NORMALIZED HARMONIC MAP HEAT FLOW

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ABSTRACT. For any closed Riemannian manifold N we propose the normalized harmonic map heat flow as a means to obtain non-constant harmonic maps $u\colon S^m \to N$, $m \geq 3$.

1. Overview

Finding non-constant harmonic maps $u\colon S^m \to N \subset \mathbb{R}^n$ for a closed target manifold N and $m \geq 3$ is a prototype of a super-critical variational problem.

El Soufi [12] showed that any non-trivial (sufficiently smooth) harmonic map $u\colon S^m \to N$ either achieves a strict maximum of the Dirichlet energy

$$D(u) = \frac{1}{2}\int_M |\nabla_g u|^2 d\mu_{g_M}$$

with respect to the action $\Gamma \ni \gamma \to u \circ \gamma$ of the Möbius group Γ of S^m on u, or is constant in the direction defined by some $a \in S^m$. Thus, given a smooth map $u_0\colon S^m \to N$, one might hope to find a harmonic map homotopic to u_0 as a critical point of D of mountain-pass type, minimizing the quantity $\sup_{\gamma\in\Gamma} D(u \circ \gamma)$ among maps u homotopic to u_0. Even though this min-max procedure at first may look similar to the study of sweep-outs in the work of Marques-Neves [21] on the Willmore conjecture, the fact that in our case the Möbius group acts from the right makes a decisive difference. Indeed, in our setting for any u each $\gamma \in \Gamma$ induces a "variation of the independent variables" while keeping the image of u fixed, whereas in the setting of Marques-Neves the map u itself is deformed. A further complication arises from the fact that when applying the harmonic map heat flow to Γ-type sweep-outs of the form $(u\circ\gamma)_{\gamma\in\Gamma}$ singularities arise instaneously, possibly destroying any topological data. Moreover, since the heat flow in general does not commute with $\gamma \in \Gamma$, the family of such Γ-type sweep-outs is not invariant under the flow; that is, flowing a Γ-type sweep-out for any time $t > 0$ in general will fail to produce a Γ-type sweep-out.

Rather than look for min-max critical points of D of the kind above we therefore should seek harmonic maps as minimizers (or more generally as "critical points" – in a sense to be defined) of the functional

$$E(u) = \sup_{\gamma\in\Gamma} D(u \circ \gamma),$$

directly. In order to achieve this, in this paper we propose the normalized harmonic map heat flow as a new tool; see Definition 4.1 below. For ease of exposition we will focus on the case when $m = 3$.

Outline. In the following Section 2 we first review the concept of harmonic maps and recall some results for the standard harmonic map heat flow in dimensions

Date: March 14, 2019.

two and higher. In Section 3 we then look at the action of the Möbius group on the Dirichlet energy of a given map $u\colon S^3 \to N$ and derive the strengthened form Theorem 3.5 of El Soufi's result [12] from an elementary estimate for the effect of dilation (in stereographic coordinates) on $D(u)$, made quantitative via our concept of δ_0-uniformly 3-dimensional map. For such maps with "small" tension field, moreover, the condition $D(u) = E(u)$, whose verification involves the whole group Γ, is shown to be equivalent to the more easily tractable condition $X(u) = 0$ for the center of mass $X(u) \in \mathbb{R}^4$, also introduced in this section.

The latter observation motivates our definition of normalized harmonic map flow for maps $u\colon S^3 \to N$ in Section 4. Standard methods allow to show existence of a smooth solution to this flow for any smooth initial data u_0 with sufficiently small tension field, and having "Möbius non-degenerate" second variation $d^2D(u_0)$, as defined in this section. As a test for our theory, in Theorem 4.5 we show that the identity map $\bar{u}\colon S^3 \to S^3$ is stable under normalized harmonic map flow (while it is an unstable rest point for the standard harmonic map heat flow), and Theorem 4.6 extends this result to a proof of global existence and convergence of the flow for geometries (and maps) close to (the identity map on) S^3. Complementing this well-posedness result, from a non-existence result of Smith [29] we deduce that in general the normalized harmonic map flow may blow up in finite or infinite time, which raises the question of a possible characterization of singularities.

In Section 5, finally, we give an outlook on a possible global theory. In particular, in Lemma 5.1 we observe that the functional E and the normalized harmonic map flow allow to detect topologically non-trivial data, which might lead to non-trivial results even in the case of blow-up.

Acknowledgement. I am indebted to Rick Schoen for pointing me to the work of El Soufi when I first told him about my ideas, and to Tobias Lamm for locating the reference [12] relevant here, which greatly encouraged me to proceed further with this research. I also thank Gerhard Huisken for helpful discussions and the anonymous referee for careful reading of the paper and useful comments.

The results below were first publicly presented in January 2018 at a conference at Abu Dhabi New York University in honor of the 60^{th} birthdays of Professors Fang-Hua Lin and Jalal Shatah, and I dedicate this paper to my friends Jalal and Fang-Hua at this occasion.

2. A quick review of harmonic maps and their flows

2.1. **Harmonic maps.** Let (M, g), (N, h) be closed (compact and without boundary) Riemannian manifolds, where $m = dim(M) \geq 2$. By Nash's theorem we may assume that N is isometrically embedded in some Euclidean $\mathbb{R}^n$. Let $\delta > 0$ such that the nearest-neighbor projection $\pi_N\colon N_\delta \subset \mathbb{R}^n \to N$ from the δ-neighborhood $N_\delta = \cup_{p \in N} B_\delta(p)$ of N to N is well-defined and smooth. For a smooth map $u\colon M \to N \hookrightarrow \mathbb{R}^n$ then let

$$D(u) = \frac{1}{2} \int_M |\nabla_g u|^2 d\mu_g$$

be the Dirichlet energy of u.

The map u is harmonic if u is a critical point of D in the sense that

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} D(\pi_N(u + \varepsilon\varphi)) = -\int_M \tau(u)\varphi \, d\mu_g = 0$$

for any variation $\varphi \in C^1(M;\mathbb{R}^n)$, where

$$\tau(u) = d\pi_N(u)\Delta u$$

is the tension field of u. Thus, u is harmonic if and only if $\tau(u) = 0$.

Note that with the help of the second fundamental form $A(p)\colon T_pN \times T_pN \to T_p^\perp N$ at any point $p \in N$ the tension field may be expressed as

$$\tau(u) = \Delta u + A(u)(\nabla u, \nabla u) = \Delta u + \sum_{i=1}^{m} A(u)(\partial_i u, \partial_i u).$$

2.2. **Harmonic map heat flow.** If N is non-positively curved, it was shown by Eells-Sampson [10] that for any given smooth map $u_0\colon M \to N$ there is a smooth harmonic map $u\colon M \to N$ homotopic to u_0, which can be obtained by solving the harmonic map heat flow

$$u_t = \tau(u) = \Delta u + A(u)(\nabla u, \nabla u) \text{ on } M \times [0, \infty[\tag{2.1}$$

with initial data $u(0) = u_0$ and letting $t \to \infty$ suitably. The harmonic map u is uniquely determined by the map u_0 when N has strictly negative sectional curvature, and u achieves the infimum of the Dirichlet energy in the homotopy class of u_0, which is strictly positive if u_0 is topologically non-trivial.

2.3. **Two dimensions.** The 2-dimensional setting is distinguished by the conformal invariance of Dirichlet's integral. Sacks-Uhlenbeck [26] showed that, given any closed surface M, for any closed target N with $\pi_2(N) = 0$ and for any smooth $u_0\colon M \to N$ there is a harmonic map homotopic to u_0. On the other hand, Eells-Wood [11] showed that there is no harmonic map $u\colon T^2 \to S^2$ of degree 1.

Note that in 2 dimensions the energy space $H^1 \hookrightarrow L^p$ for any $p < \infty$, but H^1 fails to embed into L^∞ or C^0. Hence even though H^1-homotopy classes of maps by work of Courant [6] or Schoen-Uhlenbeck [27] are well-defined, they are not weakly closed in H^1 and the direct methods cannot be applied, as the result of Eells-Wood demonstrates.

For any closed surface M and without any curvature assumptions on the target, a global analogue of the Eells-Sampson result [10] was obtained by the author [30]. In fact, for any smooth $u_0\colon M \to N$ in [30] the existence of a unique, global, partially regular solution to the harmonic map heat flow with non-increasing energy was established, and it was shown that the at most finitely many singular points in space-time of the flow are characterized by the "bubbling-off" of non-constant smooth harmonic maps $S^2 \to N$ that arise as limits from the suitably rescaled flow.

The energy identities subsequently derived by Qing [22], Wang [34], Ding-Tian [7], Lin-Wang [18], Qing-Tian [23] further sharpen these results.

Chang-Ding-Ye [1] gave an example showing that the flow indeed may blow up in finite time. Freire [14] showed that the solution obtained in [30] is unique even among energy class weak solutions with non-increasing energy. Surprisingly, the latter condition is essential for uniqueness, as Topping's [33] examples of "backwards bubbling" show. Rupflin [24], finally, showed that this is the only mechanism through which uniqueness may be lost.

