Galimberti we are not able to decide whether the metrics $\hat{g}_k = e^{2\hat{v}_k} g_b = \alpha_k e^{2v_k} g_b$ on the torus converge to a limit metric on M_∞ or vanish as $k \to \infty$.

Proof. By (4.9) and Lemma 4.3 for a subsequence $k \to \infty$ we have

$$K^+_{g_k} e^{2v_k}\, d\mu_{g_b} \rightharpoonup K_+ + \sum_{i \in I} \gamma_i \delta_{p^{(i)}_\infty}$$

weakly-$*$ in the sense of measures, with a measure $K_+ \ge 0$ on (M, g_b) having no atoms and with at most countably many atoms of weight $\gamma_i > 0$, $i \in I \subset \mathbb{N}$. Note that (4.9) gives the bound $\sum_i \gamma_i \le 8\pi$. Hence after relabelling there is $i_0 \in \mathbb{N}$ such that $\gamma_i \ge \pi/2$ if and only if $1 \le i \le i_0$.

Also note that for any p with $f_0(p) < 0$ for suitable $r > 0$ we have

$$\int_{B_r(p)} K^+_{g_k} d\mu_{g_k} \le \|h_k e^{2v_k}\|_{L^2(M,g_b)} \to 0 \text{ as } k \to \infty;$$

thus, necessarily there holds $f_0(p^{(i)}_\infty) = 0$, $1 \le i \le i_0$.

i) Given any open set $\Omega \subset \overline{\Omega} \subset M_\infty := M \setminus \{p^{(i)}_\infty;\ 1 \le i \le i_0\}$ there exists a radius $R > 0$ such that for any $p_0 \in \Omega$ in Euclidean coordinates around $p_0 = 0$ we have

$$\int_{B_R(0)} K^+_{g_k}\, d\mu_{g_k} < \pi \tag{4.15}$$

for sufficiently large $k \in \mathbb{N}$.

Clearly we may assume that Ω is connected and is so large that $\int_\Omega f_0\, d\mu_{g_b} < 0$. Covering $\overline{\Omega}$ with finitely many balls $B_{R/2}(p_i)$, $1 \le i \le I$, and splitting $v_k = v^{(0)}_k + v^{(+)}_k + v^{(-)}_k$ on each $B = B_i = B_R(p_i)$ as in the proof of Lemma 4.3, with $\gamma = 1/20$ we then have $\|v^{(\pm)}_k\|_{W^{1,3/2}(B)} \le C$ as well as the uniform bounds

$$\|e^{v^{(+)}_k|}\|_{L^p(B)} + \|e^{-\gamma v^{(-)}_k|}\|_{L^p(B)} + \le C \quad \text{for any } 1 \le p < 4,$$

and a subsequence $(v^{(\pm)}_k)$ converges weakly in $W^{1,3/2}(B)$. Moreover, again arguing as in the proof of Lemma 4.3, we have uniform bounds $v^+_k \le C$, $v^{(0)}_k \le C_1$ on $B_{2R/3}(p_i)$, and either for a subsequence we have $|v^{(0)}_k| \le C$ on each $B_{R/2}(p_i)$ for all $k \in \mathbb{N}$, or there holds $v_k \to -\infty$ uniformly on Ω as $k \to \infty$. (Here we use that Ω by assumption is connected.)

Suppose that for a subsequence we have $|v^{(0)}_k| \le C$ uniformly on every ball $B_{R/2}(p_i)$, $1 \le i \le I$. Since $v^{(0)}_k$ is harmonic, then a subsequence $(v^{(0)}_k)$ converges locally smoothly on $B_{R/2}(p_i))$, and it follows that $v_k \to v_\infty$ weakly in $W^{1,3/2}(\Omega)$ and pointwise almost everywhere. By Fatou's lemma then

$$\liminf_{k\to\infty} \int_\Omega |f_{\lambda_k}| e^{2v_k}\, d\mu_{g_b} \ge \int_\Omega |f_0| e^{2v_\infty}\, d\mu_{g_b} > 0,$$

and similar to our argument in the proof of Lemma 4.3 from the fact that $\alpha_k \to \infty$ as $k \to \infty$, we then derive a contradiction by estimating

$$\int_\Omega \alpha_k |f_{\lambda_k}| e^{2v_k}\, d\mu_{g_b} \le \int_M |K_{g_k}|\, d\mu_{g_k} + \|h_k e^{2v_k}\|_{L^2} \le 16\pi + o(1).$$

Thus $v_k \to -\infty$ uniformly on Ω as $k \to \infty$, proving the first claim.

ii) Next, consider any point $p_0 = p_\infty^{(i)}$, $1 \le i \le i_0$. In order to be able to follow our reasoning in [3] as closely as possible, we split $v_k = w_k + z_k$, where

$$-\Delta_{g_b} z_k = (h_k - \bar{h}_k)e^{2v_k}$$

with $\int_M z_k \, d\mu_{g_b} = 0$ and with

$$\bar{h}_k = \int_M h_k e^{2v_k} \, d\mu_{g_b} \ \text{ satisfying } |\bar{h}_k| \le \varepsilon_k \to 0 \text{ as } k \to \infty.$$

Since

$$\|(h_k - \bar{h}_k)e^{2v_k}\|_{L^2} \le 2\|h_k e^{2v_k}\|_{L^2} = 2\varepsilon_k \to 0 \ \text{ as } k \to \infty,$$

from elliptic regularity theory we have

$$\|z_k\|_{L^\infty} \le C\|z_k\|_{H^2} \le C\|\Delta_{g_b} z_k\|_{L^2} \to 0 \text{ as } k \to \infty. \tag{4.16}$$

For any $R > 0$ now there holds

$$\sup_{B_R(p_0)} w_k \to \infty \text{ as } k \to \infty;$$

otherwise we have

$$\begin{aligned}\int_{B_R(p_0)} K_{g_k}^+ e^{2v_k} \, d\mu_{g_b} &\le \alpha_k \lambda_k e^{2\|z_k\|_{L^\infty}} \int_{B_R(p_0)} e^{2w_k} \, d\mu_{g_b} + \varepsilon_k \\ &\le CR^2 + \varepsilon_k < \pi/4\end{aligned}$$

for sufficiently small $R > 0$ and sufficiently large $k \in \mathbb{N}$, contrary to our assumption about p_0.

Thus, in view of part i) of this proof there exists $R > 0$ and $p_k \to p_0$ such that w_k for sufficiently large $k \in \mathbb{N}$ attains its maximum on $B_R(p_0)$ at p_k. In local Euclidean coordinates x near $p_0 = 0$ then $p_k = x_k$ and $\Delta w_k(x_k) \le 0$. From

$$-\Delta w_k = \alpha_k f_{\lambda_k} e^{2v_k} + \bar{h}_k e^{2v_k} \tag{4.17}$$

it then follows that

$$\alpha_k f_{\lambda_k}(x_k) + \bar{h}_k = \alpha_k (f_0(x_k) + \lambda_k) + \bar{h}_k \ge 0,$$

and from (4.14) we conclude that

$$-f_0(x_k) \le \lambda_k \big(1 + (\alpha_k \lambda_k)^{-1} \bar{h}_k\big) \le 2\lambda_k$$

for sufficiently large $k \in \mathbb{N}$. Since p_0 by assumption is non-degenerate, there exists a constant $C_1 > 0$ such that $-f_0(x) > 2\lambda_k$ for $|x|^2 \ge C_1 \lambda_k$, $|x| \le R$. It follows that $|x_k|^2 \le C_1 \lambda_k$; moreover, with error $o(1) \to 0$ as $k \to \infty$ the bound $\gamma_i \ge \pi/2$ and (4.16) give

$$\pi/2 \le \int_{B_R(0)} K_{g_k}^+ \, d\mu_{g_k} + o(1) \le C_1 \pi \alpha_k \lambda_k^2 e^{2w_k(x_k)} + o(1). \tag{4.18}$$

We are then left with two cases. First consider the case when $\alpha_k \lambda_k^2 e^{2w_k(x_k)} \to \infty$. Let $r_k^2 \alpha_k \lambda_k e^{2w_k(x_k)} = 1$ with $r_k^2/\lambda_k \to 0$ as $k \to \infty$, and rescale

$$\tilde{w}_k(x) = w_k(x_k + r_k x) - w_k(x_k) \le \tilde{w}_k(0) = 0, \ \tilde{z}_k(x) = z_k(x_k + r_k x)$$

on $D_k = \{x;\ |x_k + r_k x| < R\}$. Note that by (4.14) we have $|w_k(x_k) + \log r_k| \le C$; moreover, as $k \to \infty$ the domains D_k exhaust $\mathbb{R}^2$.

