A LIOUVILLE-TYPE RESULT FOR A FOURTH ORDER EQUATION IN CONFORMAL GEOMETRY

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ABSTRACT. We show that there are no conformal metrics $g = e^{2u}g_{\mathbb{R}^4}$ on $\mathbb{R}^4$ induced by a smooth function $u \le C$ with $\Delta u(x) \to 0$ as $|x| \to \infty$ having finite volume and finite total Q-curvature, when $Q(x) = 1 + A(x)$ with a negatively definite symmetric 4-linear form $A(x) = A(x, x, x, x)$. Thus, in particular, for suitable smooth, non-constant $f_0 \le \max f_0 = 0$ on a four-dimensional torus any "bubbles" arising in the limit $\lambda \downarrow 0$ from solutions to the problem of prescribed Q-curvature $Q = f_0 + \lambda$ blowing up at a point p_0 with $d^k f_0(p_0) = 0$ for $k = 0, ..., 3$ and with $d^4 f_0(p_0) < 0$ are spherical, similar to the two-dimensional case.

1. BACKGROUND AND RESULTS

In order to put our results into context we first quickly review some related results for prescribed curvature problems on surfaces.

1.1. "Bubbling" metrics on closed surfaces.

Let (M, g_0) be a closed Riemann surface with smooth background metric g_0 and Euler charcteristic $\chi_M < 0$. Also let $f_0 \colon M \to \mathbb{R}$ be a smooth, non-constant function with $\max_{p \in M} f_0(p) = 0$, and for any $\lambda \in \mathbb{R}$ let $f_\lambda = f_0 + \lambda$. Then, if all maximum points p_0 of f_0 where $f_0(p_0) = 0$ are non-degenerate, it was shown by this author jointly with Borer and Galimberti [1] that for any sufficiently small $\lambda > 0$ there exists a conformal metric $g_\lambda = e^{2u_\lambda}g_0$ on M of prescribed Gauss curvature f_λ, which, as $\lambda \downarrow 0$ suitably exhibits "bubbling" with finite volume and finite total curvature. Galimberti in [6] extended this result to the case when $\chi_M = 0$. Finally, again in the case when $\chi_M = 0$, in [11] a similar "bubbling" behavior in the limit regime when $\lambda \downarrow 0$ was also observed for the corresponding prescribed curvature flow by this author.

The "bubbles" produced in this way are of the form $g = e^{2u}g_{\mathbb{R}^2}$, where $u \le C$ either solves the standard Liouville equation

$$-\Delta u = e^{2u} \text{ on } \mathbb{R}^2 \tag{1.1}$$

or the equation

$$-\Delta u = (1 + (Ax, x))e^{2u} \text{ on } \mathbb{R}^2, \tag{1.2}$$

where A is a negatively definite symmetric matrix given by $A = \frac{1}{2}Hess_{p_0}f_0$ for some $p_0 \in M$ with $f_0(p_0) = 0$. Moreover, they have finite volume and finite total curvature

$$\int_{\mathbb{R}^2} e^{2u}\,dx < \infty, \quad \int_{\mathbb{R}^2} (1 + (Ax, x))e^{2u}\,dx \in \mathbb{R}. \tag{1.3}$$

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All solutions u with (1.3) to the Liouville equation (1.1) have been classified by Chen-Li [4] and they uniquely correspond to the solution obtained by pull-back of the round spherical metric on S^2 under stereographic projection and its rescalings. Moreover, in [11] this author was able to rule out the existence of solutions $u \le C$ to equation (1.2) satisfying (1.3). Thus, in $n = 2$ dimensions all "bubbles" resulting from blow-up are spherical.

1.2. **"Bubbling" metrics of prescribed Q-curvature.** Similar results may be expected to hold true in the 4-dimensional case for the problem of prescribed Q-curvature. Let (M, g_0) be a closed manifold of dimension $n = 4$ with smooth background metric g_0 and suppose that the Paneitz operator P_0 is non-negative with kernel consisting of constant functions only and with total Q-curvature $k_{P_0} < 0$. By a result of Chang-Yang [3], later generalized by Djadli-Malchiodi[5], we then may assume that (M, g_0) has constant Q-curvature. For this case, and for smooth non-constant $f_0 \le 0 = \max_M f_0$, $f_\lambda = f_0 + \lambda$ as above, Galimberti [7] established the existence of "bubbling" metrics as $\lambda \downarrow 0$ analoguous to our results in [1], and Ngo-Zhang [9] obtained "bubbling" results for the case $k_{P_0} = 0$ which are the analogue of Galimberti's [6] results for the 2-torus. Ngo-Zhang [9] also studied the corresponding prescribed Q-curvature flow and showed "bubbling" analoguous to the results in [11]

