VARIATIONAL PRINCIPLES FOR MINIMAL SURFACES

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ABSTRACT. Extending the variational approach to Plateau's problem developped in [4], we show that similar results can be obtained for surfaces normalized by a 3-point condition. Moreover, we supply further details for the regularity results in [4].

1. BACKGROUND

In [4] a general theory of critical points and a Morse theory was developped for the classical Plateau problem for parametric minimal surfaces X of the type of the 2-disc

$$B = \{w = (u, v) \in \mathbb{R}^2; u^2 + v^2 < 1\},$$

spanning a curve Γ in $\mathbb{R}^n$.

By definition, a surface $X \colon B \to \mathbb{R}^n$ is a minimal surface spanning Γ if $X \in C^2(B;\mathbb{R}^n) \cap C^0(\overline{B};\mathbb{R}^n)$, $\Delta X = 0$ in B (componentwise), $X|_{\partial B} \colon \partial B \to \Gamma$ is a homeomorphism, and if X is conformal; that is, if

$$|X_u|^2 = |X_v|^2, \langle X_u, X_v\rangle = 0 \text{ in } B, \tag{1}$$

where $X_u = \frac{\partial X}{\partial u}$, etc., and where $\langle \cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^n$.

If Γ is smooth ($\Gamma \in C^r$ for some $r \geq 3$ is enough), as shown in [4], minimal surfaces X spanning Γ equivalently can be characterized as follows. Let $\gamma \colon S^1 \cong \mathbb{R}/2\pi \to \Gamma$ be a smooth (C^r-) parametrization of Γ with $\gamma' \neq 0$. Identifying $S^1 = \mathbb{R}/2\pi \cong \partial B$ and letting h denote harmonic extension from ∂B to B, admissible comparison surfaces can be represented via γ as

$$X = h(\gamma \circ x) = X(x), \tag{2}$$

where $x \colon \mathbb{R} \to \mathbb{R}$ is continuous, non-decreasing, and satisfies $x(\phi + 2\pi) = x(\phi) + 2\pi$. That is, in polar coordinates $w = re^{i\phi}$ on B, we have

$$X(e^{i\phi}) = \gamma(x(\phi)) \text{ on } \partial B,$$

and X is harmonic. Minimal surfaces of finite area spanning Γ then are precisely those maps $X = X(x)$ with finite Dirichlet integral

$$D(X) = \frac{1}{2}\int_B |\nabla X|^2\, dw$$

which are conformal. By Hildebrandt's regularity result [2], any such minimal surface X is smooth on $\overline{B}$; therefore, finally, (1) is equivalent to orthogonality

$$\langle X_r, X_\phi\rangle = 0 \text{ on } \partial B. \tag{3}$$

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In [4] now, minimal surfaces are shown to uniquely correspond to critical points of the functional

$$E(x) = D(X(x))$$

on the closed convex set

$$M = \{x = id + \xi \colon \mathbb{R} \to \mathbb{R}\ ;\ x \text{ monotone}, \xi \in H^{1/2,2} \cap C^0(\mathbb{R}/2\pi)\},$$

endowed with the metric inherited from $H^{1/2,2} \cap C^0(\mathbb{R}/2\pi)$, where

$$|\xi|^2_{H^{1/2}} = \int_0^{4\pi} \int_0^{4\pi} \frac{|\xi(\phi) - \xi(\phi')|^2}{|\phi - \phi'|^2}\, d\phi d\phi'.$$

However, since $D(X \circ g) = D(X)$ for any X and any conformal diffeomorphism $g \colon B \to B$, in order to ensure the compactness necessary for a satisfactory theory of critical points, admissible comparison maps need to be normalized with respect to the action of the Möbius group

$$\mathcal{G} = \{g \colon w \mapsto e^{i\phi_0} \frac{a + w}{1 + \overline{a}w}; a \in \mathbb{C}, |a| < 1, \phi_\circ \in \mathbb{R}\}$$

of the disc.

In contrast to the classical 3-point normalization [1], leading to the class

$$M^* = \{x \in M;\ x(\frac{2\pi k}{3}) = \frac{2\pi k}{3}, \forall k \in \mathbb{Z}\},$$

in [4] the normalized class

$$M^\dagger = \{x \in M; \int_0^{2\pi} \xi(\phi) d\phi = \int_0^{2\pi} \xi(\phi) \sin\phi\ d\phi = \int_0^{2\pi} \xi(\phi) \cos\phi\ d\phi = 0\}$$

was proposed and it was shown that generalized Lusternik-Schnirelman and Morse theories hold for E in the class $M^\dagger$, and that the critical points of E on $M^\dagger$ uniquely correspond to the minimal surfaces spanning Γ, normalized with respect to $\mathcal{G}$.

In this note we show that the latter bi-unique correspondence with minimal surfaces spanning Γ also holds for the critical points of E in M^*; see Theorem 2.1 below. As is clear form [4], Proposition 2.9, this assertion is equivalent to a regularity result for critical points of E in M^*; see Proposition 2.2. However, this result does not resolve the fundamental difficulties for setting up a Morse theory for E in M^*, pointed out in [4], p. 62.

Our second aim is to provide further details for the proof of Proposition 2.2 in the case $M^\dagger$, as in [3], in particular, concerning the normalization of testing functions with respect to $\mathcal{G}$, which was only sketched in [4].

2. Results

As shown in [4] for any $x \in M$ the functional E extends to a C^1-functional on the space $x + H^{1/2,2} \cap L^\infty(\mathbb{R}/2\pi)$ and

$$\langle dE(x), \xi \rangle = \int_{\partial B} \langle \partial_n X, \frac{d}{d\phi}\gamma(x) \cdot \xi \rangle do$$

for all $\xi \in H^{1/2,2} \cap L^\infty(\mathbb{R}/2\pi)$, where n is the unit outward normal on ∂B.

By definition, moreover, $x \in M$ is critical for E if

$$\langle dE(x), x - y \rangle \leq 0 \tag{4}$$

for all $y \in M$ with $||x-y||_T < 1$. Observe that for any $y \in M$ by convexity of M for any $t \in [0,1]$ also $y_t = (1-t)x + ty \in M$ with $x - y_t = t(x-y)$ and $\langle dE(x), x-y\rangle = t^{-1}\langle dE(x), x-y_t\rangle$. Choosing $t_0 = (1+||x-y||_T)^{-1}$, we conclude that x is critical for E in M if and only if (4) holds for all $y \in M$.

Similarly, $x \in M^*$ (respectively $M^\dagger$) is critical for E in M^* (respectively $M^\dagger$) if and only if (4) holds for all $y \in M^*$ (respectively $M^\dagger$). Observe that critical points of E in M that belong to M^* (respectively $M^\dagger$) are also critical for E in M^* (respectively $M^\dagger$).

Theorem 2.1. *Suppose x is critical for E in either M, M^*, or $M^\dagger$. Then $X = X(x)$ is a minimal surface spanning Γ.*

It is clear that, conversely, any (suitably normalized) minimal surface X corresponds to a critical point x of E in M (or M^*, or $M^\dagger$) such that $X(x) = X$.

As explained in [4], Proposition 2.9, Theorem 2.1 is a consequence of the following regularity result.

Proposition 2.2. *Suppose x is critical for E in either M, M^*, or $M^\dagger$. Then $X = X(x) \in H^{2,2}(B, \mathbb{R}^n)$.*

For the cases M and $M^\dagger$ the above assertions were established in [4], Propositions 2.9 and 2.10. Their proof is achieved by inserting

$$y = x - \tau^2(x - x(\phi_0)), \tag{5}$$

respectively

$$y = x + \varepsilon\partial_{-h}\partial_h x \tag{6}$$

as testing functions in (4), where $\tau \in C_0^\infty(B_{2r}(e^{i\phi_0}))$ is a cut-off function such that $0 \le \tau \le 1, \tau = 1$ on $B_r(e^{i\phi_0}), \phi_0 \in \mathbb{R}$, and where $\partial_h x(\phi) = h^{-1}(x(\phi+h) - x(\phi))$. In the case of M^* and $M^\dagger$ these comparison functions have to be normalized by composition with some $g \in \mathcal{G}$ to achieve $y \circ g \in M^*, M^\dagger$, respectively.

