1. De Giorgi's theorem

1.1 Hilbert's 19^{th} problem

At the 1900 ICM Hilbert presented a list of 23 problems as a challenge to his fellow mathematicians, among which

Problem 19: Are the solutions of regular problems in the Calculus of Variations always necessarily analytic?

Of course, some terms here need to be explained:

- What is a "regular problem" in the Calculus of Variations?

- Should one perhaps substitute "smooth" for "analytic"?

1.2 A model problem

Let $\Omega \subset\subset \mathbb{R}^n$ be a smoothly bounded open set, $u_0 \in C^\infty(\bar{\Omega})$ (or $u_0 \in C^{2,\alpha}(\Omega)$).

Also let $f \in C^\infty(\mathbb{R}^n)$ be such that the following conditions hold:

- Coerciveness and quadratic growth:

$\exists\, c_0 > 0,\ C_1 \geq 0\ \forall p \in \mathbb{R}^n\ \forall i,j \in \{1,\dots,n\}$:

$$(1.1)\quad c_0|p|^2 - C_1 \leq f(p) \leq C_1|p|^2 + C_1,$$

$$(1.2)\quad |f_{p_i}(p)| \leq C_1|p| + C,\quad |f_{p_ip_j}(p)| \leq C_1.$$

- Uniform convexity

$\exists\, \lambda > 0\ \forall p \in \mathbb{R}^n,\ \xi = (\xi_i) \in \mathbb{R}^n$:

$$(1.3)\quad \lambda|\xi|^2 \leq f_{p_ip_j}(p)\,\xi_i\,\xi_j \leq C_1|\xi|^2.$$

Here, we tacitly sum over $1 \leq i,j \leq n$, and we let $f_{p_i}(p) = \frac{\partial f}{\partial p_i}(p)$, etc.

In fact, (1.3) implies (1.1) and (1.2).

Then consider the variational problem to find a function $u \in C^\infty(\bar{\Omega})$ (or $u \in C^{2,\alpha}(\bar{\Omega})$) such that

$$(1.4) \qquad u = u_0 \quad \text{on } \partial\Omega$$

and so that

$$(1.5) \quad E(u) = \int_\Omega f(\nabla u)\,dx = \min_{v = u_0 \text{ on } \partial\Omega} E(v) .$$

<u>Theorem</u> 1.1. Given u_0 as above, there exists $u \in H^1(\Omega)$ satisfying (1.4) in the sense that

$$u - u_0 \in H^1_0(\Omega)$$

and such that

$$E(u) = \inf_{v - u_0 \in H^1_0(\Omega)} E(v) .$$

Moreover, u weakly solves the Euler-Lagrange equation

$$(1.6) \qquad -\frac{\partial}{\partial x_i}\left(f_{p_i}(\nabla u)\right) = 0$$

associated with (1.5).

Proof: i) The space

$$M = \{v \in H^1(\Omega);\ v - u_0 \in H_0^1(\Omega)\}$$
$$= \{u_0\} + H_0^1(\Omega) \subset H^1(\Omega)$$

is a closed and convex subset of the Hilbert space

$$X = H^1(\Omega);$$

hence, M also is weakly sequentially closed. Moreover, $E(u) < \infty$ for $u \in M$.

By coerciveness, a minimizing sequence $(u_k)_{k \in \mathbb{N}} \subset M$ with

$$E(u_k) \to \alpha = \inf_{v \in M} E(v) \geq -\infty$$

is bounded with

$$c_0 \|\nabla u_k\|_{L^2}^2 \leq \int_\Omega f(u_k)\,dx + C_1|\Omega| \leq C,$$

and with

$$\|u_k\|_{L^2} \leq \|u_k - u_0\|_{L^2} + \|u_0\|_{L^2}$$
$$\leq C\|\nabla u_k\|_{L^2} + C\|u_0\|_{H^1} \leq C$$

by Poincaré's inequality. A sub-sequence $u_k \xrightarrow[(k\to\infty,\, k\in\Lambda)]{w} u$ in $H^1(\Omega)$, and $u \in M$ since M is weakly sequentially closed.

Finally, by convexity of f it follows that E is w.s.l.s.c.; in particular,

$$\alpha \leq E(u) = \int_\Omega f(\nabla u)\,dx \leq \liminf_{\substack{k\to\infty \\ k\in\Lambda}} E(u_k) = \alpha\ ^{1)},$$

and

$$E(u) = \alpha,$$

as claimed.

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1) By Mazur's lemma (Satz 4.6.3, FA I), given k_0, suitable convex linear combinations

$$v_\ell = \sum_{k=k_0}^{\ell} a_{k\ell}\, u_k \xrightarrow{(\ell\to\infty)} u \text{ in } H^1(\Omega)$$

strongly. By convexity and continuity of f,

$$E(u) = \lim_{\ell\to\infty} E(v_\ell) \leq \lim_{\ell\to\infty} \Big(\sum_{k=k_0}^{\ell} a_{k\ell}\, E(u_k)\Big) \leq \sup_{k\geq k_0} E(u_k),$$

and the inequality follows as $k_0 \to \infty$.

ii) By minimality, for any $\varphi \in H_0^1(\Omega)$ we have
$$E(u) \leq E(u+\varepsilon\varphi), \quad \varepsilon \in \mathbb{R}.$$

Hence
$$0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} E(u+\varepsilon\varphi) = \int_\Omega \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f(\nabla u+\varepsilon\nabla\varphi)\,dx$$

$$= \int_\Omega f_{p_i}(\nabla u)\frac{\partial}{\partial x_i}\varphi\,dx, \quad \forall \varphi \in H_0^1(\Omega),$$

which is the weak form of (1.6). $\square$

In view of Thm. 1.1 we are thus faced with the problem of showing that any weak solution $u \in H^1(\Omega)$ of equation (1.6) is a classical solution of (1.6). This is very similar to what we have done in FA II, but with an important difference, as we will see.

Lemma 1.2. Let $u \in \{u_0 + H_0^1(\Omega)\}$ be any weak solution of (1.6), such as obtained e.g. in Thm. 1.1. Then $u \in H^2_{loc}(\Omega)$, and for any $1 \leq l \leq n$ the function $u^{(l)} = \frac{\partial}{\partial x_l} u \in H^1_{loc}(\Omega)$ weakly solves the equation

$$(1.7) \quad -\frac{\partial}{\partial x_i}\left(f_{p_i p_j}(\nabla u)\,\frac{\partial u^{(l)}}{\partial x_j}\right) = 0 \text{ in } \Omega.^{1)}$$

Proof. i) Fix $1 \leq l \leq n$. For $h \neq 0$ with $|h| < h_0 \ll 1$ let

$$\partial_l^h u(x) = \frac{u(x + h e_l) - u(x)}{h}$$

be the difference quotient in direction x_l.

