# THE SUPERCRITICAL LANE-EMDEN EQUATION AND ITS GRADIENT FLOW

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ABSTRACT. The following text is a summary of the author's talk at the conference on "Nonlinear problems with singular data" at the Accademia dei Lincei, Rome, Novemver 26, 2013.

## 1. The Lane-Emden equation

1.1. **A question of Paul Rabinowitz.** Consider for $p > 2$ the boundary value problem

$$-\Delta u = |u|^{p-2}u \text{ on } \Omega \subset\subset \mathbb{R}^n, n \geq 3,\ u = 0 \text{ on } \partial\Omega. \tag{1.1}$$

Recall that for $2 < p < 2^* := \frac{2n}{n-2}$ there exists a solution $u > 0$ of equation (1.1), either obtained from $0 \leq v \in H_0^1(\Omega)$ with $\|v\|_{L^p} = 1$ and minimizing the Sobolev quotient

$$Q(v) = \frac{\|\nabla v\|_{L^2}^2}{\|v\|_{L^p}^2} = \min_{0 \leq w \in H_0^1(\Omega)} Q(w) = S_p(\Omega),$$

or obtained as a mountain-pass critical point of the functional

$$E_p(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{p}\int_\Omega |u|^p dx,\ u \in H_0^1 \cap L^p(\Omega).$$

For $p = 2^*$ a dichotomy occurs. On the one hand, by a result of Pohozaev [30] on any star-shaped domain $\Omega \subset\subset \mathbb{R}^n$ any (smooth) solution $u$ of (1.1) necessarily vanishes. Moreover, the Sobolev quotient $S_{2^*}(\Omega)$ is never attained on a domain $\Omega \neq \mathbb{R}^n$.

On the other hand, when $\Omega = B_{R_2} \setminus B_{R_1}(0)$ for some $0 < R_1 < R_2$, then there exists a radial solution $u > 0$ of (1.1), which can be obtained as a minimizer of the Sobolev quotient among radially symmetric functions. In addition we have the following general existence result of Coron [7].

**Theorem 1.1.** *Suppose $0 \notin \Omega \supset B_{R_2} \setminus B_{R_1}(0)$, where $R_2/R_1 \geq R >> 1$. Then there exists a solution $u > 0$ of* (1.1).

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Coron's proof uses Struwe's [33] quantization of the energy levels where the Palais-Smale condition for $E_{2^*}$ fails, coupled with a clever minimax argument. See also [34] for further background.

Prompted by Coron's result, as recalled by Brezis [4], Rabinowitz asked: Suppose $\Omega$ is as in Coron's theorem. Does there exist a solution $u > 0$ of (1.1) for *every* $p > 2^*$? – Partial results were obtained by Dancer, del Pino, Felmer, Ge, Jing, Molle, Musso, Pacard, Passaseo, Wei, and others (see [8], [9], [10], [15], [24], [29]), but so far there is no general theory.

1.2. **Good notions of "solution".** Should one insist on classical solutions or should one also admit solutions with "mild" singularities? – Note that when $n \geq 7$ by a result of Jäger-Kaul [19] the singular weakly harmonic map $\bar{u}\colon x \mapsto x/|x|$ is absolutely energy-minimizing among maps $u\colon B_1^n(0) \to S^n = \partial B_1^{n+1}(0)$ with boundary data $u(x) = x$ for $|x| = 1$, where we let $B_r^n(x_0) = \{x \in \mathbb{R}^n;\ |x| < r\}$ for any $r > 0$, $x_0 \in \mathbb{R}^n$.

Jointly with Melanie Rupflin [31] we investigated this question in the simple case of radially symmetric solutions of (1.1). Refining results on the asymptotic decay of radial solutions of (1.1) due to Fowler [12] and Ni-Serrin [26] we showed that on the one hand the weak solution (with a suitable constant $b_* = b_*(n,p) > 0$)

$$u_*(x) = b_*|x|^{-\frac{2}{p-2}} \in H^1_{loc} \cap L^p_{loc}(\mathbb{R}^n)$$

of (1.1) on $\mathbb{R}^n$ is in the $H^1_{loc} \cap L^p_{loc}$-closure of the set of smooth solutions.

