Advanced Nonlinear Studies **x** (xxxx), xxx–xxx

Dedicated to Antonio Ambrosetti on the occasion of his 68^{th} *birthday*

Supercritical elliptic equations

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Received 31 August 2012

Communicated by Andrea Malchiodi

Abstract

For an elliptic model equation with supercritical power non-linearity we give a complete description of radial solutions and discuss self-similar blow-up solutions.

1991 Mathematics Subject Classification. 35J15, 58J05.

Key words. Supercritical elliptic equations, radial symmetry, selfsimilar solutions

1 Introduction

Not much is known about non-trivial solutions u to the equation

$$-\Delta u = |u|^{p-2}u \text{ in } \Omega \tag{1.1}$$

on a smoothly bounded domain $\Omega \subset \mathbb{R}^n$ when $n \geq 3$ and when the exponent p in the nonlinear term is in the "super-critical" range $p > 2^* = \frac{2n}{n-2}$.

Here, in a first attempt to understand this equation more thoroughly, we focus on solutions of (1.1) on $\mathbb{R}^n \setminus \{0\}$ which either are radially symmetric or self-similar in the sense that $u = u_R$ for all $R > 0$, where

$$u_R(x) = R^a u(Rx),\ R > 0, \tag{1.2}$$

with $a := \frac{2}{p-2}$. Note that this scaling of a solution u of equation (1.1) on $\mathbb{R}^n \setminus \{0\}$ again produces a solution u_R to (1.1), and $u = u_R$ for all $R > 0$ if and only if

$$u(x) = |x|^{-a} v(\frac{x}{|x|}),\ x \neq 0, \tag{1.3}$$

where v solves the equation

$$-\Delta_{S^{n-1}} v + b \cdot v = |v|^{p-2}\, v \text{ on } S^{n-1} \tag{1.4}$$

with $b := a(n-2-a)$. Such solutions also might arise from blow-up of singular solutions of (1.1) on a domain $\Omega \subset \mathbb{R}^n$ which are stationary in the sense of Pacard [6].

Conversely, if $p > 2^*$ any solution $v \neq 0$ to (1.4) of class $H^1 \cap L^p$ via (1.3) induces a solution $u \in H^1_{loc} \cap L^p_{loc}(\mathbb{R}^n)$ of (1.1) blowing up at the origin. Equation (1.4) is similar to equation (1.1), but – unless $p = 2^+ := \frac{2(n-1)}{n-3}$, the critical Sobolev exponent in $n-1$ dimensions – no longer has any of the invariance properties of the latter. When $p \leq 2^+$ any solution v to (1.4) of class H^1 is smooth, and solutions $v > 0$ have been classified completely. In contrast, there is a vast abundance of sign-changing self-similar solutions; see Section 3. When $p > 2^+$ we expect equation (1.4) to admit also singular solutions $0 < v \in H^1 \cap L^p$, inducing solutions $u \in H^1$ of (1.1) blowing up on a higher-dimensional singular set. In a sequel [8] to this paper we plan to study this latter case more closely.

Our results thus are most conclusive for solutions $u > 0$ and may be summarized, as follows. On the one hand, in Theorem 2.1 we characterize the self-similar solution

$$u^*(r) = b^{\frac{1}{p-2}} r^{-a} \in H^1_{loc} \cap L^p_{loc}(\mathbb{R}^n) \tag{1.5}$$

as the strong limit $u^* = \lim_{R\to\infty} u_R$ of rescalings of any smooth, radial entire solution u of (1.1) with $u(0) > 0$. On the other hand, as a consequence of a result of Gidas and Spruck [4], for mildly super-critical exponents $2^* < p < 2^+$ the solution u^* also is the unique positive, self-similar solution of (1.1) of class $H^1_{loc} \cap L^p_{loc}$ on $\mathbb{R}^n$.

Moreover, for arbitrary $p > 2^*$ the function u^* is the only radially symmetric solution of (1.1) with a mild singularity at $r = 0$; indeed, by Theorems 2.1 and 2.3 any radial solution $u \neq \pm u^*$ which is singular at $r = 0$ changes sign infinitely many times and blows up at the origin with a predetermined rate which, in particular, implies that $u \notin H^1_{loc}$.

