SCATTERING FOR A CRITICAL NONLINEAR WAVE EQUATION IN TWO DIMENSIONS

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ABSTRACT. We show that the solutions to the Cauchy problem for a wave equation with critical exponential nonlinearity in 2 space dimensions scatter for arbitrary smooth, compactly supported initial data.

1. INTRODUCTION

Consider the initial value problem for the equation

$$u_{tt} - \Delta u + u\big(e^{u^2} - 1 - u^2\big) = 0 \text{ on } \mathbb{R} \times \mathbb{R}^2 . \tag{1}$$

with smooth Cauchy data

$$(u, u_t)_{|t=0} = (u_0, u_1) \in C_c^\infty(\mathbb{R}^2). \tag{2}$$

Observe that for a classical solution u of (1), (2) the energy

$$E(u(t)) = \frac{1}{2} \int_{\{t\} \times \mathbb{R}^2} \big(|u_t|^2 + |\nabla u|^2 + F(u)\big) dx \tag{3}$$

is conserved, where $F(u) = e^{u^2} - 1 - u^2 - u^4/2$ is a primitive of the nonlinear term $f(u) = u\big(e^{u^2} - 1 - u^2\big)$.

For the related problem when $f(u)$ is replaced by the nonlinearity $n(u) = ue^{u^2}$ Ibrahim, Majdoub, and Masmoudi in [3] showed that whenever the corresponding initial energy is at most 2π the Cauchy problem (1), (2) admits a global smooth solution. Together with Nakanishi, in [5] the same authors also showed that when $f(u)$ is replaced by $l(u) = u(e^{u^2} - u^2)$ the solution scatters, again assuming the associated initial energy to be bounded by 2π. The constant 2π is related to the best constant in the Moser-Trudinger inequality [6], [10], which defines the limit case of Sobolev's embedding of the space $H^1(\mathbb{R}^2)$. It was conjectured in [5] that this number also marks an energy threshold for the onset of "super-critical" behavior in (1) and its variants. This conjecture was partially confirmed through the examples given in [4], showing that the solutions no longer depend in a locally uniformly continuous fashion on the data when the initial energy exceeds the value 2π.

In contrast with these expectations, however, Struwe [9] showed that the initial value problem for equation (1) has a global smooth solution for smooth Cauchy data (u_0, u_1) with arbitrarily large energy. This result was originally demonstrated

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when $f(u)$ is replaced by the nonlinearity $n(u) = ue^{u^2}$ but the proof is valid also for all the above variants of equation (1).

Moreover, by building on the techniques developed in [9], Sack [7] was able to show scattering for any solution u of (1), (2) for arbitrarily large smooth, compactly supported data with rotational symmetry. Here, by definition, a solution u to (1) scatters if for the solution v to the homogeneous linear wave equation

$$v_{tt} - \Delta v = 0 \quad \text{on } \mathbb{R} \times \mathbb{R}^2 \tag{4}$$

for suitable "scattering data"

$$(v, v_t)_{|t=0} = (v_0, v_1) \in \dot{H}^1 \times L^2(\mathbb{R}^2) \tag{5}$$

there holds

$$||Du(t) - Dv(t)||_{L^2(\mathbb{R}^2)} \to 0 \quad \text{as } t \to \infty, \tag{6}$$

where $Du = (u_t, \nabla u)$ is the space-time differential of u.

Combining the insights of [7] and [9], in the present paper we now establish scattering in the general (non-symmetric) case.

Theorem 1.1. *For any $u_0, u_1 \in C_c^\infty(\mathbb{R}^2)$ there exist $(v_0, v_1) \in \dot{H}^1 \times L^2(\mathbb{R}^2)$ such the solution u to (1), (2) scatters to the solution v of (4), (5) in the sense of (6).*

For the proof of Theorem 1.1, as in [7] it suffices to show finiteness of the scattering norm

$$\|u_{tt} - \Delta u\|_{L^{1,2}_{t,x}} = \|f(u)\|_{L^{1,2}_{t,x}} = \int_0^\infty \|f(u(t))\|_{L^2(\mathbb{R}^2)} dt$$

of the solution u to (1), (2) for given data. In [7] this already was partially achieved by applying the techniques of [9] to the function U obtained from u through conformal inversion, which satisfies an equation similar to (1). Conformal inversion also is a key element in the proof of Theorem 1.1 in the present paper, and we crucially exploit the fact that the wave operator and nonlinear terms of degree 5 and higher are well-behaved under this transformation. Even though our proof therefore cannot be extended to the case when $f(u)$ is replaced by the nonlinearity $l(u) = u(e^{u^2} - u^2)$, it is to be expected that the analogue of Theorem 1.1 also holds in this case, since scattering properties should only improve in the presence of a mass term. However, it is not clear if scattering holds when $f(u)$ is replaced by the nonlinearity $n(u) = ue^{u^2}$ since the cubic term seems difficult to treat even in the small energy regime.

Note that also when $f(u)$ is replaced by either $l(u)$ or $n(u)$, by [9] the solutions to the Cauchy problem (1), (2) for smooth data always are globally regular.

2. Preliminaries

2.1. **Energy identity.** Multiplying (1) by u_t we obtain the identity

$$0 = \frac{d}{dt} e(u) - div(\nabla u \cdot u_t) \tag{7}$$

for the energy density

$$e(u) = \frac{1}{2}\big(|u_t|^2 + |\nabla u|^2 + F(u)\big).$$

and the density of momentum $m(u) = \nabla u \cdot u_t$.