3. Normalization

The Möbius group $\mathcal{G}$ acts on M, as follows. For $g \in \mathcal{G}$ there exists a smooth map $\hat{g}\colon \mathbb{R} \to \mathbb{R}$ such that

$$g(e^{i\phi}) = e^{i\hat{g}(\phi)}, \phi \in \mathbb{R}. \tag{7}$$

Observe that $\hat{g}$ is uniquely determined by g up to an integer multiple of 2π; moreover, $\hat{g}$ is non-decreasing and satisfies $\hat{g}(\phi+2\pi) = \hat{g}(\phi)+2\pi$ for all $\phi \in \mathbb{R}$. Finally, for $x \in M, g \in \mathcal{G}$ with corresponding $X = X(x), \hat{g}\colon \mathbb{R} \to \mathbb{R}$, we have $X \circ g = X(x \circ \hat{g})$. Usually, we will indiscriminately write g also for $\hat{g}$.

In this section we establish the following two results.

Proposition 3.1. *For any $x \in M$ there exist conformal transformations $g^*, g^\dagger \in M$ such that $x \circ g^* \in M^*, x \circ g^\dagger \in M^\dagger$, respectively.*

Moreover, let $x \in M^*$ or $M^\dagger$, respectively, and let y be either of the two comparison functions defined in (5) or (6) above.

Proposition 3.2. *i) There is $r_0 = r_0(x) > 0$ such that for $0 < r < r_0$ and any $w_0 \in \partial B$ the function y defined by (5) may be normalized by composition with a suitable element $g \in \mathcal{G}$ so that $y \circ g \in M^*$ or $M^\dagger$, respectively, and there exists C such that $|g - id| \le Cr$ as $r \to 0$.*
ii) Similarly, if $x \in M^*$ there is $h_0 = h_0(x) > 0$ such that for $0 < h < h_0$ and $0 < 2\varepsilon \le h^2$ the function y defined by (6) may be normalized by composition with some $g \in \mathcal{G}$ so that $y \circ g \in M^*$ and $|g - id| \le Ch$ as $h \to 0$.

Observe that for $x \in M^\dagger$ and $0 < 2\varepsilon \le h^2$ the function y defined by (6) automatically belongs to $M^\dagger$; see [4], p. 67.

Proof of Proposition 3.1 i) The case M^* is classical. Indeed, let $\phi_1 < \phi_2 < \phi_3 < \phi_1 + 2\pi$ be points such that $x(\phi_k) = \frac{2\pi k}{3}, k = 1, 2, 3$. The unique element $g^* \in \mathcal{G}$ with

$$g^*(e^{i\frac{2\pi k}{3}}) = e^{i\phi_k}, k = 1, 2, 3$$

then achieves the desired normalization.

ii) To handle the case $M^\dagger$ denote as $\overline{F} = (F^1, F^2, F^3)^t \colon M \times \mathcal{G} \to \mathbb{R}^3$ the map

$$\overline{F}(x, g) = \int_0^{2\pi} (x \circ g - id) \begin{pmatrix} 1 \\ \sin\phi \\ \cos\phi \end{pmatrix} d\phi,$$

and for $\phi_0 \in \mathbb{R}, a = \alpha + i\beta \in B \subset \mathbb{C}$ denote as $g_{\phi_0,a} \in \mathcal{G}$ the map given by

$$g_{\phi_0,a}(w) = e^{i\phi_0} \frac{a + w}{1 + \overline{a}w}, w \in B.$$

Observe that, given $y \in M$ we may simply choose

$$\phi_y = -\frac{1}{2\pi} \int_0^{2\pi} (y - id)\, d\phi$$

to achieve the first normalization condition

$$F^1(y, g_{\phi_y,0}) = \int_0^{2\pi} (y(\phi + \phi_y) - \phi)\, d\phi = \int_0^{2\pi} (y - id)\, d\phi + 2\pi\phi_y = 0.$$

Moreover, for any $\phi_0 \in \mathbb{R}$ there holds

$$\begin{aligned} \begin{pmatrix} F^2(y, g_{\phi_0,0}) \\ F^3(y, g_{\phi_0,0}) \end{pmatrix} &= \int_0^{2\pi} (y(\phi + \phi_0) - \phi) \begin{pmatrix} \sin\phi \\ \cos\phi \end{pmatrix} d\phi \\ &= \int_0^{2\phi} (y - id) \begin{pmatrix} \sin(\phi - \phi_0) \\ \cos(\phi - \phi_0) \end{pmatrix} d\phi = \begin{pmatrix} \cos\phi_0 & -\sin\phi_0 \\ \sin\phi_0 & \cos\phi_0 \end{pmatrix} \begin{pmatrix} F^2(y, id) \\ F^3(y, id) \end{pmatrix}. \end{aligned}$$

Hence it suffices to find $a \in B$ solving the equations

$$F^2(x, g_{0,a}) = F^3(x, g_{0,a}) = 0;$$

the desired normalization will then be achieved by $g^\dagger = g_{0,a} \circ g_{\phi_y,0}$, where $y = x \circ g_{0,a} \in M$. To simplify the notation, let $F \colon B \to \mathbb{R}^2 \cong \mathbb{C}$ be the map given by

$$F(a) = \begin{pmatrix} F^2(x, g_{0,a}) \\ F^3(x, g_{0,a}) \end{pmatrix} = \int_0^{2\pi} (x \circ g_{0,a} - id) \begin{pmatrix} \sin\phi \\ \cos\phi \end{pmatrix} d\phi.$$

In Lemma 3.3 below we establish that F continuously extends to $\overline{B}$ with $F(a) = 2\pi a$ for $a \in \partial B$. Thus, the rescaled map $\frac{1}{2\pi} F_{|\partial B}$ induces a homeomorphism of ∂B of topological degree 1. The existence of an element $a \in B$ solving the equation

$F(a) = 0$ then follows from degree theory; see for instance [Deimling], p. ??? .

$\square$

Lemma 3.3. *The above map F continuously extends to $\overline{B}$ with $F(a) = 2\pi a, a \in \partial B$.*

Proof. By conformality, for any $g \in \mathcal{G}$ we have
$$D(g) = \frac{1}{2}\int_B |\nabla g|^2\, dw = A(g) = \pi.$$
Hence $\mathcal{G}$ is bounded in $H^{1,2}(B;\mathbb{R}^2)$. By compactness of the trace operator $H^{1,2}(B;\mathbb{R}^2) \ni g \mapsto g_{|\partial B} \in L^2(\partial B;\mathbb{R}^2)$ and continuity of x, therefore the set of maps $(x \circ g_{0,a})_{a\in B}$ is relatively compact in $L^2(\partial B;\mathbb{R}^2)$. Moreover, as $a \to a_0 = e^{i\phi_0} \in \partial B$ we have
$$g_{0,a}(w) = \frac{a+w}{1+\overline{a}w} \to a_0$$
locally uniformly on $\overline{B}\setminus\{-a_0\}$, and therefore $\hat{g}_{0,a}(\phi) \to \phi_0$, locally uniformly in $\phi \in]-\phi_0, 2\pi-\phi_0[$. Here we choose a representant of $\hat{g}_{0,a}$ such that $\hat{g}_{0,a}(\phi_0) \to \phi_0$ as $a \to a_0$.

We can thus continuously extend F to $\overline{B}$ and
$$\begin{aligned}
F(e^{i\phi_0}) &= \int_0^{2\pi-\phi_0} (x(\phi_0)-\phi)\begin{pmatrix}\sin\phi\\ \cos\phi\end{pmatrix} d\phi + \int_{2\pi-\phi_0}^{2\pi} (x(2\pi+\phi_0)-\phi)\begin{pmatrix}\sin\phi\\ \cos\phi\end{pmatrix} d\phi\\
&= \int_0^{2\pi} (x(\phi_0)-\phi)\begin{pmatrix}\sin\phi\\ \cos\phi\end{pmatrix} d\phi + 2\pi\int_{2\pi-\phi_0}^{2\pi} \begin{pmatrix}\sin\phi\\ \cos\phi\end{pmatrix} d\phi\\
&= 2\pi\begin{pmatrix}\cos\phi_0\\ \sin\phi_0\end{pmatrix} = 2\pi e^{i\phi_0},
\end{aligned}$$
as claimed. $\square$

For the proof of Proposition 3.2 we need the following expansion for $\hat{g}_{\phi_0,\alpha+i\beta}$ in terms of ϕ_0, α, and β near $\phi_0 = \alpha = \beta = 0$.

