

# CALCULUS OF VARIATIONS

— Lecture notes by Michael Struwe —

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# Introduction

Greeks's search for "optimal form" (as in isoperimetric problems, e.g. problem of Dido) is part of our classical heritage.

Fermat's postulate that light rays travel along an optically shortest path is the first analytic expression of variational principle, allowing to deduce the laws of reflection and refraction.

In 1744 Euler published the first textbook on the Calculus of Variations, where he expresses his belief that "every effect in nature follows a maximum or minimum rule". At the same time Maupertuis published his famous "least action principle" with the same claim.

However, extremal properties of solutions to the equations considered were usually deduced "a posteriori" after solving the Euler-Lagrange equations. Conversely, in the 19th century a solution of the Dirichlet problem

$$\begin{aligned} -\Delta u &= f & \text{in } \Omega \subset \mathbb{R}^n \\ u &= 0 & \text{on } \partial\Omega, \end{aligned}$$

arising in Riemann's proof of his famous uniformization theorem ("Riemann mapping theorem") was obtained by invoking Dirichlet's principle assuming the existence of a minimizer of the associated energy

$$E(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} f u dx.$$

This latter reasoning was finally validated by Weierstrass, Hilbert, and others.

The search for closed geodesics led to the development of tools for finding also non-minimizing critical points of variational integrals in the work of Birkhoff and later Lusternik-Schnirelman and Morse.

But a rigorous analytic foundation for a general critical point theory only became available in the 1960's through the work of Palais-Smale, which then opened the way to a very fruitful and increasingly successful development.

# 1 The direct method

## 1.1 A general existence result

"Dirichlet's principle" can be deduced from the following result

**Theorem 1.1.1.** *Let  $X$  be a reflexive Banach space,  $M \subset X$  a weakly sequentially closed (w.s.c.) subset, and let  $E : M \rightarrow \mathbb{R} \cup \{\infty\}$ ,  $E \not\equiv \infty$ . Also suppose that  $E$  is coercive in the sense*

$$E(u) \rightarrow \infty \quad (\|u\|_X \rightarrow \infty, u \in M).$$

*Finally, assume that  $E$  is weakly sequentially lower semi-continuous (w.s.l.s.c.), that is,*

$$E(u) \leq \liminf_{k \rightarrow \infty} E(u_k)$$

*whenever  $(u_k) \subset M$  satisfies  $u_k \xrightarrow{w} u$  ( $k \rightarrow \infty$ ).*

*Then there exists  $u \in M$  such that*

$$E(u) = \inf_{v \in M} E(v) < \infty.$$

*Proof.* Let  $(u_k) \subset M$  be a minimizing sequence, satisfying

$$E(u_k) \rightarrow \inf_{v \in M} E(v) =: \alpha < \infty.$$

By coerciveness of  $E$ ,  $(u_k)$  is bounded. Since  $X$  is reflexive, a subsequence  $u_k \xrightarrow{w} u$  ( $k \rightarrow \infty, k \in \Lambda$ ) exists.

Since  $(u_k) \subset M$ , and since  $M$  is (w.s.c.),  $u \in M$ , and

$$\alpha \leq E(u) \leq \liminf_{k \rightarrow \infty} E(u_k) = \alpha$$

by (w.s.l.s.c.) of  $E$ . Thus,

$$\alpha = \inf_{v \in M} E(v) = E(u) > -\infty.$$

□

## 1.2 Dirichlet's principle.

Let  $\Omega \subset \mathbb{R}^n$  be smoothly bounded, and let  $f \in C^\infty(\overline{\Omega})$  (for simplicity).

We seek a (sufficiently smooth) function  $u : \Omega \rightarrow \mathbb{R}$  (say,  $u \in C^2(\overline{\Omega})$ ) such that

$$-\Delta u = - \sum_{i=1}^n \frac{\partial^2 u}{(\partial x_i)^2} = f \quad \text{in } \Omega, \tag{1.2.1}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{1.2.2}$$

Introduce the energy

$$E(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 dx - \int_{\Omega} f v dx, \quad v \in H_0^1(\Omega).$$

Note that we can expand

$$E(v + \varphi) = E(v) + \int_{\Omega} (\nabla v \cdot \nabla \varphi - f\varphi) dx + \frac{1}{2} \int_{\Omega} |\nabla \varphi|^2 dx \quad (1.2.3)$$

for any  $v, \varphi \in H_0^1(\Omega)$ , so that

$$\int_{\Omega} (\nabla v \cdot \nabla \varphi - f\varphi) dx = \langle dE(v), \varphi \rangle_{H^{-1} \times H_0^1(\Omega)}$$

is the Fréchet derivative of  $E$  at  $v \in H_0^1(\Omega)$ , evaluated on  $\varphi \in H_0^1(\Omega)$ .

**Proposition 1.2.1.** *Suppose that  $u \in C^2(\overline{\Omega}) \hookrightarrow H_0^1(\Omega)$  solves (1.2.1), (1.2.2). Then there holds*

$$E(u) = \min_{v \in H_0^1(\Omega)} E(v).$$

*Proof.* For any  $\varphi \in H_0^1(\Omega)$ , upon integrating by parts from (1.2.1) we have

$$\int_{\Omega} (\nabla u \cdot \nabla \varphi - f\varphi) dx = - \int_{\Omega} (\Delta u + f)\varphi dx = 0,$$

and (1.2.3) gives

$$E(u + \varphi) = E(u) + \frac{1}{2} \int_{\Omega} |\nabla \varphi|^2 dx \geq E(u),$$

as claimed.  $\square$

The argument to conversely solve (1.2.1), (1.2.2) by minimizing  $E$  is more subtle. In a first step we employ Theorem 1.1.1 to find a "weak solution" of (1.2.1), (1.2.2).

**Proposition 1.2.2.** *For any  $f \in L^2(\Omega)$  there exists  $u \in H_0^1(\Omega)$  such that*

$$E(u) = \min_{v \in H_0^1(\Omega)} E(v).$$

*Proof.* We verify the conditions in Theorem 1.1.1.

- i) The space  $X = H_0^1(\Omega)$  is a Hilbert space, hence also a reflexive Hilbert space;
- ii) The set  $M = X$  is (w.s.c.);
- iii) Since  $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ , by Hölder's inequality for any  $v \in H_0^1(\Omega)$  we have

$$\left| \int_{\Omega} f v dx \right| \leq \|f\|_{L^2} \|v\|_{L^2} \leq L \|f\|_{L^2} \|\nabla v\|_{L^2},$$

and  $E(v)$  is finite for every  $v \in H_0^1(\Omega)$ , where we use that by Poincaré's inequality

$$\|v\|_{L^2} \leq L \|\nabla v\|_{L^2}, \quad v \in H_0^1(\Omega),$$

with  $L > 0$  such that  $\Omega \subset ]0, L[ \times \mathbb{R}^{n-1}$ .

- iv) For  $v \in H_0^1(\Omega)$  then also  $\|v\|_{H_0^1} \leq (1 + L) \|\nabla v\|_{L^2}$  and

$$E(v) \geq \frac{1}{2} \|\nabla v\|_{L^2}^2 - L \|f\|_{L^2} \|\nabla v\|_{L^2} \rightarrow \infty$$

if  $\|v\|_{H_0^1} \rightarrow \infty$ , and  $E$  is coercive.

v) If  $u_k \xrightarrow{w} u$  in  $H_0^1(\Omega)$  as  $k \rightarrow \infty$ , by Rellich's theorem we may deduce that  $u_k \rightarrow u$  in  $L^2(\Omega)$  and

$$\int_{\Omega} u_k f \, dx \longrightarrow \int_{\Omega} u f \, dx \quad (k \rightarrow \infty).$$

Moreover, expanding with  $\varphi_k := u_k - u \xrightarrow{w} 0$  ( $k \rightarrow \infty$ ), we have

$$\begin{aligned} \int_{\Omega} |\nabla u_k|^2 \, dx &= \int_{\Omega} |\nabla(u + \varphi_k)|^2 \, dx \\ &= \int_{\Omega} |\nabla u|^2 \, dx + 2 \underbrace{\int_{\Omega} \nabla u \cdot \nabla \varphi_k \, dx}_{o(1) \rightarrow 0 \, (k \rightarrow \infty)} + \underbrace{\int_{\Omega} |\nabla \varphi_k|^2 \, dx}_{\geq 0} \\ &\geq \int_{\Omega} |\nabla u|^2 \, dx + o(1), \end{aligned}$$

and  $E$  is (w.s.l.s.c.).

Theorem 1.1.1 now yields the claim. □

**Proposition 1.2.3.** *A minimizer  $u \in H_0^1(\Omega)$  weakly solves (1.2.1), (1.2.2) in the sense that*

$$\forall \varphi \in H_0^1(\Omega) : \int_{\Omega} (\nabla u \cdot \nabla \varphi - f \varphi) \, dx = 0.$$

*Proof.* By (1.2.3) for any  $\varphi \in H_0^1(\Omega)$  and any  $\varepsilon \in \mathbb{R}$  we have

$$\begin{aligned} E(u) &\leq E(u + \varepsilon \varphi) \\ &= E(u) + \varepsilon \int_{\Omega} (\nabla u \cdot \nabla \varphi - f \varphi) \, dx + \frac{\varepsilon^2}{2} \int_{\Omega} |\nabla \varphi|^2 \, dx. \end{aligned}$$

Differentiating, we find

$$0 = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} E(u + \varepsilon \varphi) = \int_{\Omega} (\nabla u \cdot \nabla \varphi - f \varphi) \, dx,$$

as claimed. □

The second step in solving (1.2.1), (1.2.2) via Dirichlet's principle then is regularity theory, allowing to assert that the above weak solution actually solves (1.2.1), (1.2.2) classically.

## 1.3 Coercive nonlinearities

For smoothly bounded  $\Omega \subset \subset \mathbb{R}^n$ ,  $f \in C^\infty(\overline{\Omega})$  and any  $1 < p < \infty$  consider the problem

$$-\Delta u + u|u|^{p-2} = f \quad \text{in } \Omega, \tag{1.3.1}$$

$$u = 0 \quad \text{on } \partial\Omega \tag{1.3.2}$$

with associated energy

$$E(u) = \int_{\Omega} \left( \frac{|\nabla u|^2}{2} + \frac{|u|^p}{p} - fu \right) \, dx \quad u \in H_0^1(\Omega) \cap L^p(\Omega).$$

Recall that  $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$  for  $1 \leq q \leq 2^* := \frac{2n}{n-2}$ , if  $n \geq 3$ . Thus, if  $n \geq 3$ , when  $p > 2^*$  we may either regard  $E : H_0^1(\Omega) \cap L^q(\Omega) \rightarrow \mathbb{R}$  or view  $E$  as  $E : H_0^1(\Omega) \rightarrow \mathbb{R} \cup \{\infty\}$ . With the latter choice  $X = H_0^1(\Omega)$  we may invoke Theorem 1.1.1, since a large part of the proof of Proposition 1.2.2 still applies in the present context.

**Proposition 1.3.1.** *There exists  $u \in H_0^1(\Omega)$  such that*

$$E(u) = \inf_{v \in H_0^1(\Omega)} E(v),$$

and  $u \in C^{2,\gamma}(\overline{\Omega})$  for some  $0 \leq \gamma < 1$  is a classical solution of (1.3.1), (1.3.2).

*Proof.* We verify the conditions of Theorem 1.1.1.  $X = H_0^1(\Omega)$  is reflexive,  $M = X$  is (w.s.c.), and  $E : H_0^1(\Omega) \rightarrow \mathbb{R} \cup \{\infty\}$  is finite for  $v \in H_0^1(\Omega) \cap L^p(\Omega)$ , hence  $E \neq \infty$ .

Moreover, by step iv) of the proof of Proposition 1.2.2

$$E(v) \geq \frac{1}{2} \int_{\Omega} |\nabla v|^2 dx - \int_{\Omega} f v dx \rightarrow \infty \quad (\|v\|_{H_0^1} \rightarrow \infty),$$

and  $E$  is coercive.

Finally, if  $u_k \xrightarrow{w} u$  in  $H_0^1(\Omega)$  as  $k \rightarrow \infty$ , by Rellich's theorem a subsequence  $u_k \rightarrow u$  in  $L^2(\Omega)$  and almost everywhere as  $k \rightarrow \infty$ ,  $k \in \Lambda$ , and Fatou's lemma gives

$$\int_{\Omega} |u|^p dx \leq \liminf_{k \rightarrow \infty} \int_{\Omega} |u_k|^p dx.$$

Together with step v) of the proof of Proposition 1.2.2 this gives that  $E$  is (w.s.l.s.c.) on  $H_0^1(\Omega)$ .

By Theorem 1.1.1 there exists  $u \in H_0^1(\Omega)$  such that

$$E(u) = \inf_{v \in H_0^1(\Omega)} E(v) < \infty;$$

in particular,  $u \in L^p(\Omega)$  as well.

For any  $\varphi \in H_0^1(\Omega) \cap L^p(\Omega)$  then

$$0 = \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} E(u + \varepsilon\varphi) = \int_{\Omega} (\nabla u \nabla \varphi + (u|u|^{p-2} - f)\varphi) dx; \quad (1.3.3)$$

that is,  $u \in H_0^1(\Omega) \cap L^p(\Omega)$  solves (1.3.1) in the sense of distributions.

**Claim:**  $u \in L^\infty$ , and  $\|u\|_{L^\infty}^{p-1} \leq \|f\|_{L^\infty}$ .

*Proof.* Fix  $L \geq 0$  with  $L^{p-1} = \|f\|_{L^\infty}$ . Note

$$\varphi := (u - L)_+ = \max\{u - L, 0\} \in H_0^1(\Omega) \cap L^p(\Omega).$$

Thus,  $\varphi$  is admissible in (1.3.3) and with

$$\nabla \varphi(x) = \begin{cases} \nabla u(x) & \text{, for a.e. } x \in \Omega \text{ with } u > L \\ 0 & \text{, else} \end{cases}$$

we obtain

$$\begin{aligned} I &:= \int_{\Omega} (u|u|^{p-2} - f) (u - L)_+ dx \\ &\leq \int_{\Omega} (\nabla u \nabla \varphi + u|u|^{p-2} \varphi - f \varphi) dx = 0. \end{aligned}$$

But  $I > 0$ , if we assume, that  $u(x) > L$  in a set of positive measure; so  $u(x) \leq L$  almost everywhere. Similarly,  $u(x) \geq L$  a.e., and thus we obtain the claim.  $\square$



In view of the claim now

$$-\Delta u = f - u|u|^{p-2} \in L^\infty;$$

hence,  $u \in W^{2,q}(\Omega)$  for any  $q < \infty$ , and  $u \in C^{1,\alpha}(\overline{\Omega})$  for every  $0 < \alpha < 1$  by the Calderon-Zygmund estimate and Sobolev's embedding.

Since  $p > 2$ , then  $u|u|^{p-2} \in C^\alpha$  for every  $0 < \alpha < 1^1$ , and Schauder estimates give  $u \in C^{2,\alpha}(\overline{\Omega})$ .  $\square$

## 1.4 A non-coercive model problem, constraints

For smoothly bounded  $\Omega \subset \mathbb{R}^n$ ,  $2 < p < \infty$ , now consider the problem

$$-\Delta u = u|u|^{p-2} \quad \text{in } \Omega, \quad (1.4.1)$$

$$u = 0 \quad \text{on } \partial\Omega \quad (1.4.2)$$

with associated energy

$$E(u) = \int_{\Omega} \left( \frac{|\nabla u|^2}{2} - \frac{|u|^p}{2} \right) dx, \quad u \in H_0^1 \cap L^p(\Omega).$$

Note that  $u_0 \equiv 0$  is a non-degenerate solution of (1.4.1), (1.4.2) in the sense that the linearized operator  $L = -\Delta - g'(0) = -\Delta$ , where

$$g(u) = u|u|^{p-2}, \quad u \in \mathbb{R},$$

has positive spectrum.

However,  $u_0$  does not achieve  $\inf(E)$ ; in fact,

$$E(su) = \frac{s^2}{2} \|\nabla u\|_{L^2}^2 - \frac{s^p}{2} \|u\|_{L^p}^p \xrightarrow{(s \rightarrow \infty)} -\infty \quad (1.4.3)$$

for every  $u \in H_0^1 \cap L^p(\Omega) \setminus \{0\}$ , and  $\inf(E) = -\infty$ .

There are now two cases. On the one hand, if  $n \geq 3$  and  $p > 2^*$ , there holds

$$\inf_{u \in C_c^\infty(\Omega)} \frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^p}^p} = 0.$$

Consequently,  $u_0 \equiv 0$  is not a local  $E$ -minimizer in  $H_0^1(\Omega)$  and it is unlikely that a non-trivial solution exists.

Note that by (1.4.3)  $u_0 \equiv 0$  is a local minimizer of  $E$  on any line  $\{su : s \in \mathbb{R}\}$ ,  $u \in H_0^1 \cap L^p(\Omega)$ , and achieves a maximum when  $s = s_u$ , where

$$\langle dE(s_u u), u \rangle = s_u \|\nabla u\|_{L^2}^2 - s_u^{p-1} \|u\|_{L^p}^p = 0,$$

with

$$E(s_u u) = s_u^2 \left( \frac{1}{2} - \frac{1}{p} \right) \|\nabla u\|_{L^2}^2 = \left( \frac{1}{2} - \frac{1}{p} \right) \left( \frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^p}^p} \right)^{\frac{2p}{p-2}}.$$

<sup>1</sup>

$$\frac{|(u|u|^{p-2})(x) - (u|u|^{p-2})(y)|}{|x - y|^\gamma} \leq \underbrace{\frac{|(u|u|^{p-2})(x) - (u|u|^{p-2})(y)|}{|u(x) - u(y)|^\gamma}}_{\leq C < \infty, \gamma = p-1} \cdot \underbrace{\left( \frac{|u(x) - u(y)|}{|x - y|} \right)^\gamma}_{\leq \|\nabla u\|_{L^\infty}}.$$

On the other hand, if  $p \leq 2^* := \frac{2n}{n-2}$  for  $n \geq 3$ , by Sobolev's embedding  $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$  there is  $C_p > 0$  with

$$\|u\|_{L^p} \leq C_p \|\nabla u\|_{L^2}, \quad u \in H_0^1(\Omega),$$

and

$$E(s_u u) \geq \left(\frac{1}{2} - \frac{1}{p}\right) C_p^{-\frac{2p}{p-2}} =: \beta_p > 0.$$

Moreover,  $u_0 \equiv 0$  is a local minimizer of  $E$  in the  $H_0^1$ -topology.

We may then hope to find a non-trivial solution  $u \neq 0$  of (1.4.1), (1.4.2) along a ray  $\{su : s \in \mathbb{R}\}$  where  $u \in H_0^1(\Omega)$  minimizes the Sobolev quotient  $\|\nabla u\|_{L^2} / \|u\|_{L^p}$ .

**Theorem 1.4.1.** *Let  $\Omega \subset \subset \mathbb{R}^n$ ,  $2 < p < 2^* = \frac{2n}{n-2}$  if  $n \geq 3$ ,  $2 < p < \infty$ , if  $n \leq 2$ . Then there exists  $0 < u \in C^2(\bar{\Omega})$  solving (1.4.1), (1.4.2).*

For the proof we apply Theorem 1.1.1 with  $X = H_0^1(\Omega)$ ,  $D(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx$ , and

$$M := \{u \in H_0^1(\Omega) : \|u\|_{L^p} = 1\}.$$

Note that  $M$  is (w.s.c.) by Rellich's theorem, and  $D$  is (w.s.l.s.c.) and coercive on  $H_0^1(\Omega)$ . By Theorem 1.1.1 thus exists  $u \in M$  with

$$D(u) = \inf_{v \in M} D(v).$$

Moreover, since  $\|\nabla |u|\|_{L^2} = \|\nabla u\|_{L^2}$  and since  $u$  and  $|u|$  have identical  $L^p$ -norm, we may assume that  $u \geq 0$ .

**Claim 1.** The function  $\bar{u} = \alpha u$  weakly solves (1.4.1), (1.4.2), where

$$\alpha = (2D(u))^{\frac{1}{p-2}} > 0.$$

*Proof.* Given  $0 \neq \varphi \in H_0^1(\Omega)$ , fix  $\varepsilon_0 = \frac{1}{2\|\varphi\|_{L^p}} > 0$ . Then for  $|\varepsilon| < \varepsilon_0$  we have

$$\|u + \varepsilon \varphi\|_{L^p} > \frac{1}{2},$$

and

$$u_\varepsilon := \frac{u + \varepsilon \varphi}{\|u + \varepsilon \varphi\|_{L^p}} \in M, \quad \varepsilon < \varepsilon_0.$$

Note that  $\varepsilon \mapsto u_\varepsilon$  is of class  $C^1$  with

$$\begin{aligned} \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} u_\varepsilon &= \varphi - u \frac{1}{p} \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \left( \int_{\Omega} |u + \varepsilon \varphi|^p dx \right) \\ &= \varphi - u \int_{\Omega} u |u|^{p-2} \varphi dx. \end{aligned}$$

By minimality of  $D(u)$  then

$$\begin{aligned} 0 &= \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} D(u_\varepsilon) = \int_{\Omega} \nabla u \cdot \nabla \left( \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} u_\varepsilon \right) dx \\ &= \int_{\Omega} \nabla u \cdot \nabla \varphi dx - \underbrace{\|\nabla u\|_{L^2}^2}_{=\alpha^{p-2}} \int_{\Omega} u |u|^{p-2} \varphi dx; \end{aligned}$$

that is,  $u$  weakly solves

$$-\Delta u = \|\nabla u\|_{L^2}^2 u |u|^{p-2} = u |\bar{u}|^{p-2} \text{ in } \Omega,$$

and our claim follows after multiplying with  $\alpha$ . □

For ease of notation, in the following we write  $u$  instead of  $\bar{u}$ .

**Claim 2.**  $u \in C^2(\bar{\Omega})$  classically solves (1.4.1), (1.4.2); moreover,  $u > 0$  in  $\Omega$ .

*Proof.* Set  $a := |u|^{p-2} \in L^{\frac{2^*}{p-2}}(\Omega)$ ,  $n \geq 3$ , where

$$s := \frac{2^*}{p-2} > \frac{2^*}{2^*-2} = \frac{n}{2}.$$

By Claim 1 then  $u \in H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$  weakly solves

$$\begin{aligned} -\Delta u &= au & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega. \end{aligned}$$

Suppose  $u \in L^{q_1}(\Omega)$  for some  $q_1 \geq 2^*$ . Then  $au \in L^r(\Omega)$ , where

$$\frac{1}{r} = \frac{1}{s} + \frac{1}{q_1},$$

and the Calderon-Zygmund inequality together with Sobolev's embedding yields  $u \in W^{2,r}(\Omega) \hookrightarrow L^{q_2}(\Omega)$ , with

$$\frac{1}{q_2} = \frac{2}{r} - \frac{2}{n} = \frac{1}{q_1} - \underbrace{\left(\frac{2}{n} - \frac{1}{s}\right)}_{=: \delta > 0},$$

or for any  $q_2 < \infty$ , if  $\frac{1}{q_1} \leq \delta$ .

Thus, either  $q_1 \geq \frac{1}{\delta}$ , or

$$q_2 = \frac{1}{\frac{1}{q_1} - \delta} = \frac{q_1}{1 - \delta q_1},$$

and after a finite number of iterations starting with  $q_1 = 2^*$  we will have achieved that  $u \in L^q(\Omega)$  for any  $u \in L^q(\Omega)$  for any  $q < \infty$  as for  $n \leq 2$ ; hence  $-\Delta u = au \in L^t(\Omega)$  for any  $t < s$ . Choosing  $\frac{n}{2} < t < s$ , we arrive at

$$u \in W^{2,t}(\Omega) \hookrightarrow C^{0,\alpha}(\bar{\Omega})$$

for some  $0 < \alpha < 1$ , and  $-\Delta u = u|u|^{p-2} \in C^{0,\alpha}(\bar{\Omega})$ . Schauder theory then yields that  $u \in C^{2,\alpha}(\bar{\Omega})$ , proving the first claim.

Since in addition  $u \geq 0$ ,  $-\Delta u = u|u|^{p-2} \geq 0$ , strict positivity of  $u$  follows from the strong maximum principle.  $\square$

Is the restriction  $p < 2^*$  needed?

**Theorem 1.4.2.** (Pohožaev) Let  $\Omega \subset\subset \mathbb{R}^n$ ,  $n \geq 3$ , be strictly star-shaped with respect to  $x_0 = 0$  in the sense that

$$\nu(x) \cdot x > 0, \quad x \in \partial\Omega,$$

where  $\nu(x)$  is the outward unit normal to  $\Omega$  at  $x \in \partial\Omega$ .

Then for  $p \geq 2^*$  problem (1.4.1), (1.4.2) only admits  $u_0 \equiv 0$  as classical solution.

*Proof.* Multiplying (1.4.1) with  $u$  and  $x \cdot \nabla u$  we obtain the identities

$$0 = (\Delta u + u|u|^{p-2}) u = \operatorname{div}(u \nabla u) - |\nabla u|^2 + |u|^p$$

and

$$\begin{aligned} 0 &= (\Delta u + u|u|^{p-2}) x \cdot \nabla u \\ &= \underbrace{\operatorname{div}(\nabla u x \cdot \nabla u)}_{=\sum_{i,j} \partial_i (\partial_i u x^j \partial_j u)} - |\nabla u|^2 - x \cdot \nabla \left( \frac{|\nabla u|^2}{2} - \frac{|u|^p}{2} \right) \\ &= \operatorname{div}(\nabla u x \cdot \nabla u) - x \left( \frac{|\nabla u|^2}{2} - \frac{|u|^p}{2} \right) + \frac{n-2}{2} |\nabla u|^2 - \frac{n}{p} |u|^p \end{aligned}$$

respectively.

Thus, we find

$$\begin{aligned} 0 &= (\Delta u + u|u|^{p-2}) \left( x \cdot \nabla u + \frac{n-1}{2} u \right) \\ &= \operatorname{div} \left( \frac{n-2}{2} u \nabla u + \nabla u x \cdot \nabla u - x \left( \frac{|\nabla u|^2}{2} - \frac{|u|^p}{2} \right) \right) \\ &\quad + \left( \frac{n-2}{2} - \frac{n}{p} \right) |u|^p. \end{aligned}$$

Integrating, and using that (1.4.2) implies that

$$\nabla u = \nu \nu \cdot \nabla u \quad \text{on } \partial\Omega,$$

we obtain

$$0 = n \left( \frac{1}{2^*} - \frac{1}{p} \right) \int_{\Omega} |u|^p dx + \int_{\partial\Omega} \underbrace{\nu \cdot x}_{>0} \frac{|\nabla u|^2}{2} dx,$$

where  $\nu \cdot x > 0$  by assumption.

If  $p > 2^*$ , then  $\|u\|_{L^p}^p = 0$ , and  $u \equiv 0$ . If  $p = 2^*$ , we find  $\nabla u \equiv 0$  on  $\partial\Omega$  and then  $u \equiv 0$  by Heinz' theorem on unique continuation [Hei55].  $\square$

## 1.5 Concentration - Compactness

P.-L. Lions' "Concentration-compactness principle" [Lio84a], [Lio84b], [Lio85a], [Lio85b] may be invoked to (partially) understand what happens in the case of problem (1.4.1), (1.4.2) when  $p = 2^*$ . We start with the locally compact setting.

### 1.5.1 The locally compact case

As a model problem consider

$$-\Delta u + au = a|u|^{p-2} \quad \text{in } \mathbb{R}^n, \quad (1.5.1)$$

$$u(x) \rightarrow 0 \quad (|x| \rightarrow \infty), \quad (1.5.2)$$

where  $2 < p < 2^* = \frac{2n}{n-2}$  if  $n \geq 3$ ,  $2 < p < \infty$  for  $n \leq 2$ , and with  $a \in C^0(\mathbb{R}^n)$  satisfying

$$a(x) \rightarrow a_{\infty} > 0 \quad (|x| \rightarrow \infty). \quad (1.5.3)$$

Following the approach to (1.4.1), (1.4.2) in Section 1.4 we seek to find a solution  $u > 0$  to (1.5.1), (1.5.2) by minimizing the energy

$$E(u) = \int_{\mathbb{R}^n} (|\nabla u|^2 + au^2) dx$$

in

$$M := \{u \in H^1(\mathbb{R}^n) : \|u\|_{L^p} = 1\}.$$

Recall that for  $n \geq 3$  we have

$$H^1(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n), L^{2^*}(\mathbb{R}^n)$$

and hence  $H^1(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$  for every  $2 < p < 2^*$  by interpolation, using Hölder's inequality.

Likewise, for  $n \leq 2$  we have  $H^1(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$  for every  $2 < p < \infty$ .

**Lemma 1.5.1.**  *$E$  is (w.s.l.s.c.) and coercive with respect to  $H^1(\mathbb{R}^n)$ .*

*Proof.* Let

$$\Omega := \left\{ x \in \mathbb{R}^n : a(x) < \frac{1}{2}a_\infty \right\}.$$

Observe that by (1.5.3) the set  $\Omega$  is bounded. For  $u \in M$  then

$$\begin{aligned} E(u) &\geq \frac{1}{2} \|\nabla u\|_{L^2}^2 + \frac{1}{4} a_\infty \|u\|_{L^2}^2 \\ &\quad + \frac{1}{2} \int_{\Omega} \left( a - \frac{1}{2} a_\infty \right) \underbrace{u^2}_{\in L^{p/2}} dx \\ &\geq \min \left\{ \frac{1}{2}, \frac{a_\infty}{4} \right\} \|u\|_{H^1}^2 - C \mu(\Omega)^{\frac{p-2}{p}} \underbrace{\|u\|_{L^p}^2}_{=1} \\ &\rightarrow \infty \quad (\|u\|_{H^1} \rightarrow \infty, u \in M), \end{aligned}$$

and  $E$  is coercive.

Moreover, splitting

$$\begin{aligned} E(u) &= \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^n \setminus \Omega} au^2 dx + \frac{1}{4} \int_{\Omega} a_\infty u^2 \\ &\quad + \frac{1}{2} \int_{\Omega} \left( a - \frac{1}{2} a_\infty \right) u^2 dx \end{aligned}$$

and observing that

$$\langle u, v \rangle_a := \int_{\mathbb{R}^n} \nabla u \nabla v dx + \int_{\mathbb{R}^n \setminus \Omega} auv dx + \int_{\Omega} \frac{a_\infty}{2} uv dx$$

defines an equivalent inner product on  $H^1(\mathbb{R}^n)$ . For  $u_k \xrightarrow{w} u$  in  $H^1(\mathbb{R}^n)$  we have

$$\begin{aligned} \|u_k\|_a^2 &= \langle u_k, u_k \rangle_a \\ &= \underbrace{\langle u_k - u, u_k - u \rangle_a}_{\geq 0} + 2 \underbrace{\langle u, u_k - u \rangle_a}_{\rightarrow 0 \text{ (} k \rightarrow \infty)} + \langle u, u \rangle_a \\ &\geq \|u\|_a^2 + o(1), \end{aligned}$$

with  $o(1) \rightarrow 0$  ( $k \rightarrow \infty$ ). Since by Rellich's theorem we may assume that  $u_k \rightarrow u$  in  $L^2(\Omega)$ , we also have

$$\int_{\Omega} \left( a - \frac{1}{2} a_{\infty} \right) u_k^2 dx \xrightarrow{(k \rightarrow \infty)} \int_{\Omega} \left( a - \frac{1}{2} a_{\infty} \right) u^2 dx,$$

and we conclude that

$$\begin{aligned} E(u) &= \frac{1}{2} \|u\|_a^2 + \frac{1}{2} \int_{\Omega} \left( a - \frac{1}{2} a_{\infty} \right) u^2 dx \\ &\leq \liminf_{k \rightarrow \infty} \left( \frac{1}{2} \|u_k\|_a^2 + \frac{1}{2} \int_{\Omega} \left( a - \frac{1}{2} a_{\infty} \right) u_k^2 dx \right) \\ &= \liminf_{k \rightarrow \infty} E(u_k). \end{aligned}$$

□

However, the set  $M$  is not (w.s.c.). To see this, for  $u \in M, x_0 \in \mathbb{R}^n$  let

$$u_{x_0}(x) = u(x + x_0), \quad x \in \mathbb{R}^n.$$

Then  $u_{x_0} \in M$  for every  $x_0 \in \mathbb{R}^n$ , but  $u_{x_0} \xrightarrow{w} 0 \notin M$  as  $|x_0| \rightarrow \infty$ . Moreover, we have after substituting  $y = x + x_0$

$$\begin{aligned} E(u_{x_0}) &= \frac{1}{2} \int_{\mathbb{R}^n} (|\nabla u|^2 + a(x - x_0)u^2) dx \\ &\xrightarrow{|x_0| \rightarrow \infty} \frac{1}{2} \int_{\mathbb{R}^n} (|\nabla u|^2 + a_{\infty}u^2) dx =: E_{\infty}(u). \end{aligned} \tag{1.5.4}$$

We call  $E_{\infty}$  the "functional at infinity". Define

$$I := \inf_{u \in M} E(u), \quad I_{\infty} := \inf_{u \in M} E_{\infty}(u).$$

**Theorem 1.5.1.** (P.-L. Lions)

- i) There holds  $I \leq I_{\infty}$ .
- ii) The condition  $I < I_{\infty}$  is necessary and sufficient for the relative compactness of all minimizing sequences  $(u_k)_{k \in \mathbb{N}} \subset M$  for  $E$ .
- iii) In particular, if  $I < I_{\infty}$  there exists a solution  $0 < u \in H^1(\mathbb{R}^n)$  of (1.5.1).

**Preparations** for the proof: For  $0 \leq \lambda \leq 1$  let

$$M_{\lambda} = \{u \in H^1(\mathbb{R}^n) : \|u\|_{L^p}^p = \lambda\}$$

and set

$$I_{\lambda} = \inf_{u \in M_{\lambda}} E(u), \quad E_{\lambda, \infty} = \inf_{u \in M_{\lambda}} E_{\infty}(u).$$

Note that by homogeneity there holds

$$E(u) = \lambda^{2/p} E\left(\frac{u}{\|u\|_{L^p}}\right), \quad u \in M_{\lambda},$$

and likewise for  $E_{\infty}$ ,  $0 \leq \lambda \leq 1$ . Hence

$$I_{\lambda} = \lambda^{2/p} I, \quad I_{\lambda, \infty} = \lambda^{2/p} I_{\infty}, \quad 0 \leq \lambda \leq 1.$$

**Lemma 1.5.2.** *The condition  $I < I_\lambda$  is equivalent to the condition*

$$\forall 0 \leq \lambda < 1 : I < I_\lambda + I_{1-\lambda, \infty}. \quad (1.5.5)$$

*Proof.* Suppose (1.5.5) holds. Set  $\lambda = 0$  to obtain  $I < I_{1, \infty} = I_\infty$ .

Conversely, suppose  $I < I_\infty$  and fix  $0 \leq \lambda < 1$ . Note that  $I_\infty > 0$ . If  $I \leq 0$  then

$$I - I_\lambda = \underbrace{(1 - \lambda^{2/p})}_{>0} I \leq 0 < (1 - \lambda)^{2/p} I_\infty = I_{1-\lambda, \infty}.$$

If  $I > 0$ , observing that the condition  $p > 2$  implies

$$1 - \lambda \leq (1 - \lambda)^{2/p}, \quad \lambda \leq \lambda^{2/p},$$

we have

$$\begin{aligned} I - I_\lambda &= (1 - \lambda^{2/\lambda}) I \leq (1 - \lambda)^{2/\lambda} I \\ &< (1 - \lambda)^{2/p} I_\infty = I_{1-\lambda, \infty}, \end{aligned}$$

as claimed. □

*Proof of Theorem 1.5.1.* i) We show  $I \leq I_\infty$ . Suppose by contradiction that  $I > I_\infty$ .

Choose  $u \in M$  with

$$E_\infty(u) < I.$$

But then by (1.5.4) we have

$$I \leq E(u_{x_0}) \rightarrow E_\infty(u) < I \quad (|x_0| \rightarrow \infty).$$

ii) **Necessity.** Suppose  $I = I_\infty$ , and let  $(u_k) \subset M$  with

$$E_\infty(u_k) \rightarrow I_\infty = I \quad (k \rightarrow \infty).$$

By (1.5.3) there is  $(x_k) \subset \mathbb{R}^n$  such that

$$|E(u_{k, x_k}) - E_\infty(u_k)| \leq \frac{1}{k}, \quad \forall k;$$

in addition, we may assume that

$$v_k := u_{k, x_k} \xrightarrow{w} 0 \quad \text{in } H^1(\mathbb{R}^n).$$

Then

$$E(v_k) = E(u_{k, x_k}) \leq E_\infty(u_k) + \frac{1}{k} \xrightarrow{(k \rightarrow \infty)} I_\infty = I,$$

but  $(v_k)$  is not relatively compact.

**Sufficiency.** Suppose  $I < I_\infty$ , and let  $(u_k) \subset M$  be an arbitrary minimizing sequence for  $E$ . Since  $E$  is coercive on  $M$ ,  $(u_k)$  is bounded, and we may assume that

$$u_k \xrightarrow{w} u \quad \text{in } H^1(\mathbb{R}^n) \text{ as } k \rightarrow \infty$$

for some  $u \in H^1(\mathbb{R}^n)$ , and pointwise a.e.

