EQUIVARIANT WAVE MAPS IN TWO SPACE DIMENSIONS

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Abstract. Singularities of co-rotational wave maps from $(1+2)$-dimensional Minkowski space into a surface N of revolution after a suitable rescaling give rise to non-constant co-rotational harmonic maps from S^2 into N. In consequence, for non-compact target surfaces of revolution the Cauchy problem for wave maps is globally well-posed.

1. Corotational wave maps

Let N be a surface of revolution with metric

$$ds^2 = d\rho^2 + g^2(\rho)d\theta^2,$$

where $\theta \in S^1$ and with $g \in C^\infty(\mathbb{R})$ satisfying $g(0) = 0$, $g'(0) = 1$. Moreover, we assume that g is odd and either

$$g(\rho) > 0 \text{ for all } \rho > 0 \tag{1}$$

with

$$\int_0^\infty |g(\rho)|\, d\rho = \infty, \tag{2}$$

or, if N is compact, that g has a first zero $\rho_1 > 0$ where $g'(\rho_1) = -1$, and that g is periodic with period $2\rho_1$. Note that in this second case assumption (2) is trivially satisfied.

The case (1) corresponds to non-compact surfaces; condition (2) is a technical assumption needed to rule out that N contains a "sphere at infinity".

We regard (ρ, θ) as polar coordinates on N. Letting (r, ϕ) be the usual polar coordinates on $\mathbb{R}^2$, we then consider equivariant wave maps $U : \mathbb{R} \times \mathbb{R}^2 \to N$ given by

$$\rho = u(t, r), \theta = \phi.$$

The wave map system

$$\Box U^a = \Gamma^a_{bc}(U)\partial_\mu U^b \partial^\mu U^c, \tag{3}$$

written in terms of a local coordinate system on N and standard Cartesian coordinates $t = x^0, x^1, x^2$ on Minkowski space, in this co-rotational case simplifies to the nonlinear scalar equation

$$\Box u + \frac{f(u)}{r^2} = 0, \tag{4}$$

where $\Box u = u_{tt} - \Delta u = u_{tt} - u_{rr} - \frac{u_r}{r}$ and with $f(u) = g(u)g'(u)$.

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In [5], Shatah and Tahvildar-Zadeh showed that the initial value problem for (3) with smooth equivariant data

$$(U, U_t)_{|t=0} = (U_0, U_1) \tag{5}$$

of finite energy admits a unique smooth solution for small time, which may be extended for all time if the target surface N is geodesically convex.

The latter is equivalent to the assumption $g'(\rho) \geq 0$ for all $\rho > 0$. This condition was later weakened by Grillakis [2] who showed that it suffices to assume $g(\rho) + g'(\rho)\rho > 0$ for $\rho > 0$. Note that this hypothesis, in particular, implies conditions (1) and (2).

Here, we further improve these results; moreover, we relate the possible appearance of singularities to the existence of (non-constant) harmonic maps $\overline{U}\colon S^2 \to N$, thereby confirming a long-standing conjecture about wave maps in this special, co-rotational case. The main new technical ingredient is Lemma 3.3 below. Also remark that recent numerical work of Bizon et al. [1] strongly supports the belief that for suitable initial data equivariant wave maps $U\colon \mathbb{R} \times \mathbb{R}^2 \to S^2$ indeed may develop singularities in finite time.

2. Results

Singularities of co-rotational maps may be detected by measuring their energy

$$E(U(t), R) = \int_{B_R(0)} |DU(t)|^2 \, dx,$$

with density $|DU|^2 = |U_t|^2 + |\nabla U|^2$. In terms of $u = u(t)$ we have

$$E(U(t), R) = 2\pi \int_0^R \left(|Du|^2 + \frac{g^2(u)}{r^2} \right) rdr.$$

We also let

$$E(U(t)) = \lim_{R\to\infty} E(U(t), R).$$

By the results of Shatah-Tahvildar-Zadeh [5] there exists a number $\varepsilon_0 = \varepsilon_0(N) > 0$ such that the Cauchy problem for co-rotational wave maps for smooth data with energy $E(U(0)) < \varepsilon_0$ admits a global smooth solution; confer also [4], Theorem 8.1. By finite speed of propagation, similarly we obtain well-posedness of the Cauchy problem for time $t \leq R$, provided $E(U(0), R) < \varepsilon_0$.

