OLGA LADYZHENSKAYA - A LIFE-LONG DEVOTION TO MATHEMATICS

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Abstract. On May 13, 2002, Professor Ladyzhenskaya was awarded the degree of "Doctor honoris causa" by the University of Bonn. The following is the text of the Laudatio in honor of Professor Ladyzhenskaya read at this occasion.

1. The beginning of a mathematical career

Olga Alexandrovna Ladyzhenskaya was born on March 7, 1922, in the town of Kologrive, in a family of old Russian nobility. Her father Alexander Ivanovich Ladyzhenski was teaching mathematics at the local school. He transmitted his passion for mathematics not only to his students but also to his daughter Olga Alexandrovna who from early childhood showed a strong talent for logical thinking. In 1939 she was admitted to the Leningrad Teachers' Training College and from 1941 to 1943 she taught mathematics to the senior classes at Kologrive Secondary School. From 1943 to 1947 she then studied mathematics at the University of Moscow. Among her teachers in Moscow were Gel'fand, Petrovskii, Sobolev, and Tikhonov. In 1947 she graduated with the highest distinction; her thesis advisor was I. G. Petrovskii. In the same year she married Andrei Alexevich Kiselev, also a mathematician who taught history of mathematics at Leningrad State University, and returned to Leningrad/St. Petersburg to continue her studies at the University of Leningrad. Here she also was influenced by Smirnov. She completed her Ph. D. in 1949 with a thesis supervised by Sobolev, and, in the spring of 1953, at Moscow State University, she handed in her thesis for the D. Sc. degree, comparable to the German "Habilitation". Finally, in 1954 she was appointed Professor at Leningrad University and in 1961 she became director of the Laboratory of Mathematical Physics at Steklov Institute (Leningrad branch) where her mathematical successes soon brought her wide recognition, both in the Soviet Union and abroad.

This is what we can read about Professor Ladyzhenskaya's youth and the beginning of her mathematical career in the account of A. D. Alexandrov, A. P. Oskolkov, N. N. Ural'tseva, and L. D. Faddeev [1] on the occasion of her 60th birthday. We get the impression of an uneventful youth passed in rural tranquility and economic security in the family of a state official and a mathematical gift whose full potential became apparent only rather late.

However, the truth is very far from this, and it could only be told after communist rule of Russia had ended. Those were difficult days for a descendent of the Russian noble class. In 1937, Professor Ladyzhenskaya's father was arrested by Stalin's men. In fact, as Alexander Solschenizyn recalls in his epic account of the "Gulag",

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Alexander I. Ladyzhenski had been warned by a peasant that his name was in "their" lists; but he stayed, he would not leave the students who depended on him; see [30], p. 23-24. In a show trial he was convicted as an "enemy of the Russian people" and sentenced to death. Olga Alexandrovna Ladyzhenskaya was lucky enough to be allowed to finish high school - unlike her two older sisters who were expelled. In 1939 she passed the entrance examinations to study at prestigious Leningrad University, at that time the best University in the Soviet Union; however, as the daughter of a "class enemy" she was not admitted. When she finally was allowed to enter Moscow University in 1943, it was only because the mother of one of her students was able to use personal contacts in her favor. Although she had completed her second thesis as early as 1951 she was not allowed to defend her thesis before 1953, after Stalin's death.

There is only one explanation why inspite of such adversity Olga Alexandrovna Ladyzhenskaya was able to rise to the top of renowned Steklov Institute and to become the uncontested head of the Leningrad school of mathematics, and this is her work.

2. Work

Professor Ladyzhenskaya has written more than 250 mathematical papers; her work covers the whole spectrum of partial differential equations, ranging from hyperbolic equations to differential equations generated by symmetric functions of the eigenvalues of the Hessian, and discussing topics ranging from uniqueness to convergence of Fourier series or finite-difference approximation of solutions. She developed the functional analytic treatment of nonlinear stationary problems by Leray-Schauder degree theory and pioneered the theory of attractors for dissipative equations.

She is author of three monographs that have greatly influenced the development of the field of partial differential equations throughout the second half of the last century; in fact, her book on "Linear and quasilinear elliptic equations" [20], that she wrote jointly with her former student Nina Ural'tseva, was one of the first mathematical monographs I ever bought. I only regret that I did not buy her book with Ural'tseva and Solonnikov [19] on parabolic equations at the same time. It has been reprinted now, but only with a soft cover which may not be entirely appropriate for a book on "hard" analysis.