Rupflin-Topping [25] introduced Teichmüller harmonic map flow, where also the conformal structure of the domain is evolved in direction of the negative gradient of the Dirichlet energy, as a means to canonically deform an initial map u_0 to a connected sum of minimal immersions.

2.4. **Higher dimensions.** For closed M of dimension $m \geq 3$ and an arbitrary closed target N, with the help of the monotonicity formula and ε-regularity results obtained by this author in [31], Chen-Struwe [3] showed existence of a global, partially regular weak solution u of the harmonic map heat flow for any smooth initial map $u_0 \colon M \to N$. The solution obtained in [3] is smooth away from a closed singular set of locally finite m-dimensional Hausdorff measure (with respect to the parabolic metric).

In fact, the solution u is smooth for all time and unique unless for some first singular time $t_s > 0$ and some $p_s \in M$ in geodesic normal coordinates x around $p_s = 0$ with a constant $C > 0$ (possibly depending on $D(u_0)$) there holds

$$\liminf_{r \downarrow 0} \left(r^{2-m} \int_{B_{Cr}(0)} |\nabla u(t_s - r^2)|^2 dx \right) > 0, \tag{2.2}$$

in which case the monotonicity formula gives subconvergence

$$\bar{u}_k(x,t) := u(r_k x, t_s + r_k^2 t) \to \bar{u}_\infty \text{ weakly in } H^1_{loc}(\mathbb{R}^m \times]-\infty, 0[) \quad (k \to \infty) \tag{2.3}$$

for suitable $r_k \to 0$, where $\bar{u}_\infty$ is self-similar, that is, satisfies $2t\frac{d}{dt}\bar{u}_\infty + x \cdot \nabla \bar{u}_\infty \equiv 0$. If the blow-up is of type I, then $(r_k)_{k\in\mathbb{N}}$ may be chosen so that $\bar{u}_k \to \bar{u}_\infty$ smoothly locally, where $\bar{u}_\infty$ is non-constant, and either

$$\bar{u}_\infty(x,t) = \bar{v}_\infty(x/\sqrt{-t}), \text{ or } \bar{u}_\infty(x,t) = \bar{w}_\infty(x/|x|) \tag{2.4}$$

with a smooth quasi-harmonic map $\bar{v}_\infty \colon \mathbb{R}^m \to N$, or with a smooth harmonic map $\bar{w}_\infty \colon S^{m-1} \to N$. Otherwise, the blow-up is said to be of type II, and there exist sequences $(x_k, t_k) \to (0, t_s)$, $r_k \to 0$ such that

$$\bar{u}_k(x,t) = u(x_k + r_k x, t_k + r_k^2 t) \to \tilde{u}_\infty \text{ smoothly locally in } \mathbb{R}^m \times \mathbb{R} \tag{2.5}$$

as $k \to \infty$, with a smooth, non-constant "eternal" solution $\tilde{u}_\infty$ of (2.1).

Coron-Ghidaglia [5] and Chen-Ding [2] gave examples showing that the heat flow may blow up in finite time. (We revisit the Chen-Ding [2] result in more detail in the next subsection.) Also for any finite energy data $u_0 \in H^1(M;N)$, the result of Chen-Struwe [3] yields a global, partially regular weak solution u of the harmonic map heat flow, which however, as shown by Coron [4], may fail to be unique among partially regular weak solutions with finite energy. It is an open question whether the monotonicity formula or Feldman's [13] notion of stationary flows can give uniqueness; see Germain-Ghoul-Miura [15] for a recent study of these questions in the equivariant (co-rotational) setting and further references. The book by Lin-Wang [20] gives a comprehensive account of harmonic maps and their heat flows with a more thorough discussion of these issues.

2.5. **Harmonic maps of spheres.** We now focus on the case $M = S^m$, $m \geq 3$. Smith [29] and Eells-Ratto [8], [9] studied symmetric or equivariant harmonic maps from S^m to a sphere or to a manifold of revolution. For general targets, however, very little is known. In particular, it is not possible to obtain existence of harmonic maps via the direct method, since for any closed N the infimum of Dirichlet energy in any homotopy class of maps $S^m \to N$ is zero, as observed first by Eells-Sampson [10], p. 130 f.; White [35] later obtained a more general version of this result.

Chen-Ding [2] used the latter observation, together with the monotonicity formula and ε-regularity results from [31], [3] to produce examples of topologically non-trivial smooth maps $u_0 \colon S^m \to S^m$, $m \geq 3$ for which the harmonic map heat flow blows up in finite time. In [32], Theorem 6, the Chen-Ding argument was simplified and the estimate $T^{\frac{m-2}{2}} \le CD(u_0)$ for the blow-up time was obtained. In particular, since for any smooth $u\colon S^m \to S^m$ we have $\inf_{\gamma\in\Gamma} D(u\circ\gamma) = 0$, given any topologically non-trivial smooth map $u_0\colon S^m \to S^m$ and any $T > 0$ there exists $\gamma \in \Gamma$ such that the heat flow with initial data $u_0 \circ \gamma$ will blow up before time T. It is therefore impossible to smoothly deform the sweep-out $(u_0 \circ \gamma)_{\gamma\in\Gamma}$ via the heat flow even for short time.

Moreover, from El Soufi's results (or (3.5) below) it follows that the identity map $\bar{u}\colon S^m \to S^m$ is unstable. Hence even for initial data u_0 close to $\bar{u}$ with probability one the standard heat flow will fail to converge to $\bar{u}$.

In the following we describe a flow that is designed to avoid these difficulties by acting as a gradient flow for the functional E, directly. Some prerequisites regarding the action of the Möbius group are needed.

3. Möbius group

We now also fix the dimension $m = 3$. However, results analogous to the ones derived below may be expected to hold for any $m \ge 3$.

3.1. **Stereographic coordinates.** Define stereographic projection $\pi\colon S^3 \subset \mathbb{R}^4 \to \mathbb{R}^3$ by

$$\pi(x) = \frac{x'}{1+x^4},\ x = (x', x^4) \in S^3 \subset \mathbb{R}^4,$$

with inverse $\psi\colon \mathbb{R}^3 \to S^3$ given by

$$\psi(x) = \frac{(2x, 1-|x|^2)}{1+|x|^2},\ x \in \mathbb{R}^3.$$

Computing

$$\psi^* g_{S^3} = \Big(\frac{2}{1+|x|^2}\Big)^2 g_{\mathbb{R}^3},$$

for any $u \in C^1(S^3; N)$ with $\hat{u} = u \circ \psi\colon \mathbb{R}^3 \to N$ we then have

$$\begin{aligned} D(u) &= \frac{1}{2}\int_{S^3} |\nabla_{g_{S^3}} u|^2 d\mu_{g_{S^3}} \\ &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla_{\psi^* g_{S^3}} \hat{u}|^2 d\mu_{\Psi^* g_{S^3}} = \int_{\mathbb{R}^3} \frac{1}{1+|x|^2}|\nabla \hat{u}|^2 dx. \end{aligned}$$

3.2. **Scaling.** For any $s > 0$, letting

$$\delta_s(x) = s^{-1}x,\ x \in \mathbb{R}^3,$$

and setting

$$\gamma_s = \psi \circ \delta_s \circ \pi\colon S^3 \to S^3,$$

we introduce the scaled map

$$u_s = u \circ \gamma_s = u \circ \psi \circ \delta_s \circ \pi\colon S^3 \to N,$$

represented by

$$\hat{u}_s = \hat{u} \circ \delta_s\colon \mathbb{R}^3 \to N,$$

with

$$\begin{aligned} D(u_s) &= \int_{\mathbb{R}^3} \frac{1}{1+|x|^2}|\nabla(\hat{u}\circ\delta_s)|^2 dx \\ &= \int_{\mathbb{R}^3} \frac{s^{-2}}{1+|x|^2}(|\nabla\hat{u}|^2 \circ \delta_s) dx = \int_{\mathbb{R}^3} \frac{s}{1+s^2|x|^2}|\nabla\hat{u}|^2 dx. \end{aligned}$$

Note that for $u \in C^1(S^3;N)$ also $\hat{u} \in C^1(\mathbb{R}^3;N)$ and the above expression immediately gives $D(u_s) \to 0$ as $s \to \infty$. In particular, for any $u \in C^1(S^3;N)$ the number $\sup_{s>0} D(u_s)$ is achieved and hence is finite.

3.3. **Möbius group and energy.** Given any $a \in S^3$ there is $R \in SO(4)$ with $Ra = -e_4$. For any $s > 0$ then let $\gamma_{a,s} = R^{-1} \circ \gamma_s \circ R$ and set

$$\Gamma = \{\gamma_{a,s};\ a \in S^3,\ 0 < s < \infty\} .$$

By slight abuse of terminology, we call Γ the Möbius group of S^3. For $u \in C^1(S^3;N)$ also let

$$E(u) = \sup_{\gamma \in \Gamma} D(u \circ \gamma) \quad \sup_{a \in S^3,\, 0<s<\infty} D(u \circ \gamma_{a,s})$$

be the energy of u, which is finite by the above. Since the set Γ forms a group, by this definition there holds $E(u) = E(u \circ \gamma_{a,s})$ for any $a \in S^3$ and any $s > 0$. In the following computations we usually fix $a = -e_4$ for simplicity, so that $\gamma_{a,s} = \gamma_s$ as defined above.