On the other hand we observe that for $2^* < p < 2\frac{2n-1}{n-2}$ there are oscillating distribution solutions $u \in W_0^{1,\frac{n}{n-1}} \cap L^{p-1}(B_1(0))$ of (1.1) on $B_1(0)$. The latter is in striking contrast with Pohozaev's [30] result. A "good" nootion of weak solution should therefore allow the former but rule out the latter behavior.

1.3. **"Stationary"solutions.** Prompted by work of Evans [11], Frank Pacard [27] proposed the following definition. A weak solution $u \in H^1 \cap L^p(\Omega)$ of (1.1) is *stationary*, if there holds

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} E_p(u \circ (id + \varepsilon\tau)) = 0,\ \forall \tau \in C_0^\infty(\Omega).$$

He obtained the following result. For given $p > 2^*$ let $\mu = \mu(p) = \frac{2p}{p-2} < n$.

**Theorem 1.2** (Pacard [27])**.** *Let $u \geq 0$ be a "stationary" weak solution of* (1.1)*, and suppose that $2^* < p < 2^+ = \frac{2(n-1)}{n-3}$. Then $u \in C^2(\Omega \setminus S)$, where the singular set $S$ is closed with $\mathcal{H}^{n-\mu}(S) = 0$.*

*Remark* 1.3. i) In [31] we observe that the weak solution $u_*(x) = b_*|x|^{-\frac{2}{p-2}}$ of (1.1) on $\mathbb{R}^n$ is stationary.

ii) Alternatively, one could restrict attention to non-negative weak solutions of class $H^1 \cap L^p(\Omega)$. Such a restriction would also seem natural from the point of view of applications in geometry such as in the Yamabe problem. See also [32].

1.4. **Scaling.** The exponent $\mu$ in Pacard's result reflects the fact that whenever $u$ is a solution of (1.1), then so is

$$u_R(x) = R^{\frac{2}{p-2}} u(x_0 + Rx),\ R > 0,\ x_0 \in \mathbb{R}^n.$$

This scaling property distinguishes a particular Morrey exponent. Recall that a function $f$ belongs to the Morrey space $L^{p,\lambda}(\Omega)$ with $0 < \lambda < n$ if

$$\|f\|^p_{L^{p,\lambda}(\Omega)} := \sup_{x_0 \in R^n,\ r>0} r^{\lambda-n} \int_{B_r(x_0)\cap\Omega} |f|^p dx < \infty.$$

Note the invariance of the Morrey norm $\|u\|^p_{L^{p,\mu}(\mathbb{R}^n)} = \|u_R\|^p_{L^{p,\mu}(\mathbb{R}^n)}$, $R > 0$.

Pacard's results in particular yield that any "stationary" solution of (1.1) lies in the space $L^{p,\mu}(\Omega)$.

1.5. **Monotonicity.** Pacard's key tool is a novel monotonicity formula; he shows that for a stationary weak solution $u$ of (1.1) the map

$$r \mapsto r^{\mu-n} \int_{B_r(x_0)} \Big(\frac{|\nabla u|^2}{2} + \frac{|u|^p}{p}\Big) dx + \frac{1}{p-2}\frac{d}{dr}\Big(r^{\mu-n}\int_{\partial B_r(x_0)} u^2 do\Big)$$

is non-decreasing. Note that the expected minus-sign in the volume integral is magically changed into a plus-sign at the expense of adding the derivative of a non-negative term of lower order.