**Lemma 1.5.3.** *Let  $u_k \xrightarrow{w} u$  in  $H^1(\mathbb{R}^n)$  and a.e. With error  $o(1) \rightarrow 0$  ( $k \rightarrow \infty$ ) then we have*

$$1 = \|u_k\|_{L^p}^p = \|u_k - u\|_{L^p}^p + \|u\|_{L^p}^p + o(1).$$

*Proof.* Since  $u_k \rightarrow 0$  a.e., for any  $0 \leq t \leq 1$  we also have

$$f_k(t, \cdot) := |u_k - tu|^{p-2}(u_k - tu)u \xrightarrow{(k \rightarrow \infty)} (1-t)^{p-1}|u|^p \quad \text{a.e.}$$

Using pointwise convergence a.e. we can bound

$$|f_k(t, \cdot)| \leq (|u_k| + |u|)^{p-1}|u| \quad (1.5.6)$$

uniformly in  $0 \leq t \leq 1$ , by dominated convergence then a.e. we have

$$\begin{aligned} |u_k|^p - |u_k - u|^p &= - \int_0^1 \frac{d}{dt} |u_k - tu|^p dt \\ &= p \int_0^1 |u_k - tu|^{p-2}(u_k - tu)u dt \\ &= p \int_0^1 f_k(t, \cdot) dt \xrightarrow{(k \rightarrow \infty)} p \int_0^1 (1-t)^{p-1}|u|^p dt \\ &= |u|^p. \end{aligned}$$

But by (1.5.6) the functions  $\int_0^1 f_k(t, \cdot) dt$  are also equi-integrable. Indeed for any measurable  $\Omega \subset \mathbb{R}^n$

$$\begin{aligned} \int_{\Omega} \left| \int_0^1 f_k(t, x) dt \right| dx &\leq \int_{\Omega} (|u_k| + |u|)^{p-1} |u| dx \\ &\leq C (\|u_k\|_{L^p}^{p-1} + \|u\|_{L^p}^{p-1}) \left( \int_{\Omega} |u|^p dx \right)^{\frac{1}{p}} < \varepsilon, \end{aligned}$$

if  $\mathcal{L}^n(\Omega) < \delta(\varepsilon)$  or if  $\Omega \subset \mathbb{R}^n \setminus B_R(0)$ ,  $R \geq R(\varepsilon)$ . The claim then follows by Vitali's theorem.  $\square$

**Lemma 1.5.4.** *Let  $u_k \xrightarrow{w} u$  in  $H^1(\mathbb{R}^n)$ . With error  $o(1) \rightarrow 0$  we have*

$$E(u_k) = E(u) + E_{\infty}(u_k - u) + o(1).$$

*Proof.* Note the identities

$$\begin{aligned} \|\nabla u_k\|_{L^2}^2 &= \|\nabla((u_k - u) + u)\|_{L^2}^2 \\ &= \|\nabla(u_k - u)\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + 2 \underbrace{\int_{\mathbb{R}^n} \nabla(u_k - u) \nabla u dx}_{\rightarrow 0 \ (k \rightarrow \infty)} \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}^n} a u_k^2 dx &= \int_{\mathbb{R}^n} a((u_k - u) + u)^2 dx \\ &= \int_{\mathbb{R}^n} a(u_k - u)^2 dx + \int_{\mathbb{R}^n} a u^2 dx + 2 \underbrace{\int_{\mathbb{R}^n} \underbrace{a u}_{\in L^2} (u_k - u) dx}_{\rightarrow 0 \ (k \rightarrow \infty)}, \end{aligned}$$



respectively. Set  $v_k = u_k - u$ , and for  $\varepsilon > 0$  let

$$\Omega_\varepsilon = \{x \in \mathbb{R}^n : |a(x) - a_\infty| > \varepsilon\} \subset \subset \mathbb{R}^n.$$

By Rellich's theorem we have that  $v_k \rightarrow 0$  in  $L^2(\Omega_\varepsilon)$  for every  $\varepsilon > 0$  as  $k \rightarrow \infty$ . Hence

$$\begin{aligned} \left| \int_{\mathbb{R}^n} a v_k^2 dx - \int_{\mathbb{R}^n} a_\infty v_k^2 dx \right| &= \left| \int_{\mathbb{R}^n} (a - a_\infty) v_k^2 dx \right| \\ &\leq \left| \int_{\mathbb{R}^n \setminus \Omega_\varepsilon} (a - a_\infty) v_k^2 dx \right| + o(1) \\ &\leq \varepsilon \|v_k\|_{L^2}^2 + o(1). \end{aligned}$$

Letting  $\varepsilon = \varepsilon_k \downarrow 0$  suitably, we obtain the claim.  $\square$

Returning to the proof of sufficiency of the condition  $I < I_\infty$ , resp. (1.5.5), for relative compactness of  $(u_k)$ , from Lemmas 1.5.3 and 1.5.4 we conclude that with  $o(1) \xrightarrow{(k \rightarrow \infty)} 0$

$$\begin{aligned} I &= E(u_k) + o(1) = E(u) + E_\infty(u_k - u) + o(1) \\ &\geq I_\lambda + I_\infty(u_k - u) + o(1), \end{aligned}$$

where

$$\lambda = \|u\|_{L^p}^p, \quad \|u_k - u\|_{L^p}^p \rightarrow 1 - \lambda \quad (k \rightarrow \infty)$$

for some  $0 \leq \lambda \leq 1$ .

Suppose  $\lambda < 1$ . Replacing  $u_k - u$  by

$$w_k = \frac{u_k - u}{\|u_k - u\|_{L^p}} (1 - \lambda)^{\frac{1}{p}} \in M_{1-\lambda},$$

we have

$$E_\infty(w_k) = E_\infty(u_k - u) + o(1)$$

and hence

$$I \geq I_\lambda + E_\infty(w_k) + o(1) \geq I_\lambda + I_{1-\lambda, \infty} + o(1),$$

contradicting (1.5.5) for large  $k$ .

Hence  $\lambda = 1$ , and  $u \in M = M_1$ . But then

$$I \leq E(u) \leq \lim_{k \rightarrow \infty} E(u_k) = E(u) + \lim_{k \rightarrow \infty} E_\infty(u_k - u) = I$$

shows that  $E(u) = I$  and  $\|u_k - u\|_{H^1}^2 \leq C E_\infty(u_k - u) \xrightarrow{(k \rightarrow \infty)} 0$ . Hence  $u_k \rightarrow u$  in  $H^1(\mathbb{R}^n)$ , as claimed.

- iii) Assume  $I < I_\infty$  and let  $(u_k) \subset M$  be a minimizing sequence. By part ii) we may assume that  $u_k \xrightarrow{(k \rightarrow \infty)} u$  in  $H^1(\mathbb{R}^n)$ , where  $u \in M$ , and

$$E(u) = \lim_{k \rightarrow \infty} E(u_k) = \inf_{v \in M} E(v).$$

Thus, following the argument for the proof of Theorem 1.4.1,  $u$  weakly solves (1.5.1) and achieves (1.5.2) in the sense that  $u \in L^2(\mathbb{R}^n)$ . Since  $p < 2^*$ , also the argument

for regularity may be carried over from Theorem 1.4.1, and  $u \in C^2(\mathbb{R}^n)$  solves (1.5.1) classically.

Observing that  $E(v) = E(|v|)$ ,  $v \in M$ , we may again assume that  $u \geq 0$ , and strict positivity follows from the maximum principle, applied on any ball  $B_R(0)$ ,  $R > 0$ .  $\square$

**Remark 1.5.1.** i) If  $a = a(|x|)$ , we may restrict our attention to functions  $u = u(|x|) \in H^1(\mathbb{R}^n)$ . If  $n \geq 3$ , then for  $2 < p < 2^*$  the set

$$M_{rad} = \{u = u(|x|) \in H^1(\mathbb{R}^n) : \|u\|_{L^p} = 1\}$$

is (w.s.c.). Hence there always exists a (positive) minimizer of  $E$  in  $M_{rad}$ , corresponding to a solution  $0 < u = u(|x|) \in H^1(\mathbb{R}^n)$  of (1.5.1), (1.5.2). In particular, for  $a \equiv a_\infty = 1$  on  $\mathbb{R}^3$ ,  $p = 4$ , there exists a ground state solution  $u > 0$  of the static nonlinear Schrödinger equation

$$-\Delta u + u = u^3 \quad \text{on } \mathbb{R}^3.$$

*Proof.* For  $u = u(|x|) \in C_c^\infty(\mathbb{R}^n)$  we have

$$\begin{aligned} |u(x)| &= \left| \int_{|x|}^\infty u_r(s) ds \right| \leq \left( \int_{|x|}^\infty |u_r(s)|^2 s^{n-1} ds \cdot \int_{|x|}^\infty \frac{ds}{s^{n-1}} \right)^{\frac{1}{2}} \\ &\leq C \|\nabla u\|_{L^2} |x|^{\frac{2-n}{2}}, \quad \forall x \neq 0. \end{aligned}$$

Hence,  $H_{rad}^1(\mathbb{R}^n) = \{u \in H^1(\mathbb{R}^n) : u = u(|x|)\} \hookrightarrow L^p(\mathbb{R}^n)$  compactly for any  $p \in ]2, 2^*[$ . Indeed, let  $u_k \xrightarrow{w} u$  in  $H_{rad}^1(\mathbb{R}^n)$ , and let  $\varepsilon > 0$  be given. Then

$$\begin{aligned} \|u_k - u\|_{L^p(\mathbb{R}^n)}^p &= \int_{B_R(0)} |u_k - u|^p dx + \int_{\mathbb{R}^n \setminus B_R(0)} |u_k - u|^p dx \\ &\leq o(1) + \|u_k - u\|_{L^2(\mathbb{R}^n)}^2 \|u_k - u\|_{L^\infty(\mathbb{R}^n \setminus B_R(0))}^{p-2} \\ &\leq CR^{-\frac{(n-2)(p-2)}{2}} + o(1) \leq \varepsilon, \end{aligned}$$

if  $R \geq R_0(\varepsilon)$ ,  $k \geq k_0(R, \varepsilon)$ .  $\square$

ii) Note that for  $a \equiv 1$  we have translation invariance of  $I$  and  $I_\infty$ . Translation invariance, however, is removed in the radial setting, and Theorem 1.5.1 does not apply.

### 1.5.2 The locally non-compact case

As an illustration of the additional difficulties arising in the limit case  $p = 2^* = \frac{2n}{n-2}$  when  $n \geq 3$  consider the following.

**Example.** Let  $0 \neq u \in H_0^1(B_1(0))$ ,  $n \geq 3$ . For  $k \in \mathbb{N}$  scale

$$u_k(x) = k^{\frac{n-2}{2}} u(kx), \quad x \in B_1(0),$$

where we extend  $u(x) = 0$  for  $|x| \geq 1$ . Note that  $u_k \in H_0^1(B_1(0))$  with

$$\begin{aligned} \|\nabla u\|_{L^2}^2 &= \|\nabla u_k\|_{L^2}^2 =: A, \\ \|u\|_{L^{2^*}}^{2^*} &= \|u_k\|_{L^{2^*}}^{2^*} =: B, \end{aligned}$$

for all  $k \in \mathbb{N}$ , and  $u_k \xrightarrow{w} 0$  in  $H_0^1(B_1(0))$  with

$$|\nabla u_k|^2 dx \xrightarrow{w^*} A \delta_{\{x=0\}}, \quad |u_k|^{2^*} dx \xrightarrow{w^*} B \delta_{\{x=0\}}$$

weakly in the sense of measures. Moreover, Sobolev's inequality

$$\mathcal{S} \|u\|_{L^{2^*}}^2 \leq \|\nabla u\|_{L^2}^2, \quad u \in C_c^\infty(\mathbb{R}^n)$$

gives the bound  $\mathcal{S} B^{2/2^*} \leq A$ .

**Sobolev's constant.** The number

$$\mathcal{S} = \inf_{0 \neq u \in H_0^1(\Omega)} \frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^{2^*}}^2}$$

is the best constant for the embedding  $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$  for any open  $\Omega \subset \mathbb{R}^n$ . By scaling as above, we see that  $\mathcal{S}$  is independent of the domain  $\Omega$ .

Similar to Section 1.5.1 we now again let  $\Omega = \mathbb{R}^n$ . Recall the definition

$$\dot{H}^1(\mathbb{R}^n) = \text{clos}(C_c^\infty(\mathbb{R}^n))$$

with respect to the norm

$$\|u\|_{\dot{H}^1} = \|\nabla u\|_{L^2}.$$

**Theorem 1.5.2.** (P.-L. Lions)

Let  $(u_k) \subset \dot{H}^1(\mathbb{R}^n)$  with  $u_k \xrightarrow{w} u$  in  $\dot{H}^1(\mathbb{R}^n)$  and suppose that

$$\mu_k = |\nabla u_k|^2 dx \xrightarrow{w^*} \mu, \quad \nu_k = |u_k|^{2^*} dx \xrightarrow{w^*} \nu \quad (k \rightarrow \infty).$$

Then there is an at most countable set  $J \subset \mathbb{N}$  and concentration points  $x^{(j)} \in \mathbb{R}^n$  with weights  $\mu^{(j)}, \nu^{(j)} > 0, j \in J$ , such that

$$\begin{aligned} \mu &\geq |\nabla u|^2 dx + \sum_{j \in J} \mu^{(j)} \delta_{\{x=x^{(j)}\}}, \\ \nu &= |u|^{2^*} dx + \sum_{j \in J} \nu^{(j)} \delta_{\{x=x^{(j)}\}}. \end{aligned}$$

Moreover, we have

$$\mathcal{S}(\nu^{(j)})^{2/2^*} \leq \mu^{(j)}, \quad \forall j \in J,$$

and

$$\sum_{j \in J} (\nu^{(j)})^{\frac{2}{2^*}} < \infty.$$

**Remark.** An analogous result holds true for the embeddings

$$\dot{W}^{k,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n), \quad kp < n, \quad \frac{1}{q} = \frac{1}{p} - \frac{k}{n}, \quad p > 1,$$

where  $\dot{W}^{k,p}(\mathbb{R}^n)$  is the homogeneous Sobolev space obtained as closure of  $C_c^\infty(\mathbb{R}^n)$  w.r.t.

$$\|u\|_{\dot{W}^{k,p}} = \sum_{|\alpha|=k} \|\partial^\alpha u\|_{L^p}, \quad u \in C_c^\infty(\mathbb{R}^n).$$

*Proof.* i) It suffices to consider  $u = 0$ . Indeed, let  $v_k = u_k - u$  with  $v_k \xrightarrow{w} 0$  in  $\dot{H}^1(\mathbb{R}^n)$ .

By Lemma 1.5.4 with error  $o(1) \xrightarrow{w^*} 0$  we have

$$\begin{aligned} |v_k|^{2^*} dx &= (|u_k|^{2^*} - |u|^{2^*}) dx + o(1) \\ &= \nu_k - |u|^{2^*} dx + o(1) \xrightarrow{w^*} \nu - |u|^{2^*} dx \geq 0, \\ |\nabla v_k|^2 dx &= (|\nabla u_k|^2 - |\nabla u|^2) dx + o(1) \\ &= \mu_k - |\nabla u|^2 dx + o(1) \xrightarrow{w^*} \mu - |\nabla u|^2 dx \geq 0, \end{aligned}$$

and it suffices to consider the case  $u = 0$ .

ii) Suppose now that  $u_k \xrightarrow{w} 0$  in  $\dot{H}^1(\mathbb{R}^n)$ , and  $u_k \rightarrow 0$  in  $L_{loc}^2(\mathbb{R}^n)$  by Rellich's theorem. For any  $\xi \in C_c^\infty(\mathbb{R}^n)$  then we obtain

$$\begin{aligned} \mathcal{S} \left( \int_{\mathbb{R}^n} |\xi|^{2^*} d\nu \right)^{\frac{2}{2^*}} &= \mathcal{S} \lim_{k \rightarrow \infty} \left( \int_{\mathbb{R}^n} |\xi|^{2^*} d\nu_k \right)^{\frac{2}{2^*}} \\ &= \mathcal{S} \lim_{k \rightarrow \infty} \left( \int_{\mathbb{R}^n} |u_k \xi|^{2^*} dx \right)^{\frac{2}{2^*}} \leq \liminf_{k \rightarrow \infty} \left( \int_{\mathbb{R}^n} |\nabla(u_k \xi)|^2 dx \right) \\ &= \liminf_{k \rightarrow \infty} \left( \int_{\mathbb{R}^n} |\xi|^2 |\nabla u_k|^2 dx \right) = \int_{\mathbb{R}^n} |\xi|^2 d\mu. \end{aligned} \quad (1.5.7)$$

Let

$$\nu = \nu_0 + \sum_{j \in J} \nu^{(j)} \delta_{\{x=x^{(j)}\}},$$

with  $\nu_0$  free of atoms and  $\nu^{(j)} > 0, j \in J$ . Since

$$\int_{\mathbb{R}^n} d\nu < \infty,$$

the set  $J$  is at most countable.

For any  $j \in J$ , upon choosing  $\xi \in C_c^\infty(\mathbb{R}^n)$  with  $0 \leq \xi \leq 1, \xi(x^{(j)}) = 1$ , from (1.5.7) we obtain

$$\mathcal{S}(\nu^{(j)})^{\frac{2}{2^*}} \delta_{\{x=x^{(j)}\}} \leq \mu;$$

hence

$$\mu \geq \sum_{j \in J} \mu^{(j)} \delta_{\{x=x^{(j)}\}}$$

with

$$\mathcal{S}(\nu^{(j)})^{\frac{2}{2^*}} \leq \mu^{(j)}, \quad j \in J.$$

In particular, we have

$$\begin{aligned} \mathcal{S} \sum_{j \in J} (\nu^{(j)})^{\frac{2}{2^*}} &\leq \sum_{j \in J} \mu^{(j)} \leq \int_{\mathbb{R}^n} d\mu \\ &\leq \limsup_{k \rightarrow \infty} \int_{\mathbb{R}^n} d\mu_k < \infty. \end{aligned}$$

The Theorem then follows once we prove that  $\nu_0 = 0$ .

**Claim.**  $\nu_0 = 0$ .

*Proof.* Let  $\Omega \subset \mathbb{R}^n$  be open, and let  $(\xi_l) \subset C_c^\infty(\Omega)$  with  $0 \leq \xi_l \leq 1$  satisfy

$$\xi_l \rightarrow \chi_\Omega \quad a.e. (k \rightarrow \infty).$$

From (1.5.7) we then obtain

$$\begin{aligned} \mathcal{S} \left( \int_{\Omega} d\nu \right)^{\frac{2}{2^*}} &= \mathcal{S} \lim_{l \rightarrow \infty} \left( \int_{\mathbb{R}^n} |\xi_l|^{2^*} d\nu \right)^{\frac{2}{2^*}} \\ &\leq \lim_{l \rightarrow \infty} \left( \int_{\mathbb{R}^n} |\xi_l|^2 d\mu \right) = \int_{\Omega} d\mu. \end{aligned} \quad (1.5.8)$$

In particular,  $\nu_0 \leq \nu$  is absolutely continuous w.r.t.  $\mu$ . By Radon-Nikodym Theorem there exists  $f \in L^1(\mathbb{R}^n, d\mu)$  such that

$$d\nu_0 = f d\mu,$$

and

$$f(x) = \lim_{r \rightarrow 0} \left( \frac{\int_{B_r(x)} d\nu_0}{\int_{B_r(x)} d\mu} \right) \quad \mu - a.e. x \in \mathbb{R}^n.$$

But by (1.5.8) and since  $\nu_0$  is free of atoms

$$\begin{aligned} \frac{\int_{B_r(x)} d\nu_0}{\int_{B_r(x)} d\mu} &\leq \frac{\left( \int_{B_r(x)} d\nu_0 \right)^{\frac{2}{2^*}} \left( \int_{B_r(x)} d\nu_0 \right)^{\frac{2^*-2}{2^*}}}{\int_{B_r(x)} d\mu} \\ &\leq \frac{1}{\mathcal{S}} \left( \int_{B_r(x)} d\nu_0 \right)^{\frac{2^*-2}{2^*}} \rightarrow 0 \quad \text{as } r \rightarrow 0 \end{aligned}$$

for every  $x \in \mathbb{R}^n$ . Thus  $f = 0$   $\mu$ -a.e., and  $\nu_0 = 0$ . □

□

**Application.** We use Theorem 1.5.2 to show that the (best) Sobolev constant for the embedding  $\dot{H}^1(\mathbb{R}^n) \hookrightarrow L^{2^*}(\mathbb{R}^n)$ ,  $n \geq 3$  is attained.

**Theorem 1.5.3.**  $\exists 0 < u \in \dot{H}^1(\mathbb{R}^n) : \mathcal{S} \|u\|_{L^{2^*}}^2 = \|\nabla u\|_{L^2}^2$ .

*Proof (following P.-L. Lions).* Let

$$M = \{u \in \dot{H}^1(\mathbb{R}^n) : \|u\|_{L^{2^*}} = 1\},$$

and let  $(u_k) \subset M$  be a minimizing sequence for

$$E(u) = \int_{\mathbb{R}^n} |\nabla u|^2 dx.$$

**Normalization.** Scale

$$\tilde{u}_k(x) = R_k^{\frac{n-2}{2}} u_k(x_k + R_k x)$$

with suitable  $x_k \in \mathbb{R}^n$ ,  $R_k > 0$ . Note that

$$\|\tilde{u}_k\|_{L^{2^*}} = \|u_k\|_{L^{2^*}} = 1, \quad k \in \mathbb{N},$$

while

$$E(\tilde{u}_k) = E(u_k), \quad k \in \mathbb{N}.$$

For suitably chosen  $x_k$  and  $R_k$  we can achieve

$$\int_{B_1(0)} |\tilde{u}_k|^{2^*} dx = \sup_{x_0 \in \mathbb{R}^n} \int_{B_1(x_0)} |\tilde{u}_k|^{2^*} dx = \frac{1}{2}. \quad (1.5.9)$$

Replacing  $u_k$  by  $\tilde{u}_k$ , if necessary, we may assume  $u_k = \tilde{u}_k$ .

**Convergence.** We may assume  $u_k \xrightarrow{w} u$  in  $\dot{H}^1(\mathbb{R}^n)$ . Let  $v_k = u_k - u \xrightarrow{w} 0$  ( $k \rightarrow \infty$ ). By Lemma 1.5.4 with error  $o(1) \rightarrow 0$  ( $k \rightarrow \infty$ ) we have

$$E(u_k) = E(u) + E(v_k) + o(1),$$

and

$$1 = \|u_k\|_{L^{2^*}}^{2^*} = \|u\|_{L^{2^*}}^{2^*} + \|v_k\|_{L^{2^*}}^{2^*} + o(1).$$

Set  $\lambda = \|u\|_{L^{2^*}}^{2^*} \in [0, 1]$ . If  $\lambda = 1$ , we have  $u \in M$ , and hence

$$\mathcal{S} \leq E(u) \leq \liminf_{k \rightarrow \infty} E(u_k) = \mathcal{S},$$

and the proof is complete.

If  $0 < \lambda < 1$ , by strict concavity of the map  $f(s) = s^{\frac{2}{2^*}}$  we have

$$\lambda^{\frac{2}{2^*}} + (1 - \lambda)^{\frac{2}{2^*}} > 1$$

and hence

$$\begin{aligned} \mathcal{S} &< \mathcal{S} \liminf_{k \rightarrow \infty} (\|u\|_{L^{2^*}}^2 + \|v_k\|_{L^{2^*}}^2) \\ &\leq \liminf_{k \rightarrow \infty} (\|\nabla u\|_{L^2}^2 + \|\nabla v_k\|_{L^2}^2) \leq \liminf_{k \rightarrow \infty} \|\nabla u_k\|_{L^2}^2 = \mathcal{S}. \end{aligned}$$

Contradiction!

Finally, if  $\lambda = 0$  we have  $u = 0$ ; that is,  $u_k \xrightarrow{w} 0$  in  $\dot{H}^1(\mathbb{R}^n)$ . By Theorem 1.5.2 we may assume that

$$|u_k|^{2^*} dx \xrightarrow{w^*} \nu = \sum_{i \in J} \nu_i \delta_{\{x=x^{(i)}\}}$$

with weights  $0 < \nu_i < 1, i \in J$ . Our normalization (1.5.9) implies

$$\sum_{|x^{(i)}| \leq 1} \nu_i \geq \frac{1}{2}, \quad 0 < \nu_i \leq \frac{1}{2}, \quad \forall i \in J.$$

For a suitable  $\varepsilon > 0$ , as will be determined below, choose  $R > 1$  such that

$$\sum_{|x^{(i)}| > R} \nu_i \leq \varepsilon.$$

Fix a cut-off function  $\varphi \in C_c^\infty(\mathbb{R}^n)$  with  $0 \leq \varphi \leq 1$ ,  $\varphi \equiv 1$  on  $B_R(0)$ ,  $|\nabla \varphi| < 1$ . Decompose  $u_k = v_k + w_k$ , where

$$v_k = u_k \varphi, \quad w_k = u_k(1 - \varphi) \xrightarrow{w} 0 \text{ in } \dot{H}^1(\mathbb{R}^n).$$

We can estimate  $\varphi^{2^*} + (1 - \varphi)^{2^*} \leq 1$  and thus also

$$\begin{aligned}
0 &\leq \|u_k\|_{L^{2^*}}^{2^*} - (\|v_k\|_{L^{2^*}}^{2^*} + \|w_k\|_{L^{2^*}}^{2^*}) \\
&= \int_{\mathbb{R}^n} |u_k|^{2^*} \left( \underbrace{1 - (\varphi^{2^*} + (1 - \varphi)^{2^*})}_{\in C_0^\infty(\mathbb{R}^n \setminus B_R(0))} \right) dx \\
&\xrightarrow{(k \rightarrow \infty)} \sum_{0 < \varphi(x^{(i)}) < 1} \nu_i (1 - (\varphi^{2^*}(x^{(i)}) + (1 - \varphi(x^{(i)}))^{2^*})) \\
&\leq \sum_{|x^{(i)}| > R} \nu_i \leq \varepsilon.
\end{aligned} \tag{1.5.10}$$

Let

$$w_0 = \liminf_{k \rightarrow \infty} \|w_k\|_{L^{2^*}}^{2^*} \leq \frac{1}{2}$$

(by (1.5.9)) and recall that

$$\begin{aligned}
|v_k|^{2^*} dx &= |u_k|^{2^*} \varphi^{2^*} dx \xrightarrow{w^*} \nu \varphi^{2^*} =: \\
&= \sum_{i \in J} \underbrace{\nu_i \varphi^{2^*}(x^{(i)})}_{=: w_i} \delta_{\{x=x^{(i)}\}}
\end{aligned}$$

with

$$\sum_{i \in J} w_i \geq \sum_{|x^{(i)}| \leq R} \nu_i \geq \frac{1}{2} \geq \nu_i \geq w_i, \quad \forall i \in J.$$

The estimate (1.5.10) implies that

$$0 \leq 1 - \left( \sum_{i \in J} w_i + w_0 \right) \leq \varepsilon;$$

that is

$$\Omega := w_0 + \sum_{i \in J} w_i \in [1 - \varepsilon, 1].$$

By Theorem 1.5.2 we also may assume that

$$|\nabla v_k|^2 dx \xrightarrow{w^*} \lambda \geq \sum_{i \in J} \lambda_i \delta_{\{x=x^{(i)}\}},$$

where

$$\mathcal{S} w_i^{2/2^*} \leq \lambda_i, \quad \forall i \in J.$$

It follows that, with  $J_0 := J \cup \{0\}$ , we have

$$\begin{aligned}
\mathcal{S} &\geq E(v_k) + E(w_k) + o(1) \\
&\geq \sum_{i \in J} \lambda_i + \mathcal{S} \|w_k\|_{L^{2^*}}^2 + o(1) \\
&\geq \mathcal{S} \sum_{i \in J_0} w_i^{2/2^*} + o(1) \\
&\geq \mathcal{S} (1 - \varepsilon)^{2/2^*} \sum_{i \in J_0} \left( \frac{w_i}{\Omega} \right)^{2/2^*} + o(1);
\end{aligned}$$

that is,

$$\mathcal{S} \geq \mathcal{S}(1 - \varepsilon)^{2/2^*} \sum_{i \in J_0} \left( \frac{w_i}{\Omega} \right)^{2/2^*}.$$

But for  $0 < \varepsilon < \frac{1}{3}$  we can split any sum of numbers  $0 < \alpha_i \leq \frac{1}{2(1-\varepsilon)} < \frac{3}{4}$ ,  $i \in J_0$ , with

$$\sum_{i \in J_0} \alpha_i = 1$$

into two separate sums with

$$\frac{1}{4} < \sum_{i \in I} \alpha_i, \quad \sum_{i \in J_0 \setminus I} \alpha_i < \frac{3}{4}.$$

(This is clear if there exists  $i_0$  with  $\alpha_{i_0} > \frac{1}{4}$ ; otherwise we collect indices  $i$  until  $\sum_{i \in I} \alpha_i$  first exceeds  $\frac{1}{4}$ .)

By strict convexity then

$$\begin{aligned} \sum_{i \in J_0} \alpha_i^{2/2^*} &= \sum_{i \in I} \alpha_i^{2/2^*} + \sum_{i \in J_0 \setminus I} \alpha_i^{2/2^*} \\ &\geq \left( \sum_{i \in I} \alpha_i \right)^{2/2^*} + \left( \sum_{i \in J_0 \setminus I} \alpha_i \right)^{2/2^*} \\ &\geq \inf_{\frac{1}{4} < s < \frac{3}{4}} (s^{2/2^*} + (1-s)^{2/2^*}) > 1 \end{aligned}$$

We conclude that for sufficiently small  $0 < \varepsilon < \frac{1}{3}$  we have

$$(1 - \varepsilon)^{2/2^*} \sum_{i \in J_0} \left( \frac{w_i}{\Omega} \right)^{2/2^*} > 1.$$

The contradiction rules out the case  $\lambda = 0$ . Thus  $\lambda = 1$ , and the proof is complete.  $\square$

**Remark.** i) The Sobolev constant  $\mathcal{S}$  is never attained on a domain  $\Omega \subsetneq \mathbb{R}^n$ ,  $n \geq 3$ .

If we suppose that for some such domain  $\Omega$  there exists  $0 < u \in H_0^1(\Omega)$  with  $\mathcal{S} \|u\|_{L^{2^*}}^2 = \|\nabla u\|_{L^2}^2$  a suitable multiple  $\tilde{u} = \alpha u$ ,  $\alpha > 0$ , solves

$$\begin{aligned} -\Delta \tilde{u} &= \tilde{u}^{2^*-1} \quad \text{in } \Omega, \\ \tilde{u} &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

From this fact, and from the fact that by scale-invariance  $\mathcal{S}(\Omega) = \mathcal{S}(\mathbb{R}^n) = \mathcal{S}$  is independent of the domain we derive a contradiction as follows.

**Case 1:** If  $\Omega$  is bounded,  $\Omega \subset B_R(0)$  for some  $R > 0$ , extending  $u(x) = 0$  for  $x \in B_R(0) \setminus \Omega$ , from  $u$  we obtain a minimizer  $0 \neq u \in H_0^1(B_R(0))$  of Sobolev's ratio, which is impossible by Pohožaev's theorem 1.4.2

**Case 2:** If  $\Omega$  is unbounded,  $\Omega \neq \mathbb{R}^n$ , by extending  $u(x) = 0$  for  $x \in \mathbb{R}^n \setminus \Omega$  we obtain a  $0 \leq \tilde{u} \not\equiv 0$  solving  $-\Delta \tilde{u} = \tilde{u}^{2^*-1}$  in  $\mathbb{R}^n$  vanishing outside  $\Omega$ , and the strong maximum principle then gives a contradiction.



ii) On  $\mathbb{R}^n$ ,  $n \geq 3$ , the Sobolev constant is attained by the Talenti functions [Tal76]

$$u(x) = \left( \frac{1}{1 + |x|^2} \right)^{\frac{n-2}{2}}$$

and their rescalings

$$\begin{aligned} u_{\varepsilon, x_0}(x) &= \varepsilon^{\frac{2-n}{2}} u\left(\frac{x - x_0}{\varepsilon}\right) \\ &= \left( \frac{\varepsilon}{\varepsilon^2 + |x - x_0|^2} \right)^{\frac{n-2}{2}}, \end{aligned}$$

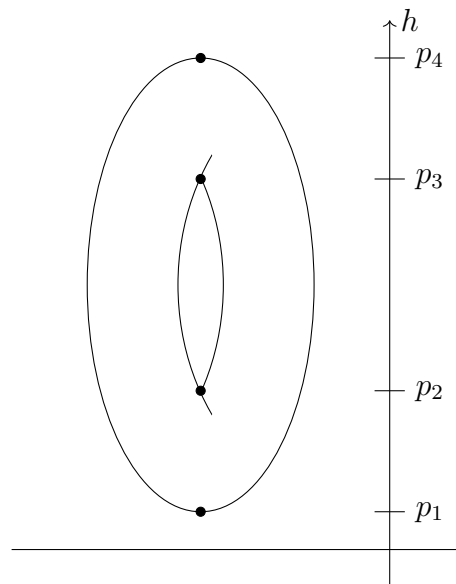
which we will re-encounter later in the context of Yamabe's theorem.

iii) Lion's proof of Theorem 1.5.3 also works in the case of the embedding  $\dot{W}^{k,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$  when  $\frac{1}{q} = \frac{1}{p} - \frac{k}{n} > 0$  and yields a minimizer of the Sobolev ratio in this case, as well.

## 2 Saddle points

### 2.1 Finite-dimensional examples

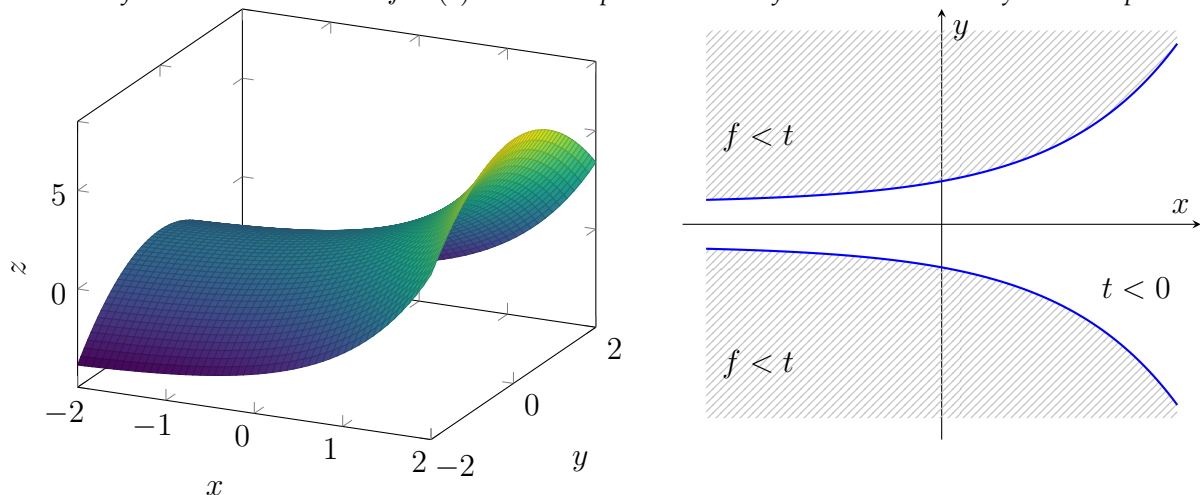
**Example 1.** The height function  $h$  on a torus as in the illustration has exactly 4 critical points  $p_i$ , where  $dh(p_i) = 0$ ,  $1 \leq i \leq 4$ , reflecting a change of topology of the level surfaces  $h^{-1}(t)$ ,  $t \in \mathbb{R}$ , or corresponding to the generators of  $\pi_k(T^2)$ ,  $k = 0, 1, 2$ .

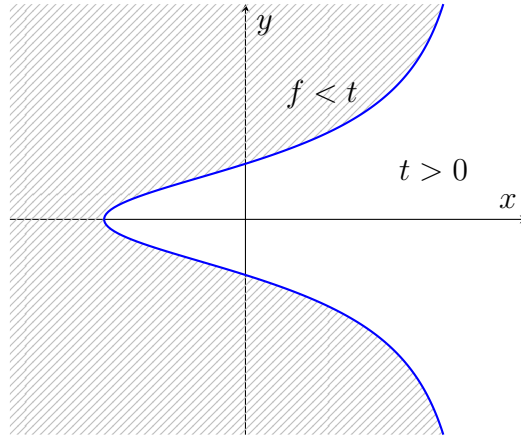


**Example 2.** Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be given by

$$f(x, y) = e^x - y^2, \quad (x, y) \in \mathbb{R}^2.$$

Note that for  $t < 0$  the level set  $f^{-1}(t)$  has 2 components while for  $t > 0$  it has only one component





However, in view of

$$df(x, y) = (e^x, 2y) \neq 0, \quad (x, y) \in \mathbb{R}^2,$$

the function  $f$  does not have any critical points! This shows, that some compactness is needed, if we want to carry over the ideas from Example 1 to the general case.

**Example 3.** A finite dimensional analogue of the famous theorem by Lusternik-Schnirelmann. In 1929 Lusternik-Schnirelman published the following result.

**Theorem 2.1.1.** (Lusternik-Schnirelman [LS29])

On any surface  $S \subset \mathbb{R}^3$  diffeomorphic to  $S^2$  there are at least 3 geometrically distinct, non-constant closed geodesics with self-intersections ("prime").

An analogous result can be obtained for a flat, convex, smoothly bounded billard. We may view this as a special limit case of the above theorem, for a sequence of convex surfaces  $S_k$  that are symmetric with respect to the  $(x, y)$ -plane with fixed intersection

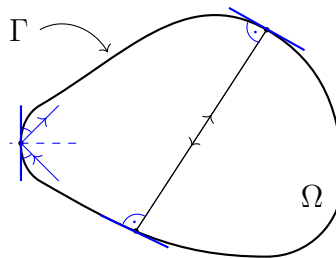
$$\Gamma = S_k \cap \{(x, y, z) \in \mathbb{R}^3 : z = 0\}, \quad k \in \mathbb{N}$$

and height above  $\{z = 0\}$  tending to zero ( $k \rightarrow \infty$ ).

**Definition 2.1.1.** i) A (convex) billard is a smoothly bounded, strictly convex domain  $\Omega \subset \subset \mathbb{R}^2$  with boundary  $\Gamma$  diffeomorphic to  $S^1$ .

ii) Geodesics in  $\Omega$  are straight lines that are reflected with equal angles in  $\Gamma = \partial\Omega$ .

iii) Prime (non self-intersecting) closed geodesics are lines through  $\Omega$  that are reflected in themselves.



The analogue of Theorem 2.1.1 for a billard then may be stated as follows:

**Theorem 2.1.2.** Any plane, convex billard  $\Omega$  contains (at least) 2 distinct prime closed geodesics.

**Remark 2.1.1.** Regarding  $\Omega$  as a (collapsed) surface  $S \subset \mathbb{R}^3$ , doubly covered by  $\Omega$ , we may regard  $\Gamma = \partial\Omega$  as a first prime closed geodesic in  $S$  and the 2 closed geodesics guaranteed by Theorem 2.1.2 as the remaining 2 prime closed geodesics as in Theorem 2.1.1 by Lusternik-Schnirelman.