For any $\eta \in C_c^\infty(\Omega)$ and $0 < |h| < h_0 = \operatorname{dist}(\operatorname{supp}(\eta), \partial\Omega)$ then we may test (1.6) with

$$\varphi = \partial_l^{-h}(\partial_l^h u\, \eta^2) \in H_0^1(\Omega)$$

to obtain the following

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1) *Exercise*: Show that for $u_0 \in H^2(\Omega)$ we have $u \in H^2(\Omega)$ for any weak solution of (1.6) satisfying (1.4).

identity

$$0=\int_\Omega f_{p_i}(\nabla u)\,\frac{\partial}{\partial x_i}\left(\partial_\ell^{-h}\left(\partial_\ell^h u\,\eta^2\right)\right)dx$$

$$=\int_\Omega \partial_\ell^h\left(f_{p_i}(\nabla u)\right)\left(\partial_\ell^h\frac{\partial u}{\partial x_i}\eta^2+2\partial_\ell^h u\,\frac{\partial\eta}{\partial x_i}\eta\right)dx\,.$$

Expanding

$$\left(\partial_\ell^h f_{p_i}(\nabla u)\right)(x)=\frac{f_{p_i}(\nabla u(x+he_\ell))-f_{p_i}(\nabla u(x))}{h}$$

$$=\int_0^1 f_{p_ip_j}\left(\nabla u(x)+t(\nabla u(x+he_\ell)-\nabla u(x)\right)dt\,.$$

$$\cdot\,\frac{\frac{\partial}{\partial x_j}u(x+he_\ell)-\frac{\partial}{\partial x_j}u(x)}{h}\,,$$

and using (1.2), (1.3) there results

$$\lambda\|\partial_\ell^h\nabla u\cdot\eta\|_{L^2}^2\leq C_1\|\partial_\ell^h\nabla u\cdot\eta\|_{L^2}\|\partial_\ell^h u\cdot\eta\nabla\eta\|_{L^2}\,,$$

so that we can bound

$$\|\partial_\ell^h\nabla u\cdot\eta\|_{L^2}^2\leq C\|\partial_\ell^h u\cdot\eta\nabla\eta\|_{L^2}\,,\quad |h|<h_0\,.$$

Estimating

$$\sup_{\substack{1\le l\le n\\ |h|<h_0}} \|\partial_l^h u \cdot \eta \nabla \eta\|_{L^2}^2 \le C\|\nabla u\|_{L^2}^2 \le C E(u) + C,$$

from Satz 8.3.1, FA II, it follows that $\nabla u \in H^1_{loc}$, as claimed.

ii) Choosing now $\varphi = \frac{\partial}{\partial x_l}\eta \in C_c^\infty(\Omega)$ as testing function, and integrating by parts with respect to x_l, we find

$$0 = \int_\Omega f_{p_i}(\nabla u)\frac{\partial}{\partial x_i}\left(\frac{\partial \eta}{\partial x_l}\right)dx$$

$$= -\int_\Omega f_{p_i p_j}(\nabla u)\frac{\partial^2 u}{\partial x_j \partial x_l}\cdot\frac{\partial \eta}{\partial x_i}\,dx,$$

which is the weak form of (1.7).

Note that by Satz 8.3.2, FA II, we have

$$\frac{\partial}{\partial x_l}\left(f_{p_i}(\nabla u)\right) = f_{p_i p_j}(\nabla u)\frac{\partial^2 u}{\partial x_j \partial x_l}$$

weakly. □

Problem: In FA II we have proved a-priori bounds and existence of solutions $u \in C^{1,\alpha}(\Omega)$, or $u \in C^{2,\alpha}(\Omega)$ of the boundary value problem

$$(1.8) \quad -\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) = 0 \quad \text{in } \Omega$$

$$u = u_0 \quad \text{on } \partial\Omega$$

for uniformly elliptic coefficients $a_{ij} = a_{ji}$ of class C^{α}, or of class $C^{1,\alpha}$, respectively.

Equation (1.7) is of the same kind as (1.8), but with coefficients

$$a_{ij}(x) = f_{p_i p_j}(\nabla u(x)) = f_{p_i p_j}(u^{(1)}, .., u^{(n)})$$

which are a-priori only bounded and measurable.

Thus there is a regularity gap:

In order to be able to apply the tools from FA II we have to show that weak solutions $u = u^{(e)}$ of (1.8) are Hölder continuous even if we only assume $a_{ij} \in L^{\infty}(\Omega)$.

1.3 De Giorgi's theorem

Def. 1.3. A function $u \in H^1(\Omega)$ satisfying (1.4) is an extremal of E, if u weakly solves (1.4), (1.6).

Recall that by Lemma 1.2 any extremal $u \in H^1(\Omega)$ of E satisfies $u \in H^2_{loc}(\Omega)$, and the components $u^{(l)}$, $1 \le l \le n$, of ∇u weakly solve (1.7), that is, equation (1.8) with uniformly elliptic coefficients

$$a_{ij} = a_{ji} = f_{p_i p_j}(\nabla u) \in L^\infty(\Omega).$$

We note that the convexity condition (1.3), which is natural from the point of view of the Calculus of Variations, also provides uniform ellipticity of (1.7), (1.8); that is,

$$\exists\, 0 < \lambda \le \Lambda \ \forall \xi = (\xi_i)_{1 \le i \le n} \in \mathbb{R}^n :$$

(1.9) $$\lambda |\xi|^2 \le a_{ij} \xi_i \xi_j \le \Lambda |\xi|^2 .$$

After much effort by many mathematicians, including, in particular, E. Hopf, Stampacchia, and Morrey, finally in the paper

"Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari", in Mem. Accad. Sci. Torino (1957),

Ennio de Giorgi proved the following result.

<u>Theorem</u> 1.4 (De Giorgi) Let $u \in H^1(\Omega)$ be a weak solution of (1.8) with uniformly elliptic coefficients $a_{ij} = a_{ji} \in L^\infty(\Omega)$. Then $u \in C^\alpha_{loc}(\Omega)$ for some $0 < \alpha \leq 1$.

Independently and almost simultaneously J. Nash obtained the same result also for weak solutions of parabolic equations in 1958.

Here, we present Moser's proof of Thm. 1.5. Later Moser also treated the parabolic case; Nash's original approach was revisited by Dan Stroock in the late 1980's.

1.4 Moser's weak sup-estimate

Let $\Omega \subset \mathbb{R}^n$ be open and bounded.

Recall that a function $u \in H^1(\Omega)$ is a <u>weak solution</u> of (1.8) if there holds

$$\forall \varphi \in C_c^\infty(\Omega): \int_\Omega a_{ij}\, \partial_i u\, \partial_j \varphi \, dx = 0.$$

<u>Theorem</u> 1.5 (Moser (1960) Let $u \in H^1(B_{2r}(0))$ be a weak solution of (1.8) with (1.9). Then $u \in L^\infty(B_r(0))$, and

$$\sup_{|x|<r} |u(x)|^2 \leq C r^{-n} \int_{B_{2r}(0)} |u|^2 dx.$$

Note: NLoG we may assume $n \geq 3$, $r = 1$.

<u>Proof</u>: We iteratively show the improved integrability/regularity $u \in H^1 \cap L^{p_k}(B_k)$ for

$$p_k = \frac{n}{n-2} \cdot p_{k-1}, \quad p_0 = 2,$$

on concentric balls

$$B_k = B_{r_k}(0), \quad r_k = (1 + 2^{-k}) r, \quad k \in \mathbb{N}_0.$$

By assumption $u \in H^1(B_{r_0}(0)) = H^1 \cap L^{p_0}(B_0)$.