Observe that $\hat{g}_{\phi_0,a} = \hat{g}_{\phi_0,0} \circ \hat{g}_{0,a}$ and $\hat{g}_{\phi_0,0}(\phi) = \phi + \phi_0$. Moreover, the expansion
$$\begin{aligned}
g_{0,\alpha+i\beta}(e^{i\phi}) = e^{i\hat{g}_{0,\alpha+i\beta}(\phi)} &= \frac{\alpha + i\beta + \cos\phi + i\sin\phi}{1+\alpha\cos\phi+\beta\sin\phi + i(\alpha\sin\phi - \beta\cos\phi)}\\
&= e^{i\phi}(1 - 2i\alpha\sin\phi + 2i\beta\cos\phi) + O(\alpha^2+\beta^2)
\end{aligned}$$
implies
$$d_a\hat{g}_{0,a}(\phi)_{|_{a=0}} = \frac{d_a g_{0,a}(e^{i\phi})_{|_{a=0}}}{ie^{i\phi}} = (-2\sin\phi, 2\cos\beta).$$
Thus, we conclude that near $\phi_0 = \alpha = \beta = 0$ there holds
$$\hat{g}^{(\phi)}_{\phi_0,\alpha+i\beta} = \phi_0 - 2\alpha\sin\phi + 2\beta\cos\beta + O(\phi_0^2+\alpha^2+\beta^2). \tag{8}$$
Introducing ϕ_0, α, and β as coordinates for $\mathcal{G}$, and identifying $\overline{F}$ with the map $\tilde{F}\colon \{M - id\} \times \mathbb{R} \times B \to \mathbb{R}^3$, given by
$$\tilde{F}(\xi,\phi_0,a) = \overline{F}(x,g) = \int_0^{2\pi} \xi\circ\hat{g}\begin{pmatrix}1\\ \sin\phi\\ \cos\phi\end{pmatrix} d\phi + \int_0^{2\pi}(\hat{g}-id)\begin{pmatrix}1\\ \sin\phi\\ \cos\phi\end{pmatrix} d\phi$$
for $\xi = x - id, x \in M, \hat{g} = \hat{g}_{\phi_0,a}, \phi_0 \in \mathbb{R}, a \in B$, we now assert

Lemma 3.4. *$\tilde{F}$ extends to a map $\tilde{F} \in C^1(H^{1/2,2}(\mathbb{R}/2\pi) \times \mathbb{R} \times B; \mathbb{R}^3)$ and at any $\xi = x - id$, where $x \in M$, any $\phi_0 \in \mathbb{R}$, any $a = \alpha + i\beta \in B$, the differential $L = (\partial_{\phi_0}\tilde{F}, \partial_\alpha \tilde{F}, \partial_\beta \tilde{F})(\xi, \phi_0, a)\colon \mathbb{R}^3 \to \mathbb{R}^3$ is regular.*

Proof. Replacing x by $\tilde{x} = x \circ g_{\phi_0,a}$, if necessary, it suffices to verify the assertion for $\phi_0 = \alpha = \beta = 0$. After a change of coordinates, for any $\phi_0 \in \mathbb{R}, a \in B$ we have

$$\tilde{F}(\xi, \phi_0, a) = \int_0^{2\pi} \xi \begin{pmatrix} 1 \\ \sin(\hat{g}^{-1}(\phi)) \\ \cos(\hat{g}^{-1}(\phi)) \end{pmatrix} \frac{d\phi}{\hat{g}'(\phi)} + \int_0^{2\pi} (\hat{g} - id) \begin{pmatrix} 1 \\ \sin\phi \\ \cos\phi \end{pmatrix} d\phi,$$

where $\hat{g} = \hat{g}_{\phi_0,a}$.

Thus, $\tilde{F}$ is an affine function of ξ and therefore extends smoothly to $H^{1/2,2}(\mathbb{R}/2\pi) \times \mathbb{R} \times B$, as claimed. Moreover, denoting $d = \left(\frac{\partial}{\partial\phi_0}, \frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\beta}\right)$ we have

$$L = d\tilde{F}(\xi, 0) = \int_0^{2\pi} \left\{ d\hat{g} \left[\begin{pmatrix} 1 \\ \sin\phi \\ \cos\phi \end{pmatrix} - \xi \begin{pmatrix} 0 \\ \cos\phi \\ -\sin\phi \end{pmatrix} \right] - d\hat{g}'\xi \begin{pmatrix} 1 \\ \sin\phi \\ \cos\phi \end{pmatrix} \right\} d\phi.$$

Using that

$$d\hat{g} = (1, -2\sin\phi, 2\cos\phi), dg' = (0, -2\cos\phi, -2\sin\phi)$$

on account of (8), we thus obtain

$$L = 2 \begin{pmatrix} \pi & 0 & 0 \\ 0 & a_1 - \pi & -a_2 \\ 0 & a_2 & a_1 + \pi \end{pmatrix},$$

where

$$a_1 = \int_0^{2\pi} \xi \sin 2\phi \, d\phi, \quad a_2 = \int_0^{2\phi} \xi \cos 2\phi \, a\phi. \tag{9}$$

Lemma 3.4 thus is an immediate consequence of the following result. □

Lemma 3.5. *For any $x = \xi + id \in M$ and with a_1, a_2 given by (9), there holds $a_1^2 + a_2^2 < \pi^2$.*

Proof. Observe that for any $\phi_0 \in \mathbb{R}$ there holds

$$\begin{aligned} \begin{pmatrix} a_1' \\ a_2' \end{pmatrix} &= \int_0^{2\pi} \xi(\phi + \phi_0) \begin{pmatrix} \sin 2\phi \\ \cos 2\phi \end{pmatrix} d\phi = \int_0^{2\pi} \xi(\phi) \begin{pmatrix} \sin 2(\phi - \phi_0) \\ \cos 2(\phi - \phi_0) \end{pmatrix} d\phi \\ &= \begin{pmatrix} \cos 2\phi_0 & -\sin 2\phi_0 \\ \sin 2\phi_0 & \cos 2\phi_0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \end{aligned}$$

By suitable choice of ϕ_0 and replacing x by $x(\cdot + \phi_0)$, if necessary, we may therefore always assume that $a_2 = 0$ and $a_1 \geq 0$. Also noting that

$$a_1 = \int_0^{2\pi} (x(\phi) - \phi) \sin 2\phi \, d\phi = \int_0^{2\pi} x(\phi) \sin 2\phi \, d\phi + \pi,$$

it therefore suffices to show that

$$\int_0^{2\pi} x(\phi) \sin 2\phi \, d\phi < 0$$

for any $x \in M$.

Using monotonicity of x and the symmetries of the function $\phi \mapsto \sin 2\phi$, the latter integral can be estimated

$$\int_0^{2\pi} x(\phi)\sin 2\phi\, d\phi = \int_0^{\pi/2} (x(\phi) - x(\phi + \frac{\pi}{2}))\sin 2\phi\, d\phi$$
$$+ \int_\pi^{3\pi/2} (x(\phi) - x(\phi + \frac{\pi}{2}))\sin 2\phi\, d\phi \le 0,$$

and equality holds if and only if

$$x(\phi) = x(\phi + \frac{\pi}{2}) \text{ for } \phi \in]0, \frac{\pi}{2}[\cup]\pi, \frac{3\pi}{2}[. \tag{10}$$

By continuity of x in the latter case we also have (10) for $\phi \in [0, \frac{\pi}{2}] \cup [\pi, \frac{3\pi}{2}]$, thus arriving at the contradiction

$$x(0) = x(\frac{\pi}{2}) = x(\pi) = x(\frac{3\pi}{2}) = x(2\pi) = x(0) + 2\pi.$$

The proof is complete. □

Proof of Proposition 3.2 i) In the case $M^\dagger$ the asserted result is an immediate consequence of Lemma 3.4 and the implicit function theorem.