**Analytic formulation.** Parametrize  $\Gamma$  by  $\gamma \in C^2(\mathbb{R}/\mathbb{Z}; \mathbb{R}^2)$  and let

$$f(s, t) := \frac{1}{2} |\gamma(s) - \gamma(t)|^2, \quad (s, t) \in \mathbb{R}^2.$$

A line  $\ell$  connecting  $\gamma(s)$  with  $\gamma(t)$  is reflected in itself at the point  $\gamma(s)$  iff

$$0 = \langle \dot{\gamma}(s), \gamma(s) - \gamma(t) \rangle_{\mathbb{R}^2} = \frac{\partial}{\partial s} f(s, t),$$

and at the point  $\gamma(t)$  iff

$$0 = \langle \dot{\gamma}(t), \gamma(s) - \gamma(t) \rangle_{\mathbb{R}^2} = -\frac{\partial}{\partial t} f(s, t).$$

**Conclusion.** A line  $\ell = \overline{\gamma(s)\gamma(t)}$  is a prime closed geodesic in  $\Omega$  iff

$$df(s, t) = 0;$$

that is, prime closed geodesics uniquely correspond to critical points of  $f$ .

**Obvious critical points.** Since our domain

$$\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \cong S^1 \times S^1 \cong T^2$$

is compact,  $f$  achieves its maximum and its minimum. However, since

$$f(s, t) \geq 0, \quad f(s, t) = 0 \Leftrightarrow s = t \pmod{\mathbb{Z}},$$

minimizers are related to the constant curves and do not give rise to prime closed geodesics in  $\Omega$ .

On the other hand, there exists  $(\bar{s}, \bar{t})$  with

$$f(\bar{s}, \bar{t}) = \max_{0 \leq s, t \leq 1} f(s, t) = \bar{\beta} > 0,$$

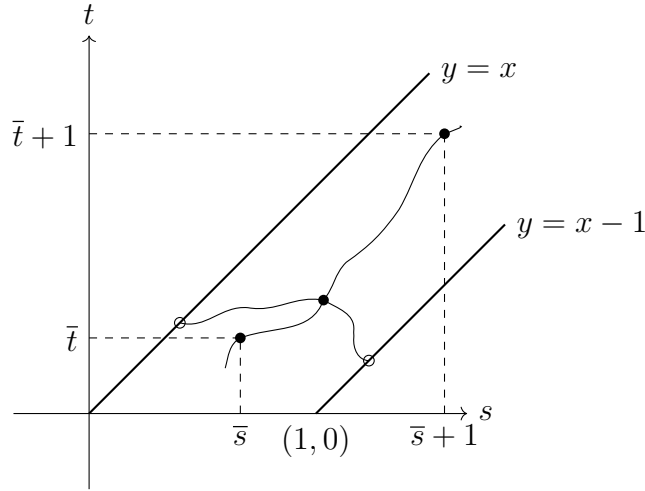
corresponding to a "longest" closed prime geodesic: the diameter of  $\Omega$ .

**Saddle points.** In the "energy landscape" defined by  $f$ , the "valleys"

$$\{(s, s) : s \in \mathbb{R}\}, \quad \{(s+1, s) : s \in \mathbb{R}\}$$

thus are separated by a "mountain ridge" with peaks at the points

$$\{(\bar{s} + k, \bar{t} + k) : k \in \mathbb{Z}\}.$$



We hope to find a second critical point at a "mountain-pass". For this, we seek a "path"  $p$  connecting two points  $x_0, x_1$  in different "valleys" and crossing the dividing "mountain ridge" at point of lowest maximal "elevation".

In the present finite-dimensional setting, "paths" may be simply compact connected sets.

**Definition 2.1.2.** A set  $\emptyset \neq p \subset T^2$  is connected, if for any open sets  $O_1, O_2 \subset T^2$  there holds

$$p \subset O_1 \cup O_2, O_1 \cap p \neq \emptyset = O_2 \cap p \Rightarrow O_1 \cap O_2 \neq \emptyset.$$

Recall that an interval  $\emptyset \neq I \subset \mathbb{R}$  is connected; moreover, continuous functions map connected sets to connected sets.

*Proof of Theorem 2.1.2.* Let

$$\beta = \inf_{p \in P} \sup_{(s,t) \in p} f(s,t) > 0,$$

where

$$P = \{p \subset T^2 : x_0 := (0,0) \in p, x_1 := (1,0) \in p, p \text{ compact and connected} \}.$$

**Claim 1.**  $\exists p \in P : \sup_{(s,t) \in p} f(s,t) = \beta.$

*Proof.* Let  $(p_k) \subset P$  with

$$\sup_{(s,t) \in p_k} f(s,t) < \beta + \frac{1}{k}, \quad k \in \mathbb{N},$$

and set

$$K_l = \overline{\bigcup_{k \geq l} p_k} \subset T^2.$$

Then  $K_l$  is compact and connected, and

$$x_{0,1} \in K_{l+1} \subset K_l \neq \emptyset, \quad l \in \mathbb{N}.$$

It follows that

$$x_{0,1} \in K := \bigcap_{l \in \mathbb{N}} K_l \subset K_l, \quad l \in \mathbb{N},$$

is compact and connected, and

$$\sup_K f \leq \sup_{K_l} f \leq \beta + \frac{1}{l}, \quad l \in \mathbb{N}.$$

Set  $p := K$ . □

**Claim 2.**  $\exists x^* = (s^*, t^*) \in p : f(s^*, t^*) = \beta, df(s^*, t^*) = 0.$

*Proof (indirect).* Suppose by contradiction that for any  $x = (s, t) \in p$  with  $f(x) = \beta$  there holds  $df(x) \neq 0$ .

By compactness of  $p$  then there exists  $\varepsilon > 0$  such that

$$\forall x \in T^2 : \text{dist}(x, p) = \inf\{|x - y| : y \in p\} < \varepsilon, |f(x) - \beta| < \varepsilon \Rightarrow |df(x)| > \varepsilon. \quad (2.1.1)$$

Without loss of generality, set  $\varepsilon < \beta$ .

Define the negative gradient flow

$$\Phi : T^2 \times [0, 1] \rightarrow T^2$$

as follows:

Let  $\varphi \in C^\infty(\mathbb{R})$  with  $0 \leq \varphi \leq 1$ ,  $\varphi(s) = 1$  for  $s \leq \frac{1}{2}$ ,  $\varphi(s) = 0$  for  $s \geq 1$  and for any  $x \in T^2$  let  $\Phi(x, t)$  solve the initial value problem

$$\frac{d}{dt}\Phi(x, t) = -\varphi\left(\frac{\beta - f(x)}{\varepsilon}\right) df(\Phi(x, t)), \quad (2.1.2)$$

$$\Phi(x, 0) = x. \quad (2.1.3)$$

Since  $\gamma \in C^2$ ,  $f \in C^2$ , the vector field on the right of (2.1.2) is of class  $C^1$ . Hence there exists a solution  $\Phi \in C^1(T^2 \times [0, 1], T^2)$  of (2.1.2), (2.1.3). (The solution exists for  $0 \leq t \leq 1$  by compactness of  $T^2$ .) Moreover, for any  $x \in p$ , any  $0 < t < 1$  such that

$$\beta \geq f(\Phi(x, t)) > \beta - \frac{\varepsilon}{2}$$

we have

$$\varphi\left(\frac{\beta - f(\Phi(x, t))}{\varepsilon}\right) = 1$$

and hence

$$\frac{d}{dt}f(\Phi(x, t)) = \left\langle df(\Phi(x, t)), \frac{d}{dt}\Phi(x, t) \right\rangle = -|df(\Phi(x, t))|^2; \quad (2.1.4)$$

in addition, letting

$$C_1 = \sup_{x \in T^2} |df(x)| < \infty$$

there holds

$$\begin{aligned} \text{dist}(\Phi(x, t), p) &\leq |\Phi(x, t) - x| \leq \int_0^t \left| \frac{d}{ds}\Phi(x, s) \right| ds \\ &\leq Ct \leq C_1 t < \varepsilon, \end{aligned}$$

if  $0 < t < \frac{\varepsilon}{C_1}$ , and hence (2.1.1) and (2.1.4) give

$$\frac{d}{dt}f(\Phi(x, t)) < -\varepsilon^2.$$

In conclusion, at time  $t_1 = \min\left\{1, \frac{\varepsilon}{C_1}\right\} > 0$  for any  $x \in p$  we either have

$$f(\Phi(x, t_1)) \leq \beta - \frac{\varepsilon}{2} < \beta,$$

or there holds

$$\beta \geq f(x) \geq f(\Phi(x, t)) \geq \beta - \frac{\varepsilon}{2}, \quad 0 \leq t \leq t_1,$$

and thus

$$\frac{d}{dt} f(\Phi(x, t)) < -\varepsilon^2, \quad 0 \leq t \leq t_1,$$

leading to the bound

$$f(\Phi(x, t_1)) \leq f(x) - \varepsilon^2 t_1 \leq \beta - \frac{\varepsilon^3}{C_1} < \beta.$$

Finally, since  $0 < \varepsilon < \beta$  by assumption, we have  $\beta - f(x_{0,1}) = \beta > \varepsilon$ , and  $\Phi(x_{0,1}, t) = x_{0,1}$  for all  $0 \leq t \leq t_1$ .

Thus,  $p_1 := \Phi(p, t_1) \in P$ . But

$$\sup_{p_1} f \leq \max \left\{ \beta - \frac{\varepsilon}{2}, \beta - \frac{\varepsilon^3}{C_1} \right\} < \beta,$$

contradicting the definition of  $\beta$ . □

If  $\beta < \bar{\beta}$  clearly  $x^* \neq \bar{x}$ , and the proof is complete.

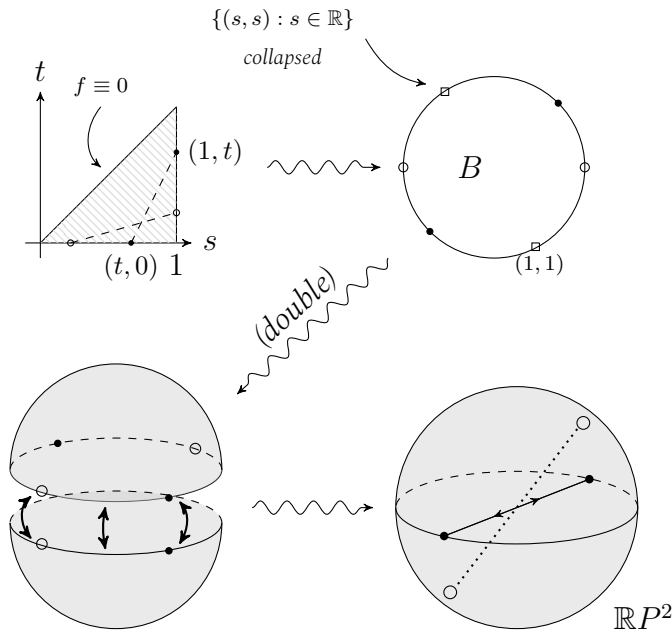
If  $\beta = \bar{\beta}$  then every  $p \in P$  contains a point  $x = (s, t)$  of maximal length, corresponding to a prime closed geodesic. Thus, in this case there even exist infinitely many such lines. □

**Remark.** i) The existence of 3 critical points in Theorem 2.1.2 again reflects the topology of the underlying space.

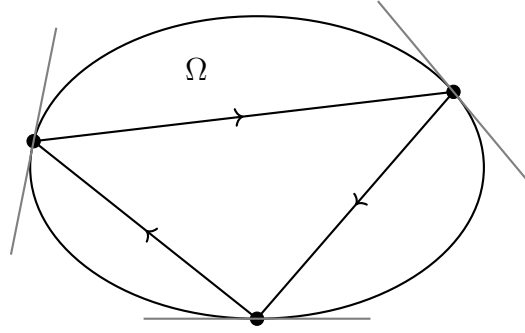
To see this, note that the symmetry

$$f(s, t) = f(t, s) = f(t, s - 1), \quad 0 \leq s, t \leq 1,$$

allows to regard  $f$  as a function on  $\mathbb{R}P^2$  by means of the identifications indicated in the picture below.

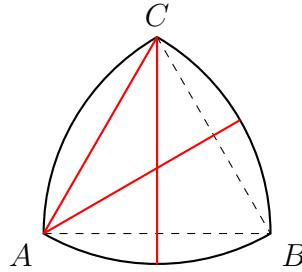


ii) There are other kinds of "closed geodesics" in a billard, as in the picture below.



These (and more complicated patterns) can also be studied with tools from the theory of dynamical systems.

- iii) If  $\beta = \beta_0$ , the billard  $\Omega$  has the same width, viewed from any direction. Is  $\Omega$  a ball?  
No! There are many curves  $\Gamma$  of constant width, including, for instance, regular polygons with circular boundary segments, such as the Reuleaux triangles.



- iv) Whereas Claim 1 in the proof of Theorem 2.1.2 is finite-dimensional in nature, the proof of Claim 2 is sufficiently flexible to be generalized to the infinite-dimensional case, provided a compactness condition holds.

## 2.2 Pseudo-gradient vector fields

Let  $X$  be a Banach space,  $E : X \rightarrow \mathbb{R}$ .

**Definition 2.2.1.** i)  $E$  is Fréchet differentiable at  $u \in X$  if there exists a linear map  $dE(u) \in X^*$  such that

$$E(u + v) - E(u) - \langle dE(u), v \rangle_{X^* \times X} = o(\|v\|_X)$$

for  $v \in X$ .

- ii)  $E$  is of class  $C^1$ , if  $E$  is Fréchet differentiable at every  $u \in X$  and if the map  $X \ni u \mapsto dE(u) \in X^*$  is continuous.

**Example 2.2.1.** i) Let  $\Omega \subset \subset \mathbb{R}^n$ ,

$$E(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} f u dx, \quad u \in H_0^1(\Omega),$$



with

$$\langle dE(u), v \rangle_{H^{-1} \times H_0^1} = \int_{\Omega} (\nabla u \nabla v - f v) dx$$

for every  $u, v \in H_0^1(\Omega)$ ; hence  $E \in C^1(H_0^1(\Omega))$ .

ii) For  $\Omega \subset \subset \mathbb{R}^n$ ,  $2 \leq p \leq 2^* = \frac{2n}{n-2}$  if  $n \geq 3$  let

$$E(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{p} \int_{\Omega} |u|^p dx, \quad u \in H_0^1(\Omega),$$

with

$$\langle dE(u), v \rangle_{H^{-1} \times H_0^1} = \int_{\Omega} (\nabla u \nabla v - u|u|^{p-2}v) dx$$

for  $u, v \in H_0^1(\Omega)$ . By Hölder's inequality, the map

$$H_0^1(\Omega) \ni u \mapsto u|u|^{p-2} \in H^{-1}(\Omega) = (H_0^1(\Omega))^*$$

is continuous; thus  $E \in C^1(H_0^1(\Omega))$ .

iii) Let  $\Omega \subset \subset \mathbb{R}^n$ ,  $a \in C^1(\mathbb{R})$  with  $|a|, |a'| \leq 1$ , and let

$$E(u) = \frac{1}{2} \int_{\Omega} a(u) |\nabla u|^2 dx, \quad u \in H_0^1(\Omega).$$

Then for  $v \in C_c^\infty(\Omega)$  there holds

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} E(u + \varepsilon v) = \int_{\Omega} (a(u) \nabla u \nabla v + a'(u) |\nabla u|^2 v) dx,$$

but

$$v \mapsto \int_{\Omega} a'(u) |\nabla u|^2 v dx$$

in general does not extend to all  $v \in H_0^1(\Omega)$  unless  $a \equiv \text{const}$ ,  $a' = 0$  or  $n = 1$ .

iv) Let  $\Omega \subset \subset \mathbb{R}^n$  and let

$$E(u) = \int_{\Omega} \sqrt{1 + |\nabla u|^2} dx, \quad u \in W_0^{1,1}(\Omega).$$

Then for any  $u, v \in W_0^{1,1}(\Omega)$  the Gateaux (directional) derivative

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} E(u + \varepsilon v) = \int_{\Omega} \frac{\nabla u \nabla v}{\sqrt{1 + |\nabla u|^2}} dx$$

exists, but  $E$  is nowhere Fréchet differentiable.

Take e.g.  $u \equiv 0$  and note

$$E(0 + v) - E(0) = \int_{\Omega} (\sqrt{1 + |\nabla v|^2} - 1) dx$$

with

$$\sup_{\|v\|_{W_0^{1,1}} = \varepsilon} \frac{\int_{\Omega} (\sqrt{1 + |\nabla v|^2} - 1) dx}{\int_{\Omega} |\nabla v| dx} = 1$$

for every  $\varepsilon > 0$ . (Take  $v = v_k = \varepsilon \varphi_k$ , where  $0 \leq \varphi_k \leq 1$ ,  $\varphi_k \xrightarrow[k \rightarrow \infty]{a.e.} \chi_B$  for a ball  $B \subset \subset \Omega$ .)

**Definition 2.2.2.** Let  $E \in C^1(X)$ .

- i) A point  $u \in X$  is critical if  $dE(u) = 0$ ; otherwise  $u$  is regular.
- ii) A number  $\beta \in \mathbb{R}$  is a critical value of  $E$ , if there exists a critical point  $u \in X$  with  $E(u) = \beta$ ; otherwise,  $\beta$  is a regular value.

For  $E \in C^1(X)$  let

$$\tilde{X} = \{u \in X : dE(u) \neq 0\}$$

be the set of regular points.

**Definition 2.2.3.** Let  $E \in C^1(X)$ . A vector field  $\tilde{e} : \tilde{X} \rightarrow X$  is a pseudo-gradient vector field (p.-g.v.f.) provided  $\tilde{e}$  is locally Lipschitz continuous and satisfies the bounds

- i)  $\|\tilde{e}(u)\|_X < 1$ ,
- ii)  $\langle dE(u), \tilde{e}(u) \rangle > \frac{1}{2} \|dE(u)\|_{X^*}$

for any  $u \in \tilde{X}$ .

**Theorem 2.2.1.** (Palais, 1966)

Every  $E \in C^1(X)$  admits a pseudo-gradient vector field.

*Proof.* i) Let  $u_0 \in \tilde{X}$ . By definition of  $\|dE(u_0)\|_{X^*}$ , there is  $v_0 = v(u_0) \in X$  such that

$$\|v_0\|_X < 1, \quad \langle dE(u_0), v_0 \rangle > \frac{1}{2} \|dE(u_0)\|_{X^*}. \quad (2.2.1)$$

Moreover, since  $dE$  is continuous, (2.2.1) holds true (with the same  $v_0$ ) for all  $u$  in a neighborhood  $U_0 = U(u_0)$  of  $u_0$  in  $\tilde{X}$ .

- ii) The above family  $(U(u))_{u \in \tilde{X}}$  clearly is an open cover of  $\tilde{X}$ . Since  $\tilde{X} \subset X$  is metric and hence paracompact, there exists a locally finite refinement  $(U_\iota)_{\iota \in I}$  of this cover so that  $U_\iota \subset U(u_\iota)$  for some  $u_\iota \in \tilde{X}$ ,  $\iota \in I$ . Let  $(\varphi_\iota)_{\iota \in I}$  be a partition of unity subordinate to  $(U_\iota)_{\iota \in I}$  with Lipschitz continuous  $0 \leq \varphi_\iota \leq 1$  satisfying

$$\text{supp}(\varphi_\iota) \subset U_\iota, \quad \sum_{\iota \in I} \varphi_\iota(x) = 1, \quad \forall x \in \tilde{X}.$$

(Note that the sum is locally finite.)

For instance, take

$$\begin{aligned} \psi_\iota(x) &= \text{dist}(x, X \setminus U_\iota) \\ &= \inf\{\|y - x\|_X : y \in X \setminus U_\iota\} \in C_0^{0,1}(U_\iota) \end{aligned}$$

and let

$$\varphi_\iota(x) = \frac{\psi_\iota(x)}{\sum_{\kappa \in I} \psi_\kappa(x)}, \quad \iota \in I.$$

Since the sum near any  $x \in \tilde{X}$  involves only a fixed finite set of indices  $\kappa \in I$ , and is uniformly positive,  $\varphi_\iota \in C_0^{0,1}(U_\iota)$  has the derived properties.

iii) Set

$$\tilde{e}(u) = \sum_{\iota \in I} \varphi_{\iota}(x) v(u_{\iota}), \quad u \in \tilde{X}.$$

Then  $\tilde{e}$  is locally Lipschitz and satisfies

$$\begin{aligned} \|\tilde{e}(x)\|_X &\leq \max_{u \in U_{\iota}} \|v(u_{\iota})\|_X < 1, \\ \langle dE(u), \tilde{e}(u) \rangle_{X^* \times X} &= \sum_{\iota \in I} \varphi_{\iota}(u) \underbrace{\langle dE(u), v(u_{\iota}) \rangle}_{> \frac{1}{2} \|dE(u)\|_{X^*} \quad \forall \iota: u \in U_{\iota}} \\ &> \frac{1}{2} \|dE(u)\|_{X^*}. \end{aligned}$$

□

## 2.3 Compactness, Palais-Smale condition

Let  $E \in C^1(X)$ ,  $\beta \in \mathbb{R}$ , and set

$$\begin{aligned} E_{\beta} &= \{u \in X : E(u) < \beta\}, \\ K_{\beta} &= \{u \in X : E(u) = \beta, dE(u) = 0\}, \end{aligned}$$

and for any  $\delta > 0$  let

$$N_{\beta, \delta} = \{u \in X : |E(u) - \beta| < \delta, \|dE(u)\|_{X^*} < \delta\}.$$

**Definition 2.3.1.** i) A sequence  $(u_k) \subset X$  is a  $(\mathcal{P}\text{-}\mathcal{S})_{\beta}$ -sequence if

$$E(u_k) \rightarrow \beta, \|dE(u_k)\|_{X^*} \rightarrow 0 \quad (k \rightarrow \infty).$$

ii)  $E$  satisfies the Palais-Smale condition at energy  $\beta$  ( $(\mathcal{P}\text{-}\mathcal{S})_{\beta}$  for short) if any  $(\mathcal{P}\text{-}\mathcal{S})_{\beta}$ -sequence is relatively compact.

Note that when  $E \in C^1(X)$  satisfies  $(\mathcal{P}\text{-}\mathcal{S})_{\beta}$  then as an analogue of (2.1.1) - where compactness of  $T^2$  was crucial - we obtain the following conclusion.

**Remark 2.3.1.** Let  $E \in C^1(X)$ ,  $\beta \in \mathbb{R}$ , and suppose  $E$  satisfies  $(\mathcal{P}\text{-}\mathcal{S})_{\beta}$ . Then

$$K_{\beta} = \emptyset \Rightarrow \exists \delta > 0 : N_{\beta, \delta} = \emptyset.$$

*Proof.* Suppose by contradiction that for every  $k \in \mathbb{N}$  there exists  $u_k \in X$  with

$$|E(u_k) - \beta| < \frac{1}{k}, \|dE(u_k)\|_{X^*} < \frac{1}{k}.$$

Then  $(u_k)$  is a  $(\mathcal{P}\text{-}\mathcal{S})_{\beta}$ -sequence. Since  $E$  by assumption satisfies  $(\mathcal{P}\text{-}\mathcal{S})_{\beta}$  a subsequence  $u_k \rightarrow u$  ( $k \rightarrow \infty, k \in \Lambda$ ). By continuity of  $E$  and  $dE$  then

$$E(u) = \beta, \quad dE(u) = 0,$$

and  $u \in K_{\beta}$ ; contradiction! □

A number of interesting examples of  $C^1$ -functionals satisfying the Palais-Smale condition can be characterized as follows.

**Theorem 2.3.1.** *Let  $E \in C^1(X)$ ,  $\beta \in \mathbb{R}$ , and suppose*

- i) *any  $(\mathcal{P}\text{-}\mathcal{S})_\beta$ -sequence is bounded,*
- ii) *there is a (fixed) linear isomorphism  $L : X \rightarrow X^*$  such that*

$$dE(u) = Lu - K(u), \quad u \in X,$$

*where  $K : X \rightarrow X^*$  is continuous and compact; that is,  $K$  maps bounded sets to relatively compact sets.*

*Then  $E$  satisfies  $(\mathcal{P}\text{-}\mathcal{S})_\beta$ .*

*Proof.* Let  $(u_k) \subset X$  be a  $(\mathcal{P}\text{-}\mathcal{S})_\beta$ -sequence. By i)  $(u_k)$  is bounded; hence by ii) a subsequence

$$K(u_k) =: v_k \rightarrow v \in X^* \quad (k \rightarrow \infty, k \in \Lambda).$$

from

$$w_k := dE(u_k) = Lu_k - v_k \rightarrow 0 \quad \text{in } X^*$$

then we obtain

$$u_k = L^{-1}(v_k + w_k) \rightarrow L^{-1}v =: u, \quad (k \rightarrow \infty, k \in \Lambda).$$

□

**Example 2.3.1.** i) *Let  $\Omega \subset\subset \mathbb{R}^n$ ,  $f \in L^2(\Omega)$ ,*

$$E(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} f u dx, \quad u \in H_0^1(\Omega),$$

*with*

$$dE(u) = -\Delta u - f.$$

*Since*

$$L : H_0^1(\Omega) \ni u \mapsto -\Delta u \in H^{-1}(\Omega)$$

*is a linear isomorphism with*

$$\|Lu\|_{H^{-1}} = \sup_{\substack{v \in H_0^1(\Omega) \\ \|v\|_{H_0^1}=1}} \underbrace{\langle Lu, v \rangle_{H^{-1} \times H_0^1}}_{= \int_{\Omega} \nabla u \nabla v dx} = \|\nabla u\|_{L^2} = \|u\|_{H_0^1}$$

*for  $u \in H_0^1(\Omega)$ , and since  $K(u) \equiv f$  defines a continuous, compact operator  $K : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ ,  $(\mathcal{P}\text{-}\mathcal{S})_\beta$ -sequences for any  $\beta \in \mathbb{R}$  are bounded and  $E$  satisfies  $(\mathcal{P}\text{-}\mathcal{S})_\beta$  for any  $\beta \in \mathbb{R}$  by Theorem 2.3.1.*

- ii) *Let  $\Omega \subset\subset \mathbb{R}^n$ ,  $2 < p < 2^* = \frac{2n}{n-2}$ , and consider*

$$E(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{p} \int_{\Omega} |u|^p dx, \quad u \in H_0^1(\Omega),$$

*with*

$$dE(u) = -\Delta u - u|u|^{p-2} = Lu - K(u), \quad u \in H_0^1(\Omega),$$

where

$$L = -\Delta, \quad K : H_0^1(\Omega) \ni u \mapsto u|u|^{p-2} \in H^{-1}(\Omega).$$

For  $p < 2^*$  the embedding  $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$  is compact. Thus, if  $(u_k) \subset H_0^1(\Omega)$  is bounded, a subsequence  $u_k \rightarrow u$  in  $L^p(\Omega)$  and

$$K(u_k) \xrightarrow{(k \rightarrow \infty, k \in \Lambda)} K(u) \quad \text{in } L^{\frac{p}{p-1}}(\Omega) \subset L^{\frac{2^*}{2^*-1}} \cong (L^{2^*}(\Omega))^* \subset H^{-1}(\Omega).$$

Thus, condition ii) of Theorem 2.3.1 holds true.

**Claim:** Any  $(\mathcal{P}\text{-}\mathcal{S})_\beta$ -sequence for  $E$  is bounded.

*Proof.* Let  $(u_k) \subset H_0^1(\Omega)$  be a  $(\mathcal{P}\text{-}\mathcal{S})_\beta$ -sequence. Compute

$$\begin{aligned} pE(u_k) - \langle dE(u_k), u_k \rangle_{H^{-1} \times H_0^1} &= \left(\frac{p}{2} - 1\right) \|\nabla u_k\|_{L^2}^2 \\ &= \frac{p-2}{2} \|u_k\|_{H_0^1}^2 \\ &= p\beta + o(1) \left(1 + \|u_k\|_{H_0^1}\right). \end{aligned}$$

The claim follows. □

Thus,  $E$  satisfies  $(\mathcal{P}\text{-}\mathcal{S})_\beta$  for any  $\beta \in \mathbb{R}$ .

iii) Let  $\Omega \subset \subset \mathbb{R}^n$ ,  $n \geq 3$ , and let  $p = 2^* = \frac{2n}{n-2}$ ,

$$E(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{2^*} \int_{\Omega} |u|^{2^*} dx, \quad u \in H_0^1(\Omega).$$

**Claim:** There is  $\beta > 0$  such that  $E$  does not satisfy  $(\mathcal{P}\text{-}\mathcal{S})_\beta$ .

*Proof.* NLoG there exists  $R > 0$  such that

$$B_R(0) \subset \Omega.$$

Choose a cut-off function  $\varphi \in C_c^\infty(B_R(0))$  with  $0 \leq \varphi \leq 1$ ,  $\varphi \equiv 1$  on  $B_{R/2}(0)$  and let

$$u_k(x) = \varphi(x) \left( \frac{ck}{1 + k^2|x|^2} \right)^{\frac{n-2}{2}} = \varphi(x) u_{\frac{1}{k}}^*(x),$$

where

$$u^*(x) = \left( \frac{c}{1 + |x|^2} \right)^{\frac{n-2}{2}}, \quad x \in \mathbb{R}^n,$$

solves

$$-\Delta u^* = (u^*)^{2^*-1} \quad \text{on } \mathbb{R}^n \tag{2.3.1}$$

and  $u_\varepsilon^*(x) = \varepsilon^{\frac{2-n}{2}} u^*\left(\frac{x}{\varepsilon}\right)$  for any  $\varepsilon > 0$  as in Section 1.5.

Then, as  $k \rightarrow \infty$ ,

$$\begin{aligned} E(u_k) &\longrightarrow \frac{1}{2} \|\nabla u^*\|_{L^2}^2 - \frac{1}{2^*} \|u^*\|_{L^{2^*}}^{2^*} \\ &= \left(\frac{1}{2} - \frac{1}{2^*}\right) \|\nabla u^*\|_{L^2}^2 =: \beta^*. \end{aligned}$$

However, by Hölder's inequality and (2.3.1),

$$\begin{aligned}
\langle dE(u_k), v \rangle_{H^{-1} \times H_0^1} &= \int_{\Omega} (\nabla u_k \nabla v - u_k |u_k|^{2^*-2} v) \, dx \\
&= \int_{\Omega} \left( \varphi \nabla u_{\frac{1}{k}}^* \nabla v + u_{\frac{1}{k}}^* \nabla \varphi \nabla v - \varphi^{2^*-1} (u_{\frac{1}{k}}^*)^{2^*-1} v \right) \, dx \\
&= \underbrace{\int_{\mathbb{R}^n} \left( \nabla u_{\frac{1}{k}}^* \nabla (v\varphi) - (u_{\frac{1}{k}}^*)^{2^*-1} v\varphi \right) \, dx}_{=0} \\
&\quad + o \left( \int_{B_R \setminus B_{\frac{R}{2}}(0)} \left( |\nabla u_{\frac{1}{k}}^*| + |u_{\frac{1}{k}}^*| \right) (|\nabla v| + |v|) \, dx \right) \\
&\quad + o \left( \int_{B_R \setminus B_{\frac{R}{2}}(0)} |u_{\frac{1}{k}}^*|^{2^*-1} |v| \, dx \right) \\
&\leq C \left( \int_{B_R \setminus B_{\frac{R}{2}}(0)} \left( |\nabla u_{\frac{1}{k}}^*|^2 + |u_{\frac{1}{k}}^*|^2 \right) \, dx \right)^{\frac{1}{2}} \cdot \|v\|_{H_0^1} \\
&\quad + C \left( \int_{B_r \setminus B_{\frac{R}{2}}(0)} |u_{\frac{1}{k}}^*|^{2^*} \, dx \right)^{\frac{2^*-1}{2^*}} \|v\|_{L^{2^*}} \\
&= o(1) \|v\|_{H_0^1}, \quad o(1) \rightarrow 0 \, (k \rightarrow \infty).
\end{aligned}$$

□

As a refinement of Remark 2.3.1 we now obtain the following result. For  $\beta \in \mathbb{R}$ ,  $\rho > 0$  also let

$$U_{\beta, \rho} = \bigcup_{u \in K_{\beta}} B_{\rho}(u; X)$$

be the norm-neighborhood of  $K_{\beta}$ .

**Lemma 2.3.1.** *Suppose  $E \in C^1(X)$  satisfies  $(\mathcal{P}.\text{-}\mathcal{S}.)_{\beta}$  for some given  $\beta \in \mathbb{R}$ . Then*

- i)  $K_{\beta}$  is compact;
- ii) the families  $(N_{\beta, \delta})_{\delta > 0}$ ,  $(U_{\beta, \rho})_{\rho > 0}$  each constitute a fundamental system of neighborhoods of  $K_{\beta}$ ; that is, given any open  $N \supset K_{\beta}$  there exist  $\rho > 0$ ,  $\delta > 0$  with

$$U_{\beta, \rho} \subset N, \quad N_{\beta, \delta} \subset N.$$

- iii) In particular, if  $K_{\beta} = \emptyset = N$ , there is  $\delta > 0$  such that  $N_{\beta, \delta} = \emptyset$ .

*Proof.* i) Let  $(u_k) \subset K_{\beta}$ . Then  $(u_k)$  is a  $(\mathcal{P}.\text{-}\mathcal{S}.)_{\beta}$ -sequence. Since  $E$  satisfies  $(\mathcal{P}.\text{-}\mathcal{S}.)_{\beta}$ , a subsequence  $u_k \rightarrow u$  ( $k \rightarrow \infty, k \in \Lambda$ ), where  $u \in K_{\beta}$  by continuity of  $E$  and  $dE$ . Thus,  $K_{\beta}$  is (sequentially) compact.

- ii) Let  $N \supset K_\beta$  be open. Suppose by contradiction that  $N_{\beta,\delta} \setminus N \neq \emptyset$  for every  $\delta > 0$  or that  $U_{\beta,\rho} \setminus N \neq \emptyset$  for every  $\rho > 0$ .

Let  $u_k \in N_{\beta,\frac{1}{k}} \setminus N$ ,  $v_k \in U_{\beta,\frac{1}{k}} \setminus N$ , respectively,  $k \in \mathbb{N}$ . Then  $(u_k)$  is a  $(\mathcal{P}.\text{-}\mathcal{S}.)_\beta$ -sequence. By  $(\mathcal{P}.\text{-}\mathcal{S}.)_\beta$ , a subsequence  $u_k \rightarrow u \in K_\beta$  ( $k \rightarrow \infty, k \in \Lambda$ ); hence  $u_k \in N$  for sufficiently large  $k \in \Lambda$ , contrary to our choice of  $(u_k)$ .

Similarly, for  $v_k \in U_{\beta,\frac{1}{k}} \setminus N$  let  $w_k \in K_\beta$  with

$$\|v_k - w_k\|_X < \frac{2}{k}, \quad k \in \mathbb{N}.$$

By i), a subsequence  $w_k \rightarrow w \in K_\beta$  ( $k \rightarrow \infty, k \in \Lambda$ ), and

$$\|v_k - w\|_X \leq \|v_k - w_k\|_X + \|w_k - w\|_X \xrightarrow{(k \rightarrow \infty, k \in \Lambda)} 0$$

shows that  $v_k \in N$  for sufficiently large  $k \in \Lambda$ . □

## 2.4 Deformation Lemma, Minimax Principle

With the help of Lemma 2.3.1 we now obtain the following result.

**Theorem 2.4.1.** (*Deformation Lemma*)

Let  $E \in C^1(X)$ ,  $\beta \in \mathbb{R}$ ,  $N$  an open neighborhood of  $K_\beta$ , and let  $\bar{\varepsilon} > 0$ .

Then, if  $E$  satisfies  $(\mathcal{P}.\text{-}\mathcal{S}.)_\beta$ , there exist  $0 < \varepsilon < \bar{\varepsilon}$  and  $\Phi \in C^0(X \times [0, 1]; X)$  such that

- i)  $\Phi(\cdot, t) : X \rightarrow X$  is a homeomorphism,  $0 \leq t \leq 1$ ;
- ii)  $\Phi(u, t) = u$ , if either  $t = 0$ , or if  $dE(u) = 0$ , or if  $|E(u) - \beta| \geq \bar{\varepsilon}$ ;
- iii)  $t \mapsto E(\Phi(u, t))$  is non-increasing,  $u \in X$ ;
- iv)  $\Phi(E_{\beta+\varepsilon}, 1) \subset E_{\beta-\varepsilon} \cup N$ , and  $\Phi(E_{\beta+\varepsilon} \setminus N, 1) \subset E_{\beta-\varepsilon}$ .

*Proof.* By Lemma 2.3.1 we can find numbers  $0 < \delta, \rho < 1$ , such that

$$N \supset U_{\beta,3\rho} \supset U_{\beta,\rho} \supset N_{\beta,\delta} \supset K_\beta.$$

Fix

$$\varepsilon = \frac{1}{4} \min\{\bar{\varepsilon}, \delta\rho\} > 0.$$

Let  $\varphi \in C^\infty(\mathbb{R})$  with  $0 \leq \varphi \leq 1$  and such that

$$\varphi(s) = \begin{cases} 1, & \text{if } |s| \leq \frac{1}{2}, \\ 0, & \text{if } |s| \geq 1. \end{cases}$$

Also let

$$\eta(u) = \min\{\rho^{-1} \text{dist}(u, U_{\beta,\rho}), 1\}$$

so that  $\eta : X \rightarrow \mathbb{R}$  with  $0 \leq \eta \leq 1$  is Lipschitz continuous with

$$\eta(u) = \begin{cases} 1, & \text{if } u \notin U_{\beta,2\rho}, \\ 0, & \text{if } u \in U_{\beta,\rho}. \end{cases}$$

With  $\tilde{X}, \tilde{e} : \tilde{X} \rightarrow X$  as in Theorem 2.2.1 then let

$$e(u) = \begin{cases} \overbrace{\eta(u) \varphi\left(\frac{E(u) - \beta}{\varepsilon}\right)}^{=: \varphi_{\varepsilon, \beta}(E(u))} \tilde{e}(u), & u \in \tilde{X}, \\ 0, & \text{else.} \end{cases}$$

**Claim 1.**  $e$  is well-defined and locally Lipschitz continuous with  $e(u) = 0$  for  $u \in X$  with  $dE(u) = 0$ , and  $\|e(u)\|_X < 1$ ,  $u \in X$

*Proof.* Note that the  $\varphi$ -factor vanishes if

$$|E(u) - \beta| \geq \varepsilon,$$

while

$$\eta(u) = 0, \quad u \in N_{\beta, \delta}.$$

Let  $u \in X$  be critical for  $E$ .