Similar to the 2-dimensional case, also in dimension $n = 4$ the "bubble" metrics $g = e^{2u} g_{\mathbb{R}^2}$ are related to solutions $u \le C$ either of the equation

$$\Delta^2 u = e^{4u} \text{ on } \mathbb{R}^4 \tag{1.4}$$

or of the equation

$$\Delta^2 u = K(x) e^{4u} \text{ on } \mathbb{R}^4, \tag{1.5}$$

where $K(x)$ results from the Taylor expansion of f_0 around some $p_0 \in M$ with $f_0(p_0) = 0$; moreover, the "bubble" metrics again have finite volume and finite total Q-curvature

$$V_0 = \int_{\mathbb{R}^4} e^{4u}\, dx < \infty, \quad K_0 = \int_{\mathbb{R}^4} K(x) e^{4u}\, dx \in \mathbb{R}. \tag{1.6}$$

However, as shown by Chang-Chen [2], in contrast to the 2-dimensional case, in dimension $n = 4$ there is an abundance of non-spherical solutions to the equation (1.4) and also of solutions to the equation (1.5) for any given, smooth function K which is positive somewhere and satisfies $|K(x)| \le C|x|^s$ for some $s < 0$, all satisfying (1.6). This prompted the question whether in $n = 4$ dimensions for certain functions f_0 non-spherical blow-up might be possible.

Here we partially resolve this question and show that, in contrast to expectation, non-spherical blow-ups do not arise at points $p_0 \in M$ with $f_0(p_0) = 0$, if $d^k f_0(p_0) = 0$ for $1 \le k \le 3$ while $d^4 f_0(p_0)(x, x, x, x) < 0$ for any $x \ne 0$. Any non-spherical blow-up solution then would be a solution of (1.5) for $K(x) = 1 + A(x, x, x, x)$ with $A = \frac{1}{24} d^4 f_0(p_0)$. A key observation is that, by an argument of Robert and this author [10], proof of Proposition 2.3, building on the work of C.-S. Lin [8], for the above blow-up limits there additionally holds the condition that

$$\Delta u(x) \to 0 \quad \text{as } |x| \to \infty. \tag{1.7}$$

With this extra information then we obtain the following result.

Theorem 1.1. *Suppose that $K(x) = 1+A(x,x,x,x)$, where A is a negative definite and symmetric 4-linear map. Then there is no solution $u \in C^\infty(\mathbb{R}^4)$ of equation (1.5) with $u \le C$ and satisfying (1.6) as well as (1.7).*

Indeed, it is not too difficult to carry over the ideas from the proof of the analogous non-existence result Theorem 5.2 in our previous work [11] to the degenerate setting of Theorem 1.1. In contrast, it seems to be an interesting question whether the above result will also hold for the non-degenerate case when $d^2 f_0(p_0) < 0$ at all maximum points p_0.

Notation. Throughout the letter C will denote a generic constant, occasionally numbered for clarity.

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2. Proof of Theorem 1.1

Let $u \in C^\infty(\mathbb{R}^4)$ be a solution of equation (1.5) with $u \le C$ and satisfying $\Delta u(x) \to 0$ as $|x| \to \infty$. Moreover, assume that (1.6) holds true. Writing $K(x)e^{4u} = (1 + A(x,x,x,x))e^{4u} =: F \in L^1(\mathbb{R}^4)$ for brevity, following C.-S. Lin [8] we introduce

$$v(x) = \frac{1}{8\pi^2}\int_{\mathbb{R}^4} \big(\log|x-y| - \log|y|\big)F(y)\,dy \tag{2.1}$$

formally solving the equation

$$\Delta^2 v = -F(y) \quad \text{on } \mathbb{R}^4. \tag{2.2}$$

Thus the function $w = u + v$ satisfies $\Delta^2 w = 0$. In fact, adapting the argument of [8] to our setting, from our assumptions (1.6) and (1.7) we see that v is indeed well-defined and that u can be represented in terms of v, as follows.

Lemma 2.1. *The function v is well-defined and with a constant $C \in \mathbb{R}$ there holds*

$$u(x) = C - v(x) = C - \frac{1}{8\pi^2}\int_{\mathbb{R}^4} \big(\log|x-y| - \log|y|\big)F(y)\,dy. \tag{2.3}$$

Proof. Since $u \le C$ the function F is locally bounded and for any x the integral

$$\int_{B_1(x)\cup B_1(0)} \big(\log|x-y| - \log|y|\big)F(y)\,dy.$$

is well-defined. It thus suffices to bound $\big(\log|x-y|-\log|y|\big)$ for $|y| \ge 1$, $|x-y| \ge 1$.

If $|y| \ge 2|x|$ and hence $|y|/2 \le |x-y| \le 2|y|$ we can estimate

$$|\log|x-y| - \log|y|| = |\log(|x-y|/|y|)| \le \log 2.$$

On the other hand, if $|y| < 2|x|$, for $|y| \ge 1$, $|x-y| \ge 1$ we can bound

$$\log|x-y| + \log|y| \le 2\log|x| + C,$$

and $v(x)$ is well-defined for every $x \in \mathbb{R}^4$.