In the case $x \in M^*$ we argue as follows. Let y be given by (5), where $w_0 = e^{i\phi_0} \in \partial B$. If $B_{2r}(w_0)$ does not contain any of the points $e^{\frac{i2\pi k}{3}}, k = 1, 2, 3$, then $y \in M^*$ and we are done. Otherwise, suppose $1 \in B_{2r}(w_0)$ and let $w_{1,2} = e^{i\phi_{1,2}}, \phi_1 < 0 < \phi_2$, be the endpoints of the arc $\partial B \cap B_{2r}(w_0)$. Note that the assumption $1 \in B_{2r}(w_0)$ implies $|\phi_0| < 3r$ and hence $|\phi_{1,2}| < |\phi_0| + 3r < 6r$. Moreover, from the relation

$$y(\phi_1) = x(\phi_1) < 0 < x(\phi_2) = y(\phi_2) \tag{11}$$

and continuity of y we conclude that there exists $\overline{\phi} \in]\phi_1, \phi_2[$ such that $y(\overline{\phi}) = 0$. Using (8) and the fact that the three vectors

$$d\hat{g}_{0,0}(0) = (1, 0, 2),\ d\hat{g}_{0,0}(2\pi/3) = (1, -\sqrt{3}, -1),\ d\hat{g}_{0,0}(4\pi/3) = (1, \sqrt{3}, -1)$$

are linearly independent, for sufficiently small $r < r_0$ we then can determine a map $g = g_{\phi_y, a} \in \mathcal{G}$ such that $\hat{g}(0) = \overline{\phi}, \hat{g}\left(\frac{2\pi k}{3}\right) = \frac{2\pi k}{3}, k = 1, 2$, with $|\phi_y| + |a| \le C|\overline{\phi}| \le Cr$, as desired.

ii) For $x \in M^*$ and y given by (6) with $0 < 2\varepsilon < h^2$, similar to (11) we have

$$\begin{aligned} y(2\pi k/3 - h) &= \left(1 - \frac{2\varepsilon}{h^2}\right) x(2\pi k/3 - h) + \frac{\varepsilon^2}{h}\Big(x(2\pi k/3) + x(2\pi k/3 - 2h)\Big) \\ &\le \frac{2\pi k}{3} \le \left(1 - \frac{2\varepsilon}{h^2}\right) x(2\pi k/3 + h) + \frac{\varepsilon^2}{h}\Big(x(2\pi k/3 + 2h) + x(2\pi k/3)\Big) \\ &\qquad = y(2\pi k/3 + h), k = 1, 2, 3. \end{aligned}$$

Thus, for sufficiently small $h < h_0$ there exists $g \in \mathcal{G}$ with $|g - id| < Ch$ such that $y \circ g \in M^\dagger$, as claimed. □

4. Estimates

Finally, we establish the regularity estimates required for the proof of Proposition 2.2. Let x be a critical point of E in M^* or $M^\dagger$, and let y be a comparison function, g an element of $\mathcal{G}$ such that $y \circ g \in M^*$ or $M^\dagger$, respectively, as determined in Proposition 3.2. Also let $X = X(x), Y = X(y)$. Observe that $X \circ g = X(x \circ g)$, and similarly for Y.

Finally, note that by harmonicity of X there holds $\int_B \langle \nabla X, \nabla \varphi \rangle \, dw = 0$ for all $\varphi \in C_0^\infty(B; \mathbb{R}^n)$. Hence for any $Z \in H^{1,2}(B; \mathbb{R}^n)$ the integral $\int_B \langle \nabla X, \nabla Z \rangle \, dw = \int_{\partial B} \langle \partial_n X, Z \rangle \, do$ depends only on the trace $Z|_{\partial B} \in H^{1/2,2}(\partial B; \mathbb{R}^n)$. We will use this observation to replace a comparison surface Y by a more convenient extension of its boundary values, whenever necessary.

Our aim is to derive the following Dirichlet growth estimate for X; see [4], Lemma 5.4.

Lemma 4.1. *There exist constants $\alpha > 0, C$ such that for all $r > 0, w_0 \in \partial B$ there holds*

$$\int_{B \cap B_r(w_0)} |\nabla X|^2 \, dw \le C r^{2\alpha}.$$

Proof. As in [4], the desired estimate will follow by inserting the function y given by (5) as comparison function.

On ∂B, for any $z \in M$, letting $Z = X(z)$ we have

$$X - Z = \gamma \circ x - \gamma \circ z = \frac{d}{d\phi}\gamma(x)(x - z) - \int_x^z \int_x^{x'} \frac{d^2}{d\phi^2}\gamma(x'') \, dx'' \, dx'; \tag{12}$$

compare [4], p. 50. In particular, for $z = y \circ g$ we obtain

$$\frac{d}{d\phi}\gamma(x)(x - y \circ g) = X - Y \circ g + \psi,$$

where

$$\psi = \int_x^{y \circ g} \int_x^{x'} \frac{d^2}{d\phi^2}\gamma(x'') \, dx'' \, dx'$$

Thus, from (4) we have

$$0 \ge \langle dE(x), x - y \circ g \rangle = I + II,$$

where

$$\begin{aligned} I &= \int_B \langle \partial_n X, X - Y \circ g \rangle \, dw \\ &= \int_B \langle \nabla X, \nabla (X - Y) \rangle \, dw + \int_B \langle \nabla X, \nabla (Y - Y \circ g) \rangle \, dw =: I_1 + I_2. \end{aligned}$$

By conformal invariance, the second term equals

$$I_2 = \int_B \langle \nabla X, \nabla (Y - Y \circ g) \rangle \, dw = \int_B \langle \nabla (X - X \circ g^{-1}), \nabla Y \rangle \, dw.$$

Adding

$$\frac{X - X \circ g^{-1}}{2} + \frac{X + X \circ g^{-1}}{2} - X = 0$$

to Y, the latter may be estimated

$$\int_B \langle \nabla(X - X\circ g^{-1}), \nabla Y\rangle\, dw = \frac{1}{2}\int_B |\nabla(X - X\circ g^{-1}|^2\, dw$$
$$+\frac{1}{2}\int_B \left(|\nabla X|^2 - |\nabla(X\circ g^{-1})|^2\right)\, dw + \int_B \langle \nabla(X - X\circ g^{-1}), \nabla(Y - X)\rangle\, dw$$
$$\geq \frac{1-\delta}{2}\int_B |\nabla(X - X\circ g^{-1})|^2\, dw - \frac{1}{2\delta}\int_B |\nabla(X-Y)|^2\, dw.$$

Here we also used the fact that by conformal invariance

$$\int_B |\nabla X|^2\, dw = \int_B |\nabla(X\circ g^{-1})|^2\, dw;$$

moreover, we employed Young's inequality $2ab \leq \delta a^2 + \delta^{-1}b^2$ for $0 < \delta, a, b \in \mathbb{R}$.

In case y is given by (5), on ∂B we have

$$X - Y = \gamma\circ x - \gamma\circ(x - \tau^2(x - x_0))$$
$$= \frac{d}{d\phi}\gamma(x)\tau^2(x - x_0) - \int_x^{x-\tau^2(x-x_0)}\int_x^{x'} \frac{d^2}{d\phi^2}\gamma(z'')dz''dz'. \tag{13}$$

Also let $X_0 = \gamma(x_0)$, satisfying

$$X - X_0 = \frac{d}{d\phi}\gamma(x)(x - x_0) - \int_x^{x_0}\int_x^{x'} \frac{d^2}{dp^2}\gamma(x'')\, dx''\, dx' \tag{14}$$

on ∂B.

Observe that, by continuity of X and x, for sufficiently small $r < r_0$ we may use (14) to define a unique extension $x \in H^{1,2}(B\cap B_{3r}(w_0))$ satisfying the estimates

$$C^{-1}|x - x_0| \leq |X - X_0| \leq C|x - x_0|, \tag{15}$$

$$C^{-1}|\nabla x| \leq |\nabla X| \leq C|\nabla x| \tag{16}$$

with a uniform constant C. Indeed multiplying (14) by $\frac{d}{d\phi}\gamma(x)/|\frac{d}{d\phi}\gamma(x)|^2$ and observing that

$$\left|\int_x^{x_0}\int_x^{x'} \frac{d^2}{d\phi^2}\gamma(x'')\, dx''\, dx'\right| \leq C|x - x_0|^2 \leq o(1)|x - x_0|,$$

where $o(1) \to 0$ as $r \to 0$, the first estimate immediately follows. The second estimate is obtained in the same way from the differentiated equation (14).