**Case i)**  $u \in N_{\beta, \frac{\delta}{2}}$ . Then  $\eta \equiv 0$  in a neighborhood of  $u$ .

**Case ii)**  $u \notin N_{\beta, \frac{\delta}{2}}$ . Then  $|E(u) - \beta| \geq \frac{\delta}{2} > 2\varepsilon$ , and the  $\varphi$ -factor vanishes in a neighborhood of  $u$ .

Thus,  $e$  is well-defined near  $u$  and locally Lipschitz. Moreover,  $\|e(u)\|_X \leq \|\tilde{e}(u)\|_X$  for  $u \in \tilde{X}$ , or  $e(u) = 0$ .  $\square$

In view of Claim 1 then the vector field  $e$  induces a flow  $\Phi : X \times \mathbb{R} \rightarrow X$  with

$$\frac{d}{dt} \Phi(u, t) = -e(\Phi(u, t)) \tag{2.4.1}$$

$$\Phi(u, 0) = u, \quad u \in X \tag{2.4.2}$$

i) By the semi-group property

$$\Phi(\cdot, s + t) = \Phi(\cdot, s) \circ \Phi(\cdot, t)$$

for all  $s, t \in \mathbb{R}$ , each  $\Phi(\cdot, t)$  is a homeomorphism with

$$(\Phi(\cdot, t))^{-1} = \Phi(\cdot, -t), \quad t \in \mathbb{R}.$$

ii) Clearly,  $\Phi(u, t) = u$  for  $t = 0$  by (2.4.2) or when  $e(u) = 0$  and any  $t \in \mathbb{R}$ . By construction,  $e(u) = 0$  for  $u$  while  $|E(u) - \beta| \geq \bar{\varepsilon} > 2\varepsilon$ , or when  $dE(u) = 0$  by Claim 1.

iii) For  $u \in X, t \in \mathbb{R}$  compute

$$\begin{aligned} \frac{d}{dt} E(\Phi(u, t)) &= \left\langle dE(\Phi(u, t)), \frac{d}{dt} \Phi(u, t) \right\rangle_{X^* \times X} \\ &= -\eta(\Phi(u, t)) \varphi\left(\frac{E(u) - \beta}{\varepsilon}\right) \langle dE(\Phi(u, t)), \tilde{e}(\Phi(u, t)) \rangle_{X^* \times X} \leq 0 \end{aligned}$$



iv) Suppose for  $u \in E_{\beta+\varepsilon}$  there holds  $\Phi(u, 1) \notin E_{\beta-\varepsilon}$  and therefore by iii)

$$|E(\Phi(u, t)) - \beta| \leq \varepsilon, \quad 0 \leq t \leq 1.$$

By choice of  $\varphi$  then  $\varphi(\frac{E(\Phi(u, t)) - \beta}{\varepsilon}) = 1$  for  $0 \leq t \leq 1$ .

**Claim 2:**  $\Phi(u, t) \in N$  for all  $0 \leq t \leq 1$ .

*Proof.* Suppose  $\Phi(u, t) \notin U_{\beta, 2\delta} \supset N_{\beta, \delta}$  for all  $0 \leq t \leq 1$ . Then

$$\eta(\Phi(u, t)) = 1, \quad \|dE(\Phi(u, t))\|_X \geq \delta$$

for all  $0 \leq t \leq 1$ , and

$$\begin{aligned} \frac{d}{dt} E(\Phi(u, t)) &\leq -\langle dE(\Phi(u, t)), \tilde{e}(\Phi(u, t)) \rangle_{X^* \times X} \\ &\leq -\frac{1}{2} \|dE(\Phi(u, t))\|_{X^*} \leq -\frac{\delta}{2} \end{aligned} \quad (2.4.3)$$

for  $0 \leq t \leq 1$ , so that

$$E(\Phi(u, 1)) \leq E(\underbrace{\Phi(u, 0)}_{=0}) - \frac{\delta}{2} < \beta + \varepsilon - 2\varepsilon = \beta - \varepsilon,$$

contrary to our assumption.

Thus, there is  $t_0 \in [0, 1]$  with  $\Phi(u, t_0) \in U_{\beta, 2\delta}$ . Suppose that there is  $t_1 \in [0, 1]$  such that  $\Phi(u, t_1) \notin U_{\beta, 3\rho}$ . Note that  $\|e(u)\|_X < 1$  implies

$$\|\Phi(u, s) - \Phi(u, t)\|_X \leq |t - s|, \quad 0 \leq s, t \leq 1.$$

Hence there is a time interval  $I \subset [0, 1]$  of length  $|I| \geq \rho$  between  $t_0$  and  $t_1$  such that

$$\Phi(u, t) \in U_{\beta, 3\rho} \setminus U_{\beta, 2\rho}, \quad t \in I,$$

and (2.4.3) implies

$$E(\Phi(u, 1)) \leq E(u) - \frac{\delta}{2} |I| < \beta + \varepsilon - \frac{\delta\rho}{2} \leq \beta - \varepsilon,$$

again contradicting our assumption. Hence  $\Phi(u, t) \in U_{\beta, 3\rho} \subset N$  for all  $t \in [0, 1]$ . □

By Claim 2, if  $\Phi(u, 1) \notin E_{\beta-\varepsilon}$  thus it follows that  $\Phi(t, 1) \in N$ .

Conversely, if  $u \notin N$  we must have  $\Phi(u, 1) \in E_{\beta-\varepsilon}$ , and the proof is complete. □

**Minimax strategies.** With the help of the pseudo-gradient flows  $\Phi$  constructed in Theorem 2.4.1 we can now characterize "minimax" critical values as follows.

**Definition 2.4.1.** Let  $M$  be a topological space,  $(\Phi(\cdot, t))_{0 \leq t \leq 1}$  a continuous family of homeomorphism  $\Phi(\cdot, t)$  of  $M$  with  $\Phi(\cdot, 0) = \text{id}$ . A Family  $\mathcal{F} \subset \mathcal{P}(M)$  is (forward)  $\Phi$ -invariant, if

$$\forall F \in \mathcal{F}, 0 \leq t \leq 1 : \Phi(F, t) \in \mathcal{F}.$$

**Example 2.4.1.** i)  $\mathcal{F} = \{M\}$ , or  $\mathcal{F} = \mathcal{P}(M)$  are  $\Phi$ -invariant for any  $\Phi$  as in Definition 2.4.1

ii) For any map  $\alpha \in C^0(S^k, M)$  the family

$$[\alpha] = \{\alpha'(S^k) \subset M : \alpha' \text{ is homotopic to } \alpha\}$$

is  $\Phi$ -invariant for any  $\Phi$  as above; similarly for maps from any topological space  $N$  into  $M$ .

**Theorem 2.4.2.** (Minimax principle, Palais)

Let  $E \in C^1(X)$ ,  $\beta_0 \geq -\infty$ . Suppose that  $\mathcal{F}$  is forward  $\Phi$ -invariant for any of the flows  $\Phi$  constructed in Theorem 2.4.1 that fixes  $E_{\beta_0}$ , and suppose that

$$\beta := \inf_{F \in \mathcal{F}} \sup_{u \in F} E(u) > \beta_0.$$

Then, if  $E$  satisfies  $(\mathcal{P}.\mathcal{S})_\beta$ , the value  $\beta$  is critical.

*Proof.* Suppose by contradiction that  $K_\beta = \emptyset$ . For  $N \neq \emptyset$ ,  $\bar{\varepsilon} = \beta - \beta_0 > 0$  let  $\Phi : X \times [0, 1] \rightarrow X$ ,  $\varepsilon > 0$  be constructed as in Theorem 2.4.1, and let  $F \in \mathcal{F}$  with

$$\sup_{u \in F} E(u) < \beta + \varepsilon.$$

Then  $\Phi(\cdot, t)$  fixes  $E_{\beta_0}$  for  $0 \leq t \leq 1$  and

$$F_1 := \Phi(F, 1) \subset E_{\beta-\varepsilon}, \quad F_1 \in \mathcal{F},$$

contradicting the definition of  $\beta$ . □

As an example of a concrete implementation of Theorem 2.4.2 we consider the following.

**Theorem 2.4.3.** (Mountain pass lemma)

Let  $E \in C^1(X)$  with  $E(0) = 0$ , and suppose there exist  $\rho > 0$ ,  $a > 0$  such that

$$\inf_{\|u\|_X = \rho} E(u) \geq a.$$

Moreover, suppose that there exists  $u_1 \in X$  with  $\|u_1\|_X > \rho$  and  $E(u_1) \leq 0$ . Then, if  $E$  satisfies  $(\mathcal{P}.\mathcal{S})_\beta$  for every  $\beta \geq \alpha$ , there is a critical value  $\beta \geq a$  of  $E$ .

*Proof.* Let

$$\mathcal{F} = \{F \subset X : F \text{ connected}, u_0 = 0, u_1 \in F\}$$

and set

$$\beta = \inf_{F \in \mathcal{F}} \sup_{u \in F} E(u) \geq a > \beta_0 = 0.$$

Then  $\mathcal{F}$  is  $\Phi$ -invariant for every  $(\Phi(\cdot, t))_{0 \leq t \leq 1}$  that fixes  $E_0$ , and Theorem 2.4.2 gives the claim. □

**Example 2.4.2.** i) Let  $\Omega \subset \subset \mathbb{R}^n$ ,  $2 < p < 2^* = \frac{2n}{n-2}$  if  $n \geq 3$ ,

$$E(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{p} \int_{\Omega} |u|^p dx, \quad u \in H_0^1(\Omega).$$

Then  $E(0) = 0$ ,

$$E(\lambda u) = \frac{\|u\|_{H_0^1}^2}{2} \cdot \lambda^2 - \frac{\|u\|_{L^p}^p}{p} \cdot \lambda^p \xrightarrow{(\lambda \rightarrow \infty)} -\infty$$

for every  $u \in H_0^1(\Omega) \setminus \{0\}$ . Moreover, by Sobolev's embedding we can bound

$$E(u) \geq \frac{1}{2} \|u\|_{H_0^1}^2 - c(p, \Omega) \|u\|_{H_0^1}^p \geq a > 0$$

for

$$\|u\|_{H_0^1} = \left( \frac{1}{4c(p, \Omega)} \right)^{\frac{1}{p-2}} =: \rho, \quad a = \frac{1}{4} \rho^2.$$

Finally,  $E$  satisfies  $(\mathcal{P}.\mathcal{S}.)_\beta$  for every  $\beta \in \mathbb{R}$  by Example 2.3.1 ii), and Theorem 2.4.3 applies.

ii) A more general case of i). Let  $\Omega \subset\subset \mathbb{R}^n$ ,  $g \in C^0(\mathbb{R})$  and consider the problem

$$-\Delta u = g(u) \quad \text{in } \Omega, \quad (2.4.4)$$

$$u = 0 \quad \text{on } \partial\Omega. \quad (2.4.5)$$

Suppose that  $g$  satisfies the following conditions:

a)  $\exists C > 0, 0 \leq s < 2^* - 1 = \frac{n+2}{n-2}$  if  $n \geq 3 \forall u \in \mathbb{R}$ :

$$|g(u)| \leq C(1 + |u|^3);$$

b)  $\frac{g(u)}{u} \rightarrow 0$  ( $0 \neq |u| \rightarrow 0$ );

c)  $\exists \theta > 2, R > 0 \forall |u| \geq R$ :

$$0 < \theta G(u) \leq g(u)u,$$

where

$$G(u) = \int_0^u g(s) ds$$

is the primitive of  $g$  with  $G(0) = 0$ .

Then there exists  $0 \neq u \in H_0^1(\Omega)$  solving (2.4.4), (2.4.5).

To see this, let

$$E(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} G(u) dx, \quad u \in H_0^1(\Omega).$$

By a) the functional  $E \in C^1(H_0^1(\Omega))$  with

$$\langle dE(u), v \rangle_{H^{-1} \times H_0^1} = \int_{\Omega} (\nabla u \nabla v - g(u)v) dx$$

for any  $u, v \in H_0^1(\Omega)$ .

Moreover, for any  $\beta \in \mathbb{R}$  let  $(u_k) \subset H_0^1(\Omega)$  be a  $(\mathcal{P}.\mathcal{S}.)_\beta$ -sequence. Then

$$\begin{aligned} \theta E(u_k) - \langle dE(u_k), u_k \rangle_{H^{-1} \times H_0^1} &= \theta \beta + o(1) (1 + \|u_k\|_{H^{-1}}) \\ &= \frac{\theta - 2}{2} \|\nabla u_k\|_{L^2}^2 - \int_{\Omega} \underbrace{(\theta G(u_k) - g(u_k)u_k)}_{\leq 0, \text{ if } |u_k| \geq R} dx \\ &\geq \frac{\theta - 2}{2} \|u_k\|_{H_0^1}^2 - C, \end{aligned}$$

and  $(u_k)$  is bounded. Since  $s < 2^* - 1$ , by Rellich's theorem the operator

$$K : H_0^1(\Omega) \ni u \mapsto g(u) \in H^{-1}(\Omega)$$

is compact. Hence,  $E$  satisfies  $(\mathcal{P}\text{-}\mathcal{S})_\beta$  for every  $\beta \in \mathbb{R}$  by Theorem 2.3.1.

Next, we note that c) implies

$$\frac{d}{du}(\log u^\theta) = \frac{\theta}{u} \leq \frac{g(u)}{G(u)} = \frac{d}{du}(\log G(u)), \quad u \geq R,$$

and similarly for  $u \leq -R$ . Hence

$$G(u) \geq G(u) \frac{|u|^\theta}{R^\theta} = C|u|^\theta, \quad |u| \geq R.$$

In particular,  $s + q \geq \theta > 2$ .

Thus, we have  $E(0) = 0$ , and for every  $u \neq 0$ ,  $u \in H_0^1(\Omega)$

$$E(\lambda u) = \frac{\lambda^2}{2} \|u\|_{H_0^1}^2 - C\lambda^\theta \|u\|_{H_0^1}^\theta \xrightarrow{(\lambda \rightarrow \infty)} -\infty,$$

so that  $E(\lambda u) \leq 0$  for  $\lambda \geq \lambda_1 > 0$ .

Finally, let  $\lambda_1 > 0$  be the first Dirichlet eigenvalue of  $-\Delta$  on  $\Omega$ . By b) there exists  $r_0 > 0$  such that

$$|g(u)| \leq \frac{\lambda_1}{2} |u| \quad \text{for } |u| \leq r_0,$$

and hence

$$G(u) = \int_0^u g(v) dv \leq \frac{\lambda_1}{4} |u|^2 \quad \text{for } |u| \leq r_0.$$

By Tchebychev's inequality, for any  $u \in H_0^1(\Omega)$  we can bound the measure

$$\begin{aligned} A(r_0) &= \mathcal{L}^n(\{x \in \Omega : |u(x)| \geq r_0\}) \leq r_0^{-2^*} \int_\Omega |u|^{2^*} dx \\ &\leq C r_0^{-2^*} \|u\|_{H_0^1}^{2^*}. \end{aligned}$$

It follows that for any  $u \in H_0^1(\Omega)$  with  $\|u\|_{H_0^1} = \rho$  we can estimate

$$\begin{aligned} E(u) &= \frac{1}{2} \|u\|_{H_0^1}^2 - \underbrace{\int_\Omega \left( G(u) - \frac{\lambda_1}{4} u^2 \right) dx}_{\leq 0, \text{ if } |u| \leq r_0} - \frac{\lambda_1}{4} \|u\|_{L^2}^2 \\ &\geq \frac{1}{4} \underbrace{\|u\|_{H_0^1}^2}_{=\rho^2} - \int_{\{x: |u(x)| \geq r_0\}} G(u) dx, \end{aligned}$$

where

$$\begin{aligned} \int_{\{x: |u(x)| \geq r_0\}} G(u) dx &\leq C \int_{\{x: |u(x)| \geq r_0\}} (1 + |u|^{s+1}) dx \\ &\leq C A(r_0) + C (A(r_0))^{\frac{2^* - (s+1)}{2^*}} \|u\|_{L^{2^*}}^{s+1} \\ &\leq C r_0^{-2^*} \|u\|_{H_0^1}^{2^*} + C r_0^{(s+1)-2^*} \|u\|_{H_0^1}^{2^*} \\ &\leq C \|u\|_{H_0^1}^{2^*} \leq C \rho^{2^*}, \end{aligned}$$

and again Theorem 2.4.3 can be applied.

## 2.5 Index Theory

Briefly returning to the finite-dimensional setting, recall that for a symmetric matrix  $A \in \mathbb{R}^{n \times n}$  the eigenvalues  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$  are characterized by the minimax principle

$$\lambda_i = \inf_{\substack{V \subset \mathbb{R}^n \\ \dim(V) \geq i}} \sup_{\substack{x \in V \\ |x|=1}} \langle Ax, x \rangle_{\mathbb{R}^n}, \quad 1 \leq i \leq n.$$

Similarly, for the Laplace operator on  $\Omega \subset \subset \mathbb{R}^n$  with homogeneous Dirichlet boundary condition we have the minimax-characterization

$$\lambda_i = \inf_{\substack{V \subset H_0^1(\Omega) \\ \dim(V) \geq i}} \sup_{u \in V, \|u\|_{L^2}=1} \|\nabla u\|_{L^2}^2, \quad i \in \mathbb{N},$$

of the  $i^{\text{th}}$  Dirichlet eigenvalue.

However, these characterizations strongly use the fact that the operators involved are linear.

In the nonlinear case, for instance, for the equation

$$\begin{aligned} -\Delta u &= u|u|^{p-2} \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

is there a concept that generalizes the notion of dimension?

In fact, in the case of symmetry with respect to some compact group action such a concept can be defined; in particular, when we are dealing with an even functional  $E \in C^1(X)$  with  $E(-u) = E(u)$ ,  $u \in X$ , with symmetry group  $\mathbb{Z}_2 = \{\text{id}, -\text{id}\}$ , or when there is an  $O(2)$ -symmetry with  $S^1$ -orbits on which the energy is constant.

For simplicity, in the following we only consider the case of a  $\mathbb{Z}_2$ -symmetry, that is, the case when  $E$  is even.

**A  $\mathbb{Z}_2$ -index:** The Krasnoselskii genus.

Let  $M \subset X$  be a submanifold of class  $C^{1,1}$ ; for instance,  $M = \{u \in X : \|u\|_X = 1\}$ . Suppose  $M = -M = \{-u : u \in M\}$  and set

$$\mathcal{A} = \{A \subset X : A \text{ closed}, A = -A\}.$$

**Definition 2.5.1.** (Coifman (1969))

For  $\emptyset \neq A \in \mathcal{A}$  let

$$\gamma(A) = \begin{cases} \inf\{m : \exists h \in C^0(A; \mathbb{R}^m \setminus \{0\}), h(-u) = h(u)\}, \\ \infty, \text{ if } \{\dots\} = \emptyset; \text{ in particular, if } 0 \in A, \end{cases}$$

and set  $\gamma(\emptyset) = 0$ .

**Remark 2.5.1.** By the Tietze extension theorem, any odd map  $h \in C^0(A; \mathbb{R}^m)$  may be extended to a map  $\tilde{h} \in C^0(X; \mathbb{R}^m)$ . Letting

$$h(u) = \frac{1}{2}(\tilde{h}(u) - \tilde{h}(-u))$$

the extension may be chosen to be odd.

The index  $\gamma(A)$  achieves the desired generalization of the concept of dimension. To see this, we need certain facts from degree theory.

**Theorem 2.5.1.** *For any open  $\Omega \subset \mathbb{R}^n$ ,  $h \in C^0(\overline{\Omega}; \mathbb{R}^n)$ , any  $y \in \mathbb{R}^n \setminus h(\partial\Omega)$  there exists a number*

$$d(h, \Omega, y) \in \mathbb{Z},$$

*the topological degree of  $h$  in  $y$  w.r.t.  $\Omega$ , having the properties*

- i)  $d(\text{id}, \Omega, y) = 1, \quad \forall y \in \Omega$  (normalization)
- ii)  $d(h, \Omega, y) \neq 0 \Rightarrow \exists x \in \Omega : h(x) = y$  (solution property)
- iii)  $d(h(t), \Omega, y(t)) = \text{const.}$  for any  $h \in C^0(\overline{\Omega} \times [0, 1], \mathbb{R}^n)$ ,  $y \in C^0([0, 1], \mathbb{R}^n)$  with  $y(t) \in h(\partial\Omega, t)$ ,  $0 \leq t \leq 1$  (homotopy invariance).

For more information, confer [Dei85].

**Theorem 2.5.2.** (Borsuk-Ulam)

*Let  $\Omega \subset \mathbb{R}^n$  be open, bounded with  $\Omega = -\Omega$ ,  $0 \in \Omega$ , and let  $h \in C^0(\overline{\Omega}; \mathbb{R}^n)$  be odd with*

$$0 \notin h(\partial\Omega), \quad h(-u) = -h(u), \quad u \in \Omega.$$

*Then*

$$d(h, \Omega, 0) \text{ is odd;}$$

*in particular,  $d(h, \Omega, 0) \neq 0$ .*

**Theorem 2.5.3.** *Let  $0 \in U$  —  $U \subset \mathbb{R}^m \subset X$  be open. Then*

$$\gamma(\partial U) = m.$$

*Proof.* Clearly,  $\text{id} : \partial U \rightarrow \mathbb{R}^m \setminus \{0\}$  is odd, hence

$$\gamma(\partial U) \leq m.$$

Suppose  $\gamma(\partial U) = k \leq m$ , and let

$$h : \partial U \rightarrow \mathbb{R}^k \setminus \{0\} \hookrightarrow \mathbb{R}^m$$

be odd. We extend  $h$  to an odd map  $h \in C^0(X; \mathbb{R}^k)$ . Since  $0 \notin h(\partial U)$ , the topological degree of  $h$  in 0 w.r.t.  $U$  is well-defined and

$$d(h, U, 0) = 1 \pmod{2}$$

for all  $y \in \mathbb{R}^m$  close to 0. By the solution property of the degree for any such  $y \in \mathbb{R}^m$  there exists  $x \in U$  with  $h(x) = y$ ; hence  $k = m$ .  $\square$

The index has the following properties

**Theorem 2.5.4.** *Let  $A, A_1, A_2 \in \mathcal{A}$ ,  $h : X \rightarrow X$  continuous and odd. Then we have*

- i)  $\gamma(A) \geq 0$ ,  $\gamma(A) = 0$  iff  $A = \emptyset$  (definiteness)
- ii)  $A_1 \subset A_2 \Rightarrow \gamma(A_1) \leq \gamma(A_2)$  (monotonicity)

- iii)  $\gamma(A_1 \cup A_2) \leq \gamma(A_1) + \gamma(A_2)$  (sub-additivity)
- iv)  $\gamma(A) \leq \gamma(\overline{h(A)})$  (supervariance)
- v)  $A$  compact,  $0 \notin A \Rightarrow \gamma(A) < \infty$  and there is a neighborhood  $N \supset A$  with  $\overline{N} \in \mathcal{A}$  s.t.

$$\gamma(\overline{N}) = \gamma(A) \quad (\text{continuity})$$

- vi)  $0 \in A$  finite  $\Rightarrow \gamma(A) \leq 1$

*Proof.* i) immediate from the definition

- ii) We may assume  $\gamma(A_2) = m < \infty$ . Let  $h : A_2 \rightarrow \mathbb{R}^m \setminus \{0\}$  be continuous and odd. Then  $h|_{A_1} : A_1 \rightarrow \mathbb{R}^m \setminus \{0\}$  has these properties, and  $\gamma(A_1) \leq m$ .

- iii) Again we may assume that  $\gamma(A_1) = m_1 < \infty, \gamma(A_2) = m_2 < \infty$  and choose  $h_i : A_i \rightarrow \mathbb{R}^{m_i} \setminus \{0\}$  continuous and odd,  $i = 1, 2$ . Extend  $h_i$  to an odd map  $h_i \in C^0(X, \mathbb{R}^{m_i})$  for  $i = 1, 2$ . Then

$$h = (h_1, h_2) \in C^0(X, \mathbb{R}^{m_1+m_2})$$

is continuous and odd and satisfies

$$h(A_1 \cup A_2) \not\supset 0.$$

- iv) If  $\gamma(\overline{h(A)}) = m < \infty$ , we may choose an odd, continuous  $l : \overline{h(A)} \rightarrow \mathbb{R}^m \setminus \{0\}$ . With  $l \circ h : A \rightarrow \mathbb{R}^m \setminus \{0\}$  as comparison map, we conclude that  $\gamma(A) \leq m$ .
- v) For any  $u \in A$  let  $0 < r = r(u) < \|u\|_X$  and define

$$h_u : \overline{B_r(u; X) \cup B_r(-u; X)} \rightarrow \mathbb{R} \setminus \{0\}$$

by letting

$$h_u = \pm 1 \quad \text{in } B_r(\pm u; X).$$

Finitely many set  $N_{u_1}, \dots, N_{u_I}$ , where

$$N_u = B_r(u; X) \cup B_r(-u; X), \quad u \in A$$

cover  $A$ ; hence  $\gamma(A) < \infty$  by iii).

Let  $m = \gamma(A)$ , and choose an odd, continuous  $h : A \rightarrow \mathbb{R}^m \setminus \{0\}$ . Since  $A$  is compact, also  $h(A)$  is compact. Hence

$$2p := \text{dist}(0, h(A)) > 0.$$

Let  $h \in C^0(X, \mathbb{R}^m)$  be an odd extension of  $h$  and set

$$N = h^{-1}(U_p(h(A))).$$

Then  $\overline{N} \in \mathcal{A}$ , and  $\text{dist}(0, h(\overline{N})) \geq p > 0$  shows that  $\gamma(\overline{N}) \leq m$ . Equality follows by ii).

- vi) If  $A = \{u_i, -u_i : 1 \leq i \leq I\} \not\supset 0$ , choose  $h(\pm u_i) = \pm 1$ . Then  $h$  is odd and continuous with  $h : A \rightarrow \mathbb{R} \setminus \{0\}$ .

□

## Equivariant pseudo-gradient vector fields and flows

For applications, we now also need variants of Theorem 2.2.1 and Theorem 2.4.1 that respect the symmetry. We phrase these results for an arbitrary compact symmetry group  $G$ .

Let  $E \in C^1(X)$ ,  $M \subset X$  a  $C^{1,1}$ -submanifold,  $G$  a compact group acting linearly on  $X$  as a group of isometries with

$$gM = \{gu : u \in M\} = M, \quad g \in G,$$

and suppose  $E$  is  $G$ -equivariant in the sense

$$E(gu) = E(u), \quad u \in X, g \in G.$$

Note that then we also have

$$\begin{aligned} \langle dE(gu), gv \rangle_{X^* \times X} &= \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} E(g(u + \varepsilon v)) \\ &= \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} E(u + \varepsilon v) \\ &= \langle dE(u), v \rangle_{X^* \times X}; \end{aligned}$$

that is,

$$g^* dE(gu) = dE(u), \quad u \in X, g \in G. \quad (2.5.1)$$

Let

$$\widetilde{M} = \{u \in M : dE(u)|_{T_u M} \neq 0\}$$

be the regular set of  $E|_M$ .

**Definition 2.5.2.** A locally Lipschitz vector field  $\tilde{e} : \widetilde{M} \rightarrow X$  is a pseudo-gradient vector field for  $E$  on  $M$  if for any  $u \in \widetilde{M}$  there holds

- i)  $\tilde{e}(u) \in T_u M$ ,  $u \in \widetilde{M}$ ,
- ii)  $\|\tilde{e}(u)\|_X < 1$ ,  $u \in \widetilde{M}$ ,
- iii) and finally

$$\begin{aligned} \langle dE(u), \tilde{e}(u) \rangle_{X^* \times X} &> \frac{1}{2} \sup_{\substack{v \in T_u M \\ \|v\|_X < 1}} \langle dE(u), v \rangle_{X^* \times X} \\ &=: \frac{1}{2} \|dE|_M(u)\|_{T_u^* M}, \quad u \in \widetilde{M}. \end{aligned}$$

In the same way as Theorem 2.2.1 we then can show the following result.

**Theorem 2.5.5.** For any  $E \in C^1(X)$ ,  $\subset X$  as above there exists a pseudo-gradient vector field  $\tilde{e} : \widetilde{M} \rightarrow X$ . The vector field  $\tilde{e}$  may be chosen to be  $G$ -equivariant with

$$\tilde{e}(gu) = g(\tilde{e}(u)), \quad u \in \widetilde{M}, g \in G, \quad (2.5.2)$$

if  $E$  and  $M$  are in accordance with (2.5.1).



*Proof.* The construction of  $\tilde{e}$  with properties i)-iii) in Definition 2.5.2 proceeds exactly as in Theorem 2.2.1

To obtain a  $G$ -equivariant vector field, for any  $u \in \widetilde{M}$  let

$$\bar{e}(u) = \int_G g^{-1} \tilde{e}(gu) dg,$$

where  $dg$  is a Haar measure on  $G$ . For any  $g \in G$  then we have

$$\begin{aligned} \bar{e}(h(u)) &= \int_G g^{-1} \tilde{e}(ghu) dg \\ &= h \int_G \underbrace{h^{-1}g^{-1}}_{=(gh)^{-1}} \tilde{e}((gh)u) \underbrace{dg}_{d(gh)} \\ &= h\bar{e}(u) \end{aligned}$$

as desired. □

Similarly to Definition 2.3.1 we say

**Definition 2.5.3.** i)  $(u_k) \subset M$  is a  $(\mathcal{P}.\text{-}\mathcal{S}.)_\beta$ -sequence for  $E|_M$  if

$$E(u_k) \rightarrow \beta, \|dE|_M(u_k)\|_{T_{u_k}^* M} \rightarrow 0 \quad (k \rightarrow \infty).$$

ii)  $E_M$  satisfies  $(\mathcal{P}.\text{-}\mathcal{S}.)_\beta$ , if every  $(\mathcal{P}.\text{-}\mathcal{S}.)_\beta$ -sequence  $(u_k)$  is relatively compact.

Setting for  $\beta \in \mathbb{R}$ ,  $\delta, \rho > 0$  the sets

$$\begin{aligned} E_\beta &= \{u \in M : E(u) < \beta\}, \\ K_\beta &= \{u \in M : E(u) = \beta, dE|_M(u) = 0\}, \\ N_{\beta,\delta} &= \{u \in M : |E(u) - \beta| < \delta, \|dE|_M(u)\|_{T_u^* M} < \delta\}, \\ U_{\beta,\delta} &= \bigcup_{u \in K_\beta} B_\rho(u) \cap M, \end{aligned}$$

then the statements of Lemma 2.3.1 hold. Note that if  $E$  and  $M$  are  $G$ -equivariant, then also the sets  $E_\beta$ ,  $K_\beta$ ,  $N_{\beta,\delta}$ , and  $U_{\beta,\delta}$  are.

We then have the analogue of Theorem 2.4.1.

**Theorem 2.5.6.** Let  $E \in C^1(X)$ ,  $M \subset X$  as above,  $\beta \in \mathbb{R}$ , and suppose  $E|_M$  satisfies  $(\mathcal{P}.\text{-}\mathcal{S}.)_\beta$ . Then for any open  $N \supset K_\beta$ , any  $\bar{\varepsilon} > 0$  there exist  $\varepsilon > 0$  and a flow  $\Phi \in C^0(M \times [0, 1]; M)$  with the properties i)-iv) of Theorem 2.4.1.

If  $E$ ,  $M$ , and  $N$  are  $G$ -equivariant, the flow  $\Phi$  may be chosen to be equivariant, as well, satisfying

$$\Phi(gu, t) = g\Phi(u, t) \tag{2.5.3}$$

for all  $g \in G, u \in M, 0 \leq t \leq 1$ .

*Proof.* With  $\tilde{e} : \widetilde{M} \rightarrow X$  as in Theorem 2.5.5,  $0 < \rho, \delta \leq 1, \varepsilon > 0, \varphi_{\varepsilon,\beta}$  and  $\eta$  as in the proof of Theorem 2.4.1, let  $\Phi : M \times \mathbb{R} \rightarrow M$  solve

$$\begin{aligned} \frac{d}{dt} \Phi(u, t) &= -e(\Phi(u, t)) \\ \Phi(u, 0) &= u, \end{aligned}$$

where

$$e(u) = \eta(u)\varphi_{\varepsilon,\beta}(E(u))\tilde{e}(u) \in T_u M, \quad u \in \widetilde{M},$$

and  $e(u) = 0$  if  $dE|_M(u) = 0$ . The unique, global solution  $\Phi \in C^0(M \times \mathbb{R}; X)$  to this initial value problem defines a flow on  $M$  with the desired properties.

If  $E$ ,  $M$  and  $N$  are  $G$ -equivariant, we may choose  $\tilde{e}$  satisfying (2.5.2), and (2.5.2) then also holds true for  $e$ . Since  $g\Phi(u, t) =: \Phi_g(u, t)$  for any  $g \in G$ ,  $u \in M$  then also satisfies

$$\begin{aligned} \frac{d}{dt}\Phi_g(u, t) &= g \frac{d}{dt}\Phi(u, t) = -ge(\Phi(u, t)) \\ &= -e(\Phi_g(u, t)), \quad t \in \mathbb{R}, \\ \Phi_g(u, 0) &= gu, \end{aligned}$$

by uniqueness we have

$$\Phi_g(u, t) = \Phi(gu, t)$$

as desired. □

### Minimax-critical points of even functionals

Combining the equivariant deformation lemma for an even functional with the notion of index, we then obtain the following result (Now  $G = \mathbb{Z}_2 = \{\text{id}, -\text{id}\}$ ):

**Theorem 2.5.7.** *Let  $E \in C^1(X)$ ,  $M \subset X \setminus \{0\}$  as above with  $E(u) = E(-u)$ ,  $u \in X$ ,  $M = -M$ , and suppose that  $E|_M$  satisfies  $(\mathcal{P}\text{-}\mathcal{S})_\beta$  for every  $\beta \in \mathbb{R}$ . Suppose that for  $K \leq \infty$  the numbers*

$$\beta_k = \inf_{\substack{A \in \mathcal{A} \\ \gamma(A) \geq k}} \sup_{u \in A} E(u)$$

*are well-defined with  $\beta_k > -\infty$ ,  $1 \leq k < K$ . Then each  $\beta = \beta_k$ ,  $1 \leq k < K$ , is critical for  $E$ ,  $\beta_1 \leq \beta_2 \leq \dots \leq \beta_k \leq \beta_{k+1} \leq \dots$ , and if*

$$\beta_k = \beta_{k+1} = \dots = \beta_{k+l} =: \beta$$

*for some  $k$  and  $l \in \mathbb{N}$  there holds  $\#K_\beta = \emptyset$ .*

*Proof.* i) Suppose  $K_\beta = \emptyset$  for some  $\beta = \beta_k$ , and let  $\Phi, \varepsilon > 0$  be as constructed in Theorem 2.5.6 for  $N = \emptyset, \bar{\varepsilon} = 1$ .

By definition of  $\beta_k$  there exists  $A \in \mathcal{A}$  with  $\gamma(A) \geq k$  such that

$$\sup_{u \in A} E(u) < \beta + \varepsilon.$$

By (2.5.2) and Theorem 2.5.4 then

$$A_1 := \Phi(A, 1) \in \mathcal{A}, \quad \gamma(A_1) \geq \gamma(A) \geq k,$$

and by Theorem 2.5.6 we have  $A_1 \subset E_{\beta-\varepsilon}$ , that is,

$$\sup_{u \in A_1} E(u) \leq \beta - \varepsilon,$$

contradicting the definition of  $\beta_k$ .

- ii) Let  $\beta = \beta_k = \beta_{k+1} = \cdots = \beta_{k+l}$  for some  $k$  and  $l \in \mathbb{N}$ . By Lemma 2.3.1,  $K_\beta$  is compact, and  $0 \notin K_\beta \in \mathcal{A}$  by (2.5.1). By Theorem 2.5.4 (v) there exists  $N \supset K_\beta$  open with  $\overline{N} \in \mathcal{A}$  and  $\gamma(\overline{N}) = \gamma(K_\beta) < \infty$ . Let  $\Phi, \varepsilon > 0$  be as constructed in Theorem 2.5.5 for  $N$  and  $\bar{\varepsilon} = 1$ , and let  $A \in \mathcal{A}$  with  $\gamma(A) \geq k + l$  such that

$$\sup_{u \in A} E(u) < \beta + \varepsilon.$$

Then  $A \subset E_{\beta+\varepsilon}$ , and  $A_1 = \Phi(A, 1) \subset E_{\beta-\varepsilon} \cup N$  with

$$k + l \leq \gamma(A) \leq \gamma(A_1) \leq \gamma(\overline{E_{\beta-\varepsilon}}) \cup \gamma(\overline{N})$$

by Theorem 2.5.4viii) and iv). But

$$\gamma(\overline{E_{\beta-\varepsilon}}) < k$$

by definition of  $\beta_k$ ; hence

$$\gamma(K_\beta) = \gamma(\overline{N}) > l \geq 1,$$

and by Theorem 2.5.4 vi) it follows that  $\#K_\beta = \infty$ , as claimed.  $\square$

**Application.** As a model problem, we can apply Theorem 2.5.7 to obtain infinitely many solutions to the boundary value problem

$$-\Delta u = u|u|^{p-2} \quad \text{in } \Omega \subset \subset \mathbb{R}^n, \quad (2.5.4)$$

$$u = 0 \quad \text{on } \partial\Omega, \quad (2.5.5)$$

where  $2 < p < 2^* = \frac{2n}{n-2}$ , if  $n \geq 3$ .

