WAVE MAPS WITH AND WITHOUT SYMMETRIES

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Introduction

Many of the results on wave maps seem highly technical and require deep results from harmonic analysis for a complete understanding. In these three lectures we present direct approaches to certain global aspects of the wave map problem, with powerful conclusions.

Lecture I: The Cauchy problem for wave maps

In this first lecture we recall the approach presented in [20] for showing global existence and uniqueness for the Cauchy problem for wave maps from the $(1+m)$-dimensional Minkowski space, $m \geq 4$, to any complete Riemannian manifold with bounded curvature, provided the initial data are small in the critical norm.

1.1. **Wave maps.** Let (N, h) be a complete Riemannian manifold of dimension k with $\partial N = \emptyset$. We denote space-time coordinates on $\mathbb{R}^{m+1}$ as $(t, x) = (x^\alpha), 0 \leq \alpha \leq m$. A wave map $u \colon \mathbb{R}^{m+1} \to N$ is a solution to the equation

$$D^\alpha \partial_\alpha u = 0, \tag{1}$$

where $\partial_\alpha = \frac{\partial}{\partial x^\alpha}$ and where we raise and lower indices with the Minkowski metric $(\eta_{\alpha\beta}) = \mathrm{diag}(-1, 1, \ldots, 1)$. We tacitly sum over repeated indices. Moreover, D is the covariant pull-back derivative in the bundle u^*TN.

The equivalent extrinsic form of equation (1) reveals that this is a quasilinear wave equation. Recall that the Nash embedding theorem permits to regard N as a submanifold of some Euclidean $\mathbb{R}^n$. Letting $u = (u^1, \ldots, u^n) \colon \mathbb{R}^{m+1} \to N \hookrightarrow \mathbb{R}^n$ be the corresponding extrinsic representation of our wave map u, equation (1) then takes the form

$$\Box u^i = -\partial^\alpha \partial_\alpha u^i = u^i_{tt} - \Delta u^i = B^i_{jk}(u) \partial_\alpha u^j \partial^\alpha u^k, 1 \leq i \leq n, \tag{2}$$

where $B(p) \colon T_pN \times T_pN \to (T_pN)^\perp$ is the second fundamental form of $N \subset \mathbb{R}^n$ at any $p \in N$. This extrinsic form of the wave map equation (1) will be very useful in the sequel.

Note that equation (2) geometrically can be interpreted simply as saying that $\Box u \perp T_uN$, which immediately gives the intrinsic form (1). Moreover, in the case when $N = S^k \hookrightarrow \mathbb{R}^{k+1}$ equation (2) takes the form $\Box u = \lambda u$ for some scalar function λ. Taking account of the fact that $|u|^2 \equiv 1$, we compute

$$\lambda = \Box u \cdot u = -\partial^\alpha(\partial_\alpha u \cdot u) + \partial_\alpha u \partial^\alpha u = \partial_\alpha u \partial^\alpha u = |\nabla u|^2 - |u_t|^2$$

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and thus find the equation

$$\Box u = u_{tt} - \Delta u = (|\nabla u|^2 - |u_t|^2)u \tag{3}$$

for a wave map $u\colon \mathbb{R}^{m+1} \to S^k \hookrightarrow \mathbb{R}^{k+1}$.

We study the Cauchy problem for wave maps with initial data

$$(u, u_t)_{|t=0} = (u_0, u_1) \in \dot{H}^{\frac{m}{2}} \times \dot{H}^{\frac{m}{2}-1}(\mathbb{R}^m; TN), \tag{4}$$

where $\dot{H}^s$ for any s denotes the homogenous Sobolev space. Note that from any solution u to equation (1) or (2), we can obtain further solutions by scaling $u^R(t,x) = u(Rt, Rx)$. In view of the invariance

$$||(u, u_t)_{|t=0}||_{\dot{H}^{\frac{m}{2}} \times \dot{H}^{\frac{m}{2}-1}(\mathbb{R}^m;TN)} = ||(u^R, u^R_t)_{|t=0}||_{\dot{H}^{\frac{m}{2}} \times \dot{H}^{\frac{m}{2}-1}(\mathbb{R}^m;TN)} \tag{5}$$

the $\dot{H}^{\frac{m}{2}} \times \dot{H}^{\frac{m}{2}-1}$-regularity is critical.

With $L^{(2m,2)}(\mathbb{R}^m) \hookrightarrow L^{2m}(\mathbb{R}^m)$ denoting the Lorentz space, the main result from [20] may now be stated, as follows.

Theorem 1.1. *Suppose N is complete, without boundary and has bounded curvature in the sense that the curvature operator R and the second fundamental form B and all their derivatives are bounded, and let $m \geq 4$. Then there is a constant $\varepsilon_0 > 0$ such that for any $(u_0, u_1) \in H^{\frac{m}{2}} \times H^{\frac{m}{2}-1}(\mathbb{R}^m; TN)$ satisfying*

$$||u_0||_{\dot{H}^{\frac{m}{2}}} + ||u_1||_{\dot{H}^{\frac{m}{2}-1}} < \varepsilon_0$$

there exists a unique global solution $u \in C^0(\mathbb{R}; H^{\frac{m}{2}}) \cap C^1(\mathbb{R}; H^{\frac{m}{2}-1})$ of (1), (4) satisfying

$$\sup_t ||du(t)||_{\dot{H}^{\frac{m}{2}-1}} + \int_{\mathbb{R}} ||du(t)||^2_{L^{(2m,2)}(\mathbb{R}^m)}\, dt \leq C\varepsilon_0 \tag{6}$$

and preserving any higher regularity of the data.

For $N = S^k$, global wellposedness of the Cauchy problem (1), (4) for initial data having small energy in the critical norm was first shown by Tao [26], [27], initially only for $m \geq 5$ and finally for all $m \geq 2$. For $m \geq 5$, by a variant of Tao's method, Klainerman-Rodnianski [10] were able to extend his results to general targets, independently and almost simultaneously with our work [20] with Shatah. Similar results are due to Nahmod - Stefanov - Uhlenbeck [16]. In the low-dimensional cases $2 \leq m \leq 3$ for wave maps $u\colon \mathbb{R}^{m+1} \to H^2$ to hyperbolic space H^2, the analogue of Theorem 1.1 was obtained by Krieger [12], [13]. Finally, Tataru [30] established well-posedness of the Cauchy problem for (1), (4) for initial data of small critical energy in the low-dimensional cases $2 \leq m \leq 3$ also for general targets. Previous work of Tataru [28], [29] already had shown the problem to be wellposed for initial data of small energy in a critical Besov space.

Whereas the methods of Tao, Klainerman-Rodnianski, Tataru, and many others working on this problem strongly rely on Littlewood-Paley theory and a sophisticated analysis of the interaction between different frequency components of a solution, the approach in [20] requires no microlocalization. It proceeds in physical space and is very direct, using as a tool essentially only the Strichartz estimate and its recent subtle improvement by Keel and Tao [9].

Terence Tao, and independently also Sergiu Klainerman and Igor Rodnianski pointed out that estimates similar to the crucial $L^1_t L^\infty_x$-estimate in Lemma 1.2

below can also be obtained from bilinear estimates for the wave equation obtained by Klainerman-Tataru [11]. Tristan Rivière has brought to our attention further applications of Lorentz spaces in gauge theory related to our use of Lorentz spaces here.

1.2. **Uniqueness and higher regularity.** The condition (6) easily yields uniqueness when we consider the extrinsic form (2) of the wave map system. Indeed, let u and v be solutions to (2) of class $H^{\frac{m}{2}}$ with $u, v \in C^0(\mathbb{R}; H^{\frac{m}{2}}) \cap C^1(\mathbb{R}; H^{\frac{m}{2}-1})$, and suppose that

$$u_{|t=0} = v_{|t=0},\ u_{t|t=0} = v_{t|t=0}.$$

Moreover, we assume (6), that is, in particular,

$$||du||^2_{L^2_t L^{2m}_x} = \int_{\mathbb{R}} ||du(t)||^2_{L^{2m}(\mathbb{R}^m)}\, dt < \infty,$$

and similarly for v. Then $w = u - v$ satisfies

$$w_{tt} - \Delta w = [B(u) - B(v)](\partial_\alpha u, \partial^\alpha u) + B(v)(\partial_\alpha u + \partial_\alpha v, \partial^\alpha w).$$

Multiplying by w_t, we obtain

$$\frac{1}{2}\frac{d}{dt}||dw(t)||^2_{L^2} = I(t) + II(t),$$

where by Sobolev's embedding $\dot{H}^1(\mathbb{R}^m) \hookrightarrow L^{\frac{2m}{m-2}}(\mathbb{R}^m)$ we can estimate

$$\begin{aligned} I(t) &= \int_{\mathbb{R}^m} \langle [B(u) - B(v)](\partial_\alpha u, \partial^\alpha u), w_t\rangle\, dx \le C \int_{\mathbb{R}^m} |du|^2 |w||dw|\, dx \\ &\le C||du||^2_{L^{2m}} ||w||_{L^{\frac{2m}{m-2}}} ||dw||_{L^2} \le C||du||^2_{L^{2m}} ||dw||^2_{L^2}. \end{aligned}$$

In order to bound the term $II(t)$, we note that orthogonality $\langle B(u)(\cdot,\cdot), u_t\rangle = 0 = \langle B(v)(\cdot,\cdot), v_t\rangle$ implies

$$\begin{aligned} &|\langle B(v)(\partial_\alpha u, \partial^\alpha w), w_t\rangle| = |\langle B(v)(\partial_\alpha u, \partial^\alpha w), u_t\rangle| \\ &\qquad = |\langle [B(v) - B(u)](\partial_\alpha u, \partial^\alpha w), u_t\rangle| \le C|du|^2|w||dw|, \end{aligned}$$

and similarly for the term involving $\partial_\alpha v$.

Thus also this term can be bounded

$$II(t) \le C(||du||^2_{L^{2m}} + ||dv||^2_{L^{2m}})||dw||^2_{L^2},$$

yielding the inequality

$$\frac{d}{dt}||dw||^2_{L^2} \le C(||du||^2_{L^{2m}} + ||dv||^2_{L^{2m}})||dw||^2_{L^2}.$$

Hence we obtain the uniform estimate

$$||dw||^2_{L^\infty_t L^2_x} \le ||dw(0)||^2_{L^2} \cdot \exp(C(||du||^2_{L^2_t L^{2m}_x} + ||dv||^2_{L^2_t L^{2m}_x})).$$

Since $dw(0) = 0$, uniqueness follows.

Higher regularity estimates for (smooth) solutions u of (2) satisfying (6) for sufficiently small $\varepsilon > 0$ can be obtained in similar fashion by differentiating the intrinsic form of the wave map equation covariantly in spatial directions and using standard energy estimates; see [20] for details.

1.3. **Moving frames and Gauge condition.** Our approach requires the construction of a suitable frame for the pull-back bundle u^*TN, as pioneered by Christodoulou-Tahvildar-Zadeh [2] and Hélein [7]. With no loss of generality, we may assume that TN is parallelizable, that is, there exist smooth vector fields $\overline{e}_1, \dots, \overline{e}_k$ such that at each $p \in N$ the collection $\overline{e}_1(p), \dots, \overline{e}_k(p)$ is an orthonormal basis for T_pN; see [2], [7]. Given a (smooth) map $u\colon \mathbb{R}^{m+1} \to N$ then the vector fields $\overline{e}_a \circ u, 1 \le a \le k$, yield a smooth orthonormal frame for the pull-back bundle u^*TN. Moreover, we may freely rotate this frame at any point $z = (t,x) \in \mathbb{R}^{m+1}$ with a matrix $(R_a^b) = (R_a^b(z)) \in SO(k)$, thus obtaining the frame

$$e_a = R_a^b \overline{e}_b \circ u, 1 \le a \le k.$$

Expressing du as

$$du = q^a e_a \tag{7}$$

with an $\mathbb{R}^k$-valued 1-form $q = q_\alpha \, dx^\alpha$, then we have

$$|du|^2 = |q|^2 = \sum_{\alpha=0}^{m} |q_\alpha|^2 .$$

In particular, for $1 \le p \le \infty$ the L^p-norm of du is well-defined, independently of the choice of "gauge" (R_a^b), and coincides with the L^p-norm of du in the extrinsic representation of u as a map $u\colon \mathbb{R}^{m+1} \to N \subset \mathbb{R}^m$. Later we will see that if the gauge R is suitably chosen, and if $\varepsilon_0 > 0$ is sufficiently small, also the norms of the derivatives of du and the derivatives of q agree up to a multiplicative constant.

