WAVE MAPS

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Abstract. In these lectures we outline the known results concerning existence, uniqueness, and regularity for the Cauchy problem for harmonic maps from $(1+m)$-dimensional Minkowski space into a Riemannian target manifold, also known as σ-models or wave maps. In particular, we mark the limits of the classical theory in high dimensions and trace recent developments in dimension $m = 2$, substantiating the conjecture that in this "conformal" case the Cauchy problem is well-posed in the energy space.

Contents

1. Local existence. Energy method

1.1. The setting. Let (M, γ) be an m-dimensional Riemannian manifold without boundary, the "domain" of our maps, and let (N, g) be a compact, k-dimensional Riemannian manifold, with $\partial N = \emptyset$, the "target". For simplicity, in these lectures we only consider the case $M = \mathbb{R}^m$; however, many of the results presented below can easily be extended to the case of a compact domain manifold, for instance, to the case $M = T^m = \mathbb{R}^m/\mathbb{Z}^m$, the flat torus. Moreover, by Nash's embedding theorem, we may assume that $N \subset \mathbb{R}^d$ isometrically for some $d > k$. We denote as $T_pN \subset T_p\mathbb{R}^d \cong \mathbb{R}^d$ the tangent space of N at a point p, and we denote as $T_p^{\perp}N$ the orthogonal complement of T_pN with respect to the inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^d$. TN, $T^{\perp}N$ will denote, respectively, the corresponding tangent and normal bundles. Moreover, since N is compact, there exists a tubular neighborhood $U_{2\delta}(N)$ of width 2δ of N in $\mathbb{R}^d$ such that the nearest neighbor projection $\pi_N \colon U_{2\delta}(N) \to N$ is well-defined and smooth.

For M and N as above we consider smooth maps $u \colon \mathbb{R} \times M \to N \hookrightarrow \mathbb{R}^d$ on the space-time cylinder $\mathbb{R} \times M$. The space-time coordinates will be denoted as $z = (t, x) = (x^\alpha)_{0 \le \alpha \le m}$ and we denote as $\frac{\partial}{\partial x^\alpha} u = \partial_\alpha u = u_{x^\alpha}$ the partial derivative of u with respect to x^α, $0 \le \alpha \le m$. Also let $D = (\frac{\partial}{\partial t}, \nabla) = (\frac{\partial}{\partial x^\alpha})_{0 \le \alpha \le m}$. $\mathbb{R} \times M$ will be endowed with the Minkowski metric $\eta = (\eta_{\alpha\beta}) = \mathrm{diag}(1, -1, \dots, -1)$ and we raise and lower indeces with the metric. By convention, we tacitly sum over repeated indeces. Thus, for example, $\partial^\alpha = \eta^{\alpha\beta}\partial_\beta$, where $(\eta^{\alpha\beta}) = (\eta_{\alpha\beta})^{-1}$ $(= (\eta_{\alpha\beta})$ in our setting). Moreover,

$$\Box = \partial^\alpha \partial_\alpha = \frac{\partial^2}{\partial t^2} - \Delta$$

is the wave operator and

$$\frac{1}{2}\langle \partial^\alpha u, \partial_\alpha u \rangle = \frac{1}{2}\big(|u_t|^2 - |\nabla u|^2\big)$$

is the Lagrangean density of u.

1.2. Wave maps. Let $u \colon \mathbb{R} \times M \to N$ be sufficiently smooth. A (compactly supported) variation of u is a family of maps $u_\epsilon \colon \mathbb{R} \times M \to N$ depending smoothly on a parameter $\epsilon \in]-\epsilon_0, \epsilon_0[$ for some $\epsilon_0 > 0$, with $u_0 \equiv u$ and such that $u_\epsilon \equiv u_0$ outside some compact region $Q \subset \mathbb{R} \times M$ for all ϵ.

Given a map $\varphi \in C_0^\infty(\mathbb{R} \times M; \mathbb{R}^d)$, an admissible variation may be obtained, for instance, by letting $u_\epsilon = \pi_N(u + \epsilon\varphi)$ for $|\epsilon| < 2\delta \|\varphi\|_{L^\infty}^{-1}$, where $\pi_N \colon U_{2\delta}(N) \to N$ is the smooth nearest neighbor projection defined in Section 1.1.

A map u then is a wave map if u is a stationary point for the Lagrangean

$$\mathcal{L}(u;Q) = \frac{1}{2}\int_Q \langle \partial^\alpha u, \partial_\alpha u\rangle \, dz$$

with respect to compactly supported variations u_ϵ, $|\epsilon| < \epsilon_0$, in the sense that

$$\frac{d}{d\epsilon}\mathcal{L}(u_\epsilon;Q) = 0,$$

where Q strictly contains the support of $u_\epsilon - u$.

In particular, for the variation $u_\epsilon = \pi_N(u+\epsilon\varphi)$ above we then obtain the equation

$$\begin{aligned} 0 = \frac{d}{d\epsilon}\mathcal{L}\big(\pi_N(u+\epsilon\varphi);Q\big) &= \int_Q \langle \partial^\alpha u, \partial_\alpha (d\pi_N(u)\cdot\varphi)\rangle \, dz \\ &= -\int_Q \langle \partial^\alpha \partial_\alpha u, d\pi_N(u)\cdot\varphi\rangle \, dz \end{aligned}$$

for all $\varphi \in C_0^\infty(\mathbb{R}\times M;\mathbb{R}^d)$; that is, $\Box u(z) \perp T_{u(z)}N$ for all $z \in \mathbb{R}\times M$, or

$$\Box u \perp T_u N$$

for short.

To understand this relation in more explicit terms, fix a point $z_0 \in \mathbb{R}\times M$ and let $\nu_{k+1}, \dots, \nu_d$ be an orthonormal frame for $T_p^\perp N$, smoothly depending on $p \in N$ for p near $p_0 = u(z_0)$. Then we can find scalar functions $\lambda^l\colon \mathbb{R}\times M \to \mathbb{R}$, $k < l \le d$, such that near $z = z_0$ there holds

$$\Box u = \lambda^l (\nu_l \circ u);$$

in fact,

$$\begin{aligned} \lambda^l &= \langle \Box u, \nu_l \circ u\rangle \\ &= \partial^\alpha \langle \partial_\alpha u, \nu_l \circ u\rangle - \langle \partial_\alpha u, \partial^\alpha (\nu_l \circ u)\rangle \\ &= -\langle \partial_\alpha u, d\nu_l(u)\cdot \partial^\alpha u\rangle = -A^l(u)(\partial_\alpha u, \partial^\alpha u) \end{aligned}$$

is given by the second fundamental form A^l of N with respect to ν_l. Thus, the wave map equation takes the form

$$\Box\, u = -A(u)(\partial_\alpha u, \partial^\alpha u) \perp T_u N, \tag{1.1}$$

where $A = A^l \nu_l$ is the second fundamental form of N. We regard the term on the right of (1.1) as a Lagrange multiplier associated with the target constraint $u(\mathbb{R}\times M) \subset N$.

1.3. Examples. In certain cases equation (1.1) takes a particularly simple form.

1.3.1. *The sphere.* For $N = S^k \subset \mathbb{R}^{k+1}$ equation (1.1) translates into the equation

$$\Box u = \big(|\nabla u|^2 - |u_t|^2\big)u.$$

Indeed, since $u \perp T_u S^k$ it suffices to check that

$$\langle \Box u, u\rangle = \partial^\alpha \langle \partial_\alpha u, u\rangle - \langle \partial_\alpha u, \partial^\alpha u\rangle = |\nabla u|^2 - |u_t|^2.$$

1.3.2. *Geodesics.* Suppose $\gamma\colon \mathbb{R} \to N$ is a geodesic parametrized by arc-length and $u = \gamma \circ v$ for some map $v\colon \mathbb{R} \times M \to \mathbb{R}$. Compute

$$\Box u = \partial^\alpha\big(\gamma'(v)\partial_\alpha v\big) = \gamma''(v)\partial_\alpha v\partial^\alpha v + \gamma'(v)\,\Box\, v.$$

Note that γ' is parallel along γ; that is, $\gamma''(s) \perp T_{\gamma(s)}N$ for all $s \in \mathbb{R}$.

Thus, u satisfies (1.1) if and only if v solves the linear, homogeneous wave equation $\Box v = 0$.

1.4. Basic questions. In view of the hyperbolic nature of equation (1.1), in particular, in view of Example 1.3.1, it is natural to study the Cauchy problem for equation (1.1) for (sufficiently) smooth initial data

$$(u, u_t)\,|_{t=0} = (u_0, u_1)\colon M \to TN. \tag{1.2}$$

The basic questions we shall ask are the following.

Local well-posedness in the smooth category: Does the initial value problem (1.1), (1.2) for smooth data always admit a unique smooth solution for small time $|t| < T$?

The smoothness hypothesis on the solution and the data may be weakened. In fact, for a function $u \in L^2_{\mathrm{loc}}(\mathbb{R} \times M; N)$ it is possible to interpret equation (1.1) in the sense of distributions provided $Du \in L^2_{\mathrm{loc}}(\mathbb{R} \times M)$.

More specifically, for $s \in \mathbb{N}_0$ we let $H^s(M;N) = \big\{v \in H^{s,2}(M;\mathbb{R}^d); v(M) \subset N\big\}$ denote the Sobolev space of maps $v\colon M \to N$ such that v possesses square integrable distributional derivatives of any order up to s. Moreover, we say that $u \in L^2_{\mathrm{loc}}(\mathbb{R} \times M;N)$ is a weak solution of (1.1), (1.2) of class H^s provided $(\frac{\partial}{\partial t})^\sigma u(t) \in H^{s-\sigma}(M)$ for all $\sigma \le s$, locally uniformly in t, and if u weakly solves (1.1) and assumes the initial data (1.2) in the sense of traces.

Then we can pose the problem of

Regularity: What is the minimal regularity of the data to ensure unique local solvability of (1.1), (1.2) in the same regularity class?

Global well-posedness: Does there exist a regularity class such that the Cauchy problem (1.1), (1.2) admits a unique solution in this class for all time?

We do not consider explicitly the issue of stability, that is, continuous dependence of solutions on the data. However, quite often stability is related to uniqueness.

1.5. Energy estimates. Let $e(u) = \frac{1}{2}|Du|^2$ be the energy density of a map $u\colon \mathbb{R} \times M \to N$, and let

$$E\big(u(t)\big) = \int_{\mathbb{R}^m} \big(e(u)\big)(t)\,dx$$

be the total energy of u at time t. Note that, if u solves (1.1), we have the conservation law

$$\begin{aligned} 0 = \langle \Box u, u_t\rangle &= \frac{d}{dt}\left(\frac{|u_t|^2}{2}\right) - \operatorname{div}\langle \nabla u, u_t\rangle + \langle \nabla u, \nabla u_t\rangle \\ &= \frac{d}{dt}e(u) - \operatorname{div}\langle \nabla u, u_t\rangle. \end{aligned}$$

Hence, if $Du(t)$ has compact support, upon integrating over $\mathbb{R}^m$ we find that

$$\frac{d}{dt}E\big(u(t)\big) = 0;$$

that is, total energy is conserved. A similar energy estimate also holds on light cones. In particular, it follows that the diameter of $\mathrm{supp}\big(Du(t)\big)$ grows with speed at most 1 and hence $Du(t)$ has compact support for all t whenever $\mathrm{supp}\big(Du(0)\big)$ is compact.

1.6. L^2-theory. The above energy inequality may be generalized to obtain a priori bounds for higher derivatives, as well. Consider the Cauchy problem

$$\Box u = f \quad \text{in} \quad \mathbb{R} \times M$$
$$u \mid_{t=0} = g,\ u_t \mid_{t=0} = h,$$

where f, g, and h are smooth functions such that $\mathrm{supp}(Du(0)) = \mathrm{supp}(h, \nabla g)$ is compact and $\mathrm{supp}(f(t))$ is compact for any t. Then we have

$$\frac{d}{dt}e(u) - \mathrm{div}(\nabla u u_t) = f u_t \le |f||u_t|$$

and hence by Hölder's inequality

$$\|Du(t)\|_{L^2} \cdot \frac{d}{dt}\|Du(t)\|_{L^2} = \frac{d}{dt}E\big(u(t)\big) \le \int_{\{t\}\times\mathbb{R}^m} |f||u_t|\, dx$$
$$\le \|f(t)\|_{L^2}\|u_t(t)\|_{L^2} \le \|f(t)\|_{L^2}\|Du(t)\|_{L^2}.$$

It follows that

$$\frac{d}{dt}\|Du(t)\|_{L^2} \le \|f(t)\|_{L^2}.$$

Similarly, for any multi-index $I = (i_0, \dots, i_m) \in \mathbb{N}_0^{1+m}$ with $|I| = \Sigma_\alpha i_\alpha$, letting $\partial^I = \Pi_\alpha \partial_\alpha^{i_\alpha}$ we obtain

$$\frac{d}{dt}\|D\partial^I u(t)\|_{L^2} \le \|\partial^I f(t)\|_{L^2} \tag{1.3}$$

for all t. Integrating in time, thus we find that for any $I \in \mathbb{N}_0^{1+m}$ there holds

$$\sup_{0\le t\le T} \|D\partial^I u(t)\|_{L^2} \le \|D\partial^I u(0)\|_{L^2} + \int_0^T \|\partial^I f(t)\|_{L^2}\, dt,$$

and similarly for $T < 0$. Note that, using the equation $\Box u = f$, we can express any derivative $D\partial^I u(0)$ in terms of spatial derivatives of g, and h of orders $|I| + 1$, and $|I|$, and space-time derivatives of f at $t = 0$ of order $|I - 1|$, respectively. For instance,

$$u_{tt} = \Box u + \Delta u = f + \Delta g.$$

Letting

$$\|v\|_{L^{\infty,2}} = \sup_{0\le t\le T} \|v(t)\|_{L^2},$$

therefore for any $s \in \mathbb{N}_0$ we obtain the estimate

$$\|D^{s+1}u\|_{L^{\infty,2}} \le T\|D^s f\|_{L^{\infty,2}} + C\big(\|D^{s-1}f(0)\|_{L^2} + \|\nabla^{s+1}g\|_{L^2} + \|\nabla^s h\|_{L^2}\big).$$

with a constant $C = C(s)$.

