PARTIAL REGULARITY FOR HARMONIC MAPS, AND RELATED PROBLEMS

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Abstract. Via gauge theory, we give a new proof of partial regularity for harmonic maps in dimensions \( m \geq 3 \) into arbitrary targets. This proof avoids the use of adapted frames and permits to consider targets of "minimal" \( C^2 \) regularity. The proof we present moreover extends to a large class of elliptic systems.

1. Introduction

In [10], the first author presented a new approach to the regularity result of Hélein [6] for weakly harmonic maps in dimension \( m = 2 \) where he succeeded in writing the harmonic map system in the form of a conservation law whose constituents satisfied elliptic equations with a Jacobian structure to which Wente’s [12] regularity results could be applied.

Consider for instance a harmonic map \( u = (u^1, \ldots, u^n) \in H^1(B; \mathbb{R}^n) \) from a ball \( B^m = B \subset \mathbb{R}^m \) to a hypersurface \( N \subset \mathbb{R}^n \) with normal \( \nu \). In this case the harmonic map equation may be written in the form

\[
-\Delta u^i = w^i \nabla w^j \cdot \nabla u^j = (w^i \nabla w^j - w^j \nabla w^i) \cdot \nabla u^j, \quad 1 \leq i \leq n,
\]

where \( w = \nu \circ u \). The key idea then is to identify the anti-symmetry of the 1-form

\[
\Omega^{ij} = (w^i dw^j - w^j dw^i), \quad 1 \leq i, j \leq n,
\]

as the essential structure of equation (1).

Interpreting \( \Omega \in L^2(B; \mathfrak{so}(n) \otimes \Lambda^1 \mathbb{R}^n) \) as a connection in the \( SO(n) \)-bundle \( u^* TN \) and following Uhlenbeck’s approach to the existence of Coulomb gauges [11], if \( m = 2 \), one succeeds in finding \( P \in H^1(B; SO(n)) \) and \( \xi \in H^1(B) \) such that

\[
P^{-1}dP + P^{-1}\Omega P = *d\xi,
\]

where \( * \) is the Hodge dual. Further algebraic manipulations then yield the existence of matrix-valued \( A, B \in H^1(B) \) with

\[
||dist(A, SO(n))||_{L^\infty} \leq C||\Omega||_{L^2}
\]

such that (1) may be written as

\[
\text{div}(A \nabla u + B \nabla^\perp u) = 0,
\]

where \( \nabla^\perp = *d \). By Hodge decomposition one then obtains \( E \) and \( D \) in \( W^{1,2}(B) \) such that

\[
A \nabla u = \nabla D + \nabla^\perp E.
\]
From (5) we see that $D$ and $E$ satisfy the equations

\begin{equation}
- \Delta D = -\text{div}(A \nabla u) = \nabla B \cdot \nabla^2 u, \\
- \Delta E = \text{curl}(A \nabla u) = \nabla^1 A \cdot \nabla u,
\end{equation}

which exhibit the desired Jacobian structure. The results in [3] then imply that $D, E \in W^{1,1}_{\text{loc}}(B)$. Provided that we restrict our attention to a domain where $\|\Omega\|_{L^2}$ is sufficiently small, from (4) we conclude that $\nabla u = A^{-1}(\nabla D + \nabla^1 E) \in W^{1,1}_{\text{loc}}(B)$ and $u \in W^{2,1}_{\text{loc}}(B) \hookrightarrow C^0(B)$, which implies full regularity.

In dimensions $m \geq 3$, the harmonic map equation is super-critical in the Sobolev space $H^1(B; \mathbb{R}^n)$ and no regularity result, not even a partial one, can be expected. In fact, in [9] the first author constructed examples of weak solutions to (1) in $H^1(B; S^2)$ for $m \geq 3$ which are nowhere continuous.

Under the further assumption that the solution $u$ lies in the homogeneous Morrey space $L^{2,m-2}_1$, which sometimes also is denoted as $M^{2,1}_1$, with

\begin{equation}
\|u\|_{L^{2,m-2}_1}^2 = \sup_{x \in B, r > 0} \left( \frac{1}{r^{m-2}} \int_{B_r(x) \cap B} |\nabla u|^2 \right) < +\infty,
\end{equation}

the harmonic map equation (1) becomes critical. More generally, this is true for any elliptic system with a nonlinearity growing quadratically in the gradient (see [5]). Assumption (8) is natural in the context of harmonic maps; in fact, it is a direct consequence of the geometric stationarity assumption described in Section 2. Observe that in dimension $m = 2$ assumption (8) corresponds exactly to the finite energy condition to higher dimensions.