Likewise we see that we may differentiate under the integral to obtain

$$\Delta v(x) = \frac{1}{4\pi^2}\int_{\mathbb{R}^4} \frac{F(y)}{|x-y|^2}\,dy$$

and $\Delta^2 v = -F$. Thus, letting $h = \Delta u + \Delta v$ we find $\Delta h = 0$. In view of our assumption (1.7) that $\Delta u(x) \to 0$ as $|x| \to \infty$, and estimating

$$\Big|\int_{B_1(x_0)} \Delta v\, dx\Big| \le \frac{1}{4\pi^2}\int_{\mathbb{R}^4}\int_{B_1(x_0)} \frac{|F(y)|}{|x-y|^2}\, dx\, dy \to 0 \quad \text{as } |x_0| \to \infty,$$

from the mean value property of harmonic functions we then see that $h \equiv 0$, and $w = u + v$ is harmonic. But estimating

$$\begin{aligned}\int_{B_1(x_0)}\int_{B_1(x)} &\big(\log|x-y| - \log|y|\big)F(y)\, dy\, dx \\ &\le C\int_{B_2(x_0)}\int_{B_1(x_0)} \big|\log|x-y|\big||F(y)|\, dx\, dy + C\log|x_0| + C \\ &\le C\log|x_0| + C,\end{aligned}$$

we obtain the bound

$$\int_{B_1(x_0)} v(x)\, dx \le C\log|x_0| + C$$

for $|x_0| \gg 1$. Thus, since $u \le C$ and again using the mean value property of harmonic functions, we find the bound $w(x) = u(x) + v(x) \le C\log(2 + |x|)$ for all $x \in \mathbb{R}^4$. We conclude that $w \equiv C$ for some $C \in \mathbb{R}$, which gives the claim. □

From the representation (2.3), Lemma 3.1 and 3.2 below then are available. Since the proof of these results is rather technical, we postpone them to a later section and proceed, following the ideas of [11].

Lemma 2.2. *There holds*

$$\frac{u(x)}{\log|x|} = \frac{C}{\log|x|} - \frac{1}{8\pi^2}\int_{\mathbb{R}^4} \frac{\log|x-y| - \log|y|}{\log|x|} F(y)\, dy \to -\alpha \tag{2.4}$$

as $|x| \to \infty$, where

$$\alpha = \frac{1}{8\pi^2}\int_{\mathbb{R}^4} F(y)\, dy = \frac{K_0}{8\pi^2}. \tag{2.5}$$

Proof. Recalling that for $|y| \ge 2|x|$ there holds

$$\big|\log|x-y| - \log|y|\big| = \big|\log(|x-y|/|y|)\big| \le \log 2,$$

we can write

$$\begin{aligned}\int_{\mathbb{R}^4} &\frac{\log|x-y| - \log|y|}{\log|x|} F(y)\, dy \\ &= \int_{|y|\le 2|x|} \frac{\log|x-y| - \log|y|}{\log|x|} F(y)\, dy + o(1)\end{aligned}$$

with error $o(1) \to 0$ as $|x| \to \infty$. Moreover, again using that $F \in L^1 \cap L^\infty_{loc}(\mathbb{R}^4)$ it is not hard to show that

$$\int_{|y|\le 2|x|} \frac{\log|y|}{\log|x|} F(y)\, dy \to 0 \quad (|x| \to \infty).$$

Also note that Lemma 3.2 below allows to bound

$$\int_{B_1(x)} \log|x-y| F(y)\, dy \le C \sup_{B_1(x)} |F(y)| \le C\int_{B_1(x)} |F(y)|\, dy,$$

where

$$\int_{B_1(x)} |F(y)|\, dy \to 0 \quad (|x| \to \infty)$$

as $|x| \to \infty$ since $F \in L^1(\mathbb{R}^4)$.

Finally, we observe that for any $y \in B_{2|x|}(0) \setminus B_1(x)$ we have the uniform bound $0 \le \log|x-y|/\log|x| \le \log 3$ while for any fixed $y \in \mathbb{R}^4$ as $|x| \to \infty$ there holds $\log|x-y|/\log|x| \to 1$. Again using that $F \in L^1(\mathbb{R}^4)$, from Lebesgue's theorem on dominated convergence we then obtain the claim. □

From (2.4) and (1.6), with error $o(1) \to 0$ for any $\mu > 4\alpha - 4$ we now can bound

$$\begin{aligned}\int_{\mathbb{R}^4\setminus B_1(0)} |x|^{-\mu}\, dx &\le C \int_{\mathbb{R}^4} |x|^4 e^{4u}\, dx + C \\ &\le C \int_{\mathbb{R}^4} |A(x,x,x,x)| e^{4u}\, dx + C = C(V_0 - K_0) + C < \infty.\end{aligned}$$