**Theorem 2.5.8.** *Let  $\Omega \subset \subset \mathbb{R}^n$ ,  $2 < p < 2^* = \frac{2n}{n-2}$ , if  $n \geq 3$ . Then there exist infinitely many distinct pairs  $(u, -u)$  of solutions  $u \in H_0^1(\Omega)$  to (2.5.4), (2.5.5).*

*Proof.* Let  $X = H_0^1(\Omega)$ , and set

$$M = \{u \in H_0^1(\Omega) : \|u\|_{L^p} = 1\}.$$

Moreover, let

$$E(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{p} \int_{\Omega} |u|^p dx, \quad u \in H_0^1(\Omega).$$

Letting  $G = \{\text{id}, -\text{id}\}$ , we then note that  $M$  and  $E$  are  $G$ -equivariant with  $M = -M$  and

$$E(u) = E(-u), \quad u \in H_0^1(\Omega).$$

Moreover,  $M = f^{-1}(\{0\})$ , where

$$f(u) = \|u\|_{L^p}^p - 1, \quad u \in H_0^1(\Omega),$$

is of class  $C^2$ , with

$$T_u M = \ker df(u) = \left\{ v \in H_0^1(\Omega) : \int_{\Omega} v u |u|^{p-2} dx = 0 \right\}$$

for  $u \in M$ . Note that the projection

$$\pi_u : H_0^1(\Omega) \ni v \mapsto \pi_u(v) = v - \varepsilon_u(v)u \in T_u M$$

with

$$\varepsilon_u(v) = \int_{\Omega} v u |u|^{p-2} dx$$

is locally Lipschitz in  $u \in M$ , as we see from

$$\begin{aligned} \|\pi_{u_1}(v) - \pi_{u_2}(v)\|_{H_0^1} &= \|\varepsilon_{u_1}(v)u_1 - \varepsilon_{u_2}(v)u_2\|_{H_0^1} \\ &\leq |\varepsilon_{u_1}(v) - \varepsilon_{u_2}(v)| \|u_1\|_{H_0^1} + |\varepsilon_{u_2}(v)| \|u_1 - u_2\|_{H_0^1} \end{aligned}$$

with

$$|\varepsilon_{u_2}(v)| \leq \|u_2\|_{L^p}^{p-1} \|v\|_{L^p} \leq C \|u_2\|_{H_0^1}^{p-1} \|v\|_{H_0^1}$$

and with

$$\begin{aligned} |\varepsilon_{u_1}(v) - \varepsilon_{u_2}(v)| &\leq \|v\|_{L^p} \|u_1|u_1|^{p-2} - u_2|u_2|^{p-2}\|_{L^{\frac{p}{p-1}}} \\ &\leq C \|v\|_{L^p} (\|u_1\|_{L^p}^{p-2} + \|u_2\|_{L^p}^{p-2}) \|u_1 - u_2\|_{L^p}. \end{aligned}$$

Finally,  $E|_M$  satisfies  $(\mathcal{P}\text{-}\mathcal{S})_{\beta}$  for any  $\beta \in \mathbb{R}$ . To see this, let  $(u_k) \subset M$  be a  $(\mathcal{P}\text{-}\mathcal{S})_{\beta}$ -sequence. Then  $(u_k) \subset M$  is bounded, and we may select a subsequence with  $u_k \xrightarrow{w} u$  in  $H_0^1(\Omega)$  and strongly in  $L^p(\Omega)$ . Thus, also  $v_k = \pi_{u_k}(u_k - u)$  is bounded, and with error  $o(1) \rightarrow 0$  there holds

$$\begin{aligned} o(1) &= \langle dE(u_k), \pi_{u_k}(u_k - u) \rangle \\ &= \int_{\Omega} \nabla u_k \nabla (u_k - u) dx - \varepsilon_{u_k}(u_k - u) \int_{\Omega} |\nabla u_k|^2 dx \\ &= \|\nabla(u_k)\|_{L^2}^2 - 2 \underbrace{E(u_k)}_{\rightarrow \beta} \int_{\Omega} \underbrace{(u_k - u) u_k |u_k|^{p-2}}_{\rightarrow 0 \text{ in } L^p(\Omega)} dx + o(1) \\ &= \|u_k - u\|_{H_0^1}^2 + o(1), \end{aligned}$$

and  $u_k \rightarrow u$  strongly in  $H_0^1(\Omega)$ , as desired. Theorem 2.5.7 thus is applicable, and yields infinitely many distinct critical points  $u_k$  of  $E|_M$  with critical values  $\beta_k$ , satisfying the equation

$$\begin{aligned} 0 &= \langle dE(u_k), \pi_{u_k}(v) \rangle_{H^{-1} \times H_0^1} \\ &= \int_{\Omega} \nabla u_k \nabla v dx - \varepsilon_{u_k}(v) \underbrace{\int_{\Omega} |\nabla u_k|^2 dx}_{= 2\beta_k + \frac{2}{p}} \\ &= \int_{\Omega} (\nabla u_k \nabla v - \alpha_k^{p-2} u_k |u_k|^{p-2} v) dx \end{aligned}$$

where  $\alpha_k = (2\beta_k + \frac{2}{p})^{\frac{1}{p-2}}$ . Then

$$\tilde{u}_k = \alpha_k u_k = \left(2\beta_k + \frac{2}{p}\right)^{\frac{1}{p-2}} u_k \in H_0^1(\Omega)$$

solves (2.5.4), (2.5.5) with

$$\begin{aligned}\tilde{\beta}_k &:= E(\tilde{u}_k) = \frac{1}{2} \|\nabla \tilde{u}_k\|_{L^2}^2 - \frac{1}{p} \|\tilde{u}_k\|_{L^p}^p \\ &= \alpha_k^2 \left( \beta_k + \frac{1}{p} \right) - \frac{1}{p} \alpha_k^p = \left( \frac{1}{2} - \frac{1}{p} \right) \alpha_k^p \\ &= \left( \frac{1}{2} - \frac{1}{p} \right) \left( 2\beta_k + \frac{2}{p} \right)^{\frac{p}{p-2}}, \quad k \in \mathbb{N}.\end{aligned}$$

□

**Remark 2.5.2.** i) We can estimate the rate of growth of the sequence  $\beta_k, k \in \mathbb{N}$ . Let

$$0 < \lambda_1 < \lambda_2 \leq \dots \leq \lambda_k \leq \lambda_{k+1} \leq \dots$$

be the Dirichlet eigenvalues of  $-\Delta$  on  $\Omega$  with  $L^2$ -orthonormal eigenfunctions  $\varphi_k$  satisfying

$$\begin{aligned}-\Delta \varphi_k &= \lambda_k \varphi_k \quad \text{in } \Omega, \\ \varphi_k &= 0 \quad \text{on } \partial\Omega, \quad k \in \mathbb{N}.\end{aligned}$$

For  $k \in \mathbb{N}$  then also let

$$X_k = \text{span}\{\varphi_1, \dots, \varphi_k\}$$

with  $L^2$ -orthogonal projection

$$\pi_k : H_0^1(\Omega) \rightarrow X_k \cong \mathbb{R}^k.$$

Note that for any  $u \in \ker \pi_{k-1}$  there holds

$$\begin{aligned}\|\nabla u\|_{L^2}^2 &= \sum_{i \geq k} (u, \varphi_i)_{L^2}^2 \overbrace{\|\nabla \varphi_i\|_{L^2}^2}^{= \lambda_i \|\varphi_i\|_{L^2}^2 = \lambda_i} \\ &\geq \lambda_k \sum_{i > k} (u, \varphi_i)_{L^2}^2 = \lambda_k \|u\|_{L^2}^2.\end{aligned}$$

Interpolating, then for  $2 < p < p^*$  we find

$$\|u\|_{L^p} \leq \|u\|_{L^2}^\alpha \|u\|_{L^{2^*}}^{1-\alpha} \leq C \lambda_k^{-\frac{\alpha}{2}} \|\nabla u\|_{L^2},$$

where

$$\frac{1}{p} = \frac{\alpha}{2} + \frac{1-\alpha}{2^*} = \frac{1}{2^*} + \frac{\alpha}{n}, \quad \alpha = n \left( \frac{1}{p} - \frac{1}{2^*} \right) = 1 - n \left( \frac{1}{p} - \frac{1}{2} \right).$$

With  $M = \{u \in H_0^1(\Omega) : \|u\|_{L^p} = 1\}$  as before, then for any  $A \in \mathcal{A}$  there holds

$$\gamma(A) \geq k \Rightarrow \pi_{k-1}(A) \ni 0;$$

hence with a constant  $c_0 > 0$  we have

$$2\beta_k + \frac{2}{p} \geq \inf_{\substack{\pi_{k-1}(u)=0 \\ u \in M}} \|\nabla u\|_{L^2}^2 \geq c_0 \lambda_k^\alpha = c_0 \lambda_k^{1-n \overbrace{\left( \frac{1}{p} - \frac{1}{2} \right)}^{\frac{p-2}{2p}}}$$

for any  $k \in \mathbb{N}$ .

The energies  $E(\tilde{u}_k)$  of the corresponding solutions then are bounded from below by

$$\tilde{\beta} = \left( \frac{1}{2} - \frac{1}{p} \right) \left( 2\beta_k + \frac{2}{p} \right) \geq c_1 \lambda_k^{\frac{p}{p-2} - \frac{n}{2}}.$$

Finally, with the Wely asymptotic formula

$$\lambda_k \cong c(\Omega) k^{\frac{2}{n}}, \quad k \in \mathbb{N},$$

we obtain the bound

$$\tilde{\beta}_k \geq c_2 k^{\frac{2p}{n(p-2)} - 1}, \quad k \in \mathbb{N},$$

with constants  $c_1, c_2, c(\Omega) > 0$ .

ii) Bahri-Lions [BL88] improved the bound in i) to

$$\tilde{\beta}_k \geq c_0 k^{\frac{2p}{n(p-2)}}, \quad k \in \mathbb{N}.$$

iii) Letting, for  $X = H_0^1(\Omega)$ ,

$$\Gamma = \{h \in C^0(X, X) : h(-u) = -h(u), h(u) = u \text{ if } E(u) \leq 0\},$$

we can compare  $\tilde{\beta}_k$  with the numbers

$$\tilde{\delta}_k := \inf_{h \in \Gamma} \sup_{u \in X_k} E(h(u)) \geq \tilde{\beta}_k, \quad k \in \mathbb{N}.$$

*Proof.* Let

$$\widetilde{M} := \left\{ u \in H_0^1(\Omega) \setminus \{0\} : 0 = \langle d\tilde{E}(u), u \rangle_{H^{-1} \times H_0^1} = \|u\|_{H_0^1}^2 - \|u\|_{L^p}^p \right\}$$

and observe that  $\widetilde{M}$  is homeomorphic to  $M$  via

$$\pi : \widetilde{M} \ni u \rightarrow \frac{u}{\|u\|_{L^p}} \in M.$$

For any  $h \in \Gamma$ , and  $k \in \mathbb{N}$ , the set  $h^{-1}(\widetilde{M}) \cap X_k$  bounds an open neighborhood  $U = -U$  of 0 in  $X_k \cong \mathbb{R}^k$ ; hence

$$\gamma(h^{-1}(\widetilde{M}) \cap X_k) = \gamma(\partial U) \geq k$$

by Theorem 2.5.3, and thus by Theorem 2.5.4 iii)

$$A := \pi(h(X_k) \cap \widetilde{M}) \in \mathcal{A}$$

satisfies

$$\gamma(A) \geq k.$$

So

$$\sup_{v \in A} E(v) \geq \beta_k.$$

Moreover, for any  $u \in h(X_k) \cap \widetilde{M}$  with  $\pi(u) = v \in M$  and  $u = \alpha v$  we have

$$\alpha^2 \|\nabla v\|_{L^2}^2 = \|\nabla u\|_{L^2}^2 = \|u\|_{L^p}^p = \alpha^p, \quad \alpha^{p-2} = \|\nabla v\|_{L^2}^2;$$

so

$$\begin{aligned} E(u) &= \left(\frac{1}{2} - \frac{1}{p}\right) \|\nabla u\|_{L^2}^2 = \left(\frac{1}{2} - \frac{1}{p}\right) \alpha^2 \|\nabla v\|_{L^2}^2 \\ &= \left(\frac{1}{2} - \frac{1}{p}\right) \alpha^p = \left(\frac{1}{2} - \frac{1}{p}\right) \|\nabla v\|_{L^2}^{\frac{2p}{p-2}} \\ &= \left(\frac{1}{2} - \frac{1}{p}\right) \left(2E(v) + \frac{2}{p}\right)^{\frac{p}{p-2}} \end{aligned}$$

and there results

$$\sup_{u \in X_k} E(h(u)) \geq \left(\frac{1}{2} - \frac{1}{p}\right) \left(2\beta_k + \frac{2}{p}\right)^{\frac{p}{p-2}} = \widetilde{\beta}_k.$$

□

## Perturbation theory

What happens, if we perturb the symmetry? Given  $f \in L^2(\Omega)$ , do we still find infinitely many solutions  $u$  of

$$-\Delta u = u|u|^{p-2} + f \quad \text{in } \Omega, \quad (2.5.6)$$

$$u = 0 \quad \text{on } \partial\Omega? \quad (2.5.7)$$

**Theorem 2.5.9.** (Struwe (1980), Bahri-Berestycki (1981), Bahri-Lions (1988))

Let  $\Omega \subset \subset \mathbb{R}^n$ ,  $f \in L^2(\Omega)$ , and suppose  $2 < p < 2^* = \frac{2n}{n-2}$ , if  $n \geq 3$  satisfies

$$\frac{2p}{n(p-2)} > \frac{p}{p-1}, \quad \text{i.e. } p < \frac{2(n-1)}{n-2}. \quad (2.5.8)$$

Then (2.5.6), (2.5.7) admits infinitely many distinct solutions.

**Remark.** If  $n = 3$ , condition (2.5.8) is satisfied for any  $2 < p < 4$ .

The proof of Theorem 2.5.9 may be considerably simplified with the help of the following abstract result

**Theorem 2.5.10.** (Rabinowitz (1982))

Suppose  $E \in C^1(X)$  satisfies  $(\mathcal{P}.\mathcal{S}.)_\beta$  for all  $\beta \in \mathbb{R}$ . Let  $W \subset X$  be a subspace with  $\dim W < \infty$ ,  $w^* \in X \setminus W$ , and set  $W^* = W \oplus \text{span}\{w^*\}$ . Also let

$$W_+^* = \{w + tw^* : w \in W, t \geq 0\}.$$

Suppose

$$i) \quad E(0) = 0$$

$$ii) \quad \exists E^* > 0 \forall u \in W^* : \|u\|_X \geq R^* \Rightarrow E(u) \leq 0.$$

Let

$$\Gamma = \{h \in C^0(X, X) : h(-u) = -h(u), h(u) = u \text{ if } \max\{E(u), E(-u)\} \leq 0\}$$

and suppose

$$\beta^* := \inf_{h \in \Gamma} \sup_{u \in W_+^*} E(h(u)) > \beta := \inf_{h \in \Gamma} \sup_{u \in W} E(h(u)) > 0.$$

Then there is  $\gamma^* \geq \beta^*$  with  $K_{\gamma^*} \neq \emptyset$ .

*Proof.* Fix some  $\gamma \in ]\beta, \beta^*[$  and let

$$\Lambda = \{h \in \Gamma : E(h(u)) \leq \gamma \ \forall u \in W\}.$$

Then  $\Lambda \neq \emptyset$  by definition of  $\beta$  and

$$\gamma^* = \inf_{h \in \Lambda} \sup_{u \in W_+^*} E(h(u)) \geq \beta^* > \gamma$$

is well-defined.

**Claim.**  $K_{\gamma^*} \neq \emptyset$ .

Suppose  $K_{\gamma^*} = \emptyset$ . For  $\gamma^*, \bar{\varepsilon} = \gamma^* - \gamma > 0$ ,  $N = \emptyset$  let  $0 < \varepsilon < \bar{\varepsilon}$ ,  $\Phi \in C^0(X \times [0, 1], X)$  be as in Theorem 2.4.1. Choose  $h \in \Lambda$  with

$$\sup_{u \in W_+^*} E(h(u)) < \gamma^* + \varepsilon.$$

Note that by choice of  $\bar{\varepsilon}$  and since  $h \in \Lambda$  we have

$$\Phi(h(-u), 1) = h(-u) = -h(u) = -\Phi(h(u), 1) \quad (2.5.9)$$

for any  $u \in W$ . Hence, letting

$$h_1(u) = \begin{cases} \Phi(h(u), 1), & \text{if } u \in W_+^*, \\ -\Phi(h(-u), 1), & \text{if } -u \in W_+^* \end{cases}$$

we obtain an odd map  $h_1 \in C^0(W^*, X)$ .

Moreover,  $h_1(u) = u$  for any  $u$  with  $\max\{E(u), E(-u)\} \leq 0$ , since  $h \in \Lambda \subset \Gamma$  and  $\Phi(\cdot, 1)$  both do, as well. Hence  $h_1$  may be extended to  $h_1 \in \Gamma$ , and  $h_1 \in \Lambda$  by (2.5.9). But

$$\sup_{u \in W_+^*} E(h_1(u)) \leq \gamma^* - \varepsilon.$$

□

*Proof of Theorem 2.5.9.* Let

$$E_f(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{p} \int_{\Omega} |u|^p dx - \int_{\Omega} f u dx, \quad u \in H_0^1(\Omega),$$

and let

$$E_0(u) = \frac{E_f(u) + E_f(-u)}{2} = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{p} \int_{\Omega} |u|^p dx.$$

Note that we have

$$dE_0(u) - dE_f(u) = f, \quad u \in H_0^1(\Omega).$$



Moreover,  $E_f$  satisfies  $(\mathcal{P}\text{-}\mathcal{S})_\beta$  for any  $\beta \in \mathbb{R}$ . Suppose by contradiction that there is  $\beta_0 > 1$  such that  $E_f$  has no critical value  $\beta > \beta_0$ . Choose  $k_0 \in \mathbb{N}$  such that

$$\tilde{\delta}_k = \inf_{h \in \Gamma} \sup_{u \in X_k} E_0(h(u)) > \beta_0$$

for  $k \geq k_0$ , where  $X_k = \text{span}\{\varphi_1, \dots, \varphi_k\}$  as in Remark 2.5.2, and where

$$\Gamma = \{h \in C^0(X, X) : h(-u) = -h(u), h(u) = u \text{ if } E_0(u) \leq \max\{E_f(u), E_f(-u)\} \leq 0\}$$

Then for  $k \geq k_0$  we also have

$$\delta_k := \inf_{h \in \Gamma} \sup_{u \in X_k} E_f(h(u)) \geq \tilde{\delta}_k > \beta_0.$$

From our assumption and Theorem 2.5.10 we then conclude that for every  $k \geq k_0$  there holds

$$\delta_k^* := \inf_{h \in \Gamma} \sup_{u \in X_{k+1}^+} E_f(h(u)) = \delta_k,$$

where

$$X_{k+1}^+ = \{w + t\varphi_{k+1} : w \in X_k, t \geq 0\}.$$

Choose  $h \in \Gamma$  so that

$$\sup_{u \in X_{k+1}^+} E_f(h(u)) \leq \delta_k + 1.$$

**Claim 1.** We may assume that  $E_f$  achieves its maximum in  $h(X_{k+1})$  in  $N_{12}$ , where

$$N_a = \{u \in X : \|dE_f(u)\|_{H^{-1}} + \|dE_f(-u)\|_{H^{-1}} < a\|f\|_{H^{-1}} = a\|dE_0(u) - dE_f(u)\|_{H^{-1}}\}.$$

Note that for  $a > 2$  the set  $N_a = -N_a$  is an open neighborhood of

$$K = \{u \in X : dE_f(u) = 0 \text{ or } dE_f(-u) = 0 \text{ or } dE_0(u) = 0\}.$$

*Proof.* Let  $\tilde{e}$  be an odd pseudo-gradient vector field for  $E_0$  and set

$$e(u) = -\varphi(\max\{E_f(u), E_f(-u)\})\eta(u)\tilde{e}(u),$$

if  $u \notin K$ ,  $e(u) = 0$  for  $u \in K$ , where  $\varphi \in C^\infty$  satisfies  $0 \leq \varphi \leq 1$ ,  $\varphi(s) = \begin{cases} 0, & s \leq 0 \\ 1, & s \geq 1 \end{cases}$ , and

where  $\eta$  is locally Lipschitz continuous with  $0 \leq \eta(u) = \eta(-u) \leq 1$  (so that  $e$  is odd) and

$$\eta(u) = \begin{cases} 1, & u \notin N_{12}, \\ 0, & u \in N_8. \end{cases}$$

Then, if  $u \notin N_{12}$  with  $\max\{E_f(u), E_f(-u)\} \geq 1$  we have

$$\begin{aligned} -\langle dE_f(u), e(u) \rangle_{H^{-1} \times H_0^1} &= \langle dE_0(u), \tilde{e}(u) \rangle_{H^{-1} \times H_0^1} - \langle f, \tilde{e}(u) \rangle_{H^{-1} \times H_0^1} \\ &\geq \frac{1}{2} \|dE_0(u)\|_{H^{-1}} - \|f\|_{H^{-1}} \geq \frac{1}{2} \|dE_f(\pm u)\|_{H^{-1}} - 2\|f\|_{H^{-1}} \\ &\geq \frac{1}{4} (\|dE_f(u)\|_{H^{-1}} + \|dE_f(-u)\|_{H^{-1}} - 8\|f\|_{H^{-1}}) \\ &\geq \|f\|_{H^{-1}}. \quad (\text{Similarly, } \langle dE_f(u), e(u) \rangle_{H^{-1} \times H_0^1} \leq 0, \text{ if } u \notin N_8.) \end{aligned}$$

That is, for the (even) flow  $\Phi$  generated by  $e$  we have

$$\frac{d}{dt} E_f(\Phi(u, t)) \leq -\|f\|_{H^{-1}}$$

away from  $N_{12}$ . Choosing  $h_t = \Phi(\cdot, t) \circ h$  for large  $t \geq 0$ , we obtain our claim.  $\square$

In view of Claim 1 then we have

$$\begin{aligned} \delta_k + 1 &\geq \sup_{u \in X_{k+1}^+} E_f(h(u)) \geq \sup_{u \in X_{k+1}} E_f(h(u)) - \sup_{\substack{u \in N_{12} \\ \min\{E_f(u), E_f(-u)\} \leq \delta_k + 1}} \overbrace{|E_f(u) - E_f(-u)|}^{=2 \int_{\Omega} f u \, dx} \\ &\geq \delta_{k+1} - C \sup_{\substack{u \in N_{12} \\ \min\{E(u), E(-u)\} \leq \delta_k + 1}} \|u\|_{L^2} \end{aligned}$$

**Claim 2.** For  $u \in N$  with  $\min\{E_f(u), E_f(-u)\} \leq \delta_k + 1$  there holds

$$\|u\|_{L^2} \leq C \delta_k^{\frac{1}{p}} + C.$$

*Proof.* Compute, using Young's inequality

$$\begin{aligned} \|u\|_{L^2}^p &\leq C \|u\|_{L^p}^p = C \left( 2E_0(u) - \langle dE_0(u), u \rangle_{H^{-1} \times H_0^1} \right) \\ &\leq C \left( 2 \min\{E_f(u), E_f(-u)\} - \langle dE_f(u), u \rangle_{H^{-1} \times H_0^1} + 3\|f\|_{L^2} \|u\|_{L^2} \right) \\ &\leq C(\delta_k + 1) + C\|f\|_{L^2} \|u\|_{L^2} \\ &\leq C\delta_k + \frac{1}{2} \|u\|_{L^2}^p + C \end{aligned}$$

The claim follows. □

Since  $\delta_k \geq \beta_k \rightarrow \infty$  ( $k \rightarrow \infty$ ) we conclude the bound

$$\begin{aligned} \delta_{k+1} &\leq \delta_k + C \delta_k^{\frac{1}{p}} + C \\ &\leq \delta_k + C \delta_k^{\frac{1}{p}} = \delta_k \left( 1 + C \delta_k^{\frac{1-p}{p}} \right), \quad k \geq k_0. \end{aligned}$$

By iteration, Remark 2.5.2, and (2.5.8) then

$$\begin{aligned} \delta_{k_0+l} &\leq \delta_{k_0+l-1} \left( 1 + C \delta_{k_0+l-1}^{\frac{1-p}{p}} \right) \leq \dots \\ &\leq \delta_{k_0} \prod_{j=0}^{l-1} \left( 1 + C \delta_{k_0+j}^{\frac{1-p}{p}} \right) \\ &= \delta_{k_0} \exp \left( \sum_{j=0}^{l-1} \log \left( 1 + C \delta_{k_0+j}^{\frac{1-p}{p}} \right) \right) \\ &\leq \delta_{k_0} \exp \left( \sum_{j=0}^{\infty} C \delta_{k_0+j}^{\frac{1-p}{p}} \right) \\ &\leq \delta_{k_0} \exp \left( C \sum_{k=k_0}^{\infty} k^{\frac{2p}{n(p-2)} \cdot \frac{1-p}{p}} \right) \\ &\leq C \delta_{k_0}, \quad \text{uniformly in } l \in \mathbb{N} \end{aligned}$$

if  $\frac{2p}{n(p-2)} < \frac{p}{p-1}$ . Contradiction! □

**Remark.** A. Bahri [Bah81] showed that for any  $p \in ]2, 2^*[$  the equation (2.5.6), (2.5.7) has infinitely many solutions for generic  $f \in L^2(\Omega)$ .

## 2.6 Lusternik-Schnirelman theory on convex sets - The Plateau problem

### 2.6.1 Abstract setting

Let  $X$  be a Banach space,  $E \in C^1(X)$ , and let  $M \subset X$  be closed and convex.

**Definition 2.6.1.** For any  $u \in M$  the number

$$g(u) := \sup_{\substack{v \in M \\ \|u-v\|_X \leq 1}} \langle dE(u), u-v \rangle_{X^* \times X}$$

is the slope of  $E$  at  $u$  relative to  $M$ .

**Remark 2.6.1.** i) If  $M = X$  there holds

$$g(u) = \|dE(u)\|_{X^*}, \quad u \in M = X.$$

ii) The map  $M \ni u \mapsto g(u) \in \mathbb{R}$  is continuous.

*Proof.* For any  $u_0 \in M$ , any  $\varepsilon > 0$  there is an open  $U_0 \subset B_\varepsilon(u_0; X)$  such that  $u_0 \in U_0$  and

$$\|dE(u) - dE(u_0)\|_{X^*} < \varepsilon, \quad u \in U_0 \cap M.$$

For  $u \in U_0 \cap M$  let  $v = v(u) \in M$  such that

$$g(u) \leq \langle dE(u), u-v \rangle_{X^* \times X} + \varepsilon, \quad \|u-v\|_X < 1 - \varepsilon.$$

Then for  $u_1, u_2 \in U_0 \cap M$  with  $v_i = v(u_i) \in M$

$$\begin{aligned} g(u_1) - g(u_2) &\leq \langle dE(u_1), u_1 - v_1 \rangle_{X^* \times X} - \langle dE(u_2), u_2 - v_1 \rangle_{X^* \times X} + \varepsilon \\ &\leq \|dE(u_1) - dE(u_2)\|_{X^*} + \|dE(u_2)\|_{X^*} \|u_1 - u_2\|_X + \varepsilon \\ &\leq C\varepsilon. \end{aligned}$$

□

**Definition 2.6.2.** i) A point  $u \in M$  is critical for  $E$  in  $M$ , if

$$g(u) = 0;$$

otherwise,  $u$  is called regular.

ii) A number  $\beta \in \mathbb{R}$  is a critical value of  $E$  in  $M$ , if there exists a critical point  $u \in M$  with

$$E(u) = \beta;$$

otherwise,  $\beta \in \mathbb{R}$  is called a regular value.

As before, for  $\beta \in \mathbb{R}, \delta > 0, \rho > 0$  we let

$$\begin{aligned} E_\beta &= \{u \in M : E(u) < \beta\}, \\ K_\beta &= \{u \in M : g(u) = 0, E(u) = \beta\}, \\ N_{\beta,\delta} &= \{u \in M : |E(u) - \beta| < \delta, g(u) < \delta\}, \\ U_{\beta,\rho} &= \{u \in M : \exists v \in K_\beta : \|u - v\|_X < \rho\}. \end{aligned}$$

Also let

$$\widetilde{M} = \{u \in M : g(u) \neq 0\}$$

be the set of regular points.

**Definition 2.6.3.** i) A sequence  $(u_k) \subset M$  is a  $(\mathcal{P}.\text{-}\mathcal{S}.)_\beta$ -sequence for  $E$  in  $M$ , if

$$E(u_k) \rightarrow \beta, g(u_k) \rightarrow 0 \quad (k \rightarrow \infty).$$

ii)  $E$  satisfies  $(\mathcal{P}.\text{-}\mathcal{S}.)_\beta$  in  $M$ , if every  $(\mathcal{P}.\text{-}\mathcal{S}.)_\beta$ -sequence in  $M$  is relatively compact.

Then we have the analogue of Lemma 2.3.1.

**Lemma 2.6.1.** Suppose  $E$  satisfies  $(\mathcal{P}.\text{-}\mathcal{S}.)_\beta$  in  $M$ . Then there holds

- i)  $K_\beta \subset M$  is compact;
- ii) The families  $(N_{\beta,\delta})_{\delta>0}, (U_{\beta,\rho})_{\rho>0}$  each are fundamental systems of neighborhoods of  $K_\beta$ ; in particular,
- iii) if  $K_\beta = \emptyset$  there exists  $\delta > 0$  with  $N_{\beta,\delta} = \emptyset$ .

*Proof.* Identical with the proof of Lemma 2.3.1. □

**Definition 2.6.4.** A locally Lipschitz continuous map  $\tilde{e} : \widetilde{M} \rightarrow X$  defines a (negative) pseudo-gradient vector field on  $M$ , if

$$i) \quad v(u) = u + \tilde{e}(u) \in M, \quad u \in \widetilde{M},$$

and if

- ii)  $\|\tilde{e}(u)\|_X = \|u - v(u)\|_X < 1, \quad u \in \widetilde{M},$
- iii)  $\frac{1}{2}g(u) < \langle dE(u), u - v(u) \rangle_{X^* \times X}, \quad u \in \widetilde{M}.$

Similar to Theorem 2.3.1 we can then show the following result.

**Theorem 2.6.1.** There exists a pseudo-gradient vector field  $\tilde{e} : \widetilde{M} \rightarrow X$ .

*Proof.* For any  $u_0 \in \widetilde{M}$  choose  $v_0 = v(u_0) \in M$  with  $\|u_0 - v_0\|_X < 1$  and

$$\frac{1}{2}g(u_0) < \langle dE(u_0), u_0 - v_0 \rangle_{X^* \times X}.$$

By continuity of  $dE$  and  $g$  there exists an open  $U_0 = U(u_0) \ni u_0$  such that

$$\forall u \in U_0 \cap M : \frac{1}{2}g(u) < \langle dE(u), u - v_0 \rangle_{X^* \times X}, \quad \|u - v_0\|_X < 1.$$

The family  $(U(u))_{u \in \widetilde{M}}$  admits a locally finite refinement  $U_\iota \subset U(u_\iota)$ ,  $\iota \in I$ , still covering  $\widetilde{M}$ . With a locally Lipschitz continuous partition of unity  $(\varphi_\iota)_{\iota \in I}$  subordinate to  $(U_\iota)_{\iota \in I}$  then let

$$v(u) = \sum_{\iota \in I} \varphi_\iota(u) v(u_\iota), \quad u \in \widetilde{M}.$$

By convexity of  $M$ , then  $v(u) \in M$  for every  $u \in \widetilde{M}$ , and

$$\widetilde{e}(u) := v(u) - u$$

has the desired properties.  $\square$

**Remark 2.6.2.** Since we assume that  $E \in C^1(X)$  is globally defined, we can extend the definition of  $g$  to the open neighborhood

$$M_1 = \bigcup_{u \in M} B_1(u; X) \subset X$$

of  $M$  and proceed as in Theorem 2.6.1 to obtain  $\widetilde{e}_1 : \widetilde{M}_1 \rightarrow X$  with the properties i) – iii) in Definition 2.6.4, extending  $\widetilde{e}$  to  $\widetilde{M}_1$ .

**Theorem 2.6.2.** Suppose  $E \in C^1(X)$  satisfies  $(\mathcal{P}.\text{-}\mathcal{S}.)_\beta$  on  $M$  for some  $\beta \in \mathbb{R}$ . Then for every  $\bar{\varepsilon} > 0$ , every open  $N \supset K_\beta$  in  $M$  there exists  $0 < \varepsilon < \bar{\varepsilon}$  and a negative pseudo-gradient flow  $\Phi \in C^0(M \times [0, 1]; M)$  such that

- i)  $\Phi(\cdot, t) : M \rightarrow M$  is continuous,  $0 \leq t \leq 1$ ;
- ii)  $\Phi(u, t) = u$  if either  $t = 0$ , or if  $g(u) = 0$ , or if  $|E(u) - \beta| \geq \bar{\varepsilon}$ ;
- iii) the map  $t \mapsto E(\Phi(u, t))$  is non-increasing for any  $u \in M$ ;
- iv)  $\Phi(E_{\beta+\varepsilon}, 1) \subset E_{\beta-\varepsilon} \cup N$ , and  $\Phi(E_{\beta+\varepsilon} \setminus N, 1) \subset E_{\beta-\varepsilon}$ .

*Proof.* Given  $N \supset K_\beta$ , with the help of Lemma 2.6.1 we find  $0 < \delta, \rho \leq 1$  with

$$N \supset U_{\beta, 3\rho} \supset U_{\beta, \rho} \supset N_{\beta, \delta} \supset K_\beta$$

as in the proof of Theorem 2.4.1. With

$$0 < \varepsilon = \frac{1}{4} \min\{\bar{\varepsilon}, \rho\delta\} < \frac{\bar{\varepsilon}}{2}$$

and locally Lipschitz  $0 \leq \varphi, \eta \leq 1$  such that

$$\begin{aligned} \varphi_{\beta, \varepsilon}(E(u)) &= \varphi\left(\frac{|E(u) - \beta|}{\varepsilon}\right), \\ \eta(u) &= \begin{cases} 0, & u \in U_{\beta, \rho}, \\ 1, & u \notin U_{\beta, 2\rho}, \end{cases} \end{aligned}$$

we then let

$$e(u) = \begin{cases} \eta(u) \varphi_{\beta, \varepsilon}(E(u)) \widetilde{e}(u), & u \in \widetilde{M}, \\ 0, & \text{if } g(u) = 0, \end{cases}$$

to obtain a locally Lipschitz continuous vector field  $e : M \rightarrow X$  with

$$\|e(u)\|_X, e(u) + u \in M, u \in M,$$

and such that

$$\langle dE(u), e(u) \rangle_{X^* \times X} \leq 0, \quad u \in M.$$

**Claim:** There exists a unique solution  $\Phi \in C^0(M \times [0, \infty[; M)$  of the initial value problem

$$\begin{aligned} \frac{d}{dt}\Phi(u, t) &= e(\Phi(u, t)), \quad t \geq 0, \\ \Phi(u, 0) &= u \end{aligned}$$

for every  $u \in M$

*Proof.* With the extension  $\tilde{e}_1 : \widetilde{M}_1 \rightarrow X$  as in Remark 2.6.2 we can extend  $e$  as

$$e_1(u) = (1 - \text{dist}(u, M))\eta(u)\varphi_{\beta, \varepsilon}(E(u))\tilde{e}_1(u)$$

to a locally Lipschitz continuous vector field  $e_1 : X \rightarrow X$ . The solution  $\Phi_1 \in C^0(X \times \mathbb{R}; X)$  to

$$\begin{aligned} \frac{d}{dt}\Phi_1(u, t) &= e_1(\Phi(u, t)), \\ \Phi_1(u, 0) &= u \end{aligned}$$

then defines a flow of homeomorphisms  $\Phi_1(\cdot, t)$  of  $X$ . Since for any  $u \in M$  we have

$$u + \left. \frac{d}{dt} \right|_{t=0} \Phi_1(u, t) = u + e_1(u) \in M,$$

the set  $M$  is forward  $\Phi_1$ -invariant and  $\Phi := \Phi_1|_{M \times [0, 1]}$  satisfies

$$\Phi(\cdot, t) : M \rightarrow M, \quad 0 \leq t \leq 1.$$

The remaining properties follow from the properties of  $e$  as in the proof of Theorem 2.4.1. □

□

The following is a variant of the "mountain-pass" lemma.

**Theorem 2.6.3.** Suppose  $E \in \mathbb{C}^1(X)$  satisfies  $(\mathcal{P}.\text{-}\mathcal{S}.)_\beta$  for any  $\beta \in \mathbb{R}$  and assume that  $u_0 \neq u_1$  are relative minimizers of  $E$  in  $M$  with  $E(u_1) \leq E(u_0) = \beta$ .

Let

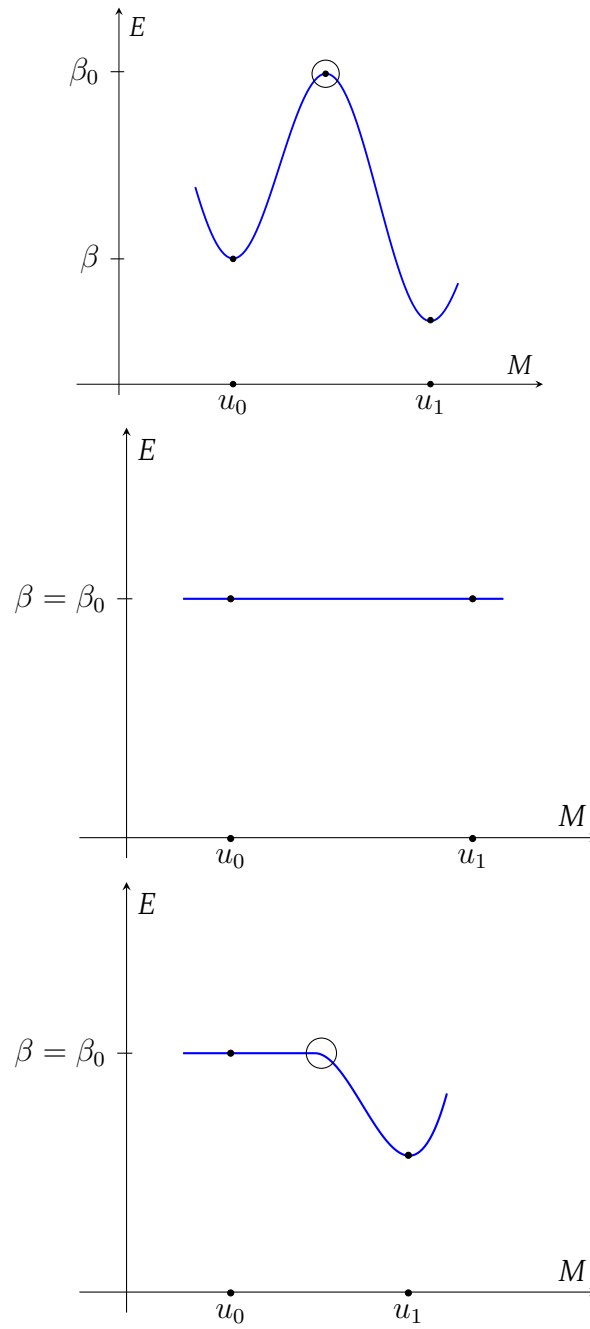
$$\beta = \inf_{\gamma \in G} \sup_{0 \leq s \leq 1} E(\gamma(s)) \geq \beta_0,$$

where

$$\Gamma = \{\gamma \in C^0([0, 1]; M) : \gamma(0) = u_0, \gamma(1) = u_1\}.$$

Then either there exists  $u \in K_\beta$  which is not a relative minimizer of  $E$ , or  $E(u_1) = \beta_0 = \beta$ , and for any open  $N_1 \supset K_\beta$  there exists  $\gamma_1 \in \Gamma$  with  $\gamma_1([0, 1]) \subset N_1$ .