Strengthening the assumption that $u \in H^1(B; \mathbb{R}^n)$ by assuming (8), one might then hope to be able to extend the approach described above to the case $m \geq 3$. However, in order to achieve (4), in dimension $m = 2$ one crucially uses that by Wente’s result mentioned above the solution $\psi \in H^1(B_2; \wedge^2 \mathbb{R}^n)$ of the equation

\begin{equation}
\begin{aligned}
\Delta \psi &= df \wedge dg & \text{in } B^m \\
\psi &= 0 & \text{on } \partial B^m
\end{aligned}
\end{equation}

for given $f$ and $g$ in $H^1(B_2)$ belongs to $L^\infty$. Unfortunately, this result does not extend to $m \geq 3$ when we replace the assumption $f, g \in H^1(B^m)$ by the condition that $f$ and $g$ belong to the Morrey space $L^{2,m-2}_1(B^m)$. Indeed, for $m = 3$, letting $f = \frac{1}{|x|}$ and choosing $g = \frac{1}{|x|^2}$, we have $f, g \in L^{2,1}_1(B^3)$ and equation (9) admits a unique solution $\psi \in L^{2,1}_1(B^3)$, but $\psi \notin L^\infty$. Thus the $L^\infty$-bound (4) does not seem to be available in dimension larger than 2 and the approach outlined above seems to fail for this reason.

However, as we presently explain, (1) - (3) in combination with standard techniques of elliptic regularity theory already suffice to conclude partial regularity, directly. In fact, via the gauge transformation $P$, from (1) we obtain the equation

\begin{equation}
-\text{div}(P^{-1} \nabla u) = (P^{-1} \nabla P + P^{-1} \Omega P) \cdot P^{-1} \nabla u = \ast \xi \cdot P^{-1} du,
\end{equation}
where the right hand side already has the structure of a Jacobian – up to the harmless (bounded) factor $P^{-1}$. Also observe that $\nabla u$ may be recovered from the term $P^{-1} \nabla u$ without any difficulty.

More generally, partial regularity results can be obtained for a large class of elliptic systems with quadratic growth that can be cast in the form

$$
-\Delta u = \Omega \cdot \nabla u \quad \text{in } B
$$

already considered in [10]. (In coordinates, equation (11) simply reads $-\Delta u^i = \Omega^{ij} \cdot \nabla u^j$.)

**Theorem 1.1.** For every $m \in \mathbb{N}$ there exists $\varepsilon(m) > 0$ such that for every $\Omega \in L^2(B^m, so(n) \otimes \wedge^1 \mathbb{R}^m)$ and for every weak solution $u \in H^1(B^m, \mathbb{R}^n)$ of equation (11), satisfying the Morrey growth assumption

$$
\sup_{x \in B, r > 0} \left( \frac{1}{r^{m-2}} \int_{B_r(x) \cap B} (|\nabla u|^2 + |\Omega|^2) \, dx \right) < \varepsilon(m),
$$

we have that $u$ is locally Hölder continuous in $B = B(m)$ with exponent $0 < \alpha = \alpha(m) < 1$.

The previous result is optimal, as shown by the standard example of the weakly harmonic map $u: B^3 \to S^2 \hookrightarrow \mathbb{R}^3$ with $u(x) = x/|x|$. We have $u \in H^1(B^3, \mathbb{R}^3)$ and, letting $\Omega = (\Omega^{ij}) := (u^i du^j - u^j du^i) \in L^2(B^m, so(n) \otimes \wedge^1 \mathbb{R}^m)$, we see that $u$ weakly satisfies the equation (11) and the condition

$$
\sup_{x \in B, r > 0} \left( \frac{1}{r^{m-2}} \int_{B_r(x) \cap B} (|\nabla u|^2 + |\Omega|^2) \, dx \right) < +\infty.
$$

The map $u$, however, is not continuous at the origin.

## 2. Stationary harmonic maps

For a smooth, closed, oriented $k$-dimensional submanifold $N \subset \mathbb{R}^n$ and a ball $B \subset \mathbb{R}^m$ let

$$
H^1(B; N) = \{ u \in H^1(B; \mathbb{R}^n); u(x) \in N \text{ for almost every } x \in B \}.
$$

Recall that a map $u \in H^1(B; N)$ is stationary if $u$ is critical for the energy

$$
E(u) = \int_B |\nabla u|^2 \, dx
$$

both with respect to variations of the map $u$ and with respect to variations in the domain.

