SCATTERING FOR A CRITICAL NONLINEAR WAVE EQUATION IN TWO DIMENSIONS

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Abstract. We show that the solutions to the Cauchy problem for a wave equation with critical exponential nonlinearity in 2 space dimensions scatter for arbitrary smooth, compactly supported initial data.

1. Introduction

Consider the initial value problem for the equation

\( u_{tt} - \Delta u + u(e^{u^2} - 1 - u^2) = 0 \) on \( \mathbb{R} \times \mathbb{R}^2 \).

with smooth Cauchy data

\( (u, u_t)_{t=0} = (u_0, u_1) \in C^\infty_c(\mathbb{R}^2) \).

Observe that for a classical solution \( u \) of (1), (2) the energy

\( E(u(t)) = \frac{1}{2} \int_{\{t\} \times \mathbb{R}^2} \left( |u_t|^2 + |\nabla u|^2 + F(u) \right) dx \)

is conserved, where \( F(u) = e^{u^2} - 1 - u^2 - u^4/2 \) is a primitive of the nonlinear term \( f(u) = u(e^{u^2} - u^2) \).

For the related problem when \( f(u) \) is replaced by the nonlinearity \( n(u) = ue^{u^2} \) Ibrahim, Majdoub, and Masmoudi in [3] showed that whenever the corresponding initial energy is at most \( 2\pi \) the Cauchy problem (1), (2) admits a global smooth solution. Together with Nakanishi, in [5] the same authors also showed that when \( f(u) \) is replaced by \( l(u) = u(e^{u^2} - u^2) \) the solution scatters, again assuming the associated initial energy to be bounded by \( 2\pi \). The constant \( 2\pi \) is related to the best constant in the Moser-Trudinger inequality [4], [10], which defines the limit case of Sobolev’s embedding of the space \( H^1(\mathbb{R}^2) \). It was conjectured in [5] that this number also marks an energy threshold for the onset of “super-critical” behavior in (1) and its variants. This conjecture was partially confirmed through the examples given in [4], showing that the solutions no longer depend in a locally uniformly continuous fashion on the data when the initial energy exceeds the value \( 2\pi \).

In contrast with these expectations, however, Struwe [9] showed that the initial value problem for equation (1) has a global smooth solution for smooth Cauchy data \( (u_0, u_1) \) with arbitrarily large energy. This result was originally demonstrated...
when \( f(u) \) is replaced by the nonlinearity \( n(u) = ue^{u^2} \) but the proof is valid also for all the above variants of equation (1).

Moreover, by building on the techniques developed in [9], Sack [7] was able to show scattering for any solution \( u \) of (1), (2) for arbitrarily large smooth, compactly supported data with rotational symmetry. Here, by definition, a solution \( u \) to (1) scatters if for the solution \( v \) to the homogeneous linear wave equation

\[
v_{tt} - \Delta v = 0 \quad \text{on } \mathbb{R} \times \mathbb{R}^2
\]

for suitable “scattering data”

\[
(v, v_t)_{t=0} = (v_0, v_1) \in \dot{H}^1 \times L^2(\mathbb{R}^2)
\]

there holds

\[
\|D\!u(t) - D\!v(t)\|_{L^2(\mathbb{R}^2)} \to 0 \quad \text{as } t \to \infty,
\]

where \( D\!u = (u_t, \nabla u) \) is the space-time differential of \( u \).

Combining the insights of [7] and [9], in the present paper we now establish scattering in the general (non-symmetric) case.

**Theorem 1.1.** For any \( u_0, u_1 \in C_c^\infty(\mathbb{R}^2) \) there exist \( (v_0, v_1) \in \dot{H}^1 \times L^2(\mathbb{R}^2) \) such the solution \( u \) to (1), (2) scatters to the solution \( v \) of (4), (5) in the sense of (6).

For the proof of Theorem 1.1 as in [7] it suffices to show finiteness of the scattering norm

\[
\|u_{tt} - \Delta u\|_{L^1_{t,2}} = \|f(u)\|_{L^1_{t,2}} = \int_0^\infty \|f(u(t))\|_{L^2(\mathbb{R}^2)} dt
\]

of the solution \( u \) to (1), (2) for given data. In [7] this already was partially achieved by applying the techniques of [9] to the function \( U \) obtained from \( u \) through conformal inversion, which satisfies an equation similar to (1). Conformal inversion also is a key element in the proof of Theorem 1.1 in the present paper, and we crucially exploit the fact that the wave operator and nonlinear terms of degree 5 and higher are well-behaved under this transformation. Even though our proof therefore cannot be extended to the case when \( f(u) \) is replaced by the nonlinearity \( l(u) = u(e^{u^2} - u^2) \), it is to be expected that the analogue of Theorem 1.1 also holds in this case, since scattering properties should only improve in the presence of a mass term. However, it is not clear if scattering holds when \( f(u) \) is replaced by the nonlinearity \( n(u) = ue^{u^2} \) since the cubic term seems difficult to treat even in the small energy regime.

Note that also when \( f(u) \) is replaced by either \( l(u) \) or \( n(u) \), by [9] the solutions to the Cauchy problem (1), (2) for smooth data always are globally regular.