For the <u>induction step</u>, suppose that $u \in H^1 \cap L^{p_k}(B_k)$ for some $k \in \mathbb{N}_0$. Set $p = p_k/2$ and let $\eta \in C_c^\infty(B_{r_k}(0))$ with $0 \le \eta \le 1$ such that

$$\eta = 1 \text{ on } B_{r_{k+1}}(0), \quad |\nabla \eta| \le \frac{2}{r_k - r_{k+1}} = \frac{2^{k+2}}{r}.$$

Then for any $l > 0$ the function

$$(1.10) \qquad \varphi = u \min\{|u|^{2p-2}, l^{2p-2}\} \eta^2 \in H_0^1(B_k)$$

is an admissible testing function, and from (1.9) we obtain

$$\lambda \int_{B_k} |\nabla u|^2 \min\{|u|^{2p-2}, l^{2p-2}\} \eta^2 \, dx$$

$$(1.11) \qquad + \lambda(2p-2) \int_{\{x \in B_k;\, |u(x)| < l\}} |\nabla u|^2 |u|^{2p-2} \eta^2 \, dx$$

$$\le 2\Lambda \int_{B_k} |\nabla u| \, |u| \min\{|u|^{2p-2}, l^{2p-2}\} \eta |\nabla \eta| \, dx.$$

$$=: I.$$

By Young's inequality, we can bound

$$I \leq \frac{\lambda}{2} \int_{B_k} |\nabla u|^2 \min\{|u|^{2p-2}, \ell^{2p-2}\} \eta^2 \, dx$$

$$+ C \int_{B_k} |u|^2 \min\{|u|^{2p-2}, \ell^{2p-2}\} |\nabla \eta|^2 \, dx .$$

Absorbing the first term on the left of (1.11), and letting

$$v_\ell = \min\{|u|^p, \ell^p\} \eta \in H^1(B_k)$$

we obtain

$$\int_{B_k} |\nabla v_\ell|^2 dx \leq p^2 \int_{\{x; |u(x)| < \ell\}} |\nabla u|^2 |u|^{2p-2} \eta^2 \, dx$$

$$+ 2 \int_{B_k} |u|^{2p} |\nabla \eta|^2 dx$$

$$\leq \left(C \frac{4p^2}{4p-3} + 2 \right) \int_{B_k} |u|^{2p} |\nabla \eta|^2 dx$$

$$\leq C\, 2^{2k} p \int_{B_k} |u|^{2p} dx,$$

uniformly in $\ell > 0$.

By Sobolev's embedding then $v_l \in L^{\frac{2n}{n-2}}(B_k)$

and

$$\left(\int_{B_k} \min\{|u|^{\frac{2np}{n-2}}, l^{\frac{2np}{n-2}}\} \eta^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}$$

$$\le \|v_l\|^2_{L^{\frac{2n}{n-2}}} \le C \|\nabla v_l\|^2_{L^2}$$

$$\le C\, 2^{2k} p \int_{B_k} |u|^{2p} dx \, .$$

Letting $l \to \infty$, by Beppo Levi's theorem we find $|u|^p \eta \in L^{\frac{2n}{n-2}}(B_k)$ with

$$\| |u|^p \eta \|^2_{L^{\frac{2n}{n-2}}(B_k)} \le \limsup_{l \to \infty} \|v_l\|^2_{L^{\frac{2n}{n-2}}(B_k)}$$

$$\le C\, 2^{2k} p\, r^{-2} \int_{B_k} |u|^{2p} dx \, ,$$

But $\eta \equiv 1$ on $B_{k+1} = B_{r_{k+1}}(0)$ and $\frac{2np}{n-2} = \frac{np_k}{n-2} = p_{k+1}$; thus $u \in L^{p_{k+1}}(B_{k+1})$ and from the previous estimate we complete the induction step with

$$\|u\|^{p_k}_{L^{p_{k+1}}(B_{k+1})} \leq C\, 2^{2k} p_k \|u\|^{p_k}_{L^{p_k}(B_k)}$$

for any $k \in \mathbb{N}$, with $p_k = \left(\frac{n}{n-2}\right)^k 2$. Taking the p_k-th root and iterating, we find

$$\|u\|_{L^{p_{k+1}}(B_{k+1})} \leq C^{\frac{1}{p_k}} 2^{\frac{2k}{p_k}} p_k^{\frac{1}{p_k}} \|u\|_{L^{p_k}(B_k)}$$

$$\leq \dots \leq C^{\sum_{j=0}^{k} \frac{1}{p_j}} 2^{\sum_{j=0}^{k} \frac{2j}{p_j}} \prod_{j=0}^{k} p_j^{\frac{1}{p_j}} \|u\|_{L^2(B_0)}$$

with exponents

$$\sum_{j=0}^{k} \frac{1}{p_j} < \frac{1}{2} \sum_{j=0}^{\infty} \left(\frac{n-2}{n}\right)^j < \infty,$$

$$\sum_{j=0}^{k} \frac{2j}{p_j} < \sum_{j=0}^{\infty} j \left(\frac{n-2}{n}\right)^j < \infty,$$

and with

$$\prod_{j=0}^{k} p_j^{\frac{1}{p_j}} = \exp\left(\sum_{j=0}^{k} \frac{1}{p_j} \log p_j\right)$$

$$\leq \exp\left(C \sum_{j=0}^{\infty} \frac{j}{p_j}\right) < \infty.$$

We conclude that $u \in \bigcap_{k \in \mathbb{N}} L^{p_k}(B_1(0))$ with

$$\sup_{k \in \mathbb{N}} \|u\|_{L^{p_k}(B_1(0))} \leq C \|u\|_{L^2(B_2(0))}.$$

Hence, $u \in L^\infty(B_1(0))$ and

$$\|u\|_{L^\infty(B_1(0))} \leq \limsup_{k \to \infty} \|u\|_{L^{p_k}(B_1(0))} \leq C \|u\|_{L^2(B_2(0))},$$

as claimed. $\square$

<u>Remark 1.6.</u> If $0 < \varepsilon \leq u \leq M < \infty$ in $B_r(0)$, for any $\eta \in C_c^\infty(B_r(0))$ the function

$$\varphi = u|u|^{2p-2}\eta^2 \in H_0^1(B_r(0))$$

is an admissible testing function for (1.8) for any $p \in \mathbb{R}$, yielding the identity

$$\begin{aligned} 0 &= \int_{B_r(0)} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_j}\, dx \\ &= (2p-1) \int_{B_r(0)} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} |u|^{2p-2} \eta^2\, dx \qquad (1.12) \\ &\quad + 2 \int_{B_r(0)} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \eta}{\partial x_j} u|u|^{2p-2} \eta\, dx. \end{aligned}$$

Using Remark 1.6 we can improve the assertion of Thm. 1.5 for non-negative weak solutions u of (1.8), as follows.

<u>Theorem</u> 1.7 (Moser). Let $0 \leq u \in H^1(B_{4r}(0))$ weakly solve (1.8) with (1.9), and let $q > 0$. Then with $C = C(q, \lambda, \Lambda, n) > 0$ there holds

$$(1.13) \quad \sup_{|x|<r} u(x) \leq C \left(r^{-n} \int_{B_{4r}(0)} u^q \, dx \right)^{\frac{1}{q}} .$$

<u>Proof</u>. We may assume $u \geq \varepsilon > 0$.

Indeed, for any $\varepsilon > 0$ the function $u_\varepsilon = u + \varepsilon$ weakly solves (1.8). If (1.13) holds true for any u_ε, we have

$$\sup_{|x|<r} u(x) \leq \sup_{|x|<r} u_\varepsilon(x)$$

$$\leq C \left(r^{-n} \int_{B_{3r}(0)} (u+\varepsilon)^q \, dx \right)^{\frac{1}{q}} , \quad \varepsilon > 0 .$$

Letting $\varepsilon \downarrow 0$, we then find (1.13).

Moreover, by Thm. 1.5 we may assume that $u \in L^\infty(B_{3r}(0))$ with $0 < \varepsilon \le u \le M < \infty$ for some $M > 0$, and for any $\eta \in C_c^\infty(B_{3r}(0))$ and any $p > 0$ the test function

$$\varphi = u|u|^{2p-2}\eta^2 \in H_0^1(B_{3r}(0))$$

is admissible in (1.8), yielding (1.12) of Rem. 1.6.