1.7. Local existence for smooth data. The results of the preceding section apply to obtain a-priori bounds for smooth solutions u to (1.1), (1.2) by letting $f := A(u)(\partial_\alpha u, \partial^\alpha u)$, etc.

The class of admissible data for $s \in \mathbb{N}_0$ is given by

$$H_c^{s+1}(M;TN) = \{(u_0,u_1) \in L^2_{\mathrm{loc}}(M;TN);\\ u_0 \in H^{s+1}(M;\mathbb{R}^d), u_1 \in H^s(M;\mathbb{R}^d), \mathrm{supp}(u_1,\nabla u_0) \subset\subset \mathbb{R}^m\}.$$

Note that by Sobolev's embedding $H^s \hookrightarrow L^\infty$ for $s > \frac{m}{2}$. Therefore, and using interpolation, whenever for some constant C_0 the estimate

$$\sup_{0\le t\le T} \|D^{s_0+1}u(t)\|_{L^2} \le C_0 \tag{1.4}$$

is satisfied for some $s_0 > \frac{m}{2}$, then for any $s \in \mathbb{N}$ we can estimate

$$\|D^s\big(A(u)(\partial_\alpha u, \partial^\alpha u)\big)\|_{L^2} \le C_s\big(1 + \|D^{s+1}u\|_{L^2}\big),$$

uniformly in $0 \le t \le T$ []. By (1.3), therefore

$$\frac{d}{dt}\|D^{s+1}u(t)\|_{L^2} \le C_s\big(1 + \|D^{s+1}u(t)\|_{L^2}\big),$$

and Gronwall's inequality yields bounds for $D^{s+1}u$ in $L^{\infty,2}$, provided $(u_0,u_1) \in H_c^{s+1}(M;TN)$. In particular, the estimate (1.4) will be valid for some $T > 0$ if we fix a constant $C_0 > \|D^{s_0+1}u(0)\|_{L^2}$, a constant depending only on u_0, u_1, and s_0.

Similarly, by the contraction principle, one can show the existence of a unique solution u of class H^{s+1} on a small time interval $0 \le t \le T$, provided $(u_0,u_1) \in H_c^{s+1}(M;TN)$ for some $s > \frac{m}{2}$. In this way we obtain

Theorem 1.1. *Fix initial data $(u_0,u_1) \in H_c^{s_0+1}(M;TN)$, where $s_0 > \frac{m}{2}$. There exists $T > 0$ and a unique solution $u\colon [0,T]\times M \to N$ of (1.1), (1.2) such that*

$$\sup_{0\le t\le T} \|D^{s_0+1}u(t)\|_{L^2} < \infty.$$

Moreover, if $(u_0,u_1) \in H_c^{s+1}(M;TN)$ for some $s > s_0$, then

$$\sup_{0\le t\le T} \|D^{s+1}u(t)\|_{L^2} < \infty.$$

In particular, if u_0, u_1 are smooth, also the solution u is smooth on $[0,T]\times M$.

For more details and references, see for instance [26] or [30].

1.8. A slight improvement. The local existence theorem in the preceding section did not use the special structure of the nonlinear term in (1.1) nor its geometric interpretation.

Using the fact that the term on the right of (1.1) is a "null-form" in the derivatives of the components of $u = (u^1, \dots, u^d)$, that is, the fact that

$$A(u)(\partial_\alpha u, \partial^\alpha u) = A_{ij}(u)\partial_\alpha u^i \partial^\alpha u^j$$

is a sum of bilinear forms whose symbol

$$\hat{A}_{ij}(u)\xi_\alpha \eta^i \xi^\alpha \eta^j = \hat{A}(u)(\eta, \eta)\xi_\alpha \xi^\alpha$$

vanishes on the null cone

$$\{\xi = (\xi_\alpha) \in \mathbb{R} \times \mathbb{R}^m; \xi_\alpha \xi^\alpha = 0\},$$

Klainerman-Machedon [31] obtained the following slight improvement of Theorem 1.1 in low dimensions.

Theorem 1.2. *Suppose $m \leq 3$. Then for any data $(u_0, u_1) \in H^2_c(M; TN)$ there exists a unique local solution u of class H^2. If $(u_0, u_1) \in H^s, s > 2$, then so is u.*

The proof in [31] uses special L^2-estimates for null forms and is quite involved. A very simple proof, based on energy estimates alone, however, can be given if one uses the *geometric* interpretation of (1.1). This observation is due to [48].

Proof. (i) First we derive local a-priori estimates for $D^2 u$ for smooth solutions u. Let ∂ be any first order derivative. Differentiating equation (1.1), we obtain

$$\Box(\partial u) = \partial A(u)(\partial_\alpha u, \partial^\alpha u) = dA(u)(\partial u, \partial_\alpha u, \partial^\alpha u) + 2A(u)(\partial_\alpha \partial u, \partial^\alpha u)$$

with data

$$(\partial u \mid_{t=0}, \partial u_t \mid_{t=0}) \in H^1_c(M; T\mathbb{R}^d).$$

Note that, by orthogonality $\langle u_t, A(u)(\cdot, \cdot)\rangle = 0$,

$$\langle \partial u_t, A(u)(\partial_\alpha \partial u, \partial^\alpha u)\rangle = -\langle u_t, dA(u)(\partial u, \partial_\alpha \partial u, \partial^\alpha u)\rangle.$$

Hence we obtain

$$\begin{aligned} \frac{d}{dt} E\big(\partial u(t)\big) &= \int_{\{t\}\times M} \langle \Box(\partial u), \partial u_t\rangle \, dx \\ &\leq C\|dA(u)\|_{L^\infty} \cdot \int_M |Du(t)|^3 |D^2 u(t)| \, dx. \end{aligned}$$

Since N is compact, dA is uniformly bounded on N. Moreover, by Sobolev's embedding, if $m = 3$ we can estimate

$$\begin{aligned} \int_M |Du(t)|^3 |D^2 u(t)| \, dx &\leq \|Du(t)\|^3_{L^6} \|D^2 u(t)\|_{L^2} \\ &\leq C\|D^2 u(t)\|^4_{L^2} = CE\big(Du(t)\big)^2. \end{aligned}$$

Similarly, if $m = 2$ we estimate

$$\int_M |Du(t)|^3 |D^2u(t)|\, dx \le C\|Du(t)\|_{L^2}\|D^2u(t)\|^3_{L^2}$$
$$\le CE\big(u(t)\big)^{1/2}E\big(Du(t)\big)^{3/2}.$$

Thus, in both cases we arrive at a Gronwall type inequality

$$\frac{d}{dt}E\big(Du(t)\big) \le CE\big(Du(t)\big)^{\gamma}$$

with some $\gamma > 1$. A local-in-time H^2-bound follows.

(ii) To show uniqueness in the class H^2 let u and v be solutions of (1.1) on $[0,T] \times M$ sharing Cauchy data

$$(u\,|_{t=0} = v\,|_{t=0}, u_t\,|_{t=0} = v_t\,|_{t=0}) \in H^2_c(M;TN).$$

Observe that, since $\langle u_t, A(u)(\cdot,\cdot)\rangle = 0$, etc. we have

$$\begin{aligned}\langle u_t - v_t, \Box\, u - \Box v\rangle &= \langle u_t - v_t, A(u)(\partial_\alpha u, \partial^\alpha u) - A(v)(\partial_\alpha v, \partial^\alpha v)\rangle \\ &= \langle u_t, \big(A(u) - A(v)\big)(\partial_\alpha v, \partial^\alpha v)\rangle - \langle v_t, \big(A(u) - A(v)\big)(\partial_\alpha u, \partial^\alpha u)\rangle \\ &\le C|u - v||D(u - v)|\big(|Du|^2 + |Dv|^2\big).\end{aligned}$$

Thus, if $m = 3$, the energy inequality and Sobolev's embedding give

$$\begin{aligned}\frac{d}{dt}E\big((u - v)(t)\big) &= \int_{\{t\}\times M} \langle u_t - v_t, \Box u - \Box v\rangle\, dx \\ &\le C\int_{\{t\}\times M} |u - v||D(u - v)|\big(|Du|^2 + |Dv|^2\big)\, dx \\ &\le C\big(\|Du(t)\|^2_{L^6} + \|Dv(t)\|^2_{L^6}\big)\|(u - v)(t)\|_{L^6}\|D(u - v)(t)\|_{L^2} \\ &\le C\big(\|D^2u(t)\|^2_{L^2} + \|D^2v(t)\|^2_{L^2}\big)\|D(u - v)(t)\|^2_{L^2} \\ &\le C\big(\|D^2u\|^2_{L^{\infty,2}} + \|D^2v\|^2_{L^{\infty,2}}\big)E\big((u - v)(t)\big),\end{aligned}$$

and uniqueness follows. For $m = 2$ the argument is similar.

(iii) An H^2-solution preserves higher regularity of the data. Indeed, by Theorem 1.1 it suffices to show this for data $(u_0, u_1) \in H^3_c(M;TN)$. Let $u\colon [0,T] \times M \to N$ be a local H^2-solution of (1.1), (1.2). We claim that u is of class H^3 on $[0,T]$ as well. For this, by Theorem 1.1 it suffices to prove an a-priori estimate for $\|D^3u(t)\|_{L^2}$. As before let ∂ be a first order differential operator. For simplicity, at first we consider only spatial derivatives.

Note that

$$\begin{aligned}\langle \partial^2 u_t, \partial^2\big(A(u)(\partial_\alpha u, \partial^\alpha u)\big)\rangle &= 2\langle \partial^2 u_t, A(u)(\partial_\alpha \partial^2 u, \partial^\alpha u) + A(u)(\partial_\alpha \partial u, \partial^\alpha \partial u)\rangle \\ &\quad + 4\langle \partial^2 u_t, dA(u)(\partial u, \partial_\alpha \partial u, \partial^\alpha u)\rangle + \langle \partial^2 u_t, d^2A(u)(\partial u, \partial u, \partial_\alpha u, \partial^\alpha u\rangle \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}.\end{aligned}$$

Clearly, by Sobolev's embedding theorem we can bound

$$\begin{aligned}\int_{\{t\}\times M}|\mathrm{III}|\,dx &\le C\int_{\{t\}\times M}|Du|^4|D^3u|\,dx\\ &\le C\|Du(t)\|_{L^\infty}\|Du(t)\|_{L^6}^3\|D^3u(t)\|_{L^2}\\ &\le C\|D^2u(t)\|_{L^2}^3\big(1+\|D^3u(t)\|_{L^2}^2\big)\le C\big(1+E\big(D^2u(t)\big)\big),\end{aligned}$$

$$\begin{aligned}\int_{\{t\}\times M}|\mathrm{II}|\,dx &\le C\int_{\{t\}\times M}|Du|^2|D^2u||D^3u|\,dx\\ &\le C\|Du(t)\|_{L^6}^2\|D^2u(t)\|_{L^6}\|D^3u(t)\|_{L^2}\\ &\le C\|D^2u(t)\|_{L^2}^2\|D^3u(t)\|_{L^2}^2\le CE\big(D^2u(t)\big).\end{aligned}$$

Before we can estimate the first term in a similar fashion we need to express I in a more convenient way. Using orthogonality $\langle u_t, A(u)(\cdot,\cdot)\rangle$ again, we have

$$\begin{aligned}\langle \partial^2 u_t,&A(u)(\partial_\alpha\partial^2u,\partial^\alpha u)\rangle = \partial\langle\partial u_t, A(u)(\partial_\alpha\partial^2 u,\partial^\alpha u)\rangle - \langle\partial u_t, A(u)(\partial_\alpha\partial^3u,\partial^\alpha u)\rangle\\ &-\langle\partial u_t, A(u)(\partial_\alpha\partial^2u,\partial^\alpha\partial u)\rangle - \langle\partial u_t, dA(u)(\partial u,\partial_\alpha\partial^2u,\partial^\alpha u)\rangle\\ &=\partial\langle\partial u_t, A(u)(\partial_\alpha\partial^2u,\partial^\alpha u)\rangle + \langle u_t, dA(u)(\partial u,\partial_\alpha\partial^3u,\partial^\alpha u)\rangle\\ &+\langle u_t, dA(u)(\partial u,\partial_\alpha\partial^2u,\partial^\alpha\partial u)\rangle - \langle\partial u_t, dA(u)(\partial u,\partial_\alpha\partial^2u,\partial^\alpha u)\rangle;\end{aligned}$$

moreover,

$$\begin{aligned}\langle u_t,&dA(u)(\partial u,\partial_\alpha\partial^3u,\partial^\alpha u)\rangle = \partial\langle u_t, dA(u)(\partial u,\partial_\alpha\partial^2u,\partial^\alpha u)\rangle\\ &-\langle\partial u_t, dA(u)(\partial u,\partial_\alpha\partial^2u,\partial^\alpha u)\rangle - \langle u_t, d^2A(u)(\partial u,\partial u,\partial_\alpha\partial^2u,\partial^\alpha u)\rangle\\ &-\langle u_t, dA(u)(\partial^2u,\partial_\alpha\partial^2u,\partial^\alpha u)\rangle - \langle u_t, dA(u)(\partial u,\partial_\alpha\partial^2u,\partial^\alpha\partial u)\rangle.\end{aligned}$$

Similarly,

$$\begin{aligned}\langle\partial^2u_t,&A(u)(\partial_\alpha\partial u,\partial^\alpha\partial u)\rangle = \partial\langle\partial u_t, A(u)(\partial_\alpha\partial u,\partial^\alpha\partial u)\rangle\\ &-\langle\partial u_t, dA(u)(\partial u,\partial_\alpha\partial u,\partial^\alpha\partial u)\rangle - 2\langle\partial u_t, A(u)(\partial_\alpha\partial^2u,\partial^\alpha\partial u)\rangle,\end{aligned}$$

and

$$\langle\partial u_t, A(u)(\partial_\alpha\partial^2u,\partial^\alpha\partial u)\rangle = -\langle u_t, dA(u)(\partial u,\partial_\alpha\partial^2u,\partial^\alpha\partial u)\rangle.$$

Thus, as above we conclude that

$$\begin{aligned}\int_{\{t\}\times M} I\,dx &\le C\int_{\{t\}\times M}\big(|Du|^2|D^2u|+|Du|^4\big)|D^3u|+|Du||D^2u|^3\,dx\\ &\le C\big(1+E\big(D^2u(t)\big)\big).\end{aligned}$$

