WELL-POSEDNESS OF THE SUPERCRITICAL LANE-EMDEN HEAT FLOW IN MORREY SPACES

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Abstract. For any smoothly bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, and any exponent $p > 2^* = 2n/(n-2)$ we study the Lane-Emden heat flow $u_t - \Delta u = |u|^{p-2}u$ on $\Omega \times]0, \infty[$ and establish local and global well-posedness results for the initial value problem with suitably small initial data $u\big|_{t=0} = u_0$ in the Morrey space $L^{2,\lambda}(\Omega)$, where $\lambda = 4/(p-2)$. We contrast our results with results on instantaneous complete blow-up of the flow for certain large data in this space, similar to ill-posedness results of Galaktionov-Vazquez for the Lane-Emden flow on $\mathbb{R}^n$.

1. Introduction

Let Ω be a smoothly bounded domain in $\mathbb{R}^n$, $n \geq 3$, and let $T > 0$. Given initial data u_0, we consider the Lane-Emden heat flow

$$u_t - \Delta u = |u|^{p-2}u \text{ on } \Omega \times [0, T[,\ u = 0 \text{ on } \partial\Omega \times [0, T[,\ u\big|_{t=0} = u_0 \tag{1.1}$$

for a given exponent $p > 2^* = 2n/(n-2)$, that is, in the "supercritical" regime.

As observed by Matano-Merle [12], p. 1048, the initial value problem (1.1) may be ill-posed for certain data $u_0 \in H_0^1 \cap L^p(\Omega)$; see also our results in Section 4 below. However, as we had shown in two previous papers [4], Section 6.5, [5], Remark 3.3, the Cauchy problem (1.1) is globally well-posed for suitably small data u_0 belonging to the Morrey space $H_0^{1,\mu} \cap L^{p,\mu}(\Omega)$, where $\mu = \frac{2p}{p-2} < n$. Here we go one step further and show that problem (1.1) even is well-posed for suitably small data $u_0 \in L^{2,\lambda}(\Omega) \supset L^{p,\mu}(\Omega)$, where $\lambda = \frac{2\mu}{p} = \frac{4}{p-2} = \mu - 2$, thus considerably improving on the results of Brezis-Cazenave [6] or Weissler [14] for initial data in L^q, $q \geq n(p-2)/2$. Our results are similar to results of Taylor [13] who demonstrated local and global well-posedness of the Cauchy problem for the equation

$$u_t - \Delta u = DQ(u) \text{ on } \Omega \times [0, T[,$$

for suitably small initial data $u\big|_{t=0} = u_0$ in a Morrey space, where D is a linear differential operator of first order and Q is a quadratic form in u as in the Navier-Stokes system. However, similar to the work of Koch-Tataru [11] on the Navier-Stokes system, in our treatment of (1.1) we are able to completely avoid the use of pseudodifferential operators in favor of simple integration by parts.

The study of the initial value problem for (1.1) for non-smooth initial data is motivated by the question whether a solution u of (1.1) blowing up at some time $T < \infty$ can be extended as a weak solution of (1.1) on a time interval $]0, T_1[$ for some $T_1 > T$. Note that if such a continuation is possible and if the extended

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solution still satisfies the monotonicity formula [5], Proposition 3.1, it follows that $u(T) \in L^{2,\lambda}(\Omega)$. Hence, the regularity assumption $u_0 \in L^{2,\lambda}(\Omega)$ is necessary from this point of view and cannot be weakened. However, our results in Section 4 show that the condition $u(T) \in L^{2,\lambda}(\Omega)$ in general is not sufficient for continuation and that a smallness condition as in our Theorems 2.1, 2.2 below is needed.

Note that the question of continuation after blow-up only is of relevance in the supercritical case when $p > 2^*$. Indeed, by work of Baras-Cohen [3] in the subcritical case $p < 2^*$ a classical solution $u \geq 0$ to (1.1) blowing up at some time $T < \infty$ always undergoes "complete blow-up" (see Section 4 for a definition), and u cannot be continued as a (weak) solution to (1.1) after time T in any reasonable way. In [8] Galaktionov und Vazquez extend the Baras-Cohen result to the critical case $p = 2^*$.

In the next section we state our well-posedness results, which we prove in Section 3. In Section 4 we then contrast these results with results on instantaneous complete blow-up of the flow for certain large data. These results are similar to ill-posedness results of Galaktionov-Vazquez for the Lane-Emden flow on $\mathbb{R}^n$; see for instance [8], Theorem 10.4. We conclude the paper with some open problems.

2. Global and local well-posedness

Recall that for any $1 \leq p < \infty$, $0 < \lambda < n$ (in Adams' [1] notation) a function $f \in L^p(\Omega)$ on a domain $\Omega \subset \mathbb{R}^n$ belongs to the Morrey space $L^{p,\lambda}(\Omega)$ if

$$\|f\|^p_{L^{p,\lambda}(\Omega)} := \sup_{x_0 \in R^n,\, r>0} r^{\lambda-n} \int_{B_r(x_0)\cap\Omega} |f|^p dx < \infty, \tag{2.1}$$

where $B_r(x_0)$ denotes the Euclidean ball of radius $r > 0$ centered at x_0. Moreover, we write $f \in L^{p,\lambda}_0(\Omega)$ whenever $f \in L^{p,\lambda}(\Omega)$ satisfies

$$\sup_{x_0 \in R^n,\, 0<r<r_0} r^{\lambda-n} \int_{B_r(x_0)\cap\Omega} |f|^p dx \to 0 \text{ as } r_0 \downarrow 0.$$