Conversely, let $U\colon [0, t_0[\times\mathbb{R}^2 \to N$ be a smooth co-rotational wave map. Then $z_0 = (t_0, 0)$ is a (first) singularity and t_0 is the blow-up time of U if and only if there holds

$$\inf_{0\leq t<t_0} E(U(t), t_0 - t) \geq \varepsilon_0 > 0. \tag{6}$$

In fact, negating condition (6) we can find a time $t < t_0$ such that

$$E(U(t), R) < \varepsilon_0$$

for some $R > t_0 - t$ and the results quoted above will allow us to extend U smoothly as a solution to (3) on a neighborhood of $z_0 = (t_0, 0)$. Observe that, by symmetry, U can only blow up at the origin.

We can now state our main result.

Theorem 2.1. *Let U be a (smooth) co-rotational solution to* (3) *blowing up at time t_0. Then there exist sequences $r_i \downarrow 0, t_i \uparrow t_0 (i \to \infty)$ such that*

$$U_i(t,x) = U(t_i + r_i t, r_i x) \to U_\infty(t,x)$$

strongly in $H^1_{\mathrm{loc}}(]-1,1[\times\mathbb{R}^2)$, where U_∞ is a non-constant, time-independent solution of (3) *giving rise to a non-constant, smooth equivariant harmonic map $\overline{U}\colon S^2 \to N$.*

As a consequence, for target manifolds that do not admit non-constant harmonic spheres we obtain global existence of smooth solutions to the Cauchy problem (3), (5) for co-rotational wave maps with smooth equivariant data. In particular, we can improve Grillakis' result as follows.

Theorem 2.2. *Suppose N is a surface of revolution with metric $ds^2 = d\rho^2 + g^2(\rho)d\theta^2$ satisfying* (1) *and* (2)*. Then for any smooth equivariant data the Cauchy problem* (3), (5) *admits a unique global smooth solution.*

Similar results also hold true in the case of radially symmetric wave maps $U = U(t,r)$ from $\mathbb{R}^{1+2}$ to a sphere [6].

3. Proofs

Let $U\colon [0,t_0[\times\mathbb{R}^2 \to N$ be a smooth co-rotational wave map blowing up at time t_0 and let $u = u(t,r)$ be the associated solution of (4).

For convenience we shift and reverse time and then scale our space-time coordinate $z = (t,x)$ so that in our new coordinates U is an equivariant solution to (3) on $]0,1] \times \mathbb{R}^2$ blowing up at the origin.

Letting

$$K^T = \{z = (t,x); 0 \le |x| \le t \le T\}$$

be the forward light cone with vertex at the origin, truncated at height T, with lateral boundary

$$M^T = \{(t,x) \in K^T; |x| = t\},$$

we also introduce the flux

$$\mathrm{Flux}(U,T) = \int_{M^T} |D^{||}U|^2\, do = 2\pi \int_0^T \left(|u_t + u_r|^2 + \frac{g^2(u)}{r^2} \right) \Big|_{t=r}\, r\, dr.$$

Here, $|D^{||}U|^2$ denotes the energy of all derivatives in directions tangent to M^T.

We recall the basic estimates for (3) from [5]; confer also [4], Chapter 8.1.

Energy inequality: For any $t, \tau, R > 0$ there holds

$$E(U(t),R) \le E(U(t+\tau), R + |\tau|). \tag{7}$$

Of course, in the present case we only consider values such that $0 < t, t+\tau \le 1$.

Flux decay: As $T \to 0$ there holds

$$\mathrm{Flux}(U,T) \to 0. \tag{8}$$

Uniform bounds. Suppose that N satisfies (2). Then for any $R > 0$ we have

$$\sup_{r<R} |u(t,r)| \le C(E(U(t),R)), \tag{9}$$

where $C(s) \to 0$ as $s \to 0$; see [5], Lemma 2.1.

Exterior energy decay: For any $0 < \lambda \le 1$ as $t \to 0$ there holds

$$E(U(t),t) - E(U(t),\lambda t) \to 0; \tag{10}$$

see [5], Lemma 2.2.

Decay of time derivatives ([5], Corollary 2.3). Suppose that N satisfies (2). Then

$$\frac{1}{T}\int_{K^T} |U_t|^2\,dz \to 0 \text{ as } T \to 0. \tag{11}$$

Finally, as is also well-known, in view of the uniform energy bounds (7) above, we have uniform Hölder continuity away from $x = 0$.