## 2. Gradient flow

For given smooth initial data $u_0 \in H^1_0 \cap L^p(\Omega)$, $T_* \le \infty$, consider the Cauchy problem

$$\begin{aligned} u_t - \Delta u &= |u|^{p-2}u \text{ on } \Omega \times [0, T_*[, \\ u &= 0 \text{ on } \partial\Omega \times [0, T_*[, \\ u\big|_{t=0} &= u_0. \end{aligned} \tag{2.1}$$

2.1. **Energy identity.** Multiplying (2.1) by $u_t$ and integrating, with $E = E_p$ we obtain the equation

$$E(u(T)) + \int_0^T \int_\Omega |u_t|^2\, dx\, dt = E(u_0),\ 0 < T < T_*. \tag{2.2}$$

This fundamental identity shows that the flow (2.1) may be regarded as the $L^2$-gradient flow for the energy $E = E_p$.

2.2. **Finite-time blow-up and global existence.** Again a dichotomy occurs. On the one hand, as observed by Kaplan [20] and Fujita [14], for data $u_0$ with $E(u_0) < 0$ the solution to (2.1) blows up in finite-time. To see this, simply multiply (2.1) by $u$ and use (2.2) to obtain the inequatlity

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2} &= -\int_{\Omega\times\{t\}} \big(|\nabla u|^2 - |u|^p\big) dx = -2E(u(t)) + \frac{p-2}{p}\|u(t)\|^p_{L^p} \\ &\ge -2E(u_0) + c_0\|u(t)\|^p_{L^2} \ge c_0\|u(t)\|^p_{L^2} \end{aligned}$$

for some constant $c_0 > 0$. Hence

$$\|u(t)\|_{L^2} \geq \left(\|u_0\|_{L^2}^{(2-p)/2} - c_0(p-2)t\right)^{-2/(p-2)},$$

and $u(t)$ must blow up at the latest at time $T = c_0^{-1}(p-2)^{-1}\|u_0\|_{L^2}^{(2-p)/2}$.

On the other hand, for data $u_0$ which are small in the $C^1$-norm the maximum principle gives global existence, as can be seen by comparing a solution $u$ of (2.1) with a sufficiently small multiple $\bar{u} = s\varphi_1$ of the first eigenfunction $\varphi_1 > 0$ of the Laplacian on $\Omega$, satisfying

$$-\Delta\varphi_1 = \mu_1\varphi_1 \text{ on } \Omega,\ \varphi_1 = 0 \text{ on } \partial\Omega.$$

Note that for sufficiently small $s > 0$ we have $s^{p-2}\|\varphi_1\|_{L^\infty}^{p-2} < \mu_1$ and there holds

$$-\Delta\bar{u} - \bar{u}^{p-1} = (\mu_1 - s^{p-2}\varphi_1^{p-2})\bar{u} > 0 \text{ on } \Omega,\ \bar{u} = 0 \text{ on } \partial\Omega.$$

Hence if $-\bar{u} \leq u_0 \leq \bar{u}$ initially, the inequality $|u| \leq \bar{u}$ will be preserved by the flow.

2.3. **"Borderline" solutions.** Following Ni-Sacks-Tavantzis [25], for data $0 \leq u_0 \in C^1(\bar{\Omega})$ not vanishing identically and any $\lambda > 0$ let $u^\lambda$ be the solution of (2.1) with $u^\lambda\big|_{t=0} = \lambda u_0$. Noting that $E(\lambda u_0) \to -\infty$ as $\lambda \to \infty$ from the results in the preceding section we then conclude

$$0 < \lambda^* := \sup\{\lambda > 0\,;\, u^\lambda \text{ is global}\} < \infty.$$

By the maximum principle, moreover, as $\lambda \uparrow \lambda^*$ we have monotone convergence $u^\lambda \uparrow u^* \leq \infty$: a "borderline" (weak) solution of (2.1).

Chou-Du-Zheng [6] showed partial regularity of these "borderline" solutions $u^*$. Their proof uses the (parabolic) monotonicity formulas of Giga-Kohn [16] for (2.1) and Struwe [34] for the heat flow of harmonic maps, respectively, and the partial regularity theory from [34].