Similarly, for any $1 \leq p < \infty$, $0 < \mu < n+2$ a function $f \in L^p(E)$ on $E \subset \mathbb{R}^n \times \mathbb{R}$ belongs to the parabolic Morrey space $L^{p,\mu}(E)$ if

$$\|f\|^p_{L^{p,\mu}(E)} := \sup_{z_0=(x_0,t_0)\in R^{n+1}, r>0} r^{\mu-(n+2)} \int_{P_r(z_0)\cap E} |f|^p dz < \infty,$$

where $P_r(x,t)$ denotes the backwards parabolic cylinder $P_r(x,t) = B_r(x) \times]t-r^2, t[$.

Given $p > 2^*$, we now fix the Morrey exponents $\mu = \frac{2p}{p-2}$ and $\lambda = \frac{4}{p-2} = \mu - 2$, which are natural for the study of problem (1.1).

Throughout the following a function u will be called a smooth solution of (1.1) on $]0, T[$ if $u \in C^1(\bar{\Omega} \times]0, T[)$ with $u_t \in L^2_{loc}(\bar{\Omega} \times [0, T[)$ solves (1.1) in the sense of distributions and achieves the initial data in the sense of traces. By standard regularity theory then u also is of class C^2 with respect to x and satisfies (1.1) classically. Schauder theory, finally, yields even higher regularity to the extent allowed by smoothness of the nonlinearity $g(v) = |v|^{p-2}v$. The function u will be called a global smooth solution of (1.1) if the above holds with $T = \infty$.

Our results on local and global well-posedness are summarized in the following theorems.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a smoothly bounded domain, $n \geq 3$. There exists a constant $\varepsilon_0 > 0$ such that for any function $u_0 \in L^{2,\lambda}(\Omega)$ satisfying $\|u_0\|_{L^{2,\lambda}} < \varepsilon_0$ there is a unique global smooth solution u to* (1.1) *on $\Omega\times]0,\infty[$.*

The smallness condition can be somewhat relaxed.

Theorem 2.2. *Let $u_0 \in L^{2,\lambda}(\Omega)$ and suppose that there exists a number $R > 0$ such that*

$$\sup_{x_0\in R^n,\, 0<r<R} r^{\lambda-n}\int_{B_r(x_0)\cap\Omega} |u_0|^2 dx \leq \varepsilon_0^2,$$

where $\varepsilon_0 > 0$ is as determined in Theorem 2.1. Then there exists a unique smooth solution u to (1.1) *on an interval $]0, T_0[$, where $T_0/R^2 = C(\varepsilon_0/\|u_0\|_{L^{2,\lambda}}) > 0$.*

In particular, for any $u_0 \in L_0^{2,\lambda}(\Omega)$ there exists a unique smooth solution u to (1.1) *on some interval $]0, T[$, where $T = T(u_0) > 0$.*

It is well-known that for smooth initial data $u_0 \in C^1(\bar{\Omega})$ there exists a smooth solution u to the Cauchy problem (1.1) on some time interval $[0, T[$, $T > 0$. By the uniqueness of the solution to (1.1) constructed in Theorem 2.1 or 2.2, the latter solution coincides with u and hence is smooth up to $t = 0$ if $u_0 \in C^1(\bar{\Omega})$.

3. Proof of Theorem 2.1

Let $n \geq 3$ and let

$$G(x,t) = (4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}},\ x \in \mathbb{R}^n, t > 0,$$

be the fundamental solution to the heat equation on $\mathbb{R}^n$ with singularity at $(0,0)$. Given a domain $\Omega \subset \mathbb{R}^n$ also let $\Gamma = \Gamma(x,y,t) = \Gamma(y,x,t)$ be the corresponding fundamental solution to the heat equation on Ω with homogeneous Dirichlet boundary data $\Gamma(x,y,t) = 0$ for $x \in \partial\Omega$. Note that by the maximum principle for any $x, y \in \Omega$, any $t > 0$ there holds $0 < \Gamma(x,y,t) \leq G(x-y,t)$.

For $x \in \Omega$, $r > 0$ we let

$$\Omega_r(x) = B_r(x) \cap \Omega;$$

similarly, for $x \in \Omega$, $r, t > 0$ we define

$$Q_r(x,t) = P_r(x,t) \cap \Omega\times]0,\infty[.$$

We sometimes write $z = (x,t)$ for a generic point in space-time. The letter C will denote a generic constant, sometimes numbered for clarity.

For $f \in L^1(\Omega)$ set

$$(S_\Omega f)(x,t) := \int_\Omega \Gamma(x,y,t) f(y)\, dy,\ t > 0,$$

so that $v = S_\Omega f$ solves the equation

$$v_t - \Delta v = 0 \text{ on } \Omega \times [0,\infty[\tag{3.1}$$

with boundary data $v(x,t) = 0$ for $x \in \partial\Omega$ and initial data $v\big|_{t=0} = f$ on Ω.

Similar to [4], Proposition 4.3, by adapting the methods of Adams [1] we can show that S_Ω is well-behaved on Morrey spaces. Recall that $\mu = \frac{2p}{p-2}$ with $2 < \mu < n$, and $\lambda = \mu - 2 = \frac{4}{p-2} > 0$.