Lemma 3.1. *For any $r_0 > 0$, any (t,r) and (s,q) with $2r_0 \le q \le s < t \le 1, 2r_0 \le r \le t$ there holds*

$$|u(t,r) - u(s,q)|^2 \le C(|r-q| + |t-s|) \tag{12}$$

with a constant C depending only on the energy $E(U(1),1)$ and r_0.

Proof. Given $r_0 > 0$, for any t and $r_0 \le r' < r \le t \le 1$ by Hölder's inequality and (7) we have

$$|u(t,r) - u(t,r')|^2 \le \left(\int_{r'}^{r} |u_r|\,dr''\right)^2 \le \frac{r-r'}{r'} \cdot \int_{r'}^{r} |u_r|^2\,r''\,dr'' \le C\frac{r-r'}{r_0},$$

while for any $s < t$ and $r_0 \le r' \le s$ we find

$$|u(s,r') - u(t,r')|^2 \le \left(\int_{s}^{t} |u_t(t',r')|\,dt'\right)^2 \le \frac{t-s}{r_0}\int_{s}^{t} |u_t(t',r')|^2\,r'\,dt'.$$

Combining these inequalities, for any (t,r) and (s,q) with $2r_0 \le q \le s < t \le 1, 2r_0 \le r \le t$ and any r' with $r_0 \le r' \le r_1 := \inf\{q,r\}$ we find

$$|u(t,r) - u(s,q)|^2 \le C\frac{r-r'+q-r'}{r_0} + 2\frac{t-s}{r_0}\int_{s}^{t} |u_t(t',r')|^2\,r'\,dt'.$$

Taking the average with respect to $r' \in [r_1 - \min\{r_0, |r-q| + |t-s|\}, r_1]$, we obtain the claim. □

For a number $\varepsilon_1 = \varepsilon_1(N) > 0$ to be determined below and $0 < t \le 1$ choose $R = R(t) > 0$ so that

$$\varepsilon_1 \le E(U(t), 6R(t)) \le 2\varepsilon_1. \tag{13}$$

Applying the energy inequality (7), for any $|\tau| \le 5R$ then we have

$$E(U(t+\tau),R) \le E(U(t+\tau), 6R - |\tau|) \le E(U(t),6R) \le 2\varepsilon_1 \tag{14}$$

and similarly

$$\varepsilon_1 \le E(U(t+\tau), 6R + |\tau|) \le E(U(t+\tau), 11R). \tag{15}$$

We may assume that $2\varepsilon_1 < \varepsilon_0$. Hence, in particular, from (13) we deduce the inequality

$$6R(t) < t \tag{16}$$

for all t. Indeed, if $6R(t) \geq t$ for some t, (13) and (7) will contradict our assumption (6).

In fact, we obtain a much stronger result.

Lemma 3.2. *$R(t)/t \to 0$ as $t \to 0$.*

Proof. Suppose by contradiction that for some sequence $t_i \downarrow 0$ $(i \to \infty)$ with associated radii $R_i = R(t_i)$ there holds $6R_i \geq \lambda t_i$ for some constant $\lambda > 0$. Then from (6) and (13) we deduce that

$$0 < \varepsilon_0 - 2\varepsilon_1 \leq E(U(t_i), t_i) - E(U(t_i), 6R_i) \leq E(U(t_i), t_i) - E(U(t_i), \lambda t_i),$$

contradicting (10) for large $i \in \mathbb{N}$. □

Consider the intervals $\Lambda_{R(t)}(t) =]t - R(t), t + R(t)[$, $0 < t \leq 1$. By Vitali's theorem we can find a countable subfamily of disjoint intervals $\Lambda_i = \Lambda_{R(t_i)}(t_i)$, $i \in \mathbb{N}$, such that $]0, 1] \subset \cup_{i=1}^{\infty} \Lambda_i^*$, where $\Lambda_i^* = \Lambda_{5R(t_i)}(t_i)$. Observe that (16) implies

$$\inf \Lambda_i^* = t_i - 5R(t_i) > R(t_i) =: R_i \tag{17}$$

for each i. Hence, passing to a subsequence, if necessary, we may assume that $t_i \to 0$ as $i \to \infty$. Indeed, for any $\tau > 0$ the interval $[\tau, 1]$ is covered by finitely many intervals Λ_i^* which, however, fail to cover $]0, 1]$ completely in view of (17). Therefore, relabelling (t_i) if necessary, we may also assume that $t_{i+1} < t_i$ for all $i \in \mathbb{N}$.