The function $\tilde{w}_k$ satisfies the equation

$$-\Delta \tilde{w}_k = r_k^2 \big(\alpha_k f_{\lambda_k}(x_k + r_k x) + \bar{h}_k\big) e^{2(\tilde{w}_k + w_k(x_k) + \tilde{z}_k)} = \tilde{f}_k e^{2\tilde{z}_k} e^{2\tilde{w}_k} \text{ on } D_k,$$

where

$$\begin{aligned}\tilde f_k(x) &= r_k^2\big(\alpha_k f_{\lambda_k}(x_k + r_k x) + \bar h_k\big)e^{2w_k(x_k)}\\ &= 1 + \lambda_k^{-1} f_0(x_k + r_k x) + (\alpha_k\lambda_k)^{-1}\bar h_k\end{aligned}$$

satisfies $\tilde f_k(0) \ge 0$. Since $(\alpha_k\lambda_k)^{-1}|\bar h_k| \le C\varepsilon_k \to 0$ as $k \to \infty$ on account of (4.14), and since in addition we have $|x_k|^2 \le C_1\lambda_k$ and $r_k^2/\lambda_k \to 0$, a subsequence $\lambda_k^{-1} f_0(x_k + r_k x) \to c_0$ for some constant $-1 \le c_0 \le 0$ and $\tilde f_k e^{2\tilde z_k} \to 1 + c_0 =: r_0^2 \ge 0$ locally uniformly on $\mathbb{R}^2$.

In view of the uniform volume bound

$$(4.19)\qquad (1 + o(1))\int_{D_k} e^{2\tilde w_k}\,dx = \int_{D_k} e^{2(\tilde w_k + \tilde z_k)}\,dx = \alpha_k\lambda_k \int_{B_R(0)} e^{2v_k}\,dx \le C,$$

from [5], Theorem 3, we then conclude that a subsequence $\tilde w_k \to \tilde w_\infty$ locally uniformly, where $\tilde w_\infty$ with $\tilde w_\infty \le 0 = \tilde w_\infty(0)$ solves the equation

$$(4.20)\qquad -\Delta \tilde w_\infty = r_0^2 e^{2\tilde w_\infty} \text{ on } \mathbb{R}^2,$$

with $\int_{\mathbb{R}^2} e^{2\tilde w_\infty}\,dx < \infty$. Hence $r_0 > 0$, and by the Chen-Li [9] classification of all solutions to this equation we have $w_\infty(x) := \tilde w_\infty(2x/r_0) + \log 2 = \log\big(\frac{2}{1+|x|^2}\big)$, and our claim follows.

In the remaining case there is $C > 0$ with $C^{-1} \le \alpha_k\lambda_k^2 e^{2w_k(x_k)} \le C$ for all $k \in \mathbb{N}$. In view of (4.14) we then have $|2w_k(x_k) + \log(\lambda_k)| \le C$. Set $r_k^2 = \lambda_k$ and rescale

$$\tilde w_k(x) = w_k(r_k x) + \log(r_k).$$

Then

$$\tilde w_k \le \tilde w_k(x_k/r_k) \le C \text{ on } D_k = \{x;\ |r_k x| < R\},\ k \in \mathbb{N}.$$

Moreover, $\tilde w_k$ satisfies the equation

$$-\Delta \tilde w_k = \tilde f_k e^{2\tilde z_k} e^{2\tilde w_k} \text{ on } D_k,$$

where a subsequence

$$\tilde f_k(x) = \alpha_k\lambda_k^2 e^{2w_k(x_k)}(1 + \lambda_k^{-1} f_0(r_k x) + (\alpha_k\lambda_k)^{-1}\bar h_k) \to r_0^2(1 + (Ax, x))$$

for some constant $r_0 > 0$ and with $A = \frac{1}{2}Hess_{f_0}(0)$. As before, from [5], Theorem 3, it follows that a subsequence $\tilde w_k \to \tilde w_\infty$ locally uniformly, where $w_\infty(x) := \tilde w_\infty(x/r_0) \le C$ for $A_0 = A/r_0^2$ satisfies the equation

$$-\Delta w_\infty = (1 + (A_0 x, x))e^{2w_\infty} \text{ on } \mathbb{R}^2,$$

with finite volume and finite total curvature

$$\int_{\mathbb{R}^2} e^{2w_\infty}\,dx < \infty,\ \int_{\mathbb{R}^2} |1 + (A_0 x, x)|e^{2w_\infty}\,dx < \infty.$$

But Theorem 4.5 below rules out this case.

iii) By the above characterization (4.20) of blow-up and (4.19), at each blow-up point $p_\infty^{(i)}$ with some constant $0 < r_0 \le 1$ we must have

$$\begin{aligned}4\pi = \int_{\mathbb{R}^2} e^{2w_\infty}\,dx &= r_0^2 \lim_{L\to\infty}\int_{B_L(0)} e^{2\tilde w_\infty}\,dx \le r_0^2 \limsup_{k\to\infty}\int_{D_k} e^{2\tilde w_k}\,dx\\ &= r_0^2 \limsup_{k\to\infty}\Big(\alpha_k\lambda_k\int_{B_R(0)} e^{2v_k}\,dx\Big) \le 8\pi r_0^2 \le 8\pi.\end{aligned}$$

Thus there can be at most $i_0 \le 2$ such blow-up points, and the proof is complete. □

4.4. **Ruling out slow blow-up.** Entire solutions of Liouville's equation with curvature functions of polynomial growth have been studied for instance by Cheng-Lin [10]. Here we establish the following non-existence result, quoted as Theorem 1.3 in the introduction.

Theorem 4.5. *Suppose A is a negative definite and symmetric 2×2-matrix. Then there is no solution $w \in C^\infty(\mathbb{R}^2)$ of the equation*

$$-\Delta w = (1+(Ax,x))e^{2w} \text{ on } \mathbb{R}^2 \tag{4.21}$$

with $w \le C$ and such that the induced metric $h = e^{2w}g_{\mathbb{R}^2}$ has finite volume and integrated curvature

$$V_0 := \int_{\mathbb{R}^2} e^{2w}\,dx < \infty,\ K_0 := \int_{\mathbb{R}^2}(1+(Ax,x))e^{2w}\,dx \in \mathbb{R} \tag{4.22}$$

and hence also has finite total curvature

$$\int_{\mathbb{R}^2} |1+(Ax,x)|e^{2w}\,dx < \infty.$$

Proof. Writing $(1+(Ax,x))e^{2w} =: F \in L^1(\mathbb{R}^2)$ for brevity, following Chen-Li [9] we introduce

$$\tilde{w}(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2}\big(\log|x-y| - \log|y|\big)F(y)\,dy \tag{4.23}$$

and note that in view of (4.22) we have $|\tilde{w}| \le C\log(2+|x|)$.

The function $v := w - \tilde{w}$ then is harmonic with $v(x) \le C + C\log(2+|x|)$. Therefore, v must be constant. Indeed, by the mean value property of harmonic functions and the divergence theorem, in view of the bound

$$|v| = 2v^+ - v \le C - v + C\log(2+|x|)$$

for any partial derivative ∂v and any $x_0 \in \mathbb{R}^2$, $R > 0$ we have

$$\begin{aligned}|\partial v(x_0)| &= \Big|\fint_{B_R(x_0)} \partial v\,dx\Big| \le CR^{-2}\int_{\partial B_R(x_0)}|v|\,do \\ &\le CR^{-1}\big(C - v(x_0) + \log(2+|x_0|+R)\big) \to 0 \text{ as } R\to\infty,\end{aligned}$$

where $\fint$ denotes the average and do denotes the one-dimensional Hausdorff measure. Hence $w = \tilde{w} + C$ for some $C \in \mathbb{R}$, as claimed.

Next, observing that for any $y \in \mathbb{R}^2$ there holds $\log|x-y|/\log|x| \to 1$ $(|x|\to\infty)$. from (4.23) as $|x|\to\infty$ we obtain

$$\begin{aligned}\frac{\tilde{w}(x)}{\log|x|} &= -\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{\log|x-y|-\log|y|}{\log|x|}F(y)\,dy \\ &\to -\frac{1}{2\pi}\int_{\mathbb{R}^2}F\,dy = -\frac{K_0}{2\pi} =: -\nu \in \mathbb{R}.\end{aligned} \tag{4.24}$$

From (4.24) for any $\mu > \nu$ we can bound $e^{2\tilde{w}} \ge |x|^{-2\mu}$ if $|x| \gg 1$ and then also

$$\begin{aligned}\int_{\mathbb{R}^2\setminus B_1(0)}|x|^{2-2\mu}\,dx &\le C\int_{\mathbb{R}^2}|x|^2e^{2\tilde{w}}\,dx + C \\ &\le C\int_{\mathbb{R}^2}|(Ax,x)|e^{2w}\,dx + C = C(V_0 - K_0) + C < \infty.\end{aligned}$$

Thus, we conclude that $\nu \ge 2$ and hence $K_0 \ge 4\pi$.