3.4. **Center of mass.** Let

$$\beta(s) = \frac{s}{1+s^2|x|^2} \quad \text{with} \quad \partial_s\big|_{s=1}\beta(s) = \frac{1-|x|^2}{(1+|x|^2)^2}.$$

For $u \in C^1(S^3;N)$ (or $u \in H^1(S^3;N)$) also define

$$\begin{aligned} X(u) &= \frac{1}{2}\int_{S^3} x|\nabla_{g_{S^3}} u|^2 d\mu_{g_{S^3}} \\ &= \frac{1}{2}\int_{\mathbb{R}^3} \psi(x)|\nabla_{\psi^* g_{S^3}} \hat{u}|^2 d\mu_{\psi^* g_{S^3}} = \int_{\mathbb{R}^3} \frac{(2x, 1-|x|^2)}{(1+|x|^2)^2}|\nabla \hat{u}|^2 dx \end{aligned}$$

the center of mass (up to the scale factor $D(u)$). Recalling the expression

$$D(u_s) = \int_{\mathbb{R}^3} \frac{s}{1+s^2|x|^2}|\nabla \hat{u}|^2 dx,$$

if we assume that $u \in C^2(S^3;N)$ we then have

$$X(u) \cdot e_4 = \partial_s\big|_{s=1} D(u_s) = -\int_{S^3} \tau(u) du \cdot \partial_s\big|_{s=1}\gamma_s \, d\mu_{g_{S^3}}. \tag{3.1}$$

Likewise for any $1 \le i \le 4$ we find

$$X(u) \cdot e_i = \partial_s\big|_{s=1} D(u \circ \gamma_{e_i,s}) = -\int_{S^3} \tau(u) du \cdot \partial_s\big|_{s=1}\gamma_{e_i,s} \, d\mu_{g_{S^3}}.$$

In particular, for $u \in C^2(S^3;N)$ to be harmonic, necessarily $X(u) = 0$.

Proposition 3.1. *For any $u \in C^2(S^3;N)$ the center of mass $X(u)$ is the unique vector $X(u) \in \mathbb{R}^4$ such that*

$$\forall \sigma \in \mathbb{R}^4 \colon X(u) \cdot \sigma = \int_{S^3} \tau(u)\xi \cdot du \, d\mu_{g_{S^3}} = -dD(u)(\xi \cdot du), \tag{3.2}$$

where for any $\sigma = (\sigma^i)_{1 \le i \le 4} \in \mathbb{R}^4$ we let $\xi = \Xi(\sigma) := -\sum_{i=1}^4 \sigma^i \frac{d}{ds}\big|_{s=1}\gamma_{e_i,s}$. Moreover, we have $X(u) = 0$ whenever u is harmonic.

3.5. **First and second variation in s.** Let $u \in C^2(S^3;N)$ (or $u \in H^2(S^3;N)$). Transforming, we obtain the expression

$$\tau_{\psi^* g_{S^3}}(\hat{u}) = \tau(u)\circ\psi = \Big(\frac{1+|x|^2}{2}\Big)^3 d\pi(u) div\Big(\frac{2}{1+|x|^2}\nabla\hat{u}\Big).$$

for the tension field in stereographic coordinates. Also note that there holds

$$(du\cdot\partial_s\big|_{s=1}\gamma_s)\circ\psi = \partial_s\big|_{s=1}\hat{u}_s = -d\hat{u}\cdot x.$$

Fix $u \in C^2(S^3;N)$ with $X(u) = 0$. Set $f(s) = D(u_s)$ with $f' = df/ds$. Similar to El Soufi [12] we then compute

$$f'(s_0) = \frac{d}{ds}\Big|_{s=s_0} D(u_{s_0}\circ\gamma_{s/s_0}) = \frac{d}{ds}\Big|_{s=s_0}\int_{\mathbb{R}^3}\frac{1}{1+|x|^2}|\nabla\hat{u}_{s_0}\circ\delta_{s/s_0}|^2 dx$$
$$= -2\int_{\mathbb{R}^3}\frac{\nabla\hat{u}_{s_0}\nabla(d\hat{u}_{s_0}\cdot x/s_0)}{1+|x|^2}dx = -2\int_{\mathbb{R}^3}\frac{\Big(\nabla\hat{u}\nabla(d\hat{u}\cdot x)\Big)\circ\delta_{s_0}}{s_0^3(1+|x|^2)}dx.$$

Substituting $y = x/s_0$, upon integrating by parts we obtain

$$\int_{\mathbb{R}^3}\frac{\Big(\nabla\hat{u}\nabla(d\hat{u}\cdot x)\Big)\circ\delta_{s_0}}{s_0^3(1+|x|^2)}dx = \int_{\mathbb{R}^3}\frac{\nabla\hat{u}\nabla(d\hat{u}\cdot x)}{1+s_0^2|x|^2}dx = \int_{\mathbb{R}^3}\frac{1+|x|^2}{1+s_0^2|x|^2}\frac{\nabla\hat{u}\nabla(d\hat{u}\cdot x)}{1+|x|^2}dx$$
$$= -\int_{\mathbb{R}^3}\frac{1+|x|^2}{1+s_0^2|x|^2}div\Big(\frac{\nabla\hat{u}}{1+|x|^2}\Big)(d\hat{u}\cdot x)dx + \int_{\mathbb{R}^3}\frac{2(s_0^2-1)|d\hat{u}\cdot x|^2}{(1+|x|^2)(1+s_0^2|x|^2)^2}dx$$

so that

$$\begin{aligned} f'(s_0) = &\int_{\mathbb{R}^3}\frac{1+|x|^2}{1+s_0^2|x|^2}\tau_{\Psi^*g_{S^3}}(\hat{u})(d\hat{u}\cdot x)d\mu_{\Psi^*g_{S^3}} \\ &- \int_{\mathbb{R}^3}\frac{(s_0^2-1)(1+|x|^2)^2}{2(1+s_0^2|x|^2)^2}|d\hat{u}\cdot x|^2 d\mu_{\Psi^*g_{S^3}}. \end{aligned} \tag{3.3}$$

From our assumption that $X(u) = 0$ and (3.2) in addition we find the equation

$$0 = f'(1) = \int_{\mathbb{R}^3}\tau_{\Psi^*g_{S^3}}(\hat{u})(d\hat{u}\cdot x)d\mu_{\Psi^*g_{S^3}},$$

and with $\frac{1+|x|^2}{1+s^2|x|^2} - 1 = \frac{(1-s^2)|x|^2}{1+s^2|x|^2}$ for any $s > 0$ there results

$$\begin{aligned} f'(s) = &\int_{\mathbb{R}^3}\frac{(1-s^2)|x|^2}{1+s^2|x|^2}\tau_{\Psi^*g_{S^3}}(\hat{u})(d\hat{u}\cdot x)d\mu_{\Psi^*g_{S^3}} \\ &- \int_{\mathbb{R}^3}\frac{(s^2-1)(1+|x|^2)^2}{2(1+s^2|x|^2)^2}|d\hat{u}\cdot x|^2 d\mu_{\Psi^*g_{S^3}}. \end{aligned} \tag{3.4}$$

Thus, at $s = 1$ we find

$$f''(1) = -\int_{\mathbb{R}^3}\Big(\frac{2|x|^2}{1+|x|^2}\tau_{\Psi^*g_{S^3}}(\hat{u})(d\hat{u}\cdot x) + |d\hat{u}\cdot x|^2\Big)d\mu_{\Psi^*g_{S^3}}. \tag{3.5}$$

Moreover, using Young's inequality to bound

$$\frac{|x|^2}{1+s^2|x|^2}|\tau_{\Psi^*g_{S^3}}(\hat{u})d\hat{u}\cdot x| \le \frac{(1+|x|^2)^2}{4(1+s^2|x|^2)^2}|d\hat{u}\cdot x|^2 + \frac{|x|^4}{(1+|x|^2)^2}|\tau_{\Psi^*g_{S^3}}|^2,$$

for any $0 < s < 1$ we obtain

$$\frac{(1-s)f'(s)}{1+s} \ge \frac{(1-s)^2}{4}\int_{\mathbb{R}^3}\big(|d\hat{u}\cdot x|^2 - 4|\tau_{\Psi^*g_{S^3}}|^2\big)d\mu_{\Psi^*g_{S^3}}. \tag{3.6}$$

In the case $s > 1$ a similar result follows from observing that $\gamma_s = \gamma_{-e_4,1/s}$.

3.6. **First conclusions.** From (3.4), (3.6) we immediately deduce El Soufi's result.

Theorem 3.2 ([12], Theorem 1.2). *Suppose that $u \in C^2(S^3;N)$ is non-constant and harmonic. Then*

$$\forall a \in S^3,\ s > 0\colon D(u) \geq D(u \circ \gamma_{a,s}),$$

with strict inequality unless $s = 1$.