If for $0 < S \le T \le T_0$ we denote as $v(y) = u(|y|, y)$ the restriction of u to the lateral boundary

$$M_S^T = \{z = (t,x); S \le t \le T, |x| = t\}$$

of the truncated forward light cone

$$K_S^T = \{z = (t,x); S \le t \le T, |x| \le t\}$$

with vertex at $z = (0,0)$, then upon integrating (7) over K_S^T we find the identity

$$E(u(S), B_S(0)) + Flux(u, M_S^T) = E(u(T), B_T(0)) \tag{8}$$

for all $0 < S < T \le T_0$, where

$$E(u(t), B_R(x_0)) := \int_{B_R(x_0)} e(u(t))dx$$

and where

$$Flux(u, M_S^T) := \frac{1}{2}\int_{B_T \setminus B_S(0)} (|\nabla v|^2 + F(v))dy$$

is the energy flux through M_S^T, as in [9]. In particular, energy will spread with speed at most 1. Identities similar to (8) hold on any region with space-like or null boundary. For $0 \le S \le T$ we also let $K^T = K_0^T$, $K_S = K_S^\infty$, and so on.

2.2. **Normalisation.** Given data $u_0, u_1 \in C_c^\infty(\mathbb{R}^2)$ with support in $B_a(0)$ for some $a > 0$, by (8) the solution u to the Cauchy problem (1), (2) in forward time is supported inside the light cone with vertex at $(-a, 0)$. We may assume that $a \ge 1/2$. Shifting time by $t_0 = 2a \ge 1$ we then may assume that u in forward time is supported inside the cone

$$K_{t_0}(a,0) = \{(t,x);\ |x| \le t - a,\ t \ge t_0\} \subset K = \{(t,x);\ |x| \le t\},$$

with initial data

$$(u, u_t)_{|t=t_0} = (u_0, u_1) \in C_c^\infty(\mathbb{R}^2)$$

and with lateral trace

$$u = 0 \text{ on } M_{t_0}(a,0),$$

where $M_{t_0}(a,0)$ is the lateral boundary of the truncated cone $K_{t_0}(a,0)$.

2.3. **Conformal inversion.** Following Grillakis [1] we let $\Phi \colon K \to K$ denote the conformal inversion given by $\Phi(t,x) = (T,X)$ with

$$T := \frac{t}{t^2 - r^2}, \quad X := \frac{x}{t^2 - r^2},$$

where $r = |x|$. Note that Φ is an involution with inverse $\Phi^{-1} = \Phi$, and that Φ maps light cones to light cones. Moreover, letting $\eta = diag(-1,1,1)$ be the Minkowski metric, we have $\Phi^*\eta = \Omega^2\eta$, where

$$\Omega = \frac{1}{t^2 - r^2} = T^2 - R^2 \text{ on } K$$

with $R = |X|$. Finally, as explained in [7], given a solution u of (1) with support in $K_{t_0}(a,0) \subset K$, the function $U = \Omega^{-1/2}(u \circ \Phi)$ solves the equation

$$U_{TT} - \Delta U = -\Omega^{-2} U(e^{\Omega U^2} - 1 - \Omega U^2) =: -g(U). \tag{9}$$

Recalling that $t_0 = 2a$, we observe that the support of U is contained in the region

$$A := \Phi(K_{t_0}(a,0)) = \{(T,X) \in K;\ T/(T^2 - R^2) \geq t_0,\ T + R \leq 2/t_0\}$$

bounded within K by a section of the hyperboloid

$$\Sigma = \{(T,X) \in K;\ (T - \frac{1}{2t_0})^2 = R^2 + \frac{1}{4t_0^2}\}$$

and the lateral boundary of an incoming light cone with vertex at the point $(2/t_0, 0)$, which meets Σ in the circle $\{(T,X) \in K;\ T = 2R = 4/(3t_0)\}$.

2.4. **Bounds for conformal energy.** Similar to (7), upon multiplying (9) with U_T we find the conservation law

$$\frac{d}{dT}\tilde{e}(U) - div(\tilde{m}(U)) = TP(U) \geq 0 \tag{10}$$

for the conformal energy density

$$\tilde{e}(U) = \frac{1}{2}\big(|U_T|^2 + |\nabla U|^2 + G(U)\big)$$

with

$$G(U) = \Omega^{-3}(e^{\Omega U^2} - 1 - \Omega U^2 - \Omega^2 U^4/2) \geq U^6/6\ ,$$

and with the density of momentum $\tilde{m}(U) = \nabla U \cdot U_T$. The fact that the lowest power term in $f(u)$ is of order u^5 is crucial for the positivity of the term

$$P(U) = \Omega^{-3}U^2(e^{\Omega U^2} - 1 - \Omega U^2) - 3\Omega^{-1}G(U) = \Omega^{-4}\sum_{k=4}^{\infty}(k-3)\frac{(\Omega U^2)^k}{k!}.$$

We have thus set the stage for the proof of Theorem 1.1.

3. Proof of Theorem 1.1

Recall the following result of Sack [7], similar to [9], Lemma 4.3.