In these books, starting from the work of DeGiorgi [6] and Nash [25], she and her co-authors provide a regularity theory for elliptic and parabolic equations in divergence form under the most general assumptions on the coefficients of the equations and prove the exact dependence of the regularity of the solution on the regularity of the data, thereby giving a complete answer to Hilbert's 19th problem for a large class of equations.

With Ural'tseva [21], [22], using the work of Krylov and Safonov [27], she later extended these results to equations in non-divergence form with bounded measurable coefficients. Previously, as an important intermediate step towards a full existence and regularity theory for such problems, in her book [20] she already derived closedness in L^2 for the corresponding elliptic operators in the case of continuous coefficients. Remarkably, in two dimensions she only needed to assume uniform ellipticity and boundedness of the coefficients, thus recovering a result of Bernstein [2].

But the problems closest to her heart, I believe, always have been the equations of hydrodynamics, in particular, the Navier-Stokes equations, to which she has made deep and lasting contributions, some of which are summarized in her third monograph on "The mathematical theory of viscuous imcompressible flow" [13].

I would like to recall some of these achievements here.

3. An interpolation inequality and applications

Right at the beginning, on the second page of Chapter 1 of her book [13] we find the inequality

$$\int_{\mathbb{R}^2} |u|^4 dx \leq 4 \left(\int_{\mathbb{R}^2} |u|^2 dx\right) \left(\int_{\mathbb{R}^2} |\nabla u|^2 dx\right) \tag{1}$$

for any smooth function $u : \mathbb{R}^2 \to \mathbb{R}$ of compact support. The beauty of this inequality lies in its simplicity, and also the proof, taken from Ladyzhenskaya's paper [14] of 1959, is particularly instructive and elegant. We reproduce it here with minor modifications.

Labelling space coordinates as $x = (x_1, x_2)$, for any function u of compact support for $i = 1, 2$ we obtain

$$u^2(x_1, x_2) = 2 \int_{-\infty}^{x_i} u \frac{\partial u}{\partial x_i} dx_i \leq 2 \int_{-\infty}^{\infty} |u||\nabla u| dx_i.$$

Hence by Fubini's theorem and Hölder's inequality we find

$$\begin{aligned}
\int_{\mathbb{R}^2} u^4(x)\, dx &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u^4(x_1, x_2)\, dx_1\, dx_2 \\
&\leq 4 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Big(\int_{-\infty}^{\infty} |u||\nabla u|\, dx_1\Big) \Big(\int_{-\infty}^{\infty} |u||\nabla u|\, dx_2\Big)\, dx_1\, dx_2 \\
&= 4\Big(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u||\nabla u|)\, dx_1\, dx_2\Big)^2 \\
&= 4\Big(\int_{\mathbb{R}^2} |u||\nabla u|\, dx\Big)^2 \leq 4\Big(\int_{\mathbb{R}^2} |u|^2\, dx\Big)\Big(\int_{\mathbb{R}^2} |\nabla u|^2\, dx\Big),
\end{aligned}$$

as claimed. The constant may be improved from 4 to 2; what is important is that the constant is independent of the size of the support of u.

Inequality (1) also appears as a special case of the interpolation inequalities that were derived by Gagliardo [9], [10] and Nirenberg [26] in the late 50's, that is, around the same time as Ladyzhenskaya. Their motivation was functional analytic and inspired by the study of linear elliptic equations, while Professor Ladyzhenskaya arrived at inequality (1) by studying the (nonlinear) Navier Stokes system; see [13], p. 203.

In fact, the analytic treatment of nonlinear equations often relies on inequalities like (1). The particular estimate (1) also played a key role in an early paper of mine, dealing with the heat flow of harmonic maps of surfaces [31]. It turned out that it was precisely the inequality (1) which was needed to conclude the argument; but I was not sure if such a "borderline" estimate could be valid. In search of help, in [19] I first came across a very general statement of interpolation inequalities in Sobolev spaces. After checking again and again all the values of n, p, q, etc., I could hardly believe my good fortune that the desired estimate, indeed, should be true. Only when I later saw the above argument in [13] was I really convinced that I had not made a mistake.

To see how the estimate (1) may be applied in hydrodynamics we need to take a look at the Navier-Stokes equations.