2. Preliminaries

2.1. Energy identity. Multiplying (1) by \( u_t \) we obtain the identity

\[
0 = \frac{d}{dt} e(u) - div(\nabla u \cdot u_t)
\]

for the energy density

\[
e(u) = \frac{1}{2} (|u_t|^2 + |\nabla u|^2 + F(u)).
\]
and the density of momentum \( m(u) = \nabla u \cdot u_t \).

If for \( 0 < S \leq T \leq T_0 \) we denote as \( v(y) = u(|y|, y) \) the restriction of \( u \) to the lateral boundary
\[
M^T_S = \{ z = (t, x); S \leq t \leq T, |x| = t \}
\]
of the truncated forward light cone
\[
K^T_S = \{ z = (t, x); S \leq t \leq T, |x| \leq t \}
\]
with vertex at \((0, 0)\), then upon integrating (7) over \( K^T_S \) we find the identity
\[
\tag{8}
E(u(S), B_S(0)) + \text{Flux}(u, M^T_S) = E(u(T), B_T(0))
\]
for all \( 0 < S < T \leq T_0 \), where
\[
E(u(t), B_R(x_0)) := \int_{B_R(x_0)} e(u(t)) dx
\]
and where
\[
\text{Flux}(u, M^T_S) := \frac{1}{2} \int_{B_T \setminus B_S(0)} (\langle \nabla v \rangle^2 + F(v)) dy
\]
is the energy flux through \( M^T_S \), as in [9]. In particular, energy will spread with speed at most 1. Identities similar to (8) hold on any region with space-like or null boundary. For \( 0 \leq S \leq T \) we also let \( K^T = K^T_0 \), \( K_S = K^T_0 \), and so on.

2.2. Normalisation. Given data \( u_0, u_1 \in C^\infty_c(\mathbb{R}^2) \) with support in \( B_a(0) \) for some \( a > 0 \), by [5] the solution \( u \) to the Cauchy problem [1], [2] in forward time is supported inside the light cone with vertex at \((-a, 0)\). We may assume that \( a \geq 1/2 \). Shifting time by \( t_0 = 2a \geq 1 \) we then may assume that \( u \) in forward time is supported inside the cone
\[
K_{t_0}(a, 0) = \{ (t, x); |x| \leq t - a, t \geq t_0 \} \subset K = \{ (t, x); |x| \leq t \},
\]
with initial data
\[
(u, u_t)|_{t=t_0} = (u_0, u_1) \in C^\infty_c(\mathbb{R}^2)
\]
and with lateral trace
\[
u = 0 \text{ on } M_{t_0}(a, 0),
\]
where \( M_{t_0}(a, 0) \) is the lateral boundary of the truncated cone \( K_{t_0}(a, 0) \).

2.3. Conformal inversion. Following Grillakis [1] we let \( \Phi: K \to K \) denote the conformal inversion given by \( \Phi(t, x) = (T, X) \) with
\[
T := \frac{t}{t^2 - r^2}, \quad X := \frac{x}{t^2 - r^2},
\]
where \( r = |x| \). Note that \( \Phi \) is an involution with inverse \( \Phi^{-1} = \Phi \), and that \( \Phi \) maps light cones to light cones. Moreover, letting \( \eta = \text{diag}(-1, 1, 1) \) be the Minkowski metric, we have \( \Phi^*\eta = \Omega^2\eta \), where
\[
\Omega = \frac{1}{t^2 - r^2} = T^2 - R^2 \text{ on } K
\]
with \( R = |X| \). Finally, as explained in [7], given a solution \( u \) of (8) with support in \( K_{t_0}(a, 0) \subset K \), the function \( U = \Omega^{-1/2}(u \circ \Phi) \) solves the equation
\[
\tag{9}
U_{TT} - \Delta U = -\Omega^{-2}U(e^{\Omega u/2} - 1 - \Omega U^2) =: -g(U).
\]
Recalling that \( t_0 = 2a \), we observe that the support of \( U \) is contained in the region

\[
A := \Phi(K_{t_0}(a, 0)) = \{(T, X) \in K; T/(T^2 - R^2) \geq t_0, \ T + R \leq 2/t_0\}
\]

bounded within \( K \) by a section of the hyperboloid

\[
\Sigma = \{(T, X) \in K; (T - \frac{1}{2t_0})^2 = R^2 + \frac{1}{4t_0^2}\}
\]

and the lateral boundary of an incoming light cone with vertex at the point \((2/t_0, 0)\), which meets \( \Sigma \) in the circle \( \{(T, X) \in K; T = 2R = 4/(3t_0)\} \).

2.4. **Bounds for conformal energy.** Similar to (7), upon multiplying (9) with \( U_T \) we find the conservation law

\[
\frac{d}{dT} \tilde{e}(U) - \text{div}(\tilde{m}(U)) = TP(U) \geq 0
\]

for the conformal energy density

\[
\tilde{e}(U) = \frac{1}{2}(|U_T|^2 + |\nabla U|^2 + G(U))
\]

with

\[
G(U) = \Omega^{-3}(e^\Omega U^2 - 1 - \Omega U^2 - \Omega^2 U^4/2) \geq U^6/6,
\]

and with the density of momentum \( \tilde{m}(U) = \nabla U \cdot U_T \). The fact that the lowest power term in \( f(u) \) is of order \( u^5 \) is crucial for the positivity of the term

\[
P(U) = \Omega^{-3}U^2(e^\Omega U^2 - 1 - \Omega U^2) - 3\Omega^{-1}G(U) = \Omega^{-4} \sum_{k=4}^{\infty} (k-3) (\Omega U^2)^k / k!.
\]

We have thus set the stage for the proof of Theorem 1.1.