In fact, from equations (3.4) and (3.6) we obtain an even stronger result. First we note the following definition.

Definition 3.3. A map $u \in H^1(S^3;N)$ is δ_0-uniformly 3-dimensional if

$$\|du \cdot \frac{d}{ds}\big|_{s=1}\gamma_{a,s}\|_{L^2(S^3,d\mu_{g_{S^3}})} = \|d\hat{u} \cdot x\|_{L^2(\mathbb{R}^3,d\mu_{\psi^* g_{S^3}})} \geq \delta_0$$

for all $a \in S^3$, where $\hat{u} = u \circ R \circ \psi$ for suitable $R = R(a) \in SO(4)$.

Remark 3.4. Note that when $u \in H^1(S^3;N)$ is δ_0-uniformly 3-dimensional then for all $\sigma = (\sigma^i)_{1\leq i\leq 4} \in \mathbb{R}^4$ with $C = 1/\delta_0$ we have

$$|\sigma| \leq C|\sigma| \inf_{a\in S^3} \|du \cdot \frac{d}{ds}\big|_{s=1}\gamma_{a,s}\|_{L^2(S^3,d\mu_{g_{S^3}})} \leq C\|du \cdot \xi\|_{L^2(S^3,d\mu_{g_{S^3}})},$$

where $\xi = \Xi(\sigma) = -\sigma^i \frac{d}{ds}\big|_{s=1}\gamma_{e_i,s}$ as above.

We then have the following extension of Theorem 3.2.

Theorem 3.5. *Suppose $u \in H^2(S^3;N)$ is δ_0-uniformly 3-dimensional for some $\delta_0 > 0$ and satisfies $2\|\tau(u)\|_{L^2} < \delta_0$. Also assume that $X(u) = 0$. Then*

$$\forall a \in S^3,\ s > 0\colon D(u) \geq D(u \circ \gamma_{a,s}),$$

with strict inequality unless $s = 1$. In particular, there holds $D(u) = E(u)$.

Proof. After a rotation of S^3 we may assume that $a = e_4$ and that $0 < s < 1$. From (3.6) then we have

$$\frac{4}{1-s^2}\frac{d}{ds}D(u_s) \geq \int_{\mathbb{R}^3} \big(|d\hat{u}\cdot x|^2 - 4|\tau_{\Psi^* g_{S^3}}|^2\big)d\mu_{\Psi^* g_{S^3}} > 0.$$

The claim follows. □

4. Normalized harmonic map flow

4.1. **Flow equation.** Fix some $0 < \alpha < 1$ and for any $T > 0$ let

$$C^{2+\alpha,1+\alpha/2}(S^3 \times [0,T];\mathbb{R}^n) = \{v \in C^1(S^3 \times [0,T];\mathbb{R}^n);\ v_t, \nabla^2 v \in C^{\alpha,\alpha/2}\},$$

where $C^{\alpha,\alpha/2}(S^3 \times [0,T])$ is the space of functions which are Hölder continuous in the parabolic metric on $S^3 \times [0,T]$ with Hölder exponent α.

Let $u_0 \in C^{2+\alpha}(S^3;N)$ satisfy $E(u_0) = D(u_0)$ and thus $X(u_0) = 0$.

Definition 4.1. A family $u = u(t) \in C^{2+\alpha,1+\alpha/2}(S^3 \times [0,T];N)$ with $u(0) = u_0$ is a solution to the *normalized harmonic map flow* on $[0,T[$ with Cauchy data u_0 if for suitable $\sigma = (\sigma^i(t))_{1\leq i\leq 4}$ with $\xi = \Xi(\sigma) = -\sum_{i=1}^4 \sigma^i \frac{d}{ds}\big|_{s=1}\gamma_{e_i,s}$ there holds

$$u_t = \tau(u) + du \cdot \xi, \tag{4.1}$$

together with

$$\frac{dX(u)}{dt} = 0 \quad \text{for all } 0 < t < T. \tag{4.2}$$

In view of (4.2) and our assumption $X(u_0) = 0$, for a solution to the normalized harmonic map flow on $[0, T[$ the necessary condition

$$X(u(t)) = 0 \tag{4.3}$$

for harmonic maps is preserved for all t. Moreover, as long as $u(t)$ is δ-uniformly 3-dimensional for some $\delta > 0$ and satisfies $2\|\tau(u(t))\|_{L^2} < \delta$ by Theorem 3.5 we have $D(u(t)) = E(u(t))$.

In fact, for such u_0 equations (4.1), (4.2) define the L^2-gradient flow for E.

4.2. **Energy identity.** Suppose u_0 with $X(u_0) = 0$ is δ_0-uniformly 3-dimensional for some $\delta_0 > 0$ and $2\|\tau(u_0)\|_{L^2} < \delta_0$, and let $u = u(t)$ solve (4.1), (4.2) for $0 \le t \le T_0$. For suitable $0 < T \le T_0$, $0 < \delta \le \delta_0$ we then may suppose that $u(t)$ is δ-uniformly 3-dimensional for some $\delta > 0$ and satisfies $2\|\tau(u(t))\|_{L^2} < \delta$ for all $t \in [0, T]$.

Computing the time derivative of the Dirichlet energy, in view of (4.1) and (3.2) we find

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} D(u(t)) &= -\int_{S^3} \tau(u) u_t \, d\mu_{g_{S^3}} = -\int_{S^3} \tau(u)(\tau(u) + du \cdot \xi) \, d\mu_{g_{S^3}} \\ &= -\int_{S^3} |\tau(u)|^2 \, d\mu_{g_{S^3}} - \sigma \cdot X(u) = -\int_{S^3} |\tau(u)|^2 \, d\mu_{g_{S^3}}, \end{aligned} \tag{4.4}$$

and (4.3) together with Theorem 3.5 guarantee that

$$D(u(t)) = E(u(t)) \quad \text{for all } 0 \le t \le T.$$

4.3. **Local existence.** For local existence of classical solutions to (4.1),(4.2) we require the data u_0 to be δ_0-uniformly 3-dimensional for some $\delta_0 > 0$.

Moreover, we need the following condition to hold.

Definition 4.2. A map $u \in C^{2+\alpha}(S^3; N)$ has *Möbius non-degenerate* second variation $d^2D(u)$, if the quadratic form

$$Q_u(\sigma, \rho) = d^2D(u)(\xi \cdot du, \eta \cdot du), \text{ with } \xi = \Xi(\sigma), \eta = \Xi(\rho) \text{ for } \sigma, \rho \in \mathbb{R}^4,$$

is non-degenerate.

We establish local existence for the flow (4.1), (4.2) for uniformly 3-dimensional data with Möbius non-degenerate second variation and small tension field.

Theorem 4.3. *Assume $u_0 \in C^{2+\alpha}(S^3; N)$ with $X(u_0) = 0$ has Möbius non-degenerate second variation $d^2D(u_0)$ and is δ_0-uniformly 3-dimensional for some $\delta_0 > 0$ with $(2 + C_0)\|\tau(u_0)\|_{L^2} < \delta_0$ for some $C_0 > 0$ depending on $d^2D(u_0)$. Then there is $T > 0$ and a unique smooth solution $u = u(t) \in C^{2+\alpha,1+\alpha/2}(S^3 \times [0, T]; N)$ of (4.1), (4.2) for suitable $\sigma = (\sigma^i(t))$ with $u(0) = u_0$, and $D(u(t)) = E(u(t))$ is non-increasing in time.*

Proof. For sufficiently small $T > 0$ and $R > 0$ to be determined let

$$V = C^{2+\alpha,1+\alpha/2}(S^3 \times [0, T]; \mathbb{R}^n)$$

and set

$$W = \{v \in V;\ v(0) = u_0,\ \|v - u_0\|_{C^{2+\alpha,1+\alpha/2}} \le R\},$$

where we extend u_0 constant in time. We may assume that $R > 0$ is so small that $\pi_N(v) = \pi_N \circ v \in C^{2+\alpha,1+\alpha/2}(S^3 \times [0,T]; N)$ is well-defined and satisfies $\|\pi_N(v) - v\|_{C^{2+\alpha,1+\alpha/2}} \le R$ for any $v \in W$.