2 Radial solutions

For any dimension $n \geq 3$ and for any given $p > 2^* = \frac{2n}{n-2}$ we consider solutions $u = u(r)$ of the equation

$$u_{rr} + \frac{n-1}{r} u_r + |u|^{p-2}\, u = 0 \text{ on }]0,\infty[, \tag{2.1}$$

corresponding to (1.1). In our first result we study entire solutions $u \in C^2([0,\infty[)$ of (2.1) with a well-defined limit

$$u(0) = u_0, u_r(0) = 0 \text{ at } r = 0, \tag{2.2}$$

giving rise to classical solutions of (1.1) on $\mathbb{R}^n$. Moreover, we prove that the function u^* defined in (1.5) is the only radial solution of (1.1) with a mild singularity at the origin.

Theorem 2.1. *i) Let $u \in C^2([0,\infty[)$ solve (2.1), (2.2) for some $u_0 > 0$. Then $u(r) > 0$, and $u'(r) < 0$ for $r > 0$. Finally, there holds*

$$r^a u(r) \to b^{\frac{1}{p-2}} \text{ as } r \to \infty,$$

for $a = \frac{2}{p-2}$ and $b = a(n-a-2)$ as above.

ii) Suppose that $u \in C^2(]0,\infty[)$ solves (2.1) with

$$\liminf_{r\to 0} r^{\frac{2p}{p-2}} \left(|u_r|^2 + |u|^p\right) < \infty. \tag{2.3}$$

Then either $u \in C^2([0,\infty[)$, and u is as in i), or $u = \pm u^*$.

From Theorem 2.1.i) we immediately obtain the characterization $u^* = \lim_{R\to\infty} u_R$ for any solution $0 < u \in C^2([0,\infty[)$ of (1.1).

Note that the regular solutions of (2.1) are uniquely determined by the choice of u_0 in (2.2). Using (1.2) and invariance of (1.1) under reflection $u \to -u$ they can be obtained by rescaling the solution satisfying (2.2) with initial data $u_0 = 1$.

Before giving the proof of Theorem 2.1, we first remark that for equation (2.1) the classical Pohozaev identity [7] takes the following form.

Theorem 2.2. *Let $u \in C^2(]\rho,R[)$ be a solution of equation (2.1). Then we have*

$$\frac{d}{dr}\left[r^n\big(\frac{1}{2}|u_r|^2 + \frac{1}{p}|u|^p + \frac{n-2}{2r}u_r u\big)\right] + \big(\frac{n-2}{2} - \frac{n}{p}\big)r^{n-1}|u|^p = 0 \text{ in }]\rho,R[. \tag{2.4}$$

A similarly useful identity results if we consider equation (2.1) in scaled coordinates. Introducing $s = \log r$, we define $v(s) = r^a u(r)$ so that $u(r) = e^{-as}v(s)$. Compute

$$\begin{aligned} r^{a+1}u_r(r) &= v'(s) - av(s),\\ r^{a+2}u_{rr}(r) &= v''(s) - (2a+1)v'(s) + a(a+1)v(s). \end{aligned}$$

Hence (2.1) translates into the equation

$$v''(s) + \mu v'(s) - b\cdot v(s) + |v|^{p-2}\, v(s) = r^{a+2}(u_{rr} + \frac{n-1}{r}u_r + |u|^{p-2}\, u) = 0 \tag{2.5}$$

for v, with $\mu = n - \frac{2p}{p-2} = n-2-2a$ and $b = a(n-2-a) = a(\mu+a)$ as defined above. Note that $\mu > 0$ if (and only if) $p > 2^*$.

Multiplying (2.5) with $2v'$ we find the identity

$$\frac{d}{ds}\big(|v'|^2 - bv^2 + \frac{2}{p}\,|v|^p\,\big) + 2\mu|v'|^2 = 0. \tag{2.6}$$

For later reference, for a solution v to (2.5) we define

$$g(s) = (|v'|^2 - bv^2 + \frac{2}{p}\,|v|^p)(s). \tag{2.7}$$

By (2.6) then for any such v the function $s \mapsto g(s)$ is non-increasing. Moreover, $g \geq -\frac{p-2}{p} b^{\frac{p}{p-2}}$ is uniformly bounded from below; we remark that this lower bound is achieved for the constant solution $v \equiv b^{\frac{1}{p-2}}$ of (2.5) which represents the selfsimilar solution u^* of (2.1). We furthermore remark that solutions of (2.1) satisfying the assumption (2.3) of Theorem 2.1.ii) correspond exactly to those solutions for which g is bounded.