Using (13) and (14) we now modify Y in B by letting

$$\tilde{Y} = X - \tau^2(X - X_0) - \tau^2\int_x^{x_0}\int_x^{x'} \frac{d^2}{d\phi^2}\gamma(x'')\, dx''\, dx'$$
$$+\int_x^{x-\tau^2(x-x_0)}\int_x^{x'} \frac{d^2}{d\phi^2}\gamma(x'')\, dx''\, dx'.$$

Observe that $Y = \tilde{Y}$ on ∂B; thus, by our above remark there holds

$$\int_B \langle \nabla X, \nabla Y\rangle\, dw = \int_B \langle \nabla X, \nabla\tilde{Y}\rangle\, dw.$$

Moreover, we have the estimates

$$|X - \tau^2(X - X_0) - \tilde{Y}| \le C\tau^2|x - x_0|^2 \le C\tau^2|X - X_0|^2$$

and

$$\begin{aligned}|\nabla(X - \tau^2(X - X_0)) - \nabla\tilde{Y}| &\le C|\nabla\tau|\tau|X - X_0|^2 + C(1-\tau^2)\tau^2|\nabla X||X - X_0| \\ &\le C|\nabla\tau|\tau|X - X_0| + C\tau^2|\nabla X||X - X_0|,\end{aligned}$$

as is easily verified.

Thus, for sufficiently small $r > 0$ we can estimate

$$\begin{aligned}\int_B \langle\nabla X, \nabla(X - Y)\rangle\, dw &= \int_B \langle\nabla X, \nabla(X - \tilde{Y})\rangle\, dw \\ &\ge \int_B (1 - \tau^2|X - X_0|)|\nabla X|^2\tau^2\, dw - C\int_B |\nabla X||X - X_0||\nabla\tau|\tau\, dw \\ &\ge \frac{7}{8}\int_B |\nabla X|^2\tau^2\, dw - C\int_B |X - X_0|^2|\nabla\tau|^2\, dw.\end{aligned}$$

Moreover, choosing $\delta = \frac{3}{4}$, we can estimate

$$\begin{aligned}\frac{1}{2\delta}\int_B |\nabla(X - Y)|^2\, dw &\le \frac{2}{3}\int_B ((1 + \tau^2|X - X_0|)|\nabla X|\tau^2 + C\tau|\nabla\tau||X - X_0|)^2\, dw \\ &\le \frac{3}{4}\int_B |\nabla X|^2\tau^2\, dw + C\int_B |X - X_0|^2|\nabla\tau|^2\, dw,\end{aligned}$$

whence

$$I \ge \frac{1}{8}\int_B |\nabla X|^2\tau^2\, dw + \frac{1}{8}\int_B |\nabla(X - X\circ g^{-1}|^2\, dw - C\int_B |X - X_0|^2|\nabla\tau|^2\, dw.$$

In order to bound II, we split

$$\begin{aligned}\psi &= \int_{x\circ g}^{y\circ g}\int_x^{x'} \frac{d^2}{d\phi^2}\gamma(x'')\, dx''\, dx' + \int_x^{x\circ g}\int_x^{x'} \frac{d^2}{d\phi^2}\gamma(x'')\, dx''\, dx' =: \psi_1 + \psi_2, \\ II &= \int_{\partial B}\langle\partial_n X, \psi\rangle\, do = \int_{\partial B}\langle\partial_n X, \psi_1\rangle\, do + \int_{\partial B}\langle\partial_n X, \psi_2\rangle\, do =: II_1 + II_2.\end{aligned}$$

Extending x to balls $B_{r_0}(w_0) \cap B$ as above and using a partition of unity, we extend x to a neighborhood of ∂B in B in such a way that (16) and the analogous estimates

$$C^{-1}|x\circ g - x| \le |X\circ g - X| \le C|x\circ g - x| \tag{17}$$

$$C^{-1}|\nabla(x\circ g - x)| \le |\nabla(X\circ g - X)| \le C|\nabla(x\circ g - x)| \tag{18}$$

hold. Also let $y = x - \tau^2(x - x_0)$. Since $\mathrm{supp}(\psi_1) \subset g^{-1}(B \cap B_{2r}(w_0))$, we may use these estimates to bound

$$|\nabla\psi_1| \le C|\nabla(y\circ g - x\circ g)||x\circ g - x| + C(|\nabla(y\circ g)| + |\nabla x|)|y\circ g - x\circ g|.$$

Hence

$$\begin{aligned}II_1 &= \int_{\partial B}\langle\partial_n X, \psi_1\rangle\, do = \int_B \langle\nabla X, \nabla\psi_1\rangle\, dw \\ &\le C\int_B |\nabla X||\nabla(y\circ g) - \nabla(x\circ g)||x\circ g - x|\, dw \\ &\quad + C\int_B |\nabla X|(|\nabla x| + |\nabla(y\circ g)|)|y\circ g - x\circ g|\, dw = II_{11} + II_{12}.\end{aligned}$$

By conformal invariance then, and since $g \to id$ uniformly as $r \to 0$ on account of Proposition 3.2 and since therefore, by continuity of x, also $x \circ g^{-1} \to x$, we can estimate

$$\begin{aligned}
II_1 &= C \int_B |\nabla(X \circ g^{-1})||\nabla(y-x)||x - x \circ g^{-1}|\, dw \\
&\leq C||x - x \circ g^{-1}||_{L^\infty(\partial B)} \Big(\int_B |\nabla X||\nabla(y-x)|\, dw \\
&\quad + \int_B |\nabla(X - X \circ g^{-1}|^2\, dw + \int_B |\nabla(y-x)|^2\, dw \Big) \\
&\leq o(1) \Big(\int_B (|\nabla X|^2 + |\nabla x|^2 \Big) \tau^2\, dw + \int_B |\nabla(X - X \circ g^{-1})|^2\, dw \\
&\quad + \int_B |x - x_0|^2 |\nabla \tau|^2\, dw \Big),
\end{aligned}$$

where $o(1) \to 0$ as $r \to 0$.

Similarly,

$$\begin{aligned}
II_{12} &\leq C \int_B |\nabla(X \circ g^{-1})|(|\nabla(x \circ g^{-1})| + |\nabla y|)|y - x|\, dw \\
&\leq C||x - x_0||_{L^\infty(B_{3r}(x_0))} \Big(\int_B |\nabla(X - X \circ g^{-1})|^2\, dw \\
&\quad + \int_{B \cap B_{3r}(w_0)} (|\nabla X|^2 + |\nabla x|^2 + |x - x_0|^2 |\nabla \tau|^2)\, dw \Big),
\end{aligned}$$

where $||x - x_0||_{L^\infty(B_{3r}(w_0))} \to 0$ as $r \to 0$.

Fix $\eta \in C_0^\infty(B)$ such that $0 \leq \eta \leq 1$ and $\eta(w) = 1$ for $|w| < 1 - r_0$ and extend ψ_2 to B as

$$\psi_2 = (1 - \eta) \int_x^{x \circ g} \int_x^{x'} \frac{d^2}{d\phi^2} \gamma(x'')\, dx''\, dx',$$

where x on $B \setminus B_{1-r_0}(0)$ is defined above.

We have

$$|\psi_2| \leq C|x \circ g - x|^2 \leq C|X \circ g - X|^2;$$

moreover, near ∂B (omitting η) we compute

$$\begin{aligned}
|\nabla \psi_2| &= |\nabla(x \circ g) \int_x^{x \circ g} \frac{d^2}{d\phi^2} \gamma(x')\, dx' - \nabla x (x \circ g - x) \frac{d^2}{d\phi^2} \gamma(x)| \\
&= |\nabla(x \circ g - x) \int_x^{x \circ g} \frac{d^2}{d\phi^2} \gamma(x')\, dx' + \nabla x \int_x^{x \circ g} (\frac{d^2}{d\phi^2} \gamma(x') - \frac{d^2}{d\phi^2} \gamma(x))\, dx'| \\
&\leq C|\nabla(x \circ g - x)||x \circ g - x| + C|\nabla x||x \circ g - x|^2 \\
&\leq C|\nabla(X \circ g - X)||X \circ g - X| + C|\nabla X||X \circ g - X|^2.
\end{aligned} \tag{19}$$

Hence

$$\begin{aligned} II_2 &= \int_{\partial B} \langle \partial_n X, \psi_2 \rangle \, do = \int_{\partial B} \langle \nabla X, \nabla \psi_2 \rangle \, dw \\ &\leq C \int_B |\nabla X||\nabla X \circ g - X||X \circ g - X| \, dw \\ &\quad + C \int_B (|\nabla X|^2 |X \circ g - X|^2 + |\nabla X||\nabla \eta||X \circ g - X|^2) \, dw = II_{21} + II_{22}. \end{aligned}$$

By Young's inequality, we have

$$\begin{aligned} II_{21} &\leq C||X \circ g - X||_{L^\infty}^2 \int_B |\nabla X|^2 \, dw + \frac{1}{16} \int_B |\nabla (X - X \circ g^{-1})|^2 \, dw, \\ II_{22} &\leq C||X \circ g - X||_{L^\infty}^2 \int_B (1 + |\nabla X|^2) \, dw. \end{aligned}$$