Letting $D = (D_\alpha)_{0 \le \alpha \le m}$ be the pull-back covariant derivative, we have

$$De_a = A_a^b e_b, 1 \le a \le k, \tag{8}$$

for some matrix-valued 1-form $A = A_\alpha dx^\alpha$. Fix a pair of space-time indices $0 \le \alpha, \beta \le m$. The curvature of D enters in the commutation relation

$$\begin{aligned} D_\alpha D_\beta e_a - D_\beta D_\alpha e_a &= D_\alpha (A_{a,\beta}^b e_b) - D_\beta (A_{a,\alpha}^b e_b) \\ &= (\partial_\alpha A_{a,\beta}^c - \partial_\beta A_{a,\alpha}^c + A_{b,\alpha}^c A_{a,\beta}^b - A_{b,\beta}^c A_{a,\alpha}^b) e_c = F_{a,\alpha\beta}^c e_c, \end{aligned}$$

or

$$\partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta] = F_{\alpha\beta} = R(\partial_\alpha u, \partial_\beta u) \tag{9}$$

for short. (The comma separates the form subscript from the vector subscript and does not indicate a differential.)

Following Hélein [7] we choose the columb gauge

$$\sum_{i=1}^{m} \partial_i A_i = 0. \tag{10}$$

This results in the equation

$$\Delta A_\beta + \partial_i [A_i, A_\beta] = \partial_i F_{i\beta} = \partial_i (R(\partial_i u, \partial_\beta u)), 0 \le \beta \le m, \tag{11}$$

where we tacitly sum over $1 \le i \le m$. Given $u\colon \mathbb{R}^{m+1} \to N$ with du having sufficiently small L^m-norm, this equation admits a unique solution A which for any fixed time we may represent as

$$A_\beta = G_i * ([A_i, A_\beta] - F_{i\beta}), \tag{12}$$

where

$$G(x) = \frac{c}{|x|^{m-2}}$$

is the fundamental solution to the Laplace operator on $\mathbb{R}^m$ and $G_i = -\partial_i G$.

Indeed, from (11) and elliptic regularity theory we have the a-priori estimate

$$\begin{aligned} ||A||_{L^m} &\leq C||A||_{\dot{W}^{1,\frac{m}{2}}} \leq C||[A,A]||_{L^{\frac{m}{2}}} + C||F||_{L^{\frac{m}{2}}} \\ &\leq C||A||^2_{L^m} + C||R||_{L^\infty}||du||^2_{L^m}; \end{aligned}$$

confer [5], Section 4.3. For sufficiently small $||A||_{L^m}$ we may absorb the first term on the right on the left hand side of this equation to obtain at any fixed time the estimate with constants C independent of t

$$||A||_{L^m} \leq C||A||_{\dot{W}^{1,\frac{m}{2}}} \leq C||du||^2_{L^m} \leq C||du||^2_{\dot{H}^{\frac{m}{2}-1}} \leq C\varepsilon_0. \tag{13}$$

For later use we derive further estimates for the connection 1-form A and the curvature F, assuming that $\varepsilon_0 > 0$ is sufficiently small. For the sake of exposition, we indicate these estimates only in the case when $m = 4$ and refer to [20] for the general case. For $1 \leq s \leq \infty$ again denote as $L^{(p,s)}(\mathbb{R}^m)$ the Lorentz space.

Lemma 1.2. *Let $m = 4$, and fix $r = 8/5$.*

(i) For any time t there holds

$$\|\nabla^2 A\|_{L^r} + \|\nabla\partial_0 A\|_{L^r} \leq C\|\nabla F\|_{L^r} \leq C\|du\|_{L^8}\|du\|_{\dot{H}^1}\,.$$

(ii) For any time t we have

$$\|A\|_{L^\infty} \leq C\|du\|^2_{L^{(8,2)}}\,.$$

Proof. (i) To estimate $\nabla^2 A$, observe that equation (11) implies

$$\|\nabla^2 A\|_{L^r} \leq C\|\nabla[A,A]\|_{L^r} + C\|\nabla F\|_{L^r}\,. \tag{14}$$

By Hölder's inequality and Sobolev's embedding we can estimate

$$\|\nabla[A,A]\|_{L^r} \leq 2\|\nabla A\|_{L^{r_1}}\|A\|_{L^m} \leq C\|\nabla^2 A\|_{L^r}\|A\|_{L^m}\,,$$

where

$$\frac{1}{r_1} = \frac{1}{r} - \frac{1}{m} = \frac{3}{8}.$$

From (13) and (14) then, for sufficiently small $\varepsilon_0 > 0$ we obtain

$$\|\nabla^2 A\|_{L^r} \leq C\|\nabla F\|_{L^r}.$$

The term ∇F only involves terms of the form $R(\nabla\partial_\alpha u, \partial_\beta u)$ and $\nabla R(\partial_\alpha u, \partial_\beta u)$ and therefore may be estimated

$$|\nabla F| \leq C(|\nabla du||du| + |du|^3).$$

Letting $q = 8 = 2m$, so that $1/r = 5/8 = 1/q + 1/2$, upon estimating

$$\|\nabla F\|_{L^r} \leq C(\|\nabla du\|_{L^2}\|du\|_{L^q} + \|du\|^2_{L^4}\|du\|_{L^q})\,,$$

from Sobolev's embedding $\|du\|_{L^4} \leq C\|du\|_{\dot{H}^1} \leq C$ we conclude that

$$\|\nabla^2 A\|_{L^r} \leq C\|\nabla F\|_{L^r} \leq C\|du\|_{L^8}\|du\|_{\dot{H}^1}.$$

To estimate $\nabla\partial_0 A$ we note that the equations

$$\partial_0 A_i = \partial_i A_0 + [A_i, A_0] + F_{0i}$$

and

$$\Delta\partial_0 A_0 + \partial_i\partial_0[A_i, A_0] = \partial_i\partial_0 F_{i0}\,,$$

from (11) make exchanging of time derivative by spatial derivative possible and thus imply the desired estimate.

(ii) By the Sobolev embedding into Lorentz spaces and i), we have

$$\|A\|_{L^{(8,2)}} \le C\|A\|_{L^{(8,\frac{8}{5})}} \le \|A\|_{\dot W^{2,\frac{8}{5}}} \le C\|du\|_{L^8}\,.$$

Therefore, and since for any $m \ge 4$ we have $G_i \in L^{(\frac{m}{m-1},\infty)}$, the dual of $L^{(m,1)}$, using the representation of A given by (12) we obtain

$$\|A\|_{L^\infty} \le C(\|[A,A]\|_{L^{(4,1)}} + \|F\|_{L^{(4,1)}}) \le C(\|A\|^2_{L^{(8,2)}} + \|du\|^2_{L^{(8,2)}}) \le C\|du\|^2_{L^{(8,2)}}\,,$$

as claimed. □

1.4. **Equivalence of Norms.** Estimate (13) implies the equivalence of the extrinsic H^ℓ-norm of du and the H^ℓ-norm of q for any ℓ, provided $\varepsilon_0 > 0$ is sufficiently small. To see this consider a vector field W in u^*TN whose coordinates in the frame $\{e_a\}$ are given by

$$W = Q^a e_a = Qe$$

with

$$\|W\|_{L^2} = \|Q\|_{L^2}\,.$$

The extrinsic partial derivative of W can be computed from the covariant derivative and the second fundamental form B as

$$D_k W = \partial_k W + B(u)(\partial_k u, W) = (\partial_k Q + AQ)e\,;$$

that is,

$$\partial_k W = (\partial_k Q + AQ)e - B(u)(\partial_k u, Qe).$$

Therefore from (12), Sobolev embedding, and boundedness of the second fundamental form B we obtain

$$\begin{aligned}\big|\, \|\partial W\|_{L^2} - \|\partial Q\|_{L^2}\big| &\le C\|AQ\|_{L^2} + \|duQ\|_{L^2}\\ &\le C(\|A\|_{L^m} + \|du\|_{L^m})\|\partial Q\|_{L^2} \le C\varepsilon_0\|\partial Q\|_{L^2}\,.\end{aligned}$$

By linearity of the map $Q \mapsto W$ and interpolation we conclude the equivalence of the H^s-norms of Q and W for all $0 \le s \le 1$. The same argument establishes the equivalence of the covariant and extrinsic H^s-norms of W for $0 \le s \le 1$. By applying this argument iteratively to $W = \nabla^\ell du$ for $\ell = 0, 1, \dots$, we then obtain the equivalence of the H^s-norm of du and H^s-norm of q for any $s \ge 0$, provided $\varepsilon_0 > 0$ is sufficiently small.

1.5. **A priori bounds.** In order to obtain the a-priori bounds from which we may derive existence, we represent a local smooth solution u of (1), (4) in terms of the 1-form q given by (7), where the frame (e_a) is in Coulomb gauge.

From (8) then we have the equations

$$0 = D_\alpha\partial_\beta u - D_\beta\partial_\alpha u = (D_\alpha q_\beta - D_\beta q_\alpha)e,$$

where we denote

$$D_\alpha q_\beta = (\partial_\alpha + A_\alpha)q_\beta; \tag{15}$$

in components, this is

$$D_\alpha(q^a_\beta e_a) = (\partial_\alpha q^c_\beta + A^c_{a,\alpha} q^a_\beta) e_c.$$

Again the comma separates the form subscript from the vector subscript and does not indicate a differential.