One might hope that for suitable $u_0$ we have convergence $u^*(t) \to u_\infty$ as $t \to \infty$, where $u_\infty$ is a non-trivial solution of (1.1). However, Matano-Merle [23] observe that radial solutions of (2.1) on a ball or on $\mathbb{R}^n$ for compactly supported initial data $u_0$ always either blow up in finite time or uniformly decay to 0 as $t \to \infty$. Since we already suspect that "interesting" solutions in general need not be smooth, we are thus led to consider also solutions $u$ to (2.1) that blow up in finite time.

## 3. Recent results

3.1. **Results for the flow** (2.1)**.** Jointly with Simon Blatt in a recent paper [2] we obtain a Pacard-type monotonicity formula for the flow (2.1), thereby using the framework of [34]. More precisely, let $\varphi = \varphi(|x|) \in C^\infty(\mathbb{R}^n)$ be a compactly supported cut-off function such that $0 \leq \varphi \leq 1$ and let

$$G(x,t) = \frac{1}{(4\pi|t|)^{n/2}}\, e^{-\frac{|x|^2}{4|t|}},\ x \in \mathbb{R}^n, t < 0,$$

be the fundamental solution to the heat equation with singularity at $(0,0)$. For any $(x_0,t_0)\in\mathbb{R}^n\times\mathbb{R}$ also set

$$G_{(x_0,t_0)}(x,t)=G(x-x_0,t-t_0).$$

Given $x_0\in\Omega$, $t_0>0$ define

$$D^\varphi(R)=D^\varphi_{(x_0,t_0)}(R)=\frac{R^\mu}{2}\int_{\Omega\times\{t_0-R^2\}}|\nabla u|^2\varphi^2(x-x_0)G_{(x_0,t_0)}\,dx$$

and for $q\geq 2$ also let

$$F^\varphi_q(R)=\frac{R^\mu}{q}\int_{\Omega\times\{t_0-R^2\}}|u|^q\varphi^2(x-x_0)G_{(x_0,t_0)}\,dx.$$

Then setting

$$H^\varphi(R)=\frac{p-2}{p+2}\big(D^\varphi(R)+F^\varphi_p(R)\big)+\frac{1}{p+2}\left(\frac{d}{dR}\big(RF^\varphi_2(R)\big)-A^\varphi_2(R)\right), \tag{3.1}$$

we show that for any $x_0\in\Omega$ with a suitable cut-off function $\varphi$ for all $R>0$ there holds

$$\begin{aligned}R\frac{d}{dR}H^\varphi(R)\geq{}&\frac{R^\mu}{4}\int_{\Omega\times\{t_0-R^2\}}\frac{|x\cdot\nabla u+2(t-t_0)u_t+au|^2}{|t_0-t|}\varphi^2G_{(x_0,t_0)}\,dx\\&+A^\varphi_0(R)+B^\varphi(R),\end{aligned}\tag{3.2}$$

where $B^\varphi$ is a boundary term and where $A^\varphi_0$, $A^\varphi_2$ are error terms induced by localization, containing derivatives of the cut-off function $\varphi$ and involving combinations of $u$ and $\nabla u$ of lower order with respect to $D^\varphi+F^\varphi_p$.

This result improves the Giga-Kohn [16] monotonicity result (which involves the difference $D^\varphi(R)-F^\varphi_p(R)$ instead of the sum) and shows conservation of the scale invariant Morrey norm up to blow-up time on domains of size proportional to remaining time. If $\Omega$ is not convex the boundary term $B^\varphi$ may also be negative. For this reason, on a general domain in [2] we derive these Morrey estimates only locally away from $\partial\Omega$; their extension up to the boundary will be treated in our forthcoming paper [3].

Moreover, by adapting the potential theoretic arguments of Adams [1] we obtain an $\varepsilon$-regularity result for weak solutions $u\in L^{p,\mu}$ of (2.1) with small Morrey norm, which improves the regularity estimates of Chou-Du-Zheng [6] for smooth solutions.