Hence there holds

$$\frac{d}{dt}E\big(\nabla^2u(t)\big)\le C\big(1+E\big(D^2u(t)\big)\big).$$

For $\partial = \frac{\partial}{\partial t}$ we use the fact that $\partial^2u_t = (\Box u)_t + \Delta u_t$. Repeating the previous computations, we can estimate

$$\int_{\{t\}\times M}\langle\Delta u_t, \partial^2A(u)(\partial_\alpha u,\partial^\alpha u)\rangle\,dx\le C\big(1+E\big(D^2u(t)\big)\big).$$

Moreover, by Young's inequality

$$\begin{aligned}\langle \Box u_t, &\partial^2\big(A(u)(\partial_\alpha u, \partial^\alpha u)\big)\rangle = \langle \partial\big(A(u)(\partial_\alpha u, \partial^\alpha u)\big), \partial^2\big(A(u)(\partial_\alpha u, \partial^\alpha u)\big)\rangle \\ &\le C\big(|Du|^3 + |Du||D^2u|\big)\big(|Du|^4 + |Du|^2|D^2u| + |D^2u|^2\big) \\ &\le C\big(|Du|^7 + |Du|^5|D^2u| + |Du|^3|D^2u|^2 + |Du||D^2u|^3\big) \\ &\le C\big(|Du|^7 + |Du||D^2u|^3\big)\end{aligned}$$

and

$$\begin{aligned}\int_{\{t\}\times M} |Du|^7\,dx &\le C\|Du(t)\|_{L^\infty}\|Du(t)\|_{L^6}^6 \\ &\le C\|D^2u(t)\|_{L^2}^6\big(1 + \|D^3u(t)\|_{L^2}\big) \le C\big(1 + E\big(D^2u(t)\big)\big)^{1/2}, \\ \int_{\{t\}\times M} |Du||D^2u|^3\,dx &\le C\|Du(t)\|_{L^6}\|D^2u(t)\|_{L^2}\|D^2u(t)\|_{L^6}^2 \\ &\le C\|D^2u(t)\|_{L^2}^2\|D^3u(t)\|_{L^2}^2 \le CE\big(D^2u(t)\big).\end{aligned}$$

Thus we also find that

$$\frac{d}{dt}E\big(u_{tt}(t)\big) \le C\big(1 + E\big(D^2u(t)\big)\big)$$

and hence that

$$\frac{d}{dt}E\big(D^2u(t)\big) \le C\big(1 + E\big(D^2u(t)\big)\big),$$

which yields the desired a-priori bound.

(iv) Existence of local H^2-solutions can now be obtained as follows. Approximate the given data $(u_0, u_1) \in H_c^2(M;TN)$ by smooth data $(u_0^n, u_1^n) \in H_c^3(M;TN)$ in the topology of $H_c^2(M;TN)$. The approximate solutions u^n exist on a uniform time interval $0 \le t \le T$ and $\|D^2u^n(t)\|_{L^2}$ is uniformly bounded for $n \in \mathbb{N}$ and $0 \le t \le T$. Hence as $n \to \infty$ a sub-sequence

$$u^n \rightharpoonup u \qquad \text{weakly in } H^2\big([0,T]\times M\big)$$

and by Rellich's theorem $u^n \to u$ strongly in $H^1\big([0,T]\times M\big)$. In particular,

$$A(u^n)(\partial_\alpha u^n, \partial^\alpha u^n) \to A(u)(\partial_\alpha u, \partial^\alpha u) \quad \text{in } L^1,$$

and hence u solves (1.1) in the distribution sense. □

It is conjectured that the initial value problem for (1.1), (1.2) is locally well-posed for data of class $H_c^{s+1}(M;TN)$, where $s \ge \frac{m-2}{2}$. In particular, in the "conformal case" $m = 2$, we expect the initial value problem to be locally well-posed for finite energy data, and hence, since energy is conserved (by classical solutions), we expect the existence of global unique solutions in this case. In Lecture 3 we will give some partial results in this regard.

1.9. Global existence, the case $m = 1$. If $m = 1$, as observed by Shatah [40] we can exchange the roles of x and t in our derivation of the conservation law

$$\frac{d}{dt}e(u) - \frac{d}{dx}\langle u_t, u_x\rangle = 0$$

in Section 1.5 to obtain

$$-\frac{d}{dx}e(u) + \frac{d}{dt}\langle u_x, u_t\rangle = \langle \Box u, u_x\rangle = 0.$$

Taking the t-derivative of the first and the x-derivative of the second equation and adding, we thus find that $e(u)$ solves the linear homogeneous wave equation

$$\Box\, e(u) = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) e(u) = 0. \tag{1.5}$$

From this fact we easily deduce:

Theorem 1.3. *Suppose $m = 1$ and let $(u_0, u_1) \in H^s_c(M; TN)$, $s \geq 2$. Then* (1.1), (1.2) *admits a unique global solution u of class H^s.*

Proof. By Sobolev's embedding, $H^1 \hookrightarrow L^\infty$ for $m = 1$. Hence, if $(u_0, u_1) \in H^2_c(M; TN)$

$$e(u)\mid_{t=0} \in H^1_c,\ \frac{d}{dt}e(u) \in L^2$$

and the energy inequality applied to (1.5) yields the a-priori bound

$$E\big(e(u)(t)\big) \leq E\big(e(u)(0)\big) < \infty,$$

uniformly in $t \in \mathbb{R}$. Hence, by Sobolev's embedding again, $e(u)$ is uniformly a-priori bounded on space-time in terms of the data. The assertion of the Theorem then follows by the same reasoning as used in the proof of Theorem 1.1. □

2. Blow-up and non-uniqueness

2.1. Overview. In Lecture 1 we convinced ourselves that the initial value problem

$$\Box\, u = -A(u)(\partial_\alpha u, \partial^\alpha u) \perp T_u N \quad \text{on } \mathbb{R} \times M, \tag{2.1}$$

$$u\mid_{t=0} = u_0, u_t\mid_{t=0} = u_1 \quad \text{on } M, \tag{2.2}$$

is locally well-posed for sufficiently regular initial data $(u_0, u_1) \in H^{s+1}_c(M; TN)$, $s > \frac{m}{2}$, see Theorem 1.1. In this lecture we now investigate the behaviour of solutions for large time. In particular, depending on the dimension m of the domain and geometric properties of the target, we shall observe a decay of regularity, blow-up of higher derivatives and non-uniqueness of weak solutions beyond such blow-up points. These results indicate the limits of a regularity theory for (2.1) in dimensions $m \geq 3$ and raise the question whether there exists a class of weak solutions for which the initial value problem for wave maps is well-posed in a global sense. For better perspective and comparison, in the next section we give a brief survey of the known regularity results for harmonic maps in the elliptic (stationary) case and for the associated parabolic flow.

2.2. Regularity in the elliptic and parabolic cases. In the elliptic case we consider weak solutions $u \in H^1_{\mathrm{loc}}(M;N)$ of the equation

$$-\Delta u = A(u)(\nabla u, \nabla u) \perp T_u N \tag{2.3}$$

with finite static energy

$$E_{st}(u) = \frac{1}{2}\int_M |\nabla u|^2\, dx < \infty.$$

Here, M may be a smooth, compact m-manifold, possibly with boundary, or $M = \mathbb{R}^m$.

Associated with (2.3) is the heat flow

$$u_t - \Delta u = A(u)(\nabla u, \nabla u) \perp T_u N, \tag{2.4}$$

$$u\,|_{t=0} = u_0, \tag{2.5}$$

which is the L^2-gradient flow for E_{st} in the space $H^1(M;N)$.

2.2.1. *Geometric constraints.* If the sectional curvature K_N of N is non-positive, the Bochner identity for (2.3) implies that

$$-\Delta e(u) \le 0$$

and hence an a-priori C^1-bound for smooth solutions. The same is true for the heat flow (2.4), (2.5). Hence the family $u(t,\cdot)$ of maps generated by (2.4), (2.5) is relatively compact in any C^l-topology and accumulates at a smooth limit $u_\infty\colon M \to N$.

Note, moreover, that (2.4) implies the identity

$$\frac{1}{2}\frac{d}{dt}|\nabla u|^2 - \mathrm{div}\langle \nabla u, u_t\rangle + |u_t|^2 = 0.$$

Upon integrating this equation over $[S,T] \times M$, we deduce that for any $T > S > 0$ there holds

$$\int_S^T \int_M |u_t|^2\, dx\, dt + E_{st}\big(u(T)\big) \le E_{st}\big(u(S)\big). \tag{2.6}$$

Letting $S \to 0$, $T \to \infty$, we find the a-priori estimate

$$\int_0^\infty \int_M |u_t|^2\, dx\, dt \le E_{st}(u_0),\ \sup_t E_{st}\big(u(t)\big) \le E_{st}(u_0); \tag{2.7}$$

in particular, $u_t \to 0$ smoothly, as $t \to \infty$.

From this observation, Eells-Sampson [10] derived their fundamental existence result:

Theorem 2.1. *Suppose $K_N \le 0$. Then for any smooth map $u_0\colon M \to N$ there exists a smooth harmonic map $u_\infty\colon M \to N$ homotopic to u_0.*

In fact, for $K_N \le 0$ *every* weakly harmonic map $u \in H^1(M;N)$ is smooth. The curvature constraint on the target can be replaced by the condition that the range of u is contained in a strictly convex geodesic ball on the target N or that the range $u(M)$ is the domain of a strictly convex function; see Hildebrandt [25] or Jost [27], [28] for a survey.

Theorem 2.1 may be false if the condition $K_N \le 0$ is violated. The following result by Lemaire [32] and Wente [49] shows that, for instance, maps $u_0\colon B^2 = B_1(0;\mathbb{R}^2) \to S^2$ of degree $\neq 0$ and which are constant on the boundary of B cannot be represented by harmonic maps.

Theorem 2.2. *If* $u \in H^1(B^2;S^2)$ *is harmonic with* $u\,|_{\partial B} \equiv$ const., *then* $u \equiv$ const.

Proof. Let $u\,|_{\partial B} \equiv p \in S^2$ and let $\pi\colon S^2 \setminus \{p\} \to \mathbb{R}^2$ denote stereographic projection from the antipodal point of p. Also let $i\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the involution $i(x) = -x$. Extend u to a map $\tilde{u} \in H^1(\mathbb{R}^2;S^2)$ by letting

$$\tilde{u}(x) = \pi^{-1}\left(i\big(\pi\big(u(\frac{x}{|x|^2})\big)\big)\right),\ |x| > 1.$$

Since $\pi^{-1} \circ i \circ \pi$ induces an isometry of S^2 and since $\tilde{u}$ and $\nabla\tilde{u}$ are (weakly) continuous along ∂B, $\tilde{u}$ is weakly harmonic and hence smooth by Hélein's result, Theorem 2.4 below.

From (2.3) it follows by direct computation that the Hopf differential associated with $\tilde{u}$,

$$\Phi(x^1 + ix^2) = |\tilde{u}_{x^1}|^2 - |\tilde{u}_{x^2}|^2 - 2i\,\tilde{u}_{x^1} \cdot \tilde{u}_{x^2}$$

is a holomorphic function on $\mathbb{R}^2 \cong \mathbb{C}$.

Moreover, by conformal invariance of Dirichlet's integral, $\Phi \in L^1(\mathbb{R}^2)$, and hence, by the mean value property of holomorphic maps, $\Phi \equiv 0$. That is, $\tilde{u}$ is conformal. Since $\tilde{u} \equiv p$ on ∂B, therefore $\nabla\tilde{u} \equiv 0$ on ∂B. But, by results of Hartman-Wintner [22], the branch points (where $\nabla\tilde{u} = 0$) of $\tilde{u}$ are isolated or $\tilde{u} \equiv$ const. □

2.2.2. *Partial Regularity.* If we drop all geometric constraints on the target, for $m \ge 3$ there is no hope of proving regularity for weakly harmonic maps $u \in H^1(M;N)$. In fact, Rivière [36] constructed examples of weakly harmonic maps with finite energy which are everywhere discontinuous.

On the other hand, for maps u that minimize E among maps $v \in H^1(M;N)$ with the same boundary data, partial regularity results are available. Indeed, by results of Schoen-Uhlenbeck [38], [39] and Giaquinta-Giusti [17], energy-minimizing maps are smooth on the complement of a closed "singular set" of finite $(m-3)$-dimensional Hausdorff measure. In particular, as was shown earlier by Morrey [35], energy-minimizing maps in dimension $m = 2$ are smooth. The example of the map $u(x) = \frac{x}{|x|}\colon M = B^m = B_1(0;\mathbb{R}^m) \to S^{m-1} = N$, which is energy minimizing if $m \ge 3$ [3], [33], shows that the above results cannot be improved.

Halfway between weakly harmonic maps and energy-minimizing maps lies the class of stationary maps $u \in H^1(M;N)$ which, by definition, are weakly harmonic and, in addition, are critical points of E_{st} also with respect to variations of the independent variables. By results of Evans [11] and Bethuel [2] these latter maps are smooth away from a singular set of dimension $\leq m-2$.

There are analogous global existence and partial regularity results for the evolution problem (2.4), (2.5). In particular in the "conformal" case $m=2$ we have the following result of Struwe [46], (which was extended to the case $\partial M = \phi$ by Chang [4]).

Theorem 2.3. *For any initial map* $u_0 \in H^1(M;N)$ *there exists a unique, global, weak solution* $u\colon [0,\infty[\times M \to N$ *of* (2.4), (2.5)*, satisfying the energy inequality* (2.6) *for all* $S<T$*, and which is smooth away from finitely many points* $(\bar{t}_i,\bar{x}_i), 1 \leq i \leq I \leq CE_{st}(u_0)$*. Each singularity* $(\bar{t},\bar{x})$ *is related to a smooth, non-constant harmonic map* $\bar{u}\colon S^2 \to N$ *in the sense that for suitable sequences* $R_n \to 0, t_n \nearrow \bar{t}$*,* $x_n \to \bar{x}$ *we have*

$$u_n(x) = u(t_n, R_n x + x_n) \to \tilde{u} \quad \text{in } H^{2,2}_{loc}(\mathbb{R}^2;N)$$

as $n\to\infty$*, where* $\tilde{u}\colon \mathbb{R}^2 \to N$ *is smooth, harmonic and extends to a smooth, non-constant harmonic map* $\bar{u}\colon S^2 \to N$*. (We refer to this "bubbling-off" process as "separation of spheres".)*

Originally, uniqueness was only established among partially regular solutions as in the statement of Theorem 2.4. By results of Freire [13], [14] and Rivière [37], in case $N=S^k$ (and probably also for general targets) uniqueness also holds among weak solutions of class H^1 satisfying the energy inequality (2.6) for all $0<S<T$.