Proposition 3.1. *There exists $T_1 > 0$ and a constant $C_1 > 0$ such that*

$$\int_{K^{T_1}} e^{4U^2} dX\, dT \leq C_1\ .$$

Proposition 3.1 allows to partially bound the scattering norm, as follows. Clearly we may assume that $T_1 \leq 1/(2t_0) \leq 1/2$. Set

$$D := \Phi^{-1}(K^{T_1}) = \{(t,x) \in K;\ t/(t^2 - r^2) \leq T_1\} \subset K,$$

and for any $t_0 \leq t_1 \leq t_2$ let

$$D(t_1) = \{(t,x) \in D;\ t = t_1\}$$

as well as

$$D_{t_1}^{t_2} = \{(t,x) \in D;\ t_1 \leq t \leq t_2\}.$$

Note that D is the part of K above the hyperboloid

$$H = \{(t,x) \in K;\ (t - \frac{1}{2T_1})^2 = r^2 + \frac{1}{4T_1^2}\};$$

therefore D together with the thickened mantle region

$$K_{t_0}(a,0) \setminus K(d,0) = \{(t,x) \in K_{t_0};\ t - d \leq |x| \leq t - a\}$$

for any $d > 1/(2T_1) \geq t_0 = 2a$ covers the support of u up to a compact domain.

By Proposition 3.1 and using that

$$e^{\Omega U^2} - 1 - \Omega U^2 \leq \Omega^2 U^4 e^{\Omega U^2},$$

we can bound

$$\int_{D_{t_1}^{t_2}} |f(u)|^2 dx\, dt = \int_{D_{t_1}^{t_2}} |u(e^{u^2} - 1 - u^2)|^2 dx\, dt$$
$$= \int_{\Phi(D_{t_1}^{t_2})} \Omega U^2 |e^{\Omega U^2} - 1 - \Omega U^2|^2 \Omega^{-3} dX\, dT \leq \int_{\Phi(D_{t_1}^{t_2})} \Omega^2 U^{10} e^{2\Omega U^2} dX\, dT.$$

But since $T_1 \leq 1$, we have that $\Omega \leq 1$ on $\Phi(D_{t_1}^{t_2}) \subset K^{T_1}$, and

$$U^{10} e^{2\Omega U^2} \leq C e^{4U^2}$$

on this region. On the other hand, using that we have $|x| \leq t - a$ throughout the support of u, we can also bound

$$\Omega = \frac{1}{(t-|x|)(t+|x|)} \leq \frac{1}{at} \tag{11}$$

on the support of u. Hence by Proposition 3.1 we find that

$$\int_{D_{t_1}^{t_2}} |f(u)|^2 dx\, dt \leq C t_1^{-2} \int_{K^{T_1}} e^{4U^2} dX\, dT \leq C t_1^{-2}.$$

After a dyadic decomposition $t_k = 2^k t_0$, $k \in \mathbb{N}$, of the time interval $[t_0, \infty[$, from Hölder's inequality we then obtain

$$\int_{t_0}^{\infty} \Big(\int_{D(t)} |f(u)|^2 dx \Big)^{1/2} dt = \sum_{k \in \mathbb{N}_0} \int_{t_k}^{t_{k+1}} \Big(\int_{D(t)} |f(u)|^2 dx \Big)^{1/2} dt$$
$$\leq \sum_{k \in \mathbb{N}_0} \big(2^k t_0\big)^{1/2} \Big(\int_{D_{t_k}^{t_{k+1}}} |f(u)|^2 dx\, dt \Big)^{1/2} \leq C \sum_{k \in \mathbb{N}_0} 2^{-k/2} < \infty.$$

To complement this estimate, we show the following bound.

Proposition 3.2. *There exists a constant $C_2 > 0$ such that*

$$\sup_{supp(U) \setminus K^{T_1}} |U| \leq C_2\ .$$

Postponing the proof of Proposition 3.2, we complete the proof of Theorem 1.1. From Proposition 3.2 and (11) we conclude that

$$|u(t,x)| = \Omega^{1/2} |U(T,X)| \leq C_2 a^{-1/2}/\sqrt{t}$$

for all $(t,x) \in K_{t_0}(a,0) \setminus D$, with $(T,X) = \Phi(t,x)$. Hence we have the bound

$$|f(u)|^2 = |u(e^{u^2} - 1 - u^2)|^2 \leq u^{10} e^{2u^2} \leq C t^{-5} \ \text{ away from } D$$

and there holds

$$\int_{t_0}^{\infty} \Big(\int_{B_t(0) \setminus D(t)} |f(u)|^2 dx \Big)^{1/2} dt$$
$$\leq \int_{t_0}^{\infty} \Big(\int_{B_t(0) \setminus D(t)} u^{10} e^{2u^2} dx \Big)^{1/2} dt \leq C \int_{t_0}^{\infty} t^{-3/2} dt < \infty.$$

Splitting $f(u) = f(u)\chi_D + f(u)\chi_{(K_{t_0}\setminus D)}$, where χ_M is the characteristic function of a set M, with the Minkowski inequality we now obtain that

$$\|f(u)\|_{L^{1,2}_{t,x}} \le \|f(u)\chi_D\|_{L^{1,2}_{t,x}} + \|f(u)\chi_{(K_{t_0}\setminus D)}\|_{L^{1,2}_{t,x}} < \infty,$$

as claimed.