It follows that $\alpha \ge 2$ and hence

$$K_0 \ge 16\pi^2 > 0. \tag{2.6}$$

Next, we multiply the terms in (1.5) with $x \cdot \nabla u$ to find, on the one hand

$$\begin{aligned}\Delta^2 u\, x \cdot \nabla u &= div\big(\nabla\Delta u\, x \cdot \nabla u) - \nabla\Delta u \nabla u - \nabla\Delta u\, x \cdot \nabla^2 u \\ &= div\big(\nabla\Delta u\, x \cdot \nabla u - \Delta u \nabla u - \Delta u\, x \cdot \nabla^2 u) + 2|\Delta u|^2 + x \cdot \nabla\big(\frac{|\Delta u|^2}{2}\big) \\ &= div\big(\nabla\Delta u\, x \cdot \nabla u - \Delta u \nabla u - \Delta u\, x \cdot \nabla^2 u + x\frac{|\Delta u|^2}{2}\big),\end{aligned} \tag{2.7}$$

and on the other

$$\begin{aligned}K(x)\, x \cdot \nabla u &= (1 + A(x,x,x,x))e^{4u}\, x \cdot \nabla u \\ &= div\big(\frac{x}{4}(1 + A(x,x,x,x))e^{4u}\big) - (1 + A(x,x,x,x))e^{4u} - A(x,x,x,x)e^{4u} \\ &= div\big(\frac{x}{4}(1 + A(x,x,x,x))e^{4u}\big) + e^{4u} - 2(1 + A(x,x,x,x))e^{4u}.\end{aligned}$$

Integrating, by finiteness of $\|F\|_{L^1}$ we have

$$R \int_{\partial B_R(0)} (1 + A(x,x,x,x))e^{4u} do \to 0$$

and thus

$$\int_{B_R(0)} (1 + A(x,x,x,x))e^{4u}\, x \cdot \nabla u\, dx \to V_0 - 2K_0 \tag{2.8}$$

as $R \to \infty$ suitably.

Proposition 2.3. *As $R \to \infty$ there holds*

$$\int_{\partial B_R(0)} \frac{x}{R} \cdot \big(\nabla\Delta u\, x \cdot \nabla u - \Delta u \nabla u - \Delta u\, x \cdot \nabla^2 u + x\frac{|\Delta u|^2}{2}\big) do \to -\frac{1}{16\pi^2} K_0^2.$$

Proof of Theorem 1.1. Combining (2.6), (2.7), (2.8), and Proposition 2.3 we obtain

$$16\pi^2 \le K_0 \le V_0 = 2K_0 - \frac{1}{16\pi^2} K_0^2 = \frac{32\pi^2 - K_0}{16\pi^2} K_0; \tag{2.9}$$

thus $K_0 = 16\pi^2 = V_0$. But from the formula for K_0 it follows that we must have $K_0 < V_0$, and the proof of Theorem 1.1 is complete. □

3. Proof of Proposition 2.3

We need the following estimate.

Lemma 3.1. *There holds*

$$\limsup_{|x|\to\infty} x\cdot\nabla u(x) = -\frac{K_0}{8\pi^2};$$

moreover, we have

$$|x||\nabla u(x)| \le C < \infty,$$

uniformly in $x\in\mathbb{R}^4$.

Proof. Differentiating (2.3) we find

$$x\cdot\nabla u(x) = -\frac{1}{8\pi^2}\int_{\mathbb{R}^4}\frac{x\cdot(x-y)}{|x-y|^2}F(y)\,dy = -\frac{K_0}{8\pi^2} + I(x), \tag{3.1}$$

with

$$I(x) := -\frac{1}{8\pi^2}\int_{\mathbb{R}^4}\frac{y\cdot(x-y)}{|x-y|^2}F(y)\,dy.$$

Also consider the term

$$II(x) = x^{\perp}\cdot\nabla u(x) = -\frac{1}{8\pi^2}\int_{\mathbb{R}^4}\frac{x^{\perp}\cdot(x-y)}{|x-y|^2}F(y)\,dy, \tag{3.2}$$

for an arbitrary rotation $P\colon x\mapsto x^{\perp}$ on $\mathbb{R}^4$ such that $x\cdot x^{\perp}=0$ for all $x\in\mathbb{R}^4$.

Given any $\varepsilon>0$ we claim that for sufficiently large $R>0$ we can estimate the error terms

$$\sup_{|x|=R}\big(|I(x)| + |II(x)| \le C\varepsilon \tag{3.3}$$

with a constant $C>0$ independent of R.