By an example due to Chang-Ding-Ye [5], singularities actually may develop in finite time, even if the initial data are smooth. Theorem 2.3 therefore is best possible.

If $m \geq 3$, the existence of global, partially regular solutions to (2.4), (2.5) was derived in [6], based on the monotonicity formula and a-priori estimates for (2.4), (2.5) from [45]. However, there is no uniqueness in the energy class [9].

Also in the time-independent case the situation improves drastically if $m=2$. In fact, we have the following beautiful result of Hélein [23].

Theorem 2.4. *Any weakly harmonic map* $u \in H^1(M;N)$ *is smooth.*

2.3. Regularity in the hyperbolic case. In short, one can say that all the problems with regularity of weakly harmonic maps and/or well-posedness of the evolution problem (2.4), (2.5) in the class of H^1-solutions are present in the hyperbolic regime, as well. Thus, contrary to the title of this section, in the sequel we will not discuss any regularity properties of wave maps. Instead, we will show the break-down of regularity and loss of uniqueness for the initial value problem (2.1), (2.2) in dimensions $m\geq 3$. The examples we discuss indicate that there is hardly any regularity to be gained (in high dimension) from geometric conditions that we may impose on the target. Moreover, in order for (2.1), (2.2) to be well-posed in a suitable class, one still needs to identify a further "entropy condition" that will ensure uniqueness of weak solutions in this class. The situation in this regard thus is analogous to the situation for the parabolic evolution problem (2.4), (2.5) in dimensions $m \geq 3$.

With techniques available at this time we can therefore only hope to prove well-posedness of the initial value problem (2.1), (2.2) in case $m = 2$, analogous to Theorem 2.3 for the parabolic problem (2.4), (2.5) in this case. Some recent results in this regard will be presented in Lecture 3 .

2.3.1. *Blow-up.* The simplest way to produce blow-up is to show the existence of self-similar solutions

$$u(t,x) = v(\frac{x}{|t|})$$

to (2.1) with non-constant smooth Cauchy data

$$u_0 = v, \quad u_1 = x \cdot \nabla v$$

at $t = -1$. Introduce similarity coordinates

$$\tau = \sqrt{t^2 - |x|^2}, \quad \xi = \frac{x}{|t|}$$

in the backward light cone from 0 and denote $|x| = r$, $|\xi| = \rho$, $x = r\omega$, $\xi = \rho\omega$ with $\omega \in S^{m-1}$. The Minkowski metric

$$ds^2 = dt^2 - dr^2 - r^2 d\omega^2$$

then can be expressed as

$$ds^2 = d\tau^2 - \tau^2 \left(\frac{d\rho^2}{(1-\rho^2)^2} + \frac{\rho^2}{1-\rho^2} d\omega^2 \right).$$

Hence u is stationary for the standard Lagrangean $\mathcal{L}$ if and only if $v(\xi) = v(\rho,\omega)$ is stationary for the reduced Lagrangean

$$\mathcal{L}_{sim}(v) = \frac{1}{2} \int \left\{ (1-\rho^2)^2 |v_\rho|^2 + \frac{1-\rho^2}{\rho^2} |v_\omega|^2 \right\} \frac{\rho^{m-1}}{(1-\rho^2)^{\frac{m+1}{2}}} \, d\rho \, d\omega$$

at $\tau = 1$. That is, u solves (2.1) if and only if v solves the equation

$$-v_{\rho\rho} - \left(\frac{m-1}{\rho} + \frac{(m-3)\rho}{1-\rho^2} \right) v_\rho + \frac{1}{\rho^2(1-\rho^2)} \Delta_\omega v \perp T_v N. \tag{2.8}$$

Remark that equation (2.8) is an *elliptic* harmonic map equation on the unit m-ball B with the hyperbolic metric

$$\frac{d\rho^2}{(1-\rho^2)^2} + \frac{\rho^2}{1-\rho^2} \, d\omega^2. \tag{2.9}$$

We seek solutions v of equation (2.8) that extend smoothly to the "boundary" $\rho = 1$ of B and hence can be continued smoothly to all of $\mathbb{R}^m$. Since information propagates with speed ≤ 1 the unique solution u of (2.1), (2.2) with initial data

$$u_0 = v, \, u_1 = x \circ \nabla v \quad \text{at } t = -1$$

then will coincide with $v(\frac{x}{|t|})$ inside the backward light cone $|x| \leq -t$ and, if $v \not\equiv$ const. on B, we obtain blow-up at $t = 0$.

2.3.1.1. *The case $m=2$.* If $m=2$, equation (2.8) becomes

$$-\big(\rho\sqrt{1-\rho^2}v_\rho\big)_\rho+\frac{1}{\rho\sqrt{1-\rho^2}}\Delta_\omega v\perp T_vN.$$

Multiplying by $\rho\sqrt{1-\rho^2}v_\rho$ and integrating with respect to $\omega\in S^1$, we obtain

$$\frac{d}{d\rho}\left(\int_{S^1}\rho^2(1-\rho^2)|v_\rho|^2\,d\omega-\int_{S^1}|v_\omega|^2\,d\omega\right)=0,$$

and hence

$$\int_{S^1}\rho^2(1-\rho^2)|v_\rho|^2\,d\omega-\int_{S^1}|v_\omega|^2\,d\omega=C_0.$$

Inspection at $\rho=0$ shows that $C_0=0$. Hence for $\rho=1$ we obtain $v_\omega=0$; that is, $v(1,\cdot)\equiv\text{const}$.

Recall that by the Riemann mapping theorem the metric (2.9) on $B=B_1(0;\mathbb{R}^2)$ is locally conformal to the standard metric. In fact, define

$$\sigma(\rho)=\exp\left(-\int_\rho^1\frac{d\rho}{\rho\sqrt{1-\rho^2}}\right)$$

and observe that the metric

$$d\sigma^2+\sigma^2d\omega^2=\sigma^2\left(\frac{d\rho^2}{\rho^2(1-\rho^2)}+d\omega^2\right)=\left(\frac{\sigma}{\rho}\right)^2(1-\rho^2)\left(\frac{d\rho^2}{(1-\rho^2)^2}+\frac{\rho^2}{1-\rho^2}\,d\omega^2\right)$$

is conformal to the metric (2.9) on B.

That is, the map $(\rho,\omega)\mapsto(\sigma,\omega)$ is a conformal diffeomorphism ψ from B, endowed with the metric (2.9), to B with the standard metric.

By conformal invariance of Dirichlet's integral and hence of the harmonic map equation (2.3) in $m=2$ dimensions, thus v induces a harmonic map $\bar v=v\circ\psi^{-1}\in H^1\cap C^0(\bar B;N)$ on the standard disc with $\bar v\,|_{\partial B}\equiv\text{const}$. By Lemaire's result Theorem 2.2, therefore we obtain the following result from [48].

Theorem 2.5. *If $m=2$ and if $u(t,x)=v(\frac{x}{|t|})$ solves (2.1) for $|x|\le|t|$, where v extends to a smooth map on a neighborhood of $\overline{B_1(0)}$, then $v\equiv\text{const}$.*

2.3.1.2 *The case $m\ge 3$.* In high dimensions, following Shatah-Tahvildar-Zadeh [44], we can obtain self-similar solutions to (2.1) as follows. We consider as target a surface of revolution N with metric

$$ds^2=dh^2+g^2(h)\,d\omega^2$$

in spherical coordinates $h>0$, $\omega\in S^{m-1}$ and we attempt to find solutions to (2.1) of the special form

$$u(t,r\omega)=h(t,r)\omega,$$

where we also express $x=r\omega\in\mathbb{R}^m$ in terms of spherical coordinates. Moreover, we make the ansatz $u(t,x)=v\big(\frac{x}{|t|}\big)$, that is, $h(t,r)=\varphi\big(\frac{r}{|t|}\big)$, $v(\xi)=\varphi(\rho)\omega$.

The reduced Lagrangean then becomes

$$\mathcal{L}_{sim}(v) = \frac{1}{2}\int (1-\rho^2)^2|\varphi_\rho|^2 + \frac{1-\rho^2}{\rho^2}(m-1)g^2(\varphi)\}\frac{\rho^{m-1}}{(1-\rho^2)^{\frac{m+1}{2}}}\,d\rho\,d\omega,$$

and equation (2.8) takes the form

$$-\varphi_{\rho\rho} - \left(\frac{m-1}{\rho} + \frac{(m-3)\rho}{1-\rho^2}\right)\varphi_\rho + \frac{(m-1)f(\varphi)}{\rho^2(1-\rho^2)} = 0. \tag{2.10}$$

where

$$f(\varphi) = g(\varphi)g'(\varphi).$$

For special target metrics g, equation (2.10) admits non-constant solutions φ for $0 < \rho < 1$ that extend smoothly to all of $\mathbb{R}_+$. In fact, we may take

$$g^2(\varphi) = \varphi^2 - \frac{1}{2}\varphi^4 \qquad \text{for } 0 < \varphi < \varphi_0$$

for some fixed number $\varphi_0 > 0$ such that $1 < \varphi_0^2 < 2$, and we extend g smoothly to $\mathbb{R}_+$. If follows that

$$f(\varphi) = g(\varphi)g'(\varphi) = \frac{1}{2}\big(g^2(\varphi)\big)' = \varphi - \varphi^3$$

for $0 < \varphi < \varphi_0$. The linear function

$$\varphi(\rho) = c\rho,$$

where $c = \sqrt{\frac{2}{m-1}}$ then solves (2.10) for $0 < \rho < c^{-1}\varphi_0 = \sqrt{\frac{m-1}{2}}\varphi_0 = \rho_0$, and $\rho_0 > 1$ if $m \geq 3$. Note that for g as above, the radius of convexity of N around 0 is

$$\varphi_* = 1,$$

which is larger than c for $m \geq 4$ and equals c if $m = 3$. By changing the metric $g(\varphi)$ on N suitably for $\varphi > c$, and by changing the initial data for h off $\overline{B_1(0)}$, we may thus construct solutions to (2.10) which blow up in finite time, with initial data having compact support and such that the target manifold is convex, if $m \geq 4$, and only slightly fails to be convex, if $m = 3$.

A more detailed analysis shows that in 3 space dimensions blow-up may occur also for more general metrics on the target surface:

Theorem 2.6 (Shatah–Tahvildar-Zadeh [44]). *Suppose $g \in C^\infty$ satisfies $g(0) = 0$, $g'(0) = 1$ and suppose g' has a smallest positive zero φ_*. Also suppose that $g''(\varphi_*) \neq 0$. Then there is a class of regular initial data such that the corresponding Cauchy problem for equivariant harmonic maps from M^{3+1} into N has a solution that blows up in finite time.*

2.3.2. *Non-uniqueness of weak solutions.* In particular, Theorem 2.6 applies to the sphere, where $g(\varphi) = \sin\varphi,\ \varphi_* = \frac{\pi}{2}$. Shatah-Tahvildar-Zadeh construct a solution φ to (2.9) on $[0,\infty[$, satisfying

$$\varphi(1) = \varphi_*$$

and having the asymptotic expansion for $\rho \to \infty$:

$$\begin{aligned}\varphi(\rho) &= a + \frac{b}{\rho} + \frac{d}{\rho^2} + O\left(\frac{1}{\rho^3}\right)\\ \varphi'(\rho) &= -\frac{b}{\rho^2} + O\left(\frac{1}{\rho^3}\right).\end{aligned}$$

They consider the corresponding function $h(t,r)\omega = \varphi\left(\frac{r}{t}\right)\omega$ as a weak solution of (2.1), that is,

$$h_{tt} - h_{rr} - \frac{2}{r}h_r + \frac{sin2h}{2r^2} = 0, \tag{2.11}$$

with singular initial data at $t = 0$, given by

$$\begin{aligned} h(0,r) &= h_0(r) = a = \lim_{t\searrow 0}\varphi\left(\frac{r}{t}\right) \qquad (r \neq 0)\ ,\\ h_t(0,r) &= h_1(r) = \frac{b}{r} = \lim_{t\searrow 0}\frac{d}{dt}\varphi\left(\frac{r}{t}\right) \qquad (r \neq 0)\ . \end{aligned} \tag{2.12}$$

Thereby, h is a weak solution of (2.11) say, on $[0,1]\times\mathbb{R}^3$, if there holds

$$\int_0^1\int_0^\infty \left\{-h_t\psi_t + h_r\psi_r + \frac{1}{2r^2}\psi\sin 2h\right\} r^2\,dr\,dt = \int_0^\infty \psi(0,r)\frac{b}{r}r^2\,dr \tag{2.13}$$

for any $\psi \in C^\infty\left([0,1]\times\mathbb{R}^3\right)$ such that $\psi(t,x) = \psi(t,r)$, $\psi(1,\cdot)\equiv 0$, and $\operatorname{supp}\psi(t) \subset B_R(0)$ for some $R > 0$. Moreover, h assumes the initial data (2.12) in the sense that

$$\begin{aligned}\|h(t,r) - a\|_{H^{1,2}_{\text{loc}}(\mathbb{R}^3)} &\to 0 \qquad (t\to 0),\\ \|h_t(t,r) - \frac{b}{r}\|_{L^2(\mathbb{R}^3)} &\to 0 \qquad (t \to 0).\end{aligned}$$

Note that $h_0 \in H^{1,2}_{\text{loc}}$, $h_1 \in L^2_{\text{loc}}$.