Given $q > 0$, we then let $p = p_k/2$, where

$$p_0 = q/2, \quad p_k = \frac{n}{n-2}\,p_{k-1}, \quad k \in \mathbb{N},$$

and iterate (1.12) on $B_k = B_{r_k}(0)$, where

$$r_k = 2 + 2^{-k}, \quad k \in \mathbb{N},$$

until $k = k_0$, with k_0 minimal such that $p_{k_0} \ge 2$. Replacing $q > 0$ with $0 < \tilde{q} < q$, if necessary, we may assume that

$$p_k \ne 1, \quad 0 \le k \le k_0.$$

Thus, from (1.12) for any k we obtain

$$\lambda \int_{B_k} |\nabla u|^2 |u|^{2p-2} \eta^2 \, dx$$

$$\leq \frac{2\Lambda}{|2p-1|} \int_{B_k} |\nabla u| \, |u|^{2p-1} |\eta| |\nabla \eta| \, dx$$

(1.14)
$$\leq \frac{\lambda}{2} \int_{B_k} |\nabla u|^2 |u|^{2p-2} \eta^2 \, dx$$

$$+ C \int_{B_k} |\nabla \eta|^2 |u|^{2p} \, dx$$

and again letting $v = u^p \eta$ as in the proof of Thm. 1.5 from Sobolev's embedding we obtain

$$\|u\|^{p_k}_{L^{p_{k+1}}(B_{k+1})} \leq \|v\|^2_{L^{\frac{2n}{n-2}}(B_k)} \leq C \|\nabla v\|^2_{L^2(B_k)}$$

$$\leq C p^2 \int_{B_k} |\nabla u|^2 |u|^{2p-2} \eta^2 \, dx$$

$$+ C \int_{B_k} |u|^{2p} |\nabla \eta|^2 \, dx$$

$$\leq C \|u\|^{p_k}_{L^{p_k}}, \quad k = 0, \dots, k_0 .$$

It follows that

$$\|u\|_{L^{p_{k_0}}(B_{k_0})} \leq C \|u\|_{L^{p_0}(B_0)},$$

and thus, since $p_{k_0} \geq 2$, $p_0 \leq q$, by Hölder's inequality

$$\|u\|_{L^2(B_{2r}(0))} \leq C \|u\|_{L^{p_{k_0}}(B_{k_0})}$$

$$\leq C \|u\|_{L^{p_0}(B_0)} \leq C \|u\|_{L^q(B_0)}.$$

Together with Thm. 1.5, this yields the claim. $\square$

Remark 1.8. Thms 1.5, 1.7 also hold for weak sub-solutions $0 \leq u \in H^1(B_{4r}(0))$ of (1.8) satisfying

(1.15) $\forall 0 \leq \varphi \in H^1_0(B_{4r}(0)): \int_{B_{4r}(0)} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_j} dx \leq 0,$

but only for $q > 1$. Indeed, for $p \geq q/2 > 1/2$ (1.12) and (1.15) give (1.14) and Thms 1.5, 1.6 hold.

1.5 Moser's weak Harnack inequality

We complement the upper bounds given by Thms 1.5 and 1.6 with a lower bound.

Def. 1.9. A function $u \in H^1(\Omega)$ is a weak super-solution of (1.8) if there holds

(1.16) $$\forall\, 0 \leq \varphi \in H^1_0(\Omega): \int_\Omega a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_j}\, dx \geq 0 ;$$

that is, if $v = -u$ is a weak sub-solution.

Theorem 1.10 (Moser) Suppose $0 \leq u \in H^1(B_{4\sqrt{n}r}(0))$ is a weak super-solution of (1.8) with (1.9). Then for suitable $0 < q < \frac{n}{n-2}$ with $C = C(\lambda, \Lambda, n) > 0$ there holds

(1.17) $$\inf_{|x| < r} u(x) \geq C^{-1} \left(r^{-n} \int_{B_{2r}(0)} u^q\, dx \right)^{\frac{1}{q}} .$$

Remark 1.11. i) The truncated fundamental solution $u(x) = \min\{|x|^{2-n}, M\}$ is weakly super-harmonic, showing that (1.17) fails for $q = \frac{n}{n-2}$.

ii) More generally, for any weak super-solution $0 < \varepsilon \leq u \in H^1 \cap L^\infty(\Omega)$ and any $f \in C^2(\mathbb{R}_+)$ which is convex and non-increasing the function

$$v = f(u) \in H^1(\Omega)$$

is a weak sub-solution.

<u>Proof</u>: For any $0 \leq \varphi \in H_0^1(\Omega)$ there holds

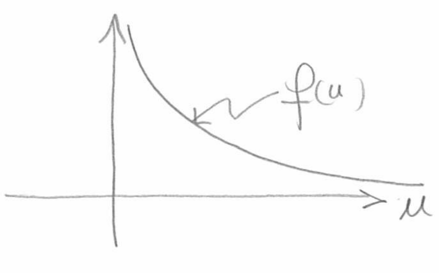
$$\int_\Omega a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial \varphi}{\partial x_j} dx = \int_\Omega a_{ij} \frac{\partial u}{\partial x_i} f'(u) \frac{\partial \varphi}{\partial x_j} dx$$

$$= \underbrace{\int_\Omega a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial}{\partial x_j}\Big(\underbrace{f'(u)\varphi}_{\leq 0}\Big) dx}_{\leq 0} - \underbrace{\int_\Omega a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \underbrace{f''(u)\varphi}_{\geq 0} dx}_{\geq 0}$$

$$\leq 0.$$

$\square$

The claim uses this with $f(u) = \frac{1}{u^{q/2}}$.



The proof of Thm. 1.10 is achieved by combining Remark 1.8, Thm. 1.5, and Thm. 1.14 below.

<u>Lemma 1.12.</u> For u as in Thm. 1.10 and any $q > 0$ there holds

$$\inf_{|x|<r} u(x) \geq \left(\frac{1}{C r^{-n} \int_{B_{2r}(0)} u^{-q} \, dx} \right)^{\frac{1}{q}}$$

with a constant $C = C(q, \lambda, \Lambda, n) > 0$.

<u>Proof:</u> As in the proof of Thm. 1.7 we may assume that $u \geq \varepsilon > 0$ for some $\varepsilon > 0$. Set $v = \left(\frac{1}{u}\right)^{q/2} \in H^1(B_{4r}(0))$.

By Remark 1.11. ii) then v is a weak sub-solution of (1.8).

<u>Proof:</u> Let $0 \leq \varphi \in H_0^1(B_{4r}(0))$ and compute

$$\int_{B_{4r}(0)} a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial \varphi}{\partial x_j} dx = -\frac{q}{2} \int_{B_{4r}(0)} a_{ij} \frac{\partial u}{\partial x_i} \frac{1}{u^{\frac{q+2}{2}}} \frac{\partial \varphi}{\partial x_j} dx$$

$$= -\frac{q}{2} \underbrace{\int_{B_{4r}(0)} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial}{\partial x_j} \left(\frac{\varphi}{u^{\frac{q+2}{2}}} \right) dx}_{\geqslant 0} - \frac{q(q+2)}{4} \underbrace{\int_{B_{4r}(0)} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\varphi}{u^{\frac{q+4}{2}}} dx}_{\geqslant 0}$$

$$\leq 0. \qquad \square$$

By Thm. 1.5 and Rem. 1.8 thus we have

$$\sup_{|x|<r} v(x) \leq C\left(r^{-n}\int_{B_{2r}(0)} v^2\,dx\right)^{\frac{1}{2}}$$

$$= C\left(r^{-n}\int_{B_{2r}(0)} u^{-q}\,dx\right)^{\frac{1}{2}};$$

moreover, there holds

$$\sup_{|x|<r} v(x) = \left(\frac{1}{\inf_{|x|<r} u}\right)^{q/2}.$$

The claim follows. □

The "jump" from exponents $q<0$ to exponents $p>0$, finally, can be achieved with the help of the following result.