3. **Proof of Theorem 1.1**

Recall the following result of Sack [7], similar to [9], Lemma 4.3.

**Proposition 3.1.** There exists \( T_1 > 0 \) and a constant \( C_1 > 0 \) such that

\[
\int_{K_{T_1}} e^{4U^2} dX \ dT \leq C_1.
\]

Proposition 3.1 allows to partially bound the scattering norm, as follows. Clearly we may assume that \( T_1 \leq 1/(2t_0) \leq 1/2 \). Set

\[
D := \Phi^{-1}(K_{T_1}) = \{(t, x) \in K; t/(t^2 - r^2) \leq T_1\} \subset K,
\]

and for any \( t_0 \leq t_1 \leq t_2 \) let

\[
D(t_1) = \{(t, x) \in D; t = t_1\}
\]

as well as

\[
D_{t_1}^{t_2} = \{(t, x) \in D; t_1 \leq t \leq t_2\}.
\]

Note that \( D \) is the part of \( K \) above the hyperboloid

\[
H = \{(t, x) \in K; (t - \frac{1}{2T_1})^2 = r^2 + \frac{1}{4T_1^2}\};
\]

therefore \( D \) together with the thickened mantle region

\[
K_{t_0}(a, 0) \setminus K(d, 0) = \{(t, x) \in K_{t_0}; t - d \leq |x| \leq t - a\}
\]
Proposition 3.2. There exists a constant $H$ such that
\[ e^{|u|^2} - 1 - |\Omega|^2 \leq \Omega^2 e^{|u|^2}, \]
we can bound
\[
\int_{D_t^{t_1}} |f(u)|^2 \, dx \, dt = \int_{D_t^{t_2}} |u(e^{v^2} - 1 - u^2)|^2 \, dx \, dt
\]
\[
= \int_{\Phi(D_t^{t_1})} |U|^2 e^{|u|^2} - 1 - |\Omega|^2 |^2 \Omega^{-3} \, dX \, dT \leq \int_{\Phi(D_t^{t_1})} \Omega^2 U^{10} e^{2|u|^2} \, dX \, dT.
\]
But since $T_1 \leq 1$, we have that $\Omega \leq 1$ on $\Phi(D_t^{t_1}) \subset K^{T_1}$, and
\[
U^{10} e^{2|u|^2} \leq C e^{2|u|^2}
\]
on this region. On the other hand, using that we have $|x| \leq t - a$ throughout the support of $u$, we can also bound
\[
(11) \quad \Omega = \frac{1}{(t - |x|)(t + |x|)} \leq \frac{1}{at}
\]
on the support of $u$. Hence by Proposition 3.1 we find that
\[
\int_{D_t^{t_1}} |f(u)|^2 \, dx \, dt \leq C t_1^{-2} \int_{K_t^{T_1}} e^{2|u|^2} \, dX \, dT \leq C t_1^{-2}.
\]
After a dyadic decomposition $t_k = 2^k t_0$, $k \in \mathbb{N}$, of the time interval $[t_0, \infty[$, from Hölder’s inequality we then obtain
\[
\int_{t_0}^{\infty} \left( \int_{D(t)} |f(u)|^2 \, dx \right)^{1/2} \, dt = \sum_{k \in \mathbb{N}_0} \int_{t_k}^{t_{k+1}} \left( \int_{D(t)} |f(u)|^2 \, dx \right)^{1/2} \, dt
\]
\[
\leq \sum_{k \in \mathbb{N}_0} (2^k)^{1/2} \left( \int_{D(t_k)} |f(u)|^2 \, dx \right)^{1/2} \leq C \sum_{k \in \mathbb{N}_0} 2^{-k/2} < \infty.
\]
To complement this estimate, we show the following bound.

Proposition 3.2. There exists a constant $C_2 > 0$ such that
\[
\sup_{\text{supp}(u) \setminus K^{T_1}} |u| \leq C_2.
\]
Postponing the proof of Proposition 3.2, we complete the proof of Theorem 1.1.

From Proposition 3.2 and (11), we conclude that
\[
|u(t, x)| = |\Omega|^{1/2} |U(T, X)| \leq C_2 a^{-1/2} \sqrt{t}
\]
for all $(t, x) \in K(a, 0) \setminus D$, with $(T, X) = \Phi(t, x)$. Hence we have the bound
\[
|f(u)|^2 = |u(e^{v^2} - 1 - u^2)|^2 \leq u^{10} e^{2|u|^2} \leq Ct^{-5}
\]
avy D and there holds
\[
\int_{t_0}^{\infty} \left( \int_{B(t) \setminus D(t)} |f(u)|^2 \, dx \right)^{1/2} \, dt
\]
\[
\leq \int_{t_0}^{\infty} \left( \int_{B(t) \setminus D(t)} u^{10} e^{2|u|^2} \, dx \right)^{1/2} \, dt \leq C \int_{t_0}^{\infty} t^{-3/2} \, dt < \infty.
\]
Splitting $f(u) = f(u)\chi_D + f(u)\chi_{(K_{t_0}\setminus D)}$, where $\chi_M$ is the characteristic function of a set $M$, with the Minkowski inequality we now obtain that

$$\|f(u)\|_{L^1_t L^2_x} \leq \|f(u)\chi_D\|_{L^1_t L^2_x} + \|f(u)\chi_{(K_{t_0}\setminus D)}\|_{L^1_t L^2_x} < \infty,$$

as claimed.