Lemma 3.1. *i) For any $p > 2^* = \frac{2n}{n-2}$ the map*

$$S_\Omega \colon L^{2,\lambda}(\Omega) \ni f \mapsto (v, \nabla v) \in L^{p,\mu} \times L^{2,\mu}(\Omega \times [0,\infty[)$$

is well-defined and bounded. Moreover, we have the bounds

$$\|v(t)\|^2_{L^\infty} \le Ct^{-\lambda/2}\|f\|^2_{L^{2,\lambda}},\ \|v(t)\|^2_{L^{2,\lambda}} \le C\|f\|^2_{L^{2,\lambda}},\ t>0. \tag{3.2}$$

ii) Let $f \in L^{2,\lambda}(\Omega)$ and suppose that for a given $\varepsilon_0 > 0$ there exists a number $R > 0$ such that

$$\sup_{x_0\in\Omega,\ 0<r<R} \Big(r^{\lambda-n}\int_{\Omega_r(x_0)} |f|^2 dx\Big)^{1/2} \le \varepsilon_0.$$

Then with a constant $C > 0$ for $v = S_\Omega f$ there holds the estimate

$$\sup_{x_0\in\Omega,\ 0<r^2\le t_0\le T_0} \Big(r^{\mu-n-2}\int_{Q_r(x_0,t_0)} |v|^p dz\Big)^{1/p} \le C\varepsilon_0,$$

where $T_0/R^2 = C(\varepsilon_0/\|f\|_{L^{2,\lambda}(\Omega)}) > 0$.

Proof. i) Let $f \in L^{2,\lambda}(\mathbb{R}^n)$ and set $v = S_\Omega f$ as above. Recall the definition of the fractional maximal functions

$$M_\alpha f(x) := \sup_{r>0} M_{\alpha,r} f(x),\ M_{\alpha,r} f(x) := r^{\alpha-n}\int_{\Omega_r(x)} |f(y)|\,dy,\ \alpha>0.$$

Note that Hölder's inequality gives the uniform bound

$$\big(M_{\lambda/2} f\big)^2 \le M_\lambda(|f|^2) \le \|f\|^2_{L^{2,\lambda}}. \tag{3.3}$$

Following the scheme outlined by Adams [1], proof of Proposition 3.1, we first derive pointwise estimates for v and bounds on parabolic cylinders $P_r(x_0,t_0)$ with radius r satisfying $0 < 2r^2 < t_0$. Using the well known estimate

$$G(x-y,t) \le C(|x-y| + \sqrt{t})^{-n}$$

for the heat kernel and recalling that $\Gamma(x,y,t) \le G(x-y,t)$, for any $t > 0$ we can bound

$$\begin{aligned}
|v(x,t)| &\le C\int_\Omega (|x-y|+\sqrt{t})^{-n}|f(y)|\,dy \\
&\le C\int_{\Omega_{\sqrt{t}}(x)} (|x-y|+\sqrt{t})^{-n}|f(y)|\,dy \\
&\quad + C\sum_{k=1}^\infty \int_{\Omega_{2^k\sqrt{t}}(x)\setminus\Omega_{2^{k-1}\sqrt{t}}(x)} (|x-y|+\sqrt{t})^{-n}|f(y)|\,dy \\
&\le C\sum_{k=0}^\infty (2^k\sqrt{t})^{-n}(2^k\sqrt{t})^{n-\lambda/2} M_{\lambda/2,2^k\sqrt{t}} f(x) \le Ct^{-\lambda/4} M_{\lambda/2} f(x).
\end{aligned}$$

Hence by (3.3) with a uniform constant $C > 0$ for any $t > 0$ there holds

$$\|v(t)\|^2_{L^\infty} \le Ct^{-\lambda/2}\|M_{\lambda/2} f\|^2_{L^\infty} \le Ct^{-\lambda/2}\|f\|^2_{L^{2,\lambda}},$$

as claimed in (3.2). Moreover, for any $x_0 \in \mathbb{R}^n$, any $t_0 > 0$ and any $0 < r < \sqrt{t_0/2}$ we obtain the bounds

$$\|v(t_0)\|^2_{L^2(\Omega_r(x_0))} \le Cr^n t_0^{-\lambda/2}\|f\|^2_{L^{2,\lambda}} \le Cr^{n-\lambda}\|f\|^2_{L^{2,\lambda}}, \tag{3.4}$$

and similarly

$$\|v\|^p_{L^p(Q_r(x_0,t_0))} \le Cr^{n+2} t_0^{-p\lambda/4}\|f\|^p_{L^{2,\lambda}} \le Cr^{n+2-\mu}\|f\|^p_{L^{2,\lambda}}, \tag{3.5}$$

where we also used that $\mu = 2p\lambda/4$.

In order to derive (3.5) also for radii $r \geq \sqrt{t_0/2}$ we need to argue slightly differently. We may assume that $x_0 = 0$. Moreover, after enlarging t_0, if necessary, we may assume that $t_0 = 2r^2$. Let $\psi = \psi_0 = \psi_0(x)$ be a smooth cut-off function satisfying $\chi_{B_r(0)} \leq \psi \leq \chi_{B_{2r}(0)}$ and with $|\nabla\psi|^2 \leq 4r^{-2}$. Set $r =: r_0$ and let $r_i = 2^i r_0$, $\psi_i(x) = \psi(2^{-i}x)$, $i \in \mathbb{N}$. For ease of notation in the following estimates we drop the index i.