Proof of Theorem 2.1. i) As observed earlier, it suffices to consider the case $u_0 = 1$. Integrating (2.4) from $\rho = 0$ to r, we find that $u_r u < 0$ on $]0, \infty[$. Since $u(0) = 1 > 0$ it follows that $u(r) > 0$, $u_r(r) < 0$ for $r > 0$.

Next, from (2.2) we see that the corresponding functions $v(s) = e^{as}u(e^s)$ and $v'(s)$ both converge to 0 as $s \to -\infty$, so

$$C_{-\infty} := \lim_{s \to -\infty} g(s) = 0. \tag{2.8}$$

Recalling that the quantity g defined in (2.7) is non-increasing and bounded from below we conclude that also the limit

$$C_\infty := \lim_{s\to\infty} g(s) = \lim_{s\to\infty} \big(|v'|^2 - bv^2 + \frac{2}{p}v^p\big) \leq 0 \tag{2.9}$$

exists; moreover, (2.6) yields that $v' \in L^2(\mathbb{R})$ and

$$C_\infty + 2\mu \int_{-\infty}^{\infty} |v'|^2 ds = C_{-\infty}. \tag{2.10}$$

It then follows that

$$\lim_{s\to\infty} v'(s) = 0,\ \lim_{s\to\infty} \big(\frac{2}{p}v^p - bv^2\big) = C_\infty, \tag{2.11}$$

because for any $\epsilon > 0$ by (2.10) we have $\inf_{[s,s+\epsilon]} |v'|^2 \to 0$ $(s \to \infty)$, and (2.5), (2.9) limit the rate of change of v'. But (2.11) implies the convergence $v(s) \to v_\infty$ as $s \to \infty$; moreover, together with (2.5) the relation (2.11) yields that also $v''(s)$ converges to a limit which must be zero as v' converges. Therefore

$$-bv + v^{p-1} \to 0 \text{ as } s \to \infty,$$

and we conclude that either $v_\infty = 0$ and $C_\infty = 0$, or $v_\infty = b^{\frac{1}{p-2}}$ and $C_\infty = -\frac{p-2}{p} b^{\frac{p}{p-2}} = \min g$.

But if $C_\infty = 0$, from (2.8) and (2.10) we conclude that $v' \equiv 0$ and hence $v \equiv \lim_{s\to\infty} v(s) = 0$, which implies that also $u \equiv 0$ and thus contradicts our assumption that $u(0) = 1$. Therefore we must have $v(s) \to b^{\frac{1}{p-2}}$ as $s \to \infty$, that is, $r^a u(r) \to b^{\frac{1}{p-2}}$ $(r \to \infty)$, as claimed.

ii) Observe the relation $b - a^2 = a(n-2-2a) = a\mu > 0$. Thus, by Young's inequality we can estimate

$$|v'|^2 - bv^2 = (r^{a+1}u_r + a \cdot r^a u)^2 - br^{2a}u^2 \le Cr^{2(a+1)}u_r^2 = Cr^{\frac{2p}{p-2}}u_r^2$$

with a constant $C = C(p,n) > 0$. If now we assume that

$$C_{-\infty} := \lim_{s\to-\infty} \left(|v'|^2 - bv^2 + \frac{2}{p}|v|^p \right) \le C \liminf_{r\to 0} r^{\frac{2p}{p-2}} (|u_r|^2 + |u|^p) < \infty, \quad (2.12)$$

similar to the above we conclude that $v'(s), v''(s) \to 0$ as $s \to -\infty$, and either $v(s) \to \pm b^{\frac{1}{p-2}}$ as $s \to -\infty$ and $C_{-\infty} = \min g$, or $v(s) \to 0$ as $s \to -\infty$. In the latter case it follows that $r^a u(r), r^{a+1}u_r(r) \to 0$ as $r \to 0$; in particular,

$$\frac{1}{r^\mu} \int_{B_r(0)} (|u_r|^2 + |u|^p)dx \to 0 \text{ as } r \to 0,$$

since $\mu = n - 2 - 2a = n - \frac{2p}{p-2} = n - ap$. From Pacard's [6] Proposition 1 and Lemma 8 then we obtain regularity for u at $x = 0$, so u must be as in i). (As observed in [1], any $p > 2^*$ is admissible for Pacard's result.)