4. Navier-Stokes equations

The mathematical problems in hydrodynamics received special attention recently, when, in May 2000, the regularity problem for the Navier-Stokes equations was named as one of the seven millenium prize problems of the Clay Institute. An award of 1 million US Dollars is promised to whoever is able to show - in one version of this problem - that for any given smooth initial velocity $u_0 = (u_0^1, u_0^2, u_0^3) : \mathbb{R}^3 \to \mathbb{R}^3$, periodic in each of the spatial variables and of vanishing divergence

$$\operatorname{div} u_0 = \sum_{i=1}^{3} \frac{\partial u_0^i}{\partial x_i} = 0,$$

there exists a global smooth velocity field $u = (u^1, u^2, u^3) : \mathbb{R}^3 \times [0, \infty[\to \mathbb{R}^3$ and an associated smooth pressure distribution $p : \mathbb{R}^3 \times [0, \infty[\to \mathbb{R}$, both also periodic in the space variables, satisfying the Navier-Stokes equations

$$\frac{\partial u^i}{\partial t} + u \cdot \nabla u^i = \Delta u^i - \frac{\partial p}{\partial x^i}, \ i = 1, ..., 3, \tag{2}$$

on $\mathbb{R}^3 \times [0, \infty[$ together with the divergence condition

$$\operatorname{div} u = 0, \tag{3}$$

and with initial data

$$u = u_0 \text{ at } t = 0. \tag{4}$$

Note that we have scaled the viscosity parameter ν usually present on the right of (2) to unity.

Alternatively, the prize will be awarded to whoever presents an example of smooth, divergence-free initial data u_0 such that the problem (2),(3), and (4) does *not* admit a smooth solution (u, p) on $\mathbb{R}^3 \times [0, \infty[$, where one is allowed to add a smooth forcing term on the right of (2), or for the answer - affirmative or negative - to the corresponding regularity question in the non-periodic case but with suitable decay conditions near spatial infinity; see the problem description by Fefferman [7] for further details.

Professor Ladyzhenskaya probably would not have agreed with this formulation of the problem. To her, I believe, as to many other people working in the field, rather than *existence and regularity* the main problem is *existence and uniqueness*.

Uniqueness as a central issue already had played a key role in her diploma thesis and again in her "habilitation" thesis in 1953. As she observed, for each of the classical initial value problems one can investigate the question of well-posedness on a whole scale of solution spaces. For the problem of determining the "correct" solution space the issues of existence and uniqueness are in conflict with one another: If one broadens the concept of solution by allowing also distribution solutions with very "bad" regularity properties it may be easy to show existence while it may be hard - if not impossible - to show uniqueness. Conversely, if one narrows the concept of solution by allowing only solutions with "good" regularity properties, it may be easy to prove uniqueness but it may be difficult to obtain the existence of a solution in the restricted class. The question then becomes whether for some choice of solution space both requirements can simultaneously be fulfilled.

For the Navier-Stokes system (2) - (4) the corresponding question was posed and answered in the affirmative - at least in two space dimensions - in the celebrated work of Kiselev-Ladyzhenskaya [12] from 1957. Estimates for the L^4-norm of a solution play a key role; in fact, uniqueness may be most easily obtained by using Ladyzhenskaya's inequality (1), as was later done by Lions-Prodi [24].

For convenience, let us look at the problem on the whole space. Our point of departure is the classical energy identity for (sufficiently smooth) solutions of (2) - (4), valid in any space dimension n.

As observed by Leray [23] and Hopf [11], when we multiply equation (2) with the test function u and take account of condition (3), the nonlinear term and the pressure term yield a term of divergence type

$$\sum_{i=1}^{n}(u\cdot\nabla u^i+\frac{\partial p}{\partial x^i})u^i=u\cdot\nabla(\frac{1}{2}|u|^2+p)=\operatorname{div}(u(p+\frac{1}{2}|u|^2)$$

that vanishes after integration. From the remaining terms, upon integrating by parts, we obtain the identity

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}|u|^2\,dx+\int_{\mathbb{R}^n}|\nabla u|^2\,dx=0\,.$$

After another integration in time there results the energy identity

$$\int_{\mathbb{R}^n}|u(t)|^2\,dx+2\int_0^t\int_{\mathbb{R}^n}|\nabla u|^2\,dx\,dt=\int_{\mathbb{R}^n}|u_0|^2\,dx=:E(u_0) \tag{5}$$

valid for (sufficiently smooth) solutions of (2). By a Galerkin approximation, and using similar arguments as leading to (5) above, we obtain a distribution solution (u,p) to (2)-(4) satisfying (5) in a weak sense, that is, with left hand side not exceeding the quantity $E(u_0)$ for any t. This is the class of Hopf solutions to (2)-(4).