Upon multipying (3.1) with $v\psi^2$ we find the equation

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(|v|^2\psi^2) - div(v\nabla v\psi^2) + |\nabla v|^2\psi^2 \\ = -2v\nabla v\psi\nabla\psi \leq \frac{1}{2}|\nabla v|^2\psi^2 + 2|v|^2|\nabla\psi|^2.\end{aligned}$$

Integrating over $\Omega\times]0,t_1[$ and using the bound $|\nabla\psi|^2 \leq 4r^{-2}$, for any $0 < t_1 < t_0$ we obtain

$$\begin{aligned}&\int_{\Omega_{2r}(0)} |v(t_1)|^2\psi^2 dx + \int_{\Omega_{2r}(0)\times]0,t_1[} |\nabla v|^2\psi^2 dxdt \\ &\leq \int_{\Omega_{2r}(0)} |f|^2\psi^2 dx + 16r^{-2}\int_{\Omega_{2r}(0)\times]0,t_1[} |v|^2 dxdt.\end{aligned} \tag{3.6}$$

For $r = r_i$, $i \in \mathbb{N}_0$, set

$$\Psi(r) := \sup_{x_0\in\Omega, 0<t<t_0} r^{\lambda-n}\int_{\Omega_r(x_0)} |v(t)|^2 dx.$$

Recalling that $\lambda = \mu - 2$, then from the previous inequality (3.6) with the uniform constants $C_1 = 2^{n-\lambda}$, $C_2 = 32C_1$ we obtain

$$\begin{aligned}\Psi(r_i) &\leq r_i^{\lambda-n}\Big(\int_{\Omega_{2r_i}(0)} |f|^2 dx + 16t_0 r_i^{-2}\sup_{0<t<t_0}\int_{\Omega_{2r_i}(0)} |v(t)|^2 dx\Big) \\ &\leq C_1\|f\|^2_{L^{2,\lambda}} + C_2 2^{-2i}\Psi(r_{i+1}).\end{aligned}$$

By iteration, for any $k_0 \in \mathbb{N}$ there results

$$\begin{aligned}\Psi(r_0) &\leq C_1\|f\|^2_{L^{2,\lambda}} + C_2\Psi(r_1) \leq C_1(1 + C_2)\|f\|^2_{L^{2,\lambda}} + C_2^2 2^{-2}\Psi(r_2) \leq \dots \\ &\leq C_1\sum_{k=0}^{k_0} C_2^k 2^{(1-k)k}\|f\|^2_{L^{2,\lambda}} + C_2^{k_0+1}2^{-k_0(k_0+1)}\Psi(r_{k_0+1}).\end{aligned}$$

Passing to the limit $k_0 \to \infty$, we obtain that $\Psi(r_1) \leq C\|f\|^2_{L^{2,\lambda}}$. Inserting this information into (3.6), where we again set $r = r_0$, then we find

$$\Psi(r) + \sup_{x_0\in\Omega} r^{\mu-2-n}\int_{\Omega_r(x_0)\times]0,t_0[} |\nabla v|^2 dxdt \leq C\|f\|^2_{L^{2,\lambda}}. \tag{3.7}$$

In particular, together with (3.4) we have now shown the bound

$$\|v(t)\|^2_{L^{2,\lambda}} \leq C\|f\|^2_{L^{2,\lambda}} \text{ for all } t > 0,$$

and thus have verified (3.2) completely.

To complete the proof of (3.5) for $r = r_0 = \sqrt{t_0/2}$, let $\psi = \psi_0$ as above and let $\tau(t) = \min\{t, t_0 - t\}$. Multiplying (1.1) with the function $v|v|^{p-2}\psi^2\tau$ then we obtain

$$\frac{1}{p}\frac{d}{dt}(|v|^p\psi^2\tau) - \frac{1}{p}\frac{d\tau}{dt}|v|^p\psi^2 - div(|v|^{p-2}v\nabla v\psi^2\tau) + (p-1)|\nabla v|^2|v|^{p-2}\psi^2\tau$$
$$= -2|v|^{p-2}v\nabla v\psi\nabla\psi\tau \geq -|\nabla v|^2|v|^{p-2}\psi^2\tau - |v|^p|\nabla\psi|^2\tau.$$

Integrating over $\Omega\times]0,t_0[$ and using that $\frac{d\tau}{dt} = 1$ for $0 < t < t_0/2$, $\frac{d\tau}{dt} = -1$ for $t_0/2 < t < t_0$, as well as the fact that the region $\Omega_{2r}(0)\times]t_0/2, t_0[$ may be covered by a collection of at most $L = L(n)$ cylinders $Q_r(x_l, t_0)$, $1 \leq l \leq L$, we find

$$\int_{Q_r(x_0,t_0/2)} |v|^p dz \leq L \sup_{1\leq l\leq L} \int_{Q_r(x_l,t_0)} |v|^p dz + Cr^{-2} \int_{\Omega_{2r}(0)\times]0,t_0[} |v|^p\tau\, dxdt$$
$$+ C\int_{\Omega_{2r}(0)\times]0,t_0[} |\nabla v|^2|v|^{p-2}\tau\, dxdt.$$