Together with our bounds for I and II_1, these estimates finally yield that for sufficiently small $r < r_0$ there holds

$$0 \geq I + II \geq \frac{1}{16} \int_B |\nabla X|^2 \tau^2 \, dw - C \int_B |X - X_0|^2 |\nabla \tau|^2 \, dw - C||X \circ g - X||_{L^\infty}^2.$$

Now by Proposition 3.2 we have $|g - id| \leq Cr$ as $r \to 0$; in fact, $|g - id| \leq o(1)r$ in case that we use the normalization $M^\dagger$. Therefore, by the Courant Lebesgue Lemma [1], p. 102, we can estimate

$$\begin{aligned} ||X \circ g - X||_{L^\infty}^2 &\leq C \sup_{|\phi - \phi'| \leq r} |X(e^{i\phi}) - X(e^{i\phi'})|^2 \\ &\leq C \sup_{w_1 \in \partial B} \int_{B_r(w_1)} |\nabla X|^2 \, dw / |lnr|. \end{aligned}$$

As in [4], p. 71 f., thus for sufficiently small $r > 0$ and all $w_0 \in \partial B$ we find

$$\begin{aligned} \int_{B \cap B_r(w_0)} |\nabla X|^2 \, dw &\leq Cr^{-2} \int_{B \cap B_{3r} \setminus B_r(w_0)} |X - X_0|^2 \, dw + \frac{1}{2} \sup_{w_1 \in \partial B} \int_{B_r(w_1)} |\nabla X|^2 \, dw \\ &\leq C \int_{B \cap B_{3r} \setminus B_r(w_0)} |\nabla X|^2 \, dw + \frac{1}{2} \sup_{w_1 \in \partial A} \int_{B_r(w_1)} |\nabla X|^2 \, dw. \end{aligned}$$

In particular, if we choose $w_0 \in \partial B$ such that

$$\int_{B \cap B_r(w_0)} |\nabla X|^2 \, dw = \sup_{w_1 \in \partial B} \int_{B_r(w_1)} |\nabla X|^2 \, dw =: \phi(r),$$

from the preceding estimate we obtain that

$$\int_{B \cap B_r(w_0)} |\nabla X|^2 \, dw \leq C_1 \int_{B \cap B_{3r} \setminus B_r(w_0)} |\nabla X|^2 \, dw. \tag{20}$$

Upon adding C_1 times the expression on the left of (20) to both sides, thus "filling the hole" on the right following an idea of Widman [5], and dividing by $1 + C_1$, we obtain the inequality

$$\phi(r) \leq \vartheta \int_{B \cap B_{3r}(w_0)} |\nabla X|^2 dw \leq \vartheta \phi(3r)$$

for any $r \leq r_0$, where $\vartheta = \frac{C_1}{1+C_1} < 1$. By iteration we then obtain the Dirichlet growth condition

$$\phi(r) \leq Cr^{2\alpha}, 0 < r < r_0, \tag{21}$$

where $\alpha = -\frac{1}{2}\frac{ln\vartheta}{ln 2} > 0$. □

Proof of Proposition 2.2

i) In the case of the normalization $M^\dagger$ the proof of Proposition 2.2 can now be completed as in [4], pp. 67–69 and p. 73 f.

ii) In the case of the normalization M^*, the comparison function y given by (6) in general does not belong to M^*. In this case we argue as follows.

By Proposition 3.1. ii) for sufficiently small $h < h_0$ and any $\varepsilon < \frac{1}{2}h^2$ there exists $g \in \mathcal{G}$ with $|g - id| \le Ch$ such that $y \circ g \in M^*$. Hence, if $x \in M^*$ is critical, we have

$$0 \ge \langle dE(x), x - y \circ g\rangle = \langle dE(x), x - y\rangle + \langle dE(x), y - y \circ g\rangle =: I + II.$$

As shown in [4], p. 67 ff., the contribution from the first term can be estimated

$$I = -\varepsilon \int_{\partial B} \left\langle \partial_n X, \frac{d}{d\phi}\gamma(x)\partial_{-h}\partial_h x\right\rangle do \ge \frac{\varepsilon}{2}\int_B |\nabla \partial_h X|^2\, dw - C\varepsilon \int_B |\nabla X|^2 |\partial_h X|^2\, dw.$$

(The ε here has a different meaning than in [4], p. 68.)

In order to bound the second term we need to derive a convenient representation for $\frac{d}{d\phi}\gamma(x)(y - y\circ g)$. In fact, by (12), applied to $z = y$ and $z = y \circ g$, on ∂B we have

$$\frac{d}{d\phi}\gamma(x)(y - y \circ g) = Y - Y \circ g + \psi_1,$$

where $Y = X(y)$ and

$$\begin{aligned}\psi_1 &= \int_x^{y\circ g}\int_x^{x'} \frac{d^2}{d\phi^2}\gamma(x'')\, dx''\, dx' - \int_x^{y}\int_x^{x'} \frac{d^2}{d\phi^2}\gamma(x'')\, dx''\, dx' \\ &= \left(\int_{x\circ g}^{y\circ g}\int_x^{x'} \frac{d^2}{d\phi^2}\gamma(x'')\, dx''\, dx' - \int_x^{y}\int_x^{x'} \frac{d^2}{d\phi^2}\gamma(x'')\, dx''\, dx'\right) \\ &\quad + \int_x^{x\circ g}\int_x^{x'} \frac{d^2}{d\phi^2}\gamma(x'')\, dx''\, dx' = \psi_{11} + \psi_{12}.\end{aligned}$$

We split $II = II_1 + II_2$, where

$$II_1 = \int_{\partial B} \langle \partial_n X, \psi_1\rangle\, do, \quad II_2 = \int_{\partial B} \langle \partial_n X, Y - Y \circ g\rangle\, do.$$

We first bound the second term II_2.

Recall that by harmonicity of X in this expression we may replace Y by any function $\tilde{Y} \in H^{1,2}(B; \mathbb{R}^n)$ such that $\tilde{Y} = Y$ on ∂B. In order to construct a suitable such extension, observe that, letting $X_h(e^{i\phi}) = X(e^{i(\phi+h)})$ on ∂B, we have

$$\varepsilon \partial_{-h}\partial_h X = \frac{\varepsilon}{h^2}(X_h - X) + \frac{\varepsilon}{h^2}(X_{-h} - X).$$

By (12), with $z = x_h = x(\cdot + h)$, we may express

$$X_h - X = \frac{d}{d\phi}\gamma(x)(x_h - x) + \int_x^{x_h}\int_x^{x'} \frac{d^2}{d\phi^2}\gamma(x'')\, dx''\, dx',$$

and similarly for X_{-h}.

On the other hand, for $z = y$ equation (12) yields

$$Y = X + \varepsilon \frac{d}{d\phi}\gamma(x)\partial_{-h}\partial_h x + \int_x^y \int_x^{x'} \frac{d^2}{d\phi^2}\gamma(x'')\, dx''\, dx'.$$

Hence we may let

$$\begin{aligned}\tilde{Y} &= X + \varepsilon\partial_{-h}\partial_h X + (1-\eta)\Big(\int_x^y \int_x^{x'} \frac{d^2}{d\phi^2}\gamma(x'')\, dx''\, dx' \\ &\quad - \frac{\varepsilon}{h^2}\int_x^{x_h}\int_x^{x'} \frac{d^2}{d\phi^2}\gamma(x'')\, dx''\, dx' - \frac{\varepsilon}{h^2}\int_x^{x_{-h}}\int_x^{x'} \frac{d^2}{d\phi^2}\gamma(x'')\, dx''\, dx' \Big) \\ &=: X + \varepsilon\partial_{-h}\partial_h X + \psi_2,\end{aligned}$$

where $\eta \in C_0^\infty(B)$ satisfies $0 \le \eta \le 1, \eta \equiv 1$ on $B_{1-r_0}(0)$, as in the proof of Lemma 4.1.