That is, we have

$$D_\alpha q_\beta - D_\beta q_\alpha = 0. \tag{16}$$

Moreover, the wave map equation (1) yields the equation

$$D^\alpha q_\alpha = 0. \tag{17}$$

Differentiating (17) with respect to x^β and using (9), (16), we derive the covariant wave equation

$$0 = D_\beta D^\alpha q_\alpha = D^\alpha D_\beta q_\alpha + F^\alpha_\beta q_\alpha = D^\alpha D_\alpha q_\beta + F^\alpha_\beta q_\alpha.$$

Expanding this identity using (15), we obtain

$$(\partial_t^2 - \Delta) q_\beta = 2A^\alpha \partial_\alpha q_\beta + (\partial^\alpha A_\alpha) q_\beta + A^\alpha A_\alpha q_\beta + F^\alpha_\beta q_\alpha =: h_\beta. \tag{18}$$

We can estimate q in terms of the initial data and h by using the Strichartz estimate for the linear wave equation

$$\Box v = h, v_{|t=0} = f, v_{t|t=0} = g. \tag{19}$$

Again denoting as $\dot{H}^\gamma = (\sqrt{-\Delta})^{-\gamma} L^2(\mathbb{R}^m)$ the homogeneous Sobolev space, and as $L^{(p,r)}(\mathbb{R}^m)$ the Lorentz space, from Keel-Tao [9], Corollary 1.3, if $h = 0$ for any $T > 0$ we have

$$\begin{aligned} ||v||_{L^2([0,T];L^{\frac{2(m-1)}{m-3}}(\mathbb{R}^m))} + ||v||_{C^0([0,T];\dot{H}^\gamma(\mathbb{R}^m))} + ||v_t||_{C^0([0,T];\dot{H}^{\gamma-1}(\mathbb{R}^m))} \\ \le C(||f||_{\dot{H}^\gamma(\mathbb{R}^m)} + ||g||_{\dot{H}^{\gamma-1}(\mathbb{R}^m)}). \end{aligned}$$

where $\gamma = \frac{m+1}{2(m-1)}$. If $m = 4$, we have $\gamma = \frac{5}{6}$ and the preceding becomes

$$\begin{aligned} ||v||_{L^2([0,T];L^6(\mathbb{R}^4))} + ||v||_{C^0([0,T];\dot{H}^{5/6}(\mathbb{R}^4))} + ||v_t||_{C^0([0,T];\dot{H}^{-1/6}(\mathbb{R}^4))} \\ \le C(||f||_{\dot{H}^{5/6}(\mathbb{R}^4)} + ||g||_{\dot{H}^{-1/6}(\mathbb{R}^4)}). \end{aligned} \tag{20}$$

By real interpolation between this estimate and the analogous estimate for derivatives of v, and using the embedding (in the notation of [9])

$$(L^2_t L^6_x, L^2_t \dot{W}^{1,6}_x)_{\frac{1}{6},2} \hookrightarrow L^2_t L^{(8,2)}_x,$$

we obtain

$$||v||_{L^2_t L^{(8,2)}_x} + ||dv||_{C^0([0,T];L^2)} \le C(||f||_{\dot{H}^1} + ||g||_{L^2}). \tag{21}$$

By Duhamel's principle, for general h it then follows that

$$||v||_{L^2_t L^{(8,2)}_x} + ||dv||_{C^0_t L^2_x} \le C(||f||_{\dot{H}^1} + ||g||_{L^2} + ||h||_{L^1_t L^2_x}). \tag{22}$$

(The crucial gain of the Lorentz exponent by real interpolation was already observed by Keel and Tao [9] but was omitted in the final statement of their theorem.)

We will apply estimate (22) to equation (18) on any time interval $[0, T]$ such that $||du||_{\dot{H}^1}$ remains sufficiently small, uniformly for $0 < t < T$. Also using the equivalence of the H^s-norms of du and q for $s \le 1$ on any such time interval, we obtain

$$\begin{aligned} ||du||_{C^0_t \dot H^1_x} + ||du||_{L^2_t L^{(8,2)}_x} &\le C(||dq||_{C^0_t L^2_x} + ||q||_{L^2_t L^{(8,2)}_x}) \\ &\le C(||dq(0)||_{L^2} + ||h||_{L^1_t L^2_x}) \le C(||du(0)||_{\dot H^1} + ||h||_{L^1_t L^2_x}) \\ &\le C(||u_0||_{\dot H^2} + ||u_1||_{\dot H^1} + ||h||_{L^1_t L^2_x}) \,. \end{aligned}$$

To estimate the various terms in h we observe that by Lemma 1.2 at any time t with $r_1 = 8/3$ we have

$$\begin{aligned} ||h||_{L^2} &\le 2\|A\partial q\|_{L^2} + \|\partial A q\|_{L^2} + \|A^2 q\|_{L^2} + \|Fq\|_{L^2} \\ &\le 2\|A\|_{L^\infty}\|q\|_{\dot H^1} + \big(\|\nabla A\|_{L^{r_1}} + \|A^2\|_{L^{r_1}} + \|F\|_{L^{r_1}}\big)\|q\|_{L^8} \,. \end{aligned}$$

But Lemma 1.2 with $r = 8/5$ implies

$$\begin{aligned} \|\nabla A\|_{L^{r_1}} + \|A^2\|_{L^{r_1}} + \|F\|_{L^{r_1}} &\le C(\|\nabla^2 A\|_{L^r} + \|\nabla(A^2)\|_{L^r} + \|\nabla F\|_{L^r}) \\ &\le C\|du\|_{L^8}\|du\|_{\dot H^1}. \end{aligned}$$

Here we also used Sobolev's embedding and (13) to bound

$$\|\nabla(A^2)\|_{L^r} \ \le C\|\nabla A\|_{L^{r_1}}\|A\|_{L^4} \le C\|\nabla^2 A\|_{L^r}.$$

From Lemma 1.2 we then obtain

$$||h||_{L^2} \le C\|q\|_{L^8}\|du\|_{L^8}\|du\|_{\dot H^1} + 2\|A\|_{L^\infty}\|q\|_{\dot H^1} \le C\|du\|^2_{L^{(8,2)}}\|du\|_{\dot H^1} \,.$$

Using these estimates, we can bound h by

$$||h||_{L^1_t L^2_x} \le C||du||^2_{L^2_t L^{(8,2)}_x}\|du\|_{L^\infty_t \dot H^1_x}$$

and we conclude that

$$||du||_{L^\infty_t \dot H^1} + ||du||_{L^2_t L^{(8,2)}_x} \le C(||u_0||_{\dot H^2} + ||u_1||_{\dot H^1} + ||du||^2_{L^2_t L^{(8,2)}_x}\|du\|_{L^\infty_t \dot H^1_x}) \,.$$

A global priori bound on $||du||_{L^\infty_t \dot H^1_x} + ||du||_{L^2_t L^8_x}$ thus follows, provided $||u_0||_{\dot H^2} + ||u_1||_{\dot H^1}$ is sufficiently small.

1.6. **Existence.** Recall that $C^\infty \times C^\infty(\mathbb{R}^m; TN)$ is dense in $H^{\frac{m}{2}} \times H^{\frac{m}{2}-1}(\mathbb{R}^m; TN)$. We can thus find smooth data $(u_0^{(k)}, u_1^{(k)}) \to (u_0, u_1)$ in $H^{\frac{m}{2}} \times H^{\frac{m}{2}-1}(\mathbb{R}^m; TN)$. The local solutions $u^{(k)}$ to the Cauchy problem for (1) with data $(u_0^{(k)}, u_1^{(k)})$ by our a-priori bounds and regularity results for sufficiently small energy

$$||u_0||^2_{\dot H^{\frac{m}{2}}} + ||u_1||^2_{\dot H^{\frac{m}{2}-1}} < \varepsilon_0$$

then may be extended as smooth solutions to (1), (4) for all time and will satisfy the uniform estimates

$$||du^{(k)}||_{C^0_t \dot H_x^{\frac{m}{2}-1}} + ||du^{(k)}||_{L^2_t L^{2m}_x} \le C(||u_0^{(k)}||_{\dot H^{\frac{m}{2}}} + ||u_1^{(k)}||_{\dot H^{\frac{m}{2}-1}}) < C\varepsilon_0$$

for sufficiently large k.

Hence as $k \to \infty$ a subsequence $u^{(k)} \rightharpoonup u$ weakly in $H^{\frac{m}{2}}_{loc}(\mathbb{R}^{m+1})$, where

$$||du||_{C^0_t \dot H^{\frac{m}{2}-1}} + ||du||_{L^2_t L^{2m}_x} \le C(||u_0||_{\dot H^{\frac{m}{2}}} + ||u_1||_{\dot H^{\frac{m}{2}-1}}) \,.$$

Since $\frac{m}{2} \ge 2$, by Rellich's theorem for a further subsequence $du^{(k)} \to du$ converges pointwise almost everywhere, and u solves (1), (4), as claimed.

Wave maps with symmetries I

The H^1-energy is the only known conserved quantity for the wave map system. The case when $m = 2$ therefore is particularly interesting, because in this dimension the H^1-energy is critical and one may hope to obtain also global results and a characterization of singularities. Indeed, this is possible in the case of symmetry.

In this second lecture, we study co-rotational wave maps from (1+2)-dimensional Minkowski space into a target surface of revolution. In the third lecture, finally, we investigate rotationally symmetric wave maps on $\mathbb{R}^{1+2}$.

2.1. **Corotational wave maps.** Let N be a surface of revolution with metric

$$ds^2 = d\rho^2 + g^2(\rho)d\theta^2,$$

where $\theta \in S^1$ and with $g \in C^\infty(\mathbb{R})$ satisfying $g(0) = 0$, $g'(0) = 1$. Moreover, we assume that g is odd and either

$$g(\rho) > 0 \text{ for all } \rho > 0 \tag{23}$$

with

$$\int_0^\infty |g(\rho)|\, d\rho = \infty, \tag{24}$$

or, if N is compact, that g has a first zero $\rho_1 > 0$ where $g'(\rho_1) = -1$, and that g is periodic with period $2\rho_1$. Note that in this second case assumption (24) is trivially satisfied. The case (23) corresponds to non-compact surfaces; condition (24) is a technical assumption needed to rule out that N contains a "sphere at infinity".

We regard (ρ, θ) as polar coordinates on N. Letting (r, ϕ) be the usual polar coordinates on $\mathbb{R}^2$, we then consider equivariant wave maps $u\colon \mathbb{R} \times \mathbb{R}^2 \to N$ given by

$$\rho = h(t, r), \theta = \phi.$$

The equation (2) for a wave map $u = (u^1, \ldots, u^n)\colon \mathbb{R}^{2+1} \to N \hookrightarrow \mathbb{R}^n$, that is

$$\Box u^i = B^i_{jk}(u)\partial_\alpha u^j \partial^\beta u^k,\ 1 \le i \le n, \tag{25}$$

in this co-rotational case simplifies to the nonlinear scalar equation

$$\Box h + \frac{f(h)}{r^2} = 0, \tag{26}$$

where

$$\Box h = h_{tt} - \Delta h = h_{tt} - \frac{1}{r}\big(rh_r\big)_r = h_{tt} - h_{rr} - \frac{h_r}{r}$$

and with $f(h) = g(h)g'(h)$. If $N = S^2$, for example, we have $g(h) = sin(h)$ and $f(h) = \frac{1}{2}sin(2h)$

In [21], Shatah and Tahvildar-Zadeh showed that the initial value problem for (25) with smooth equivariant data

$$(u, u_t)_{|t=0} = (u_0, u_1) \tag{27}$$

of finite energy admits a unique smooth solution for small time, which may be extended for all time if the target surface N is geodesically convex.

The latter condition is equivalent to the assumption $g'(\rho) \geq 0$ for all $\rho > 0$. This condition was later weakened by Grillakis [4] who showed that it suffices to assume

$$(g(\rho)\rho)' = g(\rho) + g'(\rho)\rho > 0 \text{ for } \rho > 0\,.$$

Note that this hypothesis, in particular, implies conditions (23) and (24).

In [23] we improve these results and show that conditions (23) and (24) already suffice for proving global well-posedness of the Cauchy problem for (26). In fact, we show that for general target surfaces N satisfying (24) the appearance of a singularity in (26) is related to the existence of a non-constant harmonic map $\overline{u}\colon S^2 \to N$, thereby confirming a long-standing conjecture about wave maps in this special, co-rotational case. But if N also satisfies (23), any co-rotational harmonic map $\overline{u}\colon S^2 \to N$ is constant, and global well-posedness follows.

On the other hand, when $N = S^2$ on the basis of numerical work of Bizon et al. [1] and Isenberg-Liebling [8] it had been conjectured that for suitable initial data equivariant wave maps $u\colon \mathbb{R} \times \mathbb{R}^2 \to S^2$ indeed may develop singularities in finite time. In a penetrating analysis, Krieger-Schlag-Tataru [14] and Rodnianski-Sterbenz [17] recently were able to confirm this conjecture also theoretically and give a rigorous proof of blow-up.