4. PROOF OF PROPOSITION 3.2

Recall that $\text{supp}(U) \subset A = \Phi(K_{t_0}(a,0))$. Fix a point $(T_0, X_0) \in A \setminus K^{T_1}$. By Duhamel’s formula we can decompose

$$U(T_0, X_0) = U_L(T_0, X_0) + U_N(T_0, X_0),$$

where $U_L$ solves the free wave equation \([4]\) and where $V := U_N$ solves the equation $V_{TT} - \Delta V + g(U) = 0$ on $A$ with vanishing Cauchy data.

Clearly we have $U_L = \Omega^{-1/2}(u_L \circ \Phi)$, where $u_L$ is the solution of the free wave equation \([4]\) with Cauchy data \([2]\) at time $t = t_0$. By the known decay estimates for this problem there exists a constant $C = C(u_0, u_1) > 0$ such that

$$\|u_L(t, x)\| \leq C/\sqrt{t}$$

for sufficiently large $t > t_0$; see for instance \([8]\), Lemma 4.2 and (4.9b). Recall that letting $d = 1/T_1$, for sufficiently large $t_1 > t_0$ we have

$$K_{t_1} \setminus D \subset K \setminus K(d, 0) = \{(t, x) \in K; t - d \leq |x| \leq t\}.$$

Hence for $(t, x) \in K \setminus D$ with $t \geq t_1$ we can estimate

$$(\Omega(t, x))^{-1} = (t - |x|)(t + |x|) \leq 2dt.$$

By smoothness of $u_L$ then we can uniformly bound $|(\Omega^{-1/2}u_L)(t, x)| \leq C$ everywhere on $K_{t_0} \setminus D$, and we obtain the bound

$$\sup_{(T_0, X_0) \in A \setminus K^{T_1}} |U_L(T_0, X_0)| \leq C.$$

By Duhamel’s formula, moreover, the component $U_N$ is given by the equation

$$U_N(T_0, X_0) = - \int_{A_0} g(U) \Gamma dX dT,$$

where $A_0 := K(T_0, X_0) \cap A$ and where

$$\Gamma(T, X) = \frac{1}{2\pi \sqrt{|T_0 - T|^2 - |X - X_0|^2}} \in L^{3/2}(A_0)$$

is the fundamental solution to the linear wave equation. Again estimating

$$|g(U)| = |\Omega^{-2}U(e^{\Omega u^2} - 1 - \Omega U^2)| \leq |U|^5 e^{\Omega u^2},$$

with a uniform constant $C > 0$ we can then bound

$$|U_N(T_0, X_0)| \leq \|\Gamma\|_{L^{3/2}(A_0)} \|g(U)\|_{L^3(A_0)}$$

$$\leq C \sup_{T \geq T_0} \|U(T)\|_{L^3(A_0(T))} \|e^{\Omega U^2}\|_{L^6(A_0)},$$

where $A_0(T) = \{(T, X) \in A_0\}$ for any $T \geq T_0$. The desired bound then will be a consequence of the following two lemmas.
Lemma 4.1. With a uniform constant $C_1 > 0$ for any $(T_0, X_0) \in A \setminus K^{T_1}$ there holds
\begin{equation}
\sup_{T \geq T_0} \|U(T)\|_{L^5(\mathcal{A}(T))}^5 \leq C_1.
\end{equation}