It follows that $u$ is weakly harmonic; that is, $u$ satisfies the equation

$$
-\Delta u = A(u)(\nabla u, \nabla u) = \sum_{l=1}^{n-k} \sum_{\alpha=1}^m \nu_l \langle \partial_\alpha \partial_\alpha u, \partial_\alpha u \rangle = \sum_{l=1}^{n-k} w_l \langle \nabla w_l, \nabla u \rangle
$$

in the sense of distributions, where $A$ is the second fundamental form of $N$, defined locally via an orthonormal frame field $\nu_l$, $1 \leq l \leq n - k$ for the normal bundle to $N$. Again we denote as $w_l = \nu_l \circ u$ the corresponding unit normal vector field along the map $u$, and we denote as $\langle \cdot, \cdot \rangle$ the Euclidean inner product.
Moreover, as a consequence of the stationarity condition with respect to variations in the domain we have the monotonicity estimate

\begin{equation}
\int_{B_r(x_0)} \frac{1}{r^{2-m}} |\nabla u|^2 \, dx \leq \int_{B_{2r}(x_0)} \frac{1}{2r^{2-m}} |\nabla u|^2 \, dx
\end{equation}

for all balls $B_R(x_0) \subset B$ and all $r \leq R$.

The following result was obtained by Evans [4] and Bethuel [1]. Note that their approach in general requires the target manifold $N^k$ to be of class $C^5$; see [6], Theorem 4.3.1 and Remark 4.3.2. As a corollary to our main result Theorem 1.1, however, we now easily obtain the following generalization of their result to manifolds of class $C^2$.

**Theorem 2.1.** Let $N^k \subset \mathbb{R}^n$ be a closed submanifold of class $C^2$. Let $m \geq 3$ and suppose $u \in H^1(B^m; N)$ is a stationary harmonic map. There exists a constant $\varepsilon_0 > 0$ depending only on $N$ with the following property. Whenever on some ball $B_R(x_0) \subset B$ there holds

\begin{equation}
R^{2-m} \int_{B_R(x_0)} |\nabla u|^2 \, dx < \varepsilon_0,
\end{equation}

then $u$ is Hölder continuous (and hence as smooth as permitted by the regularity of $N$) on $B_{R/2}(x_0)$. In particular, $u$ is regular in $B$ away from a singular set $S$ with $\mathcal{H}^{m-2}(S) = 0$.

**Proof.** As in (1), equation (15) equivalently may be written in the form

\begin{equation}
-\Delta u^i = \Omega^{ij} \cdot \nabla w^j,
\end{equation}

where $\Omega \in L^2(B; so(n) \times \wedge^1 \mathbb{R}^n)$ in view of our assumption on $N$, with components locally given by

\begin{equation}
\Omega^{ij} = \Omega^{ij}_a dx^a = \sum_{i=1}^{n-k} (w^j dw^i - w^i dw^j), \quad 1 \leq i, j \leq n.
\end{equation}

Note that (16) and (17) imply that $\Omega$ belongs to the Morrey space $L^{2, m-2}(B)$ with

\begin{equation}
\|\Omega\|^2_{L^{2, m-2}} = \sup_{x_0 \in B} \int_{B_{2r}(x_0) \cap B} |\Omega|^2 \, dx \leq C \sup_{x_0 \in B} \int_{B_{r}(x_0) \cap B} |\nabla u|^2 \, dx \leq C \varepsilon_0.
\end{equation}

The result now is an immediate consequence of Theorem 1.1. \qed

3. **Proof of Theorem 1.1**

We may assume that condition (12) is satisfied on $B = B_1(0)$. As in (3), we obtain the existence of a suitable gauge transformation $\Phi$, transforming $\Omega$ into Coulomb gauge by applying the following lemma. The bound (12) also yields corresponding estimates for $P$ and $\xi$.
Lemma 3.1. Suppose that condition (12) is satisfied on $B$. There exists $P \in H^1(B; SO(n))$ and $\xi \in H^1(B, so(n) \otimes \Lambda^m \mathbb{R}^n)$ such that
\begin{equation}
P^{-1}dP + P^{-1}\Omega P = *d\xi \text{ on } B, \quad d(*\xi) = 0 \text{ on } B, \quad \text{and } \xi = 0 \text{ on } \partial B.
\end{equation}
Moreover, $dP$ and $d\xi$ belong to $L^{2,m-2}(B)$ with
\begin{equation}
||dP||^2_{L^{2,m-2}} + ||d\xi||^2_{L^{2,m-2}} \leq C(||\Omega||^2_{L^{2,2}} + ||du||^2_{L^{2,2}}) \leq C\varepsilon(m).
\end{equation}

The proof of this lemma will be given in the next section.