Thus we may assume that $C_{-\infty} = \min g$ and thus that $s \mapsto g(s)$ is constant. It follows that $v \equiv \lim_{s\to-\infty} v(s) = \pm b^{\frac{1}{p-2}}$, and $u = \pm u^*$. □

From Theorem 2.1 we know that any solution $u \in C^2(]0,\infty[)$ of (2.1) that vanishes at some point $R > 0$ must be strongly singular at $x = 0$. In fact, we show in the next result that also the converse is true. Furthermore we obtain that statement ii) of the previous result remains valid under much weaker regularity assumptions.

Theorem 2.3. *Suppose that* $u \in C^2(]0,\infty[)$ *solves equation* (2.1) *with*

$$\liminf_{r\to 0} r^{\frac{2p}{p-2}} \left(|u_r|^2 + |u|^p \right) = \infty. \quad (2.13)$$

Then u *changes sign infinitely many times on any interval* $]0,R[$, $R > 0$, *and there exists a number* $c_0 > 0$ *such that*

$$|u_r|^2 + \frac{2}{p}|u|^p = c_0 r^{-(n+d)} + O(r^{-n}) \text{ as } r \to 0, \quad (2.14)$$

where $d = \frac{2p}{p+2}(n-1) - n > 0$.

Since $n + d > n > \frac{2p}{p-2}$ for $p > 2^*$, we thus see that the only radially symmetric solutions of (1.1) that are locally in H^1 are the selfsimilar solutions $\pm u^*$ and the regular solutions $u \in C^2([0,\infty[)$ described in Theorem 2.1.

Proof. i) Let u be as above. For brevity in the following we use the shorthand notation $u' = u_r$ also with reference to r. Consider the quantity

$$\Phi_0(r) := r^n(|u'|^2 + \frac{2}{p}|u|^p)$$

from Theorem 2.2 which satisfies

$$\Phi_0'(r) = nr^{-1}\Phi_0(r) + 2r^n(u'' + |u|^{p-2}u)\cdot u' = nr^{-1}\Phi_0(r) - 2(n-1)|u'|^2 \cdot r^{n-1}.$$

While not monotone itself, a slight modification of Φ_0 similar to (2.4) will give us a monotone quantity that can be used to study the asymptotic behaviour of u near zero.

Writing equation (2.1) in divergence form

$$(r^{n-1}u')' = -r^{n-1}|u|^{p-2}u \tag{2.15}$$

and letting

$$\Phi(r) := \Phi_0(r) + c \cdot r^{n-1}u' \cdot u,$$

with a constant $c \in \mathbb{R}$ to be determined, we find

$$\begin{aligned}\Phi'(r) &= \Phi_0'(r) + cr^{n-1}(|u'|^2 - |u|^p)\\ &= nr^{-1}\Phi_0 - \big[(2n-2-c)|u'|^2 + \frac{cp}{2}\cdot\frac{2}{p}|u|^p\big]r^{n-1}.\end{aligned}$$

For $c = \frac{4}{2+p}(n-1)$ chosen so that $(2n-2-c) = \frac{cp}{2}$ we then obtain the identity

$$\Phi'(r) = -dr^{-1}\Phi_0(r) \tag{2.16}$$

for the positive number d defined in the statement of Theorem 2.3.

Remark that for solutions of (2.1) satisfying (2.13) the difference $\Phi - \Phi_0$ is small compared with Φ. Indeed, for any number $\varepsilon \in]0,1[$ we may estimate

$$|\Phi - \Phi_0| \le C\Phi_0^{\frac{2+p}{2p}} r^{n-1-\frac{2+p}{2p}\cdot n} = C(r^d\Phi_0)^{\frac{2+p}{2p}} \le \frac{1}{8}\varepsilon\Phi_0 + C_\varepsilon r^\mu, \tag{2.17}$$

so that by assumption (2.13)

$$|\Phi - \Phi_0| \le \frac{1}{4}\varepsilon\Phi_0 \le \frac{\varepsilon}{4-\varepsilon}\Phi$$

for sufficiently small $r > 0$. Hence we also have

$$|\Phi - \Phi_0| \le \frac{\varepsilon}{d}\Phi \tag{2.18}$$

for $0 < r < r_\varepsilon$ small enough. Equation (2.16) thus gives the differential inequality

$$-(d+\varepsilon)r^{-1}\Phi(r) \le \Phi'(r) \le -(d-\varepsilon)r^{-1}\Phi(r)$$

which results in estimates of the form

$$C_1 \cdot r^{-d+\varepsilon} \le \Phi(r), \Phi_0(r) \le C_2 \cdot r^{-d-\varepsilon}, \tag{2.19}$$

on $]0, r_\varepsilon[$, with numbers $C_{1,2} > 0$ depending on ε.