Def. 1.13. Let $Q_0 \subset \mathbb{R}^n$ be a cube, $u \in L^1(Q_0)$. Then u has bounded mean oscillation, $u \in BMO(Q_0)$, if there is $K \geq 0$ such that for any parallel sub-cube $Q \subset Q_0$ with a constant $a_Q \in \mathbb{R}$ there holds

$$⨍_Q |u-a_Q|\,dx = \frac{1}{|Q|}\int_Q |u-a_Q|\,dx \leq K.$$

__Theorem 1.14__ (John–Nirenberg (1961))

Suppose $w \in BMO(Q_0)$ with

$$\forall Q \subset Q_0 : \fint_Q |w - a_Q| \, dx \leq 1 .$$

Then $u \in \bigcap_{p \geq 1} L^p(Q_0)$, and with constants $\alpha, \beta > 0$ depending only on n there holds

$$\int_{Q_0} e^{\alpha |w - w_0|} \, dx \leq \beta^2 ,$$

where

$$w_0 = \fint_{Q_0} w \, dx .$$

In particular, then there holds

$$\int_{Q_0} e^{\alpha w} dx \cdot \int_{Q_0} e^{-\alpha w} dx$$

$$\leq \int_{Q_0} e^{\alpha(w - w_0)} dx \cdot \int_{Q_0} e^{-\alpha(w - w_0)} dx \leq \beta^2 . \tag{1.18}$$

Postponing the proof of Thm. 1.14, we show how Thm. 1.10 follows from Lemma 1.12 and Thm. 1.14.

Proof of Thm. 1.10: Using (1.12), (1.16) with $\eta \in C_c^\infty(B_{4r}(0))$ and $p=0$ we obtain

$$0 \leq -\int_{B_{4r}(0)} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} u^{-2} \eta^2 dx$$

$$+ 2 \int_{B_{4r}(0)} a_{ij} \frac{\partial u}{\partial x_i} u^{-1} \frac{\partial \eta}{\partial x_j} \eta \, dx .$$

Thus, with $v = \log u$ we find

$$\lambda \int_{B_{4r}(0)} |\nabla v|^2 \eta^2 dx \leq$$

$$\leq 2 \Lambda \int_{B_{4r}(0)} |\nabla v| \eta |\nabla \eta| dx ,$$

and then

$$\int_{B_{4r}(0)} |\nabla v|^2 \eta^2 dx \leq C \int_{B_{4r}(0)} |\nabla \eta|^2 dx .$$

Let $Q_0 =]-2r, 2r[^n$. For any $Q = \{x_0\} +]-\delta, \delta[^n \subset Q_0$ with $Q \subset \tilde{Q} = \{x_0\} +]-2\delta, 2\delta[^n \subset \tilde{Q}_0 \subset B_{4\sqrt{n}\,r}(0)$ then let $\eta \in C_c^\infty(\tilde{Q})$ with

$$0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ on } Q, \quad |\nabla \eta| < 2/\delta$$

to obtain

$$\int_Q |\nabla v|^2 dx \leq C \int_Q |\nabla \eta|^2 dx \leq C\, \delta^{n-2}.$$

Poincaré's inequality the gives that $v \in BMO(B_{3r}(0))$ with

$$\left(\fint_Q |v - v_Q| dx \right)^2 \leq \fint_Q |v - v_Q|^2 dx$$

$$\leq C\, \delta^{2-n} \int_Q |\nabla v|^2 dx \leq C =: \gamma^2$$

for any cube Q as above.

By Thm. 1.14, applied to $w = v/\gamma$, and observing that $B_{2r}(0) \subset Q_0$ as above, we find

$$\int_{B_{2r}(0)} e^{\alpha w} dx \cdot \int_{B_{2r}(0)} e^{-\alpha w} dx \leq C \beta^2;$$

that is, with $q = \alpha/\gamma > 0$ there holds

$$\int_{B_{2r}(0)} u^q dx \leq \frac{C}{\int_{B_{2r}(0)} u^{-q} dx},$$

and Thm. 1.10 follows from Lemma 1.12. $\square$

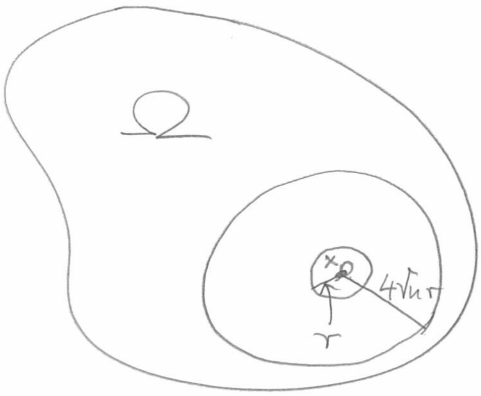
For weak solutions $u \in H^1(\Omega)$ we can combine Thm.s 1.7 and 1.10 to obtain the following result.

Corollary 1.15 (Harnack inequality) Let $0 \leq u \in H^1(\Omega)$ weakly solve (1.8) with (1.9). Then for any $B_r(x_0) \subset B_{4\sqrt{n}r}(x_0) \subset \Omega$ with $C = C(\lambda, \Lambda, n) > 0$ there holds

$$\sup_{B_r(x_0)} u \leq C \inf_{B_r(x_0)} u .$$

Proof: By Thm.s 1.7 and 1.10 for suitable $q > 0$ we have

$$\sup_{B_r(x_0)} u \leq C \left(r^{-n} \int_{B_{4r}(x_0)} u^q dx \right)^{\frac{1}{q}} \leq C \inf_{B_r(x_0)} u . \qquad \square$$



1.6 Hölder continuity

We now have all ingredients needed to complete the proof of De Giorgi's theorem.

Proof of Thm. 1.4:

Let $u \in H^1(\Omega)$ weakly solve (1.8) with (1.9), and let $\Omega' \subset\subset \Omega$ be relatively compact.

Set $L = 4\sqrt{n}$, and let $r_0 > 0$ with

$$2Lr_0 \leq \operatorname{dist}(\Omega', \partial\Omega).$$

For any $x_0 \in \Omega'$, any $0 < r < r_0$ let

$$M(r) = \sup_{B_r(x_0)} u, \quad m(r) = \inf_{B_r(x_0)} u.$$

By Corollary 1.15, applied to $\tilde{u} = u - m(Lr)$ with $0 \leq \tilde{u} \in H^1(B_{Lr}(x_0))$, with a constant

$$C = C(\lambda, \Lambda, n) > 0$$

there holds

$$M(r) - m(Lr) = \sup_{B_r(x_0)} \tilde{u} \leq C \inf_{B_r(x_0)} \tilde{u}$$

$$= C\big(m(r) - m(Lr)\big).$$

That is, we have

$$C\big(M(r)-m(r)\big) = (C-1)\big(M(r)-m(Lr)\big)$$
$$\leq (C-1)\big(M(Lr)-m(Lr)\big), \quad r<r_0.$$

With

$$\operatorname*{osc}_{B_r(x_0)} u := M(r)-m(r), \quad \theta = \frac{C-1}{C} < 1,$$

thus we find

$$\operatorname*{osc}_{B_r(x_0)} u \leq \frac{C-1}{C} \operatorname*{osc}_{B_{Lr}(x_0)} u = \theta \operatorname*{osc}_{B_{Lr}(x_0)} u$$
$$\leq \theta^2 \operatorname*{osc}_{B_{L^2r}(x_0)} u \leq \dots \leq \theta^k \operatorname*{osc}_{B_{L^kr}(x_0)} u$$

for $1 \leq k \leq k_0$ with k_0 maximal such that

$$L^{k_0} r \leq L r_0 ;$$

that is,
$$L^{k_0-1} \leq \frac{r_0}{r} < L^{k_0}, \quad k_0 \geq \frac{\log(\frac{r_0}{r})}{\log L},$$

and

$$\operatorname*{osc}_{B_r(x_0)} u \leq \theta^{k_0} \operatorname*{osc}_{B_{Lr_0}(x_0)} u \leq \theta^{k_0} \cdot \underbrace{2 \sup_{dist(x,\partial\Omega)\geq Lr_0} |u(x)|}_{= C(u,\Omega') < \infty}.$$

Estimating

$$\Theta^{k_0} = \exp\left(k_0 \log \Theta\right) \leq \exp\left(\frac{\log\left(\frac{r_0}{r}\right)}{\log L} \cdot \log \Theta\right)$$

$$= \exp\left(\frac{\log r_0 \cdot \log \Theta}{\log L}\right) \cdot r^{|\log \Theta| / \log L}$$

then for any $x_1 \in B_r(x_0)$ with $|x_1 - x_0| > r/2$ we obtain

$$|u(x_0) - u(x_1)| \leq \operatorname*{osc}_{B_r(x_0)} u \leq C r^{\alpha} \leq 2^{\alpha} C |x_0 - x_1|^{\alpha},$$

where

$$0 < \alpha = \frac{|\log \Theta|}{\log L}.$$

The claim follows.