Multiplying (4.21) with $x \cdot \nabla w$ we find the identity

$$\begin{aligned} div\big(\nabla w\, x \cdot \nabla w - \frac{x}{2}|\nabla w|^2\big) + div\big(\frac{x}{2}(1 + (Ax, x))e^{2w}\big) \\ = (1 + (Ax, x))e^{2w} + (Ax, x)e^{2w} = 2(1 + (Ax, x))e^{2w} - e^{2w}. \end{aligned}$$

Integrating over a ball $B_R(0)$, we note that by finiteness of $\|F\|_{L^1}$ we have

$$R \int_{\partial B_R(0)} (1 + (Ax, x))e^{2w} do \to 0$$

as $R \to \infty$ suitably. Also writing

$$\int_{\partial B_R(0)} \frac{x}{|x|} \cdot \big(\nabla w\, x \cdot \nabla w - \frac{x}{2}|\nabla w|^2\big)do = \int_{\partial B_R(0)} \frac{|x \cdot \nabla w|^2 - |x^\perp \cdot \nabla w|^2}{2R} do,$$

where for any $x \in \mathbb{R}^2 \cong \mathbb{C}$ we denote as $x^\perp = ix \in \mathbb{C}$ the vector x rotated by 90 degrees, we then obtain

$$\frac{1}{2R} \int_{\partial B_R(0)} \big(|x \cdot \nabla w|^2 - |x^\perp \cdot \nabla w|^2\big)do + V_0 - 2K_0 \to 0 \tag{4.25}$$

as $R \to \infty$ suitably.

Differentiating (4.23) we find

$$x \cdot \nabla w(x) = x \cdot \nabla \tilde{w}(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x \cdot (x - y)}{|x - y|^2} F(y)\, dy = -\frac{K_0}{2\pi} + I(x), \tag{4.26}$$

with

$$I(x) := -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{y \cdot (x - y)}{|x - y|^2} F(y)\, dy,$$

while

$$\begin{aligned} x^\perp \cdot \nabla w(x) = x^\perp \cdot \nabla \tilde{w}(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x^\perp \cdot (x - y)}{|x - y|^2} F(y)\, dy \\ = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{y^\perp \cdot (x - y)}{|x - y|^2} F(y)\, dy =: II(x). \end{aligned} \tag{4.27}$$

We can estimate the error terms

$$\frac{1}{2R} \int_{\partial B_R(0)} \big(|I(x)|^2 + |II(x)|^2\big)do \to 0 \tag{4.28}$$

as $R \to \infty$ suitably. Postponing the proof of (4.28), from (4.26) and (4.25) upon letting $R \to \infty$ suitably we then have

$$0 = \frac{K_0^2}{4\pi} + V_0 - 2K_0 = \frac{K_0 - 8\pi}{4\pi} K_0 + V_0, \tag{4.29}$$

and in view of $4\pi \le K_0 \le V_0$ we conclude that $K_0 = 4\pi = V_0$. Hence $A = 0$, which contradicts our assumptions and proves the claim.

It remains to show (4.28). Given any $\varepsilon > 0$ there exists $R_0 > 1$ such that

$$\int_{\mathbb{R}^2 \setminus B_{R_0}(0)} |F(y)|\, dy < \varepsilon.$$

Let $|x| = 2R \ge 2R_0$. Observing that $|y| \le |x - y| + |x|$ gives

$$\frac{|y|}{|x - y|} \le 3 \;\text{ for } y \notin B_R(x),$$

we can bound

$$\big|2\pi I(x) + \int_{B_R(x)} \frac{y\cdot(x-y)}{|x-y|^2}F(y)\,dy\big| + \big|2\pi II(x) + \int_{B_R(x)} \frac{y^\perp\cdot(x-y)}{|x-y|^2}F(y)\,dy\big|$$
$$\le C\int_{\mathbb{R}^2\setminus(B_R(x)\cup B_{R_0}(0))} \frac{|y|}{|x-y|}|F(y)|\,dy + C\frac{R_0}{2R-R_0}\int_{B_{R_0}(0)}|F(y)|\,dy$$
$$\le C\varepsilon + \frac{CR_0}{2R-R_0} \le C\varepsilon$$

if $R \ge R_0$ is sufficiently large. Moreover, we have

$$\big|\int_{B_R(x)} \frac{y\cdot(x-y)}{|x-y|^2}F(y)\,dy - \int_{B_R(x)} \frac{x\cdot(x-y)}{|x-y|^2}F(y)\,dy\big| \le \int_{B_R(x)}|F(y)|\,dy \le \varepsilon$$

and

$$\int_{B_R(x)} \frac{y^\perp\cdot(x-y)}{|x-y|^2}F(y)\,dy = \int_{B_R(x)} \frac{x^\perp\cdot(x-y)}{|x-y|^2}F(y)\,dy$$

Next, changing coordinates to $z = x - y$ we obtain

$$\int_{B_R(x)} \frac{x\cdot(x-y)}{|x-y|^2}F(y)\,dy = \int_{B_R(0)} \frac{x\cdot z}{|z|^2}F(x-z)\,dz$$
$$= \frac{1}{2}\int_{B_R(0)} \frac{x\cdot z}{|z|^2}(F(x-z)-F(x+z))\,dz,$$

where we observe that

$$\int_{B_R(0)} \frac{x\cdot z}{|z|^2}(F(x-z)+F(x+z))\,dz = 0$$

by symmetry. Similarly we have

$$\int_{B_R(x)} \frac{x^\perp\cdot(x-y)}{|x-y|^2}F(y)\,dy = \frac{1}{2}\int_{B_R(0)} \frac{x^\perp\cdot z}{|z|^2}(F(x-z)-F(x+z))\,dz.$$

Expanding

$$(A(x\pm z), x\pm z) = (Ax,x) + (Az,z) \pm 2(Ax,z)$$

we then have

$$F(x-z) - F(x+z)$$
$$= (1 + (A(x-z,x-z))e^{2w(x-z)} - (1 + (A(x+z,x+z))e^{2w(x+z)}$$
$$= \big((1 + (Ax,x) + (Az,z))\big)\big(e^{2w(x-z)} - e^{2w(x+z)}\big)$$
$$- 2(Ax,z)\big(e^{2w(x-z)} + e^{2w(x+z)}\big).$$

Note that we can bound

$$\int_{B_R(0)} \frac{|x\cdot z|}{|z|^2}\big(|(Ax,z)| + |(Az,z)|\big)\big(e^{2w(x-z)} + e^{2w(x+z)}\big)\,dz$$
$$\le C\int_{B_R(0)} |x|^2\big(e^{2w(x-z)} + e^{2w(x+z)}\big)\,dz \le C\int_{B_R(x)}|F(y)|\,dy \le C\varepsilon.$$

From (4.24) we also can bound $e^{2\tilde{w}} \le |x|^{-2\mu}$ for any $\mu < \nu$ when $|x| \gg 1$ is sufficiently large. Recalling that $\nu \ge 2$ and choosing $\mu = 3/2$ we find

$$\int_{B_R(0)} \frac{|x\cdot z|}{|z|^2}\big(e^{2w(x-z)} + e^{2w(x+z)}\big)\,dz \le R^{1-2\mu}\int_{B_R(0)}\frac{dz}{|z|} \le CR^{2-2\mu} \to 0$$

as $|x| = 2R \to \infty$. Thus we obtain

$$\big|4\pi I(x) + \int_{B_R(0)} \frac{x\cdot z}{|z|^2}(Ax,x)\big(e^{2w(x-z)} - e^{2w(x+z)}\big)\,dz\big| \le C\varepsilon,$$

and similarly

$$\big|4\pi II(x) + \int_{B_R(0)} \frac{x^\perp\cdot z}{|z|^2}(Ax,x)\big(e^{2w(x-z)} - e^{2w(x+z)}\big)\,dz\big| \le C\varepsilon,$$

if $R \ge R_0$ is sufficiently large. But letting $z = r\zeta$ with $\zeta = e^{i\theta} \in S^1$ and integrating for each $z_0 = re^{i\theta_0} \in B_R(0)$ from θ_0 to $\theta_0+\pi$ along a semi-circle with radius r, we have

$$\begin{aligned}\big|e^{2w(x-z_0)} - e^{2w(x+z_0)}\big| &\le \big|\int_0^\pi 2z^\perp\cdot\nabla w(x+z)e^{2w(x+z)}\,d\theta\big|\\ &\le C\sup_{|z|=r}|\nabla w(x+z)|\int_{\partial B_r(0)} e^{2w(x+z)}\,do.\end{aligned}$$