On the other hand, also the function

$$\tilde{h}(t,r) = \begin{cases}\varphi\left(\frac{r}{t}\right), & r > t\\ \varphi_*, & r \le t\end{cases}$$

weakly satisfies (2.11), (2.12) on $[0,1]\times\mathbb{R}^3$, with $D\tilde{h} \in L^\infty\left([0,1]; L^2(B_R(0))\right)$ for any $R > 0$, showing that weak solutions are in general not unique. To verify that $\tilde{h}$ solves (2.13), for any ψ we split

$$\begin{aligned}&\int_0^1\int_0^\infty \left\{-h_t\psi_t + h_r\psi_r + \frac{1}{2r^2}\psi\sin 2h\right\} r^2\,dr\,dt - \int_0^\infty \psi(0,r)\frac{b}{r}r^2\,dr\\ &= \left\{\int_0^1\int_t^\infty \{\dots\}\,r^2\,dr\,dt - \int_0^\infty \psi(0,r)\frac{b}{r}r^2\,dr\right\} + \int_0^1\int_0^t \{\dots\}\,r^2\,dr\,dt = I + II.\end{aligned}$$

Clearly, since $D\tilde{h}(t,r) \equiv 0$ for $r \le t$, the second integral $II = 0$. Moreover, since $\tilde{h} \equiv h$ for $r \ge t$, and since h satisfies (2.13) the first integral reduces to the boundary term

$$I = \frac{1}{\sqrt{2}} \int_0^1 (h_t(t,t) + h_r(t,t))\, \psi(t,t) t^2 \, dt$$

which also vanishes on account of

$$\begin{aligned} h_t + h_r &= -\frac{r}{t^2}\varphi'\left(\frac{r}{t}\right) + \frac{1}{t}\varphi'\left(\frac{r}{t}\right) \\ &= \frac{1}{t}\left(1 - \frac{r}{t}\right)\varphi'\left(\frac{r}{t}\right) = 0 \qquad \text{for } r = t. \end{aligned}$$

Observe that $\tilde{h}$ induces a solution $\tilde{u}$ of (2.1) with $E(\tilde{u}(t); B_1(0)) > E(u(t); B_1(0))$ for any $t \in]0,1]$, where u is the solution corresponding to h. Hence there may be a chance of restoring uniqueness by some entropy principle.

3. The conformal case $m = 2$

3.1. Overview. The results presented in Lecture 2 leave little hope for the development of a satisfactory theory of existence, uniqueness, and stability for wave maps in high dimensions $m \ge 3$, even under very stringent geometric conditions on the target and/or very restrictive symmetry assumptions on the maps involved.

By contrast, as is illustrated by the absence of self-similar solutions, Theorem 2.5, the situation seems to be much better in dimension $m = 2$, due to conformal invariance of Dirichlet's integral in this dimension. Thus, we are tempted to conjecture that a result analogous to Theorem 2.3 for the "heat flow" related to harmonic maps of surfaces M also holds for the Cauchy problem for wave maps $u\colon \mathbb{R} \times M \to N$.

In this last of three lectures we will show that this conjecture is true for equivariant maps to surfaces of revolution and we sketch some recent developments towards a general theorem of well-posedness of the Cauchy problem in dimension 2.

3.2. The equivariant case. The results that follow are mostly due to Shatah-Tahvildar-Zadeh [43], [44] and Shatah-Struwe [41], [42].

3.2.1. *Co-rotational maps.* As in Lecture 2, Section 2.3, again we consider as target a surface of revolution N with metric

$$ds^2 = dh^2 + g^2(h)\, d\omega^2$$

written in terms of polar coordinates $h > 0$, $\omega \in S^1$. We assume that g is smooth with $g(-h) = -g(h)$ and $g'(0) = 1$. Moreover, we either suppose that

$$g(h) > 0 \qquad \text{if } h > 0 \tag{3.1}$$

and

$$\int_0^h |g(s)|\, ds \to \infty \qquad \text{as } h \to \infty, \tag{3.2}$$

or that there exists $q_1 > 0$ such that

$$g(q_1) = 0, \quad g(h) > 0 \qquad \text{for } 0 < h < q_1 \tag{3.3}$$

and g is odd around q_1 as well as around $q_0 = 0$ (and hence periodic of period $2q_1$). In the latter case, for ease of exposition only, in the following we will also assume that g is even around $\frac{q_1}{2}$.

Case (3.1), (3.2) corresponds to a non-compact target surface N, including the standard plane $g(h) \equiv h$ or metrics of negative curvature like $g(h) = sinh(h)$; condition (3.2) rules out sharp cusps "at infinity". Case (3.3) corresponds to a compact target, including the standard sphere $g(h) = \sin(h)$. Remark that (3.3) also implies (3.2).

Moreover, we consider maps $u\colon \mathbb{R}\times\mathbb{R}^2 \to N$ such that, expressing a point $x \in \mathbb{R}^2$ in polar coordinates $x = r\omega$, the angle $\omega \in S^1$ is preserved by u and $h(t,x) = h(t,r)$. Such maps will be called co-rotational.

For such u we have

$$\mathcal{L}(u) = \frac{1}{2}\int \left\{ |h_t|^2 - |h_r|^2 - \frac{g^2(h)}{r^2} \right\} r\,dr\,d\omega\,dt$$

and u is stationary for $\mathcal{L}$ if and only if $h\colon \mathbb{R}\times [0,\infty[\to \mathbb{R}$ satisfies

$$h_{tt} - h_{rr} - \frac{1}{r}h_r + \frac{f(h)}{r^2} = 0, \tag{3.4}$$

where

$$f(h) = g(h)g'(h).$$

Moreover, for smooth solutions the energy (scaled with a factor 2π)

$$\frac{1}{2\pi}E\big(u(t)\big) = \frac{1}{2}\int_0^\infty \left\{ |h_t|^2 + |h_r|^2 + \frac{g^2(h)}{r^2} \right\} r\,dr = E_{equi}\big(h(t)\big)$$

is conserved.

Lemma 3.1. *If u is a co-rotational map with $E\big(u(t)\big) \le$const. for all $t \in \mathbb{R}$, then the associated map h is continuous on $\mathbb{R}\times\,]0,\infty[$ and $h(t,\cdot)$ extends continuously to $[0,\infty[$ for every $t \in \mathbb{R}$, where $g\big(h(t,0)\big) = 0$ for all t.*

Proof. Since for any $r_0 > 0$ the integral

$$\int_{r_0}^\infty |h_r(t,r)|^2\,dr \le 2r_0^{-1}E_{equi}\big(h(t)\big)$$

is uniformly bounded, by Sobolev's embedding $H^{1,2} \hookrightarrow C^{1/2}$ we conclude that $h(t,\cdot)$ is locally Hölder continuous on $]0,\infty[$, uniformly in $t \in \mathbb{R}$. Since $h_t \in L^2_{\text{loc}}\big(\mathbb{R}\times]0,\infty[\big)$, moreover, $h(\cdot,r)$ is continuous in t for almost every $r > 0$. Hence, the map $t \mapsto h(t,\cdot) \in C^0(]0,\infty[)$ is continuous by the theorem of Arzéla-Ascoli.

In order to prove continuity at $r = 0$, let

$$G(h) = \int_0^h |g(s)|\,ds.$$

Then, by Hölder's inequality, for any $t \in \mathbb{R}$ we have

(3.5)
$$\int_0^{r_0} |G(h)_r|\,dr \le \int_0^{r_0} |g(h)||h_r|\,dr \le \Big(\int_0^{r_0} \frac{|g(h)|^2}{r}\,dr\Big)^{1/2} \Big(\int_0^{r_0} |h_r|^2 r\,dr\Big)^{1/2}$$
$$\le 2E_{equi}\big(h(t);B_{r_0}(0)\big) \to 0 \quad (r_0 \to 0).$$

It follows that $\lim_{r\to 0} G\big(h(t,r)\big)$ exists for any t and hence, by strict monotonicity of G, that $\lim_{r\to 0} h(t,r) = h(t,0)$ exists for any t. Finally, since

$$\int_0^\infty \frac{g^2(h)}{r}\,dr < \infty,$$

it follows that $g\big(h(t,0)\big) = 0$ for any t. □

In case (3.1) Lemma 3.1 implies the boundary condition

$$h(t,0) = 0 \qquad \text{for all } t. \tag{3.6}$$

In case of assumption (3.3), from Lemma 3.1 we only deduce that $h(t,0) = kq_1$ for some $k \in \mathbb{Z}$.

Lemma 3.2. *There is a constant $\epsilon_0 > 0$ with the following property. If there exists $r_0 > 0$ such that*

$$E\big(u(t);B_{r_0}(0)\big) < \epsilon_0 \qquad \textit{for all } t, \tag{3.7}$$

then $h(t,0)$ is constant in time.

Proof. From (3.5) we deduce that

$$|G\big(h(t,0)\big) - G\big(h(t,r_0)\big)| \le C\epsilon_0$$

for all t. Since $h(t,r_0)$ depends continuously on t and since $G\big(h(t,0)\big) = k\,G(q_1)$ for some $k = k(t) \in \mathbb{Z}$, if $C\epsilon_0 < G(q_1)$ it follows that $h(t,0)$ is constant in time. □

Finiteness of $E\big(u(t)\big)$ and finite propagation speed for (3.4) also implies the asymptotic boundary condition

$$h(t,r) \equiv q_0 \in \mathbb{R} \qquad \text{for } r \ge R_0 + |t| \tag{3.8}$$

for some number $R_0 > 0$ and some $q_0 \in \mathbb{R}$ such that $g(q_0) = 0$. We may normalize $q_0 = 0$. Together, (3.5) and (3.8) imply the uniform bound

$$\sup_r G\big(h(t,r)\big) \le \lim_{r_0\to\infty} G\big(h(t,r_0)\big) + 2E_{equi}\big(h(t)\big) = G(0) + 2E_{equi}\big(h(t)\big).$$

From assumption (3.2) we then obtain

Lemma 3.3. *If $h\colon \mathbb{R}\times[0,\infty[\to \mathbb{R}$ corresponds to a co-rotational wave map u with data $(u_0,u_1) \in H^1_c$, then h is uniformly bounded.*

For later reference we also introduce the set

$$H^1_c\big([0,\infty[\big) = \Big\{(h_0,h_1); h_0 \in H^{1,2}_{\text{loc}} \cap C^0\big([0,\infty[\big), h_0(0) = d_0 q_1 \quad \text{for some } d_0 \in \mathbb{Z},$$
$$h_0(r), h_1(r) \equiv 0 \quad \text{for large } r\Big\}.$$

The initial value problem for co-rotational wave maps u with data $(u_0, u_1) \in H^1_c(\mathbb{R}^2; TN)$ at $t = 0$ then corresponds to the initial-boundary value problem (3.4), (3.6) for data $\big(h\,|_{t=0}, h_t\,|_{t=0}\big) \in H^1_c\big([0,\infty[\big)$.

Due to energy conservation and the semi-linear character of (3.4) it is not hard to show the existence of a global weak solution to the initial-boundary value problem (3.4), (3.6) of class H^1_c. However, it is not clear whether this solution is unique and whether the solution preserves any additional regularity properties of the data.

Fortunately, we can transform equation (3.4) to a form where these questions can be answered.

3.2.2. *The transformed equation.* To eliminate the singularity in (3.4), instead of h we introduce the map

$$\varphi(t,r) = \frac{h(t,r) - h(t,0)}{r}.$$

Recall that, by Lemma 3.2, $h(t,0) = dq_1$, where $d \in \mathbb{Z}$ is independent of $t \in \mathbb{R}$ for maps h corresponding to wave maps u which depend continuously on time in the H^1-topology.

Note that formally φ satisfies

$$\varphi_{tt} - \varphi_{rr} - \frac{3}{r}\varphi_r + \frac{f(r\varphi) - r\varphi}{r^3} = 0.$$

Expanding

$$\begin{aligned} f(r\varphi) = g(r\varphi)g'(r\varphi) &= \big(r\varphi + \frac{g'''(0)}{6}(r\varphi)^3 + \dots\big)\big(1 + \frac{g'''(0)}{2}(r\varphi)^2 + \dots\big) \\ &= r\varphi + \frac{2}{3}g'''(0)(r\varphi)^3 + \dots\,, \end{aligned}$$

we can express the nonlinear term as

$$\frac{f(r\varphi) - r\varphi}{r^3} = -K(r\varphi)\varphi^3,$$

where K is smooth, $K(h) = K(-h)$, and

$$K(0) = -\frac{2}{3}g'''(0)$$

equals $2/3$ the curvature of N at 0.

Also regarding

$$\varphi_{tt} - \varphi_{rr} - \frac{3}{r}\varphi_r = \Box\varphi,$$

as the wave operator on $\mathbb{R}\times\mathbb{R}^4$, acting on the radially symmetric function $\varphi(t,x) = \varphi(t,r)$ for $x \in \mathbb{R}^4$ with $|x| = r$, thus we arrive at the equation

$$\Box\,\varphi - K(r\varphi)\varphi^3 = 0 \qquad \text{on } \mathbb{R}\times\mathbb{R}^4. \tag{3.9}$$

In case (3.1), moreover, (3.8) translates into the boundary condition

$$\varphi(t,r) = 0 \qquad \text{for } r \geq R_0 + |t|. \tag{3.10}$$

Finally, in case (3.1), (3.2) hold, using boundedness of h (Lemma 3.3) we infer that there exists a constant C, possibly depending on h, such that

$$h(t,r) \leq Cg\big(h(t,r)\big).$$

Hence

$$\begin{aligned} E_{equi}\big(h(t)\big) &\geq \frac{1}{2}\int_0^\infty \left\{|h_t|^2 + |h_r|^2 + \frac{h^2}{C^2r^2}\right\} r\,dr \\ &= \frac{1}{2}\int_0^\infty \left\{|\varphi_t|^2 + |\varphi_r + \frac{\varphi}{r}|^2 + \frac{\varphi^2}{C^2r^2}\right\} r^3\,dr \end{aligned}$$

and it follows that (φ, φ_t) is bounded in the energy space $H^1_c(\mathbb{R}^4; T\mathbb{R})$.

Thus, solutions h of (3.4), (3.6) of class H^1 with data $(h_0, h_1) \in H^1_c$ correspond to solutions φ of (3.9) of class H^1 with data in H^1_c, and conversely.

For a compact target surface, with g satisfying (3.3), condition (3.8) translates into

$$\varphi(t,r) = -\frac{dq_1}{r} \qquad \text{for } r \geq R_0 + |t|. \tag{3.11}$$

Remark that $d = d_0 \in \mathbb{Z}$ corresponds to the degree of the initial map

$$h_0 \colon \mathbb{R}^2 \cong S^2 \to N \cong S^2.$$

Hence for initial degree $d_0 \neq 0$, from (3.11) we conclude that

$$\varphi_r(t,r) = \frac{d_0q_1}{r^2} \notin L^2\big([0,\infty[\big)$$

for any t. H^1-solutions h of (3.4), (3.6) with $d_0 \neq 0$ therefore do *not* correspond to solutions φ of (3.9) of class H^1, but only to solutions which are *locally* of class H^1.