Let $\varepsilon>0$ be given. There exists $R_0>1$ such that

$$\int_{\mathbb{R}^4\setminus B_{R_0}(0)}|F(y)|\,dy<\varepsilon.$$

Let $|x|=2R\ge 2R_0$. Observing that $|y|\le|x-y|+|x|$ gives

$$\frac{|y|}{|x-y|}\le 3 \ \text{ for } y\notin B_R(x),$$

we can bound

$$\begin{aligned}
&\big|8\pi^2 I(x) + \int_{B_R(x)}\frac{y\cdot(x-y)}{|x-y|^2}F(y)\,dy\big| \\
&\le \int_{\mathbb{R}^4\setminus(B_R(x)\cup B_{R_0}(0))}\frac{|y|}{|x-y|}|F(y)|\,dy + \frac{R_0}{2R-R_0}\int_{B_{R_0}(0)}|F(y)|\,dy \\
&\le C\varepsilon + \frac{CR_0}{2R-R_0}\le C\varepsilon
\end{aligned}$$

if $R\ge R_0$ is sufficiently large. Moreover, for $R\ge R_0$, $|x|=2R$ we have

$$\big|\int_{B_R(x)}\frac{y\cdot(x-y)}{|x-y|^2}F(y)\,dy - \int_{B_R(x)}\frac{x\cdot(x-y)}{|x-y|^2}F(y)\,dy\big| \le \int_{B_R(x)}|F(y)|\,dy\le\varepsilon.$$

Similarly, observing that

$$8\pi^2 II(x) = -\int_{\mathbb{R}^4} \frac{x^\perp \cdot (x-y)}{|x-y|^2} F(y)\,dy = -\int_{\mathbb{R}^4} \frac{y^\perp \cdot (x-y)}{|x-y|^2} F(y)\,dy,$$

we can bound

$$\begin{aligned}
&\big|8\pi^2 II(x) + \int_{B_R(x)} \frac{y^\perp \cdot (x-y)}{|x-y|^2} F(y)\,dy\big| \\
&\quad \le \int_{\mathbb{R}^4 \setminus (B_R(x) \cup B_{R_0}(0))} \frac{|y|}{|x-y|} |F(y)|\,dy + \frac{R_0}{2R-R_0} \int_{B_{R_0}(0)} |F(y)|\,dy \\
&\quad \le C\varepsilon + \frac{CR_0}{2R-R_0} \le C\varepsilon
\end{aligned}$$

if $R \ge R_0$ is sufficiently large, and we have

$$\int_{B_R(x)} \frac{y^\perp \cdot (x-y)}{|x-y|^2} F(y)\,dy = \int_{B_R(x)} \frac{x^\perp \cdot (x-y)}{|x-y|^2} F(y)\,dy.$$

Thus, it suffices to bound the integrals

$$\int_{B_R(x)} \frac{x \cdot (x-y)}{|x-y|^2} F(y)\,dy \quad \text{and} \quad \int_{B_R(x)} \frac{x^\perp \cdot (x-y)}{|x-y|^2} F(y)\,dy,$$

respectively. Changing coordinates to $z = x - y$ we obtain

$$\begin{aligned}
\int_{B_R(x)} \frac{x \cdot (x-y)}{|x-y|^2} F(y)\,dy &= \int_{B_R(0)} \frac{x \cdot z}{|z|^2} F(x-z)\,dz \\
&= \frac{1}{2} \int_{B_R(0)} \frac{x \cdot z}{|z|^2} (F(x-z) - F(x+z))\,dz,
\end{aligned}$$

where we observe that

$$\int_{B_R(0)} \frac{x \cdot z}{|z|^2} (F(x-z) + F(x+z))\,dz = 0$$

by symmetry. Expanding

$$\begin{aligned}
A(x \pm z, x \pm z, x \pm z, x \pm z) =& A(x,x,x,x) \pm 4A(x,x,x,z) \\
&+ 6A(x,x,z,z) \pm 4A(x,z,z,z) + A(z,z,z,z)
\end{aligned}$$

we then have

$$\begin{aligned}
F(x-z) - F(x+z) &= (1 + A(x-z,x-z,x-z,x-z))e^{4u(x-z)} \\
&\quad - (1 + A(x+z,x+z,x+z,x+z))e^{4u(x+z)} \\
&= \big((1 + A(x,x,x,x) + 6A(x,x,z,z) + A(z,z,z,z)\big)\big(e^{4u(x-z)} - e^{4u(x+z)}\big) \\
&\quad - 4(A(x,x,x,z) + A(x,z,z,z))\big(e^{4u(x-z)} + e^{4u(x+z)}\big).
\end{aligned}$$

Note that we can bound

$$\begin{aligned}
&\int_{B_R(0)} \frac{|x \cdot z|}{|z|^2} \big(|A(x,x,x,z) + A(x,z,z,z)|\big)\big(e^{4u(x-z)} + e^{4u(x+z)}\big)\,dz \\
&\quad \le C \int_{B_R(0)} |x|^4 \big(e^{4u(x-z)} + e^{4u(x+z)}\big)\,dz \le C \int_{B_R(x)} |F(y)|\,dy \le C\varepsilon.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}\int_{B_R(0)} \frac{|x\cdot z|}{|z|^2}\big|6A(x,x,z,z)+A(z,z,z,z)\big|\big(e^{4u(x-z)}+e^{4u(x+z)}\big)\,dz\\ \le C\int_{B_R(0)}|x|^4\big(e^{4u(x-z)}+e^{4u(x+z)}\big)\,dz \le C\int_{B_R(x)}|F(y)|\,dy\le C\varepsilon.\end{aligned}$$