4. Proof of Proposition 3.2.

Recall that $supp(U) \subset A = \Phi(K_{t_0}(a,0))$. Fix a point $(T_0, X_0) \in A \setminus K^{T_1}$. By Duhamel's formula we can decompose

$$U(T_0, X_0) = U_L(T_0, X_0) + U_N(T_0, X_0),$$

where U_L solves the free wave equation (4) and where $V := U_N$ solves the equation $V_{TT} - \Delta V + g(U) = 0$ on A with vanishing Cauchy data.

Clearly we have $U_L = \Omega^{-1/2}(u_L \circ \Phi)$, where u_L is the solution of the free wave equation (4) with Cauchy data (2) at time $t = t_0$. By the known decay estimates for this problem there exists a constant $C = C(u_0, u_1) > 0$ such that

$$|u_L(t,x)| \le C/\sqrt{t}$$

for sufficiently large $t > t_0$; see for instance [8], Lemma 4.2 and (4.9b). Recall that letting $d = 1/T_1$, for sufficiently large $t_1 > t_0$ we have

$$K_{t_1} \setminus D \subset K \setminus K(d,0) = \{(t,x) \in K;\ t - d \le |x| \le t\}.$$

Hence for $(t,x) \in K \setminus D$ with $t \ge t_1$ we can estimate

$$(\Omega(t,x))^{-1} = (t - |x|)(t + |x|) \le 2dt.$$

By smoothness of u_L then we can uniformly bound $|(\Omega^{-1/2}u_L)(t,x)| \le C$ everywhere on $K_{t_0} \setminus D$, and we obtain the bound

$$\sup_{(T_0,X_0)\in A\setminus K^{T_1}} |U_L(T_0,X_0)| \le C. \tag{12}$$

By Duhamel's formula, moreover, the component U_N is given by the equation

$$U_N(T_0,X_0) = -\int_{A_0} g(U)\Gamma\, dX\, dT,$$

where $A_0 := K(T_0,X_0) \cap A$ and where

$$\Gamma(T,X) = \frac{1}{2\pi\sqrt{|T_0 - T|^2 - |X - X_0|^2}} \in L^{3/2}(A_0)$$

is the fundamental solution to the linear wave equation. Again estimating

$$|g(U)| = |\Omega^{-2}U(e^{\Omega U^2} - 1 - \Omega U^2)| \le |U|^5 e^{\Omega U^2},$$

with a uniform constant $C > 0$ we can then bound

$$\begin{aligned} |U_N(T_0,X_0)| &\le \|\Gamma\|_{L^{3/2}(A_0)} \|g(U)\|_{L^3(A_0)} \\ &\le C \sup_{T\ge T_0} \|U(T)\|^5_{L^{30}(A_0(T))} \|e^{\Omega U^2}\|_{L^6(A_0)}, \end{aligned} \tag{13}$$

where $A_0(T) = \{(T,X) \in A_0\}$ for any $T \ge T_0$. The desired bound then will be a consequence of the following two lemmas.

Lemma 4.1. *With a uniform constant $C_1 > 0$ for any $(T_0, X_0) \in A \setminus K^{T_1}$ there holds*

$$\sup_{T \geq T_0} \|U(T)\|^5_{L^{30}(A_0(T))} \leq C_1 \ . \tag{14}$$

Proof. For any $T > 0$ also let $A(T) = \{(T, X) \in A\}$, where $A = \Phi(K_{t_0}(a, 0))$ as above. Set $\delta_0 = 1/(3t_0)$. Note that

$$\Omega(T, X) = (T - R)(T + R) \geq \delta_0 T_1$$

is uniformly bounded away from zero at points $(T, X) \in A \setminus K^{T_1}$ with $R \leq T - \delta_0$. All such points are mapped under Φ to a fixed bounded region of space-time. Since u is smoothly bounded on any compact region, we see that also $U = \Omega^{-1/2}(u \circ \Phi)$ is smoothly bounded at all such points. Fix

$$T_2 = 7/(6t_0) = 1/t_0 + 1/(6t_0).$$

Our choice $\delta_0 = 1/(3t_0)$ guarantees that for all $T \geq T_2$ we have $A(T) \subset K(\delta_0, 0)$; hence we have

$$\sup_{T \geq T_2} \|U(T)\|^5_{L^{30}(A(T))} < \infty \ .$$

Recalling that U vanishes in a neighborhood of the mantle section

$$\{(T, X) \in K;\ T + R = 2/t_0,\ T > 1/t_0\},$$

from the energy inequality (10) and in view of smoothness of the Cauchy data for U on Σ we also have

$$\sup_{T_1 \leq T \leq T_2} \tilde{E}(U(T); A(T)) = \sup_{T_1 \leq T \leq T_2} \int_{A(T)} \tilde{e}(U(T))\, dX < \infty.$$