Inserting the bound (2.19) in (2.17) we can now improve (2.18) to obtain

$$|\Phi - \Phi_0| \le C(r^d\Phi_0)^{\frac{2+p}{2p}} \le Cr^{-\frac{2+p}{2p}\varepsilon} \le Cr^{-\varepsilon}. \tag{2.20}$$

This bound in turn gives the improved differential inequality

$$\left|(r^d\Phi(r))'\right| = r^d\left|\Phi' + dr^{-1}\Phi\right| = dr^{d-1}\left|\Phi - \Phi_0\right| \le Cr^{d-1-\varepsilon} \text{ on }]0, r_\varepsilon[.$$

Integrating this inequality for a fixed number $\varepsilon \in]0, d[$ we conclude that there exists

$$c_0 = \lim_{r\to 0} r^d\Phi(r) = \lim_{r\to 0} r^d\Phi_0(r).$$

By (2.17) then $|\Phi - \Phi_0| \le C$, and we find the improved bound $\left|(r^d\Phi(r))'\right| \le Cr^{d-1}$ from which we obtain that

$$\left|r^d\Phi_0(r) - c_0\right| \le Cr^d + \left|r^d\Phi(r) - c_0\right| \le Cr^d,$$

as claimed. Observe that the limit c_0 must be positive; otherwise estimate (2.19) would be violated.

ii) In a second step we now show that any solution u of (2.1) satisfying (2.13), and therefore also satisfying (2.14), changes sign infinitely many times as $r \downarrow 0$. Assume by contradiction that some such u does not change sign on an interval $]0, r_1[$ for some $r_1 > 0$, say $u(r) \ge 0$ for $0 < r < r_1$.

We first claim that u must be decreasing in $]0, r_1[$. Indeed, since the maximum principle implies that solutions of (2.1) cannot achieve non-negative local minima, the only alternative would be that u is increasing on an interval $]0, r_2[$ for some $0 < r_2 < r_1$. However the resulting bound $0 \le u \le u(r_2)$ and the asymptotics of Φ_0 shown above then imply that $u'(r) \ge c \cdot r^{-\frac{n+d}{2}}$ as $r \to 0$, which contradicts our assumption that $u \ge 0$.

Now recall that $c_0 = \lim_{r\to 0}\Phi_0(r)r^d > 0$ and consider the sets

$$\begin{aligned} U_1(R) &:= \{r \in [R, 2R]; |u'(r)|^2 \ge \frac{1}{4}c_0 r^{-n-d}\}, \\ U_2(R) &:= \{r \in [R, 2R]; \frac{2}{p}u(r)^p \ge \frac{1}{4}c_0 r^{-n-d}\}, \end{aligned} \tag{2.21}$$

whose union contains the whole interval $[R, 2R]$, provided $R > 0$ is sufficiently small.

Since $u' \leq 0$ and $u > 0$ near zero, for the 1-dimensional Lebesgue measure $\mu(U_1(R))$ we obtain the inequality

$$u(R) \geq -\int_R^{2R} u'(r)dr \geq c_1 R^{-(n+d)/2} \cdot \mu(U_1(R)),$$

with a constant $c_1 > 0$ depending on c_0 but independent of R. In view of the estimate

$$u(R) \leq C(R^{-n}\Phi_0(R))^{1/p} \leq CR^{-(n+d)/p}$$

we then find the upper bound

$$\mu(U_1(R)) \leq CR^\gamma, \text{ with } \gamma = (n+d) \cdot \frac{p-2}{2p} > 1.$$

On the other hand, upon integrating equation (2.15) we find that

$$-\int_0^{2R} (u'r^{n-1})'dr = \int_0^{2R} u^{p-1}r^{n-1}dr \geq cR^{n-1-(n+d)\cdot\frac{p-1}{p}}\mu(U_2(R)).$$