$\square$

1.7 Proof of Thm 1.14.

It suffices to consider $Q_0 =]-1,1[^n$.

Thm. 1.14 follows from the next result.[1]

Theorem 1.16. Let $u \in L^1(Q_0)$, and suppose that for any parallel cube $Q \subset Q_0$ we have

$$(1.19) \qquad \fint_Q |u - \bar{u}_Q|\,dx = \frac{1}{|Q|}\int_Q |u - \bar{u}_Q|\,dx \leq 1,$$

where

$$\bar{u}_Q = \fint_Q u(x)\,dx.$$

Then there are $A, \gamma > 0$ such that

$$\forall \sigma > 0: \mu(\sigma) := |S(\sigma)| \leq A e^{-\gamma\sigma},$$

where

$$S(\sigma) = \{x \in Q_0;\ |u(x) - \bar{u}_{Q_0}| > \sigma\}.$$

Proof of Thm. 1.14. By classical results in real analysis, for u as in Thm. 1.14 and $f \in C^1(\mathbb{R})$ with $f(0) = 0$ and $f \equiv$ const on $]M, \infty[$ for some $M > 0$ we have

---

1) Note that if $\fint_Q |u - a_Q|\,dx \leq K$ for some $a_Q \in \mathbb{R}$, then we also have the bound $|\bar{u}_Q - a_Q| \leq K$, and $\fint_Q |u - \bar{u}_Q|\,dx \leq 2K$.

the identity

$$\int_{Q_0} f(|u-u_{Q_0}|)\,dx = -\int_0^\infty f(s)\,\mu'(s)\,ds$$

$$= \int_0^\infty \mu(s) f'(s)\,ds \le A\int_0^\infty f'(s)\, e^{-\gamma s}\,ds\,.$$

The assertions in Thm. 1.14 follow when we consider (truncated versions of) the functions

$$f(s) = s^p, \quad f(s) = e^{\alpha s}, \quad \alpha < \gamma.$$

$\square$

<u>Ex.</u> 1.17. i) The function

$$u(x) = \log\left(\frac{1}{|x|}\right) \in BMO_{loc}(\mathbb{R}^n);$$

in particular, then $BMO(Q_0) \not\hookrightarrow L^\infty(Q_0)$

but $BMO(Q_0) \subset \bigcap_{p\geq 1} L^p(Q_0)$.

ii) In FA II we have shown that for $n=2$ we have $u(x) = \log\log\frac{1}{|x|} \in H^1_0(B_{1/e}(0)) \xhookrightarrow{\text{(Poincaré)}} BMO(B_{1/e}(0))$.

The proof of Thm. 1.16 rests on the following result, obtained via a Calderón-Zygmund decomposition of Q_0 relative to $\tilde{u} = u - \bar{u}_{Q_0}$

Lemma 1.17. Let $u \in L^1(Q_0)$, $s \geq \fint_{Q_0} |u|\, dx > 0$.
Then there is a family of at most countably many open, disjoint parallel subcubes $Q_k \subset Q_0$, $k \in \mathbb{N}$, with the properties

i) $|u| \leq s$ a.e. in $Q_0 \setminus \bigcup_{k=1}^{\infty} Q_k$,

ii) $\forall k: \fint_{Q_k} |u|\, dx \leq 2^n s$,

iii) $\sum_{k=1}^{\infty} |Q_k| \leq s^{-1} \int_{Q_0} |u|\, dx$.

Proof: Iteratively divide Q_0 into parallel sub-cubes, as follows.
Divide Q_0 by halving the edges into 2^n parallel sub-cubes of equal volume, and let Q_{1j}, $j \in J_1 \subset \{1, \dots, 2^n\}$, be those sub-cubes with

$$\fint_{Q_{1j}} |u|\, dx \geq s.$$

Then for the selected cubes we have

$$s|Q_{1j}| \leq \int_{Q_{1j}} |u|\,dx \leq |Q_0| \fint_{Q_0} |u|\,dx \leq 2^n s |Q_{1j}| .$$

At each step then iterate the division on each of the remaining sub-cubes to obtain disjoint parallel sub-cubes Q_{ij} with

$$s|Q_{ij}| \leq \int_{Q_{ij}} |u|\,dx \leq 2^n s |Q_{ij}| , \quad i \in \mathbb{N},\ j \in J_i \subset \{1, \dots, 2^n\}$$

Then ii) holds, and we have

$$s \sum_{i,\, j\in J_i} |Q_{ij}| \leq \int_{\bigcup_{i,j} Q_{ij}} |u|\,dx \leq \int_{Q_0} |u|\,dx ,$$

showing iii).

Finally, for a.e. $x \in Q_0 \setminus \bigcup_{i, j\in J_i} Q_{ij}$ there are sub-cubes Q_{ij_i} with

$$\fint_{Q_{ij_i}} |u|\,dx \leq s, \quad i \in \mathbb{N};$$

so

$$|u(x)| \leq s \quad \text{a.e. } x \in Q_0 \setminus \bigcup_{i,\, j\in J_i} Q_{ij}$$

by Lebesgue differentiation.

$\square$

For any $\sigma > 0$ now let

$$F(\sigma) = \inf \{ C > 0;\ \forall u \in L^1(Q_0) : u \text{ satisfies } (1.19)$$
$$\Rightarrow |S(\sigma)| \leq C \int_{Q_0} |u - \bar{u}_{Q_0}|\, dx \}.$$

By Tchebychev's inequality $F(\sigma) \leq \frac{1}{\sigma}$.

<u>Lemma 1.18.</u> For any $\sigma \geq 2^n$ there holds

$$\forall s \in [1, 2^{-n}\sigma] : F(\sigma) \leq \frac{1}{s} F(\sigma - 2^n s).$$

<u>Proof</u>: For any $u \in L^1(Q_0)$ satisfying (1.19) with $\bar{u}_{Q_0} = 0$, we have

$$\int_{Q_0} |u|\, dx \leq 1 \leq s \leq 2^{-n}\sigma.$$

By Lemma 1.17, for any such u and a.e. $x \in Q_0$ where $|u(x)| > \sigma \geq s$ there is $k \in \mathbb{N}$ with $x \in Q_k$. Since $|\bar{u}_{Q_k}| \leq 2^n s$ by ii), we have

$$|S(\sigma)| = |\{x \in Q_0 ; |u(x)| > \sigma\}|$$
$$\leq \sum_k |\{x \in Q_k ; |u(x) - \bar{u}_{Q_k}| > \sigma - 2^n s\}|.$$

But by definition of F and (1.19) we have

$$|\{x\in Q_k;\ |u(x)-\bar{u}_{Q_k}|>\sigma-2^n s\}|$$

$$\leq F(\sigma-2^n s)\int_{Q_k}|u-\bar{u}_{Q_k}|\,dx$$

$$\leq F(\sigma-2^n s)\,|Q_k|.$$

Hence

$$|S(\sigma)|\leq F(\sigma-2^n s)\sum_k |Q_k|$$

(1.20) $$\leq \frac{F(\sigma-2^n s)}{s}\int_{Q_0}|u|\,dx.$$

Given $u\in L^1(Q_0)$ satisfying (1.19), considering $\tilde{u}=u-\bar{u}_{Q_0}$ with $\int_{Q_0}\tilde{u}\,dx=0$, from (1.20) we then obtain the claim.