2.2. **Results.** By the results of Shatah-Tahvildar-Zadeh [21] singularities of co-rotational maps may be detected by measuring their energy

$$E(u(t), R) = \frac{1}{2}\int_{B_R(0)} |Du(t)|^2\, dx,$$

with $|Du|^2 = |u_t|^2 + |\nabla u|^2$. In terms of $h = h(t)$ we have

$$E(u(t), R) = \pi \int_0^R \big(|Dh|^2 + \frac{g^2(h)}{r^2}\big)\, rdr.$$

We also let

$$E(u(t)) = \lim_{R\to\infty} E(u(t), R).$$

By [21] there exists a number $\varepsilon_0 = \varepsilon_0(N) > 0$ such that the Cauchy problem for co-rotational wave maps for smooth data with energy $E(u(0)) < \varepsilon_0$ admits a global smooth solution; confer also [19], Theorem 8.1. By finite speed of propagation, similarly we obtain well-posedness of the Cauchy problem for time $t \leq R$, provided $E(u(0), R) < \varepsilon_0$.

Conversely, let $u\colon [0, t_0[\times\mathbb{R}^2 \to N$ be a smooth co-rotational wave map. Then $z_0 = (t_0, 0)$ is a (first) singularity and t_0 is the blow-up time of u if and only if there holds

$$\inf_{0\leq t<t_0} E(u(t), t_0 - t) \geq \varepsilon_0 > 0. \tag{28}$$

In fact, for any map u satisfying (28) the space-time gradient Du cannot be bounded near the origin $(0, 0)$. On the other hand, negating condition (28) we can find a time $t < t_0$ such that

$$E(u(t), R) < \varepsilon_0$$

for some $R > t_0 - t$ and the results quoted above will allow us to extend u smoothly as a solution to (25) on a neighborhood of $z_0 = (t_0, 0)$. Observe that, by symmetry, u can only blow up at the origin.

We can now state our main result.

Theorem 2.1. *Let u be a smooth co-rotational solution to* (25) *blowing up at time t_0. Then there exist sequences $R_i \downarrow 0, t_i \uparrow t_0 (i \to \infty)$ such that*

$$u_i(t,x) = u(t_i + R_i t, R_i x) \to u_\infty(t,x)$$

strongly in $H^1_{loc}(]-1,1[\times\mathbb{R}^2)$, where u_∞ is a non-constant, time-independent solution of (25) *giving rise to a non-constant, smooth co-rotational harmonic map $\overline{u}\colon S^2 \to N$.*

As a consequence, for target manifolds that do not admit non-constant co-rotational harmonic spheres we obtain global existence of smooth solutions to the Cauchy problem (25), (27) for smooth co-rotational data. In particular, we can improve Grillakis' result as follows.

Theorem 2.2. *Suppose N is a surface of revolution with metric $ds^2 = d\rho^2 + g^2(\rho)d\theta^2$ satisfying* (23) *and* (24). *Then for any smooth co-rotational data the Cauchy problem* (25), (27) *admits a unique global smooth solution.*

As we shall see in Lecture 3, similar results also hold true in the case of radially symmetric wave maps $u = u(t,r)$ from $\mathbb{R}^{1+2}$ to an arbitrary closed target manifold; confer [24], [25].

2.3. **Notation.** Let $u\colon [0,t_0[\times\mathbb{R}^2 \to N$ be a smooth co-rotational wave map blowing up at time t_0 and let $h = h(t,r)$ be the associated solution of (26).

For convenience we shift and reverse time and then scale our space-time coordinate $z = (t,x)$ so that in our new coordinates u is an equivariant solution to (25) on $]0,1] \times \mathbb{R}^2$ blowing up at the origin.

Letting

$$K^T = \{z = (t,x); 0 \le |x| \le t \le T\}$$

be the forward light cone with vertex at the origin, truncated at height T, with lateral boundary

$$M^T = \{(t,x) \in K^T; |x| = t\},$$

we also introduce the flux

$$\mathrm{Flux}(u,T) = \frac{1}{2}\int_{M^T} |D^{||}u|^2\, do = \pi \int_0^T \left(|h_t + h_r|^2 + \frac{g^2(h)}{r^2}\right)\Big|_{t=r} r\, dr.$$

Here, $|D^{||}u|^2$ denotes the energy of all derivatives in directions tangent to M^T.

2.4. **Basic estimates.** We recall the energy bounds and decay estimates for (25) from [21]; these can also be found in [19], Chapter 8.1. Since $B(u)(v,w) \perp T_uN$ from (25) we obtain the conservation law

$$0 = \Box u \cdot u_t = \frac{\partial}{\partial t} e - div\, m \tag{29}$$

for the densities

$$e = \frac{1}{2}(|\nabla u|^2 + |u_t|^2),\ m = \nabla u \cdot u_t$$

of energy and momentum. Observe that $|m| \le e$. Integrating (29) over a truncated cone $K^{T_0} \setminus K^T$ for $0 < T \le T_0 \le 1$ we then find the identity

$$\int_{\{T\}\times B_T(0)} e\,dx + \frac{1}{2}\int_{M^{T_0}\setminus M^T} |D^{||}u|^2\,do = \int_{\{T_0\}\times B_{T_0}(0)} e\,dx\,.$$

From this we deduce the *energy inequality*

$$E(u(t),R) \le E(u(t+\tau),R+|\tau|). \tag{30}$$

for any $t,\tau,R > 0$. (Of course, in the present case we only consider values such that $0 < t, t+\tau \le 1$.)

Moreover, we conclude that

$$\lim_{T\downarrow 0}\int_{\{T\}\times B_T(0)} e\,dx$$

exists and we have *decay of the flux*

$$\text{Flux}(u,T) \to 0 \text{ as } T \downarrow 0. \tag{31}$$

Condition (24) together with the energy inequality implies the *uniform bounds*

$$\sup_{r<R}|h(t,r)| \le C(E(u(t),R)) \text{ for any } R > 0 \tag{32}$$

for the function h associated with u, where $C(s) \to 0$ as $s \to 0$. Indeed, let

$$G(s)) = \int_0^s |g(\rho)|\,d\rho.$$

Since (24) implies that $G(s) \to \infty$ as $s \to \infty$ it then suffices to estimate

$$\begin{aligned} G(|h(t,R)|) &= \int_0^R (G(|h(t,r)|)_r\,dr \le \int_0^R |g(h(t,r))||h_r(t,r)|\,dr \\ &\le \frac{1}{2}\int_0^R \big(|h_r|^2 + \frac{g^2(h(t,r))}{r^2}\big) r dr \le CE(u(t),R)\,. \end{aligned}$$

Moreover we have *exterior energy decay:* For any $0 < \lambda \le 1$ as $t \to 0$ there holds

$$E(u(t),t) - E(u(t),\lambda t) \to 0. \tag{33}$$

An immediate consequence of (33) is the *decay of time derivatives:* Suppose that N satisfies (24). Then

$$\frac{1}{T}\int_{K^T} |u_t|^2\,dz \to 0 \text{ as } T \to 0. \tag{34}$$

These estimates seem particular to the rotationally symmetric setting. The (lengthy) proof of (33) and the derivation of (34) are given in the appendix.

Finally, as is also well-known, in view of the uniform energy bounds (30) above, we have uniform Hölder continuity away from $x = 0$.

Lemma 2.3. *For any $r_0 > 0$, any (t,r) and (s,q) with $2r_0 \le q \le s < t \le 1, 2r_0 \le r \le t$ there holds*

$$|h(t,r) - h(s,q)|^2 \le C(|r-q| + |t-s|) \tag{35}$$

with a constant C depending only on the energy $E(u(1),1)$ and r_0.

Proof. Given $r_0 > 0$, for any t and $r_0 \le r' < r \le t \le 1$ by Hölder's inequality and (30) we have

$$|h(t,r) - h(t,r')|^2 \le \left(\int_{r'}^{r} |h_r| \, dr'' \right)^2 \le \frac{r - r'}{r'} \cdot \int_{r'}^{r} |h_r|^2 \, r'' \, dr'' \le C \frac{r - r'}{r_0},$$

while for any $s < t$ and $r_0 \le r' \le s$ we find

$$|h(s,r') - h(t,r')|^2 \le \left(\int_{s}^{t} |h_t(t',r')| \, dt' \right)^2 \le \frac{t-s}{r_0} \int_s^t |h_t(t',r')|^2 \, r' \, dt'.$$

Combining these inequalities, for any (t,r) and (s,q) with $2r_0 \le q \le s < t \le 1, 2r_0 \le r \le t$ and any r' with $r_0 \le r' \le r_1 := \inf\{q,r\}$ we find

$$|h(t,r) - h(s,q)|^2 \le C \frac{r - r' + q - r'}{r_0} + 2 \frac{t-s}{r_0} \int_s^t |h_t(t',r')|^2 \, r' \, dt'.$$

Taking the average with respect to $r' \in [r_1 - \min\{r_0, |r-q| + |t-s|\}, r_1]$, we obtain the claim. □

2.5. **Proofs of Theorems 2.1 and 2.2.** Fix a number $\varepsilon_1 = \varepsilon_1(N) > 0$ to be determined below. For $0 < t \le 1$ then choose $R = R(t) > 0$ so that

$$\varepsilon_1 \le E(u(t), 6R(t)) \le 2\varepsilon_1. \tag{36}$$

Applying the energy inequality (30), for any $|\tau| \le 5R$ we have

$$E(u(t+\tau), R) \le E(u(t), 6R) \le 2\varepsilon_1 \tag{37}$$

and similarly

$$\varepsilon_1 \le E(u(t+\tau), 6R + |\tau|) \le E(u(t+\tau), 11R). \tag{38}$$

We will choose ε_1 so that $2\varepsilon_1 < \varepsilon_0$. Then, in particular, from (28) and (36) we deduce the inequality

$$6R(t) < t \tag{39}$$

for all t. In fact, we obtain a much stronger result.

Lemma 2.4. *$R(t)/t \to 0$ as $t \to 0$.*

Proof. Suppose by contradiction that for some sequence $t_i \downarrow 0$ $(i \to \infty)$ with associated radii $R_i = R(t_i)$ there holds $6R_i \ge \lambda t_i$ for some constant $\lambda > 0$. Then from (28) and (36) we deduce that

$$0 < \varepsilon_0 - 2\varepsilon_1 \le E(u(t_i), t_i) - E(u(t_i), 6R_i) \le E(u(t_i), t_i) - E(u(t_i), \lambda t_i),$$

contradicting (33) for large $i \in \mathbb{N}$. □

The following lemma is the main new technical ingredient in our work [23].

Consider the intervals $\Lambda_{R(t)}(t) =]t - R(t), t + R(t)[, 0 < t \le 1$. By Vitali's theorem we can find a countable subfamily of disjoint intervals $\Lambda_i = \Lambda_{R(t_i)}(t_i), i \in \mathbb{N}$, such that $]0,1] \subset \cup_{i=1}^{\infty} \Lambda_i^*$, where $\Lambda_i^* = \Lambda_{5R(t_i)}(t_i)$. Observe that (39) implies

$$\inf \Lambda_i^* = t_i - 5R(t_i) > R(t_i) =: R_i \tag{40}$$

for each i. For any $\tau > 0$ the interval $[\tau, 1]$ is covered by finitely many intervals Λ_i^* which, however, fail to cover $]0,1]$ completely in view of (40). Therefore, we may assume that $t_i \to 0$ as $i \to \infty$.