Since $u'(r)r^{n-1} \to 0$ as $r \to 0$ the left hand side is equal to $-u'(2R)(2R)^{n-1}$ and thus bounded from above by $C \cdot R^{n-1-(n+d)/2}$. We conclude that also the measure of the second set is bounded by $\mu(U_2(R)) \leq CR^\gamma$ and thus in total

$$\mu(U_1(R)) + \mu(U_2(R)) \leq CR^\gamma << R = \mu([R, 2R])$$

for small $R > 0$, which contradicts the fact that $[R, 2R] \subset U_1(R) \cup U_2(R)$. □

Thus, sign-changing solutions are necessarily oscillatory near the origin. On the other hand, we now show that the sign of any solution to (2.1) stabilizes near infinity.

Theorem 2.4. *Let $u \in C^2(]0,\infty[)$ solve (2.1). Then there exists $R > 0$ such that $u(r) \neq 0$ for $r > R$, and either $\frac{u(r)}{u^*(r)} \to \pm 1$ as $r \to \infty$, or there exists a number $c_1 \neq 0$ such that*

$$u(r) = r^{-(n-2)}(c_1 + O(r^{-(2+\mu(p-2))})). \tag{2.22}$$

Proof. Let $v(s) = r^a u(r)$ be defined as above, satisfying (2.5). As in the proof of Theorem 2.2, from (2.6) we conclude that the limit

$$C_\infty := \lim_{s\to\infty} g(s) = \lim_{s\to\infty} \left(|v'|^2 - bv^2 + \frac{2}{p}|v|^p\right) < \infty$$

exists and that $v' \in L^2([0,\infty[)$ with

$$C_\infty + 2\mu \int_0^\infty |v'|^2 ds = g(0) < \infty.$$

Still following the proof of Theorem 2.2, we then conclude that

$$v'(s) \to 0 \text{ as } s \to \infty,$$

and either $v(s) \to 0$ as $s \to \infty$, or $v(s) \to \pm b^{\frac{1}{p-2}}$. In the latter case, $\frac{u(r)}{u^*(r)} \to \pm 1$ and clearly our claim is true. In the following we therefore may assume that $\lim_{s\to\infty} v(s) = 0$.

Let now $s_1 \in \mathbb{R}$ be such that $|v|^{p-2} < b$ for $s > s_1$. Then on $[s_1, \infty)$ the expression $v''(s) + \mu v'(s) = (b - |v|^{p-2})v$ has the same sign as $v(s)$, so v can neither achieve a positive maximum nor a negative minimum on $[s_1, \infty)$. So if there is any point $s_0 > s_1$ with $v(s_0) = 0$ we thus find that v is monotone on $[s_0, \infty)$; in particular, $v(s) \neq 0$ for $s > s_0$. If no such s_0 exists we define $s_0 := s_1$ so that again $v(s) \neq 0$ for $s > s_0$.

To see also the claimed asymptotic behavior, for $s > s_0$ we divide (2.5) by v to find the identity

$$\Big(\frac{v'}{v}\Big)' + \Big(\frac{v'}{v}\Big)^2 + \mu\frac{v'}{v} = \frac{v''}{v} + \mu\frac{v'}{v} = b - |v|^{p-2} \tag{2.23}$$

for $b = a(\mu + a)$ as above.

For $f := v'/v$ this gives the equation

$$f' + f^2 + \mu f - b = f' + (f - a)(f + \mu + a) = -|v|^{p-2}. \tag{2.24}$$

We claim that $\lim_{s\to\infty} f(s) = -(\mu + a)$. First of all, if $f(s_2) < -(\mu + a)$ for some s_2 then f would be decreasing on $[s_2, \infty)$ and thus satisfy $f'(s) \le -\varepsilon f^2(s)$ for some $\varepsilon > 0$ so that $f(s) \to -\infty$ as $s \to s_-$ for some $s_- < \infty$ which is impossible as $v(s) \neq 0$ for $s > s_0$.

On the other hand, inspection of the sign of the nonlinearity combined with the assumption that $v(s) \to 0$ shows that if $\limsup_{s\to\infty} f(s) > -(\mu + a)$ then f tends to $a = \frac{2}{p-2}$ as $s \to \infty$. But then $v' \ge v/(p-2)$ for large s, and $v(s) \to \infty$ as $s \to \infty$, contrary to our assumption. We thus conclude that $f(s) \to -(\mu + a)$ as $s \to \infty$.