Similar to our estimate for I_2 in the proof of Lemma 4.1 then we have

$$\begin{aligned}II_2 &= \int_{\partial B} \langle \partial_n X, \tilde{Y} - \tilde{Y}\circ g\rangle\, do = \int_B \langle \nabla X, \nabla(\tilde{Y} - \tilde{Y}\circ g)\rangle\, dw \\ &= \int_B \langle \nabla(X - X\circ g^{-1}), \nabla\tilde{Y}\rangle\, dw \\ &= \frac{1}{2}\int_B |\nabla(X - X\circ g^{-1}|^2\, dw + \int_B \langle \nabla(X - X\circ g^{-1}), \nabla(\tilde{Y} - X)\rangle\, dw \\ &= \frac{1}{2}\int_B |\nabla(X - X\circ g^{-1})|^2\, dw + \int_B \langle \nabla(X - X\circ g^{-1}), \nabla(\varepsilon\partial_{-h}\partial_h X)\rangle\, dw \\ &\quad + \int_B \langle \nabla(X - X\circ g^{-1}), \nabla\psi_2\rangle\, dw \\ &\ge \frac{1}{4}\int_B |\nabla(X - X\circ g^{-1})|^2\, dw - 2\varepsilon^2\int_B |\nabla\partial_{-h}\partial_h X|^2\, dw - 2\int_B |\nabla\psi_2|^2\, dw \\ &= II_{21} + II_{22} + II_{23}.\end{aligned}$$

But

$$\int_B |\nabla\partial_{-h}\partial_h X|^2\, dw \le 2h^{-2}\int_B |\nabla\partial_h X|^2\, dw$$

and, choosing $\varepsilon \le \frac{1}{8}h^2$, we can estimate

$$|II_{22}| \le \frac{\varepsilon}{4}\int_B |\nabla\partial_h X|^2\, dw.$$

Moreover, we easily verify that

$$|\psi_2| \le C|\varepsilon\partial_{-h}\partial_h x|^2 + C\varepsilon\left(|\partial_h x|^2 + |\partial_{-h}x|^2\right) \le C\varepsilon\left(|\partial_h x|^2 + |\partial_{-h}x|^2\right).$$

Near ∂B, where $\eta = 0$, we also have

$$\begin{aligned}|\nabla\psi_2| &\le C(|\nabla y| + |\nabla x|)|y - x| \\ &\quad + C\frac{\varepsilon}{h^2}(|\nabla x_h| + |\nabla x_{-h}| + |\nabla x|)(|x - x_h| + |x - x_{-h}|) \\ &\le \frac{C\varepsilon}{h}(|\nabla x_h| + |\nabla x_{-h}| + |\nabla x|)(|\partial_h x| + |\partial_{-h}x|),\end{aligned}$$

and hence, observing that $\varepsilon \leq \frac{1}{8}h^2$,

$$|II_{23}| \leq 2\int_B |\nabla\psi_2|^2\,dw \leq C\varepsilon^2 \int_B (|\nabla\eta|^2)(|\partial_h x|^2 + |\partial_{-h}x|^2)\,dw$$
$$+ C\varepsilon \int_B (1-\eta)^2(|\nabla x_h|^2 + |\nabla x_{-h}|^2 + |\nabla x|^2)(|\partial_h x|^2 + |\partial_{-h}x|^2)\,dw.$$

Similar to our bounds for II in the proof of Lemma 4.1 we now estimate

$$II_1 = \int_{\partial B} \langle \partial_n X, \psi_1\rangle\,do = \int_{\partial B} \langle \partial_n X, \psi_{11}\rangle\,do + \int_{\partial B} \langle \partial_n X, \psi_{12}\rangle\,do = II_{11} + II_{12}.$$

Again we use the extension of x to an r_0-neighborhood of ∂B and truncate with $1-\eta$ to define extensions of ψ_{11}, ψ_{12} to B.

Then we have

$$\begin{aligned}|\psi_{11}| &\leq C(|y\circ g - x\circ g||x\circ g - x| + |y\circ g - x\circ g|^2 + |y-x|^2)\\ &\leq C\varepsilon|\partial_{-h}\partial_h x\circ g||x\circ g - x| + C\varepsilon^2(|\partial_{-h}\partial_h x\circ g|^2 + |\partial_{-h}\partial_h x|^2),\\ &\leq o(1)\varepsilon(|\partial_{-h}\partial_h x\circ g| + |\partial_{-h}\partial_h x|),\end{aligned}$$

where we observe that

$$||x\circ g - x||_{L^\infty} + ||\varepsilon\partial_{-h}\partial_h x||_{L^\infty} = o(1) \to 0$$

as $h \to 0, \varepsilon \leq \frac{1}{8}h^2$.

Moreover,

$$\begin{aligned}|\nabla\psi_{11}| = |(\nabla y\circ g - \nabla x\circ g)\int_x^{x\circ g} \frac{d^2}{d\phi^2}\gamma(x')\,dx'&\\ + \nabla y\circ g\int_{x\circ g}^{y\circ g} \frac{d^2}{d\phi^2}\gamma(x')\,dx' - \nabla y\int_x^y \frac{d^2}{d\phi^2}\gamma(x')\,dx'&\\ - \nabla x[(y\circ g - x\circ g) - (y-x)]\frac{d^2}{d\phi^2}\gamma(x)|&\\ \leq C\varepsilon|\nabla\partial_{-h}\partial_h x\circ g||x\circ g - x| + C|\nabla y\circ g - \nabla y||y\circ g - x\circ g|&\\ + C\varepsilon|\nabla\partial_{-h}\partial_h x|(|y\circ g - x\circ g| + |y-x|)&\\ + C|\nabla x|(|y\circ g - x\circ g|^2 + |y-x|^2 + |x\circ g - x|^2),&\end{aligned}$$

where we have expanded

$$-\nabla y\int_x^y \frac{d^2}{d\phi^2}\gamma(x')\,dx' = -\nabla y\int_{x\circ g}^{y\circ g}\ldots + \nabla y\left(\int_{x\circ g}^{y\circ g}\ldots - \int_x^y\ldots\right)$$

and then combined the second term with the last term in the expansion of $\nabla\psi_{11}$ to obtain

$$\begin{aligned}
&\left|\nabla y\left(\int_{x\circ g}^{y\circ g}\frac{d^2}{d\phi^2}\gamma(x')\,dx'-\int_x^y\frac{d^2}{d\phi^2}\gamma(x')\,dx'\right)\right.\\
&\quad\left.-\nabla x\Big((y\circ g-x\circ g)-(y-x)\Big)\frac{d^2}{d\phi^2}\gamma(x)\right|\\
&\le C|\nabla y-\nabla x|(|y\circ g-x\circ g|+|y-x|)\\
&\quad+\left|\nabla x\int_{x\circ g}^{y\circ g}\left(\frac{d^2}{d\phi^2}\gamma(x')-\frac{d^2}{d\phi^2}(x)\right)\,dx'\right|+\left|\nabla x\int_x^y\left(\frac{d^2}{d\phi^2}\gamma(x')-\frac{d^2}{d\phi^2}\gamma(x)\right)\,dx'\right|\\
&\le C\varepsilon|\nabla\partial_{-h}\partial_h x|(|y\circ g-x\circ g|+|y-x|)\\
&\quad+C|\nabla x||y\circ g-x\circ g|(|y\circ g-x\circ g|+|x\circ g-x|)+C|\nabla x||y-x|^2.
\end{aligned}$$

Observing that $|\nabla\partial_{-h}\partial_h x|\le h^{-1}(|\nabla\partial_h x|+|\nabla\partial_{-h}x|), h^{-1}|y-x|\le\varepsilon h^{-2}(|\partial_h x|+|\partial_{-h}x|)\le|\partial_h x|+|\partial_{-h}x|$, and recalling that $\varepsilon\le\frac{1}{8}h^2$, thus we find

$$\begin{aligned}
|\nabla\psi_{11}|\le\;& C\varepsilon(|\nabla\partial_h x\circ g|+|\nabla\partial_{-h}x\circ g|)|h^{-1}(x\circ g-x)|\\
&+C\varepsilon(|\partial_h x\circ g|+|\partial_{-h}x\circ g|)|h^{-1}(\nabla y\circ g-\nabla y)|\\
&+C\varepsilon(|\nabla\partial_h x|+|\nabla\partial_{-h}x|)(|\partial_h x\circ g|+|\partial_{-h}x\circ g|+|\partial_h x|+|\partial_{-h}x|)\\
&+C\varepsilon|\nabla x|\left(|\partial_h x\circ g|^2+|\partial_{-h}x\circ g|^2+|\partial_h x|^2+|\partial_{-h}x|^2+|h^{-1}(x\circ g-x)|^2\right).
\end{aligned}$$