Now suppose that $n=2$. Let u and v be Hopf solutions on some time interval $[0,T]$ for the same data $u=v=u_0$ at $t=0$, and set $w=v-u$. We intend to show that $w=0$, thus establishing existence and uniqueness within the Hopf class of solutions. (For the following argument we need not worry about the pressure p.) Taking the difference of equations (2) for v and u and multiplying by w, then similar to our derivation of (5) above, upon integrating by parts we obtain the identity (in the distribution sense on $[0,T]$ and with $(a,b)=\sum_i a^ib^i$ for $a=(a^1,a^2)$, $b=(b^1,b^2)\in\mathbb{R}^2$)

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}|w|^2\,dx+\int_{\mathbb{R}^2}|\nabla w|^2\,dx=-\int_{\mathbb{R}^2}(w,w\cdot\nabla u)\,dx.$$

The remaining terms vanish because u and v, and therefore also w, are divergence-free. By Hölder's inequality we can estimate the term on the right

$$|\int_{\mathbb{R}^2}(w, w\cdot\nabla u)\,dx| \le \Big(\int_{\mathbb{R}^2}|\nabla u|^2\,dx\Big)^{1/2}\Big(\int_{\mathbb{R}^2}|w|^4\,dx\Big)^{1/2},$$

which yields the estimate

$$\frac{1}{2}\frac{d}{dt}||w||^2_{L^2} + ||\nabla w||^2_{L^2} \le |\int_{\mathbb{R}^2}(w, w\cdot\nabla u)\,dx| \le ||w||^2_{L^4}||\nabla u||_{L^2}. \tag{6}$$

Observing that we are in $n = 2$ space dimensions, we may now use (1) to bound

$$||w||^2_{L^4} \le 2||w||_{L^2}||\nabla w||_{L^2}.$$

With the familiar estimate $2|ab| \le a^2 + b^2$ we proceed to estimate the right hand side in (6) above

$$||w||^2_{L^4}||\nabla u||_{L^2} \le 2||w||_{L^2}||\nabla w||_{L^2}||\nabla u||_{L^2} \le ||w||^2_{L^2}||\nabla u||^2_{L^2} + ||\nabla w||^2_{L^2}.$$

From (6) then we conclude the differential inequality

$$\frac{d}{dt}||w||^2_{L^2} \le 2||w||^2_{L^2}||\nabla u||^2_{L^2}.$$

Recalling (5), we therefore obtain the uniform bound

$$||w(t)||^2_{L^2} \le ||w(0)||^2_{L^2}\exp\Big(2\int_0^t ||\nabla u(s)||^2_{L^2}\,ds\Big) \le ||w(0)||^2_{L^2}\exp(||u_0||^2_{L^2})$$

for any $t \le T$ and therefore $w = 0$, as desired. The above formal argument may be justified by mollifying the function w in time. Inequality (1) ensures that all integrals are well-defined.

The solution class originally considered by Kiselev-Ladyzhenskaya was slightly more narrow, involving also square integrable time and mixed second space-time derivatives. This illustrates the rich variety of possible choices between the class of Leray-Hopf solutions on the one extreme and smooth, classical solutions on the other.

In fact, in two space dimensions, as was shown by Serrin [28], [29], for smooth initial data u_0 all these choices are equivalent as any Hopf solution is smooth and smooth solutions are easily seen to be unique. The previously made distinction between the question of regularity and the question of uniqueness thus may seem artificial.

However, there are parabolic equations with a structure not too much different from that of the Navier-Stokes equations where this distinction *is* meaningful. For example, in my result [31] that I already briefly mentioned in relation to inequality (1) above, in 1985 I showed the existence of a unique solution to the heat flow of harmonic maps of surfaces whose energy is monotonically decreasing and which is regular away from finitely many points in space-time. Many people believed that the solution would actually be smooth until Chang-Ding-Ye [5] in 1992 gave an example where the solution indeed blows up from smooth data after a finite time.