Moreover, for $T_1 \leq T \leq T_2$ the Sobolev inequality

$$\|U(T)\|_{L^{30}(A(T))} \leq C_0(\|\nabla U\|_{L^2(A(T))} + \|U\|_{L^6(A(T))}) \leq C\tilde{E}(U(T); A(T))$$

holds with a uniform constant $C_0 > 0$. Since for any $(T_0, X_0) \in A \setminus K^{T_1}$ and any $T \geq T_0$ there holds $A_0(T) \subset A(T)$, we then obtain the uniform bound

$$\sup_{T \geq T_0} \|U(T)\|^5_{L^{30}(A_0(T))} \leq \sup_{T \geq T_1} \|U(T)\|^5_{L^{30}(A(T))} =: C_1 < \infty \ ,$$

as desired. □

For the statement of the second lemma we introduce the characteristic coordinates

$$\xi = t + r, \quad \eta = t - r \ .$$

Fixing

$$d = 1/T_1, \tag{15}$$

then for any $\xi_1 \geq 2t_0$ we let

$$\Gamma_1(\xi_1) = \{(t, x) \in K;\ \xi = t + |x| = \xi_1,\ 0 < \eta = t - |x| < d\} \ . \tag{16}$$

Lemma 4.2. *There exists a constant $C_2 > 0$ such that for any $\xi \geq 2t_0$ there holds*

$$\int_{\Gamma_1(\xi)} e^{6u^2} do \leq C_2 \xi \ . \tag{17}$$

The bound (17) is analoguous to the bound on p. 1822, l. 3-4, in [9], proof of Lemma 4.3. In the present context, however, we need to show (17) in the asymptotic regime when $\xi \to \infty$ instead of $\xi \downarrow 0$, as in [9]. For completeness, we therefore give the proof of (17) in detail in the following section.

Proof of Proposition 3.2 (completed). Recall that there exists a compact set D_0 such that for any $(T_0, X_0) \in A \setminus K^{T_1}$ there holds $\Phi(A_0) \subset D_0 \cup_{\xi \geq 2t_0} \Gamma_1(\xi)$. By Lemma 4.2 and using (11) as well as the bound $t \leq \xi \leq 2t$ then we can estimate

$$\int_{A_0} e^{6\Omega U^2} dX\, dT = \int_{\Phi(A_0)} e^{6u^2}\Omega^3 dx\, dt \leq C \int_{\Phi(A_0)} t^{-3} e^{6u^2} dx\, dt$$
$$\leq C + C \sup_{\xi \geq 2t_0} \Big(\xi^{-1} \int_{\Gamma_1(\xi)} e^{6u^2} do\Big) \int_{t_0}^{\infty} \frac{dt}{t^2} \leq C,$$

uniformly in $(T_0, X_0) \in A \setminus K^{T_1}$. From (13) and Lemma 4.1 we hence obtain the uniform bound $|U_N(T_0, X_0)| \leq C$. Together with (12) this completes the proof of Proposition 3.2. □

5. Proof of Lemma 4.2

The proof of Lemma 4.2 relies on the following multiplier estimates analogous to the ones in [9]. Observe that in polar coordinates (r, ϕ) the conservation law (7) may be written in the form

$$\partial_t(re) - \partial_r(rm) = r^{-1}\partial_\phi(u_t u_\phi)\ , \tag{18}$$

where now

$$e = e(u) = \frac{1}{2}(u_t^2 + u_r^2 + r^{-2}u_\phi^2 + F(u)),\ m = m(u) = u_t u_r\ .$$

We also have the following analogues of the identities (15) and (16) in [9]. Multiplying (1) by $x \cdot \nabla u$ we obtain

$$0 = \frac{d}{dt}\big(u_t\, x \cdot \nabla u\big) - div\big(\nabla u\, x \cdot \nabla u - \frac{x}{2}(|\nabla u|^2 - |u_t|^2 + F(u))\big) + |u_t|^2 - F(u)\ ,$$

which in polar coordinates and with the notation

$$q = q(u) = r^{-2}u_\phi^2 + F(u)\ .$$

reads

$$\partial_t(r^2 m) - \partial_r(r^2(e - q)) + r(u_t^2 - F(u)) = \partial_\phi(u_r u_\phi)\ . \tag{19}$$

Finally, when we multiply (1) by u we find

$$0 = \frac{d}{dt}\big(u_t u)\big) - div\big(u\nabla u\big) + |\nabla u|^2 - |u_t|^2 + u^2(e^{u^2} - 1 - u^2)\ ,$$

that is,

$$\partial_t\big(ru_t u\big) - \partial_r\big(ru_r u\big) + r\big(|\nabla u|^2 - |u_t|^2 + u^2(e^{u^2} - 1 - u^2)\big) = r^{-1}\partial_\phi\big(uu_\phi\big)\ . \tag{20}$$

From these identities we deduce the conservation laws

$$\begin{aligned}&\partial_t\big(\frac{r^2}{t}(e+m+u_t\frac{u}{2r}+\frac{|u|^2}{4rt})\big)-\partial_r\big(\frac{r^2}{t}(e-q+m+u_r\frac{u}{2r})\big)\\&\quad+\frac{r}{t}\big((1+\frac{r}{t})(e+m)+\frac{|u|^2}{2t^2}+\frac{u^2}{2}(e^{u^2}-1-u^2)-\frac{3}{2}F(u)\big)\\&=t^{-1}\partial_\phi\big((u_r+u_t+\frac{u}{2r})u_\phi\big)\end{aligned}\tag{21}$$

and

$$\begin{aligned}&\partial_t\big(\frac{r^2}{t}(m-e+u_t\frac{u}{2r}+\frac{|u|^2}{4rt})\big)-\partial_r\big(\frac{r^2}{t}(e-q-m+u_r\frac{u}{2r})\big)\\&\quad+\frac{r}{t}\big((1-\frac{r}{t})(e-m)+\frac{|u|^2}{2t^2}+\frac{u^2}{2}(e^{u^2}-1-u^2)-\frac{3}{2}F(u)\big)\\&=t^{-1}\partial_\phi\big((u_r-u_t+\frac{u}{2r})u_\phi\big)\ .\end{aligned}\tag{22}$$

similar to equations (20) and (21) in [9].