For any $v \in W$ set $w = \pi_N(v)$ and let $\sigma = (\sigma^i) \in \mathbb{R}^4$ with corresponding $\xi = \Xi(\sigma)$ as in Proposition 3.1 be defined such that for any $\rho \in \mathbb{R}^4$ with corresponding $\eta = \Xi(\rho)$ we have

$$d^2D(w)(\tau(w) + \xi \cdot dw, \eta \cdot dw) = \int_{S^3} \tau(w)\, \eta \cdot d(\tau(w) + \xi \cdot dw)\; d\mu_{g_{S^3}}. \tag{4.5}$$

Note that this definition only requires $\tau(w) \in L^2(S^3)$. Indeed, similar to (3.1), for $f \in L^2(S^3)$ with $\hat{f} = f \circ \psi$, for $\rho = (0, \dots, 0, 1)$ we have

$$\begin{aligned} 2\int_{S^3} f\, \eta \cdot df\; d\mu_{g_{S^3}} &= \partial_s\big|_{s=1}\Big(\int_{\mathbb{R}^3} \big(\frac{2}{1+|x|^2}\big)^3 |\hat{f} \circ \delta_s|^2\, dx\Big) \\ &= 3\int_{\mathbb{R}^3} \psi^4(x)\big(\frac{2}{1+|x|^2}\big)^3 |\hat{f}|^2\, dx = 3\int_{S^3} p^4 |f(p)|^2\, d\mu_{g_{S^3}}(p), \end{aligned}$$

where $p = (p^i) \in S^3 \subset \mathbb{R}^4$, and similarly for arbitrary ρ.

Since $d^2D(u_0)$ by assumption is Möbius non-degenerate, there exist $C_0 > 0$, $R > 0$ such that whenever $C_0\|\tau(u_0)\|_{L^2} < \delta_0$ the equation (4.5) uniquely determines $\sigma = \sigma(v) \in C^{\alpha/2}([0,T]; \mathbb{R}^4)$ for any $v \in W$, and $\xi = \Xi(\sigma) \in C^{\alpha,\alpha/2}$.

In view of the Campanato estimates for the heat equation, the map $\Phi\colon W \to V$ is well-defined, where for any $v \in W$ the function $u = \Phi(v)$ solves the equation

$$u_t - \Delta_{g_{S^3}} u = A(\pi_N(v))(\nabla \pi_N(v), \nabla \pi_N(v)) + \xi \cdot d(\pi_N(v)) \tag{4.6}$$

Moreover, for sufficiently small $R > 0$, $T = T(R) > 0$ there holds $\Phi\colon W \to W$ and Φ is contracting. Indeed, the first term on the right of (4.6) only depends on $w = \pi_N(v)$ and ∇w, and for $v_1, v_2 \in W$ there holds

$$\|w_1 - w_2\|_{C^{\alpha,\alpha/2}} + \|\nabla w_1 - \nabla w_2\|_{C^{\alpha,\alpha/2}} \le CT^{\frac{1-\alpha}{2}} \|v_1 - v_2\|_{C^{2+\alpha,1+\alpha/2}}$$

Likewise, for any $\varepsilon > 0$, any $t > 0$ by Ehrling's lemma we have

$$\begin{aligned} \|\sigma(v_1) - \sigma(v_2)\|_{L^\infty} &\le C\|w_1 - w_2\|_{H^2} \le C\|w_1 - w_2\|_{C^2} \le C\|v_1 - v_2\|_{C^2} \\ &\le \varepsilon\|v_1 - v_2\|_{C^{2+\alpha}} + C(\varepsilon)\|v_1 - v_2\|_{C^{1+\alpha}}. \end{aligned}$$

Since $v_1 = v_2 = u_0$ at $t = 0$ for any $\varepsilon > 0$ for sufficiently small $T > 0$ there holds

$$\|v_1 - v_2\|_{C^{1+\alpha}} \le \varepsilon\|v_1 - v_2\|_{C^{2+\alpha,1+\alpha/2}}.$$

Thus, for any $\varepsilon > 0$ for sufficiently small $T > 0$ we have

$$\|\xi(v_1) - \xi(v_2)\|_{C^{\alpha,\alpha/2}} \le \varepsilon\|v_1 - v_2\|_{C^{2+\alpha,1+\alpha/2}}.$$

Since W is complete there exists a unique fixed point $u = \Phi(u) \in W$, and $u(x,t) \in N$ for all (x,t), as the following lemma shows. By (3.2) and (4.5) then

$$\begin{aligned} \frac{d}{dt}(X(u) \cdot \rho) &= \int_{S^3} \tau(u)\, \eta \cdot du_t\; d\mu_{g_{S^3}} - d^2D(u)(u_t, \eta \cdot du) \\ &= \int_{S^3} \tau(u)\, \eta \cdot d(\tau(u) + \xi \cdot du)\; d\mu_{g_{S^3}} - d^2D(u)(\tau(u) + \xi \cdot du, \eta \cdot du) = 0 \end{aligned}$$

for all $t \ge 0$, and u solves (4.1), (4.2). □

Lemma 4.4. *For $u = \Phi(u)$ there holds $u(x,t) \in N$ for all $(x,t) \in S^3 \times [0,T]$.*

Proof. For ease of notation only, assume that $N \subset \mathbb{R}^4$ is a smooth oriented hypersurface with a smooth unit normal vector field $\nu\colon N \to \mathbb{R}^4$ such that $\nu(p) \perp T_pN$ for every $p \in N$. (For the general case, we may take a smooth local orthonormal frame $\nu_1, \dots, \nu_k$ for the normal bundle $T^\perp N$.) Then we have

$$u - \pi_N(u) = \nu(u)\langle \nu(u), u - \pi_N(u)\rangle_{\mathbb{R}^n}, \tag{4.7}$$

where we write $\nu(u) = \nu(\pi_N(u))$ for brevity, and there holds

$$A(v)(\nabla v, \nabla v) = \nu(v)\langle d\nu(v)\nabla v, \nabla v\rangle_{\mathbb{R}^n}$$

for $v = \pi_N(u) \in C^1(S^3; N)$.

By orthogonality $\langle d(\pi_N(u)), \nu(u)\rangle_{\mathbb{R}^n} \equiv 0$, the function

$$f = f(x,t) = |u - \pi_N(u)|^2/2$$

then satisfies

$$\frac{d}{dt}f = \langle u_t - d\pi_N(u)u_t, u - \pi_N(u)\rangle_{\mathbb{R}^n} = \langle u_t, \nu(u)\rangle_{\mathbb{R}^n}\langle \nu(u), u - \pi_N(u)\rangle_{\mathbb{R}^n}.$$

Writing $\Delta f = \Delta_{g_{S^3}} f$ and so on, we also have

$$\Delta f = \langle u - \pi_N(u), \nu(u)\rangle_{\mathbb{R}^n}\Delta\langle \nu(u), u - \pi_N(u)\rangle_{\mathbb{R}^n} + |\nabla\langle \nu(u), u - \pi_N(u)\rangle_{\mathbb{R}^n}|^2.$$

Observing that $\langle u - \pi_N(u), \nabla(\nu(u))\rangle_{\mathbb{R}^n} \equiv 0$ gives

$$\begin{aligned}\Delta\langle u - \pi_N(u), \nu(u)\rangle_{\mathbb{R}^n} &= div\langle \nabla(u - \pi_N(u)), \nu(u)\rangle_{\mathbb{R}^n}\\ &= div\langle \nabla u, \nu(u)\rangle_{\mathbb{R}^n} = \langle \Delta u, \nu(u)\rangle_{\mathbb{R}^n} + \langle \nabla u, d\nu(u)\nabla(\pi_N(u))\rangle_{\mathbb{R}^n},\end{aligned}$$

we then find

$$\begin{aligned}\frac{d}{dt}f - \Delta f \le{}& \langle u_t - \Delta u, \nu(u)\rangle_{\mathbb{R}^n}\langle \nu(u), u - \pi_N(u)\rangle_{\mathbb{R}^n}\\ &- \langle \nabla u, d\nu(u)\nabla(\pi_N(u))\rangle_{\mathbb{R}^n})\langle \nu(u), u - \pi_N(u)\rangle_{\mathbb{R}^n}.\end{aligned}$$

Letting $\xi = \Xi(\sigma)$ and recalling that $\langle (d\pi_N(u)) \cdot \xi, \nu(u)\rangle_{\mathbb{R}^n} = 0$, since u solves equation (4.6) with $v = u$ we then find

$$\begin{aligned}\langle u_t - \Delta u, \nu(u)\rangle_{\mathbb{R}^n} &= \langle A(\pi_N(u))(\nabla(\pi_N(u)), \nabla(\pi_N(u))), \nu(u)\rangle_{\mathbb{R}^n}\\ &= \langle \nabla\pi_N(u), d\nu(u)\nabla(\pi_N(u))\rangle_{\mathbb{R}^n}\end{aligned}$$

and thus

$$\frac{d}{dt}f - \Delta f \le -\langle \nabla(u - \pi_N(u)), d\nu(u)\nabla(\pi_N(u))\rangle_{\mathbb{R}^n}\langle \nu(u), u - \pi_N(u)\rangle_{\mathbb{R}^n}.$$

But recalling that $\langle \nu(u), d\nu(u)\rangle_{\mathbb{R}^n} \equiv 0$, $\nu(u) = \nu(\pi_N(u))$ from (4.7) we have

$$\begin{aligned}&\langle \nabla(u - \pi_N(u)), d\nu(u)\nabla(\pi_N(u))\rangle_{\mathbb{R}^n}\\ &\quad= \langle \nabla(\langle u - \pi_N(u)), \nu(u)\rangle_{\mathbb{R}^n}\nu(u)), d\nu(u)\nabla(\pi_N(u))\rangle_{\mathbb{R}^n}\\ &\quad= \langle u - \pi_N(u), \nu(u)\rangle_{\mathbb{R}^n}|\nabla\nu(\pi_N(u)))|^2,\end{aligned}$$

and

$$\frac{d}{dt}f - \Delta f \le -|\nabla\nu(u))|^2|\langle \nu(u), u - \pi_N(u)\rangle_{\mathbb{R}^n}|^2 \le 0.$$

Since $f\big|_{t=0} = 0$, from the maximum principle we then conclude that $f \equiv 0$, as claimed. $\square$

4.4. **Test case.** By (4.4), if $u(t) \equiv u_0 \in C^{2+\alpha}(S^3;N)$ with $E(u_0) = D(u_0)$ is a time-independent solution of the flow (4.1), (4.2), satisfying the hypotheses of Theorem 4.3, then u_0 is harmonic and $\sigma \equiv 0$. Conversely, If $u_0 \in C^{2+\alpha}(S^3;N)$ is harmonic, clearly the constant family $u(t) \equiv u_0$ with $\sigma \equiv 0$ is a global, smooth solution to (4.1), (4.2).

