ADDENDUM TO OUR PAPER SUPERCRITICAL ELLIPTIC EQUATIONS

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After publication of our paper we became aware of some results related to our analysis of the equation

$$-\Delta u = |u|^{p-2}u \text{ on } \mathbb{R}^n,\ n \geq 3, \tag{1.1}$$

that we would like to briefly discuss and place into context with regard to the focus of our work. Let $2^* = \frac{2n}{n-2}$ and set $a = \frac{2}{p-2}$, $b = a(n-a-2)$ for convenience.

As pointed out by Bonforte et al. [1] in their analysis of equation (1.1) in hyperbolic space, the asymptotic decay $u(x) = O(|x|^{-a})$ of any radial solution u to equation (1.1) was already shown by Ni-Serrin [5], Theorem 2.2, (2.5). Ni-Serrin [5], p.234, also remark that the asymptotic behavior of such u can be inferred more precisely from Fowler's [2] analysis of the equation $y'' + t^{\alpha-q}y^q = 0$, which results from the radial equation (1.1) after a change of variables, even though "because of the technical nature of this literature it is not generally an easy task to translate the conclusions to our circumstances".

In our work we not only establish the existence of $\lim_{r\to\infty} r^a u(r) = b^{\frac{1}{p-2}}$ as part i) of our Theorem 2.1, but – perhaps more importantly – we also interpret this result as showing that the singular weak solution $u^*(x) = b^{\frac{1}{p-2}}|x|^{-a}$ belongs to the closure (in the natural topology given by $H^1_{loc} \cap L^p_{loc}(\mathbb{R}^n)$) of the set of smooth, classical solutions to equation (1.1). Thus, we naturally encounter weak solutions with "mild" singularities of this equation. Also note that u^* is a "stationary" solution of (1.1) in the sense of Pacard [6]; that is, $u = u^*$ satisfies the condition

$$\int_{\mathbb{R}^n} \Big(\frac{\partial u}{\partial x^i}\frac{\partial u}{\partial x^j}\frac{\partial \phi^j}{\partial x^i} - \big(\frac{1}{2}|\nabla u|^2 - \frac{1}{p}|u|^p\big)\frac{\partial \phi^i}{\partial x^i}\Big)dx = 0 \text{ for all } \phi \in C_0^\infty(\mathbb{R}^n;\mathbb{R}^n).^1 \tag{1.2}$$

In contrast, in view of our Theorems 2.1.ii) and Theorem 2.3 any oscillating radial solution of (1.1) on $\mathbb{R}^n \setminus \{0\}$ fails to be in the space $H^1_{loc} \cap L^p_{loc}(\mathbb{R}^n)$. However, the estimate

$$|u_r|^2 + \frac{2}{p}|u|^p = c_0 r^{-\frac{2p}{p+2}(n-1)} + O(r^{-n}) \text{ as } r \to 0 \tag{1.3}$$

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[1] For a general weak solution $u \in H^1_{loc} \cap L^p_{loc}$ with isolated singularities x_i, $i \in \mathbb{N}$, the Morrey-condition $r^{-1}\int_{B_r(x_i)}(|\nabla u|^2 + |u|^p)dx \to 0$ $(r \to 0)$, $i \in \mathbb{N}$, is sufficient for stationarity. Even though for $u = u^*$ this condition holds true at $x_0 = 0$ only in case $p > 2^+ = \frac{2(n-1)}{n-3}$, for a rotationally symmetric solution u stationarity already follows when $u \in H^1_{loc} \cap L^p_{loc}(\mathbb{R}^n)$. Indeed, for $r > 0$ split a given $\phi \in C_0^\infty(\mathbb{R}^n)$ as $\phi = \phi\tau + \phi(1-\tau)$, where $\chi_{B_r(0)} \leq \tau = \tau(|x|) \leq \chi_{B_{2r}(0)}$ with $|\nabla\tau| \leq 2r^{-1}$. Further splitting $\phi\tau = (\phi - \phi(0))\tau + \phi(0)\tau$ we observe that the contribution to (1.2) from $\phi(1-\tau)$ vanishes because u classically solves (1.1) away from $x = 0$, and the contribution from $\phi(0)\tau$ vanishes by symmetry. Finally, using that $u \in H^1_{loc} \cap L^p_{loc}(\mathbb{R}^n)$ we see that also the remaining contribution vanishes in the limit as $r \downarrow 0$.

in our Theorem 2.3, and similar results that can be recovered from [2], Theorem VII and parts of section 11, case II, show that such oscillating solutions u still belong to $W^{1,\frac{n}{n-1}}_{loc}$, and $u \in L^{p-1}_{loc}$ if $2^* < p < 2\frac{2n-1}{n-2}$. Since $\{0\}$ has vanishing $W^{1,n}$-capacity, these solutions will then be distribution solutions of (1.1), which nicely complements Pohozaev's uniqueness result for classical solutions of (1.1) vanishing on the boundary of some ball.

Finally, in view of Pacard's [6] monotonicity formula, any blow-up profile obtained from blowing up a "stationary" weak solution of (1.1) around a singular point corresponds to a self-similar weak solution $\bar{u}(x) = |x|^{-a} v(\frac{x}{|x|})$ of (1.1), where v solves the equation

$$-\Delta_{S^{n-1}} v + b \cdot v = |v|^{p-2}\, v \text{ on } S^{n-1}. \tag{1.4}$$

The rigidity result for positive solutions v of (1.4) in case $2^* < p < 2^+ := \frac{2(n-1)}{n-3}$ that we infer from the work of Gidas-Spruck [4] then further point to the set of positive, "stationary" weak solutions $u \in H^1 \cap L^p(\Omega)$ as the appropriate functional analytic setting for studying equation (1.1) on a domain. (If $n = 3$ we let $2^+ = \infty$.)

This latter part of our work makes contact with a result of Henghui Zou [7] who for $2^* < p < 2^+$ showed radial symmetry of any smooth solution $u > 0$ to equation (1.1) decaying at the rate $u(x) = O(|x|^{-a})$ as $|x| \to \infty$, thus generalizing the famous Gidas-Ni-Nirenberg [3] result to this case. A key step in Zou's argument is first to show that the limit $v(x) = \lim_{r\to\infty} r^a u(rx)$ exists for any $x \in S^{n-1}$ and that v solves equation (1.4), and then to invoke the method of Gidas-Spruck [4] to conclude from this that $v \equiv const$. (In our work, we observe that Theorems B.1 and B.2 of [4] directly give the latter result.)

References

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