It is now crucial to observe that

$$u^2(e^{u^2}-1-u^2)-3F(u)=\sum_{k=4}^{\infty}(k-3)\frac{|u|^{2k}}{k!}\geq 0\ .\tag{23}$$

We then obtain the following analogue of [9], Lemma 3.1.

Lemma 5.1. *With a constant $C>0$ we have*

$$\int_{K_{t_0}}\Big((1\pm\frac{r}{t})(e\pm m)+\frac{|u|^2}{2t^2}\Big)\frac{dx\ dt}{t}\leq CE(u(0))\ .$$

Proof. Integrating the identities (21), (22), respectively, over a truncated cone K_S^T for $t_0\leq S\leq T<\infty$ we obtain

$$I_\pm:=\int_{K_S^T}\Big((1\pm\frac{r}{t})(e\pm m)+\frac{|u|^2}{2t^2}\Big)\frac{dx\ dt}{t}\leq II_\pm+III_\pm+VI+V,$$

with $II_\pm$, $III_\pm$, and IV corresponding to the top, lower, and lateral boundary terms, and with 'error' term

$$V=-\frac{1}{2}\int_{K_S^T}\big(u^2(e^{u^2}-1-u^2)-3F(u)\big)\frac{dx\ dt}{t}\leq 0$$

on account of (23). As in [9] we can bound $|II_\pm|+|III_\pm|\leq CE_0$ with a uniform constant independent of S and T. Moreover, since $supp(u)\subset K_{t_0}(a,0)$ the contribution IV from the lateral boundary vanishes. Setting $S=t_0$ and letting $T\uparrow\infty$, we then obtain the claim. □

In particular, given $0<\varepsilon<1$ we can find $t_\varepsilon>t_0$ so that

$$\int_{K_{t_\varepsilon}}\Big((1\pm\frac{r}{t})(e\pm m)+\frac{|u|^2}{2t^2}\Big)\frac{dx\ dt}{t}\leq\varepsilon\ .\tag{24}$$

Given $\xi_0\geq 2t_\varepsilon$, we set $\eta_0=\xi_0/8$. After increasing t_ε, if necessary, we may assume that $\eta_0\geq 2d$, as defined in (15). For $\xi_1\in[\xi_0,4\xi_0]$ then we let

$$\Gamma_0(\xi_1)=\{(t,x)\in K^{T_\varepsilon};\ \xi=t+|x|=\xi_1,\ \eta=t-|x|<\eta_0\}\supset\Gamma_1(\xi_1),$$

as defined in (16), and set

$$Q(\xi_1) := \int_{\Gamma_0(\xi_1)} \big(q + \frac{|u|^2}{t^2}\big) do \ .$$

Note that

$$\frac{r}{t} = \frac{\xi - \eta}{\xi + \eta} = 1 - \frac{2\eta}{\xi + \eta} \geq 1 - \frac{2\eta_0}{\xi_0} = \frac{3}{4} \tag{25}$$

for any $(t, x) \in \Gamma_0(\xi_1)$ whenever $\xi_1 \geq \xi_0 \geq 2t_\varepsilon$.

Finally, setting

$$\Gamma(\xi_1) = \{(t, x) \in K;\ \xi = t + |x| = \xi_1\},$$

and integrating (7) over the region

$$\{(t, x) \in K_{t_0};\ \xi = t + |x| \leq \xi_1\} \ ,$$

for any $\xi_1 \geq 2t_\varepsilon$ we also obtain the bound

$$2 \int_{\Gamma(\xi_1)} u_\eta^2 \, do \leq \int_{\Gamma(\xi_1)} (e - m) \, do \leq E(u(0), B_{t_0}(0)) =: E_0 \tag{26}$$

as a useful variant of the energy inequality (9).

Now note that $t \geq t_\varepsilon$ for any $(t, x) \in K$ with $\xi = t + |x| \geq 2t_\varepsilon$. Changing variables $(t, x) \mapsto (\xi = t + |x|, x)$, then from (24) we see that for any $\xi_0 \geq 2t_\varepsilon$ with an absolute constant C there holds

$$\inf_{\xi_0 < \xi < 2\xi_0} Q(\xi) \leq \xi_0^{-1} \int_{\xi_0}^{2\xi_0} Q(\xi) d\xi \leq C \int_{K_{t_\varepsilon}} \big((1 + \frac{r}{t})(e + m) + \frac{|u|^2}{2t^2}\big) \frac{dx\ dt}{t} < C\varepsilon \ .$$

Thus, given $\xi_0 \geq 2t_\varepsilon$ we can choose a number $\xi_1 \in [\xi_0, 2\xi_0]$ such that

$$Q(\xi_1) \leq 2 \inf_{\xi_0 < \xi < 2\xi_0} Q(\xi) < C\varepsilon \ . \tag{27}$$

In fact, we can show uniform smallness of Q, similar to [9], Lemma 4.1.