By Lemma 2.2 we also can bound $e^{4u(x\pm z)} \le R^{-4\mu}$ for any $\mu<2$ when $|x| = 2R \ge 2|z|$ with sufficiently large $R \gg 1$. Choosing $\mu = 3/2$ we find

$$\int_{B_R(0)} \frac{|x\cdot z|}{|z|^2}\big(e^{4u(x-z)}+e^{4u(x+z)}\big)\,dz \le CR^{1-4\mu}\int_{B_R(0)}\frac{dz}{|z|}\le CR^{4-4\mu}\to 0$$

as $|x| = 2R \to \infty$.

Thus we obtain

$$\Big|8\pi^2 I(x)+\frac{1}{2}\int_{B_R(0)}\frac{x\cdot z}{|z|^2}A(x,x,x,x)\big(e^{4u(x-z)}-e^{4u(x+z)}\big)\,dz\Big|\le C\varepsilon,$$

and by the same type of reasoning also that

$$\Big|8\pi^2 II(x)+\frac{1}{2}\int_{B_R(0)}\frac{x^\perp\cdot z}{|z|^2}A(x,x,x,x)\big(e^{4u(x-z)}-e^{4u(x+z)}\big)\,dz\Big|\le C\varepsilon$$

if $R \ge R_0$ is sufficiently large.

But letting $i,j,k\in S^3$ be the usual imaginary quaternions with $i^2=j^2=k^2=ijk=-1$, and integrating for each z_0 with $|z_0|=r<R$ from z_0 to $-z_0$ along a semi-circle $S(z_0)$ of radius r, parametrized by $\theta\in[0,\pi]$, with tangent vector iz at any $z=z(\theta)\in S(z_0)$, we have

$$\begin{aligned}\big|e^{4u(x-z_0)}-e^{4u(x+z_0)}\big| &\le \Big|\int_0^\pi 4iz\cdot\nabla u(x+z)e^{4u(x+z)}\,d\theta\Big|\\ &\le C\sup_{|z|=r}|\nabla u(x+z)|\int_{S(z_0)}e^{4u(x+z)}\,ds,\end{aligned}$$

where ds denotes arc-length. Hence for any $|x|=2R$ and sufficiently large $R\ge R_0$ there results

$$\begin{aligned}|I(x)|+\sup_{P\in O(4)}|II(x)| &\le C\sup_{|z|\le R}|\nabla u(x+z)|\int_{B_R(0)}|x|^5e^{4u(x+z)}\,dz+C\varepsilon\\ &\le C\sup_{y\in B_R(x)}|y||\nabla w(y)|\int_{B_R(x)}|F(y)|\,dy+C\varepsilon \qquad (3.4)\\ &\le C\varepsilon\sup_{y\in B_R(x)}|y||\nabla w(y)|+C\varepsilon.\end{aligned}$$

But for any $\mu<2$ for any sufficiently large $|x|=2R\ge 2R_1=2R_1(\mu)\ge 2R_0\ge 2$ and any $y\in B_R(x)$ we can bound $|F(y)|\le C|y|^4e^{4u(y)}\le CR^{4-4\mu}$ to obtain

$$\begin{aligned}|\nabla u(x)| = \Big|\int_{\mathbb{R}^4}\frac{x-y}{|x-y|^2}F(y)\,dy\Big| &\le \int_{\mathbb{R}^4\setminus B_R(x)}\frac{|F(y)|}{|x-y|}\,dy+\int_{B_R(x)}\frac{|F(y)|}{|x-y|}\,dy\\ &\le CR^{-1}+CR^{7-4\mu}\le CR^{7-4\mu}.\end{aligned}$$

Thus, for any $0<\nu<1$ the number $\sup_{y\in\mathbb{R}^4}|y|^\nu|\nabla u(y)|$ is attained.