Lemma 2.5. *With the above notations there holds*

$$\liminf_{i\to\infty} \frac{1}{R_i} \int_{\Lambda_i} \int_{B_t(0)} |u_t|^2 \, dx \, dt = 0.$$

Proof. Negating the assertion, we can find a number $\delta > 0$ and an index $i_0 \in \mathbb{N}$ such that

$$\int_{\Lambda_i} \int_{B_t(0)} |u_t|^2 \, dx \, dt \geq \delta R_i \quad \text{for } i \geq i_0. \tag{41}$$

Given $0 < T < \inf \cup_{i<i_0} \Lambda_i^*$, let $I_0 = \{i \,;\, \inf \Lambda_i^* < T\} \subset \{i_0, i_0+1, \dots\}$. Observe that

$$]0, T[\subset \cup_{i\in I_0} \Lambda_i^* \,.$$

By (40) we have

$$R_i < \inf \Lambda_i^* = t_i - 5R_i < T$$

and therefore

$$t_i + R_i < T + 6R_i < 7T$$

for all $i \in I_0$. It follows that

$$\cup_{i\in I_0} \Lambda_i \subset]0, 7T]. \tag{42}$$

By choice of I_0, our assumption (41), and in view of (42) we now obtain that

$$\begin{aligned} \delta T &\leq \delta \sum_{i\in I_0} \operatorname{diam} \Lambda_i^* = 10\,\delta \sum_{i\in I_0} R_i \leq 10 \sum_{i\in I_0} \int_{\Lambda_i} \int_{B_t(0)} |u_t|^2 \, dx \, dt \\ &= 10 \int_{\cup_{i\in I_0}\Lambda_i} \int_{B_t(0)} |u_t|^2 \, dx \, dt \leq 10 \int_{K^{7T}} |u_t|^2 \, dz, \end{aligned} \tag{43}$$

where we also used the fact that the intervals Λ_i are disjoint. But for small $T > 0$ this contradicts (34), thus proving the lemma. □

Proof of Theorem 2.1. i) Letting

$$u_i(t,x) = u(t_i + R_i t, R_i x), i \in \mathbb{N},$$

from Lemma 2.5 for a suitable subsequence we obtain

$$\int_{-1}^{1} \int_{B_{r_i}(0)} |\partial_t u_i|^2 \, dx \, dt \to 0 \text{ as } i \to \infty, \tag{44}$$

where $r_i = t_i/R_i - 1 \to \infty$ as $i \to \infty$ on account of Lemma 2.4. Relabelling, we may assume that (44) holds true for the original sequence (u_i).

Moreover, the energy inequality (30) implies the uniform bound

$$E(u_i(t), r_i) \leq E(u(1), 1) =: E_0 \tag{45}$$

for all $i \in \mathbb{N}$ and $|t| \leq 1$.

Hence we may extract a further subsequence such that $u_i \rightharpoonup u_\infty$ weakly in H^1_{loc} and locally uniformly away from $x = 0$ on $[-1,1] \times \mathbb{R}^2$ as $i \to \infty$, and similarly for the associated functions h_i. Their limit h_∞ then is associated with u_∞ and is a time-independent solution of (26) away from $x = 0$. It follows that $u_\infty(t,x) = \overline{u}(x)$ is a time-independent solution of (25) on $]-1,1[\times(\mathbb{R}^2\setminus\{0\})$; that is, $\overline{u}\colon \mathbb{R}^2\setminus\{0\} \to N$ is a smooth, co-rotational harmonic map with finite energy

$$E(\overline{u}) = \int_{\mathbb{R}^2} |\nabla u|^2\,dx \le \liminf_{i\to\infty} \sup_{|t|\le 1} E(u_i(t), r_i) \le E_0.$$

By [18] then $\overline{u}$ extends to a smooth harmonic map $\overline{u}\colon \mathbb{R}^2 \to N$. Since $\mathbb{R}^2$ is conformal to $S^2 \setminus \{p_0\}$ by stereographic projection from any point $p_0 \in S^2$ and since the composition of a harmonic map with a conformal transformation again yields a harmonic map with the same energy, we may thus regard $\overline{u}$ as a harmonic map from $S^2 \setminus \{p_0\}$ to N. Finally, recalling that $E(\overline{u}) < \infty$ and again using [18], we see that the map $\overline{u}$ extends to a smooth equivariant harmonic map $\overline{u}\colon S^2 \to N$.

ii) To show that $\overline{u}$ is non-constant we now establish strong convergence

$$u_i \to u_\infty \text{ in } H^1_{loc}(]-1,1[\times\mathbb{R}^2)$$

as $i \to \infty$. Recalling (37), we have

$$E(u_i(t),1) \le 2\varepsilon_1, \quad E(u_\infty(t),1) \le 2\varepsilon_1$$

uniformly in i and $|t| \le 1$. Hence, from (32) for sufficiently small $\varepsilon_1 > 0$ the images of $B_1(0)$ under $u_i(t)$ or u_∞ are all contained in a fixed coordinate system around the center of symmetry $O \in N$. In addition, we can achieve that

$$\sup_{|t|,|x|\le 1} |B(u_i)||u_i - u_\infty| \le \frac{1}{4} \tag{46}$$

uniformly in $i \in \mathbb{N}$, provided $\varepsilon_1 > 0$ is chosen sufficiently small.

For any $\varphi \in C_0^\infty(]-1,1[\times\mathbb{R}^2)$ with $0 \le \varphi \le 1$ then, upon multiplying the equation (25) for u_i by $(u_i - u_\infty)\varphi$ and integrating by parts we obtain

$$\int_{\mathbb{R}^{1+2}} |D(u_i - u_\infty)|^2 \varphi\,dz \le \int_{\mathbb{R}^{1+2}} |B(u_i)||Du_i|^2|u_i - u_\infty|\varphi\,dz + I, \tag{47}$$

with error

$$\begin{aligned} |I| &\le C \int_{\mathbb{R}^{1+2}} (|\partial_t u_i|^2\varphi + |Du_i||u_i - u_\infty||D\varphi|)\,dz \\ &\quad + \sum_\alpha |\int_{\mathbb{R}^{1+2}} \partial_\alpha u_\infty \partial_\alpha (u_i - u_\infty)\varphi\,dz| \to 0 \text{ as } i \to \infty \end{aligned}$$

in view of (44) and since $u_i \to u_\infty$ strongly in L^2_{loc} by Rellich's theorem.

Now we estimate

$$|Du_i|^2 \le 2|D(u_i - u_\infty)|^2 + 2|Du_\infty|^2$$

and observe that

$$\int_{\mathbb{R}^{1+2}} |Du_\infty|^2 |u_i - u_\infty|\varphi\,dz \to 0$$

as $i \to \infty$ by bounded almost everywhere convergence $u_i \to u_\infty$ and Lebesgue's theorem on dominated convergence. Also recalling (46), we thus may absorb the first term on the right of (47) on the left to obtain that

$$\int_{\mathbb{R}^{1+2}} |D(u_i - u_\infty)|^2 \varphi\,dz \to 0$$

as $i \to \infty$. Since φ as above is arbitrary, this yields the desired convergence $u_i \to u_\infty$ in $H^1_{loc}(]-1,1[\times\mathbb{R}^2)$.

But, recalling (38), we also have the uniform lower bound

$$\varepsilon_1 \le E(u_i(t), 11)$$

for all $i \in \mathbb{N}$ and $|t| \le 1$ and we conclude that $u_\infty \not\equiv const$, as claimed. Therefore, also $\overline{u}\colon S^2 \to N$ is non-constant, and the proof of Theorem 2.1 is complete. □

Proof of Theorem 2.2. In view of Theorem 2.1 it suffices to show that any co-rotational harmonic map $\overline{u}\colon S^2 \to N$ with finite energy is constant. Let $\overline{u}$ be such a map, viewed as a map $\overline{u}\colon \mathbb{R}^2 \to N$. Also consider the associated distance function $\rho = \overline{h}(r)$, a time-independent solution of (26). The image $\overline{u}(S^2)$ being compact there exists $r_0 > 0$ such that

$$|\overline{h}(r_0)| = \max_{r>0} |\overline{h}(r)|.$$

Hence $\overline{h}_r(r_0) = 0$ and therefore $\overline{u}_r(x) = 0$ for any $x \in \partial B_{r_0}(0)$.

Since any harmonic map $\overline{u}\colon \mathbb{R}^2 \to N$ with finite energy is conformal, the vanishing of $\overline{u}_r$ implies that also $\overline{u}_\phi$ vanishes along $\partial B_{r_0}(0)$, and we conclude that $\overline{u} \equiv const$ on $\partial B_{r_0}(0)$. Equivariance of $\overline{u}$ then implies that $g(\overline{h}(r_0)) = 0$ and hence $\overline{h}(r_0) = 0$ on account of (23). But then $\overline{h} \equiv 0$ by choice of r_0, and $\overline{u} \equiv const \equiv O$, as desired. □

Wave maps with symmetries II

In this final lecture we show that the Cauchy problem for radially symmetric wave maps $u(t,x) = u(t,|x|)$ from the $(1+2)$-dimensional Minkowski space to an arbitrary smooth, compact Riemannian manifold without boundary is globally well-posed for arbitrary smooth, radially symmetric data.

3.1. **The result.** Again let N be a smooth, compact Riemmanian k-manifold without boundary, isometrically embedded in $\mathbb{R}^n$. Given smooth, radially symmetric data $(u_0,u_1) = (u_0(|x|), u_1(|x|))\colon \mathbb{R}^2 \to TN$, by a result of Christodoulou-Tahvildar-Zadeh [2] there is a unique smooth solution $u = (u^1,\dots,u^n) = u(t,|x|)$ for small time to the Cauchy problem for the equation

$$\Box u = u_{tt} - \Delta u = B(u)(\partial_\alpha u, \partial^\alpha u) \perp T_u N, \tag{48}$$

with initial data

$$(u, u_t)_{|t=0} = (u_0, u_1). \tag{49}$$

Here B again denotes the second fundamental form of N.

As shown by Christodoulou-Tahvildar-Zadeh [2], the solution may be extended globally, if the energy of u is small or if the range of u is contained in a convex part of the target N. Either condition, however, turns out to be unnecessary. In fact, by using the blow-up analysis from [23] that we presented in the second lecture, in [24], [25] we showed that the local solution may be extended globally for any target manifold.

Theorem 3.1. *Let $N \subset \mathbb{R}^n$ be a smooth, compact Riemannian manifold without boundary. Then for any radially symmetric data $(u_0, u_1) = (u_0(|x|), u_1(|x|)) \in C^\infty(\mathbb{R}^2; TN)$ there exists a unique, smooth solution $u = u(t, |x|)$ to the Cauchy problem* (48), (49)*, defined for all time.*

The regularity requirements on the data may be relaxed; we consider smooth data mainly for ease of exposition.

Summarizing the ideas of the proof, as in the co-rotational symmetric setting of [23] that we described in the second lecture, again we argue indirectly. Thus, we suppose that the local solution u to (48), (49) becomes singular in finite time. As before we then obtain a sequence of rescaled solutions u_l on the region $]-1,1[\times\mathbb{R}^2$ with energy bounds and such that $\partial_t u_l \to 0$ in $L^2_{loc}(]-1,1[\times\mathbb{R}^2)$. Finally, rephrasing the wave map equation intrinsically as described in the first lecture, and imposing the exponential gauge, we establish energy decay. But this contradicts the blow-up criterion of Christodoulou and Tahvildar-Zadeh [2] and completes the proof.

I would like to thank Jalal Shatah for suggesting the use of the exponential gauge.