**Remark 2.6.3.** Theorem 2.6.3 covers the following 3 cases, illustrated in the pictures below.



*Proof of Theorem 2.6.3.* In case  $\beta > \beta_0$  let  $\bar{\varepsilon} = \beta - \beta_0$ ; else, let  $\bar{\varepsilon} = 1$ . Suppose that  $K_\beta$  consists only of relative minimizers of  $E$  in  $M$ . Then  $K_\beta$  is both open and closed in

$$\bar{E}_\beta = \{u \in M : E(u) \leq \beta\},$$

and there exists an open  $N_0 \supset K_\beta$  such that

$$N_0 \cap E_\beta = \emptyset.$$

Given any open  $N_1 \supset K_\beta$  we then let  $0 < \varepsilon < \bar{\varepsilon}$ ,  $\Phi$  be as given by Theorem 2.6.2 for  $N = N_0 \cap N_1$ . Choose  $\gamma \in \Gamma$  with

$$\sup_{0 \leq s \leq 1} E(\gamma(s)) < \beta + \varepsilon.$$

Since  $g(u_0) = 0 = g(u_1)$ , then  $\gamma_1 := \Phi(\cdot, 1) \circ \gamma \in \Gamma$  and

$$\gamma_1([0, 1]) \subset \Phi(E_{\beta+\varepsilon}, 1) \subset E_{\beta-\varepsilon} \cup N.$$

But

$$(E_{\beta-\varepsilon} \cap N) \subset (E_{\beta-\varepsilon} \cap N_0) = \emptyset,$$

and we cannot have  $\gamma_1([0, 1]) \subset E_{\beta-\varepsilon}$  by definition of  $\beta$ .

Thus,  $\gamma_1([0, 1]) \subset N \subset N_1$ , and the proof is complete.  $\square$

### 2.6.2 The classical Plateau problem

Let

$$B = B_1(0) \subset \mathbb{R}^2 \cong \mathbb{C}$$

be the unit disc.

**Definition 2.6.5.** A map  $u \in C^1(B; \mathbb{R}^3)$  is conformal, if the Hopf differential

$$\Phi_u(z) = (\partial u)^2 = \frac{1}{4} (|u_x^2 - |u_y|^2 - 2i\langle u_x, u_y \rangle_{\mathbb{R}^3})$$

vanishes, where

$$z = x + iy \in V \subset \mathbb{C}, \partial u = \frac{1}{2}(\partial_x u - i\partial_y u), u_x = \partial_x u,$$

etc. for brevity.

For conformal  $u \in C^2(B; \mathbb{R}^3)$  there holds

$$-\Delta u = 2H(u) u_x \wedge u_y \quad \text{in } B$$

away from branch points where  $u_x = 0 = u_y$ . Here,  $H = H(u(z))$  is the mean curvature of the surface  $\mathcal{S} = u(B)$  at  $p = u(z)$ .

**Definition 2.6.6.**  $u \in C^2(B; \mathbb{R}^3)$  is minimal, if  $H(u) \equiv 0$ .

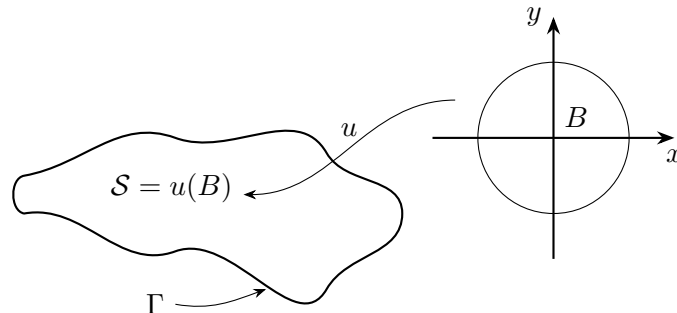
Since for every  $u \in C^1(B; \mathbb{R}^3)$  we may introduce conformal parameters, Plateau's problem for disc-type minimal surfaces then is to determine all solutions  $u \in C^2(B; \mathbb{R}^3) \cap C^1(\bar{B}; \mathbb{R}^3)$  of the problem

$$-\Delta u = 0 \quad \text{in } B, \tag{2.6.1}$$

$$\Phi_u = (\partial u)^2 = \frac{1}{4} (|u_x^2 - |u_y|^2 - 2i\langle u_x, u_y \rangle_{\mathbb{R}^3}) = 0 \quad \text{in } B, \tag{2.6.2}$$

$$u|_{\partial B} : \partial B \rightarrow \Gamma \quad \text{is a homeomorphism,} \tag{2.6.3}$$

where  $\Gamma \subset \mathbb{R}^3$  is a given, smooth, closed curve in  $\mathbb{R}^3$ .





**Remark 2.6.4.** i) Conditions (2.6.1)-(2.6.3) are conformally invariant: If  $u$  satisfies these conditions then so does every map  $u \circ g$ , where  $g \in C^\infty(B; B)$  is conformal.

ii) The Möbius group  $\mathcal{G}$  of conformal transformations of  $B$  is given by

$$\mathcal{G} = \left\{ g(z) = e^{i\theta} \frac{a - z}{1 - \bar{a}z} : \theta \in \mathbb{R}, a \in B \right\}.$$

iii) Fixing 3 distinct point  $q_k \in \Gamma$ ,  $k = 1, 2, 3$ , we may normalize admissible maps  $u$  by imposing the 3 point condition,

$$u\left(e^{i\frac{2\pi k}{3}}\right) = q_k, \quad 1 \leq k \leq 3. \quad (2.6.4)$$

Let

$$\mathcal{C}(\Gamma) = \{u \in H^1(B; \mathbb{R}^3) : u|_{\partial B} : \partial B \rightarrow \Gamma \text{ is cont., weakly monotone parametrization of } \Gamma\}$$

and set

$$\mathcal{C}^*(\Gamma) = \{u \in \mathcal{C}(\Gamma) : u \text{ satisfies (2.6.4)}\}.$$

The first important result (which was rewarded with a fields medal for J. Douglas) is the following theorem.

**Theorem 2.6.4.** (Douglas (1930), Radò (1931))

For every smoothly embedded closed curve  $\Gamma \cong S^1$  in  $\mathbb{R}^3$  there exists  $u \in \mathcal{C}^*$  with

$$D(u) := D(u; B) = \frac{1}{2} \int_B |\nabla u|^2 dz = \min_{v \in \mathcal{C}^*(\Gamma)} D(v),$$

and  $u$  satisfies (2.6.1), (2.6.2).

The key for showing compactness of a minimizing sequence is the following result.

**Lemma 2.6.2.** (Courant-Lebesgue)

Any sequence  $(u_k) \subset \mathcal{C}^*(\Gamma)$  with

$$\sup_k D(u_k) < \infty$$

has equi-continuous traces  $(u_k|_{\partial B})$ .

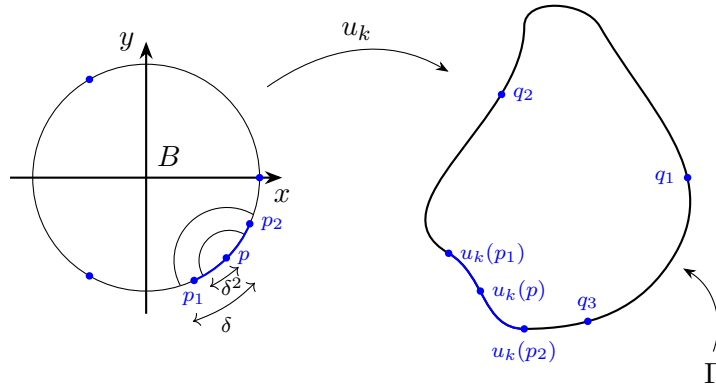
*Proof.* Let  $(u_k) \subset \mathcal{C}^*(\Gamma)$  with  $\sup_k D(u_k) = C_0 < \infty$ . For  $p = e^{i\theta} \in \partial B$  and  $0 < \delta < \frac{1}{2}$ , by Hölder's and Fubini's theorem for any  $u = u_k$  we can bound

$$\begin{aligned} & \inf_{\delta^2 < r < \delta} \left( \int_{\partial B_r(p) \cap B} |\nabla u| dx \right)^2 \\ & \leq \frac{\int_{\delta^2}^{\delta} \left( \int_{\partial B_r(p) \cap B} |\nabla u| dx \right)^2 \frac{dr}{r}}{\int_{\delta^2}^{\delta} \frac{dr}{r}} \\ & \leq \frac{2\pi \int_{\delta^2}^{\delta} \int_{\partial B_r(p) \cap B} |\nabla u|^2 ds dr}{\log(\frac{1}{\delta})}, \end{aligned}$$

where  $ds$  is the element of length along  $\partial B_r(p)$ . Hence, letting  $p_1(r), p_2(r)$  be the points of intersection in  $\partial B_r(p) \cap \partial B$  we have

$$\begin{aligned} \inf_{\delta^2 < r < \delta} |u(p_1(r)) - u(p_2(r))|^2 &\leq \inf_{\delta^2 < r < \delta} \left( \int_{\partial B_r(p) \cap B} |\nabla u| ds \right)^2 \\ &\leq \frac{4\pi D(u)}{\log(\frac{1}{\delta})} \\ &\leq \frac{4\pi C_0}{\log(\frac{1}{\delta})} \xrightarrow{\delta \downarrow 0} 0. \end{aligned}$$

Since the arc  $\partial B \cap B_{\frac{1}{2}}(p)$  contains at most one point  $e^{\frac{2\pi il}{3}}, l \in \{1, 2, 3\}$ , and  $u_k \in \mathcal{C}^*(\Gamma)$ , the arc  $\partial B \cap B_{\delta^2}(p)$  is mapped by each  $u_k$  to a subspace of  $\Gamma$  containing at most one point  $q_l$ , whose diameter shrinks to 0 as  $\delta \downarrow 0$ , uniformly in  $k$ . The claim follows. □



*Proof of Theorem 2.6.4.* Let  $(u_k) \subset \mathcal{C}^*(\Gamma)$  be a minimizing sequence with

$$D(u_k) \rightarrow \inf_{v \in \mathcal{C}(\Gamma)} D(v) \quad (k \rightarrow \infty).$$

Then  $(u_k) \subset H^1(B; \mathbb{R}^3)$  is bounded in view of

$$\|u\|_{L^2(B)} \leq C (\|\nabla u\|_{L^2} + \|u|_{\partial B}\|_{L^2(\partial B)}),$$

for  $u \in H^1(B; \mathbb{R}^3)$ , and  $u_k \xrightarrow{w} u$  in  $H^1(B; \mathbb{R}^3)$  as  $k \rightarrow \infty$ ,  $k \in \Lambda$ .

Moreover, by Lemma 2.6.2 and Arzela-Ascoli's theorem for a further subsequence we have  $u_k|_{\partial B} \rightarrow u|_{\partial B}$  in  $C^0(\partial B; \mathbb{R}^3)$ , and  $u \in \mathcal{C}^*(\Gamma)$  satisfies

$$D(u) = \inf_{v \in \mathcal{C}^*(\Gamma)} D(v).$$

For any  $\varphi \in C_c^\infty(B; \mathbb{R}^3)$  then

$$D(u) \leq D(u + \varepsilon \varphi), \quad \varepsilon \in \mathbb{R}.$$

Hence

$$0 = \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} D(u + \varepsilon \varphi) = \int_B \nabla u \nabla \varphi \, dz,$$

and

$$-\Delta u = 0 \quad \text{in } B.$$

In order to also see (2.6.2) let  $\tau \in C^1(\overline{B}; \mathbb{R}^2)$  and for  $|\varepsilon| \ll 1$  consider the diffeomorphism

$$\text{id} + \varepsilon\tau : B \rightarrow \Omega_\varepsilon := (\text{id} + \varepsilon\tau)(B).$$

For sufficiently small  $|\varepsilon|$  then by the Riemann mapping theorem there exists a conformal diffeomorphism  $\psi_\varepsilon : B \rightarrow \Omega_\varepsilon$ . Composing with a suitable  $g_\varepsilon \in \mathcal{G}$ , the conformal group of the disc, moreover, we may assume that

$$u_\varepsilon := u \circ (\text{id} + \varepsilon\tau)^{-1} \circ \psi_\varepsilon \circ g_\varepsilon \in \mathcal{C}^*(\Gamma)$$

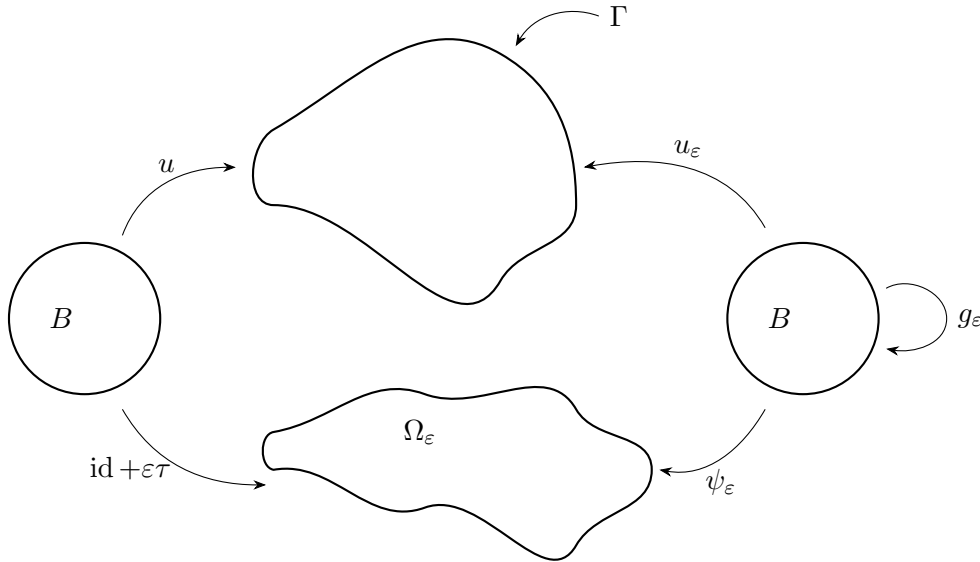
with

$$D(u_\varepsilon) = D(u \circ (\text{id} + \varepsilon\tau)^{-1}; \Omega_\varepsilon)$$

by conformal invariance of  $D$ .

By minimality of  $D(u) = D(u_0)$  then we have

$$0 = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} D(u \circ (\text{id} + \varepsilon\tau)^{-1}; \Omega_\varepsilon).$$



Changing variables  $\zeta = (\text{id} + \varepsilon\tau)(z)$  and using

$$\begin{aligned} d(\text{id} + \varepsilon\tau)^{-1} &= (\mathbf{1} + \varepsilon d\tau)^{-1} \circ (\text{id} + \varepsilon\tau)^{-1} \\ &= \mathbf{1} - \varepsilon d\tau \circ (\text{id} + \varepsilon\tau)^{-1} + o(\varepsilon^2) \end{aligned}$$

as well as

$$\det(\mathbf{1} + \varepsilon d\tau) = 1 + \varepsilon \text{trace } d\tau + o(\varepsilon^2)$$

we find

$$\begin{aligned} D(u \circ (\text{id} + \varepsilon\tau)^{-1}; \Omega_\varepsilon) &= \frac{1}{2} \int_{\Omega_\varepsilon} |du \cdot (\mathbf{1} - \varepsilon d\tau) \circ (\text{id} + \varepsilon\tau)^{-1}|^2 d\zeta + o(\varepsilon^2) \\ &= \frac{1}{2} \int_B |du \cdot (\mathbf{1} - \varepsilon d\tau)|^2 |\det(\mathbf{1} + \varepsilon d\tau)| dz + o(\varepsilon^2), \end{aligned}$$

and there results

$$\begin{aligned}
& \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} D(u \circ (\text{id} + \varepsilon\tau)^{-1}; \Omega_\varepsilon) \\
&= - \int_B \langle du \cdot d\tau, du \rangle_{\mathbb{R}^2 \times \mathbb{R}^3} dz + \frac{1}{2} \int_B |du|^2 \text{tr } d\tau dz \\
&= \int_B \left( \frac{1}{2} |du|^2 (\tau_x^1 + \tau_y^2) - \langle u_x \tau_x^1 + u_y \tau_x^2, u_x \rangle - \langle u_x \tau_y^1 + u_y \tau_y^2, u_y \rangle \right) dz \\
&= - \frac{1}{2} \int_B (|u_x|^2 (\tau_x^1 - \tau_y^2) + |u_y|^2 (\tau_y^2 - \tau_x^1) + 2 \langle u_x, u_y \rangle (\tau_y^1 + \tau_x^2)) dz \\
&= \text{Re} \int_B \Phi^{\mathbb{C}\text{-mult.}} \underbrace{\bar{\partial}\tau}_{=\frac{1}{2}(\partial_x + i\partial_y)\tau} dw = 0
\end{aligned}$$

for any  $\tau \in C^1(\bar{B}; \mathbb{C})$ . But for every  $f \in C_c^\infty(B; \mathbb{C})$  there exists

$$\tau(z) = \int_B \frac{f(\xi)}{z - \xi} d\xi$$

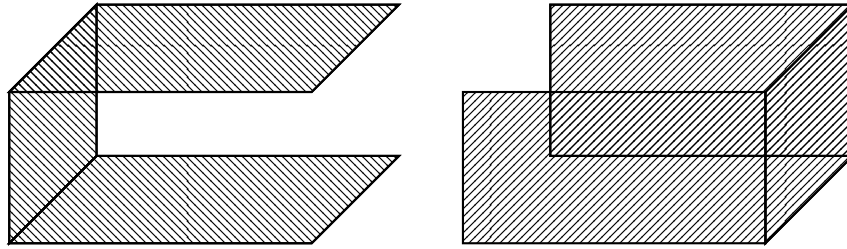
with

$$\bar{\partial}\tau = f.$$

Hence  $\Phi \equiv 0$ , and the proof is complete.  $\square$

### 2.6.3 The mountain-pass lemma for minimal surfaces

There are curves  $\Gamma \subset \mathbb{R}^3$  that span (at least) two different solutions of Plateau's problem, for instance, due to symmetry, as in the following Example.



Already in 1939 Morse-Tompkins set up a Morse theory for the problem, predicting the existence of an unstable third solution of saddle-type in this case. Working in the  $C^0$ -topology, they sought to identify minimal surfaces with "homotopy critical" points in their theory. This approach was later criticized by Tromba, who pointed out that even smoothly non-degenerate solutions of Plateau's problem might not be "homotopy-critical", similar to our observation in Section 1.4, and developed a version of degree theory to overcome this difficulty.

The full Morse theory was derived by Struwe (1982) within the framework developed in 2.6.1.

Given a smoothly embedded closed curve  $\Gamma \cong S^1$  in  $\mathbb{R}^3$ , we choose a smooth reference parametrization

$$\gamma : S^1 = \mathbb{R}/2\pi \rightarrow \Gamma.$$

For any smooth  $\varphi : \partial B \rightarrow \mathbb{R}^3$  also let  $h(\varphi) : B \rightarrow \mathbb{R}^3$  be the harmonic extension of  $\varphi$  to  $B$ . Note that the Dirichlet energy of  $h(\varphi)$  may be expressed in terms of Douglas' integral

$$D(h(\varphi)) = \frac{1}{16\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{|\varphi(\theta) - \varphi(\theta')|^2}{\sin^2\left(\frac{\theta - \theta'}{2}\right)} d\theta d\theta', \quad (2.6.5)$$

which is equivalent to the  $H^{\frac{1}{2}}$ -norm of  $\varphi$  on  $\partial B \cong \mathbb{R}/2\pi$ . Hence, the map  $h$  extends to a continuous map

$$h : H^{\frac{1}{2}}(\partial B; \mathbb{R}^3) \rightarrow H^1(B; \mathbb{R}^3).$$

We then may recast the setting for the solution of Plateau's problem, as follows.

Let

$$M = \{u \in H_{loc}^{\frac{1}{2}} \cap C^0(\mathbb{R}) : u(t + 2\pi) = u(t) + 2\pi, t \in \mathbb{R}, u \text{ non-decreasing} \}$$

be the set of admissible re-parametrizations such that the map  $h(\gamma \circ u) \in H^1 \cap C^0(\overline{B}; \mathbb{R}^3)$  satisfies the Plateau boundary condition (2.6.3).

The set

$$M' = \{v = u - \text{id} : u \in M\}$$

then is a closed, convex subset of the Banach space

$$X = H^{\frac{1}{2}} \cap C^0(\mathbb{R}/2\pi);$$

for ease of notation, however, we prefer to work with the set  $M$ , which then is a closed, convex subset of the affine space  $\{\text{id}\} + X$ .

Also let

$$M^* = \left\{ u \in M : u\left(\frac{2\pi k}{3}\right) = \frac{2\pi k}{3}, k \in \mathbb{Z} \right\}$$

be the set of reparametrizations of  $\gamma$  such that  $\gamma \circ u$  satisfies (2.6.4) for  $q_k = \gamma(\frac{2\pi k}{3}) \in \Gamma, 1 \leq k \leq 3$ .

Finally, setting

$$E(u) = D(h(\gamma \circ u)) = \frac{1}{16\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{|\gamma(u(s)) - \gamma(u(t))|^2}{\sin^2\left(\frac{t-s}{2}\right)} ds dt$$

we obtain a functional of class  $C^1$  on  $\{\text{id}\} + X$  to which we can apply the abstract theory developed in Section 2.6.1 above.

In fact, for any  $u \in M$ , any  $\xi \in X$ , letting  $U = h(\gamma \circ u)$  be the harmonic extension of the parametrization  $\gamma \circ u$  of  $\Gamma$ , we have

$$\begin{aligned} \langle dE(u), \xi \rangle_{X^* \times X} &= \langle d_u D(h(\gamma \circ u)), \xi \rangle_{X^* \times X} \\ &= \int_B \nabla u \nabla \left( \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} h(\gamma(u + \varepsilon \xi)) \right) dz \\ &= \int_{\partial B} \langle \partial_u U, \gamma'(u) \rangle_{\mathbb{R}^3} \xi ds \end{aligned}$$

with

$$\gamma'(u)\xi \in H^{\frac{1}{2}} \cap C^0(\partial B; \mathbb{R}^3) \cong X$$

and outward normal derivative

$$\partial_k u \in H^{-\frac{1}{2}}(\partial B; \mathbb{R}^3) \subset X^*.$$

Here we use that  $X$  is an algebra with

$$\|\xi\eta\|_X \leq \|\xi\|_X \|\eta\|_X, \quad \xi, \eta \in X,$$

as is most easily seen by considering  $\xi, \eta \in X$  as traces  $\xi = \Xi|_{\partial B}, \eta = Y|_{\partial B}$  where  $\Xi, Y \in H^1 \cap C^0(\bar{B})$ , and observing that

$$\|\nabla(\Xi Y)\|_{L^2} \leq \|\nabla \Xi\|_{L^2} \|Y\|_{L^\infty} + \|\Xi\|_{L^\infty} \|\nabla Y\|_{L^2}.$$

In particular, as in Definition 2.6.1 we let

$$g(u) = \sup_{\substack{v \in M \\ \|u-v\|_X \leq 1}} \langle dE(u), u-v \rangle_{X^* \times X}, \quad u \in M.$$

Then we have the following key result.

**Proposition 2.6.1.** *For any  $u \in M$  there holds  $g(u) = 0$  iff  $U := h(\gamma \circ u)$  solves (2.6.1)-(2.6.3).*

*Proof.* By definition of  $h$  and  $M$ , any surface  $U = h(\gamma \circ u)$  with  $u \in M$  solves (2.6.1) and satisfies the Plateau boundary condition (2.6.3). Thus, only condition (2.6.2) is an issue. For this note the identity for  $z = x + iy = re^{i\phi} \in B$

$$\begin{aligned} 4z^2(\partial U)^2 &= [(x + iy)(U_x - iU_y)]^2 \\ &= (rU_r - iU_\phi)^2 \\ &= r^2|U_r|^2 - |U_\phi|^2 - 2ir\langle U_r, U_\phi \rangle_{\mathbb{R}^3}. \end{aligned} \tag{2.6.6}$$

From

$$\bar{\partial}(z^2(\partial U)^2) = 2z^2 \overbrace{(\bar{\partial}\partial U)}^{=\frac{1}{4}\Delta U} \partial U = 0$$

we then conclude analyticity of

$$\Psi = r^2|U_r|^2 - |U_\phi|^2 - 2ir\langle U_r, U_\phi \rangle_{\mathbb{R}^3}.$$

Moreover,  $U$  satisfies (2.6.2) iff  $\Psi \equiv 0$ , and the latter holds iff  $\langle U_r, U_\phi \rangle_{\mathbb{R}^3} \equiv 0$  on  $\partial B$ . Indeed, if  $\langle U_r, U_\phi \rangle_{\mathbb{R}^3} \equiv 0$  on  $\partial B$ , by harmonicity  $\text{Im } |\Psi| \equiv 0$  in  $B$  and  $\Psi(re^{i\phi}) \equiv \text{const} = \Phi(0) = 0$  by the Cauchy-Riemann equations.

- i) Suppose  $U = h(\gamma \circ u)$  solves (2.6.1)-(2.6.3). By the regularity theory of Hildebrandt and Nitsche, and since  $\Gamma$  is smooth, then  $U \in C^1(\bar{B}; \mathbb{R}^3)$  and (2.6.2) gives

$$\langle U_r, \gamma'(u) \rangle_{\mathbb{R}^3} \equiv 0;$$

so

$$\langle dE(u), \xi \rangle_{X^* \times X} = \int_{\partial B} \langle U_r, \gamma'(u) \rangle_{\mathbb{R}^3} \xi \, ds = 0$$

for any  $\xi \in X$ . Hence  $g(u) = 0$ .

- ii) To see the converse, we also require a regularity result.

**Lemma 2.6.3.** *Let  $u \in M$  with  $g(u) = 0$ . Then  $U = h(\gamma \circ u) \in H^2(B; \mathbb{R}^3)$ .*

See Struwe [Str82], [Str88]; Prop. II.5.1, p. 66 *ff.*, Imbusch-Struwe [IS99] for the analogous result in  $M^*$ .

Suppose  $u \in M$  satisfies  $g(u) = 0$  and let  $U = h(\gamma \circ u)$  as above. For any  $\xi \in C^1(\mathbb{R}/2\pi)$  and any  $|\varepsilon| < \|\xi'\|_{L^\infty}^{-1}$  the map  $\text{id} + \varepsilon\xi \in C^1$  is a diffeomorphism on  $\mathbb{R}$ , and

$$u_\varepsilon = u \circ (\text{id} + \varepsilon\xi) \in M.$$

Moreover, regularity  $U \in H^2(B; \mathbb{R}^3)$  implies

$$U_\phi = \gamma'(u) \cdot \frac{du}{d\phi} \in H^{\frac{1}{2}}(\partial B; \mathbb{R}^3),$$

and thus that  $\frac{du}{d\phi} \in H^{\frac{1}{2}}$ , since  $|\gamma'(u)| \geq c_0 > 0$ . It then follows that  $\|u_\varepsilon - u\|_X \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , and we may choose  $\varepsilon_0 > 0$  such that

$$\|u_\varepsilon - u\|_X < 1 \text{ for } |\varepsilon| < \varepsilon_0.$$

Thus, from our assumption that  $g(u) = 0$  we deduce

$$0 = g(u) \geq \langle dE(u), u - u_\varepsilon \rangle_{X^* \times X}, \quad |\varepsilon| < \varepsilon_0.$$

It follows that for every  $\xi \in C^1(\mathbb{R}/2\pi)$  there holds

$$\begin{aligned} 0 &\geq \lim_{\varepsilon \downarrow 0} \frac{\langle dE(u), u - u_\varepsilon \rangle_{X^* \times X}}{\varepsilon} \\ &= \int_{\partial B} \langle \partial_u U, \gamma'(u) \rangle_{\mathbb{R}^3} \lim_{\varepsilon \downarrow 0} \frac{u - u_\varepsilon}{\varepsilon} ds \\ &= - \int_{\partial B} \langle \partial_u U, \gamma'(u) \rangle_{\mathbb{R}^3} \frac{du}{d\phi} \cdot \xi ds = - \int_{\partial B} \langle \partial_r U, \partial_\phi U \rangle \cdot \xi ds. \end{aligned}$$

Replacing  $\xi$  with  $-\xi$ , we then see that the latter, in fact, equals 0.

But  $\partial_r U, \partial_\phi U \in H^{\frac{1}{2}}(\partial B; \mathbb{R}^3) \hookrightarrow L^p(\partial B; \mathbb{R}^3)$  for every  $p < \infty$ . Choosing  $p = 4$  and observing that  $C^1(\mathbb{R}/2\pi)$  is dense in  $L^2(\mathbb{R}/2\pi)$  we then obtain

$$\langle U_r, U_\phi \rangle_{\mathbb{R}^3} \equiv 0 \quad \text{on } \partial B,$$

and

$$\Psi = r^2 |U_r|^2 - |U_\phi|^2 - 2ir \langle U_r, U_\phi \rangle_{\mathbb{R}^3} \equiv 0,$$

as shown above, which is equivalent to (2.6.2). The proof is complete.  $\square$

Finally, we verify the Palais-Smale condition.

**Proposition 2.6.2.** *Let  $(u_k) \subset M^*$  such that*

$$E(u_k) \rightarrow \beta, \quad g(u_k) \rightarrow 0 \quad (k \rightarrow \infty).$$

*Then there is  $u \in K_\beta$  and a subsequence  $\Lambda \subset \mathbb{N}$  such that  $\|u_k - u\|_X \rightarrow 0$  ( $k \rightarrow \infty, k \in \Lambda$ ).*

*Proof.* Let  $U_k = h(\gamma \circ u_k) \in \mathcal{C}^*(\Gamma)$ ,  $k \in \mathbb{N}$ . Since

$$D(U_k) = E(u_k) \rightarrow \beta \quad (k \rightarrow \infty),$$

by Lemma 2.6.2 the family  $(U_k|_{\partial B} = \gamma \circ u_k)$  is equi-continuous and bounded in  $H^{\frac{1}{2}}$  on  $\partial B$ . But then also

$$u_k = \gamma^{-1} \circ (\gamma \circ u_k)$$

is bounded in  $H^{\frac{1}{2}}$ , and a subsequence  $u_k \xrightarrow{w} M^*$  in  $H^{\frac{1}{2}}$  and uniformly. For suitable  $0 < r \leq 1$  then

$$\|r(u_k - u)\|_X < 1, \quad k \in \mathbb{N},$$

and

$$v_k = u_k + r(u - u_k) = (1 - r)u_k + ru \in M$$

satisfies  $\|v_k - u_k\|_X < 1$ ; so

$$\begin{aligned} o(1) &= g(u_k) \geq \langle dE(u_k), u_k - v_k \rangle_{X^* \times X} \\ &= r \int_{\partial B} \langle \partial_u U_k, \gamma'(u_k) \rangle_{\mathbb{R}^3} (u_k - u) ds. \end{aligned}$$

Expanding

$$\begin{aligned} U_k - U &= \gamma \circ u_k - \gamma \circ u \\ &= \gamma'(u_k)(u_k - u) - \int_0^1 \int_0^t \gamma''(u_k + r(u - u_k)) dr dt (u - u_k)^2 \end{aligned}$$

we then have

$$\left\| \int_0^1 \int_0^t \gamma''(u_k + r(u - u_k)) dr dt (u - u_k)^2 \right\|_{H^{\frac{1}{2}}} \leq C \|u - u_k\|_{L^\infty} \rightarrow 0 \quad (k \rightarrow \infty, k \in \Lambda).$$

Thus, with error  $o(1) \rightarrow 0$ ,

$$\begin{aligned} o(1) &\geq \int_{\partial B} \langle \partial_u U_k, U - k - U \rangle_{\mathbb{R}^3} ds - o(1) \\ &= \int_B \nabla U_k \nabla (U_k - U) dz - o(1) \\ &= \|\nabla (U_k - U)\|_{L^2(B)}^2 - o(1), \end{aligned}$$

and  $U_k \rightarrow U$  in  $H^1$ ,  $U_k|_{\partial B} = \gamma \circ u_k \rightarrow \gamma \circ u$  in  $H^{\frac{1}{2}}$ . Again using that  $\gamma$  is a diffeomorphism, we then obtain  $u_k \rightarrow u$  in  $H^{\frac{1}{2}}$ , and thus

$$\|u_k - u\|_X \rightarrow 0 \quad (k \rightarrow \infty, k \in \Lambda).$$

□

**Remark.** In order to see the relative compactness in  $C^0$  of Palais-Smale sequences, we may also use the following observation.

Let  $(u_k) \subset M^*$  be a  $(\mathcal{P}\text{-}\mathcal{S})_\beta$ -sequence. By monotonicity we have the uniform  $W^{1,1}$ -bound

$$\int_0^{2\pi} |u'_k| ds = \int_0^{2\pi} u'_k ds = u_k(2\pi) - u_k(0) = 2\pi, \quad k \in \mathbb{N}.$$



Hence a subsequence  $u_k \rightarrow u$  in  $L^1_{loc}(\mathbb{R})$  and almost everywhere as  $k \rightarrow \infty, k \in \Lambda$ , where  $u : [0, 2\pi] \rightarrow [0, 2\pi]$  is non-decreasing with

$$u|_{\left[\frac{2\pi k}{3}, \frac{2\pi(k+1)}{3}\right]} \subset \left[\frac{2\pi k}{3}, \frac{2\pi(k+1)}{3}\right], \quad k \in \mathbb{Z}. \quad (2.6.7)$$

Moreover, by a theorem of Polya, the convergence is uniform if  $u \in C^0$ .

But if  $u \notin C^0$ , by (2.6.7) there would be a point  $s_0$  with  $u(s_0^-) < u(s_0^+) \leq u(s_0^-) + \frac{4\pi}{3}$ , and  $D(h(\gamma \circ u)) = \infty$ , contradicting

$$D(U) \leq \liminf_{k \rightarrow \infty} D(U_k) = \beta < \infty,$$

where  $U = h(\gamma \circ u)$ ,  $U_k = h(\gamma \circ u_k)$ ,  $k \in \mathbb{N}$ .

## 3 Limits cases

### 3.1 The Yamabe problem

#### 3.1.1 The Yamabe constant

Let  $(M, g_0)$  be a closed<sup>2</sup> Riemannian  $n$ -manifold,  $n \geq 3$ . A conformal metric

$$g = u^{\frac{4}{n-2}} g_0, \quad u > 0$$

has scalar curvature

$$R_g = u^{1-2^*} (-c(n)\Delta_{g_0}u + R_{g_0}u),$$

where

$$\Delta_{g_0}u = \frac{1}{\sqrt{\det g_0}} \partial_i \left( g_0^{ij} \sqrt{\det g_0} \partial_j u \right), \quad (g_0^{ij}) = g_0^{-1}$$

is the Laplace-Beltrami operator in the metric  $g_0$ , and where

$$c(n) = \frac{4(n-1)}{n-2}.$$

**Definition 3.1.1.** *The operator given by*

$$L_g u = -c(n)\Delta_g u + R_g u$$

*is the conformal Laplacian on  $(M, g)$ .*

Note the conformal transformation rule:

For

$$h = v^{\frac{4}{n-2}} g = (uv)^{\frac{4}{n-2}} g_0, \quad c > 0,$$

we have

$$R_h = v^{1-2^*} L_g v = (uv)^{1-2^*} L_{g_0}(uv);$$

that is, there holds

$$L_g v = u^{1-2^*} L_{g_0}(uv). \quad (3.1.1)$$

**Definition 3.1.2.** *A conformal metric  $g = u^{\frac{4}{n-2}} g_0$  is a Yamabe metric on  $(M, g_0)$  if  $R_g \equiv \text{const}$ ; that is, if  $u > 0$  solves the equation*

$$L_{g_0} u = r u^{2^*-1} \quad \text{on } M \quad (3.1.2)$$

for some constant  $r \in \mathbb{R}$ .

**Remark 3.1.1.** i) Scaling  $\tilde{u} = \alpha u$  with  $\alpha = |r|^{\frac{1}{2^*-2}} > 0$  we can achieve  $r \in \{-1, 0, 1\}$  in (3.1.2).

ii) Similar to Section 1.5 we can attempt to find a solution of (3.1.2) by minimizing

$$E_{(M, g_0)}(u) = \int_M (|\nabla u|_{g_0}^2 + c(n)^{-1} R_{g_0} u^2) d\mu_{g_0}$$

---

<sup>2</sup>compact,  $\partial M = \emptyset$

over the set

$$M = \{u \in H^1(M, g_0) : \|u\|_{L^{2^*}(M, g_0)} = 1\},$$

where, in local coordinates,

$$|\nabla u|_{g_0}^2 = g_0^{ij} \partial_i u \partial_j u, \quad d\mu_{g_0} = \sqrt{\det g_0} \, dx.$$

Note that for  $g = u^{\frac{4}{n-2}} g_0$  we have

$$\sqrt{\det g} = u^{2^*} \sqrt{\det g_0}, \quad d\mu_g = u^{2^*} d\mu_{g_0},$$

so

$$\|u\|_{L^{2^*}(M, g_0)}^{2^*} = \int_M d\mu_g = \text{vol}(M, g).$$

Equivalently, we can try to find a minimizer of the Sobolev-Yamabe ratio

$$\mathcal{S}_{(M, g_0)}(u) = \frac{E_{(M, g_0)}(u)}{\|u\|_{L^{2^*}(M, g_0)}^{2^*}}.$$

iii) For any  $u, v > 0$  with  $g = u^{\frac{4}{n-2}} g_0, h = v^{\frac{4}{n-2}} g$  by (3.1.1) there holds

$$\begin{aligned} c(n)E_{(M, g)}(v) &= \int_M (c(n)|\nabla v|_g^2 + R_g v^2) \, d\mu_g \\ &= \int_M L_g v \cdot v \, d\mu_g = \int_M u^{1-2^*} L_{g_0}(uv) \cdot v u^{2^*} \, d\mu_{g_0} \\ &= \int_M (uv) L_{g_0}(uv) \, d\mu_{g_0} = c(n)E_{(M, g_0)}(uv); \end{aligned}$$

moreover,

$$\|v\|_{L^{2^*}(M, g)}^{2^*} = \int_M v^{2^*} \, d\mu_g = \int_M (uv)^{2^*} \, d\mu_{g_0} = \|uv\|_{L^{2^*}(M, g_0)}^{2^*}.$$

In particular, the Yamabe constant

$$\mathcal{S}_{(M, g_0)} = \int_{(uv)>0} \mathcal{S}_{(M, g_0)}(uv) = \inf_{v>0} \mathcal{S}_{(M, g)}(v)$$

is a conformal invariant.