Lemma 3.3. *With the above notations there holds*

$$\liminf_{i \to \infty} \frac{1}{R_i} \int_{\Lambda_i} \int_{B_t(0)} |U_t|^2 \, dx \, dt = 0.$$

Proof. Negating the assertion, we can find a number $\delta > 0$ and an index $i_0 \in \mathbb{N}$ such that

$$\int_{\Lambda_i} \int_{B_t(0)} |U_t|^2 \, dx \, dt \geq \delta R_i \quad \text{for } i \geq i_0. \tag{18}$$

For $0 < T < \inf \cup_{i < i_0} \Lambda_i^*$ let

$$i_1 = \max\{j;]0, T] \subset \cup_{i \geq j} \Lambda_i^*\} \geq i_0.$$

Then by maximality of i_1 and (16), (17) we have

$$R_{i_1} < \inf \Lambda_{i_1}^* = t_{i_1} - 5R_{i_1} < T$$

and therefore

$$6R_i \leq t_i \leq t_{i_1} < T + 5R_{i_1} < 6T$$

for all $i \geq i_1$. It follows that

$$\cup_{i \geq i_1} \Lambda_i \subset]0, 7T]. \tag{19}$$

Since $]0, T] \subset \cup_{i \geq i_1} \Lambda_i^*$ by choice of i_1, from (18) and (19) we now obtain that

$$\begin{aligned} \delta T &\leq \delta \sum_{i \geq i_1} \operatorname{diam} \Lambda_i^* = 10\, \delta \sum_{i \geq i_1} R_i \leq 10 \sum_{i \geq i_1} \int_{\Lambda_i} \int_{B_t(0)} |U_t|^2 \, dx \, dt \\ &= 10 \int_{\cup_{i \geq i_1} \Lambda_i} \int_{B_t(0)} |U_t|^2 \, dx \, dt \leq 10 \int_{K^{7T}} |U_t|^2 \, dz, \end{aligned} \tag{20}$$

where we also used the fact that the intervals Λ_i are disjoint. But for small $T>0$ this contradicts (11), thus proving the lemma. $\square$

Proof of Theorem 2.1. i) Letting
$$U_i(t,x)=U(t_i+R_it,R_ix), i\in\mathbb{N},$$
from Lemma 3.3 for a suitable subsequence we obtain
$$\int_{-1}^{1}\int_{B_{r_i}(0)}|\partial_t U_i|^2\,dx\,dt\to 0 \text{ as } i\to\infty, \tag{21}$$
where $r_i=t_i/R_i-1\to\infty$ as $i\to\infty$ on account of Lemma 3.2. Relabelling, we may assume that (21) holds true for the original sequence (U_i).

Moreover, the energy inequality (7) implies the uniform bound
$$E(U_i(t),r_i)\le E(U(1),1)=:E_0 \tag{22}$$
for all $i\in\mathbb{N}$ and $|t|\le 1$.

Hence we may extract a further subsequence such that $U_i\rightharpoonup U_\infty$ weakly in H^1_{loc} and locally uniformly away from $x=0$ on $[-1,1]\times\mathbb{R}^2$ as $i\to\infty$, and similarly for the associated functions u_i. Their limit u_∞ then is associated with U_∞ and is a time-independent solution of (4) away from $x=0$. It follows that $U_\infty(t,x)=\overline{U}(x)$ is a time-independent solution of (3) on $]-1,1[\times(\mathbb{R}^2\setminus\{0\})$; that is, $\overline{U}\colon\mathbb{R}^2\setminus\{0\}\to N$ is a smooth, co-rotational harmonic map with finite energy
$$E(\overline{U})=\int_{\mathbb{R}^2}|\nabla U|^2\,dx\le\liminf_{i\to\infty}\sup_{|t|\le 1}E(U_i(t),r_i)\le E_0.$$
By [3] then $\overline{U}$ extends to a smooth harmonic map $\overline{U}\colon\mathbb{R}^2\to N$. Since $\mathbb{R}^2$ is conformal to $S^2\setminus\{p_0\}$ by stereographic projection from any point $p_0\in S^2$ and since the composition of a harmonic map with a conformal transformation again yields a harmonic map with the same energy, we may thus regard $\overline{U}$ as a harmonic map from $S^2\setminus\{p_0\}$ to N. Finally, recalling that $E(\overline{U})<\infty$ and again using [3], we see that the map $\overline{U}$ extends to a smooth equivariant harmonic map $\overline{U}\colon S^2\to N$.