3.2. **Basic estimates.** Let $u = u(t, |x|)\colon [0, t_0[\times\mathbb{R}^2 \to N \subset \mathbb{R}^n$ be a smooth radially symmetric wave map blowing up at time t_0. Necessarily, blow-up occurs at $x = 0$. As before, upon shifting and reversing time and then scaling our space-time coordinates suitably, we may assume that u is a smooth radial solution to (48) on $]0,1] \times \mathbb{R}^2$ blowing up at the origin. Again let

$$K^T = \{z = (t, x); 0 \le |x| \le t \le T\}$$

be the truncated forward light cone from the origin with lateral boundary

$$M^T = \{(t, x) \in K^T; |x| = t\}.$$

Denoting as

$$e = \frac{1}{2}|Du|^2 = \frac{1}{2}(|u_t|^2 + |u_r|^2), \quad f = \frac{1}{2}|D^{||}u|^2 = \frac{1}{2}|u_t + u_r|^2$$

the energy and flux density of u, and letting

$$E(u, R) = \int_{B_R(0)} e\, dx, \quad \text{Flux}(u, T) = \int_{M^T} f\, do$$

be the local energy and the flux through M^T, then from [2], [21] we have the following results just as in the co-rotational setting. The identity (29) again leads to the *energy inequality:* For any $t, \tau, R > 0$ there holds

$$E(u(t), R) \le E(u(t+\tau), R + |\tau|). \tag{50}$$

(Again, we only consider values such that $0 < t, t+\tau \le 1$. Together with [2] this yields the *blow-up criterion:* There exists $\varepsilon_0 = \varepsilon_0(N) > 0$ such that

$$E(u(t), t) \ge \varepsilon_0 \text{ for all } 0 < t \le 1\,. \tag{51}$$

Moreover, we have *flux decay:*

$$Flux(u, T) \to 0 \text{ as } T \to 0. \tag{52}$$

As shown in the Appendix, similar to (33) and (34) we also have *exterior energy decay* and *decay of time derivatives:* For any $0 < \lambda \le 1$ as $t \to 0$ there holds

$$E(u(t), t) - E(u(t), \lambda t) \to 0, \tag{53}$$

and

$$\frac{1}{T}\int_{K^T} |u_t|^2\, dz \to 0 \text{ as } T \to 0. \tag{54}$$

Moreover, as shown in Lemma 2.3, the function u is locally uniformly Hölder continuous on $]0,1] \times B_1(0)$ away from $x = 0$.

Fix a number $0 < \varepsilon_1 = \varepsilon_1(N) < \varepsilon_0/2$ as determined below. For $0 < t \le 1$ we again choose $R = R(t)$ so that

$$\varepsilon_1 \le E(u(t), 6R) \le 2\varepsilon_1. \tag{55}$$

Then from (50) for any $|\tau| \le 5R$ we have

$$E(u(t+\tau), R) \le E(u(t), 6R) \le 2\varepsilon_1 < \varepsilon_0 \tag{56}$$

and similarly

$$\varepsilon_1 \le E(u(t+\tau), 6R + |\tau|) \le E(u(t+\tau), 11R). \tag{57}$$

In particular, combining (51) and (55) we deduce the inequality

$$6R(t) \le t \tag{58}$$

for all t. In fact, from (51), (53), and (55) as in Lemma 2.4 we even obtain that

$$R(t)/t \to 0 \text{ as } t \downarrow 0. \tag{59}$$

As in Lemma 2.5 we consider the intervals $\Lambda_{R(t)}(t) =]t - R(t), t + R(t)[, 0 < t \le 1$. An application of Vitali's covering theorem and (54) then yields a sequence $t_l \to 0$ with corresponding radii $R_l = R(t_l)$ such that

$$\frac{1}{R_l}\int_{\Lambda_l}\left(\int_{B_t(0)} |u_t|^2\, dx\right) dt \to 0$$

as $l \to \infty$, where $\Lambda_l = \Lambda_{R_l}(t_l), l \in \mathbb{N}$. Rescale, letting

$$u_l(t,x) = u(t_l + R_l t, R_l x), l \in \mathbb{N}.$$

Observe that u_l solves (48) on $[-1,1] \times \mathbb{R}^2$ with

$$\int_{-1}^{1}\left(\int_{D_l(t)} |\partial_t u_l|^2\, dx\right) dt \to 0 \text{ as } l \to \infty, \tag{60}$$

where

$$D_l(t) = \{x; R_l|x| \le t_l + R_l t\}$$

exhausts $\mathbb{R}^2$ as $l \to \infty$ uniformly in $|t| \le 1$ on account of (59).

Moreover, from (50), (51), (56), and (57) we have the uniform energy estimates

$$\frac{1}{2}E(u_l(t), 1) \le \varepsilon_1 \le E(u_l(t), 11) \tag{61}$$

and

$$\varepsilon_0 \le \frac{1}{2}\int_{D_l(t)} |Du_l|^2\, dx = E(u(t_l + R_l t), t_l + R_l t) \le E(u(1), 1) =: E_0, \tag{62}$$

uniformly for $|t| \le 1$ and sufficiently large $l \in \mathbb{N}$. Hence, we may assume that $u_l \rightharpoonup u_\infty$ weakly in $H^1_{loc}(]-1,1[\times\mathbb{R}^2)$ and locally uniformly away from $x = 0$, where $u_\infty(t,x) = u_\infty(|x|)$ is a time-independent radial map $u_\infty\colon \mathbb{R}^2 \to N$ with finite energy $E(u_\infty) \le E_0$.

Lemma 3.2. *We have $u_\infty \equiv const$, and $Du_l \to 0$ in $L^2_{loc}(]-1,1[\times(\mathbb{R}^2 \setminus \{0\})$ as $l \to \infty$.*

Proof. We claim that u_∞ is smooth and harmonic. Indeed, fix any function $\varphi \in C_0^\infty(]-1,1[\times\mathbb{R}^2)$ vanishing near $x = 0$. Upon multiplying (48) by $(u_l - u_\infty)\varphi$ and integrating by parts, we then have

$$\int_{\mathbb{R}^{1+2}} |D(u_l - u_\infty)|^2 \varphi\, dz = \int_{\mathbb{R}^{1+2}} \langle B(u_l)(\partial_\alpha u_l, \partial^\alpha u_l), u_l - u_\infty\rangle \varphi\, dz + I,$$

where

$$\begin{aligned} |I| \le 2 \int_{\mathbb{R}^{1+2}} |\partial_t u_l|^2 \varphi\, dz + \int_{\mathbb{R}^{1+2}} |Du_l||u_l - u_\infty||D\varphi|\, dz \\ + \left| \int_{\mathbb{R}^{1+2}} Du_\infty \cdot D(u_l - u_\infty)\varphi\, dz \right| \to 0 \end{aligned}$$

as $l \to \infty$. Observing that $(u_l - u_\infty)\varphi \to 0$ uniformly, moreover, we have

$$\int_{\mathbb{R}^{1+2}} \langle B(u_l)(\partial_\alpha u_l, \partial^\alpha u_l), u_l - u_\infty\rangle \varphi\, dz \to 0$$

as $l \to \infty$, and $u_l \to u_\infty$ strongly in $H^1_{loc}(]-1,1[\times\mathbb{R}^2 \setminus \{0\})$. Thus, we may pass to the distribution limit in equation (48) for u_l and find that u_∞ is weakly harmonic on $\mathbb{R}^2 \setminus \{0\}$. Since u_∞ has finite energy, by results of [18] then u_∞ is smooth and extends to a smooth, radially symmetric harmonic map $u_\infty\colon \mathbb{R}^2 \to N$.

Next recall that a harmonic map $u_\infty\colon \mathbb{R}^2 \to N$ with finite energy is conformal; in particular, there holds $|\partial_r u_\infty| = \frac{1}{r}|\partial_\phi u_\infty| \equiv 0$, and u_∞ must be constant. □

Finally we note the following estimate similar to [2], Lemma 4.

Lemma 3.3. *For any $\psi = \psi(t) \in C_0^\infty(]-1,1[)$ there holds*

$$\int_{-1}^{1} \int_{B_1(0)} |\partial_t u_l|^2 \psi |\log|x||\, dx\, dt = \int_{-1}^{1} \int_{B_1(0)} e(u_l)\psi\, dx\, dt + o(1),$$

where $o(1) \to 0$ as $l \to \infty$.

Proof. In radial coordinates $r = |x|$, equation (48) for $u = u_l$ may be written in the form

$$u_{tt} - \frac{1}{r}\, \partial_r(ru_r) \perp T_u N. \tag{63}$$

Multiplying by $u_r \psi r^2 \log r$, we obtain

$$\begin{aligned} 0 = \frac{d}{dt}\left(\langle u_t, u_r\rangle \psi r^2 \log r\right) - \frac{d}{dr}\left(\frac{|u_t|^2 + |u_r|^2}{2} \psi r^2 \log r\right) \\ + |u_t|^2 \psi r \log r - \langle u_t, u_r\rangle \psi_t r^2 \log r + e(u) r \psi. \end{aligned}$$

Upon integrating this identity over the domain $0 < r < 1, |t| < 1$ and observing that the boundary terms vanish, we find

$$\int_{-1}^{1}\int_{0}^{1} |u_t|^2 \psi r \log r\, dr\, dt + \int_{-1}^{1}\int_{0}^{1} e(u) r\psi\, dr\, dt = \int_{-1}^{1}\int_{0}^{1} \langle u_t, u_r\rangle \psi_t r^2 \log r\, dr\, dt.$$

In view of (60), (62), and Hölder's inequality the last term may be estimated

$$\begin{aligned}\Big|\int_{-1}^{1}\int_{0}^{1} \langle u_t, u_r\rangle \psi_t r^2 \log r\, dr\, dt\Big|^2 &= \Big|\frac{1}{2\pi}\int_{-1}^{1}\int_{B_1(0)} \langle u_t, u_r\rangle \psi_t r \log r\, dx\, dt\Big|^2 \\ &\le C\int_{-1}^{1}\int_{B_1(0)} |u_t|^2\, dx\, dt \cdot \int_{-1}^{1}\int_{B_1(0)} |u_r|^2\, dx\, dt \to 0 \text{ as } l \to \infty,\end{aligned}$$

proving the claim. □

3.3. **Intrinsic setting.** Recalling the set-up from our first lecture, in terms of the pull-back covariant derivative D in u^*TN we may write equation (63) as

$$D_t u_t - \frac{1}{r} D_r(r u_r) = 0. \tag{64}$$

Again we may assume that TN is parallelizable and we let $\overline{e}_1, \dots, \overline{e}_k$ be a smooth orthonormal frame field such that at any point $p \in N$ the vectors $\overline{e}_1(p), \dots, \overline{e}_k(p)$ form an orthonormal basis for T_pN. From $(\overline{e}_i)_{1\le i\le k}$ we then obtain a frame $e_i = R_i^j(\overline{e}_j \circ u), 1 \le i \le k$, for the pull-back bundle, where $R = R(t,r) = (R_i^j)$ is a smooth map from $\mathbb{R}^{1+2}$ into $SO(k)$.

Denoting

$$De_i = A_i^j e_j$$

with a matrix-valued connection 1-form $A = A_0\, dt + A_1\, dr$, we compute the curvature F of D via the commutation relation (9), or, more concisely,

$$dA + \frac{1}{2}[A, A] = F.$$

Moreover, we now impose the "exponential gauge" condition $A_1 = 0$. This yields the relation

$$*dA = -\partial_r A_0 = F_{01}.$$

If we normalize $A_0(t, 1) = 0$ for all t, from this relation we obtain

$$A_0 = \int_r^1 F_{01}\, ds.$$

Observing that

$$|F_{01}| \le C|du|^2, \tag{65}$$

from (61) we then deduce the estimate

$$|A_0| \le a_0 := \int_r^1 |F_{01}|\, ds \le C\int_r^1 |du|^2\, ds \le C\varepsilon_1 r^{-1}.$$

Note that in the exponential gauge for any fixed time t the frame field $e = e(t,r)$ is obtained by parallel transport along the curve $\gamma(r) = u(t,r)$ from the frame $e(t,1)$ at $r = 1$.