3.2.3. *Well-posedness, a model case.* To set the stage for the general result first consider the model case

$$g(h) = h + \frac{1}{4}h^3.$$

Equation (3.9) in this case reads

$$\Box\,\varphi + \varphi^3 = 0 \qquad \text{on } \mathbb{R}\times\mathbb{R}^4 \tag{3.12}$$

with Cauchy data

$$\varphi\mid_{t=0} = \varphi_0, \quad \varphi_t\mid_{t=0} = \varphi_1, \tag{3.13}$$

where $(\varphi_0, \varphi_1) \in H^1_c(\mathbb{R}^4; T\mathbb{R})$.

Equation (3.12) is a special case of a class of semi-linear wave equations involving critical growth exponents for which a full theory of existence, uniqueness and regularity has been developed in the past years, starting with the work of Struwe [47] on radial solutions of the equation

$$\Box\varphi + \varphi^5 = 0 \qquad \text{on } \mathbb{R}\times\mathbb{R}^3$$

in 1988. The symmetry condition in [47] was removed by Grillakis [19], still in 3 space dimensions.

The insight how to treat the higher-dimensional cases, in particular the case $m = 4$ which is relevant here, came from the work of Kapitanskii [29] who pointed out the use of the Strichartz inequalities for the analysis of semilinear wave equations like (3.12).

Grillakis [20] then was the first to realize that the Strichartz estimates and the crucial decay estimate from [47] could be combined to prove regularity for the equation

$$\Box\,\varphi + \varphi|\varphi|^{2^*-2} = 0 \qquad \text{in } \mathbb{R}\times\mathbb{R}^m \tag{3.14}$$

in dimensions $m \le 5$, where $2^* = \frac{2m}{m-2}$ is the "Sobolev exponent" in dimension m.

More efficient use of the Strichartz estimates was then made by Shatah-Struwe [41], [42] who extended the regularity results for (3.14) to dimensions $m \le 7$ and, moreover, proved that the initial value problem for (3.14) with finite energy data $(\varphi_0, \varphi_1) \in H_c^1$ is well-posed in all dimensions $m \ge 3$.

In particular, the result from [42] applies to the Cauchy problem for (3.12), (3.13).

Theorem 3.4. *For any $(\varphi_0, \varphi_1) \in H_c^1$ there exists a unique solution φ of* (3.12), (3.13) *such that $(\varphi, \varphi_t) \in C^0(\mathbb{R}; H_c^1) \cap L^q(\mathbb{R}; \dot{B}_q^{1/2} \times \dot{B}_q^{-1/2})$, where $q = \frac{10}{3}$ and $L^q(\mathbb{R}; \dot{B}_q^{1/2})$ is the Besov space of functions φ with "half a spatial derivative" in $L^q(\mathbb{R}\times\mathbb{R}^4)$. If $(\varphi_0, \varphi_1) \in C^\infty$, then $\varphi \in C^\infty$, as well.*

Remark that by Sobolev's embedding $\varphi \in L^\infty(\mathbb{R}; L^4(\mathbb{R}^4)) \cap L^{\frac{10}{3}}(\mathbb{R}; L^{\frac{40}{7}}(\mathbb{R}^4))$ and hence by interpolation

$$\varphi \in L^5(\mathbb{R}\times\mathbb{R}^4).$$

The *proof of uniqueness* now follows easily from the Strichartz estimate

$$\|\psi\|_{L^{\frac{10}{3}}(\mathbb{R}\times\mathbb{R}^4)} \le C\|\,\Box\,\psi\|_{L^{\frac{10}{7}}(\mathbb{R}\times\mathbb{R}^4)} \tag{3.15}$$

for $\psi\colon \mathbb{R}\times\mathbb{R}^4$ with vanishing Cauchy data

$$\psi\,|_{t=0} = 0 = \psi_t\,|_{t=0}\,.$$

For simplicity we assume that the initial data have small energy. Then the square of the L^5-norm of any solution φ of (3.12), (3.13) as in Theorem 3.4 is bounded by the energy of the initial data. Let $\varphi, \tilde{\varphi}$ be two solutions of (3.12) as in Theorem 3.4 sharing Cauchy data $(\varphi_0, \varphi_1) \in H_c^1$ at $t = 0$. Then $\psi = \varphi - \tilde{\varphi}$ satisfies

$$|\Box\psi| = |\varphi^3 - \tilde{\varphi}^3| \le C|\psi|(\varphi^2 + \tilde{\varphi}^2)$$

and hence by (3.15) and Hölder's inequality

$$\begin{aligned}\|\psi\|_{L^{\frac{10}{3}}(\mathbb{R}\times\mathbb{R}^4)} &\le C\|\psi\|_{L^{\frac{10}{3}}(\mathbb{R}\times\mathbb{R}^4)}\Big(\|\varphi\|^2_{L^5(\mathbb{R}\times\mathbb{R}^4)}+\|\tilde\varphi\|^2_{L^5(\mathbb{R}\times\mathbb{R}^4)}\Big)\\ &\le CE\big(u(0)\big)\|\psi\|_{L^{\frac{10}{3}}(\mathbb{R}\times\mathbb{R}^4)},\end{aligned}$$

which implies that $\psi=0$, if $E(u(0))$ is sufficiently small. □

The slightly improved space-time integrability of the solutions obtained in Theorem 3.4 (that is, $u \in L^5$ instead of L^4, only) also suffices to propagate further regularity of the data; in particular, φ is smooth if the data are smooth; see also [18] for corresponding results in higher dimensions.

The Strichartz inequality can be localized to light cones; moreover, for small energy the estimates relevant for the proof of Theorem 3.4 in the model case (3.12) continue to be valid for the general equation (3.9) with data $(\varphi_0,\varphi_1) \in H^1_c$; see [41]. Hence it seems that this problem and therefore the Cauchy problem for co-rotational wave maps into surfaces of revolution satisfying (3.1), (3.2) is globally well-posed for small initial energy. If the energy is large, it is conceivable that concentration may occur at a finite number of points $(t_i,0)$, just as in the case of the heat flow for harmonic maps of surfaces; see Theorem 2.3. However, a curvature condition or a "small range" condition can prevent energy from concentrating; see [7], [21], [41], [43].

Under assumption (3.3), energy estimates for φ always need to be localized; moreover, the degree of the corresponding map u may change at concentration points.

We close this section by stating the following result implied by this reasoning.

Theorem 3.5. *i) Suppose N satisfies* (3.1), (3.2)*. Then for any co-rotational data $(h_0,h_1)\in H^1_c$ corresponding to data $(\varphi_0,\varphi_1)\in H^1_c$ for* (3.12)*, there exists a unique global weak solution $h(t,r)=r\varphi(t,r)$ of* (3.4)*, respectively* (3.12)*, in the following class:*
There are numbers $0=t_0<t_1<t_2<\ldots<t_I<t_{I+1}=\infty$, such that $(\varphi,\varphi_t)\in C^0\big([t_i,t_{i+1}[;H^1_c\big)\cap L^q_{loc}\big([t_i,t_{i+1}[;\dot B^{1/2}_q(\mathbb{R}^4)\big)$ for $0\le i\le I$, and

$$E_{equi}\big(h(t)\big)\equiv E_i \qquad \text{in } [t_i,t_{i+1}[,$$

where $E_{i+1}\le E_i-\epsilon_0$ for each i for some $\epsilon_0=\epsilon_0(N)>0$. In particular, $I\le E_0/\epsilon_0$. (Similarly for negative time.)

ii) If N is compact, the same result holds; however, the estimates for φ are only local in space. At each singularity, the topological degree of the corresponding map u may change.

Details will be given in the forthcoming thesis of Wilhelmy. We conjecture that concentration, as described in Theorem 3.5, can in fact only occur for compact targets as in part ii). Whether concentration of energy ever occurs in finite time, however, still remains as one of the most challenging open problems in this field.

3.3. Towards well-posedness for general targets. Now that we have found confidence that the Cauchy problem for wave maps may, indeed, be well-posed in the energy space for a $(1+2)$–dimensional space-time domain, we drop the symmetry assumption on N and the map u and return to the general setting. Thus, for given Cauchy data

$$(u_0, u_1) \in H^1_c(M; TN)$$

we hope to show the existence of a unique, global, weak solution $u\colon \mathbb{R} \times M \to N$ of the Cauchy problem

$$\Box u = -A(u)(\partial_\alpha u, \partial^\alpha u) \perp T_u N, \tag{3.16}$$

$$u \mid_{t=0} = u_0,\ u_t \mid_{t=0} = u_1. \tag{3.17}$$

While we cannot yet solve this problem, we will discuss an approximation method to prove existence and we present some partial results on the relevant convergence problem.

3.4. Approximate solutions.

3.4.1. *Penalty method.* Suppose as in Section 1 that N is compact, isometrically embedded in $\mathbb{R}^d$, and there is a tubular neighborhood $U_{2\delta}(N)$ of width 2δ of N with smooth nearest neighbor projection $\pi_N : U_{2\delta}(N) \to N$. Also let $\chi \in C^\infty(\mathbb{R})$ be a function such that $\chi(s) = s$ for $s \le \delta^2$, $\chi(s) \equiv$ const. for $s \ge 2\delta^2$, $\chi' \ge 0$, $\chi'' \le 0$, and let $\mathrm{dist}(p, N)$ be the distance from a point $p \in \mathbb{R}^d$ to its nearest neighbor on N. Note that

$$p \mapsto \frac{1}{2}\chi\big(\mathrm{dist}^2(p, N)\big)$$

then extends to a smooth function on $\mathbb{R}^d$ whose gradient is given by

$$\chi'\big(\mathrm{dist}^2(p, N)\big)\big(p - \pi_N(p)\big),$$

if $p \in \mathcal{U}_{2\delta}(N)$, and 0 otherwise.

For $L \in \mathbb{N}$ consider solutions $u^L\colon \mathbb{R} \times M \to \mathbb{R}^d$ of the Cauchy problem

$$\Box\, u^L + L\chi'\big(\mathrm{dist}^2(u^L, N)\big)\big(u^L - \pi_N(u^L)\big) = 0 \qquad \text{on } \mathbb{R} \times M \tag{3.18}$$

with data $(u_0, u_1) \in H^1_c(M; TN)$ at $t = 0$.

Equation (3.18) implies the conservation law

$$\frac{d}{dt}\left(e(u^L) + \frac{L}{2}\chi\big(\mathrm{dist}^2(u^L, N)\big)\right) - \mathrm{div}\langle \nabla u^L, u^L_t\rangle = 0, \tag{3.19}$$

where

$$e(u^L) = \frac{1}{2}|Du^L|^2$$

is the energy density for the free wave equation. Let

$$e_L(u^L) = e(u^L) + \frac{L}{2}\chi\big(\mathrm{dist}^2(u^L, N)\big)$$

and let

$$E_L\big(u^L(t)\big) = \int_M e_L\big(u^L(t)\big)\, dx.$$

Upon integrating (3.19) over M, thus we find that

$$\frac{d}{dt}E_L\big(u^L(t)\big) = 0;$$

in particular,

$$E_L\big(u^L(t)\big) = E_L\big(u^L(0)\big) = E\big(u^L(0)\big) = \frac{1}{2}\int_M \big\{|u_1|^2 + |\nabla u_0|^2\big\}\, dx$$

for all L and all t. It follows that, as $L \to \infty$, a sub-sequence

$$Du^L \rightharpoondown Du \quad \text{weakly}-* \text{ in } L^\infty\big(\mathbb{R}; L^2(M)\big).$$

Moreover, since $u^L(0) = u_0$ for all L, and in view of the compactness of the restriction (trace) operator $H^{1,2}(\mathbb{R}\times M) \hookrightarrow L^2\big(\{t\}\times M\big)$ for any t,

$$u^L \to u \qquad \text{in } L^2(M), \text{ locally}$$

uniformly in time. Hence, by Fatou's lemma, also

$$\begin{aligned}\int_{\{t\}\times M} \chi\big(\mathrm{dist}^2(u,N)\big)\, dx &\le \liminf_{L\to\infty} \int_{\{t\}\times M} \chi\big(\mathrm{dist}^2(u^L,N)\big)\, dx \\ &\le \frac{1}{L}E_L\big(u^L(t)\big) \to 0 \quad (L\to\infty)\end{aligned}$$

for any t, and it follows that $u\colon \mathbb{R}\times M \to N$. Finally, since (3.18) has propagation speed ≤ 1, for data $(u_0,u_1) \in H^1_c$ also $Du^L(t)$ has uniformly compact support for any t and thus

$$(u,u_t) \in L^\infty\big(\mathbb{R}; H^1_c(M;TN)\big).$$

However, while (3.18) implies that

$$\Box u^L \perp T_{\pi_N(u^L)}N$$

at all points in space-time, it is not clear that this relation, and hence (3.16), persists in the limit $L \to \infty$. In special cases, the analysis is, in fact, quite simple.

3.4.2. *The sphere.* We slightly modify the approximation scheme if $N = S^k$.

For $L \in \mathbb{N}$, following Shatah [40], we consider solutions $u^L\colon \mathbb{R}\times M \to \mathbb{R}^d$, $d = k+1$, of the equation

$$\Box\, u^L + L\big(|u^L|^2 - 1\big)u^L = 0 \qquad \text{on } \mathbb{R}\times M \tag{3.20}$$

with Cauchy data $(u_0,u_1) \in H^1_c(M;TN)$ at $t=0$. The initial value problem for (3.20) admits global weak solutions u^L such that

$$\begin{aligned}\tilde{E}_L\big(u^L(t)\big) &= E\big(u^L(t)\big) + \frac{L}{4}\int_M \big||u^L|^2 - 1\big|^2\, dx = \tilde{E}_L\big(u^L(0)\big) \\ &= E\big(u^L(0)\big) = \frac{1}{2}\int_M \Big\{|u_1|^2 + |\nabla u_0|^2\Big\}\, dx,\end{aligned}$$

uniformly in t, for all L. A sub-sequence (u^L) hence converges to a limit u in the sense that, as $L \to \infty$,

$$\begin{aligned} u^L &\to u && \text{in } L^2(M) \text{ for all } t, \\ Du^L &\rightharpoondown Du && \text{weakly-}* \text{ in } L^\infty(\mathbb{R}; L^2(M)), \end{aligned}$$

and for any t, by Fatou's lemma.

$$\int_M \big(|u|^2 - 1\big)^2 \, dx \le \liminf_{L\to\infty} \int_M \big(|u^L|^2 - 1\big)\, dx = 0.$$

Hence $u\colon \mathbb{R} \times M \to S^k$.