Hence for any $|x| = 2R$ and sufficiently large $R \ge R_0$ there results

$$\begin{aligned}|I(x)| + |II(x)| &\le C\sup_{|z|\le R}|\nabla w(x+z)|\int_{B_R(0)}|x|^3e^{2w(x+z)}\,dz + C\varepsilon\\ &\le C\sup_{y\in B_R(x)}|y||\nabla w(y)|\int_{B_R(x)}|F(y)|\,dy + C\varepsilon \qquad (4.30)\\ &\le C\varepsilon\sup_{y\in B_R(x)}|y||\nabla w(y)| + C\varepsilon.\end{aligned}$$

But from (4.24) for any $\alpha < 1$ for any sufficiently large $|x| = 2R \ge 2R_1 = 2R_1(\alpha)$ and any $y \in B_R(x)$ we can bound $|F(y)| \le C|y|^2e^{2\tilde w(y)} \le CR^{2-2\nu+1-\alpha}$ to obtain

$$\begin{aligned}|\nabla w(x)| = |\nabla\tilde w(x)| &= \big|\int_{\mathbb{R}^2}\frac{x-y}{|x-y|^2}F(y)\,dy\big|\\ &\le \int_{\mathbb{R}^2\setminus B_R(x)}\frac{|F(y)|}{|x-y|}\,dy + \int_{B_R(x)}\frac{|F(y)|}{|x-y|}\,dy \le CR^{-1} + CR^{4-2\nu-\alpha} \le CR^{-\alpha}.\end{aligned}$$

Thus, for any $0<\alpha<1$ the number $\sup_{y\in\mathbb{R}^2}|y|^\alpha|\nabla w(y)|$ is attained. But from (4.26), (4.27), and (4.30) for any $0<\alpha<1$ with a constant $C_2>0$ independent of α we have

$$\begin{aligned}|x|^\alpha|\nabla w(x)| &\le \big(\frac{K_0}{2\pi} + |I(x)| + |II(x)|\big)|x|^{\alpha-1}\\ &\le \big(\frac{K_0}{2\pi} + C\varepsilon\sup_{y\in B_R(x)}|y||\nabla w(y)| + C\varepsilon\big)|x|^{\alpha-1} \qquad (4.31)\\ &\le \big(\frac{K_0}{2\pi} + C_2\varepsilon\big)|x|^{\alpha-1} + C_2\varepsilon\sup_{|y|\ge R}|y|^\alpha|\nabla w(y)|,\end{aligned}$$

where $\varepsilon \to 0$ as $|x| = 2R \to \infty$. Choose $R_2 > R_0$ such that $C_2\varepsilon < 1/2 \le \frac{K_0}{4\pi}$ for $R \ge R_2$ and set

$$\Phi_\alpha(R) = \sup_{|y|\ge R}|y|^\alpha|\nabla w(y)|,\ R \ge R_2.$$

Then, upon taking the supremum with respect to $|x| \ge 2R_2$ we obtain

$$\Phi_\alpha(2R_2) \le \frac{3K_0}{4\pi} + \frac12\Phi_\alpha(R_2) \le \frac{3K_0}{4\pi} + \frac12\Phi_\alpha(2R_2) + \frac12\sup_{R_2\le|y|\le 2R_2}|y|^\alpha|\nabla w(y)|.$$

Absorbing the second term on the right on the left side of this inequality and passing to the limit $\alpha \uparrow 1$, finally, we find the uniform bound

$$|y||\nabla w(y)| \le \lim_{\alpha\uparrow 1} \Phi_\alpha(2R_2) \le \frac{3K_0}{2\pi} + \sup_{R_2\le|y|\le 2R_2} |y||\nabla w(y)| < \infty$$

for every $|y| \ge 2R_2$, which in view of (4.30) concludes the proof of (4.28). ☐

5. "Bubbling" along the flow

Of course it is unreasonable to expect that Theorem 4.4 also holds true for non-minimizing critical points or even for Palais-Smale sequences of E in $\mathcal{C}^*_\lambda$ for small $\lambda > 0$. However, our derivation of the bounds (4.7) for total curvature is sufficiently flexible to allow obtaining similar bounds for metrics evolving under the prescribed curvature flow for f_λ when $t \to \infty$ in this limit regime. Moreover, the bounds from Lemmas 2.5 and 3.1 yield precisely the control on the error terms resulting from the presence of the time derivative that is required in Theorem 4.4.

5.1. **Bounds for total curvature along the flow.** Let $f_0 \le 0$ be a smooth, non-constant function with $\max f_0 = 0$ having only non-degenerate maxima p_0 where $f_0(p_0) = 0$, and for $0 < \lambda < \lambda_0$ let $f_\lambda = f_0 + \lambda$ as above where $\lambda_0 > 0$ is such that f_{λ_0} changes sign and satisfies (1.3).

For any $0 < \lambda < \lambda_0$ and any $\delta \in]-\delta_0, 0[$, with a number $\delta_0 = \delta_0(\lambda) > 0$ to be determined below, we then consider initial data $u^\delta_{0\lambda} \in \mathcal{C}^*_\lambda$ with

$$E(u^\delta_{0\lambda}) \le \beta_\lambda + \delta^2, \tag{5.1}$$

and let $u^\delta_\lambda = u^\delta_\lambda(t)$ with $\alpha^\delta_\lambda = \alpha^\delta_\lambda(t)$ be the smooth solution of (1.6), (1.7) for f_λ with data $u^\delta_\lambda(0) = u^\delta_{0\lambda}$ guaranteed by Theorem 1.4. Also let $g^\delta_\lambda = e^{2u^\delta_\lambda} g_b$.

Lemma 5.1. *We have*

$$\begin{aligned}\liminf_{\lambda\downarrow 0} \limsup_{\delta\uparrow 0} \limsup_{t\to\infty} \int_{M\times\{t\}} |K_{g^\delta_\lambda}|\, d\mu_{g^\delta_\lambda} \\ \le 2 \liminf_{\lambda\downarrow 0} \limsup_{\delta\uparrow 0} \limsup_{t\to\infty} (\lambda \alpha^\delta_\lambda(t)) \le 4 \liminf_{\lambda\downarrow 0} (\lambda|\beta'_\lambda|) \le 16\pi.\end{aligned}$$

Proof. We claim that for any $0 < \lambda < \lambda_0$, any $\delta \in]-\delta_0(\lambda), 0[$, similar to the proof of Lemma 4.1 above, for each $t \ge 0$ we have $u^\delta_\lambda + \delta f_\lambda \in \mathcal{C}_\mu$ for some $\mu = \mu(t) > \lambda$, where $C^{-1}|\delta| \le |\mu - \lambda| \le C|\delta|$ when $\delta_0 = \delta_0(\lambda) > 0$ is sufficiently small.

Indeed, for each $t \ge 0$ by (1.4), (1.12), and the mean value theorem there are numbers $\delta', \delta'' \in]\delta, 0[$ such that

$$\int_M f_\lambda e^{2(u^\delta_\lambda + \delta f_\lambda)}\, d\mu_{g_b} = 2\delta \int_M f^2_\lambda e^{2(u^\delta_\lambda + \delta' f_\lambda)}\, d\mu_{g_b}$$

and

$$\begin{aligned}\int_M e^{2(u^\delta_\lambda + \delta f_\lambda)}\, d\mu_{g_b} &= 1 + 2\delta \int_M f_\lambda e^{2u^\delta_\lambda}\, d\mu_{g_b} + 2\delta^2 \int_M f^2_\lambda e^{2(u^\delta_\lambda + \delta'' f_\lambda)}\, d\mu_{g_b} \\ &= 1 + 2\delta^2 \int_M f^2_\lambda e^{2(u^\delta_\lambda + \delta'' f_\lambda)}\, d\mu_{g_b}.\end{aligned}$$

Thus, $u^\delta_\lambda + \delta f_\lambda \in \mathcal{C}_\mu$ with $\mu = \mu(t)$ given by

$$\mu = \lambda - \frac{2\delta \int_M f^2_\lambda e^{2u^\delta_\lambda + \delta' f_\lambda)}\, d\mu_{g_b}}{1 + 2\delta^2 \int_M f^2_\lambda e^{2(u^\delta_\lambda + \delta'' f_\lambda)}\, d\mu_{g_b}} > \lambda. \tag{5.2}$$