Expressing du as

$$du = u_t\,dt + u_r\,dr = q^i e_i,$$

where $q = q_0\,dt + q_1\,dr$ is a vector-valued 1-form with coefficients $q = (q^i)_{1\le i\le k}$, and using the notation

$$D_\alpha \partial_\beta u = D_\alpha(q^i_\beta e_i) = (\partial_\alpha q^j_\beta + A^j_{i\alpha} q^i_\beta)e_j =: (D_\alpha q_\beta)^j e_j$$

from our first lecture, we then may write equation (64) in the form

$$D_t q_0 - \frac{1}{r}D_r(rq_1) = \partial_t q_0 + A_0 q_0 - \frac{1}{r}\partial_r(rq_1) = 0. \tag{66}$$

Moreover, we have the commutation relation $D_r q_0 = D_t q_1$; that is,

$$\partial_r q_0 = \partial_t q_1 + A_0 q_1. \tag{67}$$

Finally there holds

$$|q_0| = |u_t|, |q_1| = |u_r|. \tag{68}$$

3.4. **Proof of Theorem 3.1.** By using Lemma 3.3 we show that (60) for sufficiently small $\varepsilon_1 > 0$ leads to a contradiction with (61).

Fix a cut-off function $0 \le \varphi = \varphi(r) \le 1$ in $C^\infty_0([0,1[)$ such that $\varphi(r) = 1$ for $r \le 1/2$. Also fix $0 \le \psi = \psi(t) \le 1$ in $C^\infty_0(]-1,1[)$ such that $\psi(t) = 1$ for $|t| \le 1/2$. For $u = u_l$ with associated 1-forms q, let

$$Q = Q_l = \int_r^1 q_1 \varphi\, ds.$$

Note that by Hölder's inequality and (61) we can estimate

$$|Q|^2 \le \left(\int_r^1 |q|\,ds\right)^2 \le \int_r^1 s|q|^2\,ds \cdot \int_r^1 \frac{ds}{s} \le 4\varepsilon_1 \log(\frac{1}{r}). \tag{69}$$

We will also use the bound

$$\left(\int_0^r s|q|\varphi\,ds\right)^2 \le \int_0^r s|q|^2\,ds \cdot \int_0^r s\,ds \le 2\varepsilon_1 r^2 \tag{70}$$

resulting from (61). Similarly, we have

$$\left(\int_0^r s|q_0||\log s|^{1/2}\varphi\,ds\right)^2 \le \frac{r^2}{2}\int_0^1 s|q_0|^2|\log s|\,ds,$$

which in view of (61), (65), and Lemma 3.3 allows to estimate

$$\begin{aligned}
&\int_{-1}^1\int_0^1 \left(\int_0^r s|q_0||\log s|^{1/2}\varphi\,ds\right)|F_{01}|\psi\,dr\,dt \\
&\quad\le \int_{-1}^1 \left(\int_0^1 s|q_0|^2|\log s|\,ds\right)^{1/2}\left(\int_0^1 r|F_{01}|\,dr\right)\psi\,dt \\
&\quad\le C\varepsilon_1\left(\int_{-1}^1\int_0^1 s|q_0|^2|\log s|\psi\,ds\ dt\right)^{1/2} \le C\varepsilon_1^{3/2}.
\end{aligned} \tag{71}$$

Also note that Lemma 3.2 implies

$$\int_{-1}^1\int_0^1 r|\log r|^{1/2}|q|\psi\,dr\,dt \le C\left(\int_{-1}^1\int_{B_1(0)} r|\log r||Du|^2\psi\,dx\,dt\right)^{1/2} \to 0 \tag{72}$$

as $l \to \infty$.

Using the function $Q\varphi\psi r$ as a multiplier, from (67) then we obtain

$$\int_{-1}^{1}\int_{0}^{1}\partial_t q_0 Q\varphi\psi r\,dr\,dt = -\int_{-1}^{1}\int_{0}^{1} q_0\Big(\int_r^1 \partial_t q_1\varphi\,ds\Big)\varphi\psi r\,dr\,dt + I$$
$$= \int_{-1}^{1}\int_{0}^{1}|q_0|^2\varphi^2\psi r\,dr\,dt + \int_{-1}^{1}\int_{0}^{1} q_0\Big(\int_r^1 A_0 q_1\varphi\,ds\Big)\varphi\psi r\,dr\,dt + II,$$

where, in view of (69), and (72),

$$|I| = |\int_{-1}^{1}\int_{0}^{1} q_0 Q\varphi\psi_t r\,dr\,dt| \le C\int_{-1}^{1}\int_{0}^{1} r|q_0||\log r|^{1/2}|\psi_t|\,dr\,dt \to 0$$

as $l\to\infty$. Similarly,

$$|II| \le |I| + |\int_{-1}^{1}\int_{0}^{1} q_0\Big(\int_r^1 q_0\partial_r\varphi\,ds\Big)\varphi\psi r\,dr\,dt|$$
$$\le C\int_{-1}^{1}\int_{0}^{1} r|q_0||\log r|^{1/2}\psi\,dr\,dt \to 0.$$

On the other hand, noting that

$$\frac{1}{r}\partial_r(rq_1)rQ = \partial_r(rq_1Q) + r|q_1|^2\varphi,$$

we obtain

$$\int_{-1}^{1}\int_{0}^{1}\frac{1}{r}\partial_r(rq_1)rQ\varphi\psi\,dr\,dt = \int_{-1}^{1}\int_{0}^{1} r|q_1|^2\varphi^2\psi\,dr\,dt + III,$$

where, by (69) and (72),

$$|III| \le \int_{-1}^{1}\int_{0}^{1} r|q_1||Q||\varphi_r|\psi\,dr\,dt \le C\int_{-1}^{1}\int_{0}^{1} r|\log r|^{1/2}|q_1|\psi\,dr\,dt \to 0$$

as $l\to\infty$. Thus, from (66) we deduce the identity

$$\int_{-1}^{1}\int_{0}^{1} r(|q_1|^2-|q_0|^2)\varphi^2\psi\,dr\,dt + o(1)$$
$$= \int_{-1}^{1}\int_{0}^{1} q_0\Big(\int_r^1 A_0q_1\varphi\,ds\Big)\varphi\psi r\,dr\,dt + \int_{-1}^{1}\int_{0}^{1} A_0q_0\Big(\int_r^1 q_1\varphi\,ds\Big)\varphi\psi r\,dr\,dt,$$

where $o(1)\to 0$ as $l\to\infty$. Using (70), (65) and repeated integration by parts, we find

$$\int_{-1}^{1}\int_{0}^{1} q_0\Big(\int_r^1 A_0q_1\varphi\,ds\Big)\varphi\psi r\,dr\,dt = \int_{-1}^{1}\int_{0}^{1}\Big(\int_0^r q_0\varphi s\,ds\Big)A_0q_1\varphi\psi\,dr\,dt$$
$$\le C\varepsilon_1^{1/2}\int_{-1}^{1}\int_{0}^{1} ra_0|q_1|\varphi\psi\,dr\,dt = C\varepsilon_1^{1/2}\int_{-1}^{1}\int_{0}^{1} r|q_1|\varphi\Big(\int_r^1|F_{01}|\,ds\Big)\psi\,dr\,dt$$
$$= C\varepsilon_1^{1/2}\int_{-1}^{1}\int_{0}^{1}\Big(\int_0^r s|q_1|\varphi\,ds\Big)|F_{01}|\psi\,dr\,dt \le C\varepsilon_1\int_{-1}^{1}\int_{0}^{1} r|F_{01}|\psi\,dr\,dt$$
$$\le C\varepsilon_1\int_{-1}^{1}\int_{0}^{1} r|du|^2\psi\,dr\,dt \le C\varepsilon_1^2.$$

Similarly, we estimate, now using (69) and (71),

$$\int_{-1}^{1}\int_{0}^{1} A_0 q_0\Big(\int_r^1 q_1\varphi\, ds\Big)\varphi\psi r\, dr\, dt \le C\varepsilon_1^{1/2}\int_{-1}^{1}\int_0^1 a_0|q_0||\log r|^{1/2}\varphi\psi r\, dr\, dt$$
$$= C\varepsilon_1^{1/2}\int_{-1}^{1}\int_0^1 r|q_0||\log r|^{1/2}\varphi\Big(\int_r^1 |F_{01}|\, ds\Big)\psi\, dr\, dt$$
$$= C\varepsilon_1^{1/2}\int_{-1}^{1}\int_0^1 \Big(\int_0^r s|q_0||\log s|^{1/2}\varphi\, ds\Big)|F_{01}|\psi\, dr\, dt \le C\varepsilon_1^2.$$

But then from (61), Lemma 3.2, and (60), with error $o(1)\to 0$ as $l\to\infty$ we obtain

$$\varepsilon_1 \le \frac{1}{2}\int_{-1}^{1}\int_{B_{11}(0)} |Du|^2\psi\, dx\, dt \le \pi\int_{-1}^{1}\int_0^1 r|q|^2\varphi^2\psi\, dr\, dt + o(1)$$
$$\le \pi\int_{-1}^{1}\int_0^1 r(|q_1|^2-|q_0|^2)\varphi^2\psi\, dr\, dt + o(1) \le C\varepsilon_1^2 + o(1),$$

which is impossible for sufficiently small $\varepsilon_1 > 0$ and large l. The proof of Theorem 3.1 is complete.

Appendix A: Exterior energy decay

In this Appendix we recall the proof of the following lemma which is fundamental for the treatment of the equivariant and rotationally symmetric case.

Lemma 4.1. *Let u be a radially symmetric solution of* (48) *or a co-rotational wave map on $K = K^1$ which is smooth away from the origin. Then for any $0 < \lambda \le 1$ as $t \to 0$ there holds*

$$E(u(t),t) - E(u(t),\lambda t) \to 0.$$

Proof. We follow the presentation in [19]. Therefore in the following we change time t to $-t$.

With the notation

$$e = \frac{1}{2}(|u_r|^2+|u_t|^2),\ m = u_r\cdot u_t,\ l = \frac{1}{2}(|u_r|^2-|u_t|^2) \tag{73}$$

for a radially symmetric solution u of (48) we compute

$$\frac{\partial}{\partial t}(rm) - \frac{\partial}{\partial r}(re) = ru_r\cdot\big(u_{tt} - \frac{1}{r}(ru_r)_r\big) + l = l, \tag{74}$$

thereby observing the geometric interpretation (63) of (48) and the fact that $u_r \in T_uN$. Moreover, recalling the equation (29) we have

$$\frac{\partial}{\partial t}(re) - \frac{\partial}{\partial r}(rm) = 0. \tag{75}$$

Similarly, for a co-rotational wave map u with associated function h solving (26) we let

$$\begin{aligned} e &= \frac{1}{2}(|u_r|^2+|u_t|^2) = \frac{1}{2}\big(|h_r|^2+|h_t|^2+\frac{g^2(h)}{r^2}\big),\ m = h_r\cdot h_t,\\ L &= \frac{1}{2}\big(|h_r|^2+\frac{g^2(h)}{r^2}-|h_t|^2\big) - \frac{2}{r}f(h)h_r \end{aligned} \tag{76}$$

and we compute

$$\partial_t(re) - \partial_r(rm) = 0,\ \partial_t(rm) - \partial_r(re) = L. \tag{77}$$

Changing coordinates to

$$\eta = t + r\,, \qquad \xi = t - r\,, \tag{78}$$

and introducing

$$\mathcal{A}^2 = r(e+m),\ \mathcal{B}^2 = r(e-m)\,,$$

identities (74), (75) turn into

$$\partial_\xi \mathcal{A}^2 = l\,,$$
$$\partial_\eta \mathcal{B}^2 = -l\,,$$

where

$$r^2 l^2 \le \mathcal{A}^2\mathcal{B}^2\,.$$

Likewise, (77) can be written as

$$\partial_\xi \mathcal{A}^2 = L\,,$$
$$\partial_\eta \mathcal{B}^2 = -L\,,$$

where now, with $F = g^2/2$, and using the fact that $|h| \le C(E_0)$ by (32) to bound $f^2(h) \le CF(h)$,

$$\begin{aligned} L^2 &\le \frac{3}{4}\left(h_t^2 - h_r^2\right)^2 + \frac{12}{r^2}h_r^2 f^2(h) + \frac{3}{r^4}F^2(h) \\ &\le C\left[\frac{1}{4}(h_t^2 - h_r^2)^2 + \frac{1}{r^2}(h_t^2 + h_r^2)F(h) + \frac{1}{r^4}F^2(h)\right] \\ &= \frac{C}{r^2}\mathcal{A}^2\mathcal{B}^2\,. \end{aligned}$$