But by (3.2) we have $|v|^{p-2}\tau \leq |v|^{p-2}t \leq C\|f\|^{p-2}_{L^{2,\lambda}}$, and from (3.7) we obtain

$$r^{-2}\int_{\Omega_{2r}(0)\times]0,t_0[} |v|^p\tau\, dxdt + \int_{\Omega_{2r}(0)\times]0,t_0[} |\nabla v|^2|v|^{p-2}\tau\, dxdt$$
$$\leq C\|f\|^{p-2}_{L^{2,\lambda}}\Big(r^{n-\lambda}\Psi(2r) + \int_{\Omega_{2r}(0)\times]0,t_0[} |\nabla v|^2 dxdt\Big) \leq Cr^{n-\lambda}\|f\|^p_{L^{2,\lambda}}.$$

Recalling that for each cylinder $Q_r(x_l, t_0)$, $1 \leq l \leq L$, there holds (3.5), we then obtain

$$\int_{Q_r(x_0,t_0/2)} |v|^p dz \leq L \sup_{1\leq l\leq L} \int_{Q_r(x_l,t_0)} |v|^p dz + Cr^{n-\lambda}\|f\|^p_{L^{2,\lambda}} \leq Cr^{n-\lambda}\|f\|^p_{L^{2,\lambda}},$$

and (3.5) follows since $\lambda = \mu - 2$.

Finally, for $t_0 \leq r^2$ and any $x_0 \in \Omega$ equation (3.6) yields the gradient bound

$$\int_{Q_r(0,t_0)} |\nabla v|^2 dz \leq \int_{\Omega_{2r}(0)} |f|^2\psi^2 dx + 16r^{-2}\int_{\Omega_{2r}(0)\times]0,t_0[} |v|^2 dxdt$$
$$\leq Cr^{n-\lambda}\big(\|f\|^2_{L^{2,\lambda}} + \Psi(2r)\big) \leq Cr^{n-\lambda}\|f\|^2_{L^{2,\lambda}}.$$

In view of (3.2) the same bound also holds for $t_0 > r^2$ as can be seen by shifting time by $t_0 - r^2$ and replacing f with the function $\tilde{f}(x) = v(x, t_0 - r^2) \in L^{2,\lambda}(\Omega)$. With $\lambda = \mu - 2$ we obtain the bound $\|\nabla v\|_{L^{2,\mu}} \leq C\|f\|_{L^{2,\lambda}}$, as desired.

ii) Set $L_0 := \|f\|_{L^{2,\lambda}}$. As before, for any $x \in \Omega$ we have the bound

$$|v(x,t)| \leq C\sum_{k=0}^{\infty}(2^k\sqrt{t})^{-\lambda/2}M_{\lambda/2,2^k\sqrt{t}}f(x).$$

By assumption for $r = 2^k\sqrt{t} \leq R$ we can estimate

$$M_{\lambda/2,r}(|f|)(x) \leq \big(M_{\lambda,r}(|f|^2)(x)\big)^{1/2} \leq \varepsilon_0,$$

whereas for any $r > 0$ we have

$$M_{\lambda/2,r}(|f|)(x) \leq \big(M_{\lambda,r}(|f|^2)(x)\big)^{1/2} \leq \|f\|_{L^{2,\lambda}} = L_0.$$

Let $k_0 \in \mathbb{N}$ such that $2^{-k_0\lambda/2}L_0 \leq \varepsilon_0$. Then for $0 < t < T := 2^{-2k_0}R^2$ we find the uniform estimate

$$|v(x,t)| \leq Ct^{-\lambda/4}\big(\sum_{k=0}^{k_0}2^{-k\lambda/2}\varepsilon_0 + \sum_{k=k_0+1}^{\infty}2^{-k\lambda/2}L_0\big) \leq Ct^{-\lambda/4}\varepsilon_0.$$

Proceeding as in part i) of the proof, for any $0 < t < T$, any $x_0 \in \Omega$, and any $0 < r < \sqrt{t/2}$ we then obtain the bound

$$\|v(t)\|^2_{L^2(\Omega_r(x_0))} \le C r^n t^{-\lambda/2} \varepsilon_0^2 \le C r^{n-\lambda} \varepsilon_0^2;$$

similarly, we find

$$\|v\|^p_{L^p(P_r(x_0,t_0))} \le C r^{n+2} t^{-p\lambda/4} \varepsilon_0^p \le C r^{n+2-\mu} \varepsilon_0^p \tag{3.8}$$

whenever $0 < 2r^2 < t_0 < T$. In order to derive the latter bound also for radii $r > 0$ with $t_0/2 \le r^2 \le t_0 \le T$ as in i) we may assume that $x_0 = 0$ and fix some numbers $0 < t_0 < T$, $r_0 \ge \sqrt{t_0/2}$. Setting

$$\Psi(r) := \sup_{0<t<t_0} r^{\lambda-n} \int_{B_r(0)} |v(t)|^2 dx,\ r > 0,$$

for $r = r_i = 2^i r_0$, $i \in \mathbb{N}_0$, from (3.6) we obtain the bound

$$\begin{aligned}\Psi(r_i) &\le r_i^{\lambda-n} \int_{B_{2r_i}(0)} |f|^2 dx + 16 C_1 t_0 r_i^{-2} \Psi(2r_i)\\ &\le C_1 M_{\lambda, r_{i+1}}(|f|^2)(0) + C_2 2^{-2i} \Psi(r_{i+1})\end{aligned}$$

for any $i \in \mathbb{N}$, with constants $C_1 = 2^{n-\lambda}$, $C_2 = 32 C_1$ as before.