**Example 3.1.1.** Stereographic projection

$$\pi : S^n \setminus \{p\} \rightarrow \mathbb{R}^n \quad \text{from } p = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix} \in \mathbb{R}^{n+1}$$

with inverse

$$\Phi : \mathbb{R}^n \ni x \mapsto \frac{1}{1+|x|^2} \begin{pmatrix} 2x \\ 1-|x|^2 \end{pmatrix} \in S^n \subset \mathbb{R}^{n+1}$$

induces the conformal metric

$$g = \Phi^* g_{S^n} = \left( \frac{2}{1 + |x|^2} \right)^2 g_{\mathbb{R}^n} = u^{\frac{4}{n-2}} g_{\mathbb{R}^n}$$

with

$$u(x) = \left( \frac{2}{1 + |x|^2} \right)^{\frac{n-2}{2}}$$

having scalar curvature

$$R_g = R_{g_{S^n}} = n(n-1) = u^{1-2^*} (-c(n)\Delta(u)).$$

That is,  $u$  solves the equation

$$-\Delta u = \frac{n(n-1)}{n(n)} u^{2^*-1} = \frac{n(n-2)}{4} u^{2^*-1} \quad \text{on } \mathbb{R}^n.$$

The scaled function

$$u^*(x) = \left( \frac{n(n-2)}{4} \right)^{\frac{1}{2^*-2}} u(x) = \frac{(n(n-2))^{\frac{n-2}{4}}}{(1 + |x|^2)^{\frac{n-2}{2}}}$$

then solves

$$-\Delta u^* = (u^*)^{2^*-1} \quad \text{on } \mathbb{R}^n.$$

Since by a theorem of Obata moreover we have

$$\mathcal{S}_{(S^n, g_{S^n})} = \mathcal{S}_{(S^n, g_{S^n})}(1),$$

from Remark 3.1.1 we obtain

$$\begin{aligned} \mathcal{S} &= \mathcal{S}_{(\mathbb{R}^n, g_{\mathbb{R}^n})} = \inf_{0 < v \in C_c^\infty(\mathbb{R}^n)} \mathcal{S}_{(\mathbb{R}^n, g_{\mathbb{R}^n})}(uv) \\ &= \inf_{0 < v \in C_c^\infty(S^n \setminus \{p\})} \mathcal{S}_{(S^n, g_{S^n})}(v) = \mathcal{S}_{(S^n, g_{S^n})} \\ &= \mathcal{S}_{(S^n, g_{S^n})}(1) = \mathcal{S}_{(\mathbb{R}^n, g_{S^n})}(u) = \frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^{2^*}}^2}. \end{aligned}$$

Hence  $u$  (and  $u^*$ ) achieve the best Sobolev constant, as claimed in Section 1.5.

In fact, the Yamabe invariant is maximal on  $S^n$ .

**Theorem 3.1.1.** For any closed  $(M^n, g_0)$ ,  $n \geq 3$ , we have

$$\mathcal{S}_{(M, g_0)} \leq \mathcal{S}_{(S^n, g_{S^n})} = \mathcal{S}.$$

*Proof.* Fix geodesic normal coordinates in a ball  $B_R(p_0)$  around  $p_0 = 0 \in M$ , and let  $\varphi \in C_c^\infty(B_R(0))$  with  $\varphi(x) = 1$  for  $|x| < \frac{R}{2}$ . Letting

$$u_\varepsilon^*(x) = \varepsilon^{\frac{2-n}{2}} u^* \left( \frac{x}{\varepsilon} \right) = \frac{[\varepsilon n(n-2)]^{\frac{n-2}{4}}}{[\varepsilon^2 + |x|^2]^{\frac{n-2}{2}}}, \quad \varepsilon > 0,$$

as in Example 2.3.1 iii), and setting

$$u_k = \varphi u_{\varepsilon_k}^*, \quad \varepsilon_k \downarrow 0,$$

we obtain

$$\mathcal{S}_{(M, g_0)} \leq \mathcal{S}_{(M, g_0)}(u_k) \xrightarrow{(k \rightarrow \infty)} \mathcal{S}_{(\mathbb{R}^n, g_{\mathbb{R}^n})} = \mathcal{S}_{(S^n, g_{S^n})},$$

and the claim follows.  $\square$

### 3.1.2 Local compactness

In view of Theorem 3.1.1 the functional  $E_{(S^n, g_{S^n})}$  - or the ratio  $\mathcal{S}_{(S^n, g_{S^n})}$  - plays the same role as  $E_\infty$  in Theorem 1.5.1. Already in 1976 Aubin established the following result, anticipating P.L. Lions' Theorem 1.5.1 from 1984.

**Theorem 3.1.2.** (Aubin)

Suppose  $\mathcal{S}_{(M, g_0)} < \mathcal{S}_{(S^n, g_{S^n})}$ . Then every minimizing sequence of normalized metrics

$$g_k = u_k^{\frac{4}{n-2}} g_0, \text{ with } \text{vol}(M, g_k) = 1, k \in \mathbb{N},$$

satisfying

$$\mathcal{S}_{(M, g_0)}(u_k) \rightarrow \mathcal{S}_{(M, g_0)} \quad (k \rightarrow \infty)$$

has an  $H^1$ -convergent subsequence  $u_k \xrightarrow{(k \rightarrow \infty, k \in \Lambda)} u$  with limit metric  $g = u^{\frac{4}{n-2}} g_0$  a Yamabe metric.

The case  $\mathcal{S}_{(M, g_0)} > 0$  is the most interesting. For this case let

$$F_{(M, g_0)}(u) = \frac{1}{2} \int_M (|\nabla u|_{g_0}^2 + c(n)^{-1} R_{g_0} u^2) d\mu_{g_0} - \frac{1}{2^*} \int_M u^{2^*} d\mu_{g_0}$$

be the "free" functional associated with (3.1.2).

As in Example 2.4.2 i), and using the correspondence between the constrained minimization problem for  $\mathcal{S}_{(M, g_0)}$  and saddle-type critical points of  $F_{(M, g_0)}$  as in the proof of Theorem 1.4.1, the existence of a Yamabe metric also is an immediate consequence of our next result, provided that Aubin's condition

$$\mathcal{S}_{(M, g_0)} < \mathcal{S}_{(S^n, g_{S^n})}$$

holds.

**Theorem 3.1.3.** (Trudinger (1968), Aubin (1976), Brezis-Nirenberg (1983), Cerami-Forunato-Struwe (1984))

Let  $(M^n, g_0)$  be closed,  $n \geq 3$ . Then the functional  $F_{(M, g_0)} \in C^1(H^1(M, g_0))$  satisfies  $(\mathcal{P}.\text{-}\mathcal{S}.)_\beta$  for every

$$\beta < \frac{1}{n} (\mathcal{S}_{(S^n, g_{S^n})})^{\frac{n}{2}} = \frac{1}{n} \mathcal{S}_{\frac{n}{2}}^n =: \beta^*.$$

**Remark.** A similar local compactness property was the basis for the proof of Rellich's conjecture (on the existence of disc-type surfaces of prescribed constant curvature), achieved independently by Struwe [Str82], [Str86] and Brezis-Coron [BC83].

*Proof of Theorem 3.1.3.* i) Let  $(u_k) \subset H^1(M, g_0)$  be a  $(\mathcal{P}.\text{-}\mathcal{S}.)_\beta$ -sequence, satisfying

$$F_{(M, g_0)}(u_k) \rightarrow \beta, \quad dF_{(M, g_0)}(u_k) \rightarrow 0 \quad (k \rightarrow \infty).$$

Computing, with error  $o(1) \rightarrow 0$  ( $k \rightarrow \infty$ ),

$$\begin{aligned} 2\beta + o(1) (\|u_k\|_{H^1} + 1) &= 2F_{(M, g_0)}(u_k) - \langle dF_{(M, g_0)}(u_k), u_k \rangle_{H^{-1} \times H^1} \\ &= \frac{2^* - 2}{2^*} \|u_k\|_{L^{2^*}}^{2^*} \end{aligned}$$

we find

$$\begin{aligned}
\|u_k\|_{H^1}^2 &\leq 2F_{(M,g_0)}(u_k) + (2c(n)^{-1}\|R_{g_0}\|_{L^\infty} + 1) \|u_k\|_{L^2}^2 + \frac{2}{2^*} \|u_k\|_{L^{2^*}}^{2^*} \\
&\leq 2\beta + C\|u_k\|_{L^{2^*}}^2 + \frac{2}{2^*} \|u_k\|_{L^{2^*}}^{2^*} + o(1) \\
&\leq C(\beta) + o(1) (\|u_k\|_{H^1} + 1), \quad k \in \mathbb{N}.
\end{aligned}$$

Thus  $(u_k) \subset H^1(M, g_0)$  is bounded and we may assume that  $u_k \xrightarrow{w} u$  in  $H^1$  and strongly in  $L^p$  for any  $p < 2^*$  as  $k \rightarrow \infty$ .

In particular, for any  $\varphi \in H^1(M, g_0)$  we have

$$\begin{aligned}
o(1) &= \langle dF_{(M,g_0)}(u_k), \varphi \rangle_{H^{-1} \times H^1} \\
&= \int_M ((\nabla u_k, \nabla \varphi)_{G_0} + c(n)^{-1} R_{g_0} u_k \varphi - |u_k|^{2^*-2} u_k \varphi) d\mu_{g_0} \\
&\xrightarrow{(k \rightarrow \infty)} \int_M ((\nabla u, \nabla \varphi)_{g_0} + c(n)^{-1} R_{g_0} u \varphi - |u|^{2^*-2} u \varphi) d\mu_{g_0} \\
&= \langle dF_{(M,g_0)}(u), \varphi \rangle_{H^{-1} \times H^1}
\end{aligned}$$

and  $u$  solves equation (3.1.2) with  $r = 1$ .

Choosing  $\varphi = u$ , moreover, we obtain

$$\begin{aligned}
2F_{(M,g_0)}(u) &= 2F_{(M,g_0)}(u) - \langle dF_{(M,g_0)}(u), u \rangle_{H^{-1} \times H^1} \\
&= \frac{2^* - 2}{2^*} \|u\|_{L^{2^*}}^{2^*} \geq 0.
\end{aligned}$$

- ii) Let  $v_k = u_k - u \xrightarrow{w} 0$  in  $H^1(M, g_0)$  and strongly in  $L^p(M, g_0)$  for any  $p < 2^*$  as  $k \rightarrow \infty$ . Lemmas 1.5.3 and 1.5.4 give

$$\begin{aligned}
F_{(M,g_0)}(u_k) &= F_{(M,g_0)}(u) + F_{(M,g_0)}(v_k) + o(1) \\
&= F_{(M,g_0)}(u) + F_{(M,g_0)}^\infty(v_k) + o(1)
\end{aligned}$$

and

$$\begin{aligned}
F_{(M,g_0)}^\infty(v_k) &= \frac{1}{2} \int_M |\nabla v_k|_{g_0}^2 d\mu_{g_0} - \frac{1}{2^*} \int_M v_k^{2^*} d\mu_{g_0} \\
&\leq F_{(M,g_0)}(u_k) - \underbrace{F_{(M,g_0)}(u)}_{\geq 0} + o(1) \leq \beta + o(1).
\end{aligned} \tag{3.1.3}$$

Similarly, we obtain

$$\begin{aligned}
o(1) &= \langle dF_{(M,g_0)}(u_k), v_k \rangle_{H^{-1} \times H^1} \\
&= \int_M \left( (\nabla u_k, \nabla v_k)_{g_0} + c(n)^{-1} R_{g_0} u_k v_k - |u_k|^{2^*-2} u_k \underbrace{v_k}_{=u_k-u} \right) d\mu_{g_0} \\
&= \|\nabla v_k\|_{L^2}^2 - (\|u_k\|_{L^{2^*}}^{2^*} - \|u\|_{L^{2^*}}^{2^*}) + o(1) \\
&= \|\nabla v_k\|_{L^2}^2 - \|v_k\|_{L^{2^*}}^{2^*} + o(1),
\end{aligned} \tag{3.1.4}$$

and we find

$$F_{(M,g_0)}^\infty(v_k) = \frac{1}{2} \|\nabla v_k\|_{L^2}^2 - \frac{1}{2^*} \|v_k\|_{L^{2^*}}^{2^*} = \underbrace{\left(\frac{1}{2} - \frac{1}{2^*}\right)}_{=\frac{1}{n}} \|v_k\|_{L^{2^*}}^{2^*} + o(1).$$

Together with (3.1.3) this gives the estimate

$$\frac{1}{n} \|v_k\|_{L^{2^*}}^{2^*} \leq c_0 < \beta^* = \frac{1}{n} \mathcal{S}^{\frac{n}{2}}, \quad k \geq k_0;$$

that is, with  $2^* \frac{2}{n} = \frac{4}{n-1} = 2^* - 2$ , the bound

$$\|v_k\|_{L^{2^*}}^{2^*-2} \leq c_1 < \mathcal{S}, \quad k \geq k_0. \quad (3.1.5)$$

iii) Finally, for any  $\varphi \in C^\infty(M, g_0)$  compute

$$\begin{aligned} o(1) &= \langle dF_{(M,g_0)}(u_k), v_k \varphi^2 \rangle_{H^{-1} \times H^1} \\ &= \int_M (|\nabla(v_k \varphi)|_{g_0}^2 - |v_k|^{2^*} \varphi^2) d\mu_{g_0} + o(1) \\ &\geq \|\nabla(v_k \varphi)\|_{L^2}^2 - \|v_k\|_{L^{2^*}}^{2^*-2} \|v_k \varphi\|_{L^{2^*}}^2 + o(1), \end{aligned}$$

similar to (3.1.4), also using Hölder's inequality in the last step. With (3.1.5) we then obtain

$$\|\nabla(v_k \varphi)\|_{L^2}^2 \leq c_1 \|v_k \varphi\|_{L^{2^*}}^2 + o(1),$$

where  $c_1 < \mathcal{S}$ . But if  $\text{diam}(\text{supp}(\varphi)) < R_0$  for sufficiently small  $R_0$  the metric  $g_0$  on  $\text{supp}(\varphi)$  is sufficiently close to  $g_{\mathbb{R}^n}$  to yield the bound

$$c_2 \|v_k \varphi\|_{L^{2^*}}^2 \leq \|\nabla(v_k \varphi)\|_{L^2}^2$$

with a constant  $c_2 > c_1$ , and we conclude that  $v_k \varphi \rightarrow 0$  in  $H^1$  for any  $\varphi$ .

Thus, if  $(\varphi_l)_{1 \leq l \leq L} \subset C^\infty(M, g_0)$  is a partition of unity with  $\text{diam}(\text{supp}(\varphi_l)) < R_0$  for each  $l \in \{1, \dots, L\}$ , we conclude that

$$\|v_k\|_{H^1} \leq \sum_{l=1}^L \|v_k \varphi_l\|_{H^1} \rightarrow 0 \quad (k \rightarrow \infty).$$

Thus,  $u_k \rightarrow u$  strongly in  $H^1(M, g_0)$ , as desired. □

**Corollary 3.1.1.** *If  $\mathcal{S}_{(M,g_0)} < \mathcal{S}$ , there exists  $u > 0$  with*

$$\mathcal{S}_{(M,g_0)}(u) = \mathcal{S}_{(M,g_0)},$$

and  $u$  solves (3.1.2).

*Proof.* i) If  $\mathcal{S}_{(M,g_0)} > 0$ , consider the functional  $F_{(M,g_0)}$  and note that for each  $u > 0$  the number

$$\beta(u) := \sup_{s>0} F_{(M,g_0)}(su) = \sup_{s>0} \left( \frac{s^2}{2} E_{(M,g_0)}(u) - \frac{s^{2^*}}{2^*} \|u\|_{L^{2^*}}^{2^*} \right)$$

is achieved for  $s$  with

$$s^{2^*-2} \|u\|_{L^{2^*}}^{2^*} = E_{(M,g_0)}(u),$$

so  $s^{2^*-2} \|u\|_{L^{2^*}}^{2^*-2} = \mathcal{S}_{(M,g_0)}(u)$ , and

$$\begin{aligned} \beta(u) &= s^2 \underbrace{\left( \frac{1}{2} - \frac{1}{2^*} \right)}_{=\frac{1}{n}} E_{(M,g_0)}(u) \\ &= \frac{1}{n} \underbrace{s^2 \|u\|_{L^{2^*}}^{2^*}}_{=(\mathcal{S}_{(M,g_0)}(u))^{\frac{2}{2^*-2}}} \mathcal{S}_{(M,g_0)}(u) \\ &= \frac{1}{n} (\mathcal{S}_{(M,g_0)}(u))^{\frac{n}{2}} \end{aligned}$$

Hence, if  $\mathcal{S}_{(M,g_0)} < \mathcal{S}$  in view of Theorem 3.1.3 we may invoke the mountain-pass lemma, Theorem 2.4.3, with  $0 <^3 \beta < \beta^*$  to obtain the conclusion.

ii) If  $\mathcal{S}_{(M,g_0)} = 0$  a minimizing sequence  $(u_k) \subset H^1(M, g_0)$  for  $E_{(M,g_0)}$  in

$$\Sigma_2 = \{u \in H^1(M, g_0) : \|u\|_{L^2} = 1\}$$

accumulates at  $u \in \Sigma_2$  with

$$0 \geq E_{(M,g_0)}(u) \geq \mathcal{S}_{(M,g_0)} \|u\|_{L^{2^*}}^2 = 0.$$

Hence

$$E_{(M,g_0)}(u) = 0 = \min_{v \in \Sigma_2} E_{(M,g_0)}(v),$$

and  $u$  solves

$$-c(n)\Delta_{g_0}u + R_{g_0}u = \lambda u$$

for some  $\lambda \in \mathbb{R}$ , where

$$\lambda = \lambda \|u\|_{L^2}^2 = c(n)E_{(M,g_0)}(u) = 0.$$

Thus,  $u$  solves (3.1.2) with  $r = 0$ .

Moreover,  $u$  also minimizes  $\mathcal{S}_{(M,g_0)}$  with

$$\mathcal{S}_{(M,g_0)}(u) = 0 = \mathcal{S}_{(M,g_0)}.$$

iii) If  $\mathcal{S}_{(M,g_0)} < 0$ , consider the coercive functional  $G$  on  $H^1(M, g_0)$ , given by

$$G_{(M,g_0)}(u) = \frac{1}{2} E_{(M,g_0)}(u) + \frac{1}{2^*} \|u\|_{L^{2^*}}^{2^*}.$$

By (w.s.l.s.c.) of Dirichlet's integral and  $\|\cdot\|_{L^{2^*}}$ , moreover,  $G_{(M,g_0)}$  is (w.s.l.s.c.) and Theorem 1.1.1 applies to yield  $u \in H^1(M, g_0)$  with

$$G_{(M,g_0)}(u) = \inf_{v \in H^1(M,g_0)} G_{(M,g_0)}(v) < 0,$$

---

<sup>3</sup>by using  $\mathcal{S}_{(M,g_0)} > 0$



satisfying (3.1.2) with  $r = -1$ .

Moreover,  $u$  achieves  $\beta = \inf_{0 \neq v \in H^1} \beta(v)$ , where

$$\begin{aligned} \beta(u) &= \inf_{s>0} G_{(M,g_0)}(su) = \inf_{s>0} \left( \frac{s^2}{2} E_{(M,g_0)}(u) + \frac{s^{2^*}}{2^*} \|u\|_{L^{2^*}}^{2^*} \right) \\ &= -\frac{1}{n} |\mathcal{S}_{(M,g_0)}(u)|^{\frac{n}{2}} \end{aligned}$$

is achieved for  $s$  with

$$s^{2^*-2} \|u\|_{L^{2^*}}^{2^*-2} = -\frac{E_{(M,g_0)}(u)}{\|u\|_{L^{2^*}}^2} = -\mathcal{S}_{(M,g_0)}(u).$$

so  $u$  also minimizes  $\mathcal{S}_{(M,g_0)}(u)$ . □

The existence of a Yamabe metric on any closed  $(M^n, g_0)$ ,  $n \geq 3$ , then follows from the next result.

**Theorem 3.1.4.** (Aubin (1976), Schoen (1984))

Suppose  $(M^n, g_0)$ ,  $n \geq 3$ , is closed and not conformally diffeomorphic to  $(S^n, g_{S^n})$ . Then

$$\mathcal{S}_{(M,g_0)} < \mathcal{S}_{S^n, g_{S^n}} = \mathcal{S}.$$

**Corollary 3.1.2.** On any closed  $(M^n, g_0)$ ,  $n \geq 3$ , there exists a Yamabe metric.

The proof of Theorem 3.1.4 is achieved via the explicit construction of suitable comparison functions by different methods in the cases

- i)  $n \geq 6$ , and  $(M, g_0)$  not locally conformally flat (Aubin [Aub76]),
- ii)  $3 \leq n \leq 5$ , or  $(M, g_0)$  locally conformally flat (Schoen [Sch84]).

We now sketch Schoen's proof in the case that either  $n = 3$  or that  $(M, g_0)$  is locally conformally flat. We may assume that  $\mathcal{S}_{(M,g_0)} > 0$ . The proof uses two key ingredients: the Green's function and the positive mass theorem.

**Green's function:** For any point  $p_0 \in M$  there exists a unique function  $G \in C^\infty(M \setminus \{p\}, g_0)$  with

$$L_{g_0} G = \gamma(n) \delta_{\{p=p_0\}},$$

where  $\gamma(n) \in \mathbb{R}$  is such that  $-\Delta(|x|^{2-n}) = \gamma(n) \delta_{\{x=0\}}$ , and  $G > 0$  since  $\mathcal{S}_{(M,g_0)} > 0$ .

**Idea:** For  $0 < \rho_k \in C^\infty(M, g_0)$  with  $\rho_k dx \xrightarrow[(k \rightarrow \infty)]{w^*} \gamma(n) \delta_{\{p=p_0\}}$  let  $G_k$  solve

$$L_{g_0} G_k = \rho_k \quad \text{on } M.$$

Then

$$\begin{aligned} \mathcal{S}_{(M,g_0)} \|G_k^-\|_{L^{2^*}}^2 &\leq E_{(M,g_0)}(G_k^-) \\ &= \int_M L_{g_0} G_k \cdot G_k^- d\mu_{g_0} \\ &= \int_M \rho_k G_k^- d\mu_{g_0} \leq 0 \end{aligned}$$

implies that  $G_k^- = \min\{G_k, 0\} = 0$ .

Moreover,  $G_k \xrightarrow{(k \rightarrow \infty)} G$  smoothly away from  $p_0$ ; see for instance Widman [Wid67].

**Positive mass theorem** (Schoen-Yau, Witten)

Suppose  $(M^n, g_0)$  is closed and not conformally diffeomorphic to  $(S^n, g_{S^n})$ ,  $n \geq 3$ .

- i) If  $(M, g_0)$  is locally conformally flat, in a suitable conformal chart around any point  $p_0 \in M$  we may introduce Euclidean coordinate so that  $g = g_{\mathbb{R}^n} = (\delta_{ij})$  near  $p_0 = 0$ , and there holds

$$G(x) = |x|^{2-n} + A + \alpha(x), \quad (3.1.6)$$

where  $\alpha$  is smooth and harmonic near  $p_0 = 0$  with  $\alpha(0) = 0$ , and where

$$A > 0. \quad (3.1.7)$$

- ii) If  $n = 3$ , the expansion (3.1.6) holds in geodesic normal coordinates  $x$  near  $p_0 = 0$  with  $\alpha(x) = o(|x|)$  and with

$$A > 0.$$

We refer to [LP87].

*Proof of Theorem 3.1.4 when  $(M, g_0)$  is locally conformally flat.* Fix  $p_0 \in M$  and introduce Euclidean coordinates  $x$  on a ball  $B_R(0)$  around  $p_0 = 0$  so that (3.1.6) holds, where  $A > 0$ . For  $\varepsilon > 0$  let

$$u_\varepsilon * (x) = \frac{[n(n-2)\varepsilon]^{\frac{n-2}{4}}}{[\varepsilon^2 + |x|^2]^{\frac{n-2}{2}}}$$

as in Example 3.1.1. For suitable

$$0 < \varepsilon_0 < \rho_0 < \frac{R_0}{2}$$

choose  $\varepsilon > 0$  so that

$$\varepsilon_0(\rho_0^{2-n} + A) = u_\varepsilon^*(\rho_0) \quad \left( \sim \varepsilon^{\frac{n-2}{2}} \rho_0^{2-n} \right). \quad (3.1.8)$$

Let

$$\psi(x) = \left( 2 - \frac{|x|}{\rho_0} \right)_+$$

and set

$$u(x) = \begin{cases} u_\varepsilon^*(x), & |x| \leq \rho_0, \\ \varepsilon_0(G(x) - \psi(x)\alpha(x)), & \rho_0 \leq |x| \leq 2\rho_0, \\ \varepsilon_0 G(x), & x \in M \setminus B_{2\rho_0}(0). \end{cases}$$

Then  $u$  is continuous and piecewise smooth, so  $u \in H^1$ . Using the equation

$$-\Delta u_\varepsilon^* = (u_\varepsilon^*)^{2^*-1}$$

we compute

$$\mathcal{S} \|u_\varepsilon^*\|_{L^{2^*}}^2 = \|\nabla u_\varepsilon^*\|_{L^2}^2 = \|u_\varepsilon^*\|_{L^{2^*}}^{2^*};$$

so

$$\|u_\varepsilon^*\|_{L^{2^*}}^{2^*-2} = \mathcal{S},$$

and Hölder's inequality yields

$$\begin{aligned} \int_{B_{\rho_0}(0)} |\nabla u|^2 dx &= \int_{B_{\rho_0}(0)} |\nabla u_\varepsilon^*|^2 dx = \int_{B_{\rho_0}(0)} (u_\varepsilon^*)^{2^*-2+2} dx + \int_{\partial B_{\rho_0}(0)} \partial_r u_\varepsilon^* u_\varepsilon^* do \\ &\leq \mathcal{S} \|u\|_{L^{2^*}(M, g_0)}^2 + \int_{\partial B_{\rho_0}(0)} \partial_r u_\varepsilon^* u_\varepsilon^* do. \end{aligned}$$

Moreover, since  $R_{g_0} = 0$  in  $B_{2\rho_0}(0)$ , we have

$$\begin{aligned} &\int_{M \setminus B_{\rho_0}(0)} (|\nabla u|^2 + c(n)^{-1} R_{g_0} u^2) d\mu_{g_0} \\ &= \varepsilon_0^2 \int_{M \setminus B_{\rho_0}(0)} (|\nabla G|^2 + c(n)^{-1} R_{g_0} G^2) d\mu_0 + \varepsilon_0^2 \int_{B_{2\rho_0} \setminus B_{\rho_0}(0)} (|\nabla(\psi\alpha)|^2 - 2\nabla G \cdot \nabla(\psi\alpha)) d\mu_{g_0} \end{aligned}$$

Note that  $|\alpha(x)| \leq C|x|$  implies the bound

$$|\nabla(\psi\alpha)| \leq C \quad \text{in } B_{2\rho_0} \setminus B_{\rho_0}(0).$$

Also observe that  $L_{g_0} G = 0$  in  $M \setminus B_{\rho_0}(0)$  gives

$$\begin{aligned} &\int_{M \setminus B_{\rho_0}(0)} (|\nabla G|^2 + c(n)^{-1} R_{g_0} G^2) d\mu_{g_0} \\ &= \int_{M \setminus B_{\rho_0}(0)} L_{g_0} G \cdot G d\mu_{g_0} - \int_{\partial B_{\rho_0}(0)} \partial_r G \cdot G do \\ &= - \int_{\partial B_{\rho_0}(0)} \partial_r G \cdot G do. \end{aligned}$$

Therefore we find

$$E_{(M, g_0)}(u) \leq \mathcal{S} \|u\|_{L^{2^*}}^2 + \int_{\partial B_{\rho_0}(0)} (u_\varepsilon^* \partial_r u_\varepsilon^* - \varepsilon_0^2 G \partial_r G) do + C\rho_0 \varepsilon_0^2.$$

The Theorem then follows from

**Claim 1.** For suitable  $0 < \varepsilon_0 \ll \rho_0 < \frac{R_0}{2}$  there holds

$$\int_{\partial B_{\rho_0}(0)} (u_\varepsilon^* \partial_r u_\varepsilon^* - \varepsilon_0^2 G \partial_r G) do + C\rho_0 \varepsilon_0^2 < 0.$$

**Proof of Claim 1:** Compute for  $|x| = \rho_0$  and suitable  $0 < \varepsilon_0 \ll \rho_0 \ll R_0$ , using that  $\varepsilon_0 \simeq \varepsilon^{\frac{n-2}{2}}$  by (3.1.8),

$$\begin{aligned}
\partial_r u_\varepsilon^* - \varepsilon_0 \partial_r G &= -(n-2) \left( \underbrace{u_\varepsilon^*(\rho_0)}_{\stackrel{(3.1.8)}{=} \varepsilon_0(\rho_0^{2-n}+A)} \frac{\rho_0}{\varepsilon^2 + \rho_0^2} - \varepsilon_0 \rho_0^{1-n} \right) + \varepsilon_0 \partial_r \alpha \\
&= -(n-2) \varepsilon_0 \rho_0^{-1} \left[ (\rho_0^{2-n} + A) \underbrace{\frac{1}{1 + (\frac{\varepsilon}{\rho_0})^2}}_{\geq 1 - (\frac{\varepsilon}{\rho_0})^2} - \rho_0^{2-n} \right] + \varepsilon_0 \underbrace{\partial_r \alpha}_{\leq C} \\
&\leq -(n-2) \varepsilon_0 \rho_0^{-1} A + C \underbrace{\varepsilon_0 \rho_0^{1-n} \left( \frac{\varepsilon}{\rho_0} \right)^2}_{= \varepsilon_0 \varepsilon^2 \rho_0^{-1-n} \leq C_\varepsilon, \text{ if } 0 < \varepsilon_0 < \rho_0} + C \varepsilon_0 \\
&\leq -\frac{n-2}{2} \varepsilon_0 \rho_0^{-1} A, \quad \text{if } 0 < \rho_0 \ll R_0.
\end{aligned}$$

Thus, with constants  $C, C_1 > 0$  there holds

$$\begin{aligned}
&\int_{\partial B_{\rho_0}(0)} (u_\varepsilon^* \partial_r u_\varepsilon^* - \varepsilon_0^2 G \partial_r G) \, do + C \varepsilon_0^2 \rho_0 \\
&= u_\varepsilon^*(\rho_0) \int_{\partial B_{\rho_0}(0)} (\partial_r u_\varepsilon^* - \varepsilon_0 \partial_r G) \, do - \varepsilon_0^2 \int_{\partial B_{\rho_0}(0)} \alpha G \partial_r G \, do + C \varepsilon_0^2 \rho_0 \\
&\leq -C_1 \varepsilon_0^2 \rho_0^{(2-n)-1+(n-1)} A + C \varepsilon_0^2 \rho_0 \\
&\leq -C_1 \varepsilon_0^2 A + C \varepsilon_0^2 \rho_0 < 0, \quad \text{if } 0 < \rho_0 \ll R_0.
\end{aligned}$$

□

## 3.2 A global compactness result

Surprisingly, in Yamabe-type equations we can also obtain compactness at energies exceeding the threshold for concentration.

We illustrate this with the following model problem. Let  $\Omega \subset \subset \mathbb{R}^n$  be a smoothly bounded domain in  $\mathbb{R}^n$ ,  $n \geq 3$ . For  $\lambda \in \mathbb{R}$  consider the boundary value problem

$$-\Delta u - \lambda u = u|u|^{2^*-2} \quad \text{in } \Omega, \quad (3.2.1)$$

$$u = 0 \quad \text{on } \partial\Omega \quad (3.2.2)$$

with associated energy

$$E_\lambda(u) = \frac{1}{2} \int_\Omega (|\nabla u|^2 - \lambda u^2) \, dx - \frac{1}{2^*} \int_\Omega |u|^{2^*} \, dx, \quad u \in H_0^1(\Omega).$$

We expect there to be an interplay between the solvability of (3.2.1), (3.2.2) and the Sobolev constant  $\mathcal{S}$  which is attained on the solutions  $u > 0$  of the equation

$$-\Delta u = u|u|^{2^*-2} \quad \text{on } \mathbb{R}^n, \quad u \in \dot{H}^1(\mathbb{R}^n). \quad (3.2.3)$$

In fact, the following result holds true, which improves Theorem 3.1.3 beyond the threshold  $\beta^*$ .

**Theorem 3.2.1.** (Struwe (1984))

Let  $(u_k) \subset H_0^1(\Omega)$  be a  $(\mathcal{P}\text{-}\mathcal{S})_\beta$ -sequence for  $E_\lambda$ .

Then there exists a solution  $u^{(0)}$  of (3.2.1), (3.2.2), and either

- i) a subsequence  $u_k \rightarrow u^{(0)}$  strongly in  $H_0^1(\Omega)$ , or
- ii) a subsequence  $u_k \xrightarrow{w} u^{(0)}$  in  $H_0^1(\Omega)$ , and there exist  $I \in \mathbb{N}$  and solutions  $u^{(i)} \in \dot{H}^1(\mathbb{R}^n)$  of (3.2.3) together with sequences  $(x_k^{(i)}) \subset \Omega$ ,  $(r_k^{(i)}) \subset \mathbb{R}_+$  satisfying  $x_k^{(i)} \rightarrow x^{(i)} \in \overline{\Omega}$ ,  $r_k^{(i)} \rightarrow 0$  ( $k \rightarrow \infty$ ) and with

$$\frac{\text{dist}(x_k^{(i)}, \partial\Omega)}{r_k^{(i)}} \xrightarrow{(k \rightarrow \infty)} \infty, \quad 1 \leq i \leq I,$$

such that<sup>4</sup>

$$u_k = u^{(0)} + \sum_{i=1}^I \underbrace{\left(r_k^{(i)}\right)^{\frac{2-n}{2}} u^{(i)} \left(\frac{x - x_k^{(i)}}{r_k^{(i)}}\right)}_{=: u_{r_k^{(i)}, x_k^{(i)}}^{(i)}(x)} + o(1) \quad (3.2.4)$$

where  $o(1) \rightarrow 0$  in  $\dot{H}^1(\mathbb{R}^n)$ , and with

$$E_\lambda(u_k) \rightarrow E_\lambda(u^{(0)}) + \sum_{i=1}^I E_0(u^{(i)}). \quad (3.2.5)$$

- iii) If  $u_k \geq o(1) \rightarrow 0$  in  $H_0^1(\Omega)$ , then  $u^{(0)} \geq 0$  and also  $u^{(i)} > 0$ ,  $1 \leq i \leq I$ .

**Remark 3.2.1.** i) If  $u_k \geq o(1)$  and hence  $u^{(i)} > 0$  we can precisely characterize all energy levels  $\beta \in \mathbb{R}$  where  $(\mathcal{P}\text{-}\mathcal{S})_\beta$  fails.

Indeed, by Gidas-Ni-Nirenberg [GNN79] if  $u^{(i)} > 0$  solves (3.2.3) then up to scaling and translation

$$u^{(i)} \text{ equals } u^*(x) = \frac{[n(n-2)]^{\frac{n-2}{4}}}{[1 + |x|^2]^{\frac{n-2}{2}}},$$

and

$$E_0(u^{(i)}) = E_0(u^*) = \beta^*, \quad 1 \leq i \leq I.$$

Hence,  $(\mathcal{P}\text{-}\mathcal{S})_\beta$  fails iff  $I \geq 1$  iff

$$\beta = E_\lambda(u) + I\beta^*$$

for some solution  $u \in H_0^1(\Omega)$  of (3.2.1), (3.2.2).

- ii) Bahri-Coron [BC87] note that the decomposition (3.2.4) is "orthogonal" in the sense that

$$\inf_{i \neq j} d_k^{(ij)} \rightarrow \infty \quad (k \rightarrow \infty),$$

where

$$d_k^{(ij)} = \frac{|x_k^{(i)} - x_k^{(j)}|}{r_k^{(j)}} + \frac{r_k^{(i)}}{r_k^{(j)}} + \frac{r_k^{(j)}}{r_k^{(i)}}.$$

<sup>4</sup>Extending any  $u \in H_0^1(\Omega)$  by  $u \equiv 0$  in  $\mathbb{R}^n \setminus \Omega$ , we may regard  $H_0^1(\Omega) \hookrightarrow \dot{H}^1(\mathbb{R}^n)$ .

Indeed, if for some  $i < j$  there holds  $d_k^{(ij)} \leq C < \infty$ , with the notation

$$r^{r,x_0}(x) = r^{\frac{n-2}{2}} u(x_0 + rx), \quad r \rightarrow 0, \quad x_0 \in \mathbb{R}^n,$$

we have

$$\begin{aligned} \tilde{u}_k^{(i)}(x) &= (u_k)^{r_k^{(i)}, x_k^{(i)}}(x) \\ &= u^{(i)}(x) + \left( \frac{r_k^{(i)}}{r_k^{(j)}} \right)^{\frac{n-2}{2}} u^{(j)} \left( \frac{x_k^{(i)} - x_k^{(j)}}{r_k^{(j)}} + \frac{r_k^{(i)}}{r_k^{(j)}} x \right) \\ &\quad + (u^{(0)})^{r_k^{(i)}, x_k^{(i)}}(x) + \sum_{\substack{l \neq i \\ l \neq j}} \left( u_{r_k^{(l)}, x_k^{(l)}}^{(l)} \right)^{r_k^{(i)}, x_k^{(i)}}(x) \\ &\xrightarrow{w} \tilde{u}^{(i)} \quad \text{in } H_{loc}^1(\mathbb{R}^n) \text{ as } k \rightarrow \infty, \end{aligned}$$

and for any  $\varphi \in C_c^\infty(\mathbb{R}^n)$  there holds

$$\begin{aligned} \langle dE_0(\tilde{u}^{(i)}), \varphi \rangle_{\dot{H}^{-1} \times \dot{H}^1(\mathbb{R}^n)} &= \lim_{k \rightarrow \infty} \langle dE_0(\tilde{u}_k^{(i)}), \varphi \rangle_{\dot{H}^{-1} \times \dot{H}^1(\mathbb{R}^n)} \\ &= \lim_{k \rightarrow \infty} \left\langle \underbrace{dE_\lambda(u_k)}_{\|\cdot\|_{H^{-1}} \rightarrow 0}, \underbrace{\varphi_{r_k^{(i)}, x_k^{(i)}}}_{=\varphi_k \xrightarrow{w} 0} \right\rangle_{H^{-1} \times H^1(\Omega)} = 0, \end{aligned}$$

but  $\tilde{u}^{(i)} \neq u^{(i)}$  and  $\tilde{u}^{(i)}$  does not satisfy (3.2.3).