ii) To show that $\overline{U}$ is non-constant we now establish strong convergence
$$U_i\to U_\infty \text{ in } H^1_{\mathrm{loc}}(]-1,1[\times\mathbb{R}^2)$$
as $i\to\infty$. Recalling (14), we have
$$E(U_i(t),1)\le 2\varepsilon_1,\quad E(U_\infty(t),1)\le 2\varepsilon_1$$
uniformly in i and $|t|\le 1$. Hence, from (9) for sufficiently small $\varepsilon_1>0$ the images of $B_1(0)$ under $U_i(t)$ or U_∞ are all contained in a fixed coordinate system around the center of symmetry $O\in N$. In addition, we can achieve that
$$\sup_{|t|,|x|\le 1}|\Gamma(U_i)||U_i-U_\infty|\le\frac{1}{4} \tag{23}$$
uniformly in $i\in\mathbb{N}$, provided $\varepsilon_1>0$ is chosen sufficiently small.

For any $\varphi\in C_0^\infty(]-1,1[\times\mathbb{R}^2)$ with $0\le\varphi\le 1$ then, upon multiplying the equation (3) for U_i by $(U_i-U_\infty)\varphi$ and integrating by parts we obtain
$$\int_{\mathbb{R}^{1+2}}|D(U_i-U_\infty)|^2\varphi\,dz\le\int_{\mathbb{R}^{1+2}}|\Gamma(U_i)||DU_i|^2|U_i-U_\infty|\varphi\,dz+I, \tag{24}$$

with error

$$|I| \leq C \int_{\mathbb{R}^{1+2}} (|\partial_t U_i|^2 \varphi + |DU_i||U_i - U_\infty||D\varphi|)\, dz$$
$$+ \sum_\alpha |\int_{\mathbb{R}^{1+2}} \partial_\alpha U_\infty \partial_\alpha (U_i - U_\infty)\varphi\, dz| \to 0 \text{ as } i \to \infty$$

in view of (21) and since $U_i \to U_\infty$ strongly in L^2_{loc} by Rellich's theorem.

Now we estimate

$$|DU_i|^2 \leq 2|D(U_i - U_\infty)|^2 + 2|DU_\infty|^2$$

and observe that

$$\int_{\mathbb{R}^{1+2}} |DU_\infty|^2 |U_i - U_\infty| \varphi\, dz \to 0$$

as $i \to \infty$ by bounded almost everywhere convergence $U_i \to U_\infty$ and Lebesgue's theorem on dominated convergence. Also recalling (23), we thus may absorb the first term on the right of (24) on the left to obtain that

$$\int_{\mathbb{R}^{1+2}} |D(U_i - U_\infty)|^2 \varphi\, dz \to 0$$

as $i \to \infty$. Since φ as above is arbitrary, this yields the desired convergence $U_i \to U_\infty$ in $H^1_{\mathrm{loc}}(]-1,1[\times\mathbb{R}^2)$.

But, recalling (15), we also have the uniform lower bound

$$\varepsilon_1 \leq E(U_i(t), 11)$$

for all $i \in \mathbb{N}$ and $|t| \leq 1$ and we conclude that $U_\infty \not\equiv$ const., as claimed.

Therefore, also $\overline{U}\colon S^2 \to N$ is non-constant, and the proof of Theorem 2.1 is complete. □

Proof of Theorem 2.2. In view of Theorem 2.1 it suffices to show that any co-rotational harmonic map $\overline{U}\colon S^2 \to N$ with finite energy is constant. Let $\overline{U}$ be such a map, viewed as a map $\overline{U}\colon \mathbb{R}^2 \to N$. Also consider the associated distance function $\rho = \overline{u}(r)$, a time-independent solution of (4). The image $\overline{U}(S^2)$ being compact there exists $r_0 > 0$ such that

$$|\overline{u}(r_0)| = \max_{r>0} |u(r)|.$$

Hence $\overline{u}_r(r_0) = 0$ and therefore $\overline{U}_r(x) = 0$ for any $x \in \partial B_{r_0}(0)$.

Any harmonic map $\overline{U}\colon \mathbb{R}^2 \to N$ with finite energy being conformal, this implies that also $\overline{U}_\phi$ vanishes along $\partial B_{r_0}(0)$, and we conclude that $\overline{U} \equiv$ const. on $\partial B_{r_0}(0)$. Equivariance of $\overline{U}$ then implies that $g(\overline{u}(r_0)) = 0$ and hence $\overline{u}(r_0) = 0$ on account of (1). But then $u \equiv 0$ by choice of r_0, and $\overline{U} \equiv$ const. $\equiv O$, as desired. □

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