Lemma 5.2. *There is a constant $C > 0$ such that for any $\xi_0 \geq 2t_\varepsilon$ there holds*

$$\sup_{2\xi_0 < \xi < 4\xi_0} Q(\xi) \leq C\sqrt{\varepsilon} \ .$$

Proof. Given any $\xi_0 \geq 2t_\varepsilon$ we determine $\xi_1 \in [\xi_0, 2\xi_0]$ as above satisfying (27). For any $\xi_2 \in [2\xi_0, 4\xi_0]$ consider the set

$$R = R(\xi_1, \xi_2) = \{(t, x) \in K^{T_\varepsilon};\ \xi_1 < \xi < \xi_2,\ 0 < \eta < \eta_0\}$$

with boundary $\partial R = \cup_{i=1}^4 \Gamma_i$, where

$$\Gamma_1 = \{(t, x);\ \xi_1 < \xi < \xi_2,\ \eta = 0\}, \quad \Gamma_2 = \Gamma_0(\xi_2),$$
$$\Gamma_3 = \{(t, x);\ \xi_1 < \xi < \xi_2,\ \eta = \eta_0\},\ \Gamma_4 = \Gamma_0(\xi_1).$$

as in the proof of Lemma 4.1 in [9]. Integrating the relation (21) over R, we find the identity

$$A_0 + A_2 + A_3 = A_1 + A_4 + V \ , \tag{28}$$

where

$$A_0 = \int_R \big((1 + \frac{r}{t})(e + m) + \frac{|u|^2}{2t^2}\big) \frac{dx\ dt}{t} \geq 2 \int_R u_\xi^2 \frac{dx\ dt}{t} \geq 0 \tag{29}$$

satisfies $A_0 \le \varepsilon$ in view of (24), and where the terms A_i, $1 \le i \le 4$ correspond to integrals over the boundary components Γ_i, $1 \le i \le 4$. As in the proof of Lemma 5.1, again the remainder

$$V = -\frac{1}{2}\int_R \big(u^2(e^{u^2} - 1 - u^2) - 3F(u)\big)\frac{dx\, dt}{t} \le 0$$

on account of (23). Since $supp(u) \subset K_{t_0}(a, 0)$, we also have $A_1 = 0$. Moreover, using (26) and (27) we can bound $|A_4| \le C\sqrt{\varepsilon}$ as in [9]. Finally, we find

$$A_3 = \int_{\Gamma_3} \big(4\frac{r}{t}u_\xi^2 + u_\xi\frac{u}{t} + \frac{|u|^2}{4t^2})\big)\, do \ge \int_{\Gamma_3} \big(u_\xi^2 + \frac{|u|^2}{8t^2}\big)\, do \ge 0\ ,$$

as we can see from writing

$$\begin{aligned}\partial_t\big(&\frac{r^2}{t}(e + m + u_t\frac{u}{2r} + \frac{|u|^2}{4rt})\big) - \partial_r\big(\frac{r^2}{t}(e - q + m + u_r\frac{u}{2r})\big)\\ &= \partial_\eta\big(\frac{r^2}{t}(2(e+m) - q + u_\xi\frac{u}{r} + \frac{|u|^2}{4rt})\big) + \partial_\xi\big(\frac{r^2}{t}(q + u_\eta\frac{u}{r} + \frac{|u|^2}{4rt})\big)\end{aligned}$$

in characteristic coordinates and estimating as in [9], formula (29), using (25).

Thus we conclude the bound

$$A_2 = \int_{\Gamma_0(\xi_2)} \big(\frac{r}{t}q + u_\eta\frac{u}{t} + \frac{|u|^2}{4t^2}\big)\, do \le C\sqrt{\varepsilon}\ .$$

But as in [9], formula (35), by using (29) we can estimate

$$\int_{\Gamma_0(\xi_2)} \frac{|u|^2}{t^2}\, do \le C\int_{\Gamma_0(\xi_1)} \frac{|u|^2}{t^2}\, do + C\int_R |u_\xi|^2\frac{dx\, dt}{t} \le CQ(\xi_1) + CA_0 \le C\varepsilon.$$

Hence together with (26) and (25) we may conclude as in [9] that

$$A_2 \ge \frac{3}{4}Q(\xi_2) - C\sqrt{\varepsilon}\ ,$$

and $Q(\xi_2) \le C\sqrt{\varepsilon}$. Since $\xi_0 \ge 2t_\varepsilon$ and $2\xi_0 \le \xi_2 \le 4\xi_0$ were arbitrary, the claim follows. □

Proof of Lemma 4.2, completed. Given $\xi_0 \ge 2t_\varepsilon$, let $0 \le \varphi_0 = \varphi_0(\eta) \le 1$ be a smooth cut-off function such that $\varphi_0(\eta) = 1$ for $|\eta| \le \eta_0/2$ and $\varphi_0(\eta) = 0$ for $|\eta| \ge \eta_0$, with $|\varphi_0'| \le 4/\eta_0$, where $\eta_0 = \xi_0/8$. Also fix a smooth cut-off function $0 \le \chi = \chi(\phi) \le 1$ satisfying $\chi(\phi) = 1$ for $|\phi| \le \pi/8$ and $\chi(\phi) = 0$ for $|\phi| \ge \pi/4$. After extending $u(\xi, \eta, \phi) = u(\xi, -\eta, \phi)$ for $\eta < 0$, for $2\xi_0 \le \xi \le 4\xi_0$ also let

$$u_k = u_k(\xi, \eta, \phi) = \varphi_0(\eta)\chi(\phi - k\pi/4)u,\ 1 \le k \le 8.$$