- iii) Theorem 3.2.1 and the above remarks i), ii) also hold true for other compact perturbation terms different from  $\lambda u$ ; moreover, it also holds true for the Yamabe energy  $F_{(M, g_0)}$  and gives rise to a bubble-tree decomposition of  $(\mathcal{P}.\text{-}\mathcal{S})_\beta$ -sequences, arising, for instance, as  $u(t_k)$  for a solution  $u = u(t)$  of the Yamabe flow and  $t_k \rightarrow \infty$ , where it is the basic pillar for the proof of unconditional convergence of the flow in Schwetlick-Struwe [SS03], Brendle [Bre05].

The Yamabe flow for conformal metrics  $g = g(t) = u^{\frac{4}{n-2}} g_0$  on a closed  $n$ -manifold  $(M, g_0)$  with unit volume is given by

$$u_t = (r_g - R_g)u = (c(n)\Delta_{g_0}u - R_{g_0}u)u^{2-2^*} + r_g u,$$

where

$$r_g = \int_M R_g \underbrace{u^{2^*}}_{d\mu_g} d\mu_{g_0}$$

so that

$$\frac{d}{dt} \text{vol}(M, g) = \frac{d}{dt} \int_M u^{2^*} dm_{g_0} = 2^* \int_M (r_g - R_g) d\mu = 0.$$

That is,  $u = u(t)$  satisfies

$$u^{2^*-2}u_t = c(n)\Delta_{g_0}u - R_{g_0}u + r_g u^{2^*-1},$$

and multiplying with  $u_t$  gives the energy identity

$$\int_M |u_t|^2 u^{2^*-2} d\mu_{g_0} = -c(n) \frac{d}{dt} E_{(M, g_0)}(u(t)).$$

Since  $E_{(M,g_0)}(u(t)) = \mathcal{S}_{(M,g_0)}(u(t)) \geq \mathcal{S}_{(M,g_0)}$  for all  $t$ , we then obtain the bound

$$\int_0^\infty \int_M |u_t|^2 u^{2^*-2} d\mu_{g_0} \leq \mathcal{S}_{(M,g_0)}(u(0)) - \mathcal{S}_{(M,g_0)}$$

and there exists  $t_k \rightarrow \infty$  such that

$$\int_{M \times \{t_k\}} |u_t|^2 u^{2^*-2} d\mu_{g_0} = \int_{M \times \{t_k\}} |R_g - r_g|^2 u^{2^*} d\mu_{g_0} \rightarrow 0 \quad (k \rightarrow \infty), \quad (3.2.6)$$

which is stronger than the condition

$$\begin{aligned} c(n) \|d\mathcal{S}_{(M,g_0)}(u_k)\|_{H^{-1}} &= \sup_{\|\varphi\|_{H^1} \leq 1} c(n) \langle d\mathcal{S}_{(M,g_0)}(u_k), \varphi \rangle_{H^{-1} \times H^1} \\ &= \sup_{\|\varphi\|_{H^1} \leq 1} \int_M \left( \underbrace{L_{g_0} u_k}_{= R_{g_k} u_k^{2^*-1}} - r_{g_k} r_k^{2^*-1} \right) \varphi d\mu_g \\ &= \sup_{\|\varphi\|_{H^1} \leq 1} \int_M (R_{g_k} - r_{g_k}) u_k^{2^*-1} \varphi d\mu_{g_0} \rightarrow 0 \end{aligned} \quad (3.2.7)$$

for a  $(\mathcal{P} - \mathcal{S})$ -sequence  $u_k > 0$ , with  $g_k = u_k^{\frac{4}{n-2}} g_0$ ,  $k \in \mathbb{N}$ .

For  $u_k > 0$  satisfying the intermediate (between (3.2.6) and (3.2.7)) condition

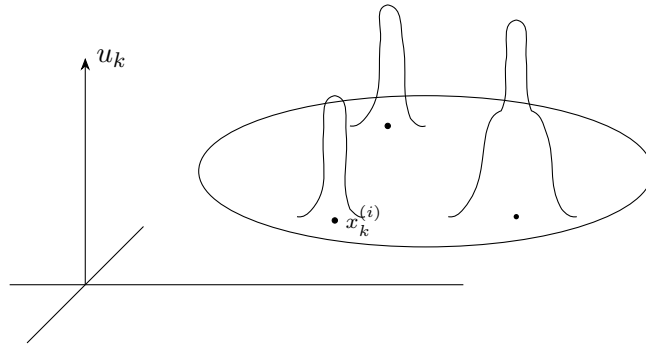
$$\begin{aligned} \sup_{\|\varphi\|_{L^{2^*}} \leq 1} \int_M (R_{g_k} - r_{g_k}) u_k^{2^*-1} \varphi d\mu_{g_0} \\ = \left( \int_M |R_{g_k} - r_{g_k}|^{2^+} u_k^{2^*} d\mu_{g_0} \right)^{\frac{1}{2^+}} \rightarrow 0 \end{aligned}$$

where

$$\frac{1}{2^+} = \frac{1}{2} + \frac{1}{n} \quad \text{with } 2^+ = (2^*)' = \frac{2^*}{2^* - 1},$$

a better approximation estimate, improving (3.2.4) was obtained by Ciraolo-Figalli-Maggi [Cir+18], Figalli - in dimensions  $3 \leq n \leq 5$ .

Can one use this to simplify the proof of convergence of the Yamabe flow?



- iv) The first results on bubble-tree decompositions are due to Sacks-Uhlenbeck [SU81] for harmonic maps of surfaces, and to Wente [Wen80] for disc-type surfaces of constant mean curvature (in the latter case only recovering one bubble). For the harmonic map heat flow,

results analogous to (3.2.4), (3.2.5) were obtained by Struwe [Str85a], Qing [Qin95], Ding-Tian [DT95], Wang [Wan96], etc. and for  $(\mathcal{P}.\text{-}\mathcal{S}.)$ -sequence satisfying the analogue of (3.2.6) by Qing-Tian [QT97].

However, examples by Parker [Par96] show that (3.2.4), (3.2.5) may fail for  $(\mathcal{P}.\text{-}\mathcal{S}.)$ -sequences for the harmonic map problem.

For surfaces of prescribed constant mean curvature the analogue of Theorem 3.2.1 (including (3.2.4), (3.2.5)) was established by Brezis-Coron [BC85], Struwe [Str85b].

- v) If  $(u_k)$  may be of different signs, also  $u^{(i)}$  may be sign-changing. For such  $u = u^{(i)}$  so long (3.2.3) we can estimate

$$E_0(u) > 2\beta^*.$$

*Proof.* Testing (3.2.3) with  $u_{\pm} = \pm \max\{\pm u, 0\} \neq 0$  we find

$$\|\nabla u_{\pm}\|_{L^2}^2 = \|u_{\pm}\|_{L^{2^*}}^{2^*} > \mathcal{S} \|u_{\pm}\|_{L^{2^*}}^2 > 0,$$

so  $E_0(u_{\pm}) > \beta^*$  (as in the proof of Corollary 3.1.1), and

$$E_0(u) = E_0(u_+) + E_0(u_-) > 2\beta^*.$$

□

- vi) Thus, if (3.2.1), (3.2.2) admits only the trivial solution  $u^{(0)} \equiv 0$ , condition  $(\mathcal{P}.\text{-}\mathcal{S}.)_{\beta}$  holds for all  $\beta \in ]\beta^*, 2\beta^*[$  (regardless of the sign of  $(u_k)$ ), and for all  $\beta \notin \beta^*\mathbb{N}$  if  $u_k \geq o(1)$ .
- vii) The decomposition (3.2.4) and the original proof of Theorem 3.2.1 in Struwe [Str84] inspired Bahri-Gérard (1997) to derive an orthogonal (in the sense of iii) nonlinear profile decomposition also for solutions of nonlinear wave equations

$$u_{tt} - \Delta u = u|u|^{p-2} \quad \text{on } \mathbb{R} \times \mathbb{R}^n;$$

however, the decomposition is exhaustive only in a weaker norm; see also Nakanishi-Schlag [NSS11].

The proof of Theorem 3.2.1 may be broken up into

- a perperation step
- a key lemma
- the iteration argument and conclusion.

*Proof of Theorem 3.2.1.* Let  $(u_k) \subset H_0^1(\Omega)$  be a  $(\mathcal{P}.\text{-}\mathcal{S}.)_{\beta}$ -sequence. Computing, with error  $o(1) \rightarrow 0$  ( $k \rightarrow \infty$ ), we obtain

$$\begin{aligned} 2\beta + o(1)(1 + \|u\|_{H^1}) &= 2E_{\lambda}(u_k) - \langle dE_{\lambda}(u_k), u_k \rangle_{H^{-1} \times H_0^1} \\ &= \frac{2^* - 2}{2^*} \|u_k\|_{L^{2^*}}^{2^*} \end{aligned}$$

and find

$$\begin{aligned} \|u_k\|_{H_0^1}^2 &= 2E_{\lambda}(u_k) + |\lambda| \cdot \|u_k\|_{L^2}^2 + \frac{1}{2^*} \|u_k\|_{L^{2^*}}^{2^*} \\ &\leq C(\beta) + o(1) \left(1 + \|u_k\|_{H_0^1}\right), \end{aligned}$$



similar to the proof of Theorem 3.1.3.

Thus,  $(u_k) \subset H_0^1(\Omega)$  is bounded, and we may assume that  $u_k \xrightarrow{w} u^{(0)}$  in  $H_0^1(\Omega)$  and strongly in  $L^p(\Omega)$ ,  $p < 2^*$ , where for any  $\varphi \in H_0^1(\Omega)$  we have

$$\langle dE_\lambda(u^{(0)}), \varphi \rangle_{H^{-1} \times H_0^1} = \lim_{k \rightarrow \infty} \langle dE_\lambda(u_k), \varphi \rangle_{H^{-1} \times H_0^1} = 0,$$

and  $u^{(0)}$  solves (3.2.1), (3.2.2). Moreover, again similar to the proof of Theorem 3.1.3 we have

$$\begin{aligned} 2E_\lambda(u^{(0)}) &= 2E_\lambda(u^{(0)}) - \langle dE_\lambda(u^{(0)}), u^{(0)} \rangle_{H^{-1} \times H_0^1} \\ &= \frac{2^* - 2}{2^*} \|u^{(0)}\|_{L^{2^*}}^{2^*} \geq 0. \end{aligned}$$

Letting  $v_k := u_k - u^{(0)} \xrightarrow{w} 0$  in  $H_0^1(\Omega)$  then by Lemmas 1.5.3 and 1.5.4 we have

$$E_0(v_k) = E_\lambda(u_k) - E_\lambda(u^{(0)}) + o(1) \leq \beta + o(1).$$

The following result is basic for the iterative argument.

**Claim 1.**  $dE_0(v_k) \xrightarrow{(k \rightarrow \infty)} 0$  in  $H^{-1}(\Omega)$ .

*Proof.* For any  $\varphi \in H_0^1(\Omega)$  we have

$$\begin{aligned} \langle dE_\lambda(v_k), \varphi \rangle_{H^{-1} \times H_0^1} &= \langle dE_\lambda(\varphi_k), \varphi \rangle_{H^{-1} \times H_0^1} - \langle dE_\lambda(u^{(0)}), \varphi \rangle_{H^{-1} \times H_0^1} \\ &\quad + \int_\Omega (u_k^{2^*-1} - v_k^{2^*-1} - (u^{(0)})^{2^*-1}) \varphi \, dx, \end{aligned}$$

where  $u^{2^*-1} := u|u|^{2^*-2}$  for brevity.

But by Vitali's theorem we have

$$\begin{aligned} u_k^{2^*-1} - v_k^{2^*-1} &= \int_0^1 \frac{d}{dt} (u_k - u^{(0)} + tu^{(0)})^{2^*-1} dt \\ &= (2^* - 1) \int_0^1 u^{(0)} |u_k - u^{(0)} + tu^{(0)}|^{2^*-2} dt \\ &\xrightarrow{(k \rightarrow \infty)} (u^{(0)})^{2^*-1} \quad \text{in } L^{2^+}(\Omega), \end{aligned}$$

where  $2^+ = (2^*)'$  as above, and our claim follows.  $\square$

For the iterative argument we then have the following key lemma.

**Lemma 3.2.1.** *Let  $v_k \xrightarrow{w} 0$  in  $H_0^1(\Omega)$  be a  $(\mathcal{P}.\text{-}\mathcal{S}.)_\beta$ -sequence for  $E_0$ . Then either*

- i)  $\beta < \beta^* = \frac{1}{n} \mathcal{S}^{\frac{n}{2}}$ , and  $\|v_k\|_{H^1} \rightarrow 0$  ( $k \rightarrow \infty$ ), or
- ii) *there exists a subsequence  $(v_k)$  and  $(x_k) \subset \Omega$ ,  $r_k \downarrow 0$  ( $k \rightarrow \infty$ ), a solution  $0 \neq v^{(0)} \in \dot{H}^1(\mathbb{R}^n)$  for (3.2.3), and a Palais-Smale sequence  $w_k \xrightarrow{w} 0$  in  $H_0^1(\Omega)$  for  $E_0$  such that with error  $o(1) \rightarrow 0$  in  $\dot{H}^1(\mathbb{R}^n)$  there holds*

$$v_k = w_k + r_k^{\frac{2-n}{2}} v^{(0)} \left( \frac{x - x_k}{r_k} \right) + o(1),$$

and

$$E_0(v_k) = E_0(w_k) + E_0(v^{(0)}) + o(1).$$

Moreover, we have

$$\frac{\text{dist}(x_k, \partial\Omega)}{r_k} \rightarrow \infty \quad (k \rightarrow \infty),$$

and  $w_k \geq o(1)$  whenever  $v_k \geq o(1) \rightarrow 0$  in  $H_0^1(\Omega)$ .

*Proof.* i) If  $\beta < \beta^*$  the proof of Theorem 3.1.3 gives convergence  $v_k \xrightarrow{(k \rightarrow \infty)} 0$  in  $H_0^1(\Omega)$ .

ii) If  $\beta \geq \beta^*$ , computing

$$\frac{1}{n} \|\nabla v_k\|_2^2 = E_0(v_k) - \frac{1}{2^*} \langle dE_0(v_k), v_k \rangle_{H^{-1} \times H_0^1} \xrightarrow{(k \rightarrow \infty)} \beta,$$

we have

$$\|\nabla v_k\|_{L^2}^2 \rightarrow n\beta \geq n\beta^* = \mathcal{S}^{\frac{n}{2}} \quad (k \rightarrow \infty).$$

**Normalization:** Fix  $L \in \mathbb{N}$  so that  $B_2(0)$  can be covered by  $L$  balls of radius 1. Choose  $x_k \in \overline{\Omega}$ ,  $r_k > 0$  such that

$$\sup_{x_0} \int_{B_{r_k}(x_0)} |\nabla v_k|^2 dx = \int_{B_{r_k}(x_k)} |\nabla v_k|^2 dx = \frac{\mathcal{S}^{\frac{n}{2}}}{2L}.$$

Note that by (3.2.7) this is possible, and  $r_k < \text{diam}(\Omega)$ . Scale

$$\tilde{v}_k(x) = (v_k)^{r_k, x_k}(x) = r_k^{\frac{n-2}{2}} v_k(x_k + r_k x) \in \dot{H}^1(\mathbb{R}^n)$$

with now

$$\sup_{x_0} \int_{B_1(x_0)} |\nabla \tilde{v}_k|^2 dx = \int_{B_1(0)} |\nabla \tilde{v}_k|^2 dx = \frac{\mathcal{S}^{\frac{n}{2}}}{2L}. \quad (3.2.8)$$

Let

$$\Omega_k = \{x \in \mathbb{R}^n : x_k + r_k x \in \Omega\}, \quad k \in \mathbb{N}.$$

We may assume that  $(\Omega_k)$  exhausts a limit domain  $\Omega_\infty$ , where either  $\Omega_\infty = \mathbb{R}^n$  or  $\Omega_\infty$  is a half-space.

In view of

$$\begin{aligned} \|\nabla \tilde{v}_k\|_{L^2(\Omega_k)}^2 &= \|\nabla v_k\|_{L^2(\Omega)}^2 \rightarrow n\beta < \infty, \\ \|dE_0(\tilde{v}_k)\|_{H^{-1}(\Omega_k)} &= \|dE_0(v_k)\|_{H^{-1}(\Omega)} \rightarrow 0 \end{aligned}$$

we may assume

$$\tilde{v}_k \xrightarrow{w} v^{(0)} \quad \text{in } \dot{H}^1(\mathbb{R}^n),$$

where  $v^{(0)} \in \dot{H}^1(\Omega_\infty) = \|\cdot\|_{\dot{H}^1} - \text{clos}(C_c^\infty(\Omega_\infty))$  solves (3.2.3) on  $\Omega_\infty$ .

Since any  $\varphi \in C_c^\infty(\Omega_\infty)$  also belongs to  $C_c^\infty(\Omega_k)$  for  $k \geq k_0(\varphi)$ , we may then also choose functions  $\tilde{v}_k^{(0)} \in H_0^1(\Omega_k)$  such that

$$\tilde{v}_k^{(0)} \xrightarrow{(k \rightarrow \infty)} v^{(0)} \quad \text{strongly in } \dot{H}^1(\mathbb{R}^n).$$

**Claim 1:**  $\tilde{v}_k \xrightarrow{(k \rightarrow \infty)} v^{(0)}$  strongly locally in  $H^1$ .

*Proof.* It suffices to show

$$\forall x_0 \in \mathbb{R}^n : \|\tilde{v}_k - \tilde{v}_k^{(0)}\|_{H^1(B_1(x_0))} \xrightarrow{(k \rightarrow \infty)} 0.$$

Fix  $x_0 \in \mathbb{R}^n$  and for  $r > 0$  let  $B_r := B_r(x_0)$ . Choose cut-off functions  $\varphi_i \in C_c^\infty(\mathbb{R}^n)$  with  $0 \leq \varphi_i \leq 1$ ,  $i = 1, 2$ , and such that

$$\begin{aligned} \varphi_1 &\equiv 1 \text{ in } B_1, \varphi_1 \equiv 0 \text{ outside } B_{\frac{3}{2}} \\ \varphi_2 &\equiv 1 \text{ in } B_{\frac{3}{2}}, \varphi_2 \equiv 0 \text{ outside } B_2. \end{aligned}$$

Let

$$\tilde{w}_k^{(i)} = (\tilde{v}_k - \tilde{v}_k^{(0)}) \varphi_i \in H_0^1(\Omega_k) \subset \dot{H}^1(\mathbb{R}^n),$$

with

$$\|\tilde{w}_k^{(i)}\|_{H^1} \leq C \|\tilde{v}_k - \tilde{v}_k^{(0)}\|_{H^1(B_2)} \leq C,$$

uniformly in  $k, i = 1, 2$ . Since  $\tilde{v}_k \xrightarrow{w} v^{(0)}$  in  $\dot{H}^1$ , and thus  $\tilde{v}_k - \tilde{v}_k^{(0)} \xrightarrow{w} 0$  in  $\dot{H}^1$ , we also have

$$\tilde{w}_k^{(i)} \xrightarrow{w} 0 \quad \text{in } \dot{H}^1(\mathbb{R}^n)$$

and strongly in  $L^p(B_2)$  for any  $p < 2^*$ .

With error  $o(1) \rightarrow 0$  ( $k \rightarrow \infty$ ), from  $dE_o(v^{(0)}) = 0$ ,

$$\begin{aligned} \|dE_0(\tilde{v}_k)\|_{H^{-1}(\Omega_k)} &= \|dE_0(v_k)\|_{H^{-1}(\Omega)} \xrightarrow{(k \rightarrow \infty)} 0, \\ \|dE_0(\tilde{v}_k^{(0)})\|_{H^{-1}(\Omega_k)} &\rightarrow \|dE_0(v^{(0)})\|_{H^{-1}(\Omega_\infty)} = 0 \end{aligned}$$

We now have

$$\begin{aligned} o(1) &= \left\langle dE_0(\tilde{v}_k) - dE_0(\tilde{v}_k^{(0)}), \overbrace{\tilde{w}_k^{(1)} \varphi_1}^{=(\tilde{v}_k - \tilde{v}_k^{(0)}) \varphi_1^2} \right\rangle_{H^{-1} \times H_0^1(\Omega_k)} \\ &= \int_{\Omega} \left( |\nabla(\tilde{v}_k - \tilde{v}_k^{(0)})|^2 - |\tilde{v}_k - \tilde{v}_k^{(0)}|^{2^*} \right) \varphi_1^2 dx + o(1) \\ &= \int_{\Omega} \left( |\nabla \tilde{w}_k^{(1)}|^2 - |\tilde{w}_k^{(2)}|^{2^*-2} |\tilde{w}_k^{(1)}|^2 \right) dx + o(1) \\ &\geq \mathcal{S} \|\tilde{w}_k^{(1)}\|_{L^{2^*}}^2 - \|\tilde{w}_k^{(2)}\|_{L^{2^*}}^{2^*-2} \|\tilde{w}_k^{(1)}\|_{L^{2^*}}^2 + o(1) \end{aligned} \tag{3.2.9}$$

similar to the preparation step.

But by (3.2.8) we can bound

$$\begin{aligned} \mathcal{S} \|\tilde{w}_k^{(2)}\|_{L^{2^*}}^2 &\leq \|\nabla \tilde{w}_k^{(2)}\|_{L^2}^2 = \int_{\mathbb{R}^n} |\nabla(\tilde{v}_k - \tilde{v}_k^{(0)})|^2 \varphi_2^2 dx + o(1) \\ &= \int_{\mathbb{R}^n} \left( |\nabla \tilde{v}_k|^2 - |\nabla \tilde{v}_k^{(0)}|^2 \right) \varphi_2^2 dx + o(1) \leq \int_{B_2} |\nabla \tilde{v}_k|^2 dx + o(1) \\ &\leq L \sup_{\tilde{x}} \int_{B_1(\tilde{x})} |\nabla \tilde{v}_k|^2 dx + o(1) \leq \frac{1}{2} \mathcal{S}^{\frac{n}{2}} + o(1) \end{aligned}$$

and

$$\begin{aligned} \|\tilde{w}\|_k^{(2)} \|_{L^{2^*}}^{2^*-2} &\leq \left( \frac{1}{2} \mathcal{S}^{\frac{n-2}{2}} \right)^{\frac{2^*-2}{2}} + o(1) \\ &= \left( \frac{1}{2} \right)^{\frac{2}{n-2}} \mathcal{S} + o(1) \leq c_0 \mathcal{S} \end{aligned}$$

for any  $c_0 \in \left[ \left( \frac{1}{2} \right)^{\frac{2}{n-2}}, 1 \right]$  and  $k \geq k_0(c_0)$ .

Hence from (3.2.9) we conclude that  $\|\tilde{w}_k^{(1)}\|_{L^{2^*}} \rightarrow 0$  and then  $\|\tilde{w}_k^{(1)}\|_{\dot{H}^1(\mathbb{R}^n)} \rightarrow 0$ , proving our claim.  $\square$

Claim 1 and (3.2.8) now yield  $v^{(0)} \not\equiv 0$ .

Since  $v^{(0)} \in \dot{H}_0^1(\Omega_\infty)$  solves (3.2.3), Pohožaev's Theorem 1.4.2 now implies that  $\Omega_\infty = \mathbb{R}^n$ , so

$$r_k^{-1} \text{dist}(x_k, \partial\Omega) \rightarrow \infty \quad (k \rightarrow \infty).$$

Moreover, letting  $\tilde{w}_k = \tilde{v}_k - \tilde{v}_k^{(0)}$  and computing

$$\begin{aligned} \|\mathrm{d}E_0(\tilde{w}_k)\|_{H^{-1}(\Omega_k)} &= \sup_{\|\varphi\|_{H_0^1} \leq 1} \left\langle \underbrace{\mathrm{d}E_0(\tilde{v}_k)}_{\rightarrow 0} - \underbrace{\mathrm{d}E_0(\tilde{v}_k^{(0)})}_{\rightarrow 0}, \varphi \right\rangle_{H^{-1} \times H_0^1(\Omega_k)} \\ &\quad + \int_{\Omega_k} \left( \tilde{v}_k^{2^*-1} - \tilde{v}_k^{(0)2^*-1} - \tilde{w}_k^{2^*-1} \right) \varphi \, dx + o(1) \\ &\rightarrow 0 \quad (k \rightarrow \infty) \end{aligned}$$

as in the preparation step, upon scaling back to  $\Omega$  we see that  $w_k = v_k - v_{r_k, x_k}^{(0)} + o(1) \in H_0^1(\Omega_k)$  is a Palais-Smale sequence with energy

$$E_0(w_k) = E_0(v_k) - \underbrace{E_0(v_{r_k, x_k}^{(0)})}_{=E_0(v^{(0)})} + o(1).$$

**Positivity.** Suppose that  $v_k \geq o(1) \rightarrow 0$  in  $H_0^1(\Omega)$ . Then  $\tilde{v}_k \geq o(1) \rightarrow 0$  in  $H_0^1(\Omega_k)$  and Claim 1 gives

$$v^{(0)} = \lim_{k \rightarrow \infty} \tilde{v}_k \geq 0$$

locally (hence everywhere) together with

$$\tilde{w}_k = \tilde{v}_k - v^{(0)} \geq o(1) \rightarrow 0 \quad \text{in } H_0^1(\Omega_k)$$

because  $v(0) \in \dot{H}^1(\mathbb{R}^n)$ . Scaling back, we find

$$w_k \geq o(1) \rightarrow 0 \quad \text{in } H_0^1(\Omega).$$

$\square$

of Theorem 3.2.1 (completed). The Theorem follows by repeatedly applying Lemma 3.2.1 after the preparation step. Since  $E_0(v^{(0)}) \geq \beta^*$  for  $v^{(0)}$  as in Lemma 3.2.1, the iteration stops after at most  $\lceil \frac{\beta}{\beta^*} \rceil$  steps.  $\square$

### 3.3 Coron's existence result

Theorem 3.2.1 may be applied to obtain solutions  $u > 0$  of the problem

$$-\Delta u = u^{2^*-1} \quad \text{in } \Omega, \quad (3.3.1)$$

$$u = 0 \quad \text{on } \partial\Omega \quad (3.3.2)$$

for suitable domains  $\Omega \subset\subset \mathbb{R}^n$ ,  $n \geq 3$ .

Recall that by Pohožaev's Theorem 1.4.2 on any star-shaped domain the problem (3.3.1), (3.3.2) only admits the trivial solution  $u \equiv 0$ .

On the other hand, when  $\Omega$  is an "annulus"  $\Omega = B_{R_2} \setminus B_{R_1}(0)$  for some  $0 < R_1 < R_2$ , by using compactness of the embedding  $H_{rad}^1(\Omega) \hookrightarrow L^p(\Omega)$  for any  $p < \infty$  as in the remark at the end of Section 1.5.1, we obtain a solution  $u > 0$  to (3.3.1), (3.3.2).

With the help of Theorem 3.2.1 Coron was able to show that solutions  $u > 0$  to (3.3.1), (3.3.2) also exist in a perturbed setting.

**Theorem 3.3.1.** (Coron [Cor84])

Let  $n \geq 3$ . There is  $R > 0$  such that for any  $\Omega \subset\subset \mathbb{R}^n$  satisfying  $0 \notin \Omega$ ,  $B_{R_2} \setminus B_{R_1}(0) \subset \Omega$  for  $0 < R_1 < R_2$  with

$$\frac{R_2}{R_1} \geq 16R^2$$

there exists a solution  $u > 0$  of (3.3.1).

*Proof.* After scaling, we may assume

$$R_1 = \frac{1}{4R} < 1 < 4R = R_2,$$

where  $R \geq 1$  is determined below.

Let

$$\Sigma = \{x \in \mathbb{R}^n : |x| = 1\}.$$

Recall that

$$\mathcal{S}(u_t^\sigma) = \frac{\|\nabla u_t^\sigma\|_{L^2}^2}{\|u_t^\sigma\|_{L^{2^*}}^2} = \mathcal{S}, \quad \forall \sigma \in \Sigma, 0 \leq t < 1.$$

Moreover, as  $t \downarrow 0$

$$u_t^\sigma \rightarrow u_0 = \left( \frac{1}{1 + |x|^2} \right)^{\frac{n-2}{2}} \quad \text{in } \dot{H}^1(\mathbb{R}^n)$$

uniformly in  $\sigma \in \Sigma$ .

Let

$$M = \{u \in H_0^1(\Omega) : \|u\|_{L^{2^*}} = 1\}$$

and set

$$v_t^\sigma = \frac{2^\sigma}{\|w_t^\sigma\|_{L^{2^*}}} \in M, \quad \sigma \in \Sigma, 0 \leq t < 1.$$

Remark 3.2.1 vi) after Theorem 3.2.1 implies the following result.

**Lemma 3.3.1.** Suppose that (3.3.1), (3.3.2) only admit  $u \equiv 0$  as a solution. Then the functional  $\mathcal{S}(u)$  satisfies  $(\mathcal{P}.\text{-}\mathcal{S})_\gamma$  on  $M$  for every  $\gamma \in ]\mathcal{S}, 2^{\frac{2}{n}}\mathcal{S}[$ .

*Proof.* By the proof of Corollary 3.1.1, if  $(v_k) \subset M$  is a  $(\mathcal{P}.\text{-}\mathcal{S}.)_\gamma$ -sequence for  $\mathcal{S}(u)$ , then the sequence

$$u_k = (\mathcal{S}(v_k))^{\frac{1}{2^*-2}} v_k \in H_0^1(\Omega)$$

is a  $(\mathcal{P}.\text{-}\mathcal{S}.)_\beta$ -sequence for

$$E_0(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{2^*} \int_{\Omega} |u|^{2^*} dx,$$

with  $\beta = \frac{1}{n} \gamma^{\frac{n}{2}}$ . By Remark 3.2.1 vi) therefore  $\mathcal{S}(u)$  satisfies  $(\mathcal{P}.\text{-}\mathcal{S}.)_\gamma$  for

$$(n\beta^*)^{\frac{2}{n}} = \mathcal{S} < \gamma < (2n\beta^*)^{\frac{2}{n}} = 2^{\frac{2}{n}} \mathcal{S}.$$

□

Fix a cut-off function  $\varphi \in C_c^\infty(\mathbb{R}^n)$  such that  $0 \leq \varphi \leq 1$  and

$$\varphi(x) = \begin{cases} 1, & \text{if } \frac{1}{2} < |x| < 2, \\ 0, & \text{if } |x| \leq \frac{1}{4} \text{ or } |x| \geq 4. \end{cases}$$

Seak

$$\varphi_R(x) = \begin{cases} \varphi(Rx), & \text{if } |x| < R^{-1}, \\ 1, & \text{if } R^{-1} \leq |x| \leq R, \\ \varphi(\frac{x}{R}), & \text{if } |x| \geq R. \end{cases}$$

For any  $\Omega \subset \subset \mathbb{R}^n$  with  $B_{4R} \setminus B_{\frac{1}{4R}}(0) \subset \Omega$  then we have

$$w_t^\sigma := u_t^\sigma \varphi_R, \quad w_0 := u_0 \varphi_R \in H_0^1(\Omega).$$

**Claim 1.**  $\sup_{\sigma \in \Sigma, 0 \leq t < 1} \mathcal{S}(w_t^\sigma) \rightarrow \mathcal{S} \ (R \rightarrow \infty).$

*Proof.* Estimate

$$\begin{aligned} & \|\nabla(w_t^\sigma - u_t^\sigma)\|_{L^2}^2 \leq \\ & \leq C \int_{\{x: |x| < \frac{1}{2R} \text{ or } |x| > 2R\}} |\nabla u_t^\sigma|^2 dx + \underbrace{CR^{-2} \int_{\{x: 2R < |x| < 4R\}} |u_t^\sigma|^2 dx}_{\leq C \left( \int_{\{x: 2R < |x|\}} |u_t^\sigma|^{2^*} dx \right)^{\frac{2}{2^*}}} + \underbrace{CR^2 \int_{\{x: |x| < \frac{1}{2R}\}} |u_t^\sigma|^2 dx}_{\leq C \left( \int_{\{x: |x| < 1/2R\}} |u_t^\sigma|^{2^*} dx \right)^{\frac{2}{2^*}}} \\ & \longrightarrow 0 \ (R \rightarrow \infty), \end{aligned}$$

uniformly in  $\sigma \in \Sigma, 0 \leq t < 1$ . The claim follows. □

**Lemma 3.3.2.** Suppose (3.3.1) only admits  $u \equiv 0$  as solution. Then for any  $\delta > 0$ , any  $\mathcal{S}_1 < 2^{\frac{2}{n}} \mathcal{S}$  there is a flow  $\Phi \in C^0(M \times [0, 1], M)$  of homeomorphisms  $\Phi(\cdot, t) : M \rightarrow M$  such that

i)  $\Phi(v, z) = v$  if  $t = 0$ , or if  $\mathcal{S}(v) < \mathcal{S} + \frac{\delta}{2}$ ,

ii)  $\Phi(M_{\mathcal{S}_1}, 1) \subset M_{\mathcal{S}+\delta}$ .

*Proof.* Cover  $[\mathcal{S} + \delta, \mathcal{S}_1]$  with finitely many intervals  $]\gamma - \varepsilon, \gamma + \varepsilon[$  as in Theorem 2.4.1 (or Theorem 2.5.6 for flows on  $M$ ), where  $0 < \varepsilon < \bar{\varepsilon} < \frac{\delta}{2}$ , and compose the corresponding flows.  $\square$

For  $v \in M$  define the center of  $v$  to be

$$F(v) := \int_{\Omega} x|v|^{2^*} dx.$$

Let  $\rho > 0$  such that

$$\Omega \cap B_{\rho}(0) = \emptyset,$$

and there exists a continuous nearest neighbor projection

$$\pi : U_{\rho}(\Omega) = \bigcup_{x \in \Omega} B_{\rho}(x) \rightarrow \Omega.$$

**Lemma 3.3.3.**  $\exists \delta > 0 \forall v \in M : \mathcal{S}(v) < \mathcal{S} + \delta \Rightarrow F(v) \in U_{\rho}(\Omega).$

*Proof (indirect).* Let  $(v_k) \subset M$  satisfy

$$\mathcal{S}(v_k) \rightarrow \mathcal{S}, \quad \text{dist}(F(v_k), \Omega) \geq \rho.$$

As in Theorem 1.5.3 there is  $(x_k) \subset \Omega, r_k > 0$  such that a subsequence

$$u_k = (v_k)_{x_k, r_k} \rightarrow \bar{u} \quad \text{in } \dot{H}^1(\mathbb{R}^n),$$

where

$$\mathcal{S}(\bar{u}) = \mathcal{S}.$$

By (3.2.3) then  $|\bar{u}| > 0$ ; hence  $r_k \rightarrow 0$ .

It follows that

$$F(v_k) = F\left(r_k^{\frac{2-n}{2}} u_k \left(\frac{x - x_k}{r_k}\right)\right) = \underbrace{x_k}_{\in \Omega} + o(1).$$

$\square$

*Proof of Theorem 3.3.1 (completed).* Suppose (3.3.1) admits only  $u \equiv 0$  as solution. Choose  $R > 0$  such that

$$\mathcal{S}_1 := \sup_{\sigma \in \Sigma, 0 \leq t < 1} \mathcal{S}(v_t^{\sigma}) = \sup_{\sigma \in \Sigma, 0 \leq t < 1} \mathcal{S}(w_t^{\sigma}) < 2^{\frac{2}{n}} \mathcal{S}.$$

For  $\delta > 0$  given by Lemma 3.3.3 and  $\mathcal{S}_1$  let  $\Phi \in C^0(M \times [0, 1], M)$  be given by Lemma 3.3.2. Then

$$F(\Phi(v_t^{\sigma}, 1)) \in U_{\rho}(\Omega), \quad \forall \sigma \in \Sigma, 0 \leq t < 1.$$

Hence the map

$$h : \Sigma \times [0, 1[ \ni (\sigma, t) \mapsto \pi(F(\Phi(v_t^{\sigma}, 1))) \in \Omega$$

is well-defined and continuous.

Moreover, since

$$\sup_{\sigma \in \Sigma} \mathcal{S}(v_t^{\sigma}) \rightarrow \mathcal{S} \quad (t \uparrow 1),$$

we have

$$\Phi(v_t^{\sigma}, 1) = v_t^{\sigma}$$

for sufficiently large  $t_1 \leq t < 1$ . Since clearly

$$F(v_t^\sigma) \rightarrow \sigma \quad (t \uparrow 1)$$

uniformly in  $\sigma$ , the map  $h$  continuously extends to  $h \in C^0(\Sigma \times [0, 1], \Omega)$  with

$$h(\sigma, 1) = \sigma, \quad \forall \sigma \in \Sigma.$$

Thus,  $h$  defines a contraction of  $\Sigma$  in  $\Omega$ . But this is impossible, since  $0 \notin \Omega$ .

Thus, (3.3.1), (3.3.2) has a solution  $u \neq 0$ . □

**Remark.** By working in the set

$$M_+ = \{u \in M : u \geq 0\}$$

one can obtain a solution  $u > 0$  to (3.3.1).

More generally, one can obtain the following result:

**Theorem 3.3.2.** (Bahri-Coron [BC87])

Let  $n = 3$ , and suppose that  $\Omega$  is not contractible. Then (3.3.1) admits a solution  $u > 0$ .



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