Note that in view of the energy bound (5.1) the argument used to prove Lemma 2.3 gives a uniform in time lower bound

$$\int_M f_\lambda^2 e^{2(u_\lambda^\delta+\delta' f_\lambda)}\, d\mu_{g_b} \geq \Big(\int_M f_\lambda\, d\mu_{g_b}\Big)^2 \Big/ \int_M e^{-2(u_\lambda^\delta+\delta' f_\lambda)}\, d\mu_{g_b} \geq c = c(\lambda,\delta) > 0$$

for any $0 < \lambda < \lambda_0$ and any $\delta < 0$, uniformly in $\delta' \in]\delta, 0[$. In addition, clearly we can bound

$$\int_M f_\lambda^2 e^{2(u_\lambda^\delta+\delta' f_\lambda)}\, d\mu_{g_b} \leq \|f_\lambda\|_{L^\infty}^2 e^{2\delta\|f_\lambda\|_{L^\infty}} \int_M e^{2u_\lambda^\delta}\, d\mu_{g_b} = \|f_\lambda\|_{L^\infty}^2 e^{2\delta\|f_\lambda\|_{L^\infty}}$$

for all such $0 < \lambda < \lambda_0$ and $\delta < 0$, uniformly in $t \geq 0$ and $\delta' \in]\delta, 0[$. In consequence, for any $0 < \lambda < \lambda_0$ we can find $\delta_0 = \delta_0(\lambda) > 0$ such that with a uniform constant $C > 0$ independent of $t \geq 0$ and $\delta \in]-\delta_0, 0[$ there holds

$$C^{-1}|\delta| \leq |\mu - \lambda| \leq C|\delta|,$$

as desired. Moreover, for any $0 < \lambda < \lambda_0$ we obtain uniform in time bounds for $\alpha_\lambda^\delta(t)$ and $u_\lambda^\delta(t)$ as in Lemmas 2.4 and 2.5 independent of $\delta \in]-\delta_0, 0[$.

Next, as in the proof of Lemma 4.1 we expand

$$E(u_\lambda^\delta + \delta f_\lambda) = E(u_\lambda^\delta) + \delta \int_M \nabla u_\lambda^\delta \cdot \nabla f_\lambda\, d\mu_{g_b} + \delta^2 E(f_\lambda)$$

with $E(f_\lambda) = E(f_0)$. Using (1.6) to write

$$\begin{aligned}\int_M \nabla u_\lambda^\delta \cdot \nabla f_\lambda\, d\mu_{g_b} &= \int_M (-\Delta_{g_b} u_\lambda^\delta) f_\lambda\, d\mu_{g_b} = \int_M K_{g_\lambda^\delta} f_\lambda e^{2u_\lambda^\delta}\, d\mu_{g_b} \\ &= \alpha_\lambda^\delta(t) \int_M f_\lambda^2 e^{2u_\lambda^\delta}\, d\mu_{g_b} - \int_M u_{\lambda,t}^\delta f_\lambda e^{2u_\lambda^\delta}\, d\mu_{g_b},\end{aligned}$$

and recalling that with $F(t) = F_\lambda^\delta(t)$ on account of Lemma 3.1 and (1.12) we can bound

$$\Big|\int_M u_{\lambda,t}^\delta f_\lambda e^{2u_\lambda^\delta}\, d\mu_{g_b}\Big| \leq \|f_\lambda\|_{L^\infty} F^{1/2}(t) \to 0 \text{ as } t \to \infty,$$

with error $o(1) \to 0$ as $t \to \infty$ for each $t \geq 0$ we find

$$\begin{aligned}\beta_\mu \leq E(u_\lambda^\delta + \delta f_\lambda) &\leq E(u_\lambda^\delta) + \delta\alpha_\lambda^\delta \int_M f_\lambda^2 e^{2u_\lambda^\delta}\, d\mu_{g_b} + \delta^2 E(f_0) + o(1) \\ &\leq \beta_\lambda + \delta\alpha_\lambda^\delta \int_M f_\lambda^2 e^{2u_\lambda^\delta}\, d\mu_{g_b} + \delta^2(1 + E(f_0)) + o(1).\end{aligned}$$

where we also used (5.1) in the last inequality. But from (5.2) we obtain

$$2\delta \int_M f_\lambda^2 e^{2u_\lambda^\delta}\, d\mu_{g_b} = \lambda - \mu + 2\delta I,$$

where $I = \int_M f_\lambda^2 h e^{2u_\lambda^\delta}\, d\mu_{g_b}$ with

$$h = 1 - \frac{e^{2\delta' f_\lambda}}{1 + 2\delta^2 \int_M f_\lambda^2 e^{2(u_\lambda^\delta+\delta'' f_\lambda)}\, d\mu_{g_b}}$$

satisfying $\|h\|_{L^\infty} \leq C\delta$ by (1.12) may be bounded

$$|I| \leq \|g\|_{L^\infty}\|f_\lambda\|_{L^\infty}^2 \leq C\delta \leq C|\mu - \lambda|$$

for a uniform constant $C > 0$ independent of $t \geq 0$ and $\delta \in]-\delta_0, 0[$.

Thus, with error $o(1) \to 0$ as $t \to \infty$ we finally arrive at the estimate

$$\beta_\mu \le \beta_\lambda - \frac{\alpha_\lambda^\delta}{2}(\mu-\lambda) + O((\mu-\lambda)^2) + o(1)$$

analoguous to the time-independent case, from which we conclude the bound

$$\limsup_{t\to\infty} \alpha_\lambda^\delta(t) \le 2\limsup_{t\to\infty}\Big(\frac{\beta_\lambda-\beta_\mu}{\mu-\lambda} + O(\mu-\lambda)\Big).$$

Letting $\delta \uparrow 0$ we have $\mu \downarrow \lambda$ uniformly in $t > 0$, and for almost every $0 < \lambda < \lambda_0$ we obtain

$$\limsup_{\delta\uparrow 0}\limsup_{t\to\infty} \alpha_\lambda^\delta(t) \le 2\lim_{\mu\downarrow\lambda}\frac{\beta_\lambda-\beta_\mu}{\mu-\lambda} = 2|\beta'_\lambda|.$$

Multiplying with $\lambda > 0$, as in (4.5) we find

$$\liminf_{\lambda\downarrow 0}\limsup_{\delta\uparrow 0}\limsup_{t\to\infty}(\lambda\alpha_\lambda^\delta(t)) \le 2\liminf_{\lambda\downarrow 0}(\lambda|\beta'_\lambda|) \le 8\pi.$$

But from (1.6) and recalling that we may bound $|f_\lambda| \le -f_0 + \lambda = -f_\lambda + 2\lambda$, with the help of (1.4) we obtain the bound

$$\int_M |K_{g_\lambda^\delta}|\,d\mu_{g_\lambda^\delta} \le \int_M |u_{\lambda,t}^\delta|\,d\mu_{g_\lambda^\delta} + \alpha_\lambda^\delta\int_M |f_\lambda|\,d\mu_{g_\lambda^\delta} \le F^{1/2}(t) + 2\lambda\alpha_\lambda^\delta(t)$$

at any time $t \ge 0$, and our claim follows. □

5.2. **"Bubbling" of the prescribed curvature flow.** We now have all the ingredients required for proving Theorem 1.5.

Proof of Theorem 1.5. By Lemma 5.1 there is a sequence $\lambda_k \downarrow 0$ such that

$$\sup_{k\in\mathbb{N}}\limsup_{\delta\uparrow 0}\limsup_{t\to\infty}\big(\lambda_k\alpha_{\lambda_k}^\delta(t) - 1/k\big) \le 8\pi.$$

We may then fix a sequence $\delta_k \uparrow 0$ satisfying

$$\sup_{k\in\mathbb{N}}\sup_{\delta_k\le\delta<0}\limsup_{t\to\infty}\big(\lambda_k\alpha_{\lambda_k}^\delta(t) - 2/k\big) \le 8\pi.$$

Choosing $\delta = \delta_k$ for each $k \in \mathbb{N}$, finally, for suitable $T_k \to \infty$ also satisfying $F_k(t) = F_{\lambda_k}^{\delta_k}(t) \le 1/k$ for $t \ge T_k$ we find the uniform bound

$$\sup_{t\ge T_k}\int_{M\times\{t\}} |K_{g_{\lambda_k}^{\delta_k}}|\,d\mu_{g_{\lambda_k}^{\delta_k}} \le 2\sup_{t\ge T_k}\big(\lambda_k\alpha_{\lambda_k}^{\delta_k}(t) + F_k(t)^{1/2}\big) \le 16\pi + 6/k \tag{5.3}$$

for all $k \in \mathbb{N}$.