Suppose that $r_{i_0} \le R$ for some $i_0 \in \mathbb{N}$. Bounding $M_{\lambda,r_i}(|f|^2)(x) \le \varepsilon_0^2$ for $i \le i_0$ and $M_{\lambda,r_i}(|f|^2)(x) \le L_0^2$ else, by iteration we then obtain

$$\begin{aligned}\Psi(r_0) &\le C_1 \varepsilon_0^2 + C_2 \Psi(r_1) \le C_1(1 + C_2)\varepsilon_0^2 + C_2^2 2^{-2} \Psi(r_2)\\ &\le C_1(1 + C_2 + C_2^2 2^{-2})\varepsilon_0^2 + + C_2^2 2^{-2} C_2 2^{-4} \Psi(r_3) \le \dots\\ &\le C_1 \sum_{i=0}^{i_0-1} C_2^i 2^{(1-i)i} \varepsilon_0^2 + C_1 \sum_{i=i_0}^{k} C_2^i 2^{(1-i)i} L_0^2 + C_2^{k+1} 2^{-k(k+1)} \Psi(r_{k+1}).\end{aligned}$$

Thus, if i_0 is such that $C_2 2^{1-i_0} \le (\varepsilon_0/L_0)^2 \le 1/2$, that is, if

$$\sqrt{2t_0} \le 2r_0 = 2^{1-i_0} r_{i_0} \le 2^{1-i_0} R \le C_2^{-1} (\varepsilon_0/L_0)^2 R,$$

upon passing to the limit $k \to \infty$ we obtain $\Psi(r_0) \le C\varepsilon_0^2$ and the analogue of (3.7) with ε_0 in place of $\|f\|_{L^{2,\lambda}}$.

Recalling the definition $T = 2^{-2k_0} R^2$ with $k_0 \in \mathbb{N}$ satisfying $2^{-k_0\lambda/2} L_0 \le \varepsilon_0$, we see that these bounds hold true for

$$0 < t_0/2 \le r_0^2 \le t_0 \le T_0 := R^2 \cdot \min\{(\varepsilon_0/L_0)^{4/\lambda}, C_2^{-2}(\varepsilon_0/L_0)^4\}.$$

Using (3.8), the remainder of the proof of (3.5) in part i) now may be copied unchanged to yield the claim. □

The assertions of Theorems 2.1 and 2.2 now are a consequence of the following result.

Lemma 3.2. *i) For any $p > 2^*$ there exists a constant $\varepsilon_0 > 0$ such that for any $u_0 \in L^{2,\lambda}(\Omega)$ with $\|u_0\|_{L^{2,\lambda}} \le \varepsilon_0$ there exists a unique solution $u \in L^{p,\mu}(\Omega \times]0,\infty[)$ to the Cauchy problem* (1.1) *such that*

$$\|u\|_{L^{p,\mu}} \le C \|u_0\|_{L^{2,\lambda}}. \tag{3.9}$$

ii) Let $u_0 \in L^{2,\lambda}(\Omega)$ and suppose that there exists a number $R > 0$ such that

$$\sup_{x_0\in\Omega,\ 0<r<R} r^{\lambda-n} \int_{\Omega_r(x_0)} |u_0|^2 dx \le \varepsilon_0^2,$$

where $\varepsilon_0 > 0$ *is as determined in i). Then there exists a unique smooth solution* u *to* (1.1) *on an interval* $]0, T_0[$*, where* $T_0/R^2 = C(\varepsilon_0^{-1}\|u_0\|_{L^{2,\lambda}(\Omega)}) > 0$.

Proof. For $u_0 \in L^{2,\lambda}(\mathbb{R}^n)$ set $w_0 = S_\Omega u_0$. For suitable $a > 0$ let

$$X := \{v \in L^{p,\mu}(\Omega\times]0,T_0[);\ \|v\|_{L^{p,\mu}} \le a\},$$

where $T_0 > 0$ in the case of the assumptions in i) may be chosen arbitrarily large and otherwise is as in assertion ii) of Lemma 3.1.

Then X is a closed subset of the Banach space $L^{p,\mu} = L^{p,\mu}(\Omega\times]0,T_0[)$. Moreover, for any $v \in X$ we have $|v|^{p-2}v \in L^{p/(p-1),\mu}$. By Lemma 4.1 in [4] there exists a unique solution $w = S(v|v|^{p-2}) \in L^{p,\mu}$ of the Cauchy problem

$$w_t - \Delta w = |v|^{p-2}v \text{ on } \Omega\times]0,T_0[,\ w\big|_{t=0} = 0,$$

with

$$\|w\|_{L^{p,\mu}} \le C\|v\|_{L^{p,\mu}}^{p-1} \le Ca^{p-1}.$$

For sufficiently small $\varepsilon_0, a > 0$ then the map

$$\Phi\colon X \ni v \mapsto w_0 + w \in X,$$

and for $v_{1,2} \in X$ with corresponding $w_i = S(v_i|v_i|^{p-2})$, $i = 1, 2$, we can estimate

$$\|\Phi(v_1) - \Phi(v_2)\|_{L^{p,\mu}} = \|w_1 - w_2\|_{L^{p,\mu}} \le C\|v_1|v_1|^{p-2} - v_2|v_2|^{p-2}\|_{L^{p/(p-1),\mu}}\ .$$