In order to pass to the limit in (3.20), observe that the nonlinear term always points in the direction of u^L. Taking the exterior product with u^L, thus from (3.20) we obtain the equation

$$\partial^\alpha(\partial_\alpha u^L \wedge u^L) = \Box u^L \wedge u^L = 0 \tag{3.21}$$

for all L. In the limit $L \to \infty$, therefore also the equation

$$\partial^\alpha(\partial_\alpha u \wedge u) = \Box u \wedge u = 0 \tag{3.22}$$

is valid in the sense of distributions, which implies that u weakly solves (3.16).

Moreover, multiplying (3.21), (3.22) with a 2-vector $\varphi \in C_0^\infty(\mathbb{R} \times M)$ and integrating by parts on $[0, \infty[\times M$, we obtain

$$\begin{aligned} \int_0^\infty \int_M &\big((\partial_\alpha u^L \wedge u^L) - (\partial_\alpha u \wedge u)\big)\partial^\alpha \varphi \, dx\, dt \\ &= \int_{\{0\}\times M} (u_1 \wedge u_0 - u_t \wedge u_0)\varphi \, dx \end{aligned}$$

In the limit $L \to \infty$, the left hand side vanishes. Since $\varphi(0, \cdot)$ is arbitrary, thus we conclude that

$$\big(u_t(0) - u_1\big) \wedge u_0 = 0;$$

that is, $u_t(0) = u_1$ in the sense of traces. Here we used the fact that both u_1 and $u_t(0)$ are tangent to S^k along u_0, that is,

$$\langle u_1, u_0\rangle = \langle u_t(0), u_0\rangle = 0.$$

Therefore, as $t \to 0$, $u_t(t, \cdot) \rightharpoondown u_1$ weakly in L^2 as $t \to 0$. On the other hand

$$\begin{aligned} \limsup_{t\to 0} \frac{1}{2}\int_M |u_t(t,\cdot)|^2 \, dx + \frac{1}{2}\int_M |\nabla u_0|^2 \, dx \ &\le \limsup_{t\to 0} E\big(u(t)\big) \\ &\le \limsup_{t\to 0} \liminf_{L\to\infty} \tilde{E}_L\big(u^L(t)\big) \\ &\le E\big(u^L(0)\big) = \frac{1}{2}\int_M \big(|u_1|^2 + |\nabla u_0|^2\big)\, dx. \end{aligned}$$

It follows that

$$\limsup_{t\to 0} \|u_t(t,\cdot)\|_{L^2} \leqq \|u_1\|_{L^2}.$$

Together with the fact that $u_t(t,\cdot) \rightharpoonup u_1$ weakly in L^2, this implies strong convergence $u_t(t,\cdot) \rightharpoonup u_1$ in L^2 as $t \to 0$. That is, u attains the prescribed initial data continuously in $H^1_c(M;TN)$. Hence we have proved:

Theorem 3.6. *Suppose $N = S^k$, and let $(u_0, u_1) \in H^1_c(M;TN)$. Then there exists a global weak solution u of* (3.16), (3.17) *of class H^1.*

Remark that u need not be unique; see Section 2.

The above method can be generalized to the orthogonal group $N = SO(n)$, as was observed by Freire [14]. The approximating equation is

$$\Box\, u + L\nabla_u F(u) = 0, \tag{3.23}$$

where

$$F(u) = \int_M |u^t u - 1|^2 \, dx,$$

and where u^t denotes the transposed matrix u and $|\cdot|$ is the norm induced by the scalar product

$$(A, B) = \operatorname{trace}(A^t B).$$

Given Cauchy data $(u_0, u_1) \in H^1_c(M, TSO(n))$ at $t = 0$, for any L there exists a solution $u^L \colon \mathbb{R} \times M \to \mathbb{R}^{n\times n}$ of (3.23) with

$$u^L \,|_{t=0} = u_0, \quad u^L_t \,|_{t=0} = u_1.$$

Moreover, (u^L, u^L_t) is bounded in $L^\infty\big(\mathbb{R}; H^1_c(M; T\mathbb{R}^{n\times n})\big)$ with $Du^L(t)$ having uniformly compact support and, as $L \to \infty$, a sub-sequence

$$\begin{aligned} u^L &\to u && \text{in } L^2(M), \text{ locally uniformly in } t, \\ Du^L &\rightharpoonup Du && \text{weakly-}{*} \text{ in } L^\infty\big(\mathbb{R}; L^2(M)\big), \end{aligned}$$

where $u \colon \mathbb{R} \times M \to SO(n)$ and $(u, u_t) \in L^\infty\big(\mathbb{R}\,; H^1_c(M; TSO(n))\big)$.

Remark that

$$T_u SO(n) = u T_1 SO(n),$$

where $T_1 SO(n) = so(n)$ denotes the Lie algebra of $SO(n)$. Recall that $so(n)$ consists precisely of the anti-symmetric matrices $A, A^t = -A$. Moreover, the orthogonal complement of $T_u SO(n)$ with respect to $(\cdot,\cdot)$,

$$T^\perp_u SO(n) = u T^\perp_1 SO(n),$$

where $T^\perp_1 SO(n) = \big(so(n)\big)^\perp$, consists precisely of the symmetric matrices $B = B^t$.

Indeed, any matrix M may be split

$$M = \frac{1}{2}(M + M^t) + \frac{1}{2}(M - M^t)$$

into its symmetric and anti-symmetric part and we have

$$\begin{aligned} (A, B) &= \operatorname{trace}(A^t B) = -\operatorname{trace}(AB) \\ &= \operatorname{trace}(B^t A) = \operatorname{trace}(BA) = \operatorname{trace}(AB) = 0 \end{aligned}$$

whenever $A = -A^t$, $B = B^t$.

Similarly, since $F(u) = F(u^t)$, we have

$$u^t \nabla_u F(u) = \big(\nabla_u F(u)\big)^t u$$

for any u. Thus, from (3.23) we obtain

$$\partial^\alpha\big((u^L)^t \partial_\alpha u^L - \partial_\alpha (u^L)^t u^L\big) = (u^L)^t \,\Box\, u^L - \Box (u^L)^t u^L = 0$$

for every L. Passing to the limit $L \to \infty$, then we find

$$\partial^\alpha (u^t \partial_\alpha u - \partial_\alpha u^t u) = u^t \,\Box\, u - \Box u^t u = 0;$$

that is, $u^t \,\Box\, u \in \big(so(u)\big)^\perp$. Since $u \in SO(u)$, we have $uu^t = 1$ and thus, finally,

$$\Box u \in u\big(so(u)\big)^\perp = T_u^\perp SO(u),$$

as desired.

It is conceivable that Theorem 3.6 extends to any homogeneous space as target.

Observe that the condition $u_1 \in T_{u_0} N$ is crucial in showing that the initial data are attained continuously in L^2. If $u_1 \notin T_{u_0} N$ one can show that as $t \to 0$ we have convergence $u_t(t) \rightharpoonup d\pi_N(u_0) u_1$, the projection of u_1 to $T_{u_0} N$, weakly in L^2. However, due to the loss of the energy of the normal component, we cannot show strong convergence.

3.5. Convergence. For general targets the problem of convergence is more difficult. Let us first consider the stationary case.

3.5.1. *The stationary case.* Suppose, for simplicity, that M is a compact surface without boundary; for instance, $M = T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Moreover, let N be a compact k-dimensional manifold, without boundary, isometrically embedded into some Euclidean $\mathbb{R}^d$, and suppose, for simplicity, that TN admits a smooth orthonormal frame field $(\bar e_1, \dots, \bar e_k)$. That is, at each point $p \in N$ the collection $\big(\bar e_1(p), \dots, \bar e_k(p)\big)$ is an orthonormal basis for $T_p N$, smoothly varying with p.

By a construction due to Hélein [24] this latter hypothesis can be made without loss of generality in the context of harmonic maps. Indeed, if the original target N does not have a parallelizable tangent bundle we can embed N as a totally geodesic submanifold of a compact manifold $\tilde N$ that has this property by taking two copies of a tubular neighborhood of N in $\mathbb{R}^d$, endowed with the product metric of $N \times \mathbb{R}^{d-k}$, and gluing them together along their boundaries. The standard basis of $\mathbb{R}^d$ then yields the desired frame field for $T\tilde N$, at least near the range N of our maps. Moreover, since $N \subset \tilde N$ is totally geodesic, for any map $u\colon M \to N \subset \tilde N \subset \mathbb{R}^{\tilde d}$ the component orthogonal to $T_u N$ of the Laplacian Δu in $T_u \tilde N$ vanishes. In particular, a harmonic map $u\colon M \to N$ will be harmonic, regarded as a map $u\colon M \to \tilde N$. Henceforth, therefore we replace N by $\tilde N$ and assume $N = \tilde N$.

Consider a sequence (u^L) of maps $u^L \in H^{1,2}(M; N)$ such that $u^L \rightharpoonup u$ weakly in H^1 and suppose

$$\Delta u^L + f^L \perp T_{u^L} N, \tag{3.24}$$

where

$$f^L \to 0 \qquad \text{strongly in } H^{-1},$$

the dual of $H^{1,2}(M;\mathbb{R}^d)$.

Theorem 3.7. *Under the above assumptions, u is (weakly) harmonic.*

This result is due to Bethuel [1]. A drastically simplified proof was recently given by Freire-Müller-Struwe [15], which we present below.

From now on, moreover, it is conventient to use the language of differential forms. Thus, we let d,δ be the exterior differential and co-differential, respectively. For a 1-form $\varphi = \varphi_\alpha\, dx^\alpha$ we have $\delta\varphi = \frac{\partial}{\partial x^1}\varphi_1 + \frac{\partial}{\partial x^2}\varphi_2$, for a 2-form $b = \beta\, dx^1 \wedge dx^2$ we have $\delta b = -\frac{\partial}{\partial x^2}\beta\, dx^1 + \frac{\partial}{\partial x^1}\beta\, dx^2$. Moreover, we define the Hodge Laplacian on forms as $\Delta = d\delta + \delta d$, acting as the standard Laplacian on the coefficients of the forms. (We always assume $M = T^2 = \mathbb{R}^2/\mathbb{Z}^2$, so that we can use the standard 1-forms dx^1, dx^2 as basis.) Finally, we contract 1-forms $\varphi = \varphi_\alpha\, dx^\alpha, \psi = \psi_\alpha\, dx^\alpha$ using the metric on $M = T^2$ by letting $\varphi \cdot \psi = \varphi_1\psi_1 + \varphi_2\psi_2$.

Let $\bar{e}_1, \dots, \bar{e}_k$ denote the orthonormal frame for TN. Then for each L the collection $\bar{e}_1 \circ u^L, \dots, \bar{e}_k \circ u^L$ is a frame for the pulled back bundle $(u^L)^{-1}TN$; that is, at each $x \in M$, the collection $\bar{e}_1\big(u^L(x)\big), \dots, \bar{e}_k\big(u^L(x)\big)$ is an orthonormal base for $T_{u^L(x)}N$. Other such frames may be obtained by rotating this frame; that is, by letting

$$e_i^L(x) = R_{ij}^L(x)\bar{e}_j\big(u^L(x)\big), \qquad \text{where } R^L = (R_{ij}^L) \in SO(k).$$

We express du (respectively du^L) in terms of e_i (respectively e_i^L) as

$$du = \theta_i e_i, \quad \theta_i = \langle du, e_i\rangle.$$

Also denote the connection 1-form of a frame field (e_i) as

$$\omega_{ij} = \langle de_i, e_j\rangle.$$

Note that

$$\langle \Delta u, e_i\rangle = \delta\theta_i - \omega_{ij}\cdot\theta_j.$$

Hence u is harmonic if and only if

$$\delta\theta_i = \omega_{ij}\cdot\theta_j \qquad \text{for any } i \tag{3.25}$$

in the distribution sense.

Note that the frames $(e_i), (e_i^L)$ are only determined up to rotation, that is, up to a gauge transformation in the bundle of frames. In particular, by choosing $e_i = R_{ij}(\bar{e}_j \circ u)$ such that (e_i) minimizes

$$\Sigma_i \int_M |\nabla e_i|^2\, dx,$$

we obtain the following result of Hélein [24].

Lemma 3.8. *For any $u \in H^{1,2}(M;N)$ there exists a frame (e_i) for $u^{-1}TN$ such that the associated connection satisfies the Coulomb gauge condition*

$$\delta\omega_{ij} = 0, \quad 1 \le i,j \le k.$$

Moreover,

$$\Sigma_i \int_M |\nabla e_i|^2 \, dx \le \Sigma_i \int_M |\nabla(\bar{e}_i \circ u)|^2 \, dx \le CE(u).$$

In the following we assume that $(e_i), (e_i^L)$ are in Coulomb gauge. Hence, in particular, (e_i^L) is bounded in H^1 and, passing to a further sub-sequence, if necessary, we may assume that $e_i^L \rightharpoonup e_i$ weakly in H^1 and $\omega_{ij}^L \rightharpoonup \omega_{ij}$ weakly in L^2, as $L \to \infty$. By Hodge decomposition, for ω_{ij} (respectively, ω_{ij}^L) we have

$$\omega_{ij} = da_{ij} + \delta b_{ij} + H_{ij},$$

where a_{ij} and b_{ij} are normalized by the condition

$$\int_M a_{ij} \, dx = \int_M b_{ij} = 0,$$

and where H_{ij} is a harmonic 1-form (a constant linear combination of dx^1, dx^2 if $M = T^2$). By mutual orthogonality,

$$\|da_{ij}\|_{L^2}^2 + \|\delta b_{ij}\|_{L^2}^2 + \|H_{ij}\|_{L^2}^2 = \|\omega_{ij}\|_{L^2}^2.$$

In particular (b_{ij}^L) is bounded in H^1, (H_{ij}^L) is bounded in any smooth topology, and we may assume that $b_{ij}^L \rightharpoonup b_{ij}$ weakly in H^1 as $L \to \infty$, while $H_{ij}^L \to H_{ij}$ smoothly.