Proof. For any $T > 0$ also let $A(T) = \{(T, X) \in A\}$, where $A = \Phi(K_{t_0}(a, 0))$ as above. Set $\delta_0 = 1/(3t_0)$. Note that
\begin{equation}
\Omega(T, X) = (T - R)(T + R) \geq \delta_0 T_1
\end{equation}
is uniformly bounded away from zero at points $(T, X) \in A \setminus K^{T_1}$ with $R \leq T - \delta_0$.
All such points are mapped under $\Phi$ to a fixed bounded region of space-time. Since $u$ is smoothly bounded on any compact region, we see that also $U = \Omega^{-1/2}(u \circ \Phi)$ is smoothly bounded at all such points. Fix
\begin{equation}
T_2 = 7/(6t_0) = 1/t_0 + 1/(6t_0).
\end{equation}
Our choice $\delta_0 = 1/(3t_0)$ guarantees that for all $T \geq T_2$ we have $A(T) \subset K(\delta_0, 0)$; hence we have
\begin{equation}
\sup_{T \geq T_2} \|U(T)\|_{L^5(\mathcal{A}(T))}^5 < \infty.
\end{equation}
Recalling that $U$ vanishes in a neighborhood of the mantle section
\begin{equation}
\{(T, X) \in K; T + R = 2/t_0, T > 1/t_0\},
\end{equation}
from the energy inequality (10) and in view of smoothness of the Cauchy data for $U$ on $\Sigma$ we also have
\begin{equation}
\sup_{T_1 \leq T \leq T_2} \int_{A(T)} \tilde{E}(U(T); A(T)) = \sup_{T_1 \leq T \leq T_2} \int_{A(T)} \tilde{e}(U(T)) dX < \infty.
\end{equation}
Moreover, for $T_1 \leq T \leq T_2$ the Sobolev inequality
\begin{equation}
\|U(T)\|_{L^5(\mathcal{A}(T))} \leq C_0\left(\|\nabla U\|_{L^2(\mathcal{A}(T))} + \|U\|_{L^5(\mathcal{A}(T))}\right) \leq C \tilde{E}(U(T); A(T))
\end{equation}
holds with a uniform constant $C_0 > 0$. Since for any $(T_0, X_0) \in A \setminus K^{T_1}$ and any $T \geq T_0$ there holds $A_0(T) \subset A(T)$, we then obtain the uniform bound
\begin{equation}
\sup_{T \geq T_0} \|U(T)\|_{L^5(\mathcal{A}(T))}^5 \leq \sup_{T \geq T_1} \|U(T)\|_{L^5(\mathcal{A}(T))}^5 =: C_1 < \infty,
\end{equation}
as desired. \hfill \Box

For the statement of the second lemma we introduce the characteristic coordinates
\begin{equation}
\xi = t + r, \quad \eta = t - r.
\end{equation}
Fixing
\begin{equation}
d = 1/T_1,
\end{equation}
then for any $\xi_1 \geq 2t_0$ we let
\begin{equation}
\Gamma_1(\xi_1) = \{(t, x) \in K; \xi = t + |x| = \xi_1, 0 < \eta = t - |x| < d\}.
\end{equation}

Lemma 4.2. There exists a constant $C_2 > 0$ such that for any $\xi \geq 2t_0$ there holds
\begin{equation}
\int_{\Gamma_1(\xi)} e^{\delta u^2} d\omega \leq C_2 \xi.
\end{equation}
The bound (17) is analogous to the bound on p. 1822, l. 3-4, in [9], proof of Lemma 4.3. In the present context, however, we need to show (17) in the asymptotic regime when $\xi \to \infty$ instead of $\xi \downarrow 0$, as in [9]. For completeness, we therefore give the proof of (17) in detail in the following section.

**Proof of Proposition 3.2 (completed).** Recall that there exists a compact set $D_0$ such that for any $(T_0, X_0) \in A \setminus K^{T_1}$ there holds $\Phi(A_0) \subset D_0 \cup \{\xi \geq 2t_0 \}$. By Lemma 4.2 and using (11) as well as the bound $t \leq \xi \leq 2t$ then we can estimate

$$\int_{A_0} e^{6u^2} dX dT = \int_{\Phi(A_0)} e^{6u^2} \Omega^3 dx dt \leq C \int_{\Phi(A_0)} t^{-3} e^{6u^2} dx dt \leq C + C \sup_{\xi \geq 2t_0} \left( \xi^{-1} \int_{\Gamma_1(\xi)} (e^{6u^2} du) \right) \int_{t_0}^{\infty} \frac{dt}{t^2} \leq C,$$

uniformly in $(T_0, X_0) \in A \setminus K^{T_1}$. From (13) and Lemma 4.1 we hence obtain the uniform bound $|U_N(T_0, X_0)| \leq C$. Together with (12) this completes the proof of Proposition 3.2. □

5. PROOF OF LEMMA 4.2

The proof of Lemma 4.2 relies on the following multiplier estimates analogous to the ones in [9]. Observe that in polar coordinates $(r, \phi)$ the conservation law (7) may be written in the form

$$\partial_t (re) - \partial_r (rm) = r^{-1} \partial_\phi (u_t u_\phi) ,$$

where now

$$e = e(u) = \frac{1}{2} (u_t^2 + u_r^2 + r^{-2} u_\phi^2 + F(u)), \quad m = m(u) = u_t u_r .$$

We also have the following analogues of the identities (15) and (16) in [9]. Multiplying (11) by $x \cdot \nabla u$ we obtain

$$0 = \frac{d}{dt} (u \cdot x) - div (\nabla u x) = \frac{1}{2} (|\nabla u|^2 - |u_t|^2 + F(u)) + |u_t|^2 - F(u) ,$$

which in polar coordinates and with the notation

$$q = q(u) = r^{-2} u_\phi^2 + F(u) .$$

reads

$$\partial_t (r^2 m) - \partial_r (r^2 (e - q)) + r (u_t^2 - F(u)) = \partial_\phi (u_t u_\phi) .$$