Recall that a function $f \in L^1(B)$ belongs to the space $BMO(B)$ if
\[ [f]_{BMO} = \sup_{x_0 \in B, r > 0} \left( \frac{1}{B_r(x_0) \cap B} \int f - \bar{f}_{x_0,r} \, dx \right) < \infty, \]
where
\[ \bar{f}_{x_0,r} = \frac{1}{B_r(x_0) \cap B} \int f \, dx \]
denotes the average of $f$ over $B_r(x_0) \cap B$, and so on. By Poincaré’s inequality, moreover, for $1 \leq p \leq m$ any function $f \in W^{1,p}(B)$ with $df \in L^{p,m-p}(B)$ belongs to $BMO(B)$ and there holds
\[ [f]^p_{BMO} \leq C||df||^p_{L^{p,m-p}} \sup_{x_0 \in B, r > 0} \left( r^{p-m} \int_{B_r(x_0) \cap B} ||df||^p \, dx \right). \]

Applying the gauge transformation $P^{-1}$ to $\nabla u$ and observing the identity $dP^{-1} = -P^{-1}dPP^{-1}$, from (11) we obtain the equation (10), that is
\begin{equation}
-\text{div}(P^{-1} \nabla u) = (P^{-1} \nabla P + P^{-1}\Omega P) \cdot P^{-1} \nabla u = *d\xi \cdot P^{-1} du.
\end{equation}

Fix a smooth cut-off function $\tau \in C_0^\infty(B)$ such that $0 \leq \tau \leq 1$, $\tau = 1$ on $B_{1/2}(0)$. Multiplying (23) by $\tau$, we obtain the equation
\begin{equation}
-\text{div}(P^{-1} \nabla (u\tau)) = *d\xi \cdot P^{-1} d(u\tau) - e,
\end{equation}
with “error” term
\begin{equation}
e = \text{div}(P^{-1}u\nabla \tau) + P^{-1} \nabla u \cdot \nabla \tau + *d\xi \cdot P^{-1} u d\tau.
\end{equation}
Since $u \in H^1 \cap L^2_{1,2-m}(B)$, we have $u \in L^p(B)$ for every $p < \infty$. Therefore, a direct application of Hölder’s inequality tells us that $u \in L^{p,m-\delta}$ for every $p < \infty$ and for every $\delta > 0$. Using this last observation and the fact that $d\xi$ is in $L^{2,m-2}$, we conclude that
\begin{equation}
\forall s \in [1,2] \quad \forall \delta > 0 : \quad \|e\|_{L^s,2-s-\delta} < \infty.
\end{equation}
We claim that $v = u\tau$ is Hölder continuous in $B$, provided the bound (12) holds with $\varepsilon(m) > 0$ sufficiently small.

Let $B_R(x_0) \subset B$ and let
\begin{equation}
P^{-1}dv = df + *dg + h
\end{equation}
be the Hodge decomposition of $P^{-1}dv$ on $B_R(x_0)$, where $f \in H^1_0(B_R(x_0))$ and where $g$ is a co-closed $m-2$-form of class $H^1(B_R(x_0))$ whose restriction to the boundary $\partial B$ also vanishes, and with a harmonic 1-form $h \in L^2(B_R(x_0))$; see [7]
Corollary 10.5.1, p. 236, for the Hodge decomposition of forms in Sobolev Spaces. Similar to (7) we have the equations
\begin{equation}
- \Delta f = -\text{div}(P^{-1}\nabla v) = *d\xi \cdot P^{-1} dv - e, \\
- \Delta g = *d(P^{-1} dv) = *(dP^{-1} \wedge dv).
\end{equation}

Fix a number $1 < p < m/(m-1)$ and let $q > m$ be the conjugate exponent with $1/p + 1/q = 1$. Since $f = 0$ on $\partial B_R(x_0)$, by duality we have
\begin{equation}
||df||_{L^p} \leq C \sup_{\varphi \in W_0^{1,q}(B_R(x_0)); ||\varphi||_{W^{1,q}} \leq 1} \int_{B_R(x_0)} df \cdot d\varphi \, dx.
\end{equation}
Here and in the following computations all norms refer to the domain $B_R(x_0)$. Note that $W_0^{1,q}(B_R(x_0)) \hookrightarrow C^{1-m/q}(B_R(x_0))$ and for all $\varphi \in W_0^{1,q}(B_R(x_0))$ with $||\varphi||_{W^{1,q}} \leq 1$ there holds
\begin{equation}
||\varphi||_{L^\infty} \leq CR^{1-m/q}||\varphi||_{W^{1,q}} \leq CR^{1-m/q}, ||d\varphi||_{L^2} \leq CR^{m/2-m/q}.
\end{equation}