It may be instructive to develop this example still a bit further. Freire [8] in 1995 showed that my solution is unique even in the class of distribution solutions with monotonically decreasing energy. On the other hand, very recently Topping [32] and Bertsch et al. [3] independently constructed examples where the heat flow admits infinitely many distinct weak solutions that all satisfy a weak version of the energy inequality similar to the Hopf solutions that we considered above.

Could something similar be the case for the Navier-Stokes system in three or more space dimensions? We do not know the answer; the right notion of "solution" remains a mystery. The class of smooth solutions may be too small. However, since it seems to be physically impossible to observe, for instance, isolated ("point") singularities of a flow, there seems to be no reason to insist that solutions be smooth but in an average sense.

On the other hand, the (weak) energy inequality alone may not be sufficient to guarantee uniqueness of weak solutions. In fact, in 1969 Professor Ladyzhenskaya gave an example showing nonuniqueness within the Hopf class of weak solutions in 3 space dimensions, however, on a space-time domain $Q_T \subset [0,T] \times \mathbb{R}^3$ which is singular at the origin [15]. Further energy or entropy constraints, that derive, for instance, from the second principle of thermodynamics, seem to be needed in order to distinguish physically realistic weak solutions from non-physical, purely mathematical ones.

A natural choice for a solution space thus could be the class of "suitable weak solutions" constructed by Caffarelli-Kohn-Nirenberg [4]. These are Hopf solutions satisfying, in addition, a local form of the energy inequality which implies a constraint similar to the monotonicity of the energy required in Freire's uniqueness result for the harmonic map heat flow. Very recently, in joint work with Seregin [18], Ladyzhenskaya has also contributed to the analysis of this class of solutions.

Professor Ladyzhenskaya considered hydrodynamics not only within the frame defined by the Navier-Stokes equations, but she has also considered alternative models to explain the multiple phenomena that we observe in fluid mechanics and she explored their mathematical properties. Likewise she developed new and mathematically extremely fruitful ideas about the nature of turbulence, that, in particular, led her to consider the notion of attractor for infinite dimensional dynamical systems [16].

Thus, we see that since the early 1950's Professor Ladyzhenskaya continues to work on the forefront of research on hydrodynamics, often providing the initial stimulus also for the fruitful investigations of others. Her ideas and the questions she raised remain relevant today, in particular, for the Navier-Stokes system in 3 space dimensions.

5. Ladyzhenskaya and the Steklov Institute

With her impressive mathematical achievements, helped by her cultured and charming personality, Professor Ladyzhenskaya has attracted a large number of excellent students to work with her at the Laboratory of Mathematical Physics of Steklov Institute and at Leningrad University, among them Solonnikov, Golovkin, Rivkind, Ivanov; her Ph. D. students include L. Faddev and N. Ural'tseva. Much of this activity is documented in Ladyzhenskaya's account [17].

Professor Ladyzhenskaya already was famous for her work worldwide when, in 1981, she was elected member of the Russian Academy of Science. She also is foreign member of numerous academies abroad, among them the Leopoldina, the oldest German academy. Until 1998 she was President of the Mathematical Society of St. Petersburg, thus a successor of Leonhard Euler in this office.

The year 1989 brought about the end of communist rule and a turn towards democracy and market economy in Russia. Russian mathematicians were allowed to travel more freely; some of them were able to visit Western countries for the first time. At the same time their economic situation deteriorated at the rate with which their salaries decreased in value in comparison with market prizes for standard goods.

Thus we can easily sympathize with those scientists, among them many leading Russian mathematicians, who accepted offers from abroad and left their country to find more favorable working conditions and a secure future for their families elsewhere. Professor Ladyzhenskaya, however, stayed and helped steer the Steklov Institute through these years of economic change and foster the careers of her coworkers there, thus remaining faithful to the legacy of her father.

She only retired officially from her position at Steklov Institute in 2000. To a large extent she uses her new freedom to travel to make contacts for the benefit of her students and for the better of the institution which has been her scientific home for over 50 years. The University of Bonn stands out prominently among the institutions with whom she has built lasting relations which have proved highly valuable for both sides.

I am grateful to be able to share this moment when these efforts and the mathematical work on which this cooperation is founded will be recognized by awarding the degree of a "Doctor honoris causa" to Professor Ladyzhenskaya.

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