The latter can be bounded

$$\|v_1|v_1|^{p-2} - v_2|v_2|^{p-2}\|_{L^{p/(p-1),\mu}} \le C\big(\|v_1\|_{L^{p,\mu}}^{p-2} + \|v_2\|_{L^{p,\mu}}^{p-2}\big)\|v_1 - v_2\|_{L^{p,\mu}}.$$

Thus for sufficiently small $a > 0$ we find

$$\|\Phi(v_1) - \Phi(v_2)\|_{L^{p,\mu}} \le Ca^{p-2}\|v_1 - v_2\|_{L^{p,\mu}} \le \frac{1}{2}\|v_1 - v_2\|_{L^{p,\mu}}.$$

By Banach's theorem the map Φ has a unique fixed point $u \in X$, and u solves the initial value problem (1.1) in the sense of distributions. Finally, for sufficiently small $a, \varepsilon_0 > 0$ we can invoke Proposition 4.1 in [4] to show that u, in fact, is a smooth global solution of (1.1). □

4. Ill-posedness for "large" data

4.1. **Minimal solutions for non-negative initial data.** In order to obtain a notion of solution of (1.1) on $\Omega\times]0,\infty[$ for arbitrary nonnegative initial data $u_0 \ge 0$, following Baras-Cohen [3] for $n \in \mathbb{N}$ we solve the initial value problem

$$u_{n,t} - \Delta u_n = f_n(u_n) = \min\{u_n^{p-1}, n^{p-1}\} \text{ on } \Omega\times]0,\infty[,\ u = 0 \text{ on } \partial\Omega\times]0,\infty[, \tag{4.1}$$

with initial data

$$u_n(x,0) = u_{0n}(x) := \min\{u_0(x), n\} \ge 0. \tag{4.2}$$

As the right-hand side $f_n(u_n)$ in (4.1) is uniformly bounded, for any $n \in \mathbb{N}$ there exists a unique global solution of (4.1), (4.2). By the maximum principle, positivity of the initial data is preserved and u_n is monotonically increasing in n. Hence, the pointwise limit $u(x,t) := \lim_{n\to\infty} u_n(x,t) \le \infty$ exists. Inspired by Baras and Cohen [3] we call this limit the *minimal solution* of problem (1.1) for the given data

u_0. Moreover, similar to their Proposition 2.1 we have $u \le v$ for any v which is an *integral solution* v of (1.1) in the sense that

$$v(t) = S_t u_0 + \int_0^t S_{t-s} v^{p-1}(s)\,ds, \tag{4.3}$$

where for brevity we now write $(S_t)_{t\ge 0}$ for the heat semigroup on Ω, defined by

$$S_t w(x) = \int_\Omega \Gamma(x,y,t)w(y)\,dy,$$

with $\Gamma > 0$ denoting the fundamental solution of the heat equation on Ω.

Indeed, by Duhamel's principle the u_n satisfy the integral equation

$$u_n(t) = S_t u_{0n} + \int_0^t S_{t-s} f_n(u_n(s))\,ds. \tag{4.4}$$

Recalling that the sequence u_n is monotonically increasing in n, from Beppo-Levi's theorem on monotone convergence we find that u satisfies (4.3). On the other hand, for each n and any integral solution v of (1.1) clearly there holds $u_n \le v$.

With these prerequisites we now show that there are initial data $u_0 \in L^{p,\mu}(\Omega)$ with even $\nabla u_0 \in L^{2,\mu}$ such that the minimal solution u to (1.1) satisfies $u \equiv \infty$ on $\Omega\times]0,\infty[$, that is, undergoes *complete instantaneous blow-up*. The following arguments are modelled on corresponding results on complete instantaneous blow-up by Galaktionov and Vazquez [8] in the case when $\Omega = \mathbb{R}^n$.

4.2. **Complete instantaneous blow-up.** It is well-known that on a bounded domain Ω equation (1.1) may be interpreted as the negative gradient flow of the energy

$$E(u) = E_\Omega(u) = \int_\Omega \big(\frac{1}{2}|\nabla u|^2 - \frac{1}{p}|u|^p\big)\,dx.$$

As observed by Ball [2], Theorem 3.2, sharpening an earlier result of Kaplan [10], for data u_0 with $E(u_0) < 0$ the solution to (1.1) blows up in finite time. Indeed, Ball [2], Theorem 3.2, observes that testing the equation (1.1) with u leads to the differential inequality

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 &= -\int_{\Omega\times\{t\}} \big(|\nabla u|^2 - |u|^p\big)\,dx = -2E(u(t)) + \frac{p-2}{p}\|u(t)\|_{L^p}^p \\ &\ge -2E(u_0) + c_0\|u(t)\|_{L^2}^p \ge c_0\|u(t)\|_{L^2}^p\end{aligned}$$

for some constant $c_0 > 0$. Hence we find

$$\|u(t)\|_{L^2} \ge \big(\|u_0\|_{L^2}^{(2-p)/2} - c_0(p-2)t\big)^{-2/(p-2)},$$

and $u(t)$ must blow up at the latest at time $T = c_0^{-1}(p-2)^{-1}\|u_0\|_{L^2}^{(2-p)/2}$.