In particular, the identity map $u_0 = id\colon S^3 \to S^3$ thus is a global, smooth solution to (4.1), (4.2). As a confirmation for the ideas that we propose we now show that this solution – while it is unstable under the standard harmonic map heat flow – is stable under the flow (4.1), (4.2).

Theorem 4.5. *Fix a number $0 < \alpha < 1$. There exists $\mu > 0$ such that for any $u_0 \in C^{2+\alpha}(S^3;S^3)$ with $\|u_0 - id\|_{C^2} < \mu$ and satisfying $X(u_0) = 0$ there is a unique, global, smooth solution $u = u(t) \in C^{2+\alpha,1+\alpha/2}(S^3 \times [0,\infty[;S^3)$ of the normalized harmonic map flow (4.1), (4.2) with data $u(0) = u_0$ for suitable $\sigma = (\sigma^i(t))$, and $u(t)$ smoothly converges to $\bar{u} = id$ as $t \to \infty$.*

In fact a stronger result holds.

Theorem 4.6. *Assume $N \subset \mathbb{R}^4$ is smoothly diffeomorphic to S^3 via a diffeomorphism $\phi_N\colon \mathbb{R}^4 \to \mathbb{R}^4$, where $\|\phi_N - id\|_{C^{2+\alpha}} < \nu$ for some $0 < \alpha < 1$, and let $u_0 \in C^{2+\alpha}(S^3;N)$ with $\|\phi_N \circ u_0 - \bar{u}\|_{C^{2+\alpha}} < \nu$, where $\bar{u} = id$ on S^3. Also suppose that $X(u_0) = 0$. Then if $\nu > 0$ is sufficiently small, there is a unique smooth solution $u = u(t) \in C^{2+\alpha,1+\alpha/2}(S^3 \times [0,T];N)$ of (4.1), (4.2) for suitable $\sigma = (\sigma^i(t))$ with $u(0) = u_0$, and a smooth harmonic map $u_\infty\colon S^3 \to N$ such that $u(t)$ smoothly converges to u_∞ as $t \to \infty$.*

Proof of Theorem 4.5. By a result of Smith [28], Example 2.12, for any $m \geq 3$ the second variation $d^2D(\bar{u})$ of the Dirichlet energy for the identity map $\bar{u} = id$ of S^m has index $m+1$ for $m \geq 3$, and for $1 \leq i \leq m+1$ the gradient of the restriction of the coordinate function z^i in the ambient Euclidean space to S^m defines an eigenfunction v_i of the operator J related to the Hessian $d^2D(\bar{u})$ such that $d^2D(\bar{u})(v_i,v_i) < 0$; moreover, if we normalize our maps with respect to isometries, the nullity of $d^2D(\bar{u})$ vanishes and $d^2D(\bar{u})$ is positively definite in the orthogonal complement of the space spanned by $(v_i)_{1 \leq i \leq m+1}$.

Indeed, any isometry may be regarded as induced by a rotation of the domain sphere. More precisely, if now $m = 3$, for maps u which are C^2-close to $\bar{u}$, having fixed any point p_0 and frame $(Y_i)_{1 \leq i \leq 3}$ for $T_{p_0}S^3 \subset \mathbb{R}^4$ in the domain sphere and a point p_1 with accompanying frame $(Z_i)_{1 \leq i \leq 3}$ for $T_{p_1}S^3 \subset \mathbb{R}^4$ in the target sphere, we can find a rotation $R \in SO(4)$ such that $u \circ R(p_0) = p_1$ and such that $du(q_0) \circ R$ takes Y_1 to a positive multiple of Z_1 and maps Y_2 to a vector $Z \in span\{Z_1, Z_2\}$ such that the triple (Z_1, Z, Z_3) has the same orientation as the triple (Z_1, Z_2, Z_3), where $q_0 = R_0(p_0)$. By the implicit function theorem, the rotation R smoothly depends on u in a C^2-neighborhood of $\bar{u}$. In this way the 6-dimensional space of Jacobi fields induced by isometries of the sphere may be identified with the image of $T_{id}SO(4) = so(4)$ under du.

With our representation $\psi(x) = \frac{(2x,1-|x|^2)}{1+|x|^2}$ of $\bar{u} = id$, the gradient of the restriction to S^3 of the coordinate function z^4 in the ambient $\mathbb{R}^4$ equals the projection

$$e_4 - \psi^4(x)\psi(x) = \big(- \frac{1-|x|^2}{1+|x|^2} \frac{2x}{1+|x|^2}, 1 - (\psi^4(x))^2 \big)$$

of $e_4 = \nabla z^4 = (0,0,0,1)$ to $T_{\psi(x)}S^3$. But this precisely coincides with the representation

$$-d\psi(x)\cdot x = -\Big(\frac{1-|x|^2}{1+|x|^2}\frac{2x}{1+|x|^2}, \frac{-4|x|^2}{(1+|x|^2)^2}\Big)$$

of $d\bar{u}\cdot\partial_s\big|_{s=1}\gamma_s$; and similarly for the remaining coordinate directions. Thus, in particular, with a constant $\bar{\lambda} > 0$ we also have that

$$d^2D(\bar{u})(\xi\cdot d\bar{u}, \xi\cdot d\bar{u}) \le -2\bar{\lambda}|\sigma|^2,$$

where for $\sigma = (\sigma^i) \in \mathbb{R}^4$ we let $\xi = \Xi(\sigma) = -\sum_{i=1}^4 \sigma^i \frac{d}{ds}\big|_{s=1}\gamma_{e_i,s}$ as defined in Proposition 3.1. Hence $u = \bar{u}$ is Möbius non-degenerate.

Fix some $\rho \in \mathbb{R}^4$ with corresponding $\eta := \Xi(\rho)$. Then as in the proof of Theorem 4.3 with tacit summation over $1 \le i \le 4$ for $u \in C^{2+\alpha}(S^3;S^3)$ we have

$$\begin{aligned}\sigma^i\frac{d}{ds}\Big|_{s=1}(X(u\circ\gamma_{e_i,s})\cdot\rho) &= \sigma^i\frac{d}{ds}\Big|_{s=1}\Big(\int_{S^3}\tau(u\circ\gamma_{e_i,s}))\,\eta\cdot d(u\circ\gamma_{e_i,s})\,d\mu_{g_{S^3}}\Big)\\ &= \int_{S^3}\tau(u)\,\eta\cdot d(\xi\cdot du)\,d\mu_{g_{S^3}} - d^2D(u)(\xi\cdot du, \eta\cdot du).\end{aligned}$$

Thus, and since $\tau(\bar{u}) = 0$, by the implicit function theorem the condition $X(u) = 0$ defines a smooth submanifold W of $C^{2+\alpha}(S^3;S^3)$ in a neighborhood of $\bar{u}$.

For any $w \in T_{\bar{u}}W$ and any $\xi = \Xi(\sigma)$, moreover, since $\xi\cdot d\bar{u}$ is an eigenfunction of the Hessian operator, for some $\lambda > 0$ there holds the equation

$$\begin{aligned}0 &= \langle d\big(X(\bar{u})\cdot\sigma\big), w\rangle = -\langle d\big(dD(\bar{u})(\xi\cdot d\bar{u})\big), w\rangle\\ &= -d^2D(\bar{u})(w, \xi\cdot d\bar{u}) = \lambda(w,\xi\cdot d\bar{u})_{L^2},\end{aligned}$$

and the spaces $T_{\bar{u}}W$ and $V_{\bar{u}} = span\{\Xi(\sigma)\cdot d\bar{u};\ \sigma\in\mathbb{R}^4\}$ are both L^2-orthogonal and orthogonal for $d^2D(\bar{u})$.