$\square$

Proof of Thm. 1.16: i) Setting $s = e$, if for some $\sigma \geq 2^n$ and $\gamma = \frac{1}{2^n e}$ there holds

$$(1.21) \quad F(\sigma) \leq A e^{-\gamma \sigma}$$

for some $A > 0$, then by Lemma 1.18 we have

$$F(\sigma + 2^n e) \leq \frac{1}{e} F(\sigma) \leq A e^{-(\gamma \sigma + 1)}$$

$$= A e^{-\gamma(\sigma + 2^n e)}.$$

Hence, if there is an interval $I \subset [2^n, \infty[$ of length $|I| \geq 2^n e$ with

$$\forall \sigma \in I: \ F(\sigma) \leq A e^{-\gamma \sigma},$$

then we conclude (1.21) for all $\sigma \geq \inf I$.

ii) A computation shows that

$$\frac{1}{\sigma} \leq a \, 2^{-n} e^{-\gamma \sigma} \quad \text{for } \sigma \in I = \left[\frac{2^n e}{e-1}, \frac{2^n e}{e-1} + 2^n e\right],$$

where

$$a = \frac{e-1}{e} e^{\frac{e}{e-1}} \simeq 1.2 .$$

Indeed for $\sigma \in I$ we have

$$\frac{1}{\sigma} \leq \frac{e-1}{2^n e}, \quad a 2^{-n} e^{-\gamma \sigma} \geq \frac{a}{2^n e^{\frac{e}{e-1}}} = \frac{e-1}{2^n e}.$$

So (1.21) holds for all $\sigma \geq \frac{2^n e}{e-1}$ with $\gamma = \frac{1}{2^n e}$ and $A = 2^{-n} a$ by Tchebychev's inequality $F(\sigma) \leq \frac{1}{\sigma}$, $\alpha > 0$. $\square$

1.8 Application: Yamabe flow

Let (M, g_0) be a closed (compact, $\partial M = \phi$) n-dimensional Riemannian manifold, $n \geq 3$.

A conformal metric

$$g = u^{\frac{4}{n-2}} g_0 \,, \quad u > 0,$$

has scalar curvature

$$(1.22) \qquad R = R_g = u^{-\frac{n+2}{n-2}} \left(-c(n) \Delta_{g_0} u + R_0 u \right),$$

where $R_0 = R_{g_0}$, $c(n) = 4 \frac{n-1}{n-2}$, and where

$$\Delta_{g_0} u = \frac{1}{\sqrt{|g_0|}} \frac{\partial}{\partial x_i} \left(\sqrt{|g_0|} \, g_0^{ij} \frac{\partial u}{\partial x_j} \right)$$

is the Laplace-Beltrami operator on (M, g_0).

After an initial (flawed) atempt by Yamabe (1960) and a first partial result by Trudinger (1968), Aubin (1976) and, finally, Schoen (1984) showed:

<u>Theorem</u> 1.19. On any (M, g_0) as above there exists a conformal metric of constant scalar curvature.

Yamabe metrics can be obtained as critical points of the Yamabe energy

$$E(u) = \int_M \left(c(n) |\nabla u|^2_{g_0} + R_0 u^2 \right) d\mu_{g_0}$$

$$= \int_M R u^{2^*} d\mu_{g_0} = \int_M R \, d\mu_g$$

subject to the constraint

$$\mathrm{vol}(M,g) = \int_M d\mu_g = \int_M u^{2^*} d\mu_{g_0} = 1,$$

where

$$2^* = \frac{2n}{n-2}$$

is the Sobolev exponent for the critical embedding $H^1(M, g_0) \hookrightarrow L^{2^*}(M, g_0)$.

They can also be characterized as rest points of the Yamabe flow

$$(1.23) \qquad u^{2^*-2} u_t - c(n) \Delta_{g_0} u + R_{g_0} u = s\, u^{2^*-1},$$

where $s = s(t)$ is determined so that the volume $\mathrm{vol}(M, g(t)) = \mathrm{vol}(M, g(0))$ for all $t \geq 0$. By (1.22), equation (1.23) is equivalent to

$$(1.24) \qquad u_t = (s - R) u.$$

For data $u(0)=u_0>0$ with $vol(M,g(0))=1$ we then have $vol(M,g(t))=1$ for all $t\geq 0$, and we may use (1.24) to compute

$$0=\frac{d}{dt}vol(M,g)=2^*\int_M u^{2^*-1}u_t\,d\mu_{g_0}$$

$$=2^*\int_M (s-R)u^{2^*}d\mu_{g_0}=2^*\Big(s-\int_M R\,d\mu_g\Big),$$

and $s=\int_M R\,d\mu_g=E(u)$. We then also have

$$\frac{d}{dt}E(u)=2\int_M\Big(c(n)(\nabla u.\nabla u_t)_{g_0}+R_0\,u\,u_t\Big)d\mu_{g_0}$$

(1.25) $$=2\int_M L_{g_0}u\cdot u_t\,d\mu_{g_0}=-2\int_M u^{2^*-2}|u_t|^2d\mu_{g_0}$$

$$=-2\int_M u^{2^*-2}|u_t|^2d\mu_{g_0}\leq 0,$$

so that

$$s=E(u)\leq E(u_0)=s_0$$

for all $t\geq 0$. Moreover, we see that the Yamabe flow is the natural L^2-gradient flow for E (in the metric $g=g(t)$) with

$$\frac{d}{dt}E(u)=-2\int_M|R-s|^2d\mu_g,\quad t\geq 0.$$

The Yamabe flow was introduced by Hamilton (1989), who also showed global existence of the flow for any smooth initial data $u(0) = u_0 > 0$. For $(M, g_0) = (S^n, g_{S^n})$ it was shown by Ye (1994) that the flow always converges as $t \to \infty$ to a Yamabe metric, and similarly for any locally conformally flat (M, g_0).

In the general case, sub-convergence of the flow (1.23) as $t \to \infty$ suitably was first shown by Schwetlick-Struwe (2003) for $3 \leq n \leq 5$ and initial data allowing at most single-point blow-up, and finally by Brendle (2005/06) for $n \geq 3$ and arbitrary initial data.

An important point in the proof is a lower bound for $u(t)$ of Harnack type, similar to Thm. 1.10.

For simplicity, we may assume $R_{g_0} \geq 0$, $R_{g(0)} \geq 0$.
Moreover, with the conformal Laplace operator

$$L_g = -c(n)\,\Delta_g + R_g$$

and geometric invariance of the expression

$$R_h = v^{-\frac{n+2}{n-2}} L_g\, v = (uv)^{-\frac{n+2}{n-2}} L_{g_0}(uv)$$

of the scalar curvature of the metric

$$h = (uv)^{\frac{4}{n-2}} g_0 = v^{\frac{4}{n-2}} g, \quad g = {}^{\frac{4}{n-2}} g_0,$$

from (1.23) we obtain the equation

$$R_t = -\frac{n+2}{n-2} R \frac{u_t}{u} + u^{-\frac{n+2}{n-2}} L_{g_0}\left(\frac{u_t}{u} \cdot u\right)$$

$$= -\frac{n+2}{n-2} R(s-R) + L_g(s-R)$$

$$= c(n)\,\Delta_g R + \frac{4}{n-2}(R-s)R$$

for the evolution of curvature $R = R_g$.
Hence $R \geq 0$ for $t \geq 0$ by the maximum principle.