Finally, when we multiply (11) by $u$ we find

$$0 = \frac{d}{dt} (u_t u) - div (u \nabla u) + |\nabla u|^2 - |u_t|^2 + u^2 (e - 1) ,$$

that is,

$$\partial_t (ru_t u) - \partial_r (ru_r u) + r (|\nabla u|^2 - |u_t|^2 + u^2 (e - 1) ) = r^{-1} \partial_\phi (u u_\phi) .$$
From these identities we deduce the conservation laws
\[
\partial_t \left( \frac{r^2}{t} (e + m + u \frac{u}{2r} + \frac{|u|^2}{4rt}) \right) - \partial_r \left( \frac{r^2}{t} (e - q + m + u \frac{u}{2r}) \right)
\]
\[
+ \frac{r}{t} \left( (1 + \frac{r}{t}) (e + m) + \frac{|u|^2}{2t^2} + \frac{u^2}{2} (e^{u^2} - 1 - u^2) - \frac{3}{2} F(u) \right)
\]
\[
= t^{-1} \partial_\phi \left( (u_r + u_t + \frac{u}{2r}) u_\phi \right)
\]
and
\[
\partial_t \left( \frac{r^2}{t} (m - e + u \frac{u}{2r} + \frac{|u|^2}{4rt}) \right) - \partial_r \left( \frac{r^2}{t} (e - q - m + u \frac{u}{2r}) \right)
\]
\[
+ \frac{r}{t} \left( (1 - \frac{r}{t}) (e - m) + \frac{|u|^2}{2t^2} + \frac{u^2}{2} (e^{u^2} - 1 - u^2) - \frac{3}{2} F(u) \right)
\]
\[
= t^{-1} \partial_\phi \left( (u_r - u_t + \frac{u}{2r}) u_\phi \right).
\]

similar to equations (20) and (21) in [9].

It is now crucial to observe that
\[
u^2 (e^{u^2} - 1 - u^2) - 3F(u) = \sum_{k=4}^{\infty} (k - 3) \frac{|u|^{2k}}{k!} \geq 0.
\]

We then obtain the following analogue of [9], Lemma 3.1.

**Lemma 5.1.** With a constant $C > 0$ we have
\[
\int_{K_{t_0}} \left( (1 \pm \frac{r}{t}) (e \pm m) + \frac{|u|^2}{2t^2} \right) \frac{dx\,dt}{t} \leq CE(u(0)).
\]

**Proof.** Integrating the identities (21), (22), respectively, over a truncated cone $K_S$ for $t_0 \leq S \leq T < \infty$ we obtain
\[
I_\pm := \int_{K_S} \left( (1 \pm \frac{r}{t}) (e \pm m) + \frac{|u|^2}{2t^2} \right) \frac{dx\,dt}{t} \leq II_\pm + III_\pm + VI + V,
\]
with $II_\pm$, $III_\pm$, and $IV$ corresponding to the top, lower, and lateral boundary terms, and with 'error' term
\[
V = -\frac{1}{2} \int_{K_S} (u^2 (e^{u^2} - 1 - u^2) - 3F(u)) \frac{dx\,dt}{t} \leq 0.
\]
on account of (23). As in [9] we can bound $|II_\pm| + |III_\pm| \leq CE_0$ with a uniform constant independent of $S$ and $T$. Moreover, since $\text{supp}(u) \subset K_{t_0}(a, 0)$ the contribution $IV$ from the lateral boundary vanishes. Setting $S = t_0$ and letting $T \uparrow \infty$, we then obtain the claim. \(

In particular, given $0 < \varepsilon < 1$ we can find $t_\varepsilon > t_0$ so that
\[
\int_{K_{t_\varepsilon}} \left( (1 \pm \frac{r}{t}) (e \pm m) + \frac{|u|^2}{2t^2} \right) \frac{dx\,dt}{t} \leq \varepsilon.
\]

Given $\xi_0 \geq 2t_\varepsilon$, we set $\eta_0 = \xi_0/8$. After increasing $t_\varepsilon$, if necessary, we may assume that $\eta_0 \geq 2d$, as defined in [15]. For $\xi_1 \in [\xi_0, 4\xi_0]$ then we let
\[
\Gamma_0(\xi_1) = \{(t, x) \in K^{T_\varepsilon}; \xi = t + |x| = \xi_1, \eta = t - |x| < \eta_0 \} \supset \Gamma_1(\xi_1),
\]
as defined in (10), and set

\[ Q(\xi) := \int_{\Gamma_0(\xi)} \left( q + \frac{|u|^2}{t^2} \right) d\nu. \]

Note that

\[ \frac{r}{t} = \frac{\xi - \eta}{\xi + \eta} = 1 - \frac{2\eta}{\xi + \eta} \geq 1 - \frac{2\eta_0}{\xi_0} = \frac{3}{4} \tag{25} \]

for any \((t, x) \in \Gamma_0(\xi_1)\) whenever \(\xi_1 \geq \xi_0 \geq 2t\epsilon\).

Finally, setting

\[ \Gamma(\xi_1) = \{(t, x) \in K; \xi = t + |x| = \xi_1\}, \]

and integrating over the region

\[ \{(t, x) \in K_{\epsilon}; \xi = t + |x| \leq \xi_1\}, \]

for any \(\xi_1 \geq 2t\epsilon\) we also obtain the bound

\[ 2 \int_{\Gamma(\xi_1)} u_\eta^2 \, d\nu \leq \int_{\Gamma(\xi_1)} (e - m) \, d\nu \leq E(u(0), B_\epsilon(0)) =: E_0 \tag{26} \]

as a useful variant of the energy inequality (9).