For such $\varphi$ then we can estimate
\begin{equation}
\begin{aligned}
\int_{B_R(x_0)} df \cdot d\varphi \, dx &= -\int_{B_R(x_0)} \Delta f \varphi \, dx \\
&= \int_{B_R(x_0)} d\xi \wedge P^{-1} dv \varphi - \int_{B_R(x_0)} e \varphi \, dx = I + II
\end{aligned}
\end{equation}
as follows. Similar to the approach introduced in [2], upon integrating by parts and using [3], Theorem II.1, we have (up to sign, which is of no importance in what follows)
\begin{equation}
I = \int_{B_R(x_0)} d\xi \wedge P^{-1} dv \varphi = \int_{B_R(x_0)} d\xi \wedge d(P^{-1} \varphi)(v - \bar{v}_{x_0,R})
\end{equation}
\begin{equation}
\leq C ||d\xi \wedge d(P^{-1} \varphi)||_{L^1}[v]_{BMO} \leq C ||d\xi||_{L^2}[d(P^{-1} \varphi)||_{L^2}||v||_{BMO}
\end{equation}
\begin{equation}
\leq C ||d\xi||_{L^2} [||dP||_{L^2}||\varphi||_{L^\infty} + ||d\varphi||_{L^2}][v]_{BMO}
\end{equation}
\begin{equation}
\leq CR^{m-1-m/q}||d\xi||_{L^2}[v]_{L^2,\infty-2} [||dP||_{L^2,\infty-2} + ||d\varphi||_{L^\infty}][v]_{BMO}
\end{equation}
\begin{equation}
\leq CR^{m/p-1}v(m)[v]_{BMO},
\end{equation}
while (25), combined with (26) and (30), for any $\delta > 0$ gives the bound
\begin{equation}
II = -\int_{B_R(x_0)} e \varphi \, dx \leq ||e||_{L^1(B_R(x_0))} ||\varphi||_{L^\infty}
\end{equation}
\begin{equation}
\leq C_\delta R^{m-1-\delta} ||e||_{L^1,BMO} ||\varphi||_{L^\infty} \leq C_\delta R^{m-m/q-\delta} = C_\delta R^{m/p-\delta}.
\end{equation}

Hence from (28) we conclude that for every $\delta > 0$ there holds
\begin{equation}
||df||_{L^p} \leq CR^{m/p-1}v(m)[v]_{BMO} + C_\delta R^{m/p-\delta}.
\end{equation}

Similarly, letting $s$ satisfy $1/2 + 1/q + 1/s = 1$, by Hölder’s inequality for an arbitrary $(m-2)$-form $\psi \in W^{1,q}(B_R(x_0), \wedge^{m-2} \mathbb{R}^m)$ vanishing on $\partial B$ and with
$\|\psi\|_{W^{1,p}} \leq 1$ in view of the decomposition $-\Delta = *d{*d + d{*d}^*$ of the Hodge-Laplacian and the equation $d(*g) = 0$ we can bound (again up to sign)

$$
\int_{B_R(x_0)} dg \cdot d\psi dx = - \int_{B_R(x_0)} \Delta g \cdot \psi dx
$$

(35)

$$
= \int_{B_R(x_0)} dP^{-1} \wedge dv \wedge \psi = \int_{B_R(x_0)} dP^{-1} \wedge d\psi(v - \bar{v}_{x_0,R})
$$

$$
\leq C \|dP\|_{L^2} \|d\psi\|_{L^2} \|v - \bar{v}_{x_0,R}\|_{L^\infty} \leq CR^{m/p-1}\varepsilon(m)\|v\|_{BMO}.
$$

By duality, we have

$$
\|dg\|_{L^p} = \sup_{k \in L^p(B_R(x_0); \Lambda^{m-1}\mathbb{R}^m)} \|k\|_{L^p} \int_{B_R(x_0)} dg \cdot k dx.
$$

(36)