We also show that at any first blow-up point $(x_0,T)\in\Omega\times]0,\infty[$ for a suitable cut-off function $\varphi$ and any sufficiently small $R>0$ there holds $H^\varphi_{(x_0,T)}(R)\geq\varepsilon_0$, where $\varepsilon_0>0$ is an absolute constant. Together with (3.2) this allows to show the existence of a non-trivial, partially regular, self-similar tangent map at any first blow-up point of type I; moreover, at any first blow-up point of type II (that is, not of type I) by arguing similar to Hamilton [17] we obtain a non-trivial tangent map which is a smooth eternal solution of the flow (2.1). This result extends results of Matano-Merle [23] for radially symmetric solutions to the general case.

3.2. **Results for the time-independent problem** (1.1)**.** By applying our $\varepsilon$-regularity result to solutions of (1.1) we are able to improve Pacard's partial regularity result as follows.

**Theorem 3.1.** *For any $2^* < p < \infty$ let $u \in H^1 \cap L^p$ be a "stationary" weak solution of* (1.1)*. Then $u \in C^2(\Omega \setminus S)$, where $S$ is closed with $\mathcal{H}^{n-\mu}(S) = 0$.*

That is, we are able to remove Pacard's additional assumption that $u \geq 0$ and his restriction on the range of admissible exponents $p$ in Theorem 1.2.

In fact, by applying the potential theoretic approach of Adams [1] to (not necessarily stationary) weak solutions $u \in H^1 \cap L^{p,\mu}$ of (1.1) we obtain the following estimates, which together with Pacard's work imply Theorem 3.1.

**Theorem 3.2.** *Let $u \in H^1 \cap L^{p,\mu}(B_2(0))$ with $\|u\|_{L^{p,\mu}(B_2(0))} \leq \varepsilon$ be a weak solution of* (1.1)*. Then $u \in C^1(B_1(0))$ with*

$$\|u\|_{L^\infty(B_1(0))} + \|\nabla u\|_{L^\infty(B_1(0))} \leq C\|u\|_{L^{p,\mu}(B_2(0))}.$$

Still using ideas and techniques from Adams [1], we also have estimates reminiscent of Sobolev's embedding and the standard $L^p$-estimates for the Laplace operator which for any weak solution $u \in H_0^1 \cap L^{p,\mu}$ of (1.1) yield the bounds

$$\|\nabla u\|_{L^{2,\mu}(\Omega)} \leq C\|\Delta u\|_{L^{\frac{p}{p-1},\mu}(\Omega)} = C\|u\|^{p-1}_{L^{p,\mu}((\Omega)} \leq C\|\nabla u\|^{p-1}_{L^{2,\mu}(\Omega)}.$$

In particular, either $u \equiv 0$ or $\|\nabla u\|_{L^{2,\mu}(\Omega)} \geq c_1 > 0$, which gives a possibly optimal threshold result for solutions $u \neq 0$ of (1.1). Related to this one might ask whether there exist minimizers of the Morrey norm ratio and if such minimizers correspond to solutions of (1.1); moreover, it would be interesting to classify all such "ground states".

## 4. Open problems

If $\Omega$ is convex, and if $u_0 \in H_0^1 \cap L^{p,\mu}(\Omega)$ with $\nabla u_0 \in L^{2,\mu}(\Omega)$ for sufficiently small $\varepsilon > 0$ and suitable $r_0 > 0$ satisfies the condition

$$\sup_{x_0 \in R^n,\, 0<r<r_0} r^{\lambda-n} \int_{\Omega_r(x_0)\cap\Omega} \big(|\nabla u_0|^2 + |u_0|^p\big) dx < \varepsilon, \tag{4.1}$$

then our results give the a-priori $L^\infty$-bound

$$\|u_k(t)\|_{L^\infty(\Omega)} \leq Ct^{-\frac{2}{p-2}} \tag{4.2}$$

on a uniform time interval $0 < t < T$ for the solutions $u_k$ of (2.1) for mollified initial data $u_{0k} \to u_0$ in $H^1 \cap L^p(\Omega)$ $(k \to \infty)$ uniformly satisfying the bound (4.1). Hence, on any convex domain the Cauchy problem for (2.1) is locally well-posed for data $u_0$ in this class. In our forthcoming work [3] we extend this result to a general domain.