Now note that \(t \geq t_\epsilon\) for any \((t, x) \in K\) with \(\xi = t + |x| \geq 2t\epsilon\). Changing variables \((t, x) \rightarrow (\xi = t + |x|, x)\), then from (24) we see that for any \(\xi_0 \geq 2t\epsilon\) with an absolute constant \(C\) there holds

\[ \inf_{\xi_0 < \xi < 2\xi_0} Q(\xi) \leq \xi_0^{-1} \int_{\xi_0}^{2\xi_0} Q(\xi) \, d\xi \leq C \int_{K_{t_\epsilon}} ((1 + \frac{r}{t})(e + m) + \frac{|u|^2}{2t^2}) \, dx \, dt \leq C\epsilon. \]

Thus, given \(\xi_0 \geq 2t\epsilon\) we can choose a number \(\xi_1 \in [\xi_0, 2\xi_0]\) such that

\[ Q(\xi_1) \leq 2 \inf_{\xi_0 < \xi < 2\xi_0} Q(\xi) < C\epsilon. \tag{27} \]

In fact, we can show uniform smallness of \(Q\), similar to (9), Lemma 4.1.

**Lemma 5.2.** There is a constant \(C > 0\) such that for any \(\xi_0 \geq 2t\epsilon\) there holds

\[ \sup_{2\xi_0 < \xi < 4\xi_0} Q(\xi) \leq C\sqrt{\epsilon}. \]

**Proof.** Given any \(\xi_0 \geq 2t\epsilon\) we determine \(\xi_1 \in [\xi_0, 2\xi_0]\) as above satisfying (27).

For any \(\xi_2 \in [2\xi_0, 4\xi_0]\) consider the set

\[ R = R(\xi_1, \xi_2) = \{(t, x) \in K^{\Gamma^*}; \xi_1 < \xi < \xi_2, 0 < \eta < \eta_0\} \]

with boundary \(\partial R = \sqcup_{i=1}^{3} \Gamma_i\), where

\[ \Gamma_1 = \{(t, x); \xi_1 < \xi < \xi_2, \eta = 0\}, \quad \Gamma_2 = \Gamma_0(\xi_2), \]

\[ \Gamma_3 = \{(t, x); \xi_1 < \xi < \xi_2, \eta = \eta_0\}, \quad \Gamma_4 = \Gamma_0(\xi_1). \]

as in the proof of Lemma 4.1 in (9). Integrating the relation (21) over \(R\), we find the identity

\[ A_0 + A_2 + A_3 = A_1 + A_4 + V, \tag{28} \]

where

\[ A_0 = \int_R ((1 + \frac{r}{t})(e + m) + \frac{|u|^2}{2t^2}) \, dx \, dt \geq 2 \int_R u_\xi^2 \, dx \, dt \geq 0 \tag{29} \]
satisfies \( A_0 \leq \varepsilon \) in view of (24), and where the terms \( A_i, 1 \leq i \leq 4 \) correspond to integrals over the boundary components \( \Gamma_i, 1 \leq i \leq 4 \). As in the proof of Lemma 5.1 again the remainder

\[
V = -\frac{1}{2} \int_R \left( u^2(e^{u^2} - 1 - u^2) - 3F(u) \right) \frac{dx dt}{t} \leq 0
\]
on account of (28). Since \( \text{supp}(u) \subset K_{t_0}(a, 0) \), we also have \( A_1 = 0 \). Moreover, using (26) and (27) we can bound \( |A_1| \leq C\sqrt{\varepsilon} \) as in [9]. Finally, we find

\[
A_3 = \int_{\Gamma_3} \left( 4 \frac{r}{t} u^2 \xi + u \xi u + \frac{|u|^2}{4t^2} \right) \text{do} \geq \int_{\Gamma_3} \left( u^2 + \frac{|u|^2}{8t^2} \right) \text{do} \geq 0 ,
\]
as we can see from writing

\[
\partial_t \left( \frac{r^2}{t}(e + m + u \frac{u}{2r} + \frac{|u|^2}{4rt}) \right) - \partial_r \left( \frac{r^2}{t}(e - q + m + u \frac{u}{2r}) \right) = \partial_q \left( \frac{r^2}{t}(2(e + m) - q + u \frac{u}{r} + \frac{|u|^2}{4rt}) \right) + \partial \left( \frac{r^2}{t}(q + u \frac{u}{r} + \frac{|u|^2}{4rt}) \right)
\]
in characteristic coordinates and estimating as in [9], formula (29), using (25).

Thus we conclude the bound

\[
A_2 = \int_{\Gamma_0(\xi_2)} \left( \frac{r}{t} q + u \frac{u}{t} + \frac{|u|^2}{4t^2} \right) \text{do} \leq C\sqrt{\varepsilon} .
\]
But as in [9], formula (35), by using (29) we can estimate

\[
\int_{\Gamma_0(\xi_2)} \frac{|u|^2}{t^2} \text{do} \leq C \int_{\Gamma_0(\xi_1)} \frac{|u|^2}{t^2} \text{do} + C \int_R \frac{|u|^2}{t^2} \frac{dx dt}{t} \leq CQ(\xi_1) + CA_0 \leq C\varepsilon .
\]
Hence together with (26) and (25) we may conclude as in [9] that

\[
A_2 \geq \frac{3}{4} Q(\xi_2) - C\sqrt{\varepsilon} ,
\]
and \( Q(\xi_2) \leq C\sqrt{\varepsilon} \). Since \( \xi_0 \geq 2t_\varepsilon \) and \( 2\xi_0 \leq \xi_2 \leq 4\xi_0 \) were arbitrary, the claim follows. \( \square \)