In order to obtain data $u_0 \in L^{p,\mu}$ leading to instantaneous complete blow-up, we combine this observation with the following well-known scaling property of equation (1.1): Whenever u is a solution of (1.1) on Ω, then for any $R > 0$, any $x_0 \in \mathbb{R}^n$ the function

$$u_{R,x_0}(x,t) = R^{-\alpha}u(R^{-1}(x-x_0), R^{-2}t) \tag{4.5}$$

with $\alpha = \frac{2}{p-2}$ is a solution of (1.1) on the scaled domain

$$\Omega_{R,x_0} := \{x \in \mathbb{R}^n;\ R^{-1}(x-x_0) \in \Omega\}.$$

Clearly we may assume that $0 \in \Omega$.

Theorem 4.1. *Let $0 \le w_0 \in C_c^\infty(B_1(0))$ with $E_{B_1(0)}(w_0) < 0$. Set*

$$M = M_{w_0} = \sup_{|y|\le 1} \big(|y|^\alpha w_0(y)\big),$$

where $\alpha = \frac{2}{p-2}$ as above. Then for every initial data $0 \le u_0 \in C^0(\Omega \setminus \{0\})$ satisfying

$$\liminf_{x\to 0} \big(u_0(x) - M|x|^{-\alpha}\big) > 0$$

the minimal solution u to (1.1) *blows up completely instantaneously.*

Proof. By Ball's above result, the solution w to (1.1) on $B_1(0)\times]0,T[$ with initial data $w(0) = w_0$ blows up after some finite time T at a point y_0.

Fix $R_0 > 0$ with $B_{R_0}(0) \subset \Omega$ and such that

$$u_0(x) > M|x|^{-\alpha} \text{ for } |x| \le R_0.$$

For $R < R_0$ and $x_0 \in \Omega$ with $|x_0| \le R_0 - R$ consider the rescaled solutions

$$w_{R,x_0}(x,t) := R^{-\alpha} w(R^{-1}(x-x_0), R^{-2}t)$$

on $B_R(x_0) \times [0, R^2T[$ that blow up at time R^2T.

Since by assumption we have

$$w_{R,0}(x,0) = R^{-\alpha} w_0(R^{-1}x) \le M|x|^{-\alpha} < u_0(x) \text{ on } B_R(0),$$

by continuity of u_0 away from $x = 0$ and continuity of w_0 there is a number $\delta = \delta(R) > 0$ such that

$$w_{R,x_0}(x,0) < u_0(x) \text{ on } B_R(x_0)$$

for all x_0 with $|x_0| < \delta$. Since in addition $u \ge 0 = w_{R,x_0}$ on $\partial B_R(x_0) \times [0, r^2T[$, by the maximum principle for any $\varepsilon > 0$, any $n \ge \|w_{R,x_0}\|_{L^\infty(B_R(x_0)\times[0,R^2T-\varepsilon])}$ there holds

$$u(x,t) \ge u_n(x,t) \ge w_{R,x_0}(x,t) \text{ on } B_R(x_0) \times [0, R^2T - \varepsilon],$$

where u_n solves (4.1) for each $n \in \mathbb{N}$. Passing to the limit $\varepsilon \to 0$, we then find

$$\begin{aligned} u(x_0 + Ry_0, R^2T) &= \Big(S_{R^2T}u_0 + \int_0^{R^2T} S_{R^2T-s} f(u(s))ds\Big)(x_0 + Ry_0) \\ &= \lim_{n\to\infty} \Big(S_{R^2T}u_{0n} + \int_0^{R^2T} S_{R^2T-s} f_n(u_n(s))ds\Big)(x_0 + Ry_0) \\ &\ge \lim_{t\uparrow R^2T} w_{R,x_0}(x_0 + Ry_0, t) = \infty \end{aligned}$$

for all $x_0 \in B_\delta(0)$.

Since $R > 0$ may be chosen arbitrarily small, we conclude that for any sufficiently small $t > 0$ there holds $\mathcal{L}^n(\{x \in \Omega; u(x,t) = \infty\}) > 0$. But then positivity of Γ and Duhamel's principle (4.3) yield

$$u(x,t) = \Big(S_t u_0 + \int_0^t S_{t-s} u^{p-1}(s)ds\Big)(x) = \infty.$$

for any $t > 0$ and any $x \in \Omega$. $\square$

4.3. **Open problems.** Can data $u(T)$ that lead to instantaneous complete blow-up arise from solutions of (1.1) with bounded energy? What is the smallest number $M > 0$ so that the conclusion of Theorem 4.1 holds true? Can one show that at least for exponents p strictly less than the Joseph-Lundgren [9] exponent

$$p_{JL} = 2 + \frac{4}{n-4-2\sqrt{n-1}} \text{ if } n \geq 11, \ p_{JL} = \infty \text{ if } n \leq 10,$$

we have $M = \alpha(n-2-\alpha) =: c_*$, where c_* appears as coefficient in the singular solution $u_*(x) := c_*|x|^{-\alpha}$ of the time-independent equation (1.1) on $\mathbb{R}^n$? (The significance of the exponent p_{JL} is illustrated for instance in Lemma 9.3 of [8].)

Hopefully, we will be able to answer some of these questions in the future.

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