Hence from (1.24), (1.25) we have $u_t \leq s_0 u$,
so

$$\|u(t)\|_{L^\infty} \leq \|u_0\|_{L^\infty} e^{s_0 t} \text{ for all } t \geq 0.$$

The following global analogue of Thm. 1.10 turns this also into a locally uniform lower bound.

<u>Theorem</u> 1.20. Suppose $0 < u \in C^\infty(M, g_0)$ satisfies

$$-c(n)\Delta_{g_0} u + R_0 u \geq 0.$$

Then for some $q > 0$ with a constant $C > 0$ there holds

$$C \inf_M u \geq \|u\|_{L^q}.$$

NLoG $q \leq 2^*$.

<u>Corollary</u> 1.21. For the solution $u = u(t)$ of (1.23) with

$$L_{g_0} u = -c(n)\Delta_{g_0} u + R_0 u = R u^{2^*-1} \geq 0$$

and $\quad vol(M, u^{\frac{4}{n-2}} g_0) = \int_M u^{2^*} d\mu_{g_0} = 1$

there holds

$$C \inf u \geq \|u\|_{L^\infty}^{1-2^*/q} \geq \left(\|u_0\|_{L^\infty} e^{s_0 t}\right)^{1-2^*/q}.$$

<u>Proof</u>: $1 = \|u\|_{L^{2^*}}^{2^*} \leq \|u\|_{L^q}^q \, \|u\|_{L^\infty}^{2^*-q}$. $\square$

On a closed (M, g_0) we can establish the weak Harnack bound of Thm. 1.20 by an iteration argument, avoiding the space BMO. The Poincaré type estimate

$$(1.26) \quad \|f\|_{L^2(M,g_0)} \leq C_1 \, \|\nabla f\|_{L^2(M,g_0)}$$

for any smooth f with $\int_M f \, d\mu_{g_0} = 0$ will play an important role. NLoG $C_1 \geq 2$.

<u>Proof</u> <u>of</u> <u>Thm</u>. 1.20. NLoG $u \geq \varepsilon > 0$. Let

$$w = \log u - K, \quad K = \int_M \log u \, d\mu_{g_0}.$$

From the estimate

$$0 \leq \int_M \left(-c(n)\Delta_{g_0} u + R_0 u\right) \frac{1}{u} \, d\mu_{g_0}$$

$$= -c(n) \int_M |\nabla w|^2_{g_0} \, d\mu_{g_0} + \int_M R_0 \, d\mu_{g_0}$$

it follows that

$$(1.27) \quad \|\nabla w\|^2_{L^2(M,g_0)} \leq c(n)^{-1} \|R_0\|_{L^\infty} =: C_2.$$

NLoG $C_2 \geq 1$.

For $p \in \mathbb{N}$ we also use $|w|^{2p}/u \geq 0$ as testing function to obtain

$$0 \leq \int_M \left(-\Delta_{g_0} u + c(u)^{-1} R_0 u\right) |w|^{2p}/u \, d\mu_{g_0}$$

$$= -\int_M |\nabla w|^2_{g_0} |w|^{2p} \, d\mu_{g_0} + 2p \int_M |\nabla w|^2_{g_0} w^{2p-1} \, d\mu_{g_0}$$

$$+ \int_M \frac{R_0}{c(u)} |w|^{2p} \, d\mu_{g_0}$$

$$\leq -\frac{1}{2} \int_M |\nabla w|^2_{g_0} |w|^{2p} \, d\mu_{g_0} + 2p^2 \int_M |\nabla w|^2_{g_0} |w|^{2p-2} \, d\mu_{g_0}$$

$$+ C_2 \int_M |w|^{2p} \, d\mu_{g_0}$$

by Young's inequality. Thus we conclude

$$\|\nabla w^{p+1}\|^2_{L^2} \leq (p+1)^2 \left(4 \|\nabla w^p\|^2_{L^2} + 2 C_2 \|w^p\|^2_{L^2}\right).$$

With (1.26) applied to w^p we can further estimate

$$\|w^p\|_{L^2} \leq C_1 \|\nabla w^p\|_{L^2} + \left|\int_M w^p \, d\mu_{g_0}\right|.$$

Hence with $(a^2+b^2)^{1/2} \leq a+b$ for $a, b \geq 0$ we find

$$\|\nabla w^{p+1}\|_{L^2} \leq (p+1)\left(2\|\nabla w^p\|_{L^2} + \sqrt{2C_2}\,\|w^p\|_{L^2}\right)$$

(1.28) $$\leq (p+1)\left(\left(2+\sqrt{2C_2}\,C_1\right)\|\nabla w^p\|_{L^2} + \sqrt{2C_2}\,\|w^p\|_{L^1}\right)$$

$$= A(p+1)\left(B\,\|\nabla w^p\|_{L^2} + \|w^p\|_{L^1}\right)$$

with $A = \sqrt{2C_2}$, $B = 2C_1 \geq \frac{2}{\sqrt{2C_2}} + C_1$.

Claim: For any $p \in \mathbb{N}$ there holds

$$\|\nabla w^p\|_{L^2} \leq A^p B^{p-1} p^p, \quad \|w^p\|_{L^2} \leq A^p B^p p^p.$$

Proof: For $p = 1$ the bounds follow from (1.27) and (1.26), recalling that $\int_M w \, d\mu_{g_0} = 0$.

Suppose the claim holds true for p. Then by (1.28) and induction hypothesis

$$\|\nabla w^{p+1}\|_{L^2} \leq A(p+1)\left(B A^p B^{p-1} p^p + A^p B^p p^p\right)$$

$$\leq A^{p+1} B^p (p+1)\, 2p^p \leq A^{p+1} B^p (p+1)^{p+1}$$

in view of Bernoulli's inequality

$$\left(\frac{p+1}{p}\right)^p = (1 + 1/p)^p \geq 2.$$

Moreover, by (1.26) similarly we can bound

$$\|w^{p+1}\|_{L^2} \leq C_1 \|\nabla w^{p+1}\|_{L^2} + \|w^{p+1}\|_{L^1}$$

$$\leq C_1 A^{p+1} B^{p} (p+1)^{p+1} + \|w^{p}\|_{L^2} \|w\|_{L^2}$$

$$\leq \frac{1}{2} A^{p+1} B^{p+1} (p+1)^{p+1} + A^{p} B^{p} p^{p} \cdot AB$$

$$\leq A^{p+1} B^{p+1} (p+1)^{p+1},$$

again using that $p^p \leq \frac{1}{2}(p+1)^p$. □

With Sterling's formula we can bound

$$p! \geq (p/e)^p, \quad p \in \mathbb{N}.$$

Thus, for any $0 < q < \frac{1}{ABe}$ we have

$$\int_M \left(e^{q|w|} - 1\right) d\mu_{g_0} = \sum_{p=1}^{\infty} \int_M \frac{(q|w|)^p}{p!} \, d\mu_{g_0}$$

$$\leq \sum_{p=1}^{\infty} \int_M \left(\frac{q e |w|}{p}\right)^p d\mu_{g_0} \leq \sum_{p=1}^{\infty} (q e AB)^p < \infty.$$

Choosing $q = \frac{1}{2ABe} > 0$ we conclude

$$\int_M u^{q} \, d\mu_{g_0} \int_M u^{-q} \, d\mu_{g_0} \leq \left(\int_M e^{q|w|} \, d\mu_{g_0}\right)^2 \leq C,$$

and the Theorem follows from Lemma 1.12.

□

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See [8] for further references on the Yamabe theorem and the Yamabe flow.