Thus, if for each $k \in \mathbb{N}$ for any $t_k \ge T_k$ we let $v_k = u_{\lambda_k}^{\delta_k}(t_k)$, satisfying (4.8) with $\alpha_k = \alpha_{\lambda_k}^{\delta_k}(t_k)$ and $h_k = u_{\lambda_k,t}^{\delta_k}(t_k)$, our Theorem 4.4 applies and we find that the flows $\big(u_{\lambda_k}^{\delta_k}(t)\big)_{t\ge T_k,\,k\in\mathbb{N}}$ exhibit (at most 2) spherical "bubbles" as $k\to\infty$. □

6. The degenerate case

Conceivably, as in the analysis [27] of "bubbling" solutions to the prescribed curvature flow on S^2, also the prescribed curvature flow for f_λ on the torus for sufficiently small $\lambda > 0$ will only exhibit a single "bubble"; moreover, similar to [27], Proposition 4.9, we expect the center of this "bubble" to move approximately in direction of the negative gradient of the prescribed curvature function f_λ. [2] If,

[2]Note, however, that in [27] we only consider flows that blow up as $t \to \infty$ and hence do not even subsequentially converge.

finally, one could establish such a result even in the case when the maxima of f_0 are allowed to be degenerate, by following the construction of Topping [29], p. 609, for the heat flow of harmonic maps one could then use this to give an example of a prescribed curvature flow with non-unique subsequential limits as $t \to \infty$ (in contrast with the analytic case studied by Ngô-Xu [21]).

6.1. **Topping's example.** Recall that Topping [29] considered the heat flow of harmonic maps from the standard 2-sphere S^2 to $\mathbb{R}^2 \times S^2$ or $T^2 \times S^2$, where the target is endowed with the warped metric $d\mu_{\mathbb{R}^2}(x)+f(x)d\mu_{S^2}(y)$ for suitable $f \geq 1$. With the fixed harmonic component $u_0 = id\colon S^2 \to S^2$ and with $z = z(t)$ satisfying $dz/dt = -\nabla f(z)$ then $u(x,t) = (z(t), u_0(x))$ is a solution to the flow. Letting $f = 1 - f_0$, where in polar coordinates (r, θ) on a Euclidean disc $B_2(0)$ the function f_0 is given by $f_0(r,\theta) = 0$ for $r \leq 1$ and by

$$f_0(r,\theta) = -e^{-\frac{1}{r-1}}\big(\sin(\frac{1}{r-1}+\theta)+2\big) \tag{6.1}$$

for $r > 1$, scaled and smoothly extended to the torus, for suitable data $z_0 = z(0)$ Topping then obtains a flow where the "center" $z(t)$ follows a "spiralling groove on a gramophone record" and thus has non-unique subsequential limits as $t \to \infty$.

6.2. **"Bubbling" in the degenerate case.** Returning to the prescribed curvature flow, given a smooth function f_0 with $f_0 \leq 0 = \max f_0$, let

$$M_0 = \{p \in M;\ f_0(p) = 0\}$$

and set $d(p) = dist(p, M_0)$. As a possible replacement of the non-degeneracy condition in Theorem 4.4 that also applies when f_0 may have degenerate maxima we propose the following condition.

Condition A: There exist $d_0 > 0$ and $A_0 > 0$ such that, letting

$$K_0 = \{x = (x^1, x^2) \in \mathbb{R}^2;\ |x^1| < x^2,\ |x| < d_0\},$$

for any $p \in M$ with $0 < d(p) < d_0$ there is a rotated copy $K_p \subset \mathbb{R}^2$ of K_0 such that in Euclidean coordinates x around $p = 0$ there holds

$$A_0 \inf_{x \in K_p} |f_0(x)| \geq |f_0(p)|. \tag{6.2}$$

Clearly, Condition A is satisfied by any function f_0 with only non-degenerate maximum points. Moreover, the function f_0 given by (6.1) satisfies Condition A (with $A_0 = 3$).

As a first step in the direction outlined above we then have the following result.

Theorem 6.1. *Let $f_0 \leq 0$ be a smooth, non-constant function with $\max f_0 = 0$ satisfying Condition A with constants $d_0, A_0 > 0$. Then for any (v_k) as in Lemma 4.3 above for suitable $i_0 \in \mathbb{N}$, $r_k^{(i)} \downarrow 0$, $p_k^{(i)} \to p_\infty^{(i)} \in M$ with $f_0(p_\infty^{(i)}) = 0$, $1 \leq i \leq i_0$, as $k \to \infty$ suitably the following holds.*

i) We have $v_k \to -\infty$ locally uniformly on $M_\infty = M \setminus \{p_\infty^{(i)};\ 1 \leq i \leq i_0\}$.

ii) For each $1 \leq i \leq i_0$ in Euclidean coordinates x around $p_k^{(i)} = 0$ we have

$$w_k^{(i)}(x) := v_k(r_k^{(i)}x) + \log r_k^{(i)} \to w_\infty(x)$$

in H^2_{loc} on $\mathbb{R}^2$, where w_∞ induces a metric $g_\infty = e^{2w_\infty} g_{\mathbb{R}^2}$ on $\mathbb{R}^2$ of locally bounded curvature, and $1 \leq i_0 \leq 4$.

Proof. Since non-degeneracy of the maxima p_0 where $f_0(p_0) = 0$ is not required for either Lemma 4.3 or (4.9) to hold, as in the proof of Theorem 4.4 we can find a subsequence $k \to \infty$ such that

$$K^+_{g_k} e^{2v_k}\, d\mu_{g_b} \rightharpoonup K_+ + \sum_{i \in I} \gamma_i \delta_{p^{(i)}_\infty}$$

weakly-$*$ in the sense of measures, with a measure $K_+ \geq 0$ on M having no atoms and with at most countably many atoms of weight $\gamma_i > 0$, $i \in I \subset \mathbb{N}$, and where $\gamma_i \geq \pi/2$ if only if $1 \leq i \leq i_0$ for some $i_0 \geq 1$. In particular, with error $o(1) \to 0$ as $k \to \infty$ we have

$$\pi/2 - o(1) \leq \alpha_k \lambda_k \leq 8\pi + o(1) \quad \text{for all } k \in \mathbb{N}. \tag{6.3}$$

analogous to (4.14).

i) The proof of statement i) above then is identical with the proof of the corresponding statement in Theorem 4.4, and we only have to show that also with the weaker Condition A on f_0 we are able to extract "bubbles" from the metrics $g_k = e^{2v_k} g_{g_b}$, $k \in \mathbb{N}$, and estimate their number.

ii) Thus let $p_0 = p^{(i)}_\infty$ for some $i \leq i_0$. With $\bar{h}_k = \int_M h_k e^{2v_k}\, d\mu_{g_b}$ satisfying $|\bar{h}_k| \leq \varepsilon_k \to 0$ $(k \to \infty)$ we split $v_k = w_k + z_k$, where

$$-\Delta_{g_b} z_k = (h_k - \bar{h}_k) e^{2v_k} \quad \text{with } \int_M z_k\, d\mu_{g_b} = 0,$$

so that $\|z_k\|_{L^\infty} \to 0$ as $k \to \infty$ as before. In Euclidean coordinates around $p_0 = 0$ then for any sufficiently small $R > 0$ we again can find points $x_k \to 0$ and radii $r_k \to 0$ such that

$$\int_{B_{r_k}(x_k)} K^+_{g_k}\, d\mu_{g_k} = \sup_{x_0 \in B_R(0)} \int_{B_{r_k}(x_0)} K^+_{g_k}\, d\mu_{g_k} = \pi/3. \tag{6.4}$$

Define

$$\tilde{w}_k(x) = w_k(x_k + r_k x) + \log r_k, \ \ \tilde{z}_k(x) = z_k(x_k + r_k x)$$

on $D_k = \{x;\ |x_k + r_k x| < R\}$, where again we note that the domains D_k exhaust $\mathbb{R}^2$ as $k \to \infty$. There holds the equation

$$-\Delta \tilde{w}_k = \big(\alpha_k f_{\lambda_k}(x_k + r_k x) + \bar{h}_k\big) e^{2(\tilde{w}_k + \tilde{z}_k)} = \tilde{f}_k e^{2\tilde{z}_k} e^{2\tilde{w}_k} \text{ on } D_k,$$

where

$$\tilde{f}_k(x) = \alpha_k f_{\lambda_k}(x_k + r_k x) + \bar{h}_k = \alpha_k \lambda_k \big(1 + \lambda_k^{-1} f_0(x_k + r_k x) + (\alpha_k \lambda_k)^{-1} \bar{h}_k\big).$$

Note that with x_k chosen such that $w_k(x_k) = \max_{B_R(p_0)} w_k$ and with r_k such that $|w_k(x_k) + \log r_k| \leq C$ as in the proof of Theorem 4.4 above, in the present degenerate case we are unable to assert a bound for r_k^2/λ_k and therefore cannot show that $\lambda_k^{-1} f_0(x_k + r_k x)$ is locally bounded from below. However, the choice (6.4) now allows to proceed with the tools developed in the proof of Lemma 4.3 and come to similar conclusions as in Theorem 4.4.