In conclusion form this, for any $\delta>0$ we can bound

$$\begin{aligned}
II_{11}&\le\int_B|\nabla X|(|\nabla\eta||\psi_{11}|+(1-\eta)|\nabla\psi_{11}|)\,dw\\
&\le\frac{\varepsilon}{16}\int_{B\setminus B_{1-r_0}(0)}|\nabla\partial_h x|^2\,dw+\delta\varepsilon\int_{B\setminus B_{1-r_0}(0)}|h^{-1}(\nabla y\circ g-\nabla y)|^2\,dw\\
&\quad+C\varepsilon\int_B|\nabla X|^2\,dw+C(\delta)\varepsilon\int_{B\setminus B_{1-r_0}(0)}(|\nabla X|^2+|\nabla x|^2)\\
&\qquad\cdot(|h^{-1}(x\circ g-x)|^2+|\partial_h x\circ g|^2+|\partial_{-h}x\circ g|^2+|\partial_h x|^2+|\partial_{-h}x|^2)\,dw
\end{aligned}$$

Similarly, we estimate

$$|\psi_{12}|\le C|x\circ g-x|^2$$

and

$$\begin{aligned}
|\nabla\psi_{12}|&=\left|\nabla\left(\int_x^{x\circ g}\int_x^{x'}\frac{d^2}{d\phi^2}\gamma(x'')\,dx''\,dx'\right)\right|\\
&=\left|(\nabla x\circ g-\nabla x)\int_x^{x\circ g}\frac{d^2}{d\phi^2}\gamma(x')\,dx'+\nabla x\int_x^{x\circ g}\left(\frac{d^2}{d\phi^2}\gamma(x')-\frac{d^2}{d\phi^2}\gamma(x)\right)\,dx'\right|\\
&\le C|\nabla(x\circ g-x)||x\circ g-x|+C|\nabla x||x\circ g-x|^2,
\end{aligned}$$

as in (19). By Young's inequality we find

$$\begin{aligned}
|II_{12}| \leq C &\int_{B\setminus B_{1-r_0}(0)} |\nabla X|(|\nabla(x\circ g - x)||x\circ g - x| + |x\circ g - x|^2)\,dw \\
&+ C\int_{B\setminus B_{1-r_0}(0)} |\nabla X||\nabla x||x\circ g - x|^2\,dw \\
&\leq \delta \int_{B\setminus B_{1-r_0}(0)} |\nabla(x\circ g - x)|^2\,dw \\
&+ C(\delta)\int_{B\setminus B_{1-r_0}(0)} \left(1 + |\nabla X|^2 + |\nabla x|^2\right)|x\circ g - x|^2\,dw \\
&\leq \delta\varepsilon \int_{B\setminus B_{1-r_0}(0)} \left|\frac{\nabla(x\circ g - x)}{h}\right|^2\,dw \\
&+ C(\delta)\varepsilon\int_{B\setminus B_{1-r_0}(0)} (1 + |\nabla X|^2 + |\nabla x|^2)|h^{-1}(x\circ g - x)|^2\,dw.
\end{aligned}$$

Observe that we may replace x by X on account of (16) - (18). Thus, we can bound

$$\begin{aligned}
|II_1| &\leq |II_{11}| + |II_{12}| \\
&\leq \frac{\varepsilon}{8}\int_B |\nabla\partial_h X|^2\,dw + \delta\varepsilon\int_B \left|\frac{\nabla(X\circ g - X)}{h}\right|^2\,dw \\
&+ C(\delta)\varepsilon\int_B (1 + |\nabla X|^2)(1 + |h^{-1}(X\circ g - X)|^2 + |\partial_h X|^2)\,dw,
\end{aligned}$$

where we simply denote $|\partial_h X|, |\partial_h X\circ g|$, etc., as $|\partial_h X|$. Together with our bounds for II_2 it then follows that

$$\begin{aligned}
0 &\geq I + II \geq I - |II_1| - |II_{22}| - |II_{23}| \\
&\geq \frac{\varepsilon}{8}\int_B |\nabla\partial_h X|^2\,dw - \delta\varepsilon\int_B \left|\frac{\nabla(X\circ g - X)}{h}\right|^2\,dw \\
&- C(\delta)\varepsilon\int_B (1 + |\nabla X|^2)(1 + |\partial_h X|^2 |h^{-1}(X\circ g - X)|^2)\,dw.
\end{aligned}$$

Dividing by $\frac{\varepsilon}{8}$ and replacing δ by $\delta/8$, thus we obtain the estimate

$$\begin{aligned}
\int_B |\nabla\partial_h X|^2\,dw \leq &\delta\int_B \left|\frac{\nabla(X\circ g - X)}{h}\right|^2\,dw \\
&+ C(\delta)\int_B (1 + |\nabla X|^2)(1 + |\partial_h X|^2 + |h^{-1}(X\circ g - X)|^2)\,dw.
\end{aligned}$$

Invoking Lemma 4.1, as in [4], p. 73 f., we can bound the last of these terms

$$\begin{aligned}
C(\delta)&\int_B (1 + |\nabla X|^2)(1 + |\partial_h X|^2 + |h^{-1}(X\circ g - X)|^2)\,dw \\
&\leq \frac{1}{2}\int_B |\nabla\partial_h X|^2\,dw + \delta\int_B \left|\frac{\nabla(X\circ g - X)}{h}\right|^2\,dw + C(\delta)\int_B (1 + |\nabla X|^2)\,dw,
\end{aligned}$$

whence we obtain the bound

$$\int_B |\nabla\partial_h X|^2\,dw \leq \delta\int_B \left|\frac{\nabla(X\circ g - X)}{h}\right|^2\,dw + C(\delta)\int_B (1 + |\nabla X|^2)\,dw \tag{22}$$

for any $\delta > 0$, uniformly in h, as $h \to 0$.

We conjecture that an estimate like (22) for a harmonic function X on the disc already implies that $X \in H^{2,2}$, with the corresponding bounds. In the absence of a rigorous result in this regard, here, however, we proceed as follows.

Introduce difference quotients with respect to $\mathcal{G}$ by letting

$$\int_B |\nabla \partial_h^{\mathcal{G}} X|^2 \, dw = \sup_{|g-id| \le h} \int_B \left| \frac{\nabla(X \circ g - X)}{h} \right|^2 \, dw.$$

Repeating the above analysis with the comparison function $y = (1 - \frac{2\varepsilon}{h^2})x + \frac{\varepsilon}{h^2}(x \circ g_h + x \circ g_h^{-1}) \in M$, where $g_h \in \mathcal{G}$ satisfies $|g_h - id| \le h$ and $0 \le \varepsilon \le \frac{1}{2}h^2$, and using the group property of $\mathcal{G}$, we arrive at the inequality analogous to (22),

$$\int_B \left| \frac{\nabla(X \circ g_h - X)}{h} \right|^2 \, dw \le \delta \int_B \left| \frac{\nabla(X \circ g - X)}{h} \right|^2 \, dw + C(\delta) \int_B (1 + |\nabla X|^2) \, dw$$

for some $g \in \mathcal{G}$ such that $|g - id| \le C_1 h$ and any $\delta > 0$, with constants $C_1, C(\delta)$ which are uniform in h as $h \to 0$. (We may assume $C_1 \in \mathbb{N}$.)

Taking the supremum with respect to g_h on the left and decomposing the first term on the right as a telescopic sum by writing $g = \tilde{g}_1 \circ \ldots \circ \tilde{g}_{C_1}$ as a C_1-fold product of transformations $\tilde{g}_k \in \mathcal{G}$ satisfying $|\tilde{g}_k - id| \le h, 1 \le k \le C_1$, we then obtain the bound

$$\int_B |\nabla \partial_h^{\mathcal{G}} X|^2 \, dw \le C\delta \int_B |\nabla \partial_h^{\mathcal{G}} X|^2 \, dw + C(\delta) \int_B (1 + |\nabla X|^2) \, dw$$

for any $\delta > 0$, uniformly in $h > 0$. (Here we also used conformal invariance of Dirichlet's integral again to simplify the representation of the telescopic sum in the second term.) Choosing $\delta > 0$ sufficiently small, there results the estimate

$$\int_B |\nabla \partial_h X|^2 \, dw \le \int_B |\nabla \partial_h^{\mathcal{G}} X|^2 \, dw \le C \int_B (1 + |\nabla X|^2) \, dw,$$

uniformly in $h \to 0$. Finally, letting $h \to 0$, we then obtain the desired conclusion. □

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