We claim that this also implies that

$$\sup_{y\in\mathbb{R}^4}|y||\nabla u(y)|<\infty.$$

Otherwise for $\nu \uparrow 1$ there exist points $x_\nu \in \mathbb{R}^4$ with $|x_\nu| =: 2R_\nu \to \infty$ such that

$$\gamma_\nu := |x_\nu|^\nu |\nabla u(x_\nu)| = \sup_{x\in\mathbb{R}^4} |x|^\nu |\nabla u(x)| \to \infty \text{ as } \nu \uparrow 1.$$

But from the definitions of I and II and estimate (3.4) above, with a constant $C_2 > 0$ independent of ν we have

$$\begin{aligned}\gamma_\nu = |x_\nu|^\nu |\nabla u(x_\nu)| &\le \big(\frac{K_0}{8\pi^2} + |I(x_\nu)| + \sup_{P\in O(4)} |II(x_\nu)|\big)|x_\nu|^{\nu-1} \\ &\le \big(\frac{K_0}{8\pi^2} + C\varepsilon \sup_{y\in B_{R_\nu}(x_\nu)} |y||\nabla u(y)| + C\varepsilon\big)|x_\nu|^{\nu-1} \\ &\le \big(\frac{K_0}{8\pi^2} + C_2\varepsilon\big)|x_\nu|^{\nu-1} + C_2\varepsilon \sup_{|y|\ge R_\nu} |y|^\nu |\nabla u(y)|\end{aligned}$$

where we may let $\varepsilon \to 0$ as $|x_\nu| = 2R_\nu \to \infty$. In particular, for sufficiently large $\nu < 1$ (and hence sufficiently large $R_\nu > 1$) we have $C_2\varepsilon < 1/2$ and we can bound

$$\gamma_\nu \le \big(\frac{K_0}{8\pi^2} + C_2\varepsilon\big)|x_\nu|^{\nu-1} + C_2\varepsilon \sup_{|y|\ge R_\nu} |y|^\nu |\nabla u(y)| \le \big(\frac{K_0}{8\pi^2} + \frac{1}{2}\big) + \frac{\gamma_\nu}{2}.$$

Thus for any $\nu < 1$ there holds

$$\gamma_\nu \le \frac{K_0}{4\pi^2} + 1.$$

It follows that for every $x \in \mathbb{R}^4$ we can bound

$$|x||\nabla u(x)| \le \limsup_{\nu\uparrow 1} |x|^\nu |\nabla u(x)| \le \limsup_{\nu\uparrow 1} \gamma_\nu \le \frac{K_0}{4\pi^2} + 1.$$

Together with (3.4) this concludes the proof. □

As an immediate consequence of Lemma 3.1 we obtain the following Harnack-type estimate.

Lemma 3.2. *There exists a constant $C > 0$ such that for any $R > 1$, any $x \in \mathbb{R}^4$ with $|x| = 2R$ there holds*

$$\sup_{y\in B_R(x)} |F(y)| \le C \inf_{y\in B_R(x)} |F(y)|.$$

Proof. For any $y \in B_R(x)$ we can bound

$$\big|\frac{d}{dt}F(x + t(y - x)\big| = |(y - x) \cdot \nabla F(x + t(y - x))| \le R|\nabla F(x + t(y - x))|.$$

Moreover, for any $z \in B_R(x)$ in view of the bound $|z| \ge |x| - R \ge R$ there holds $R|\nabla A(z)| \le 4|A(z)|$, where we denote $A(z, z, z, z)$ as $A(z)$ for brevity. Thus, by Lemma 3.1 for any $z \in B_R(x)$ we have

$$R|\nabla F(z)| \le 4\big(1 + |z||\nabla u(z)|\big)|F(z)| \le C|F(z)|.$$

It follows that

$$|\frac{d}{dt}F(x + t(y - x)| \le C|F(x + t(y - x))|$$

and

$$|F(y)/F(x)| + |F(x)/F(y)| \le C.$$

The claim follows. □

Proof of Proposition 2.3. Differentiating (2.3) further, for $x \in \partial B_R(0)$ we also find the representations

$$\Delta u(x) = -\frac{1}{4\pi^2}\int_{\mathbb{R}^4}\frac{F(y)}{|x-y|^2}\,dy = -\frac{1}{4\pi^2R^2}K_0 + III(x)$$

with

$$III(x) := \frac{1}{4\pi^2}\int_{\mathbb{R}^4}\Big(\frac{1}{|x|^2} - \frac{1}{|x-y|^2}\Big)F(y)\,dy,$$

as well as

$$(x\cdot\nabla^2 u(x), x) = -\frac{1}{8\pi^2}\int_{\mathbb{R}^4}\frac{|x|^2|x-y|^2 - 2|x\cdot(x-y)|^2}{|x-y|^4}F(y)\,dy,$$

and

$$x\cdot\nabla\Delta u(x) = \frac{1}{2\pi^2}\int_{\mathbb{R}^4}\frac{x\cdot(x-y)F(y)}{|x-y|^4}\,dy = \frac{1}{2\pi^2R^2}K_0 + IV(x),$$

where

$$IV(x) := \frac{1}{2\pi^2}\int_{\mathbb{R}^4}\Big(\frac{x\cdot(x-y)}{|x-y|^4} - \frac{1}{|x|^2}\Big)F(y)\,dy.$$