Indeed, setting $\gamma = 1/20$ and splitting $\tilde{w}_k = \tilde{w}^{(+)}_k + \tilde{w}^{(0)}_k + \tilde{w}^{(-)}_k$ on any ball $B = B_1(x_0)$, where $\Delta \tilde{w}^{(0)}_k = 0$ in B with $\tilde{w}^{(0)}_k = w_k$ on ∂B, and where $\tilde{w}^{(\pm)}_k \in H^1_0(B)$ with $\Delta \tilde{w}^{(\pm)}_k = (\Delta \tilde{w}_k)^\pm$, similar to the proof of Lemma 4.3 from (6.4) and (4.9) we obtain uniform bounds

$$\|e^{\tilde{w}^{(+)}_k|}\|_{L^p(B)} + \|e^{-\gamma \tilde{w}^{(-)}_k}\|_{L^p(B)} \leq C \quad \text{for any } 1 \leq p < 4,$$

as well as

$$\|\tilde{w}_k^{(\pm)}\|_{W^{1,p}(B)} \le C \quad \text{for any } 1 \le p < 2.$$

Moreover, estimating $\tilde{w}_k^{(0)} \le \tilde{w}_k - \tilde{w}_k^{(-)}$, for any disc $D \subset B$ we again find a uniform bound

$$exp\big(\fint_D \gamma\tilde{w}_k^{(0)}\, dx\big) \le \fint_D e^{\gamma\tilde{w}_k^{(0)}}\, dx \le C\Big(\int_B e^{2\gamma\tilde{w}_k}\, dx \cdot \int_B e^{-2\gamma\tilde{w}_k^{(-)}}\, dx\Big)^{1/2} \le C.$$

Continuing to argue as in the proof of Lemma 4.3 then we have a uniform bound $\tilde{w}_k^{(0)} \le C_1$ on $B_{2/3}(x_0)$, and either $|\tilde{w}_k^{(0)}| \le C$ on $B_{1/2}(x_0)$ for all $k \in \mathbb{N}$, or there exists a subsequence such that $\tilde{w}_k^{(0)} \to -\infty$ uniformly on $B_{1/2}(x_0)$ as $k \to \infty$ suitably. In any event, for any ball B, upon covering B with finitely many balls $B_{1/2}(x_j)$, $1 \le j \le J$, and estimating $\tilde{w}_k \le \tilde{w}_k^{(+)} + \tilde{w}_k^{(0)} \le \tilde{w}_k^{(+)} + C_1$ on each ball, we find

$$\|e^{\tilde{w}_k}\|_{L^p(B)} \le C \sum_{1\le j\le J} \|e^{\tilde{w}_k^{(+)}}\|_{L^p(B_{1/2}(x_j))} \le C \quad \text{for any } 1 \le p < 4.$$

Choosing $p = 3$, as before we conclude that $\tilde{w}_k^{(+)} \le C\|\tilde{w}_k^{(+)}\|_{W^{2,3/2}} \le C$ uniformly on any B, if $k \in \mathbb{N}$ is sufficiently large.

In particular, letting $B = B_1(0)$ and bounding $\tilde{w}_k \le \tilde{w}_k^{(+)} + \tilde{w}_k^{(0)} \le C + \tilde{w}_k^{(0)}$ on each $B_{1/2}(x_j)$ in our cover of B, if we suppose that $\tilde{w}_k^{(0)} \to -\infty$ uniformly on some $B_{1/2}(x_{j_0})$ we find that also $\tilde{w}_k \to -\infty$ uniformly on $B_{1/2}(x_{j_0})$ and hence on every $B_{1/2}(x_j)$. But then $\tilde{w}_k \to -\infty$ uniformly also on B, and in view of (6.3) with error $o(1) \to 0$ as $k \to \infty$ there results

$$\begin{aligned}\int_{B_{r_k}(x_k)} K_{g_k}^+\, d\mu_{g_k} &\le \int_{B_1(0)} \alpha_k\lambda_k e^{2\tilde{w}_k}\, dx + o(1)\\ &\le C\int_{B_1(0)} e^{2\tilde{w}_k}\, dx + o(1) \to 0,\end{aligned} \tag{6.5}$$

contradicting (6.4). Hence we have $|\tilde{w}_k^{(0)}| \le C$ on $B_{1/2}(0)$ and therefore on every ball $B_{1/2}(x_0)$ for sufficiently large $k \in \mathbb{N}$. By harmonicity we then may assume that a subsequence $\tilde{w}_k^{(0)}$ converges smoothly on every ball $B_{1/2}(x_0)$. Since we also have that $\|\tilde{w}_k^{(\pm)}\|_{W^{1,3/2}(B)} \le C$ on every $B = B_1(x_0)$, we conclude that a subsequence $\tilde{w}_k \rightharpoonup \tilde{w}_\infty$ weakly in $W^{1,3/2}_{loc}$ and almost everywhere on $\mathbb{R}^2$, where $\tilde{w}_\infty \le C$ is bounded from above.

Fatou's lemma now yields that for any $L \in \mathbb{N}$ we have

$$\int_{B_L(0)} e^{2\tilde{w}_\infty}\, dx \le \liminf_{k\to\infty} \int_{B_L(0)} e^{2(\tilde{w}_k+\tilde{z}_k)}\, dx \le \liminf_{k\to\infty}\int_M e^{2v_k}\, d\mu_{g_b} = 1.$$

Passing to the limit $L \to \infty$ we find $e^{2\tilde{w}_\infty} \in L^1(\mathbb{R}^2)$ with

$$\int_{\mathbb{R}^2} e^{2\tilde{w}_\infty}\, dx = \lim_{L\to\infty}\int_{B_L(0)} e^{2\tilde{w}_\infty}\, dx \le 1. \tag{6.6}$$

For $k \in \mathbb{N}$ set $\tilde{f}_{0k}(x) := \lambda_k^{-1} f_0(x_k + r_k x) \le 0$ so that

$$\tilde{f}_k(x) = \alpha_k f_{\lambda_k}(x_k + r_k x) + \bar{h}_k = \alpha_k\lambda_k\big(1 + \tilde{f}_{0k} + (\alpha_k\lambda_k)^{-1}\bar{h}_k\big).$$

By (6.3) we may assume that $\alpha_k\lambda_k \to \mu > 0$ as $k \to \infty$. Recalling that $|\bar{h}_k| \to 0$ and $\|z_k\|_{L^\infty} \to 0$ as $k \to \infty$, and writing $|\tilde{f}_{0k}| = -\tilde{f}_{0k} = 1 - (1 + \tilde{f}_{0k})$, by Fatou's lemma and (1.4) for any $L \in \mathbb{N}$ we have

$$\begin{aligned}
&\mu \int_{B_L(0)} \liminf_{k\to\infty} |\tilde{f}_{0k}| e^{2\tilde{w}_\infty}\, dx \le \liminf_{k\to\infty} \int_{B_L(0)} \alpha_k\lambda_k |\tilde{f}_{0k}| e^{2\tilde{w}_k}\, dx \\
&= \liminf_{k\to\infty} \int_{B_L(0)} (\alpha_k\lambda_k - \tilde{f}_k) e^{2\tilde{w}_k}\, dx \le \liminf_{k\to\infty} \int_M (\alpha_k\lambda_k - \alpha_k f_{\lambda_k}) e^{2v_k}\, d\mu_{g_b} \qquad (6.7)\\
&= \liminf_{k\to\infty} \int_M \alpha_k\lambda_k e^{2v_k}\, d\mu_{g_b} = \mu \le 8\pi < \infty.
\end{aligned}$$

We can use this result together with Condition A to show that $(\tilde{f}_{0k})$ is locally bounded. Indeed, suppose that for some sequence $y_k \to y_0$ in $\mathbb{R}^2$ there holds $|\tilde{f}_{0k}(y_k)| = |f_0(q_k)|/\lambda_k \to \infty$ as $k \to \infty$, where $q_k = x_k + r_k y_k$, $k \in \mathbb{N}$. Then $q_k \to 0$; hence $d(q_k) \to 0$ as $k \to \infty$, and we may assume that $d(q_k) < d_0$ for all k. By Condition A there exist cones K_{q_k} with vertex q_k such that

$$A_0 \inf_{y\in Q_k} |\tilde{f}_{0k}(y)| = A_0\lambda_k^{-1} \inf_{q\in K_{q_k}} |f_0(q)| \ge \lambda_k^{-1}|f_0(q_k)| = |\tilde{f}_{0k}(y_k)| \to \infty \qquad (6.8)$$

as $k \to \infty$, where the cones (with a suitable labelling of coordinates)

$$\begin{aligned}
Q_k &= \{y;\ x_k + r_k y \in K_{q_k}\} \\
&= \{y = (y^1, y^2);\ |y^1 - y_k^1| < y^2 - y_k^2,\ |y - y_k| < d_0/r_k\}
\end{aligned}$$

as $k \to \infty$ suitably exhaust the cone $Q_0 = \{y = (y^1, y^2);\ |y^1 - y_0^1| < y^2 - y_0^2\}$. From (6.7) and (6.8) there then results a contradiction.