For large data $u_0 \in H_0^1 \cap L^{p,\mu}(\Omega)$ a-priori bounds like (4.2) are not yet available. Thus we may ask if (2.1) is locally well-posed for data $u_0 \in H_0^1 \cap L^{p,\mu}(\Omega)$ in general. See [5] for a related study.

Moreover, it would seem desirable to understand and exhaustively classify the different ways how solutions to equation (2.1) can blow up, extending the work of Friedman-McLeod [13], Herrero-Velazquez [18], Matano-Merle [21], [22], Troy [36], Weissler [37], and others.

## References


[1] D. R. Adams: *A note on Riesz potentials*. Duke Math. J. 42 (1975), no. 4, 765-778.

[2] Blatt, Simon; Struwe, Michael: *An analytic framework for the supercritical Lane-Emden equation and its gradient flow*, preprint (2013).

[3] Blatt, Simon; Struwe, Michael: *Boundary regularity for the supercritical Lane-Emden flow*, to appear (2014).

[4] Brezis, Haim: *Elliptic equations with limiting Sobolev exponents the impact of topology*. Frontiers of the mathematical sciences: 1985 (New York, 1985). Comm. Pure Appl. Math. 39 (1986), suppl., S17-S39.

[5] Brezis, Haim; Cazenave, Thierry: *A nonlinear heat equation with singular initial data*. J. Anal. Math. 68 (1996), 277-304.

[6] Chou, Kai-Seng; Du, Shi-Zhong; Zheng, Gao-Feng: *On partial regularity of the borderline solution of semilinear parabolic problems*. Calc. Var. Partial Differential Equations 30 (2007), no. 2, 251-275.

[7] Coron, Jean-Michel: *Topologie et cas limite des injections de Sobolev*. (French) [Topology and limit case of Sobolev embeddings] C. R. Acad. Sci. Paris Sér. I Math. 299 (1984), no. 7, 209-212.

[8] Dancer, E. N.; Wei, Juncheng: *Sign-changing solutions for supercritical elliptic problems in domains with small holes*. Manuscripta Math. 123 (2007), no. 4, 493-511.

[9] del Pino, Manuel; Felmer, Patricio; Musso, Monica: *Two-bubble solutions in the super-critical Bahri-Coron's problem*. Calc. Var. Partial Differential Equations 16 (2003), no. 2, 113-145.

[10] del Pino, Manuel; Wei, Juncheng: *Problèmes elliptiques supercritiques dans des domaines avec de petits trous*. (English, French summary) [Supercritical elliptic problems in domains with small holes], Ann. Inst. H. Poincaré Anal. Non Linéaire 24 (2007), no. 4, 507-520.

[11] Evans, Lawrence C.: *Partial regularity for stationary harmonic maps into spheres*. Arch. Rational Mech. Anal. 116 (1991), no. 2, 101-113.

[12] Fowler, R.H.: *Further studies of Emden's and similar differential equations*. Q. J. Math. (Oxford Series) 2 (1931), 259-288.

[13] Friedman, Avner; McLeod, Bryce: *Blow-up of positive solutions of semilinear heat equations*. Indiana Univ. Math. J. 34 (1985), no. 2, 425-447.

[14] Fujita, H.: *On the blowing up of solutions of the Cauchy Problem for* $u_t = \Delta u + u^{1+\alpha}$. J. Fac. Sci. Univ. Tokyo Sect. I, 13 (1996), 109-124.

[15] Ge, Yuxin; Jing, Ruihua; Pacard, Frank: *Bubble towers for supercritical semilinear elliptic equations*. J. Funct. Anal. 221 (2005), no. 2, 251-302.

[16] Giga, Yoshikazu; Kohn, Robert V.: *Asymptotically self-similar blow-up of semilinear heat equations*. Comm. Pure Appl. Math. 38 (1985), no. 3, 297-319.

[17] Hamilton, Richard S.: *Three-manifolds with positive Ricci curvature*. J. Differential Geom., 17 (1982), no. 2, 255-306.