Thus, for any given $\varepsilon > 0$, also for $u \in W$ with $\|u-\bar{u}\|_{C^2} < \nu$ for sufficiently small $\nu = \nu(\varepsilon) > 0$ the spaces T_uW and $V_u = span\{\Xi(\sigma)\cdot du;\ \sigma\in\mathbb{R}^4\}$ are approximately orthogonal in the sense that there holds

$$|(w,\xi\cdot du)_{L^2}| \le \varepsilon\|\xi\cdot du\|_{L^2}\|w\|_{L^2},\ |d^2D(u)(w,\xi\cdot du)| \le \varepsilon\|\xi\cdot du\|_{H^1}\|w\|_{H^1} \tag{4.8}$$

for all $w \in T_uW$ and $\xi = \Xi(\sigma)$. Smith's result then moreover yields that for all $u \in W$ such that $\|u-\bar{u}\|_{C^2} < \nu$ for sufficiently small $\nu > 0$ with a uniform constant $c_0 > 0$ there holds

$$d^2D(u)(w,w) \ge c_0\|w\|_{H^1} \text{ for every } w \in T_uW. \tag{4.9}$$

For $\varepsilon > 0$, $0 < \mu \ll \nu = \nu(\varepsilon)$ to be determined, let u, $\xi = \Xi(\sigma)$ be the solution of (4.1), (4.2) with data $u(0) = u_0 \in W$, where u_0 satisfies $\|u_0 - \bar{u}\|_{C^2} < \mu$. Then there is $T > 0$ such that $u(t) \in W$ and $\|u(t) - \bar{u}\|_{C^2} < \nu$ for $0 \le t \le T$. It follows that $u_t \in T_uW$, and from (4.1), (4.3), and (4.8), by estimating

$$\begin{aligned}0 = \sigma\cdot X(u) &= \int_M \tau(u)\,\xi\cdot du\,d\mu_{g_{S^3}} = \int_M u_t\xi\cdot du\,d\mu_{g_{S^3}} - \int_M |\xi\cdot du|^2\,d\mu_{g_{S^3}}\\ &\le \varepsilon\|\xi\cdot du\|_{L^2}\|u_t\|_{L^2} - \|\xi\cdot du\|^2_{L^2},\end{aligned}$$

from Remark 3.4 we find the bound

$$|\sigma| \le C\|\xi\cdot du\|_{L^2} \le C\varepsilon\|u_t\|_{L^2}. \tag{4.10}$$

Parabolic regularity theory now gives uniform smooth bounds for u and u_t for $0 \le t \le T$. Also note that (4.3) implies

$$\begin{aligned}\|u_t\|^2_{L^2} &= \|\tau(u)\|^2_{L^2} + \|\xi \cdot du\|^2_{L^2} + 2\int_M \tau(u)\,\xi \cdot du\, d\mu_{g_{S^3}} \\ &= \|\tau(u)\|^2_{L^2} + \|\xi \cdot du\|^2_{L^2} \le \|\tau(u)\|^2_{L^2} + C\varepsilon^2\|u_t\|^2_{L^2};\end{aligned}$$

in particular, for sufficiently small $\varepsilon > 0$ we obtain

$$\|\tau(u)\|^2_{L^2} \le \|u_t\|^2_{L^2} \le 2\|\tau(u)\|^2_{L^2}$$

from which it also follows that $T \to \infty$ as $\mu \downarrow 0$.

Noting that we can compute

$$\begin{aligned}\frac{d}{dt}\|\tau(u)\|^2_{L^2} &= \lim_{h\downarrow 0}\frac{\|\tau(u(t))\|^2_{L^2} - \|\tau(u(t-h))\|^2_{L^2}}{h} \\ &= \lim_{h\downarrow 0}\frac{\big(\tau(u(t)) - \tau(u(t-h)), \tau(u(t)) + \tau(u(t-h))\big)_{L^2}}{h} \\ &= \lim_{h\downarrow 0}\frac{\big(dD(u(t-h)) - dD(u(t))\big)\big(\tau(u(t)) + \tau(u(t-h))\big)}{h} \\ &= -2d^2D(u)(u_t, \tau(u)),\end{aligned}$$

from (4.1), (4.8), and (4.9) for sufficiently small $\varepsilon > 0$ we then conclude

$$\begin{aligned}\frac{d}{dt}\|\tau(u)\|^2_{L^2} &= -2d^2D(u)(u_t, u_t) + 2d^2D(u)(u_t, \xi \cdot du) \\ &\le -2c_0\|u_t\|^2_{H^1} + 2\varepsilon\|\xi \cdot du\|_{H^1}\|u_t\|_{H^1} \le -2c_0\|u_t\|^2_{H^1} + C\varepsilon|\sigma|\|u_t\|_{H^1} \\ &\le -c_0\|u_t\|^2_{H^1} \le -c_0\|u_t\|^2_{L^2} \le -c_0\|\tau(u)\|^2_{L^2},\end{aligned}$$

for some $c_0 > 0$ and all $0 \le t \le T$, and

$$|\sigma(t)|^2 \le \|u_t\|^2_{L^2} \le 2\|\tau(u(t))\|^2_{L^2} \le 2e^{-c_0 t}\|\tau(u_0)\|^2_{L^2}$$

for all such $t > 0$.

Interpolating we also have smooth exponential decay of u_t. For sufficiently small $0 < \mu \ll \nu$ and $\|u_0 - \bar{u}\|_{C^2} < \mu$ the bound $\|u(t) - \bar{u}\|_{C^2} < \nu$ then persists for all $t > 0$, and as $t \to \infty$ we have smooth convergence $u(t) \to u_\infty$, where u_∞ is harmonic. But by non-degeneracy of $d^2D(\bar{u})$, for sufficiently small $\nu > 0$ the map $\bar{u} = id$ is the unique harmonic map u with $\|u - \bar{u}\|_{C^2} < \nu$, and our claim follows. □

Proof of Theorem 4.6. The bounds in the preceding proof are stable with respect to perturbations, and for sufficiently small $0 < \mu \ll \nu$ for any diffeomorphism φ_N of $\mathbb{R}^4$, any u_0 with $\|\varphi_N - id\|_{C^{2+\alpha}} + \|\varphi_N \circ u_0 - \bar{u}\|_{C^{2+\alpha}} < \mu \ll \nu$ we obtain a global solution $u = u(t)$ to (4.1), (4.2) such that $\|\varphi_N \circ u(t) - \bar{u}\|_{C^2} < \nu$ for all $t > 0$, and $u(t) \to u_\infty$ smoothly as $t \to \infty$, where u_∞ is harmonic. □

In a certain sense the above results are best possible and we can only hope for global existence of solutions to the normalized harmonic map flow in a perturbative setting. Indeed, for certain geometries it can be shown that the normalized harmonic map heat flow fails to converge, as we shall see below.

4.5. **The co-rotational case.** If $N \subset \mathbb{R}^4$ is symmetric around the e_4-axis, and if u is co-rotational in the sense that

$$u \circ R = R \circ u \text{ for all } R \in SO(4) \text{ with } Re_4 = e_4,$$

we have $X(u) = X(u \circ R) = RX(u)$ for all $R \in SO(4)$ with $Re_4 = e_4$ and thus $X(u) = (0, X^4(u))$. For co-rotational initial data u_0 we then can determine a co-rotational solution $u = u(t)$ of (4.1), (4.2) by solving the equaiton

$$u_t = \tau(u) + \sigma du \cdot \frac{d}{ds}\Big|_{s=1}\gamma_s \tag{4.11}$$

together with

$$\frac{dX^4(u)}{dt} = 0 \quad \text{ for all } 0 < t < T, \tag{4.12}$$

to achieve (4.3) for all $0 \le t \le T$. In fact, by uniqueness, the solution u of (4.11), (4.12) will then be the unique solution of (4.1), (4.2).

Note that for co-rotational u the condition of Möbius non-degeneracy of $d^2D(u)$ in Theorem 4.3 may be replaced by the single requirement that $f''(1) \neq 0$, where $f(s) = D(u \circ \gamma_s)$. Since by (3.5) we have

$$\begin{aligned} f''(1) &= -\int_{\mathbb{R}^3} \frac{2|x|^2}{1+|x|^2}\tau_{\Psi^* g_{S^3}}(\hat{u})(d\hat{u} \cdot x)d\mu_{\Psi^* g_{S^3}} - \int_{\mathbb{R}^3} |d\hat{u} \cdot x|^2 d\mu_{\Psi^* g_{S^3}} \\ &\le \frac{1}{2}\int_{\mathbb{R}^3} \big(4|\tau_{\Psi^* g_{S^3}}|^2 - |d\hat{u} \cdot x|^2\big)d\mu_{\Psi^* g_{S^3}}, \end{aligned}$$

the latter condition in particular will be satisfied if u is δ_0-uniformly 3-dimensional with $2\|\tau(u)\|_{L^2} < \delta_0$.

4.6. **Blow-up.** Now if

$$N = E(b) = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R};\ b^2|x|^2 + |y|^2 = b^2\}$$

is an ellipsoid with half axes 1 and $b > 0$, by a result of Smith [29], Theorem 9.3, for sufficiently large $b > 1$ there is no co-rotational harmonic map $u\colon S^3 \to N$. Thus, for co-rotational data $u_0 = \Sigma id$, as defined by Smith, the normalized harmonic map heat flow either fails to converge or has to blow up in finite time.