Note that we can combine the terms

$$\begin{aligned} &x\cdot\big(x\frac{|\Delta u|^2}{2} - \Delta u\nabla u - \Delta u\, x\cdot\nabla^2 u\big)\\ &\qquad = \Delta u\big(|x|^2\frac{\Delta u}{2} - x\cdot\nabla u - (x\cdot\nabla^2u(x),x)\big)\end{aligned}$$

with

$$\begin{aligned} &|x|^2\frac{\Delta u}{2} - x\cdot\nabla u - (x\cdot\nabla^2 u(x),x)\\ &\quad= \frac{1}{8\pi^2}\int_{\mathbb{R}^4}\Big(\frac{x\cdot(x-y)}{|x-y|^2} - \frac{2|x\cdot(x-y)|^2}{|x-y|^4}\Big)F(y)\,dy\\ &\quad= -\frac{1}{8\pi^2}K_0 - I(x) + V(x),\end{aligned}$$

where I as in the proof of Lemma 3.1 satisfies $I(x)\to 0$ as $|x|\to\infty$, and where

$$V(x) := \frac{1}{4\pi^2}\int_{\mathbb{R}^4}\Big(1 - \frac{|x\cdot(x-y)|^2}{|x-y|^4}\Big)F(y)\,dy.$$

Replacing R by $2R$ for later convenience, we now claim that we can estimate the error terms

$$\sup_{|x|=2R}\big(|x|^2|III(x)| + |x|^2|IV(x)| + |V(x)|\big)\to 0 \tag{3.5}$$

as $R\to\infty$. From this the asserted convergence follows.

Fix $\varepsilon > 0$ and let $R_0 > 1$ such that

$$\int_{\mathbb{R}^4\setminus B_{R_0}(0)}|F(y)|\,dy < \varepsilon \tag{3.6}$$

as in the proof of Lemma 3.1. For $|x| = 2R$, $|y|\le R_0$, for sufficiently large $R\ge R_0$ we can bound

$$\Big|\frac{|x|^2 - |x-y|^2}{|x-y|^2}\Big| = \frac{\big|2x\cdot y - |y|^2\big|}{|x-y|^2} \le \frac{2RR_0 + R_0^2}{(2R-R_0)^2} \le \frac{3R_0}{2R-R_0} < \varepsilon.$$

Similarly, we have

$$\Big|\frac{|x|^2 x\cdot(x-y)-|x-y|^4}{|x-y|^4}\Big| \le C\frac{R^3R_0}{(2R-R_0)^4} \le \frac{CR_0}{2R-R_0} < \varepsilon$$

and

$$\Big|\frac{|x\cdot(x-y)|^2-|x-y|^4}{|x-y|^4}\Big| \le C\frac{R^3R_0}{(2R-R_0)^4} < \varepsilon$$

for sufficiently large $R \ge R_0$. Moreover, we observe that for $y \notin B_R(x)$ the terms on the left in the above three formulas are uniformly bounded. Thus for any x with $|x| = 2R$ by (3.6) the contributions on $\mathbb{R}^4 \setminus B_R(x)$ to the error terms (3.5) are bounded by $C\varepsilon$.

Finally, for $y \in B_R(x)$ we can estimate

$$\Big|\frac{|x|^2-|x-y|^2}{|x-y|^2}\Big| \le \frac{|x|^2}{|x-y|^2}+1 \le 2\frac{|x|^2}{|x-y|^2} \le 2\frac{|x|^3}{|x-y|^3},$$

while

$$\Big|\frac{|x|^2 x\cdot(x-y)-|x-y|^4}{|x-y|^4}\Big| \le \frac{|x|^3}{|x-y|^3}+1 \le 2\frac{|x|^3}{|x-y|^3},$$

and

$$\Big|\frac{|x\cdot(x-y)|^2-|x-y|^4}{|x-y|^4}\Big| \le \frac{|x|^2}{|x-y|^2}+1 \le 2\frac{|x|^2}{|x-y|^2} \le 2\frac{|x|^3}{|x-y|^3}$$

to see that

$$|x|^2|III(x)| + |x|^2|IV(x)| + |V(x)| \le C\int_{B_R(x)} \frac{|x|^3}{|x-y|^3}|F(y)|\,dy + C\varepsilon.$$

But by Lemma 3.2 for $|x| = 2R$ we can bound

$$\begin{aligned}\int_{B_R(x)} \frac{|x|^3}{|x-y|^3}|F(y)|\,dy &\le \sup_{y\in B_R(x)}|F(y)|\int_{B_R(x)}\frac{|x|^3}{|x-y|^3}\,dy\\ &\le CR^4\sup_{y\in B_R(x)}|F(y)| \le CR^4\inf_{y\in B_R(x)}|F(y)| \le C\int_{B_R(x)}|F(y)|\,dy \le C\varepsilon,\end{aligned}$$

and

$$|x|^2|III(x)| + |x|^2|IV(x)| + |V(x)| \le C\varepsilon,$$

as claimed. □

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