Given any $\epsilon > 0$ we then find that for sufficiently large $s > s_0(\epsilon)$

$$-(\mu + a) \le \frac{v'}{v} \le (-\mu + a + \varepsilon) \tag{2.25}$$

which gives

$$v(s_0)e^{-(\mu+a)(s-s_0)} \le v(s) \le v(s_0)e^{-(\mu+a-\varepsilon)(s-s_0)} \tag{2.26}$$

as a first estimate on the asymptotics of v. Since $\mu + 2a = n - 2$ this translates into the bound

$$u(r_0)(r/r_0)^{-(n-2)} \le u(r) \le u(r_0)(r/r_0)^{-(n-2-\epsilon)} \tag{2.27}$$

for sufficiently large $r > r_0(\epsilon)$.

To obtain the more precise asymptotic expansion claimed in Theorem 2.4 we now consider the function $h(s) := f(s) + \mu + a = f(s) + n - 2 > 0$ which satisfies

$$h' - (n-2)h = -h^2 - |v|^{p-2} \ge -\varepsilon h - |v|^{p-2} \tag{2.28}$$

for $r \ge r_0(\varepsilon)$ sufficiently large. Inserting the bound on v derived in (2.26) we find that

$$e^{(n-2-\varepsilon)s} \cdot \big(h(s) \cdot e^{-(n-2-\varepsilon)s}\big)' \ge -ce^{-(p-2)(\mu+a-\varepsilon)s} \tag{2.29}$$

for s large. Since $h(s) \to 0$ as $s \to \infty$ this gives the estimate

$$h(s) \leq Ce^{-(p-2)(\mu+a-\varepsilon)s} \tag{2.30}$$

which in turn allows us to improve the differential inequality (2.25) for v to

$$0 \leq v' + (\mu + a)v \leq vh \leq Ce^{-(p-1)(\mu+a-\varepsilon)s}$$

so that

$$\left(e^{(\mu+a)s}v(s)\right)' \leq Ce^{-(p-2)(\mu+a-\varepsilon)s}.$$

Since $p > 2$ we conclude that the limit

$$c_1 = \lim_{s\to\infty} e^{(\mu+a)s}v(s) = \lim_{r\to\infty} r^{n-2}u(r)$$

exists, and thus

$$v(s) \leq Ce^{-(\mu+a)s} \text{ for large } s. \tag{2.31}$$

Repeating the above argument with the bound

$$\begin{aligned} h' - (n-2)h = -h^2 - |v|^{p-2} &\geq -Ce^{-2(p-2)(\mu+a-\varepsilon)s} - Ce^{-(p-2)(\mu+a)s} \\ &\geq -Ce^{-(p-2)(\mu+a)s}, \; s >> 1, \end{aligned}$$

resulting from (2.30) and (2.31) we obtain the estimate claimed in (2.22). □

3 Self-similar solutions

Recall that a solution u of (1.1) on $\mathbb{R}^n \setminus \{0\}$ is self-similar if

$$u(x) = u_R(x) = R^a u(Rx) \text{ for all } x \neq 0 \text{ and any } R > 0. \tag{3.1}$$

As we have seen in Section 2, equation (1.1) admits the self-similar radial solution u^*, and $u^* \in H^1_{loc} \cap L^p_{loc}(\mathbb{R}^n)$ when $p > 2^*$. By Pacard's work [6] we also expect to find not necessarily radial self-similar weak solutions of (1.1) on $\mathbb{R}^n$ as limits of blow-ups of singular "stationary" weak solutions u of equation (1.1) on a domain, and the question of classifying these singularity profiles is of interest. Self-similar solutions to (1.1) also may arise as asymptotic profiles of stationary solutions at infinity, as described in our paper [8], or as singularity profiles of solutions of the parabolic evolution problem associated with (1.1) at blow-up points of Type I, as defined in the forthcoming work [1] of Simon Blatt and the second author, which extends the work of Giga-Kohn [3] to the critical and super-critical range and also removes the restriction on the range of admissible exponents p in the results of Pacard [6].