[18] Herrero, Miguel A.; Velázquez, Juan J. L.: *Explosion de solutions d'quations paraboliques semilinaires supercritiques*. [Blowup of solutions of supercritical semilinear parabolic equations] C. R. Acad. Sci. Paris Sér. I Math. 319 (1994), no. 2, 141-145.

[19] Jäger, Willi; Kaul, Helmut: *Rotationally symmetric harmonic maps from a ball into a sphere and the regularity problem for weak solutions of elliptic systems*. J. Reine Angew. Math. 343 (1983), 146-161.

[20] Kaplan, S.: *On the growth of solutions of quasi-linear parabolic equations*. Comm. Pure. Appl. Math. 16 (1963), 305-330.

[21] Matano, Hiroshi; Merle, Frank: *On nonexistence of type II blowup for a supercritical nonlinear heat equation*. Comm. Pure Appl. Math. 57 (2004), no. 11, 1494-1541.

[22] Matano, Hiroshi; Merle, Frank: *Classification of type I and type II behaviors for a supercritical nonlinear heat equation*. J. Funct. Anal. 256 (2009), no. 4, 992-1064.

[23] Matano, Hiroshi; Merle, Frank: *Threshold and generic type I behaviors for a supercritical nonlinear heat equation*. J. Funct. Anal. 261 (2011), no. 3, 716-748.

[24] Molle, Riccardo; Passaseo, Donato: *Positive solutions of slightly supercritical elliptic equations in symmetric domains*. Ann. Inst. H. Poincaré Anal. Non Linéaire 21 (2004), no. 5, 639-656.

[25] Ni, Wei-Ming; Sacks, P. E.; Tavantzis, J.: *On the asymptotic behavior of solutions of certain quasilinear parabolic equations*. J. Differential Equations 54 (1984), no. 1, 97-120.

[26] Ni, W.M., Serrin, J.: *Existence and nonexistence theorems for ground states for quasilinear partial differential equations. The anomalous case*. Accad. Naz. Dei Lincei Atti Dei Convegni 77 (1986), 231-257.

[27] Pacard, Frank: *Partial regularity for weak solutions of a nonlinear elliptic equation*. Manuscripta Math. 79 (1993), no. 2, 161-172.

[28] Passaseo, Donato: *New nonexistence results for elliptic equations with supercritical nonlinearity*, Differential and Integral Equations, 8, no. 3 (1995), 577-586.

[29] Passaseo, Donato: *Nontrivial solutions of elliptic equations with supercritical exponent in contractible domains*. Duke Math. J. 92 (1998), no. 2, 429-457.

[30] Pohozaev, S. I.: *On the eigenfunctions of the equation* $\Delta u + \lambda f(u) = 0$. Dokl. Akad. Nauk SSSR 165 1965 36-39.

[31] Rupflin, M.; Struwe, M.: *Supercritical elliptic equations* Advanced Nonlinear Studies 12 (2012), 877-887; Addendum: Advanced Nonlinear Studies 13 (2013), 795-797.

[32] Rébai, Yomna: *Solutions of semilinear elliptic equations with one isolated singularity*. Differential Integral Equations 12 (1999), no. 4, 563-581.

[33] Struwe, Michael: *A global compactness result for elliptic boundary value problems involving limiting nonlinearities*. Math. Z. 187 (1984), no. 4, 511-517.

[34] Struwe, Michael: *On the evolution of harmonic maps in higher dimensions*. J. Differential Geom. 28 (1988), no. 3, 485-502.

[35] Struwe, Michael: *Variational methods. Applications to nonlinear partial differential equations and Hamiltonian systems*. Fourth edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 34. Springer-Verlag, Berlin, 2008.

[36] Troy, William C.: *The existence of bounded solutions of a semilinear heat equation*. SIAM J. Math. Anal. 18 (1987), no. 2, 332-336.

[37] Weissler, Fred B.: *An* $L^\infty$ *blow-up estimate for a nonlinear heat equation*. Comm. Pure Appl. Math. 38 (1985), no. 3, 291-295.

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