Hence we may assume that $\tilde{f}_k e^{2\tilde{w}_k} dx \rightharpoonup \mu(1 + \tilde{f}_{0\infty}) e^{2\tilde{w}_\infty} dx$ weakly-$*$ in the sense of measures, where $\tilde{f}_{0\infty} \le 0$ is locally bounded from below. Setting

$$\tilde{F}_\infty := \mu(1 + \tilde{f}_{0\infty}) e^{2\tilde{w}_\infty}$$

with $\int_{\mathbb{R}^2} |\tilde{F}_\infty| dx \le 16\pi$ by (4.7), then $\tilde{w}_\infty$ weakly solves the equation $-\Delta \tilde{w}_\infty = \tilde{F}_\infty$ on $\mathbb{R}^2$.

In order to obtain a lower bound on the volume of the metric $\tilde{g}_\infty = e^{2\tilde{w}_\infty} g_{\mathbb{R}^2}$ we once more follow Chen-Li [9] and set

$$w_\infty = -\frac{1}{2\pi} \int_{\mathbb{R}^2} (\log|x - y| - \log|y|) \tilde{F}_\infty(y)\, dy$$

with $|w_\infty| \le C \log(2 + |x|)$ and satisfying

$$\frac{w_\infty(x)}{\log|x|} = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{\log|x - y| - \log|y|}{\log|x|} \tilde{F}_\infty(y)\, dy \to -\frac{1}{2\pi} \int_{\mathbb{R}^2} \tilde{F}_\infty\, dy =: -\nu \qquad (6.9)$$

as $|x| \to \infty$ for some $|\nu| \le 8$. The mean value property of harmonic functions and the bound $\tilde{w}_\infty - w_\infty \le C + C\log(2 + |x|)$ then again yield the existence of a constant $C \in \mathbb{R}$ such that $w_\infty = \tilde{w}_\infty + C$. Since $e^{2\tilde{w}_\infty} \in L^1(\mathbb{R}^2)$ in view of (6.6), we then also have $e^{2w_\infty} \in L^1(\mathbb{R}^2)$. It follows that $\nu \ge 1$, and we can estimate

$$\mu \int_{\mathbb{R}^2} e^{2\tilde{w}_\infty}\, dx \ge \int_{\mathbb{R}^2} \tilde{F}_\infty\, dx \ge 2\pi.$$

Since $\mu \le 8\pi$ we conclude that $\int_{\mathbb{R}^2} e^{2\tilde{w}_\infty}\, dx \ge 1/4$. But the volumes of all bubbles obtained by rescaling the normalized metrics can add up to at most 1. Hence there can be at most four points where bubbles form, and $i_0 \le 4$. □

References

[1] Bartolucci, Daniele; De Marchis, Francesca; Malchiodi, Andrea: *Supercritical conformal metrics on surfaces with conical singularities*, Int. Math. Res. Not. IMRN 2011, no. 24, 5625-5643.

[2] Bartolucci, D.; Tarantello, G.: *Liouville type equations with singular data and their applications to periodic multivortices for the electroweak theory*, Comm. Math. Phys. 229 (2002), no. 1, 3-47.

[3] Borer, Franziska; Galimberti, Luca; Struwe, Michael: *"Large" conformal metrics of prescribed Gauss curvature on surfaces of higher genus*, Comment. Math. Helv. 90 (2015), no. 2, 407-428.

[4] Brendle, Simon: *Global existence and convergence for a higher order flow in conformal geometry*, Ann. of Math. (2) 158 (2003), no. 1, 323-343.

[5] Brezis, Haim; Merle, Frank: *Uniform estimates and blow-up behavior for solutions of* $-\Delta u = V(x)e^u$ *in two dimensions*. Comm. Partial Differential Equations 16 (1991), no. 8-9, 1223-1253.

[6] Buzano, Reto; Schulz, Mario; Struwe, Michael: *Variational methods in geometric analysis*, to appear.

[7] Carlotto, Alessandro; Malchiodi, Andrea: *Weighted barycentric sets and singular Liouville equations on compact surfaces*, J. Funct. Anal. 262 (2012), no. 2, 409-450.

[8] Chang, Sun-Yung Alice: *Non-linear elliptic equations in conformal geometry*, Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2004.

[9] Chen, Wen Xiong; Li, Congming: *Classification of solutions of some nonlinear elliptic equations*, Duke Math. J. 63 (1991), no. 3, 615-622.

[10] Cheng, Kuo-Shung; Lin, Chang-Shou: *On the asymptotic behavior of solutions of the conformal Gaussian curvature equations in* $\mathbb{R}^2$. Math. Ann. 308 (1997), 119-139.

[11] Chow, Bennett: *The Ricci-Hamilton flow on the 2-sphere*, J. Differential Geom. 33 (1991), no. 2, 325-334.

[12] del Pino, Manuel; Román, Carlos: *Large conformal metrics with prescribed sign-changing Gauss curvature*, Calc. Var. Partial Differential Equations 54 (2015), no. 1, 763-789.

[13] Ding, Wei Yue; Liu, Jiaquan: *A note on the prescribing Gaussian curvature on surfaces*, Trans. Amer. Math. Soc. 347 (1995), 1059-1066.

[14] Escobar, José F.; Schoen, Richard M.: *Conformal metrics with prescribed scalar curvature*, Invent. Math. 86 (1986), no. 2, 243-254.

[15] Galimberti, Luca: *Compactness issues and bubbling phenomena for the prescribed Gaussian curvature equation on the torus*, Calc. Var. Partial Differential Equations 54 (2015), no. 3, 2483-2501.

[16] Hamilton, Richard S.: *The Ricci flow on surfaces*, Mathematics and general relativity (Santa Cruz, CA, 1986), 237-262, Contemp. Math., 71, Amer. Math. Soc., Providence, RI, 1988.

[17] Kazdan, Jerry L.; Warner, F. W.: *Curvature functions for compact 2-manifolds*. Ann. of Math. (2) 99 (1974), 14-47.

[18] Kazdan, Jerry L.; Warner, F. W.: *Scalar curvature and conformal deformation of Riemannian structure*. J. Differential Geometry 10 (1975), 113-134.

[19] Malchiodi, Andrea; Ruiz, David: *New improved Moser-Trudinger inequalities and singular Liouville equations on compact surfaces*, Geom. Funct. Anal. 21 (2011), no. 5, 1196-1217.

[20] Moser, Jürgen: *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** (1971), 1077–1092.

[21] Ngô, Qu´ôc Anh; Xu, Xingwang: *Slow convergence of the Gaussian curvature flow on closed Riemannian surfaces with vanishing Euler characteristic*, preprint (2015).

[22] Ngô, Qu´ôc Anh; Xu, Xingwang: *private communication*.

[23] Struwe, Michael: *Critical points of embeddings of* $H_0^{1,n}$ *into Orlicz spaces*. Ann. Inst. H. Poincaré Anal. Non Linéaire 5 (1988), no. 5, 425-464.

[24] Struwe, Michael: *The existence of surfaces of constant mean curvature with free boundaries*. Acta Math. 160 (1988), no. 1-2, 19-64.

[25] Struwe, Michael: *Une estimation asymptotique pour le modèle de Ginzburg-Landau*. [An asymptotic estimate for the Ginzburg-Landau model] C. R. Acad. Sci. Paris Sr. I Math. 317 (1993), no. 7, 677-680.

[26] Struwe, Michael: *Curvature flows on surfaces*, Annali Sc. Norm. Sup. Pisa, Ser. V, 1 (2002), 247-274.

[27] Struwe, Michael: *A flow approach to Nirenberg's problem*, Duke Math. J. 128 (2005), no. 1, 19-64.

[28] Tarantello, Gabriella: *Analytical, geometrical and topological aspects of a class of mean field equations on surfaces*, Discrete Contin. Dyn. Syst. 28 (2010), no. 3, 931-973.

[29] Topping, Peter Miles: *Rigidity in the harmonic map heat flow*, J. Differential Geom. 45 (1997), no. 3, 593-610.

[30] Troyanov, Marc: *Prescribing curvature on compact surfaces with conical singularities*, Trans. Amer. Math. Soc. 324 (1991), no. 2, 793-821.

[31] Trudinger, Neil S.: *On embeddings into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), 473-484.

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