Moreover, the Coulomb gauge condition implies

$$\delta\omega_{ij} = \Delta a_{ij} = 0;$$

hence $a_{ij} = 0$, and similarly for a_{ij}^L.

Consider the term

$$\delta b_{ij}^L \cdot \theta_j^L - \delta b_{ij} \cdot \theta_j.$$

Let us fix, say, $i = 1$ and consider only the term involving $j = 2$ in this sum. For brevity we write θ, θ^L instead of θ_1, θ_1^L, etc. Let

$$b = \beta \, dx^1 \wedge dx^2, \quad b^L = \beta^L \, dx^1 \wedge dx^2.$$

Then

$$\delta b^L \cdot \theta^L = \langle \frac{\partial}{\partial x^1} \beta^L \frac{\partial}{\partial x^2} u^L - \frac{\partial}{\partial x^2} \beta^L \frac{\partial}{\partial x^1} u^L, e_2^L \rangle$$

has the structure of a Jacobian determinant. Due to this particular structure, a special weak compactness property holds. In fact, from [34], Lemma IV. 3 we have the following lemma.

Lemma 3.9.

$$\delta b^L \cdot \theta^L \rightharpoonup \delta b \cdot \theta + \Sigma_{j \in J} \nu_j \delta_{x_j}$$

weakly in the sense of distributions, where J is an at most countable set.

Since J is countable, the capacity of the set $X = \{x_j\}_{j\in J}$ vanishes and there exists a sequence of functions $\varphi_l \in C^\infty(M)$, $0 \le \varphi_l \le 1$, such that $\varphi_l \equiv 0$ in a neighborhood of X for each l and $\varphi_l \to 1$ in $H^{1,2}(M)$ as $l \to \infty$.

Hence for any $\varphi \in C^\infty$ we have

$$\int_M (\delta\theta_i - \omega_{ij}\cdot\theta_j)\varphi\,dx = \lim_{l\to\infty}\int_M (\delta\theta_i - \omega_{ij}\cdot\theta_j)\varphi\varphi_l\,dx$$

and for the proof of (3.25) it suffices to show that

$$\int_M (\delta\theta_i - \omega_{ij}\cdot\theta_j)\varphi\,dx = 0 \tag{3.26}$$

for any $\varphi \in C^\infty$ vanishing near X.

Now we use our assumption that

$$\delta\theta_i^L - \omega_{ij}^L\cdot\theta_j^L = \langle \Delta u^L, e_i^L\rangle = \langle f^L, e_i^L\rangle \rightharpoonup 0$$

as $L \to \infty$ in the sense of distributions.

Moreover, since $\theta_i^L \rightharpoonup \theta_i$ weakly in L^2, we also have weak convergence $\delta\theta_i^L \rightharpoonup \delta\theta_i$ in the distribution sense. Thus, with error terms $o(1) \to 0$ as $L \to \infty$, we have

$$\begin{aligned}\int_M (\delta\theta_i - \omega_{ij}\cdot\theta_j)\varphi\,dx &= \int_M (\delta\theta_i^L - \omega_{ij}\cdot\theta_j)\varphi\,dx + o(1)\\ &= \int_M (\omega_{ij}^L\cdot\theta_j^L - \omega_{ij}\cdot\theta_j)\varphi\,dx + o(1)\\ &= \int_M (\delta b_{ij}^L\cdot\theta_j^L - \delta b_{ij}\cdot\theta_j)\varphi\,dx + o(1)\end{aligned}$$

and the latter tends to 0 as $L \to \infty$ by Lemma 3.9, on account of the fact that φ is supported away from X. This proves (3.26) and the Theorem.

3.5.2. *The time-dependent case.* We consider the following model situation. Let (u^L) be a sequence of wave maps $u^L\colon \mathbb{R}\times M \to N$ with $E(u^L(t)) \le E(u^L(0)) \le C$, uniformly in t and L. We may assume that, as $L \to \infty$,

$$\begin{aligned} u^L &\to u && \text{in } L^2(M)\text{, locally uniformly in time}\\ Du^L &\rightharpoonup Du && \text{weakly-}*\text{ in } L^\infty\big(\mathbb{R}; L^2(M)\big).\end{aligned}$$

Then, by a result of Freire-Müller-Struwe [15] there holds

Theorem 3.10. *Under the above assumptions, the limit map $u\colon \mathbb{R}\times M \to N$ weakly solves the wave map equation* (3.16).

Below, we indicate the main steps in the proof of Theorem 3.10.

Again, as in the stationary case, we may assume that TN is parallelizable. Let $\bar e_1, \dots, \bar e_k$ be an orthonormal frame field for TN and for u, respectively u^L, consider corresponding rotated frames for the pull-back bundle $u^{-1}TN$, given by

$$e_i(z) = R_{ij}(z)\bar e_j\big(u(z)\big) \qquad \text{for } z = (t,x) \in \mathbb{R}\times M,$$

where

$$R = (R_{ij})\colon \mathbb{R}\times M \to SO(k).$$

Note that (3.16) is equivalent to the relation

$$\langle \Box u, e_i\rangle = \partial^\alpha \theta_{i,\alpha} - \omega_{ij}^{\alpha}\theta_{j,\alpha} = 0$$

in the distribution sense. (Recall that we raise indeces with the Minkowski metric.)

That is, for the proof of Theorem 3.10 we have to show that for any $\varphi \in C_0^\infty$ and any $\epsilon > 0$ there holds

$$|\int_{\mathbb{R}\times M} (\partial^\alpha \theta_{i,\alpha} - \omega_{ij}^{\alpha}\theta_{j,\alpha})\varphi\, dz| < \epsilon. \tag{3.27}$$

Fix such φ and $\epsilon > 0$. The energy inequality for (3.16) implies:

Lemma 3.11. *There is a sub-sequence* (u^L) *such that the* ϵ-concentration set *of* (u^L),

$$S_\epsilon = \Big\{ z_0 = (t_0, x_0); \forall R > 0\colon \limsup_{L\to\infty} \int_{B_R(x_0;\mathbb{R}^2)} |Du^L(t_0)|^2\, dx \ge \epsilon \Big\}$$

has vanishing $H^{1,2}$*-capacity; that is, there exists a sequence of cut-off functions* $\varphi_l \in H^{1,2}\cap L^\infty(\mathbb{R}\times M)$ *such that* $0 \le \varphi_l \le 1$, $\varphi_l \equiv 0$ *in a neighborhood of* S_ϵ *and* $\varphi_l \to 1$ *in* $H^{1,2}$ *as* $l \to \infty$.

Since

$$\int_{\mathbb{R}\times M} (\partial^\alpha \theta_{i,\alpha} - \omega_{ij}^{\alpha}\theta_{j,\alpha})\varphi_l\varphi\, dz \to \int_{\mathbb{R}\times M} (\partial^\alpha \theta_{i,\alpha} - \omega_{ij}^{\alpha}\theta_{j,\alpha})\varphi\, dz$$

as $l \to \infty$, it hence suffices to prove (3.27) for testing functions φ that vanish in a neighborhood of S_ϵ. Scaling suitably, we may assume that the support of φ is contained in a fundamental domain Q for $T^3 = \mathbb{R}^3/\mathbb{Z}^3$. Extending u^L, e_i^L, etc. suitably outside Q, we may also regard u^L, e_i^L, etc. as functions on T^3. (The modified functions u^L, of course, only satisfy (3.16) in Q.) On T^3 we impose the Coulomb gauge condition (with respect to the *Euclidean* background metric) by choosing $R^L\colon T^3 \to SO(k)$ such that

$$\Sigma_i \int_{T^3} |De_i^L|^2\, dz = \min_R \Sigma_i \int_{T^3} |D\big(R_{ij}(\bar e_j \circ u^L)\big)|^2\, dz.$$

In this gauge, we have

$$\partial_\alpha \omega_{ij,\alpha} = \delta_{eucl}\omega_{ij} = 0$$

and (e_i^L) is bounded in $H^{1,2}(T^3)$ with

$$\Sigma_i \int_Q |De_i^L|^2\, dz \le \Sigma_i \int_Q |D(\bar e_i \circ u^L)|^2\, dz \le CE\big(u^L(0)\big) \le C.$$

Hence we may assume that $e_i^L \rightharpoonup e_i$ weakly in $H^{1,2}_{\mathrm{loc}}(\mathbb{R}\times M)$ and

$$\theta_i^L = \langle du^L, e_i^L\rangle = \theta_{i,\alpha}^L\, dx^\alpha \rightharpoonup \theta_i = \langle du, e_i\rangle,$$
$$\omega_{ij}^L = \langle de_i^L, e_j^L\rangle = \omega_{ij,\alpha}^L\, dx^\alpha \rightharpoonup \omega_{ij} = \langle de_i, e_j\rangle$$

weakly in L^2 as $L \to \infty$.

Moreover, by a simple measure-theoretic argument, the set of concentration points z_0 of (e_i), satisfying

$$\limsup_{r\to 0} r^{-1} \int_{B_r(z_0)} \Sigma_i |De_i|^2 \, dz > 0,$$

has vanishing $H^{1,2}$-capacity and we also may assume that φ vanishes near such points.

Finally, since u^L solves (3.16) and since

$$\partial^\alpha \theta^L_{i,\alpha} \rightharpoonup \partial^\alpha \theta_{i,\alpha} \qquad (L \to \infty)$$

in the distribution sense, for the proof of Theorem 3.10 it suffices to show:

Lemma 3.12. *For $L \in \mathbb{N}$ sufficiently large there holds*

$$|\int_{\mathbb{R}\times M} (\omega^{L,\alpha}_{ij} \theta^L_{j,\alpha} - \omega^\alpha_{ij} \theta_{j,\alpha}) \varphi \, dz| \le \epsilon.$$

Proof. By weak convergence $\omega^L_{ij} \rightharpoonup \omega_{ij}$ in $L^2(Q)$,

$$\liminf_{L\to\infty} \int_{T^3} (\omega^{L,\alpha}_{ij} \theta^L_{j,\alpha} - \omega^\alpha_{ij} \theta_{j,\alpha}) \varphi \, dz = \liminf_{L\to\infty} \int_{T^3} \omega^{L,\alpha}_{ij} (\theta^L_{j,\alpha} - \theta_{j,\alpha}) \varphi \, dz$$

In the following, again we consider some fixed pair of indices i, j and we omit these indices for brevity.

Next, let

$$\omega^L = da^L + \delta_{eucl} b^L + H^L$$

be the Hodge decomposition of $\omega^L (= \omega^L_{12})$, normalized by the requirement that

$$\int_{T^3} a^L \, dz = \int_{T^3} b^L \wedge dx^\alpha = 0, \quad \alpha = 0, 1, 2.$$

By mutual orthogonality

$$\|\delta_{eucl} b^L\|^2_{L^2} + \|H^L\|^2_{L^2} \le \|\omega^L\|^2_{L^2} \le C.$$

It follows that $H^L \to H$ smoothly as $L \to \infty$.

Moreover, the Coulomb gauge condition implies $a^L = 0$. Finally,

$$\Delta b^L = d\omega^L = de^L_1 \wedge de^L_2$$

exhibits the crucial determinant structure. □

In the time-dependent setting we will need the following result on Jacobian determinants, due to Coifman-Lions-Meyer-Semmes [8].

Lemma 3.13. *If $\varphi, \psi \in H^{1,2}$ then $d\varphi \wedge d\psi$ belongs to the Hardy space $\mathcal{H}^1$ and $\|d\varphi \wedge d\psi\|_{\mathcal{H}^1} \le C \|d\varphi\|_{L^2} \|d\psi\|_{L^2}$.*

Decompose

$$\begin{aligned}(\theta^L - \theta)\varphi &= \langle d\big((u^L - u)\varphi\big), e^L\rangle + o(1)\\ &= \langle d(u^L - u)\varphi, e^L - e\rangle + d\langle (u^L - u)\varphi, e\rangle + o(1)\\ &= A_1^L + A_2^L + o(1),\end{aligned}$$

where $o(1) \to 0$ in L^2 as $L \to \infty$.

Denote by $a_i^L, i = 1, 2$, the solution to

$$\Delta a_i^L = \delta_{eucl}^* A_i^L,$$

where δ_{eucl}^* is the adjoint of δ_{eucl} with respect to the Minkowski metric. By using a result of Campanato and Giaquinta [16], we show that the functions $(a_{1,2}^L)$ are bounded in $\mathrm{BMO}(T^3)$. In fact, using precise estimates in Morrey spaces and our definition of S_ϵ, we can show that

Lemma 3.14.

$$\limsup_{L\to\infty} \|a_1^L\|_{BMO} \le C\epsilon,$$
$$\limsup_{L\to\infty} \|a_2^L\|_{BMO} \le C\sqrt{\epsilon}.$$

By $\mathcal{H}^1$-BMO duality (Fefferman-Stein [12]) then we have, with error $o(1) \to 0$ as $L \to \infty$,

$$\begin{aligned}\int_{T^3} \omega^L \cdot (\theta^L - \theta)\varphi\, dz &= \int_{T^3} \delta b^L \cdot (A_1^L + A_2^L) dz + o(1)\\ &= \int_{T^3} b^L \cdot \Delta(a_1^L + a_2^L) dz + o(1)\\ &= \int_{T^3} de_1^L \wedge de_2^L \cdot (a_1^L + a_2^L) dz + o(1)\\ &\le C\|de_1^L \wedge de_2^L\|_{\mathcal{H}^1}\big(\|a_1^L\|_{\mathrm{BMO}} + \|a_2^L\|_{\mathrm{BMO}}\big) + o(1)\\ &\le C\sqrt{\epsilon} + o(1),\end{aligned}$$

as desired.

Concluding remark: Observe that the proof of Theorem 3.10 might be adapted to show that the weak H^1-limit of approximate solutions u^L to (3.16) with range on N is a wave map. However, for the sequence (u^L) defined by the penalty method in Section 4.1 it is difficult to control the energy of the functions u^L in direction normal to N. For this reason we cannot (yet) use Theorem 3.10 to obtain existence of global weak solutions.

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