Differentiating relation (3.1) in R for fixed $x \in S^{n-1} = \{x \in \mathbb{R}^n;\ |x| = 1\}$, we find the relations

$$u_r + au/r = 0, u_{rr} = a(a+1)u/r^2 \text{ for } r = |x| > 0. \tag{3.2}$$

Moreover choosing $R = |x|^{-1}$ for each $x \neq 0$, we obtain the representation (1.3) with a function $v\colon S^{n-1} \to \mathbb{R}$.

Thus, evaluating (1.1) at a point $x \in S^{n-1}$, from (3.2) we obtain the equation

$$\begin{aligned} 0 &= \Delta u + |u|^{p-2}u = u_{rr} + (n-1)u_r + \Delta_{S^{n-1}}u + |u|^{p-2}u \\ &= \Delta_{S^{n-1}}v - a(n-2-a)v + |v|^{p-2}v; \end{aligned}$$

that is,

$$-\Delta_{S^{n-1}}v + b \cdot v = |v|^{p-2}\, v \text{ on } S^{n-1}, \tag{3.3}$$

or (1.4). We conclude this paper with the following remarks characterizing the solutions of (3.3) in the range $2^* < p \leq 2^+ := \frac{2(n-1)}{n-3}$, the critical exponent in $n-1$ dimensions (which almost equals the range covered by Pacard's [6] results). Observe that by standard regularity results for $p \leq 2^+$ any solution $v \in H^1$ of (3.3) is smooth. If $n = 3$ we set $2^+ = \infty$ and allow any finite number $p > 2^* = 6$.

If $n \geq 4$, by using the method of Ding [2] one can show that for any $p < \frac{2m}{m-2}$, where $n = m + k$ with $2 \leq k \leq m$, equation (3.3) possesses infinitely many distinct solutions of changing sign, giving rise to infinitely many distinct sign-changing solutions of (1.1) which blow up at the origin at the rate $|x|^{-a}$. Choosing the minimal number m with the above property, for even dimensions n we have $m = \frac{n}{2}$ and $\frac{2m}{m-2} = \frac{2n}{n-4} > 2^+$; likewise $\frac{2m}{m-2} = \frac{2(n+1)}{n-3} > 2^+$ when $n = 2m-1$ is odd. Thus, in general there is a huge multitude of self-similar solutions of (1.1). If $n = 3$ standard minimax arguments give the same conclusion for any $p > 2$.

However, the picture changes if we restrict our attention to positive solutions. In fact, by a result of Gidas and Spruck [4] for $p < 2^+$ any positive solution of (3.3) is constant. Indeed, for $n = 3$ and any $p > 6$ we can set $\beta = \frac{2}{p-3}$ to satisfy condition (B.2b) of Theorem B.1 in [4] and then check that with this choice also their condition (B.2a) holds true. If $n \geq 4$, we define $\gamma > 0$ as in (B.11) in the Remark following [4], Theorem B.2, thereby taking care to replace n by $n-1$. Recalling that there holds $R_{ij} = (n-2)g_{ij}$ for the Ricci curvature of S^{n-1} in the standard metric $g = (g_{ij})$, condition (B.10) in the Remark mentioned above then is satisfied, and the conclusion follows from [4], Theorem B.2.

Finally, in the critical case $p = 2^+ = \frac{n-1}{n-3}$, $n \geq 4$, we have

$$b = \frac{(n-1)(n-3)}{4} = \frac{1}{c(n-1)} R_{g_{S^{n-1}}}$$

with R_h the scalar curvature and $c(k) = 4\frac{k-1}{k-2}$ the constant appearing in the conformal Laplace operator

$$L_h = -c(k)\Delta_h + R_h$$

on a k-dimensional manifold (M, h), and (3.3) is the Yamabe equation on S^{n-1}, with infinitely many positive solutions u exactly arising as the coefficient in the conformal factor $u^{\frac{4}{n-3}} g_{S^{n-1}}$ of the metric obtained by pulling back the standard spherical metric $g_{S^{n-1}}$ with a Möbius transformation; see Obata [5].

In a sequel [8] to this paper we plan to investigate (1.4) also in the super-critical range $p > 2^+$ and, in particular, construct weak solutions $0 < v \in H^1$ depending only on the distance $\rho(q) = dist(q, N)$ of the point $q \in S^{n-1}$ from the North pole N and blowing up at the rate $v(\rho) \sim \rho^{-a}$ as $\rho \to 0$, similar to u^*.

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