Decomposing any $k \in L^q(B_R(x_0); \Lambda^{m-2}\mathbb{R}^m)$ as $k = d\psi + *d\rho + \eta$ with $\eta$ satisfying $d\eta = 0$, $d(*\eta) = 0$ and with $\psi = 0$ on $\partial B$, as in [7], Corollary 10.5.1, and recalling that the restriction of $g$ to $\partial B_R(x_0)$ vanishes, we then arrive at the estimate

$$
\|dg\|_{L^p} \leq C \sup_{\psi \in W^{1,q}(B_R(x_0); \Lambda^{m-2}\mathbb{R}^m)} \|\psi\|_{L^q} \int_{B_R(x_0)} dg \cdot d\psi
$$

$$
\leq CR^{m/p-1}\varepsilon(m)\|v\|_{BMO}.
$$

(37)

From the Campanato estimates for harmonic functions, as in Giaquinta [5], proof of Theorem 2.2, p.84 f., we thus conclude that for any $r < R$ there holds

$$
\int_{B_r(x_0)} |dv|^p dx \leq C \int_{B_r(x_0)} |h|^p dx + C \int_{B_r(x_0)} (|df|^p + |dg|^p) dx
$$

$$
\leq C \left(\frac{r}{R}\right)^m \int_{B_r(x_0)} |h|^p dx + C \int_{B_r(x_0)} (|df|^p + |dg|^p) dx
$$

$$
\leq C \left(\frac{r}{R}\right)^m \int_{B_r(x_0)} |dv|^p dx + C \int_{B_r(x_0)} (|df|^p + |dg|^p) dx
$$

$$
\leq C \left(\frac{r}{R}\right)^m \int_{B_r(x_0)} |dv|^p dx + CR^{m-p}\varepsilon(m)\|v\|_{BMO}^p + C_\delta R^{m-\delta p},
$$

for any $\delta > 0$. Set

$$
\Phi(x_0, r) = r^{p-m} \int_{B_r(x_0)} |dv|^p dx
$$

and define

$$
\Psi(R) = \sup_{x_0 \in B, 0 < r < R} \Phi(x_0, r).
$$

Then we can bound

$$
\sup_{x_0 \in B} \|v\|_{BMO(B_r(x_0))} \leq C\Psi(R).
$$

Fixing $\delta < 1 - 1/p$, thus from (38) we have

$$
\Phi(x_0, r) \leq C \left(\frac{r}{R}\right)^p \Phi(x_0, R) + C \left(\frac{r}{R}\right)^{p-m} \varepsilon(m)\Psi(R) + C_\delta \left(\frac{r}{R}\right)^{p-m} R^{\delta p}
$$

(39)

$$
\leq C_1 \left(\frac{r}{R}\right)^p \left(1 + \left(\frac{r}{R}\right)^{-m} \varepsilon(m)\right) \Psi(R) + C \left(\frac{r}{R}\right)^{p-m} R
$$
with a uniform constant $C_1$. Also fixing $r/R = \gamma$ such that $C_1 \gamma^{(p-1)/2} \leq 1/2$, and choosing $\varepsilon(m) = \gamma^m$, for any $R_0 \in (0, R)$ we obtain

$$\Phi(x_0, R) \leq C_1 \gamma^p (1 + \varepsilon(m) \gamma^{-m}) \Phi(R) + C \gamma^{-m} R$$

$$\leq \gamma^{(p+1)/2} \Psi(R) + CR \leq \gamma^{(p+1)/2} \Psi(R_0) + CR_0.$$ 

Upon passing to the supremum with respect to $x_0$ and $R < R_0$ we deduce that

$$\Psi(\gamma R_0) \leq \gamma^{(p+1)/2} \Psi(R_0) + CR_0$$

for any $R_0 \in [0, 1]$.

Finally, for any $r \in [0, \gamma]$, letting $k \in \mathbb{N}$ be such that $\gamma^{k+1} < r \leq \gamma^k$ and iterating as in Giaquinta [5], proof of Lemma 2.1, p.86, we conclude that

$$\Psi(r) \leq \Psi(\gamma^k) \leq \gamma^{k(p+1)/2} \Psi(1) + C \gamma^k \sum_{j=1}^{\infty} \gamma^{j(p-1)/2} \leq C r.$$ 

Hence $v \in C^{1/p}(B)$ and therefore also $u \in C^{1/p}(B_{1/2}(0))$, as claimed.

4. Proof of Lemma 3.1

For the proof of Lemma 3.1 we follow [8], where Uhlenbeck’s [11] construction of a local Coulomb gauge in Sobolev spaces was generalized to Morrey spaces. Due to the fact that the space $L^{1, m-2}$ defined earlier does not embed into $C^0$, the inverse mapping $P \to P^{-1}$ is not smooth from the space $L^{1, m-2}$ into itself. In order to avoid this difficulty, similar to [11] we first construct the local Coulomb gauge under slightly more stringent regularity assumptions.