Note that for any fixed $\xi \in [2\xi_0, 4\xi_0]$ we have

$$u_k(\xi, \cdot, \cdot) \in H_0^1([-\eta_0, \eta_0] \times [(k-1)\pi/4, (k+1)\pi/4]),$$

and by (26) and Lemma 5.2 there holds

$$\begin{aligned}\int_{-\eta_0}^{\eta_0}\int_{(k-1)\pi/4}^{(k+1)\pi/4} &\big(|\partial_\eta u_k|^2 + r^{-2}|\partial_\phi u_k|^2\big)r\, d\phi\, d\eta\\ &\le C\int_{\Gamma_0(\xi)} \big(e - m + \frac{|u|^2}{t^2}\big)\, do \le C(1 + E_0),\ 1 \le k \le 8\ .\end{aligned}$$

In addition, Lemma 5.2 yields the bound

$$\int_{-\eta_0}^{\eta_0}\int_{(k-1)\pi/4}^{(k+1)\pi/4}\big(r^{-2}|\partial_\phi u_k|^2\big)r\,d\phi\,d\eta \le C\int_{\Gamma_0(\xi)}\big(q+\frac{|u|^2}{t^2}\big)\,do = CQ(\xi)\le C\sqrt{\varepsilon}\;,$$

uniformly in $\xi\in[2\xi_0,4\xi_0]$ for any $\xi_0\ge 2t_\varepsilon$, $1\le k\le 8$. After scaling

$$v_k(\xi,\cdot)=v_k(\xi,\eta,\phi)=u_k(\xi,\lambda\eta,\lambda\phi),$$

where $\lambda=\sqrt{\pi\eta_0}$, and applying the improved Moser-Trudinger inequality in [9], Lemma 4.2, to the functions $v_k(\xi,\cdot)$ defined on a domain Ω_k of unit area, with

$$\int_{\Omega_k}\big(\xi_0|\partial_\eta v_k|^2+\xi_0^{-1}|\partial_\phi v_k|^2\big)d\phi\,d\eta\le C(1+E_0)$$

and with

$$\int_{\Omega_k}\xi_0^{-1}|\partial_\phi v_k|^2 d\phi\,d\eta\le C\sqrt{\varepsilon}\;,$$

for sufficiently small $\varepsilon>0$ we deduce the uniform bound

$$\sup_{2\xi_0\le\xi\le 4\xi_0}\Big(\xi_0^{-1}\int_{\Gamma_0(\xi)}e^{6u_k^2}do\Big)=C\sup_{2\xi_0\le\xi\le 4\xi_0}\int_{\Omega_k}e^{6v_k^2}do\le C$$

for each k. Since $d\le\eta_0/2$ we can pointwise estimate $u^2\le\max_{1\le k\le 8}u_k^2$ on $\Gamma_1(\xi)$. Hence we can bound

$$\xi^{-1}\int_{\Gamma_1(\xi)}e^{6u^2}do\le\sum_{1\le k\le 8}\Big(\xi^{-1}\int_{\Gamma_0(\xi)}e^{6u_k^2}do\Big)\le C \tag{30}$$

uniformly in $\xi\in[2\xi_0,4\xi_0]$ for any $\xi_0\ge 2t_\varepsilon$, with a constant $C>0$ independent of ξ_0, and therefore for any $\xi\ge 4t_\varepsilon$. Since u is smooth and therefore is locally bounded, (30) also holds for all remaining $\xi\ge 2t_0$, possibly with a larger constant $C>0$. The claim follows. □

References

[1] M. G. Grillakis: *Regularity and asymptotic behaviour of the wave equation with a critical nonlinearity*, Ann. of Math. (2) 132 (1990), no. 3, 485-509.

[2] S. Ibrahim, R. Jrad: *Strichartz type estimates and the wellposedness of an energy critical 2d wave equation in a bounded domain,* J. Differential Equations 250 (2011), 3740–3771.

[3] S. Ibrahim, M. Majdoub, N. Masmoudi: *Global solutions for a semilinear, two-dimensional Klein-Gordon equation with exponential-type nonlinearity*, Comm. Pure Appl. Math. 59 (2006), no. 11, 1639–1658.

[4] S. Ibrahim, M. Majdoub, N. Masmoudi: *Well and ill-posedness issues for energy supercritical waves*, preprint (2009).

[5] S. Ibrahim, M. Majdoub, N. Masmoudi, K. Nakanishi: *Scattering for the two-dimensional energy-critical wave equation*, Duke Math. J. 150.2 (2009), 287–329.

[6] J. Moser: *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** (1971), 1077–1092.

[7] M. Sack: *Scattering for a massless critical nonlinear wave equation in* 2 *space dimensions*, Analysis and PDE (2014).

[8] J. Shatah, M. Struwe: *Geometric wave equations*, Courant Lecture Notes 2, New York University (1998), 2nd edition: AMS (2000).

[9] M. Struwe: *The critical nonlinear wave equation in 2 space dimensions*, J. European Math. Soc. 15 (2013), 1805-1823.
[10] N.S. Trudinger: *On embeddings into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), 473–484.

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