**Proof of Lemma 4.2 completed.** Given \( \xi_0 \geq 2t_\varepsilon \), let \( 0 \leq \varphi_0 = \varphi_0(\eta) \leq 1 \) be a smooth cut-off function such that \( \varphi_0(\eta) = 1 \) for \( |\eta| \leq \eta_0/2 \) and \( \varphi_0(\eta) = 0 \) for \( |\eta| \geq \eta_0 \), with \( |\varphi_0'| \leq 4/\eta_0 \), where \( \eta_0 = \xi_0/8 \). Also fix a smooth cut-off function \( 0 \leq \chi = \chi(\phi) \leq 1 \) satisfying \( \chi(\phi) = 1 \) for \( |\phi| \leq \pi/8 \) and \( \chi(\phi) = 0 \) for \( |\phi| \geq \pi/4 \).

After extending \( u(\xi, \eta, \phi) = u(\xi, -\eta, \phi) \) for \( \eta < 0 \), for \( 2\xi_0 \leq \xi \leq 4\xi_0 \) also set

\[
u_k = u_k(\xi, \eta, \phi) = \varphi_0(\eta) \chi(\phi - k\pi/4)u, \ 1 \leq k \leq 8.
\]

Note that for any fixed \( \xi \in [2\xi_0, 4\xi_0] \) we have

\[
u_k(\xi, \cdot, \cdot) \in H_0^1([-\eta_0, \eta_0] \times [(k-1)\pi/4, (k+1)\pi/4]),
\]
and by (26) and Lemma 5.2 there holds

\[
\int_{-\eta_0}^{\eta_0} \int_{(k-1)\pi/4}^{(k+1)\pi/4} \left( |\partial_\eta u_k|^2 + r^{-2} |\partial_\phi u_k|^2 \right) r d\phi d\eta \leq C \int_{\Gamma_0(\xi)} \left( e - m + \frac{|u|^2}{t^2} \right) \text{do} \leq C(1 + E_0), \ 1 \leq k \leq 8 .
\]
In addition, Lemma [5.2] yields the bound
\[
\int_{\eta_0}^{\eta} \int_{(k-1)\pi/4}^{(k+1)\pi/4} (r^{-2} |\partial_\phi u_k|^2) r \, d\phi \, d\eta \\
\leq C \int_{\Gamma_0(\xi)} (q + \left| \frac{|u|^2}{t^2} \right|) \, da = CQ(\xi) \leq C\sqrt{\varepsilon},
\]
uniformly in \( \xi \in [2\xi_0, 4\xi_0] \) for any \( \xi_0 \geq 2t_\varepsilon \), \( 1 \leq k \leq 8 \). After scaling \( v_k(\xi, \cdot) = v_k(\xi, \eta, \phi) = u_k(\xi, \lambda \eta, \lambda \phi) \),
where \( \lambda = \sqrt{\pi \eta_0} \), and applying the improved Moser-Trudinger inequality in [9], Lemma 4.2, to the functions \( v_k(\xi, \cdot) \) defined on a domain \( \Omega_k \) of unit area, with
\[
\int_{\Omega_k} \left( \xi_0^{-1} |\partial_\eta v_k|^2 + \xi_0^{-1} |\partial_\phi v_k|^2 \right) d\eta d\phi \leq C(1 + E_0)
\]
and with
\[
\int_{\Omega_k} \xi_0^{-1} |\partial_\phi v_k|^2 d\phi d\eta \leq C\sqrt{\varepsilon},
\]
for sufficiently small \( \varepsilon > 0 \) we deduce the uniform bound
\[
\sup_{2\xi_0 \leq \xi \leq 4\xi_0} \int_{\Gamma_0(\xi)} e^{6u_k^2} \, da = C \sup_{2\xi_0 \leq \xi \leq 4\xi_0} \int_{\Omega_k} e^{6u_k^2} \, da \leq C
\]
for each \( k \). Since \( d \leq \eta_0/2 \) we can pointwise estimate \( u^2 \leq \max_{1 \leq k \leq 8} u_k^2 \) on \( \Gamma_1(\xi) \). Hence we can bound
\[
(30) \quad \xi^{-1} \int_{\Gamma_1(\xi)} e^{6u^2} \, da \leq \sum_{1 \leq k \leq 8} \left( \xi^{-1} \int_{\Gamma_0(\xi)} e^{6u_k^2} \, da \right) \leq C
\]
uniformly in \( \xi \in [2\xi_0, 4\xi_0] \) for any \( \xi_0 \geq 2t_\varepsilon \), with a constant \( C > 0 \) independent of \( \xi_0 \), and therefore for any \( \xi \geq 4t_\varepsilon \). Since \( u \) is smooth and therefore is locally bounded, \((30)\) also holds for all remaining \( \xi \geq 2t_\varepsilon \), possibly with a larger constant \( C > 0 \). The claim follows. \( \square \)

References


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