Lemma 4.1. There exists $\varepsilon(m, n) > 0$ and $C > 0$ such that, on $B = B^m$ for every $\alpha > 0$ and every $\Omega \in L^{m-2+\alpha}(B, so(n) \otimes \Lambda^1 \mathbb{R}^m)$ with

$$\|\Omega\|_{L^{2, m-2}}^2 \leq \varepsilon(n, m)$$

there exist $P \in L^{2, m-2+\alpha}(B; SO(n))$ and $\xi \in L^{2, m-2+\alpha}(B, so(n) \otimes \Lambda^{m-2} \mathbb{R}^m)$ such that

$$P^{-1} dP + P^{-1} \Omega P = \ast d\xi \text{ on } B, \ d(\ast \xi) = 0 \text{ on } B, \text{ and } \xi = 0 \text{ on } \partial B.$$ 

Moreover, $dP$ and $d\xi$ satisfy

$$\|dP\|_{L^{2, m-2+\alpha}}^2 + \|d\xi\|_{L^{2, m-2+\alpha}}^2 \leq C\|\Omega\|_{L^{2, m-2+\alpha}}^2,$$

and

$$\|dP\|_{L^{2, m-2}}^2 + \|d\xi\|_{L^{2, m-2}}^2 \leq C\|\Omega\|_{L^{2, m-2}}^2 \leq C \varepsilon(n, m).$$

Proof of Lemma 3.1. Let $\Omega$ be in $L^{2, m-2}$ and suppose that $\|\Omega\|_{L^{2, m-2}} < \varepsilon$ for some number $\varepsilon > 0$ to be fixed below. Although smooth functions are not dense in $L^{2, m-2}$, it is not difficult to show that the mollified forms $\Omega_\delta = \Omega \ast \chi_\delta$ obtained from $\Omega$ by convoluting $\Omega$ with a standard mollifier satisfy the uniform estimate $\|\Omega_\delta\|_{L^{2, m-2}} \leq C\|\Omega\|_{L^{2, m-2}}$. By choosing $\varepsilon > 0$ sufficiently small, we can then achieve the uniform bound $\|\Omega_\delta\|_{L^{2, m-2}} \leq \varepsilon(m, n)$, where $\varepsilon(m, n)$ is given in Lemma 4.1, to obtain the existence of $\xi_\delta$ and $P_\delta$ satisfying (41), (42), and (43) for $\Omega_\delta$ instead of $\Omega$. The uniform bound given by (43) permits to pass to the limit $\delta \to 0$ in (41), and the assertion of Lemma 3.1 follows. \[\square\]
Proof of Lemma 4.1. For \( \alpha > 0 \) introduce the set
\[
\mathcal{U}_{\mathcal{C},C} := \left\{ \Omega \in L^{2,m-2+\alpha}(B^m, so(n)) ; \; \| \Omega \|_{L^{2,m-2}} \leq \varepsilon , \; \text{and} \right. \\
\left. \text{there exist } P \text{ and } \xi \text{ satisfying (41), (42), (43)} \right\}
\]
Since clearly \( \Omega = 0 \in \mathcal{U}_{\mathcal{C},C} \), the set \( \mathcal{U}_{\mathcal{C},C} \) is not empty. The proof therefore will be complete once we show that, for small enough and \( C \) large enough, \( \mathcal{U}_{\mathcal{C},C} \) is both open and closed in the star-shaped and hence path-connected set
\[
\mathcal{V}_{\mathcal{C}} := \left\{ \Omega \in L^{2,m-2+\alpha}(B^m, so(n)) \otimes \wedge^1 \mathbb{R}^m ; \; \| \Omega \|_{L^{2,m-2}} \leq \varepsilon \right\}.
\]
The proof of closedness is similar to the proof of Lemma 3.1 given above. To see that \( \mathcal{U}_{\mathcal{C},C} \) is open in \( \mathcal{V}_{\mathcal{C}} \), observe that for \( \alpha > 0 \) the space \( L^{2,m-2+\alpha} \) embeds continuously into \( C^0 \) and the inverse mapping \( P \to P^{-1} \) from the space \( L^{2,m-2+\alpha}(B, SO(n)) \) into itself is smooth. Therefore the argument of [11] can be applied to show that, for sufficiently small \( \varepsilon > 0 \) and sufficiently large \( C \), for every \( \Omega \) in \( \mathcal{U}_{\mathcal{C},C} \) there exists \( \eta_0 > 0 \) with the property that for every \( \omega \in L^{2,m-2+\alpha} \) satisfying the bound \( \| \omega \|_{L^{2,m-2+\alpha}} \leq \eta_0 \) and \( \| \Omega + \omega \|_{L^{2,m-2}} < \varepsilon \) we can find \( \xi_\omega \in L^{2,m-2+\alpha}(B, so(n) \otimes \wedge^m \mathbb{R}^m) \) and \( P_\omega \in L^{2,m-2+\alpha}(B, SO(n)) \), respectively, satisfying (41). The openness of \( \mathcal{U}_{\mathcal{C},C} \) then may be obtained as in [10] from the following lemma. This completes the proof.