It may be interesting to speculate about the types of singularities that may arise under the normalized harmonic map flow in the co-rotational case, or about a possible characterization of the maximal value $b_* > 1$ such that for $1 < b < b_*$ we have global existence and convergence of the flow. In fact, in view of the singularity models for the standard heat flow, one might hope to obtain global existence and convergence of the flow for all $b > 1$ such that $D(u_0) < 4\pi^2$, where $u_0 = \Sigma id$ as above, and where $4\pi^2$ is the energy of the map $u\colon S^3 \to S^2$ mapping any 2-sphere of constant latitude in S^3 to the equatorial sphere $S^2 \subset E(b)$. Note that for $b = 1$, $u_0 = \bar{u} = id_{S^3}$ we have $D(u_0) = 3\pi^2$, as an elementary computation shows.

Observe that in the case when $u_0(x, y) = (x, \varphi_0(y))$ with some function φ_0 satisfying $\varphi_0(-y) = -\varphi_0(y)$ for $(x, y) \in S^3 \subset \mathbb{R}^3 \times \mathbb{R}$, as, for example, in the case when $u_0 = \Sigma id$, by uniqueness the solution $u(t)$ to the standard heat flow will be of the form $u(x, y, t) = (x, \varphi(y, t))$ with a function φ such that $\varphi(-y, t)) = -\varphi(y, t))$ for all $t > 0$; hence $X(u(t)) \equiv 0$, and, by uniqueness of the normalized flow, the normalized flow coincides with the standard heat flow in this case.

5. Global theory

Unlike the Dirichlet energy, the functional E can detect topologically non-trivial data. The reason for this is that $E(u)$ bounds a certain Morrey norm of ∇u, which allows to define weak homotopy classes.

5.1. **Morrey bounds.** Note that for any $u \in H^1(S^3;N)$ we have

$$s\int_{B_{1/s}(0)} |\nabla\hat{u}|^2 dx \le \int_{B_{1/s}(0)} \frac{2s}{1+s^2|x|^2}|\nabla\hat{u}|^2 dx \le 2D(u_s) \le 2E(u)$$

for every $s > 0$, and similarly for any $a \in S^3$, giving rise to the bound for the Morrey norm

$$\|\nabla u\|^2_{L^{2,1}} = \sup_{a\in S^3,\, 0<r<1} \frac{1}{r}\int_{B_r(a)} |\nabla_{g_{S^3}} u|^2 \, d\mu_{g_{S^3}} \le C_0 E(u)$$

for every $u \in H^1(S^3;N)$ with an absolute constant $C_0 > 0$. Define

$$H^1_0(S^3;N) = \{u \in H^1(S^3;N);\ \forall a \in S^3\colon \lim_{s\to 0} D(u_{a,s}) = \lim_{s\to\infty} D(u_{a,s}) = 0\}.$$

Note that we have $C^1(S^3;N) \subset H^1_0(S^3;N)$.

5.2. **Weak homotopy classes.** Following Schoen-Uhlenbeck [27], we observe that any $u \in H^1(S^3;N)$ having uniformly small local Morrey norm in the sense that with a suitably small number $\gamma = \gamma(N) > 0$ there holds

$$\exists r > 0\ \forall p \in S^3\colon \|\nabla_{g_{S^3}} u\|_{L^{2,1}(B_r(p))} < \gamma, \tag{5.1}$$

may be approximated in H^1 by smooth maps from S^3 to N.

Indeed, recall that there is $\delta > 0$ such that the nearest-neighbor projection $\pi_N\colon U_\delta(N) \to N$ from the δ-neighborhood $N_\delta = \cup_{p\in N} B_\delta(p;\mathbb{R}^n)$ of N in $\mathbb{R}^n$ is well-defined and smooth.

Let $0 \le \rho \in C^\infty_c(B_1(0))$ be a standard mollifying kernel with $\|\rho\|_{L^1} = 1$. For $0 < \varepsilon \ll 1$ we set $\rho_\varepsilon(x) = \varepsilon^{-3}\rho(x/\varepsilon)$ and define $u_\varepsilon = u * \rho_\varepsilon$ as usual, where the convolution is defined in local geodesic normal coordinates on S^3. As shown by Schoen-Uhlenbeck [27], p. 316, for $u \in H^1(S^3;N)$ satisfying condition (5.1) for some $r = r(u) > 0$ and sufficiently small $0 < \gamma \le \gamma(N)$, for $0 < \varepsilon \le \min\{r, 1\}$ we have that

$$u_\varepsilon = u * \rho_\varepsilon\colon S^3 \to U_\delta(N),$$

and $u^\varepsilon := \pi_N \circ u_\varepsilon \in C^\infty(S^3;N)$ is well-defined; moreover, for any $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_0$ the maps u^{ε_1}, u^{ε_2} are homotopic via $(u^\varepsilon)_{\varepsilon_1\le\varepsilon\le\varepsilon_2}$, and we can define the homotopy class of u to be the homotopy class of u^ε for sufficiently small $\varepsilon > 0$. Note that condition (5.1) holds, in particular, whenever $u \in H^1_0(S^3;N)$.

In order to illustrate this concept with an example, assume there is a diffeomorphism $\phi_N\colon N \to S^3$ and let $u \in H^1_0(S^3;N)$. The maps

$$\phi_N \circ u^\varepsilon\colon S^3 \to S^3$$

then have a well-defined topological degree. Recalling that $\pi_3(S^3) \cong \mathbb{Z}$, we may define the topological degree $deg(u) \in \mathbb{Z}$ of u as the topological degree of $\phi_N \circ u^\varepsilon$ for any $0 < \varepsilon < \varepsilon_0$.

For any $i \in \mathbb{Z}$ set

$$\mathcal{C}_i = \{u \in H^1_0(S^3;N);\ deg(u) = i\}$$

and let

$$\mathcal{C} = \{u \in H^1_0(S^3; N);\ deg(u) \neq 0\} = \cup_{i \in \mathbb{Z} \setminus \{0\}} \mathcal{C}_i.$$

Then we have the following lower bound for energy .

Lemma 5.1. *There holds $\beta := \inf_{v \in \mathcal{C}} E(v) > 0$. In particular, we have*

$$\beta_i := \inf_{v \in \mathcal{C}_i} E(v) > 0 \quad \textit{for each } i \in \mathbb{Z} \setminus \{0\}.$$

Proof. Suppose by contradiction that there is $(u_k)_{k \in \mathbb{N}} \subset \mathcal{C}$ with $E(u_k) \to 0$ as $k \to \infty$. Then for sufficiently large $k \in \mathbb{N}$ and $\gamma = \gamma(N) > 0$ as above the condition (5.1) is satisfied for all radii $0 < r < 1$. Fixing $0 < \varepsilon < r < 1$, let $u^\varepsilon_k := \pi_N \circ (u_k * \rho_\varepsilon) \in C^\infty(S^3; N)$ as above. A subsequence $(u^\varepsilon_k)_{k \in \mathbb{N}}$ then converges smoothly to a constant map. But for sufficiently small $\varepsilon > 0$ we have $deg(u^\varepsilon_k) = deg(u_k) \neq 0$ for each sufficiently large k, which is a contradiction. □

5.3. **Global theory, non-smooth flows.** It would be nice if, using our flow (4.1), (4.2), we could show that for certain values $i \in \mathbb{Z}$ of the degree the number β_i is attained. Note that the non-existence result of Smith alluded to at the end of Section 4 seems to preclude a general existence result for all $i \in \mathbb{Z}$ of this nature.

Moreover, it would be nice if a characterization of all possible singularities of the normalized harmonic map flow could be achieved. Here, in addition to singularities as may arise in the standard harmonic map heat flow, we may have to expect new types of singularities to occur.

Indeed, if $u_0 \in H^2 \cap H^1_0(S^3; N)$ does not have a tension field which is small in L^2-norm, when compared with any $\delta_0 > 0$ such that u_0 is δ_0-uniformly 3-dimensional, in particular, if $u_0 \in H^1_0(S^3; N)$ only has finite energy, we cannot hope to be able to achieve (4.3) via a smooth choice of $\sigma = \sigma(t)$, $t > 0$. Instead, in order to maintain the condition $X(u) = 0$ for all time, we may have to allow for discontinuous changes of u to $u \circ \gamma$ such that $D(u \circ \gamma) = E(u)$ at a given time $t > 0$. Alternatively, we may also consider γ such that $D(u \circ \gamma)$ only corresponds to a local maximum in the Möbius group.

Finding the correct definition of a weak solution, and then constructing a global weak solution $u = u(t)$ of the normalized harmonic map flow for general data $u_0 \in H^1_0(S^3; N)$ may be a rather challenging problem; however, the task might be feasible for initial data u_0 whose energy is small (and which therefore are homotopically trivial).

For "large" data, the issues of classifying all possible singularities and achieving both a qualitative and a quantitative control of the parameter $\sigma(t)$ governing the Möbius shift of $u(t)$ can perhaps best be explored in the co-rotational or, more generally, in the equivariant setting.

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