Thus in both cases we get the inequalities

$$|\partial_\xi \mathcal{A}| \le \frac{C}{r}\mathcal{B}\,, \quad |\partial_\eta \mathcal{B}| \le \frac{C}{r}\mathcal{A}\,. \tag{79}$$

Upon integrating (79) on a rectangle $\Gamma = [\eta, 0] \times [\xi_0, \xi]$, as shown in Figure 1, we obtain

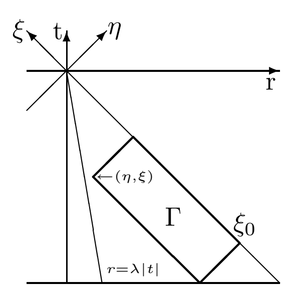

FIGURE 1. Domain of integration Γ.

$$\mathcal{A}(\eta,\xi) \le \mathcal{A}(\eta,\xi_0) + C\int_{\xi_0}^{\xi} \frac{\mathcal{B}(0,\xi')}{\eta - \xi'}\,d\xi' + C^2\int_{\xi_0}^{\xi}\int_{\eta}^{0} \frac{\mathcal{A}(\eta',\xi')}{(\eta-\xi')(\eta'-\xi')}\,d\eta'\,d\xi'\,.$$

First we estimate the second term on the right.

$$\begin{aligned}\int_{\xi_0}^{\xi} \frac{\mathcal{B}(0,\xi')}{\eta-\xi'}\,d\xi' &\le \left(\int_{\xi_0}^{\xi} \mathcal{B}^2(0,\xi')\,d\xi'\right)^{1/2} \left(\int_{\xi_0}^{\xi} \frac{d\xi'}{(\eta-\xi')^2}\right)^{1/2} \\ &= (\mathrm{Flux}(\xi_0)-\mathrm{Flux}(\xi))^{1/2}\sqrt{\frac{1}{\eta-\xi}-\frac{1}{\eta-\xi_0}} \\ &\le C\sqrt{\frac{\mathrm{Flux}(\xi_0)}{|\eta-\xi|}}\,.\end{aligned}$$

Letting

$$a(\eta,\xi) = \sup_{\eta\le\eta'\le 0} \sqrt{\eta'-\xi}\,\mathcal{A}(\eta',\xi)\,, \tag{80}$$

the third term may be bounded

$$\begin{aligned}\int_{\xi_0}^{\xi}\int_{\eta}^{0} \frac{\mathcal{A}(\eta',\xi')}{(\eta-\xi')(\eta'-\xi')}\,d\eta'\,d\xi' &\le \int_{\xi_0}^{\xi}\int_{\eta}^{0} \frac{a(\eta,\xi')}{(\eta-\xi')(\eta'-\xi')^{3/2}}\,d\eta'\,d\xi' \\ \le \int_{\xi_0}^{\xi} \frac{a(\eta,\xi')}{\eta-\xi'}\left(\frac{1}{\sqrt{\eta-\xi'}}-\frac{1}{\sqrt{-\xi'}}\right)d\xi' &\le \int_{\xi_0}^{\xi} a(\eta,\xi')\frac{\eta}{\xi'(\eta-\xi')^{3/2}}\,d\xi'\,.\end{aligned} \tag{81}$$

Also observing that

$$\sup_{\eta\le\eta'\le 0}\sqrt{\eta'-\xi}\,\mathcal{A}(\eta',\xi_0) \le \sup_{\eta\le\eta'\le 0}\frac{\sqrt{\eta'-\xi}}{\sqrt{\eta'-\xi_0}}a(\eta,\xi_0) = \frac{\sqrt{-\xi}}{\sqrt{-\xi_0}}\,a(\eta,\xi_0) \tag{82}$$

with constants C_1, C_2 we then obtain

$$a(\eta,\xi) \le \frac{\sqrt{-\xi}}{\sqrt{-\xi_0}}\,a(\eta,\xi_0) + C_1\sqrt{\mathrm{Flux}(\xi_0)} + C_2\int_{\xi_0}^{\xi} a(\eta,\xi')\frac{\eta}{\xi'(\eta-\xi')}\,d\xi'\,.$$

Setting

$$\rho(\xi') = \frac{\eta}{\xi'(\eta-\xi')}, \tag{83}$$

and letting

$$F(\xi) = \int_{\xi_0}^{\xi} a(\eta,\xi')\rho(\xi')\,d\xi',\ G(\xi) = \frac{\sqrt{-\xi}}{\sqrt{-\xi_0}}\,a(\eta,\xi_0) + C_1\sqrt{\mathrm{Flux}(\xi_0)}\,, \tag{84}$$

for any fixed η we then find the differential inequality

$$F' \le G\rho + C_2F\rho \text{ in } [\xi_0,\lambda'\eta]\,, \tag{85}$$

where $\lambda' = (1+\lambda)/(1-\lambda) > 1$. Applying Gronwall's lemma we obtain

$$F(\xi) \le \int_{\xi_0}^{\xi} G(\xi')\rho(\xi')e^{C_2\int_{\xi'}^{\xi}\rho(\xi'')d\xi''}\,d\xi'\,. \tag{86}$$

But for $\xi_0 \le \xi' \le \xi = \lambda'\eta$ we have

$$\int_{\xi'}^{\xi}\rho(\xi'')d\xi'' = \int_{\xi'}^{\xi}\frac{\eta}{\xi''(\eta-\xi'')}\,d\xi'' = \log\frac{\xi(\eta-\xi')}{\xi'(\eta-\xi)} = \log\frac{\xi(\xi-\lambda'\xi')}{\xi'(\xi-\lambda'\xi)} \le \log\frac{\lambda'}{\lambda'-1}\,.$$

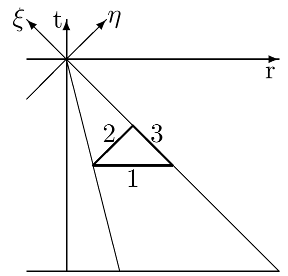

FIGURE 2. Triangular region Δ.

Hence we can estimate

$$
\begin{aligned}
a(\eta,\xi) &\le G + C_2 F \\
&\le \frac{\sqrt{-\xi}}{\sqrt{-\xi_0}}\, a(\eta,\xi_0) + C_1\sqrt{\mathrm{Flux}(\xi_0)} \\
&\quad + C_3 \int_{\xi_0}^{\xi} \left(\frac{\sqrt{-\xi'}}{\sqrt{-\xi_0}}\, a(\eta,\xi_0) + C_1\sqrt{\mathrm{Flux}(\xi_0)} \right) \frac{\eta}{\xi'(\eta-\xi')}\, d\xi' , \qquad (87)
\end{aligned}
$$

where $C_3 = e^{C_2 \log \frac{\lambda'}{\lambda'-1}}$. We also know that

$$a(\eta,\xi_0) \le \sup_{\eta\le\eta'\le 0} \sqrt{\eta'-\xi_0} \sup_{\eta\le\eta'\le 0} \mathcal{A}(\eta',\xi_0) \le C(\xi_0)\sqrt{-\xi_0}\,,$$

because u is assumed to be regular away from the origin, implying that $\mathcal{A}$ is bounded by a constant depending on ξ_0. Now, given $\epsilon > 0$, we can fix $\xi_0 < 0$ small enough such that $C\sqrt{\mathrm{Flux}(\xi_0)} < \epsilon$. Then,

$$
\begin{aligned}
a(\xi/\lambda',\xi) &\le C(\xi_0)\sqrt{-\xi} + \epsilon + C(\xi_0) \int_{\xi_0}^{\xi} \frac{\xi/\lambda'}{\sqrt{-\xi'}(\xi/\lambda'-\xi')}\, d\xi' + C\epsilon \\
&\le C(\xi_0)\sqrt{-\xi} + C\epsilon \le C\epsilon
\end{aligned}
$$

for $\xi < 0$ small enough. Therefore,

$$\mathcal{A}(\eta,\xi) \le \frac{a(\xi/\lambda',\xi)}{\sqrt{\eta-\xi}} \le \frac{C\epsilon}{\sqrt{\eta-\xi}}$$

for (η,ξ) small enough inside $K^{\lambda}_{\mathrm{ext}}$. Hence,

$$\int_{\eta}^{0} \mathcal{A}^2(\eta',\xi)d\eta' \le C\epsilon^2 \int_{\xi/\lambda'}^{0} \frac{d\eta'}{\eta'-\xi} = C\epsilon^2 \log\frac{1}{(\lambda'-1)} = C\epsilon^2\,.$$

Finally, if we integrate the energy identity (75)) on the triangle Δ (as shown in Figure 2 with vertices at (η,ξ), $(0,\xi)$, and $(0,\eta+\xi)$, with $\eta = \xi/\lambda'$ as before), we obtain

$$0 = -\int_{\lambda|t|}^{|t|} e(r,t) r\, dr - \int_{\eta}^{0} r(e+m)d\eta' + \int_{\xi+\eta}^{\xi} r(e-m)d\xi' = \mathrm{I} + \mathrm{II} + \mathrm{III}\,.$$

As $t \to 0$ we proved that $\mathrm{II} \to 0$; moreover, $\mathrm{III} \to 0$ because it is the flux, and therefore $\mathrm{I} \to 0$. □

As consequence we obtain the decay of time derivatives.

Corollary 4.2. *Let u be a radially symmetric solution of* (48) *or a co-rotational wave map on $K = K^1$ which is smooth away from the origin. In the latter case also suppose that N satisfies* (24)*. Then*

$$\frac{1}{T}\int_{K^T} |u_t|^2\, dz \to 0 \text{ as } T \to 0.$$

Proof. Again we change time t to $-t$. Multiply the identity (74), (77), respectively, by r and integrate on the truncated cone

$$K_T^{-\epsilon} = \{(t,x); t \le -\epsilon, |x| \le -t \le -T\},$$

and let $\epsilon \to 0$ to obtain

$$\Big| \iint_{K_T^0} u_t^2 r\, dr\, dt - \int_0^{|T|} (u_t u_r)\big|_{t=T} r^2\, dr \Big| \le C|T|\, \mathrm{Flux}(T)\,.$$

Therefore, for any $\lambda \in]0,1[$ we have

$$\begin{aligned}
\frac{1}{|T|}\int_T^0 \int_0^{-t} u_t^2 r\, dr\, dt &\le \frac{1}{|T|}\int_0^{|T|} |(u_t u_r)\big|_{t=T}| r^2\, dr + C\, \mathrm{Flux}(T) \\
&\le \frac{C}{|T|}\int_0^{|T|} e(T,r) r^2\, dr + C\, \mathrm{Flux}(T) \\
&\le \frac{C}{|T|}\left(\int_0^{\lambda|T|} e(T,r) r^2\, dr + \int_{\lambda|T|}^{|T|} e(T,r) r^2 dr\right) + C\, \mathrm{Flux}(T) \\
&\le C(\lambda E_0 + E^\lambda_{\mathrm{ext}}(T) + \mathrm{Flux}(T))\,.
\end{aligned}$$

Given $\epsilon > 0$ we then may choose $\lambda > 0$ such that the first term on the right is less then $\epsilon/3$. By Lemma 4.1 and by decay of the flux the second and third terms also will be less than $\epsilon/3$ for T sufficiently close to 0. □

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