Lemma 4.2. There exists \( \delta > 0 \) with the following property. Suppose that for \( \Omega \in \mathcal{V}_{\mathcal{C}} \) there exist \( \xi \in L^{2,m-2+\alpha}(B^m, so(n) \otimes \wedge^m \mathbb{R}^m) \), \( P \in L^{2,m-2+\alpha}(B^m, SO(n)) \) satisfying (41) and the estimate
\[
\| d\xi \|_{L^{2,m-2}} + \| dP \|_{L^{2,m-2}} \leq \delta.
\]
Then (42) and (43) hold for some \( C \) independent of \( \Omega \in \mathcal{V}_{\mathcal{C}} \).

Proof. In view of (41), the \((m-2)\)-form \( \xi \) satisfies
\[
\begin{cases}
\Delta \xi = * (dP^{-1} \wedge dP) + *d(P^{-1}\Omega) & \text{in } B, \\
\xi = 0 & \text{on } \partial B
\end{cases}
\]
We decompose \( \xi = u + w \) in two forms \( u \) and \( w \) solving, respectively,
\[
\begin{cases}
\Delta u = * (dP^{-1} \wedge dP) & \text{in } B \\
u = 0 & \text{on } \partial B
\end{cases}
\]
and
\[
\begin{cases}
\Delta w = *d(P^{-1}\Omega) & \text{in } B \\
w = 0 & \text{on } \partial B
\end{cases}
\]
From [5], Theorem III.2.2, for \( s \in \{2 - \alpha, 2\} \) first we obtain the bound
\[
\| du \|_{L^{2,m-s}(B)} \leq C \| \Omega \|_{L^{2,m-s}(B)}.
\]
Following the strategy of the proof of Theorem III.2.2 in [5], likewise for \( s \in \{2 - \alpha, 2\} \) we obtain that
\[
\| du \|_{L^{2,m-s}(B)} \leq C \| P \|_{BMO(B)} \| dP \|_{L^{2,m-s}(B)} \leq C \| dP \|_{L^{2,m-s}(B)} \leq C \delta \| dP \|_{L^{2,m-s}(B)}.
\]
The only modification required with respect to [5], p. 85, is that we use [3], Theorem II.1, to estimate, with $v$ as defined in [5],

$$
\|d(u-v)\|^2_{L^2(B_R(x_0)\cap B)} = -\int_{B_R(x_0)\cap B} \Delta(u-v) \wedge (u-v)
$$

(50)

$$
= \int_{B_R(x_0)\cap B} dP^{-1} \wedge dP \wedge (u-v)
$$

$$
= \int_{B_R(x_0)\cap B} dP^{-1}(P - \bar{P}_{x_0,R}) \wedge d(u-v)
$$

$$
\leq \|d(u-v)\|_{L^2(B_R(x_0)\cap B)} \|P\|_{BMO(B)} \|dP^{-1}\|_{L^2(B_R(x_0)\cap B)}.
$$

Combining (48) and (49) we then conclude

$$
\|d\xi\|_{L^{2,m-\epsilon}(B)} \leq C\delta \|dP\|_{L^{2,m-\epsilon}(B)} + C \|\Omega\|_{L^{2,m-\epsilon}(B)}.
$$

(51)

Moreover, from (41) we have

$$
\|dP\|_{L^{2,m-\epsilon}(B)} \leq \|d\xi\|_{L^{2,m-\epsilon}(B)} + \|\Omega\|_{L^{2,m-\epsilon}(B)}.
$$

(52)

Putting the estimates (51) and (52) together, upon choosing $\delta > 0$ small enough we then obtain (42) and (43). The proof is complete. \qed

References


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